

Null space, Row space, Projectors

Control Theory, Lecture 6

by Sergei Savin

Spring 2021

- Motivating questions
- Four Fundamental Subspaces
- Null space
 - ▶ Definition
 - ▶ Calculation
- Null space projection
- Closest element from a linear subspace
- Orthogonality, definition
- Projection
- Vectors in Null space, Row space
- Row and Null spaces in linear equations

MOTIVATING QUESTIONS

You have a linear operator \mathbf{A} . Try to answer the following questions:

- What are all vectors this operator can produce as outputs?
How to find them?
- Are there two inputs that make it produce the same output?
- Are there inputs that produce zero as an output?
- Are there outputs that cannot be produced?
- What is the smallest vector \mathbf{x} that produces given output \mathbf{y} ?

These questions are directly related to the idea of fundamental subspaces of a linear operator.

FOUR FUNDAMENTAL SUBSPACES

One of the key ideas in the linear algebra is that every linear operator has four fundamental subspaces:

- Null space
- Row space
- Column space
- Left null space

Our goal is to understand them. The usefulness of this understating is enormous.

NULL SPACE

Definition

Consider the following task: find all solutions to the system of equations $\mathbf{A}\mathbf{x} = \mathbf{0}$.

It can be re-formulated as follows: find all elements of the *null space* of \mathbf{A} .

Definition 1

Null space of \mathbf{A} is the set of all vectors \mathbf{x} that \mathbf{A} maps to $\mathbf{0}$

We will denote null space as $\mathcal{N}(\mathbf{A})$. In the literature, it is often denoted as $\ker(\mathbf{A})$ or $\text{null}(\mathbf{A})$.

NULL SPACE

Calculation

Now we can find all solutions to the system of equations $\mathbf{Ax} = \mathbf{0}$ by using functions that generate an orthonormal *basis* in the null space of \mathbf{A} . In MATLAB it is function `null`, in Python/Scipy - `null_space`:

- `N = null(A).`

- `N = scipy.linalg.null_space(A).`

That is it! Space of solutions of $\mathbf{Ax} = \mathbf{0}$ is the span of the columns of \mathbf{N} , and all solutions \mathbf{x}^* can be represented as $\mathbf{x}^* = \mathbf{Nz}$; for any \mathbf{z} we get a unique solution, and for any solution - a unique \mathbf{z} .

NULL SPACE PROJECTION

Local coordinates

Let \mathbf{N} be the orthonormal basis in the null space of matrix \mathbf{A} . Then, if a vector \mathbf{x} lies in the null space of \mathbf{A} , it can be represented as:

$$\mathbf{x} = \mathbf{N}\mathbf{z} \tag{1}$$

where \mathbf{z} are coordinates of \mathbf{x} in the basis \mathbf{N} .

However, there are vector which not only are not lying in the null space of \mathbf{A} , but the closest vector to them in the null space is zero vector.

CLOSEST ELEMENT FROM A LINEAR SUBSPACE

Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Its null space has orthonormal basis $\mathbf{N} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

■ $\begin{bmatrix} -2 \\ 0 \end{bmatrix} = -2\mathbf{N}$, $\begin{bmatrix} 10 \\ 0 \end{bmatrix} = 10\mathbf{N}$, - both are in the null space.

■ for $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ the closest vector in the null space is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

■ for $\mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ the closest vector in the null space is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

ORTHOGONALITY, DEFINITION

Definition

If for a vector \mathbf{x} , the closest vector to it from a linear subspace \mathcal{L} is zero vector, \mathbf{x} is called *orthogonal* to the subspace \mathcal{L} . We denote it as $\mathbf{x} \in \mathcal{L}^\perp$.

Definition

The space of all vectors \mathbf{x} , for which closest vector to them from a linear subspace \mathcal{L} is zero vector, is called *subspace, orthogonal to \mathcal{L}* and is denoted as \mathcal{L}^\perp .

PROJECTION

Part 1

Let \mathbf{L} be an orthonormal basis in a linear subspace \mathcal{L} . Take vector $\mathbf{a} = \mathbf{x} + \mathbf{y}$, where \mathbf{x} lies in the subspace \mathcal{L} , and \mathbf{y} is orthogonal to \mathcal{L} .

