Introduction, linear system representations

Control Theory, Lecture 1

by Sergei Savin

Spring 2021

Content

- Motivation
- Ordinary differential equations
  - 1st order
  - n-th order
- Linear differential equations
  - 1st order
  - n-th order
  - ...are what we will study
- Changing n-th order ODE to a State-Space form
- Read more

What is control?

The first obvious question is, what is control theory? The easiest strategy to answer this question is to bring examples of systems that you can *learn how to control*:

Figure 1: Drone

Figure 2: Robot arms

But beware, this is not the whole answer!

Why Control?

The second most natural question to ask is - why do we need to study Control Theory? *Why do Computer scientists need Control Theory?*

The easy answer is:

it is very useful in case you will work in robotics, industrial automation, self-driving vehicles, drones, aerospace, etc.

But!

this answer does not tell the main part of the story - what about people who are NOT going to work in the listed areas?

Control as an applied problem

We propose to view Control Theory as not only yet-another-subject. Instead we can try to see Control Theory course as **an application of your combined skills as a CS student**

Control as an applied problem

Skills you will learn and practice

In this course we provide you with learning and practical tasks that require:

- Linear Algebra, Differential Equations, Computational methods
- Dynamical systems, Stability (concept build on top of Theory of Ordinary Differential Equations).
- Simulation of dynamical systems (closely related to computational methods in Differential Equations), as a programming problem.
- Development of experiments in Google Colab, using Python, mathematical libraries, solving concrete, real world-related math-oriented problems.
- Representation (parametrization) of equations as a tool in both mathematical analysis and simulation, software development and problem solving.
- ...and many other things.

...so, why Control?

Control Theory, as given here, is focused on:

1. Giving you challenge to simultaneously learn a new concepts, new general and subject-specific math, and new programming tools.
2. Providing you with clear outcomes in terms of *understanding* and ability to *solve well-defined and meaningful real-world problems*.
3. Being very useful for those who will proceed to work in robotics, automation, self-driving vehicles, drones, etc.

See it as a test case for your abilities as a CS specialist.

Enough for the motivation

Now that we know (kinda) why we do it:

Let's start with the content of the course!

Ordinary differential equations

1st order

Let us remember the normal form of first-order *ordinary differential equations (ODEs):*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where $\mathbf{x} = \mathbf{x}(t)$ is the solution of the equation and t is a free variable.

Definition

We can call this equation (same as any other ODE) a *dynamical system*, and $\mathbf{x}$ is called the *state* of the dynamical system.

Example

$$\dot{x} = -3x^3 - 7 \tag{2}$$

Ordinary differential equations

n-th order

The normal form of an *n-th order* ordinary differential equation is:

$$\mathbf{x}^{(n)} = \mathbf{f}(\mathbf{x}^{(n-1)}, \mathbf{x}^{(n-2)}, ..., \ddot{\mathbf{x}}, \dot{\mathbf{x}}, \mathbf{x}, t) \tag{3}$$

where $\mathbf{x} = \mathbf{x}(t)$ is the solution of the equation. Same as before, it is a *dynamical system*, but this time the set $\{\mathbf{x},\ \dot{\mathbf{x}}\ ...,\ \mathbf{x}^{(n-1)}\}$ is called the *state* of the dynamical system.

Example

$$\ddot{x} = cos(2\dot{\mathbf{x}}) - 10x + 7 \tag{4}$$

Example

$$\begin{cases} \dddot{x}_1 = \dot{x}_1 + x_1 + x_2^2 - 4 \\ \dddot{x}_2 = 10x_1^3 + \ddot{x}_2 \end{cases} \tag{5}$$

Linear differential equations

1st order

Linear ODEs of the first order have normal form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b} \tag{6}$$

Example

$$\begin{cases} \dot{x}_1 = -20x_1 + 7x_2 + 17 \\ \dot{x}_2 = 10.5x_1 - 3x_2 - 5 \end{cases} \tag{7}$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -8 & 5 & 2 \\ 0.5 & -10 & -2 \\ 1 & -1 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 \\ 10 \\ -5 \end{bmatrix} \tag{8}$$

Linear differential equations

n-th order

A single linear ODE of the n-th order are often written in the form:

$$a_n x^{(n)} + a_{(n-1)} x^{(n-1)} + ... + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = b \tag{9}$$

Example

$$12\,\dddot{x} - 3\ddot{x} + 5.5\dot{x} + 2x = 10.5 \tag{10}$$

Example

$$5\ddot{x} - 2\dot{x} + 10x = 2 \tag{11}$$

Linear differential equations

...are what we will study

In this course we will focus entirely on linear dynamical systems. In particular, we will take a good use of the following two forms:

$$a_n x^{(n)} + a_{(n-1)} x^{(n-1)} + ... + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = b \tag{12}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b} \tag{13}$$

the last one is called *state-space representation*.

Good news:

Both of those can be used to express any linear system, hence we can change one into the other.

Changing n-th order ODE to a State-Space form

Consider eq. $\dddot{x} + a_2\ddot{x} + a_1\dot{x} + a_0 x = b$.

Make a substitution: $z_1 = x$, $z_2 = \dot{x}$, $z_3 = \ddot{x}$. Therefore:

$$\begin{cases} \dot{z}_1 = \dot{x} = z_2 \\ \dot{z}_2 = \ddot{x} = z_3 \\ \dot{z}_3 = -a_2\ddot{x} - a_1\dot{x} - a_0 x + b = -a_2 z_3 - a_1 z_2 - a_0 z_1 + b \end{cases} \tag{14}$$

Which can be directly put in the state-space form:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} \tag{15}$$

Read more

- State Space Representations of Linear Physical Systems
  lpsa.swarthmore.edu/Representations/SysRepSS.html
- Transformation: Differential Equation to State Space
  lpsa.swarthmore.edu/.../DE2SS.html

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.