Observers

Control Theory, Lecture 10

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Content

- Hamilton-Jacobi-Bellman equation

Before we considered systems and control laws of the following type:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{u} = \mathbf{K}\mathbf{x} \end{cases} \tag{1}$$

But when we implement that control law, how do we know the current value of $\mathbf{x}$? Previously we always took it from simulation.

In practice, we take it from *measurement*.

There are a number of reasons why we can not directly measure the state of the system. Here are some:

- Digital measurements are done in discrete time intervals;
- Unpredicted events (faults, collisions, etc.);
- Un-modelled kinematics or dynamics (links bending, gear box backlash, friction, etc.) making the very definition of the state disconnected from reality;
- Lack of sensors;
- Imprecise, nonlinear and biased sensors;
- Physics, quantum-scale effects and alike;

Let us introduce new notation. Assume we have an LTI system of the following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \mathbf{u} = \mathbf{g}(\hat{\mathbf{x}}, t) \\ \hat{\mathbf{x}} = \mathbf{o}(\cdot) \end{cases} \tag{2}$$

Then:

- $\mathbf{x}$ is the state (actual or true state)
- $\mathbf{y}$ is the output (actual or true output)
- $\hat{\mathbf{x}}$ is the estimated (observed) state
- $\hat{\mathbf{y}}$ is the estimated (observed) output

Notice that we never know true state $\mathbf{x}$, and therefore for the control purposes we have to use the estimated state $\hat{\mathbf{x}}$.

Let us consider autonomous dynamical system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{3}$$

with measurements $\mathbf{y}$. We want to get as good estimate of the state $\hat{\mathbf{x}}$ as we can.

First note: dynamics should also hold for our observed state:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} \tag{4}$$

Therefore if we know the initial conditions of our system exactly, and we know our model exactly, we can find exact state of the system without using measurement $\tilde{\mathbf{y}}$.
This we can call open loop observation. Unfortunately, we never know neither model nor initial conditions exactly.

We propose *observer* that takes into account measurements in a linear way:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{5}$$

with measurements $\mathbf{y}$. With this observer, we want to get as good estimate of the state $\hat{\mathbf{x}}$ as we can.

Let's define state estimation error as $\varepsilon = \hat{\mathbf{x}} - \mathbf{x}$. We can subtract (3) from (5), to get *observer error dynamics*:

$$\dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{A}\mathbf{x} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{6}$$

$$\dot{\varepsilon} = (\mathbf{A} - \mathbf{L}\mathbf{C})\varepsilon \tag{7}$$

The observer $\dot{\varepsilon} = (\mathbf{A} - \mathbf{LC})\varepsilon$ is *stable* (i.e., the state estimation error tends to zero), as long as the following matrix is negative-definite:

$$\mathbf{A} - \mathbf{LC} < 0$$

We need to find $\mathbf{L}$. Let us observe the key difference between observer design and controller design:

- Controller design: find such $\mathbf{K}$ that $\mathbf{A} - \mathbf{BK} < 0$.
- Observer design: find such $\mathbf{L}$ that: $\mathbf{A} - \mathbf{LC} < 0$

We have instruments for finding $\mathbf{K}$, what about $\mathbf{L}$?

In general, we can observe that if $\mathbf{A} - \mathbf{LC}$ is negative-definite, then $(\mathbf{A} - \mathbf{LC})^\top$ is negative-definite too (by definition of the negative-definiteness).

Therefore, we can solve the following *dual problem*:

- find such $\mathbf{L}$ that $\mathbf{A}^\top - \mathbf{C}^\top \mathbf{L}^\top < 0$.

The dual problem is *equivalent* to the control design problem. We can solve it by producing and solving algebraic Riccati equation, as in the LQR formulation. In pseudo-code it can be represented the following way:

$\mathbf{L}^\top$ `= lqr`$(\mathbf{A}^\top, \mathbf{C}^\top, \mathbf{Q}, \mathbf{R})$.
where $\mathbf{Q}$ and $\mathbf{R}$ are weight matrices, determining the "sensitivity" or "aggressiveness" of the observer.

Thus we get dynamics+observer combination:

$$
\begin{cases}
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*(t)) + \mathbf{u}^*(t) \\
\mathbf{y} = \mathbf{C}\mathbf{x} \\
\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}})
\end{cases}
\tag{8}
$$

where $\mathbf{A} - \mathbf{B}\mathbf{K} < 0$ and $\mathbf{A}^\top - \mathbf{C}^\top\mathbf{L}^\top < 0$.

Affine case is almost the same:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \\ \mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*(t)) + \mathbf{u}^*(t) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{c} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \end{cases} \tag{9}$$

where $\mathbf{A} - \mathbf{B}\mathbf{K} < 0$ and $\mathbf{A}^\top - \mathbf{C}^\top\mathbf{L}^\top < 0$.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.