Definition

We call such vector \mathbf{x} a *projection* of \mathbf{a} onto subspace \mathcal{L} , and such vector \mathbf{y} a projection of \mathbf{a} onto subspace \mathcal{L}^\perp

Projection of \mathbf{a} onto \mathcal{L} can be found as:

$$\mathbf{x} = \mathbf{L}\mathbf{L}^+ \mathbf{a} \tag{2}$$

Since \mathbf{L} is orthonormal, this is the same as $\mathbf{x} = \mathbf{L}\mathbf{L}^\top \mathbf{a}$

PROJECTION

Part 2

Since $\mathbf{a} = \mathbf{x} + \mathbf{y}$, and $\mathbf{x} = \mathbf{L}\mathbf{L}^+\mathbf{a}$, we can write:

$$\mathbf{a} = \mathbf{L}\mathbf{L}^+\mathbf{a} + \mathbf{y} \quad (3)$$

from which it follows that the projection of \mathbf{a} onto \mathcal{L}^\perp can be found as:

$$\mathbf{y} = (\mathbf{I} - \mathbf{L}\mathbf{L}^+)\mathbf{a} \quad (4)$$

where \mathbf{I} is an identity matrix. Since \mathbf{L} is orthonormal, this is the same as $\mathbf{y} = (\mathbf{I} - \mathbf{L}\mathbf{L}^\top)\mathbf{a}$

ROW SPACE

Definition

Definition

Let \mathcal{N} be null space of \mathbf{A} . Then orthogonal subspace \mathcal{N}^\perp is called *row space* of \mathbf{A} .

Definition

Row space of \mathbf{A} is the space of all inputs to \mathbf{A} that produce non-zero outputs (plus zero vector, which is included in all linear subspaces).

We will denote row space as \mathcal{R} .

VECTORS IN NULL SPACE, ROW SPACE

Given vector \mathbf{x} , matrix \mathbf{A} and its nulls space basis \mathbf{N} , and we check if \mathbf{x} is in the null space of \mathbf{A} .

We do it by noticing that if it is in the null space of \mathbf{A} , it will have zero projection onto the row space of \mathbf{A} . So, the condition is as follows:

$$(\mathbf{I} - \mathbf{N}\mathbf{N}^+)\mathbf{x} = 0 \quad (5)$$

By the same logic, condition for being in the row space is as follows:

$$\mathbf{N}\mathbf{N}^+\mathbf{x} = 0 \quad (6)$$

ROW AND NULL SPACES IN LINEAR EQUATIONS

Part 1

Consider another task: find all solutions to the system of equations $\mathbf{Ax} = \mathbf{y}$.

Assume we have two solutions to the system: \mathbf{x}_1 and \mathbf{x}_2 . We know that $\mathbf{Ax}_1 = \mathbf{Ax}_2 = \mathbf{y}$, hence $\mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$. In other words, the difference between any two solutions lies in the null space of \mathbf{A} .

On the other hand, let \mathbf{x}^* be a solution, and $\mathbf{x}^N \in \mathcal{N}(\mathbf{A})$ be a vector in the null space of \mathbf{A} . Then $\mathbf{x}^* + \mathbf{x}^N$ is also a solution, since $\mathbf{A}(\mathbf{x}^* + \mathbf{x}^N) = \mathbf{Ax}^* + \mathbf{Ax}^N = \mathbf{Ax}^* = \mathbf{y}$.

Therefore, the solution space is given by a single partial solution $\mathbf{x}^p \notin \mathcal{N}(\mathbf{A})$ and the whole null space of \mathbf{A} .

ROW AND NULL SPACES IN LINEAR EQUATIONS

Part 2

There are infinitely many ways to choose \mathbf{x}^p , since if $\mathbf{x}^p \notin \mathcal{N}(\mathbf{A})$, then $(\mathbf{x}^p + \mathbf{x}^N) \notin \mathcal{N}(\mathbf{A})$, if $\mathbf{x}^N \in \mathcal{N}(\mathbf{A})$. However:

Statement 1

The smallest-norm \mathbf{x}^p will lie in the row space of \mathbf{A} .

We can prove it by observing that there can be only one $\mathbf{x}^p \in \mathcal{R}(\mathbf{A})$ and adding to it any vector $\mathbf{x}^N \in \mathcal{N}(\mathbf{A})$ can only increase its magnitude, as \mathbf{x}^p and \mathbf{x}^N are orthogonal.

ROW AND NULL SPACES IN LINEAR EQUATIONS

Part 3

If we have \mathbf{x}^* , which is a solution to $\mathbf{A}\mathbf{x} = \mathbf{y}$, we can find the particular solution $\mathbf{x}^p \in \mathcal{R}(\mathbf{A})$ as a projection:

$$\mathbf{x}^p = (\mathbf{I} - \mathbf{N}\mathbf{N}^+)\mathbf{x}^* \quad (7)$$

where \mathbf{N} is the null space basis for \mathbf{A} . Alternatively, we can simply find it as:

$$\mathbf{x}^p = \mathbf{A}^+\mathbf{y} \quad (8)$$

All solutions to $\mathbf{A}\mathbf{x} = \mathbf{y}$ are then given as:

$$\mathbf{x}^* = \mathbf{A}^+\mathbf{y} + \mathbf{N}\mathbf{z}, \quad \forall \mathbf{z} \quad (9)$$

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.