

Stability

Control Theory, Lecture 2

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CRITICAL POINT (NODE)

Consider the following ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \quad (1)$$

Let \mathbf{x}_0 be such a state that:

$$\mathbf{f}(\mathbf{x}_0, t) = 0 \quad (2)$$

Then such state \mathbf{x}_0 is called a *node* or a *critical point*.

Node \mathbf{x}_0 is called *stable* iff for any constant δ there exists constant ε such that:

$$\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta \longrightarrow \|\mathbf{x}(t) - \mathbf{x}_0\| < \varepsilon \quad (3)$$

Think of it as "for any initial point that lies at most δ away from \mathbf{x}_0 , the rest of the trajectory $\mathbf{x}(t)$ will be at most ε away from \mathbf{x}_0 ".

Or, more picturesque, think of it as "the solutions with different initial conditions do not diverge from the node"

Node \mathbf{x}_0 is called *asymptotically stable* iff for any constant δ it is true that:

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \longrightarrow \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}_0 \quad (4)$$

Think of it as "for any initial point that lies at most δ away from \mathbf{x}_0 , the trajectory $\mathbf{x}(t)$ will asymptotically approach the point \mathbf{x}_0 ".

Or, more picturesque, think of it as "the solutions with different initial conditions converge to the node"

STABILITY VS ASYMPTOTIC STABILITY

Example

Consider dynamical system $\dot{x} = 0$, and solution $x = 7$. This solution is stable, but not asymptotically stable (other solutions do not diverge from $x = 7$, but do not converge to it either).

Example

Consider dynamical system $\dot{x} = -x$, and solution $x = 0$. This solution is stable and asymptotically stable (other solutions converge to $x = 0$).

Example

Consider dynamical system $\dot{x} = x$, and solution $x = 0$. This solution is unstable (other solutions diverge from $x = 0$).

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (5)$$

This is called a *linear time-invariant system*, or *LTI*.

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (6)$$

This is also an LTI, but it is also called an *autonomous system*, since its evolution depends only on the state of the system.

STABILITY OF AUTONOMOUS LTI

Real eigenvalues

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (7)$$

where \mathbf{A} can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, where \mathbf{D} is a diagonal matrix.

$$\dot{\mathbf{x}} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x} \quad (8)$$

Multiply it by $\mathbf{V}^{-1} \longrightarrow \mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x}$.

Define $\mathbf{z} = \mathbf{V}^{-1}\mathbf{x} \longrightarrow \dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$.

Since elements of \mathbf{D} are real, we can clearly see, that iff they are *all negative* will the system be asymptotically stable. If they are non-positive, the system is stable. And those elements are eigenvalues of \mathbf{A} .

STABILITY OF AUTONOMOUS LTI

Complex eigenvalues, 2-dimensional case (1)

Let us consider the following system:

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \quad (9)$$

The eigenvalues of the system are $\alpha \pm i\beta$. We denote $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \mathbf{x}$.

We start by claiming that the system will be stable iff the $\dot{\mathbf{x}}^\top \mathbf{x} < 0$. Indeed, vector $\dot{\mathbf{x}}$ can always be decomposed into two components, $\dot{\mathbf{x}}_{\parallel}$ parallel to \mathbf{x} , and $\dot{\mathbf{x}}_{\perp}$ perpendicular to \mathbf{x} . By definition $\dot{\mathbf{x}}_{\perp}^\top \mathbf{x} = 0$, and is responsible for the change in orientation of \mathbf{x} . The value of $\dot{\mathbf{x}}_{\parallel}$ is responsible for the change in the length of \mathbf{x} ; the length would shrink iff $\dot{\mathbf{x}}_{\parallel}$ is of opposite direction to \mathbf{x} , giving negative value of the dot product $\dot{\mathbf{x}}^\top \mathbf{x}$.

STABILITY OF AUTONOMOUS LTI

Complex eigenvalues, 2-dimensional case (2)

Let us compute $\dot{\mathbf{x}}^\top \mathbf{x}$:

$$\dot{\mathbf{x}}^\top \mathbf{x} = [\mathbf{x}_1 \quad \mathbf{x}_2] \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \quad (10)$$

$$\dot{\mathbf{x}}^\top \mathbf{x} = \alpha(\mathbf{x}_1^2 + \mathbf{x}_2^2) \quad (11)$$

From this it is clear that the product $\dot{\mathbf{x}}^\top \mathbf{x} < 0$ is negative iff $\alpha < 0$.

Definition

As long as the *real parts of the eigenvalues* of the system are *strictly negative*, the system is *asymptotically stable*. If the real parts of the eigenvalues of the system are zero, the system is *marginally stable*.

STABILITY OF AUTONOMOUS LTI

Complex eigenvalues, 2-dimensional case (3)

Vector field of $\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$ is shown below:



STABILITY OF AUTONOMOUS LTI

General case (1)

Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where \mathbf{A} can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{U}\mathbf{C}\mathbf{U}^{-1}$, where \mathbf{C} is a complex-valued diagonal matrix and \mathbf{U} is a complex-valued invertible matrix.

We multiply both sides by \mathbf{U}^{-1} , then define $\mathbf{z} = \mathbf{U}^{-1}\mathbf{x}$ to arrive at:

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} \quad (12)$$

which falls into a set of independent equations, with complex coefficients c_j :

$$\dot{z}_j = c_j z_j \quad (13)$$

STABILITY OF AUTONOMOUS LTI

General case (2)

Expanding $c_j = \alpha + i\beta$, and $z_j = u + iv$ (we dismiss subscripts for clarity), we find that $\dot{z}_j = c_j z_j$ can be expanded as:

$$\dot{u} + i\dot{v} = \dot{z}_j = c_j z_j = (\alpha + i\beta)(u + iv) \quad (14)$$

$$\dot{u} + i\dot{v} = \alpha u + i\beta u + i\alpha v - \beta v \quad (15)$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (16)$$

As we can see, $\dot{z}_j = c_j z_j$ is asymptotically stable iff $\operatorname{Re}(c_j) < 0$, and marginally stable if $\alpha = \operatorname{Re}(c_j) = 0$. Same is true for $\dot{\mathbf{z}} = \mathbf{C}\mathbf{z}$ and hence, for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, as \mathbf{U} is invertible.

STABILITY OF AUTONOMOUS LTI

Condition

Consider an autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (17)$$

Definition

Eq. (17) is stable iff real parts of eigenvalues of \mathbf{A} are non-positive.

Definition

Eq. (17) is asymptotically stable iff real parts of eigenvalues of \mathbf{A} are negative.

STABILITY OF AUTONOMOUS LTI

Illustration

Here is an illustration of *phase portraits* of two-dimensional LTIs with different types of stability:



Figure 1: phase portraits for different types of stability

Credit: staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf



- Control Systems Design, by Julio H. Braslavsky
staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.

