

# Control for systems with explicit constraints

## Control Theory, Lecture 13

by Sergei Savin

Spring 2021

- Explicit and no constraints
- Explicit and implicit constraints
- Examples of systems with constraints
- Typical reasons why explicit constraints arise
- Ways to control systems with explicit constraints

LTI systems we studied before have the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where  $\mathbf{x} \in \mathbb{R}^n$  is the state of the system. This form has no explicit constraints.

Let us introduce one of the forms of a linear dynamical system with explicit constraints:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\lambda \\ \mathbf{G}\dot{\mathbf{x}} = \mathbf{0} \end{cases}$$

where  $\mathbf{G}$  is a constraint matrix,  $\lambda \in \mathbb{R}^k$  is the constraints reaction forces, and  $\mathbf{F}$  is the reaction force Jacobian.

# EXPLICIT AND IMPLICIT CONSTRAINTS

## Example

Consider a two mass system with a spring:

$$\begin{cases} \ddot{x}_1 + \mu\dot{x}_1 + k(x_1 - x_2) = 0 \\ \ddot{x}_2 + \mu\dot{x}_2 + k(x_2 - x_1) = 0 \end{cases}$$

We can add a constraint  $x_2 = 10$ . This implies that  $\ddot{x}_2 = 0$ .

Corresponding system of equations is:

$$\begin{cases} \ddot{x}_1 + \mu\dot{x}_1 + k(x_1 - x_2) = 0 \\ \ddot{x}_2 + \mu\dot{x}_2 + k(x_2 - x_1) = \lambda \\ \ddot{x}_2 = 0 \end{cases}$$

But that is the same as:

$$\ddot{x}_1 + \mu\dot{x}_1 + k(x_1 - 10) = 0$$

Thus we transformed the system with *explicit constraints* into a system with *implicit constraints*

# EXAMPLES OF SYSTEMS WITH CONSTRAINTS



Figure 1: Walking robots



Figure 2: Polishing with industrial arms

# TYPICAL REASONS WHY EXPLICIT CONSTRAINTS ARISE

Explicit constraints are usually not a necessity and not a physical property of the problem. However, they are often encountered in practice. Typical situations when they are encountered are as follows:

- Systems with contact interactions.
- Hybrid systems (two or more different dynamics which switch between one-another).
- Nonholonomic constraints in the dynamics (dynamics of a unicycle, bicycle, etc.).
- Dynamics is more clear and easy to work with when non-minimal representation is used.

# WAYS TO CONTROL SYSTEMS WITH EXPLICIT CONSTRAINTS

There are basic ways to deal with such systems:

- Reduce to a system with implicit constraints and control that system instead.
- Treat reaction forces as a yet another external force.
- Design control law based on the explicit representation of constraints.

Consider equations in the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \quad (1)$$

where  $\mathbf{A}$  is the state matrix,  $\mathbf{B}$  is the control matrix and  $\mathbf{c}$  is the affine term of the affine dynamics model.

For systems with constraints the same linearization takes form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{S}\lambda + \mathbf{c} \\ \mathbf{G}\dot{\mathbf{x}} = 0 \end{cases} \quad (2)$$

where  $\mathbf{S}$  is linearized constraint Jacobian and  $\mathbf{G} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \dot{\mathbf{F}} & \mathbf{F} \end{bmatrix}$ .



# MINIMAL REPRESENTATION

## Part 1

We can observe that constraint  $\mathbf{G}\dot{\mathbf{x}} = 0$  implies that all feasible state velocities  $\dot{\mathbf{x}}$  lie in the null space of  $\mathbf{G}$ . This means that we can introduce a new lower dimensional variable  $\mathbf{z}$  to describe  $\dot{\mathbf{x}}$  (assuming initial value of  $\mathbf{x}$  lies in the column space of  $\mathbf{N}$ ):

$$\mathbf{N}\dot{\mathbf{z}} = \dot{\mathbf{x}} \tag{3}$$

where  $\mathbf{N} = \text{null}(\mathbf{G})$  - orthonormal basis in the null space of  $\mathbf{G}$ .

# MINIMAL REPRESENTATION

## Part 2

Let us re-express dynamics (2) in terms of  $\mathbf{z}$  by multiplying it by  $\mathbf{N}^\top$  on the left:

$$\mathbf{N}^\top \dot{\mathbf{x}} = \mathbf{N}^\top \mathbf{A} \mathbf{x} + \mathbf{N}^\top \mathbf{B} \mathbf{u} + \mathbf{N}^\top \mathbf{S} \lambda + \mathbf{N}^\top \mathbf{c} \quad (4)$$

We can prove that  $\mathbf{N}^\top \mathbf{S} = 0$  for all mechanical systems (for example, by observing that mechanical constraints do not do work) or check that our particular  $\mathbf{S}$  lies in the row space of our  $\mathbf{G}$ .

