# Stabilizing Control

## Control Theory, Lecture 4-5

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# Content

# Changing stability

Here are two LTIs:

$$\dot{x} = 2x \tag{1}$$

$$\dot{x} = 2x + u \tag{2}$$

First one is unstable. Second one - we don't know until we know what $u$ is.

If we pick $u = 0$, the result is an unstable equation. But we can also pick $u$ such that the resulting dynamics is stable, such as $u = -3x$:

$$\dot{x} = 2x + u = 2x - 3x = -x \tag{3}$$

So, we can use *control input* $u$ to change stability of the system!

**Definition**

The problem of finding control law $\mathbf{u}$ that make a certain solution $\mathbf{x}^*$ of dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ stable is called *stabilizing control problem*

This is true for both linear and non-linear systems. But for linear systems we can get a lot more details about this problem, if we restrict our choice of control law.

Consider an LTI system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{4}$$

and let us chose control as a linear function of the *state* $x$:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \tag{5}$$

Thus, we know how the system is going to look when the control is applied:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} \tag{6}$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \tag{7}$$

Note that (7) is an autonomous system. We call this (a system that was not autonomous, but became one after substituting control law) a *closed loop* system.

Observing the system $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$ we obtained, we can notice that we already have the tools to analyse its stability. Namely:

**Stability condition for LTI closed-loop system**

The real parts of the eigenvalues of the matrix $(\mathbf{A} - \mathbf{BK})$ should be negative for asymptotic stability, or non-positive for stability in the sense of Lyapunov

So, all you need to do is to find such $\mathbf{K}$ that $(\mathbf{A} - \mathbf{BK})$ has eigenvalues with negative real parts, and you made a stable closed-loop system!

We don't have to limit ourselves to just this $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{u} = -\mathbf{K}\mathbf{x}$ pair.

In fact, this pair mostly works for the simple case when the solution we want to stabilize is trivial $\mathbf{x}^*(t) = 0$.

Let us consider a slightly more complicated system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \tag{8}$$

This is called *affine system*, because of the constant term $\mathbf{c}$. What is the control that stabilizes this system? Let us propose an *affine control law*:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^* \tag{9}$$

where $\mathbf{u}^*$ is a constant term.

Thus, from $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c}$ and $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$ we get the following closed-loop system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} + \mathbf{B}\mathbf{u}^* + \mathbf{c} \tag{10}$$

And as long as we can choose such $\mathbf{u}^*$ that $\mathbf{B}\mathbf{u}^* = -\mathbf{c}$, we will get back to the previously seen form $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$.

**Existence of the stabilizing control**

Notice that same as it is possible that there exists no such $\mathbf{K}$ that $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$ is stable, same there might exist no such $\mathbf{u}^*$ that $\mathbf{B}\mathbf{u}^* = -\mathbf{c}$

Let us now consider an arbitrary solution $\mathbf{x}^* = \mathbf{x}^*(t)$ for the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{11}$$

and try to find a stabilizing control for it.

Our first step is to notice that, if $\mathbf{x}^* = \mathbf{x}^*(t)$ is a solution, that means that it satisfies the ODE (11):

$$\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \tag{12}$$

where $\mathbf{u}^* = \mathbf{u}^*(t)$ is some control law, for which the solution $\mathbf{x}^* = \mathbf{x}^*(t)$ is obtained.

If we are not given $\mathbf{u}^* = \mathbf{u}^*(t)$, we can compute it as:

$$\mathbf{u}^* = \mathbf{B}^+(\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^*) \tag{13}$$

where $\mathbf{B}^+$ is a pseudo-inverse, and the solution to this least-squared problem will have to have no residual (since $\mathbf{x}^* = \mathbf{x}^*(t)$ is a solution).

$$||\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^* - \mathbf{B}\mathbf{B}^+(\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^*)|| = 0 \tag{14}$$

Now, let us introduce the concept of *control error* $\mathbf{e}$:

$$\mathbf{e} = \mathbf{x} - \mathbf{x}^* \tag{15}$$

**Control error and stability**

If control error goes to zero asymptotically, every solution goes to $\mathbf{x}^*$.

