

# Hamilton-Jacobi-Bellman eq., Riccati eq., Linear Quadratic Regulator

Control Theory, Lecture 9

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Spring 2021

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# HAMILTON-JACOBI-BELLMAN EQUATION

## Definitions

Let us define dynamics:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \quad (1)$$

with initial conditions  $\mathbf{x}(0)$ .

Additionally we define *control policy* as:

$$\mathbf{u} = \pi(\mathbf{x}, t) \quad (2)$$

To connect with the previous ways we talked about control, we can say that choosing different control gains and different feed-forward control amounts to choosing a different control policy.

# HAMILTON-JACOBI-BELLMAN EQUATION

## Cost, optimal cost

Let  $J$  be an additive cost function:

$$J(\mathbf{x}_0, \pi(\mathbf{x}, t)) = \int_0^\infty g(\mathbf{x}, \mathbf{u}) dt \quad (3)$$

where  $g(\mathbf{x}, \mathbf{u})$  is instantaneous cost and  $\mathbf{x}_0 = \mathbf{x}(0)$  is the initial conditions. Notice that  $J$  depends on  $\mathbf{x}_0$  rather than  $\mathbf{x}(t)$ , since initial conditions and control policy completely define the trajectory of the system  $\mathbf{x}(t)$ .

Let  $J^*$  be the optimal (lowest possible) cost. In other words:

$$J^*(\mathbf{x}_0) = \inf_{\pi} J(\mathbf{x}_0, \pi(\mathbf{x}, t)) \quad (4)$$

Optimal cost is attained when optimal policy is attained:

$$\pi = \pi^*(\mathbf{x}, t)$$

# HAMILTON-JACOBI-BELLMAN EQUATION

## Differentiating optimal cost

Since  $J^*(\mathbf{x}_0)$  does not depend on  $t$ , its full derivative is zero:

$$\frac{dJ^*(\mathbf{x}_0)}{dt} = 0 \quad (5)$$

At the same time, we can expand the full derivative as follows:

$$\frac{dJ^*}{dt} = \frac{\partial J^*}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial J^*}{\partial t} = 0 \quad (6)$$

Observe that  $\frac{\partial J^*}{\partial t} = g(\mathbf{x}, \mathbf{u})$ , and  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ . Therefore:

$$\frac{\partial J^*}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) + g(\mathbf{x}, \mathbf{u}) = 0 \quad (7)$$

With this, we can formulate *Hamilton-Jacobi-Bellman equation* (HJB):

$$\min_{\mathbf{u}} \left[ g(\mathbf{x}, \mathbf{u}) + \frac{\partial J^*}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) \right] = 0 \quad (8)$$

This can be loosely interpreted as follows: the value in square brackets is  $\dot{J}(\mathbf{x}_0, \pi)$ , which is equal to 0 when  $\pi = \pi^*(\mathbf{x}, t)$ , and is positive otherwise (in the small vicinity of  $\pi^*$ ), as  $J(\mathbf{x}_0, \pi^*)$  is smaller than any  $J(\mathbf{x}_0, \pi)$ ,  $\pi^* \neq \pi$ .

We can find control that delivers minimum to the function (8):

$$u^* = \arg \min_{\mathbf{u}} \left[ g(\mathbf{x}, \mathbf{u}) + \frac{\partial J^*}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) \right] \quad (9)$$

For LTI, dynamics is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (10)$$

We can choose quadratic cost:

$$g(\mathbf{x}, \mathbf{u}) = \mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{u}^\top \mathbf{R}\mathbf{u} \quad (11)$$

Then HJB becomes:

$$\min_{\mathbf{u}} [\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{u}^\top \mathbf{R}\mathbf{u} + \frac{\partial J^*}{\partial \mathbf{x}}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})] = 0 \quad (12)$$

where  $\mathbf{Q} = \mathbf{Q}^\top \geq 0$  and  $\mathbf{R} = \mathbf{R}^\top > 0$ .

# ALGEBRAIC RICCATI

## HJB for LTI, part 2

There is a theorem that says that for LTI with quadratic cost,  $J^*$  has the form:

$$J^* = \mathbf{x}^\top \mathbf{S} \mathbf{x} \quad (13)$$

where  $\mathbf{S} = \mathbf{S}^\top \geq 0$ .

Then HJB becomes:

$$\min_{\mathbf{u}} \left[ \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{u}^\top \mathbf{R} \mathbf{u} + \mathbf{x}^\top \mathbf{S} (\mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u}) + (\mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u})^\top \mathbf{S} \mathbf{x} \right] = 0$$

Simplifying, we get:

$$\min_{\mathbf{u}} \left[ \mathbf{u}^\top \mathbf{R} \mathbf{u} + \mathbf{x}^\top (\mathbf{Q} + \mathbf{S} \mathbf{A} + \mathbf{A}^\top \mathbf{S}) \mathbf{x} + \mathbf{x}^\top \mathbf{S} \mathbf{B} \mathbf{u} + \mathbf{u}^\top \mathbf{B}^\top \mathbf{S} \mathbf{x} \right] = 0$$



# ALGEBRAIC RICCATI

## Linear Quadratic Regulator

Finding partial derivative of the HJB with respect to  $\mathbf{u}$  and setting it to zero (as it is an extreme point) we get:

$$2\mathbf{u}^\top \mathbf{R} + 2\mathbf{x}^\top \mathbf{S}\mathbf{B} = 0 \quad (14)$$

This expression can be transposed and  $\mathbf{u}$  separated:

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{S}\mathbf{x} \quad (15)$$

This is the desired control law. We can see that it is *proportional*. We can re-write it as:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \quad (16)$$

where  $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\top \mathbf{S}$  is the controller gain. This control law is called Linear Quadratic Regulator (LQR).

Substituting found control law into the HJB, we find:

$$\min_{\mathbf{u}} [\mathbf{x}^\top (\mathbf{Q} + \mathbf{S}\mathbf{A} + \mathbf{A}^\top \mathbf{S})\mathbf{x} + \mathbf{x}^\top \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{R}\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{S}\mathbf{x} - \mathbf{x}^\top \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{S}\mathbf{x} - \mathbf{x}^\top \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{S}\mathbf{x}] = 0 \quad (17)$$

Simplifying, we get:

$$\mathbf{x}^\top (\mathbf{Q} + \mathbf{S}\mathbf{A} + \mathbf{A}^\top \mathbf{S} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{S})\mathbf{x} = 0 \quad (18)$$

which would hold for all  $\mathbf{x}$  iff:

$$\mathbf{Q} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{A}^\top \mathbf{S} = 0 \quad (19)$$

This is the *Algebraic Riccati equation*.

# ALGEBRAIC RICCATI

## Numerical methods

There are a number of ways to solve LQR:

- In MATLAB there is a function  $[K, S, P] = \text{lqr}(A, B, Q, R)$ , where  $P = \text{eig}(A - B * K)$
- In Python, there is  $S = \text{scipy.linalg.solve\_continuous\_are}(A, B, Q, R)$
- In Drake (by MIT and Toyota Research) there is a function  $(K, S) = \text{LinearQuadraticRegulator}(A, B, Q, R)$

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Control-Theory-Slides-Spring-2021](https://github.com/SergeiSa/Control-Theory-Slides-Spring-2021)

Check Moodle for additional links, videos, textbook suggestions.