Lyapunov Theory, Lyapunov equations

Control Theory, Lecture 9

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Content

Asymptotic stability criteria

Autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is assymptotically stable, if there exists a scalar function $V = V(\mathbf{x}) > 0$, whose time derivative is negative $\dot{V}(\mathbf{x}) < 0$, except $V(\mathbf{0}) = 0$, $\dot{V}(\mathbf{0}) = 0$.

Marginal stability criteria

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is stable in the sense of Lyapunov, $\exists V = V(\mathbf{x}) \geq 0$, $\dot{V}(\mathbf{x}) \leq 0$.

Definition

Function $V = V(\mathbf{x}) > 0$ in this case is called *Lyapunov function*.

This is not the only type of stability as you remember, you are invited to study criteria for other stability types on your own.

Take dynamical system $\dot{x} = -x$.

We propose a *Lyapunov function candidate* $V(x) = x^2 \geq 0$. Let's find its derivative:

$$\dot{V}(x) = \frac{\partial V}{\partial x}(-x) = 2x(-x) = -x^2 \leq 0 \qquad (1)$$

This satisfies the Lyapunov criteria, so the system is stable. It is in fact asymptotically stable, because $V(x) \neq 0$ if $x \neq 0$.

Consider oscillator $\ddot{q} = f(q, \dot{q}) = -\dot{q}$.

We propose a *Lyapunov function candidate* $V(q, \dot{q}) = T(q, \dot{q}) = \frac{1}{2}\dot{q}^2 \geq 0$, where $T(q, \dot{q})$ is kinetic energy of the system. Let's find its derivative:

$$\dot{V}(q, \dot{q}) = \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}} f(q, \dot{q}) = \dot{q}(-\dot{q}) = -\dot{q}^2 \leq 0 \tag{2}$$

This satisfies the Lyapunov criteria, so the system is stable. It is in fact not proven to be asymptotically stable, because $V(q, \dot{q}) = 0$ for any q as long as $\dot{q} = 0$, and $\dot{V}(q, \dot{q}) = 0$ for any q as long as $\dot{q} = 0$.

Example 3

Consider pendulum $\ddot{q} = f(q, \dot{q}) = -\dot{q} - sin(q)$.

We propose a *Lyapunov function candidate* $V(q, \dot{q}) = E(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + 1 - cos(q) \geq 0$, where $E(q, \dot{q})$ is total energy of the system. Let's find its derivative:

$$\dot{V}(q, \dot{q}) = \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}} f(q, \dot{q}) = \dot{q} sin(q) + \dot{q}(-\dot{q} - sin(q)) = -\dot{q}^2 \leq 0 \tag{3}$$

This satisfies the Lyapunov criteria, so the system is stable. It is in fact asymptotically stable, because $V(q, \dot{q}) \neq 0$ if $q \neq 0$ and $\dot{q} \neq 0$.

Linear case

Part 1

As you saw, Lyapunov method allows you to deal with nonlinear systems, as well as linear ones. But for linear ones there are additional properties we can use.

Observation 1

For a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ we can always pick Lyapunov function candidate in the form $V = \mathbf{x}^\top \mathbf{S} \mathbf{x} \geq 0$, where $\mathbf{S}$ is a positive semidefinite matrix.

Next slides will shows where this leads us.

Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and $V = \mathbf{x}^\top \mathbf{S} \mathbf{x} \geq 0$, let's find its derivative:

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^\top \mathbf{S} \mathbf{x} + \mathbf{x}^\top \mathbf{S} \dot{\mathbf{x}} \tag{4}$$

$$\dot{V}(\mathbf{x}) = (\mathbf{A}\mathbf{x})^\top \mathbf{S} \mathbf{x} + \mathbf{x}^\top \mathbf{S} \mathbf{A} \mathbf{x} = \mathbf{x}^\top (\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A}) \mathbf{x} \tag{5}$$

Notice that $\dot{V}(x)$ should be negative for all $\mathbf{x}$ for the system to be stable, meaning that $\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A}$ should be negative semidefinite. A more strict form of this requirement is *Lyapunov equation*:

$$\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} = -\mathbf{Q} \tag{6}$$

where $\mathbf{Q}$ is a positive-definite matrix.

Marginal stability criteria, discrete case

Given $\mathbf{x}_{i+1} = \mathbf{f}(\mathbf{x}_i)$, if $V(\mathbf{x}_i) \geq 0$, and $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) \leq 0$, the system is stable.

Same as before, for linear systems we will be choosing *positive semidefinite quadratic forms* as Lyapunov function candidates.

Consider dynamics $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$ and $V = \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i \geq 0$, let's find $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i)$:

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) = (\mathbf{A}\mathbf{x}_i)^\top \mathbf{S}\mathbf{A}\mathbf{x}_i - \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i \tag{7}$$

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) = \mathbf{x}_i^\top (\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S})\mathbf{x}_i \tag{8}$$

Notice that $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i)$ should be negative for all $\mathbf{x}_i$ for the system to be stable, meaning that $\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S}$ should be negative semidefinite. A more strict form of this requirement is *Discrete Lyapunov equation*:

$$\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S} = -\mathbf{Q} \tag{9}$$

where $\mathbf{Q}$ is a positive-definite matrix.

In practice, you can easily use Lyapunov equations for stability verification. Python and MATLAB have built-in functionality to solve it:

- scipy: `linalg.solve_continuous_lyapunov(A, Q)`
- MATLAB: `lyap(A,Q)`

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.