Discrete Dynamics

Control Theory, Lecture 5 (continuation)

by Sergei Savin

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Content

- Discrete Dynamics
- Stability of the Discrete Dynamics
- Discretization
  - Finite difference
  - Finite difference in an autonomous LTI
- Zero order hold
- ZOH and other types of discretization
  - Zero order hold vs First order hold
  - Exact discretization
- Read more

The following dynamical system is called *discrete*:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \tag{1}$$

Note that those:

- have no derivatives in the equation;
- are easily simulated.

The affine control for this system can be given as:

$$\mathbf{u}_i = -\mathbf{K}\mathbf{x}_i + \mathbf{u}_i^* \tag{2}$$

Let us consider stability of the discrete dynamical system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$.

We will attack the problem in the same way as before, first assuming that $\mathbf{A} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}$, where $\mathbf{D}$ is a diagonal matrix with eigenvalues of $\mathbf{A}$ on its diagonal:

$$\mathbf{x}_{i+1} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}\mathbf{x}_i \tag{3}$$

Multiplying both sides of the equation by $\mathbf{V}$ and defining $\mathbf{z}_i = \mathbf{V}\mathbf{x}_i$, we get:

$$\mathbf{V}\mathbf{x}_{i+1} = \mathbf{V}\mathbf{V}^{-1}\mathbf{D}\mathbf{V}\mathbf{x}_i \tag{4}$$

$$\mathbf{z}_{i+1} = \mathbf{D}\mathbf{z}_i \tag{5}$$

Now, considering $\mathbf{z}_{i+1} = \mathbf{D}\mathbf{z}_i$ we can see that the norm of the state $\mathbf{z}_i$ would not increase iff the norm of the elements of $\mathbf{D}$ (which as eigenvalues of $\mathbf{A}$) are smaller than 1.

Stability criterion

In general, discrete systems $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$ are stable as long as the eigenvalues of $\mathbf{A}$ are smaller than 1 by absolute value: $|\lambda_i(\mathbf{A})| \leq 1, \ \forall i$. This is true for complex eigenvalues as well.

Consider linear time-invariant autonomous system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

The time derivative $\dot{\mathbf{x}}$ can be replaces with a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + \Delta t) - \mathbf{x}(t)) \tag{7}$$

Note that we could have also used other definitions of a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + 0.5\Delta t) - \mathbf{x}(t - 0.5\Delta t)) \tag{8}$$

or

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t) - \mathbf{x}(t - \Delta t)) \tag{9}$$

We can introduce notation:

$$\begin{cases} \mathbf{x}_0 = \mathbf{x}(0) \\ \mathbf{x}_1 = \mathbf{x}(\Delta t) \\ \mathbf{x}_2 = \mathbf{x}(2\Delta t) \\ ... \\ \mathbf{x}_n = \mathbf{x}(n\Delta t) \end{cases} \tag{10}$$

We say that $\mathbf{x}_i$ is the value of $\mathbf{x}$ at the time step i. Then the finite difference can be written, for example, as follows:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) \tag{11}$$

We can rewrite our original autonomous LTI as follows:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_i \tag{12}$$

Isolating $\mathbf{x}_{i+1}$ on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{A}\Delta t + \mathbf{I})\mathbf{x}_i \tag{13}$$

Or alternatively:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_{i+1} \tag{14}$$

Isolating $\mathbf{x}_{i+1}$ on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{I} - \mathbf{A}\Delta t)^{-1}\mathbf{x}_i \tag{15}$$

Defining *discrete state space matrix* $\bar{\mathbf{A}}$ and *discrete control matrix* $\bar{\mathbf{B}}$ as follows:

$$\bar{\mathbf{A}} = \mathbf{A}\Delta t + \mathbf{I} \tag{16}$$

$$\bar{\mathbf{B}} = \mathbf{B}\Delta t \tag{17}$$

We get discrete dynamics:

$$\mathbf{x}_{i+1} = \bar{\mathbf{A}}\mathbf{x}_i + \bar{\mathbf{B}}\mathbf{u}_i \tag{18}$$

This way of defining discrete dynamics is called *zero order hold (ZOH)*.

Graphically, we can understand what zero order hold is, by comparing it to the first order hold:

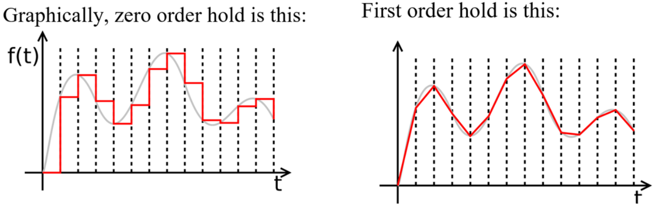

Figure 1: Different types of discretization

Let the discrete state $\mathbf{x}_i$ correspond to continuous state $\mathbf{x}$ at the moment of time t_i. Then, we can say that the discretization is *exact* the following holds for any solution $\mathbf{x}(t)$

$$\mathbf{x}_0 = \mathbf{x}(t_0) \rightarrow \mathbf{x}_i = \mathbf{x}(t_i), \ \forall i \tag{19}$$

We can compute the exact discretization as follows:

$$\bar{\mathbf{A}} = e^{\mathbf{A}\Delta t} \tag{20}$$

$$\bar{\mathbf{B}} = \mathbf{B} \int_{t_0}^{t_0+\Delta t} e^{\mathbf{A}s} ds \tag{21}$$

Read more

- Automatic Control 1 Discrete-time linear systems, Prof. Alberto Bemporad, University of Trento

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.