

Observers

Control Theory, Lecture 10

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- Hamilton-Jacobi-Bellman equation

MEASUREMENT

How do we know the state?

Before we considered systems and control laws of the following type:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{u} = \mathbf{Kx} \end{cases} \quad (1)$$

But when we implement that control law, how do we know the current value of \mathbf{x} ? Previously we always took it from simulation.

In practice, we take it from *measurement*.

MEASUREMENT

Why information is imperfect?

There are a number of reasons why we can not directly measure the state of the system. Here are some:

- Digital measurements are done in discrete time intervals;
- Unpredicted events (faults, collisions, etc.);
- Un-modelled kinematics or dynamics (links bending, gear box backlash, friction, etc.) making the very definition of the state disconnected from reality;
- Lack of sensors;
- Imprecise, nonlinear and biased sensors;
- Physics, quantum-scale effects and alike;

MEASUREMENT

Definition

Let us introduce new notation. Assume we have an LTI system of the following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \mathbf{u} = \mathbf{g}(\hat{\mathbf{x}}, t) \\ \hat{\mathbf{x}} = \mathbf{o}(\cdot) \end{cases} \quad (2)$$

Then:

- \mathbf{x} is the state (actual or true state)
- \mathbf{y} is the output (actual or true output)
- $\hat{\mathbf{x}}$ is the estimated (observed) state
- $\hat{\mathbf{y}}$ is the estimated (observed) output

Notice that we never know true state \mathbf{x} , and therefore for the control purposes we have to use the estimated state $\hat{\mathbf{x}}$.

OBSERVATION

Using the knowledge about dynamics

Let us consider autonomous dynamical system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \quad (3)$$

with measurements \mathbf{y} . We want to get as good estimate of the state $\hat{\mathbf{x}}$ as we can.

First note: dynamics should also hold for our observed state:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} \quad (4)$$

Therefore if we know the initial conditions of our system exactly, and we know our model exactly, we can find exact state of the system without using measurement $\tilde{\mathbf{y}}$.

This we can call open loop observation. Unfortunately, we never know neither model nor initial conditions exactly.

We propose *observer* that takes into account measurements in a linear way:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \quad (5)$$

with measurements \mathbf{y} . With this observer, we want to get as good estimate of the state $\hat{\mathbf{x}}$ as we can.

Let's define state estimation error as $\varepsilon = \hat{\mathbf{x}} - \mathbf{x}$. We can subtract (3) from (5), to get *observer error dynamics*:

$$\dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{A}\mathbf{x} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \quad (6)$$

$$\dot{\varepsilon} = (\mathbf{A} - \mathbf{L}\mathbf{C})\varepsilon \quad (7)$$

OBSERVATION

Observer gains

The observer $\dot{\varepsilon} = (\mathbf{A} - \mathbf{LC})\varepsilon$ is *stable* (i.e., the state estimation error tends to zero), as long as the following matrix is negative-definite:

$$\mathbf{A} - \mathbf{LC} < 0$$

We need to find \mathbf{L} . Let us observe the key difference between observer design and controller design:

- Controller design: find such \mathbf{K} that $\mathbf{A} - \mathbf{BK} < 0$.
- Observer design: find such \mathbf{L} that: $\mathbf{A} - \mathbf{LC} < 0$

We have instruments for finding \mathbf{K} , what about \mathbf{L} ?

OBSERVER DESIGN

General case: design via Riccati eq.

In general, we can observe that if $\mathbf{A} - \mathbf{LC}$ is negative-definite, then $(\mathbf{A} - \mathbf{LC})^\top$ is negative-definite too (by definition of the negative-definiteness).

Therefore, we can solve the following *dual problem*:

■ find such \mathbf{L} that $\mathbf{A}^\top - \mathbf{C}^\top \mathbf{L}^\top < 0$.

The dual problem is *equivalent* to the control design problem. We can solve it by producing and solving algebraic Riccati equation, as in the LQR formulation. In pseudo-code it can be represented the following way:

$$\mathbf{L}^\top = \text{lqr}(\mathbf{A}^\top, \mathbf{C}^\top, \mathbf{Q}, \mathbf{R}).$$

where \mathbf{Q} and \mathbf{R} are weight matrices, determining the "sensitivity" or "aggressiveness" of the observer.

OBSERVATION AND CONTROL

LTI

Thus we get dynamics+observer combination:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*(t)) + \mathbf{u}^*(t) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \end{cases} \quad (8)$$

where $\mathbf{A} - \mathbf{BK} < 0$ and $\mathbf{A}^\top - \mathbf{C}^\top \mathbf{L}^\top < 0$.

OBSERVATION AND CONTROL

Affine case

Affine case is almost the same:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \\ \mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*(t)) + \mathbf{u}^*(t) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \hat{\dot{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{c} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \end{cases} \quad (9)$$

where $\mathbf{A} - \mathbf{BK} < 0$ and $\mathbf{A}^\top - \mathbf{C}^\top \mathbf{L}^\top < 0$.

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.