

Stability

Control Theory, Lecture 2

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CRITICAL POINT (NODE)

Consider the following ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

Let \mathbf{x}_0 be such a state that:

$$\mathbf{f}(\mathbf{x}_0, t) = 0 \tag{2}$$

Then such state \mathbf{x}_0 is called a *node* or a *critical point*.

Node \mathbf{x}_0 is called *stable* iff for any constant δ there exists constant ε such that:

$$\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta \longrightarrow \|\mathbf{x}(t) - \mathbf{x}_0\| < \varepsilon \quad (3)$$

Think of it as "for any initial point that lies at most δ away from \mathbf{x}_0 , the rest of the trajectory $\mathbf{x}(t)$ will be at most ε away from \mathbf{x}_0 ".

Or, more picturesque, think of it as "the solutions with different initial conditions do not diverge from the node"

Node \mathbf{x}_0 is called *asymptotically stable* iff for any constant δ it is true that:

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \longrightarrow \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}_0 \quad (4)$$

Think of it as "for any initial point that lies at most δ away from \mathbf{x}_0 , the trajectory $\mathbf{x}(t)$ will asymptotically approach the point \mathbf{x}_0 ".

Or, more picturesque, think of it as "the solutions with different initial conditions converge to the node"

STABILITY VS ASYMPTOTIC STABILITY

Example

Consider dynamical system $\dot{x} = 0$, and solution $x = 7$. This solution is stable, but not asymptotically stable (other solutions do not diverge from $x = 7$, but do not converge to it either).

Example

Consider dynamical system $\dot{x} = -x$, and solution $x = 0$. This solution is stable and asymptotically stable (other solutions converge to $x = 0$).

Example

Consider dynamical system $\dot{x} = x$, and solution $x = 0$. This solution is unstable (other solutions diverge from $x = 0$).

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (5)$$

This is called a *linear time-invariant system*, or *LTI*.

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (6)$$

This is also an LTI, but it is also called an *autonomous system*, since its evolution depends only on the state of the system.

STABILITY OF AUTONOMOUS LTI

Example: real eigenvalues

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (7)$$

where \mathbf{A} can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, where \mathbf{D} is a diagonal matrix.

$$\dot{\mathbf{x}} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x} \quad (8)$$

Multiply it by $\mathbf{V}^{-1} \longrightarrow \mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x}$.

Define $\mathbf{z} = \mathbf{V}^{-1}\mathbf{x} \longrightarrow \dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$.

Since elements of \mathbf{D} are real, we can clearly see, that iff they are *all negative* the system will be asymptotically stable. If they are non-positive, the system is stable. And those elements are eigenvalues of \mathbf{A} .

STABILITY OF AUTONOMOUS LTI

Example: complex eigenvalues, part 1

Assume that \mathbf{A} can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{U}\mathbf{C}\mathbf{U}^{-1}$, where \mathbf{C} is a complex-valued diagonal matrix and \mathbf{U} is a complex-valued invertible matrix.

We can perform the same steps (multiply by \mathbf{U}^{-1} , then define $\mathbf{z} = \mathbf{U}^{-1}\mathbf{x}$) to arrive at:

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} \tag{9}$$

which falls into a set of independent equations, with complex coefficients c_i :

$$\dot{z}_i = c_i z_i \tag{10}$$

The solution is:

$$z_i = k_0 e^{c_i t} \tag{11}$$

STABILITY OF AUTONOMOUS LTI

Example: complex eigenvalues, part 2

The solution $z_i = k_0 e^{c_i t}$, where $c_i = \alpha_i + i\beta_i$, can be decomposed using Euler's identity:

$$z_i = k_0 e^{c_i t} = k_0 e^{(\alpha_i + i\beta_i)t} = k_0 e^{\alpha_i t} e^{i\beta_i t} = k_0 e^{\alpha_i t} (\cos(\beta_i t) + i \sin(\beta_i t))$$

As you can see, brackets $(\cos(\beta_i t) + i \sin(\beta_i t))$ has a constant norm, $\|(\cos(\beta_i t) + i \sin(\beta_i t))\| = 1$. Therefore, norm of z_i depends entirely on the norm of $e^{\alpha_i t}$, which is:

- ❶ constant if $\alpha_i = 0$, hence the system is stable.
- ❷ decreasing if $\alpha_i < 0$, hence the system is asymptotically stable.
- ❸ increasing if $\alpha_i > 0$, hence the system is unstable.

STABILITY OF AUTONOMOUS LTI

General case

Consider an autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (12)$$

Definition

Eq. (12) is stable iff real parts of eigenvalues of \mathbf{A} are non-positive.

Definition

Eq. (12) is asymptotically stable iff real parts of eigenvalues of \mathbf{A} are negative.

STABILITY OF AUTONOMOUS LTI

Illustration

Here is an illustration of *phase portraits* of two-dimensional LTIs with different types of stability:



Figure 1: phase portraits for different types of stability

Credit: staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf

- Control Systems Design, by Julio H. Braslavsky
staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.