Noting that  $\dot{\mathbf{z}} = \mathbf{N}^\top \dot{\mathbf{x}}$  and  $\mathbf{x} = \mathbf{N} \mathbf{z}$  we get:

$$\dot{\mathbf{z}} = \mathbf{N}^\top \mathbf{A} \mathbf{N} \mathbf{z} + \mathbf{N}^\top \mathbf{B} \mathbf{u} + \mathbf{N}^\top \mathbf{c} \quad (5)$$

Defining  $\mathbf{A}_N = \mathbf{N}^\top \mathbf{A} \mathbf{N}$ ,  $\mathbf{B}_N = \mathbf{N}^\top \mathbf{B}$  and  $\mathbf{c}_N = \mathbf{N}^\top \mathbf{c}$  we get:

$$\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{B}_N \mathbf{u} + \mathbf{c}_N \quad (6)$$

# MINIMAL REPRESENTATION

## Part 3

Since we achieved that our constrained dynamics is written in the standard LTI form:

$$\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{B}_N \mathbf{u} + \mathbf{c}_N, \quad (7)$$

we can use standard LTI control methods on it, for example finding optimal feedback gains via pole placement or LQR:

$$\mathbf{K}_N = \text{lqr}(\mathbf{A}_N, \mathbf{B}_N, \mathbf{Q}, \mathbf{R}) \quad (8)$$

where  $\mathbf{Q}$  and  $\mathbf{R}$  are matrices defining cost function for the LQR problem.

# INVERSE DYNAMICS

## LTI

For any LTI system, including the LTI form of a constrained system we saw previously, inverse dynamics can be solved precisely by a pseudo-inverse, as long as there exist a solution. The following condition verifies it:

$$(\mathbf{I} - \mathbf{B}\mathbf{B}^+)(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c}) = 0, \quad (9)$$

The condition checks if vector  $(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c})$  lies in the column space of  $\mathbf{B}$ . If it holds, precise solution to inverse kinematics can be found as:

$$\mathbf{u}_{ID} = \mathbf{B}^+(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c}). \quad (10)$$

For a constrained mechanical system we can solve inverse dynamics without the need for linearization. Consider the following dynamics:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{F}^\top \lambda \quad (11)$$

We can represent constraint Jacobian  $\mathbf{F}^\top$  as its QR decomposition:  $\mathbf{F}^\top = \mathbf{Q} \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}$ , where  $\mathbf{Q}^\top \mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$  and  $\mathbf{R}$  is convertible.

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{Q} \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \lambda \quad (12)$$

Let us multiply the equation by  $\mathbf{Q}^\top$ :

$$\mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{Q}^\top \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{Q}^\top \mathbf{T}\mathbf{u} + \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \lambda \quad (13)$$

Introducing switching variables (to divide upper and lower part of the equations)  $\mathbf{S}_1 = [\mathbf{I} \ \mathbf{0}]$  and  $\mathbf{S}_2 = [\mathbf{0} \ \mathbf{I}]$  and multiplying equations by one and the other we get two systems:

$$\begin{cases} \mathbf{S}_1 \mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{Q}^\top \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{S}_1 \mathbf{Q}^\top \mathbf{T}\mathbf{u} + \mathbf{R}\lambda \\ \mathbf{S}_2 \mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{Q}^\top \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{S}_2 \mathbf{Q}^\top \mathbf{T}\mathbf{u} \end{cases} \quad (14)$$

The main advantage we achieved is that now we can calculate both  $\mathbf{u}$  and  $\lambda$

Resulting expression for  $\mathbf{u}$  is:

$$\mathbf{u} = (\mathbf{S}_2 \mathbf{Q}^\top \mathbf{T})^+ \mathbf{S}_2 \mathbf{Q}^\top (\mathbf{H} \ddot{\mathbf{q}} + \mathbf{Q}^\top \mathbf{C} \dot{\mathbf{q}} + \mathbf{g}) \quad (15)$$

Expression for  $\lambda$  is:

$$\lambda = \mathbf{R}^{-1} \mathbf{S}_1 \mathbf{Q}^\top (\mathbf{H} \ddot{\mathbf{q}} + \mathbf{Q}^\top \mathbf{C} \dot{\mathbf{q}} + \mathbf{g} - \mathbf{T} \mathbf{u}) \quad (16)$$

We can notice a pseudo-inverse, implying that the no-residual solution does not have to exist.

We can easily write inverse dynamics as a QP:

$$\begin{array}{ll} \underset{\mathbf{u}, \lambda}{\text{minimize}} & ||\mathbf{u}||, \\ \text{subject to} & \begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{F}^\top \lambda \\ \mathbf{F}\ddot{\mathbf{q}} + \dot{\mathbf{F}}\dot{\mathbf{q}} = 0 \end{cases} \end{array} \quad (17)$$

If there are some constraints or limits on the control input (torque limits, for instance) or the reaction forces are restricted (by friction cones, for instance), those can be directly added.



- Mason, S., Righetti, L. and Schaal, S., 2014, November. Full dynamics LQR control of a humanoid robot: An experimental study on balancing and squatting. In 2014 IEEE-RAS International Conference on Humanoid Robots (pp. 374-379). IEEE.
- Mason, S., Rotella, N., Schaal, S. and Righetti, L., 2016, November. Balancing and walking using full dynamics LQR control with contact constraints. In 2016 IEEE-RAS 16th International Conference on Humanoid Robots (Humanoids) (pp. 63-68). IEEE. - [arxiv.org/pdf/1701.08179](https://arxiv.org/pdf/1701.08179)
- Mistry, M., Buchli, J. and Schaal, S., 2010, May. Inverse dynamics control of floating base systems using orthogonal decomposition. In 2010 IEEE international conference on robotics and automation (pp. 3406-3412). IEEE. - [cite-seerx.ist.psu.edu/viewdoc/download?doi=10.1.1.212.3601&rep=rep1](https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.212.3601&rep=rep1)

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Control-Theory-Slides-Spring-2021](https://github.com/SergeiSa/Control-Theory-Slides-Spring-2021)



Check Moodle for additional links, videos, textbook suggestions.