Remember that we have two simultaneous equations:
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^*$. We can now subtract one from the other to get:

$$\dot{\mathbf{x}} - \dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u} - \mathbf{B}\mathbf{u}^* \tag{16}$$

in other words:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \tag{17}$$

where $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$

We arrived at a new dynamical system $\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v}$, which is an LTI, and we are interested in stabilizing the solution $\mathbf{e}^* = 0$. We know how to do it with a linear control law:

$$\mathbf{v} = -\mathbf{K}\mathbf{e} \tag{18}$$

Now remember that $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$ and $\mathbf{e} = \mathbf{x} - \mathbf{x}^*$, this will become:

$$\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{u}^* \tag{19}$$

# Error dynamics

## Part 6

This control law $\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{u}^*$ can be thought of as consisting of two parts:

- Feedback control $\mathbf{u}_{FB} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*)$, which depends on the control error (which requires a feedback about the current state of your system)
- Feed-forward control $\mathbf{u}_{FF} = \mathbf{u}^*$, which depends only on the trajectory and the equations of dynamics of your system, but not on your current state

What we just did - stabilization of the arbitrary trajectory $\mathbf{x}^* = \mathbf{x}^*(t)$ - is also called *trajectory tracking control*, or *trajectory stabilization*. The solution to stabilize is called *trajectory*.

Just for completeness, let's consider the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \tag{20}$$

and stabilize trajectory $\mathbf{x}^* = \mathbf{x}^*(t)$.

We start by observing that, as before, our solution gives us equality:

$$\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* + \mathbf{c} \tag{21}$$

and after introducing control error and subtracting (21) from the original dynamics (20), we get:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \tag{22}$$

where $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$, which we already saw before. The only difference is that now $\mathbf{u}^*$ is found as:

$$\mathbf{u}^* = \mathbf{B}^+(\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^* - \mathbf{c}) \tag{23}$$

What if we want to move our system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ from initial condition to some desired state $\mathbf{x}^*$. The difference is that we do not have a solution that we used before, only the desired *node* is given. This can be called *point-to-point control*

Let us start by giving the form of the control law:

$$\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{u}^* \tag{24}$$

And thus we can re-write the dynamics as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{B}\mathbf{u}^* \tag{25}$$

Let us consider how the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{B}\mathbf{u}^*$ will behave at the point $\mathbf{x}^*$. We know that $\dot{\mathbf{x}}^* = 0$:

$$0 = \mathbf{A}\mathbf{x}^* - \mathbf{B}\mathbf{K}(\mathbf{x}^* - \mathbf{x}^*) + \mathbf{B}\mathbf{u}^* \tag{26}$$

$$0 = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \tag{27}$$

which we saw before. It provides us solution to the forward dynamics:

$$\mathbf{u}^* = -\mathbf{B}^+\mathbf{A}\mathbf{x}^* \tag{28}$$

Subtracting solution (27) from the original dynamics (25), we get familiar error dynamics $\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{e}$.

Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and the desired state $\mathbf{x}^*$ we can do a point-to-point control with the following *pure state feedback control*:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^* \tag{29}$$

We can re-write the dynamics as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} + \mathbf{B}\mathbf{u}^* \tag{30}$$

As before, we know that at the node, $\dot{\mathbf{x}}^* = 0$:

$$0 = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \tag{31}$$

Thus we can solve for $\mathbf{u}^*$

$$\mathbf{u}^* = -\mathbf{B}^+(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* \tag{32}$$

The rest is the same. Error dynamics is $\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{e}$.

Note that when $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$, we got feed-forward control in the form:

$$\mathbf{u}^* = -\mathbf{B}^+(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^*$$

But when we had $\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{u}^*$, our feed-forward control was

$$\mathbf{u}^* = -\mathbf{B}^+\mathbf{A}\mathbf{x}^*$$

The difference has to do with how the two control methods behave at the node.

# Read more

- Richard M. Murray Control and Dynamical Systems California Institute of Technology Optimization-Based Control
- Dynamic Simulation in Python

# Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.