Observers

Control Theory, Lecture 10

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Content

- Hamilton-Jacobi-Bellman equation

Before we considered systems and control laws of the following type:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{u} = \mathbf{K}\mathbf{x} \end{cases} \tag{1}$$

But when we implement that control law, how do we know the current value of $\mathbf{x}$? Previously we always took it from simulation.

In practice, we take it from *measurement*.

There are a number of reasons why we can not directly measure the state of the system. Here are some:

- Digital measurements are done in discrete time intervals;
- Unpredicted events (faults, collisions, etc.);
- Un-modelled kinematics or dynamics (links bending, gear box backlash, friction, etc.) making the very definition of the state disconnected from reality;
- Lack of sensors;
- Imprecise, nonlinear and biased sensors;
- Physics, quantum-scale effects and alike;

Let us introduce new notation. Assume we have an LTI system of the following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \mathbf{u} = \mathbf{g}(\hat{\mathbf{x}}, t) \\ \hat{\mathbf{x}} = \mathbf{o}(\cdot) \end{cases} \tag{2}$$

Then:

- $\mathbf{x}$ is the state (actual or true state)
- $\mathbf{y}$ is the output (actual or true output)
- $\hat{\mathbf{x}}$ is the estimated (observed) state
- $\hat{\mathbf{y}}$ is the estimated (observed) output

Notice that we never know true state $\mathbf{x}$, and therefore for the control purposes we have to use the estimated state $\hat{\mathbf{x}}$.

Let us consider autonomous dynamical system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{3}$$

with measurements $\mathbf{y}$. We want to get as good estimate of the state $\hat{\mathbf{x}}$ as we can.

First note: dynamics should also hold for our observed state:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} \tag{4}$$

Therefore if we know the initial conditions of our system exactly, and we know our model exactly, we can find exact state of the system without using measurement $\tilde{\mathbf{y}}$.
This we can call open loop observation. Unfortunately, we never know neither model nor initial conditions exactly.

We propose *observer* that takes into account measurements in a linear way:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{5}$$

with measurements $\mathbf{y}$. With this observer, we want to get as good estimate of the state $\hat{\mathbf{x}}$ as we can.

Let's define state estimation error as $\varepsilon = \hat{\mathbf{x}} - \mathbf{x}$. We can subtract (3) from (5), to get *observer error dynamics*:

$$\dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{A}\mathbf{x} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{6}$$

$$\dot{\varepsilon} = (\mathbf{A} - \mathbf{L}\mathbf{C})\varepsilon \tag{7}$$

The observer $\dot{\varepsilon} = (\mathbf{A} - \mathbf{LC})\varepsilon$ is *stable* (i.e., the state estimation error tends to zero), as long as the following matrix is negative-definite:

$$\mathbf{A} - \mathbf{Lc} < 0$$

We need to find $\mathbf{L}$. Let us observe the key difference between observer design and controller design:

- Controller design: find such $\mathbf{K}$ that $\mathbf{A} - \mathbf{BK} < 0$.
- Observer design: find such $\mathbf{L}$ that: $\mathbf{A} - \mathbf{Lc} < 0$

We have instruments for finding $\mathbf{K}$, what about $\mathbf{L}$?

In general, we can observe that if $\mathbf{A} - \mathbf{Lc}$ is negative-definite, then $(\mathbf{A} - \mathbf{Lc})^\top$ is negative-definite too (by definition of the negative-definiteness).

Therefore, we can solve the following *dual problem*:

- find such $\mathbf{L}$ that $\mathbf{A}^\top - \mathbf{c}^\top \mathbf{L}^\top < 0$.

The dual problem is *equivalent* to the control design problem. We can solve it by producing and solving algebraic Riccati equation, as in the LQR formulation. In pseudo-code it can be represented the following way:

$\mathbf{L}^\top$ `= lqr(`$\mathbf{A}^\top$, $\mathbf{c}^\top$, $\mathbf{Q}$, $\mathbf{R}$`)`.
where $\mathbf{Q}$ and $\mathbf{R}$ are weight matrices, determining the "sensitivity" or "aggressiveness" of the observer.

Thus we get dynamics+observer combination:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \mathbf{u} = -\mathbf{K}(\hat{\mathbf{x}} - \mathbf{x}^*(t)) + \mathbf{u}^*(t) \end{cases} \tag{8}$$

where $\mathbf{A} - \mathbf{B}\mathbf{K} < 0$ and $\mathbf{A}^\top - \mathbf{c}^\top \mathbf{L}^\top < 0$.

When we want to stabilize the origin, the dynamics+observer combination can be represented as a single LTI:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \mathbf{u} = -\mathbf{K}\hat{\mathbf{x}} \end{cases} \tag{9}$$

In matrix form it becomes:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{C} & (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} \tag{10}$$

We can't directly reason about eigenvalues of this matrix. Next slide will show a way to do it with a change of variables.

Let us use the following substitution: $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, which implies $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{e}$:
Our system had form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}(\mathbf{C}\mathbf{x} - \mathbf{C}\hat{\mathbf{x}}) \end{cases} \tag{11}$$

Since $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}$, we get:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} - (\mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}(\mathbf{C}\mathbf{x} - \mathbf{C}\hat{\mathbf{x}}))$$

$$\dot{\mathbf{e}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) - \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}})$$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}$$

Equation for $\dot{\mathbf{x}}$ takes form:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e}$$

Collecting $\dot{\mathbf{x}}$ and $\dot{\mathbf{e}}$ we get:

$$\begin{cases} \dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BKe} \\ \dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC})\mathbf{e} \end{cases} \tag{12}$$

In matrix form it becomes:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} (\mathbf{A} - \mathbf{BK}) & \mathbf{BK} \\ 0 & (\mathbf{A} - \mathbf{LC}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{13}$$

Eigenvalues of a upper block-triangular matrices equal to the union of the eigenvalues of the blocks on the main diagonal. Hence here, the eigenvalues of the system are equal to the union of eigenvalues of $(\mathbf{A} - \mathbf{BK})$ and $(\mathbf{A} - \mathbf{LC})$.

Since the eigenvalues of the system are equal to the union of eigenvalues of $(\mathbf{A} - \mathbf{BK})$ and $(\mathbf{A} - \mathbf{LC})$, we can make the following observation:

Separation principle

As long as the observer and the controller are stable independently, the overall system is stable too. This is called *separation principle*.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.