

# Column space, Left Null space, control applications

Control Theory, Lecture 7

by Sergei Savin

Spring 2021

- Column space
- Column space basis
- Column space and null space
- Projector onto column space
- Left null space
- Finding fixed points
- Checking fixed points
- Correcting fixed points
- Finding fixed points for affine systems

Consider the following task: find all vectors  $\mathbf{y}$  such that  $\mathbf{y} = \mathbf{A}\mathbf{x}$ .

It can be re-formulated as follows: find all elements of the *column space* of  $\mathbf{A}$ .

## Definition 1

*Column space* of  $\mathbf{A}$  is the set of all outputs of the matrix  $\mathbf{A}$ , for all possible inputs

We will denote column space as  $\mathcal{C}(\mathbf{A})$ . In the literature, it is often called an *image* of  $\mathbf{A}$ .

The problem of finding orthonormal basis in the column space of a matrix is often called *orthonormalization* of that matrix. Hence in MATLAB and Python/Scipy the function that does it is called `orth`:

- `C = orth(A).`

- `C = scipy.linalg.orth(A).`

That is how one finds all the outputs of the matrix  $\mathbf{A}$ : as  $\{\mathbf{C}\mathbf{z} : \forall \mathbf{z}\}$ .

Notice that  $\{\mathbf{A}\mathbf{x} : \forall \mathbf{x}\}$  might contain repeated entries if  $\mathbf{A}$  has a non-trivial null space.

# COLUMN SPACE AND NULL SPACE

Let  $\mathbf{A}$  be a square matrix, a map from  $\mathbb{X} = \mathbb{R}^n$  to  $\mathbb{Y} = \mathbb{R}^n$ . Notice that if it has a non-trivial null space, it follows that multiple unique inputs are being mapped by it to the same output:

$$\begin{aligned}\mathbf{y} &= \mathbf{A}\mathbf{x}_r = \mathbf{A}(\mathbf{x}_r + \mathbf{x}_n), \\ \mathbf{x}_r &\in \mathcal{R}(\mathbf{A}) \\ \forall \mathbf{x}_n &\in \mathcal{N}(\mathbf{A})\end{aligned}\tag{1}$$

In fact, if null space of  $\mathbf{A}$  has  $k$  dimensions, it implies that an  $n$ -dimensional subspace of  $\mathbb{X}$  is mapped to a single element of  $\mathbb{Y}$ .

It follows that in this case the dimensionality of the column space could not exceed  $n - k$ .

# PROJECTOR ONTO COLUMN SPACE

Given vector  $\mathbf{y}$  and matrix  $\mathbf{A}$ , let us find projector of  $\mathbf{y}$  onto the column space of  $\mathbf{A}$ .

This is done in the same manner as we did with the null space:

$$\mathbf{y}_c = \mathbf{A}\mathbf{A}^+\mathbf{y} \in \mathcal{C}(\mathbf{A}) \quad (2)$$

This feels nice, as it only requires the matrix itself, no need for the orthonormal basis as before. However, you need to remember that the pseudoinverse is based on SVD decomposition, same as operations of finding a basis in the null space or column space.

# PROJECTOR ONTO ROW SPACE

In the same we can define a projector onto row space. Given vector  $\mathbf{x}$  and matrix  $\mathbf{A}$ , let us find projector of  $\mathbf{x}$  onto the row space of  $\mathbf{A}$ :

$$\mathbf{x}_r = \mathbf{A}^+ \mathbf{A} \mathbf{x} \in \mathcal{R}(\mathbf{A}) \quad (3)$$

You can think of this in the following terms: first we find what output  $\mathbf{x}$  makes, then we find the smallest norm vector that produces this same output, and this vector has to 1) have the same row space projector (because output is the same), 2) has to lie in the row space, hence it is the row space projector of  $\mathbf{x}$ .

The subspace, orthogonal to the column space is called *left null space*.

## Definition

Space of all vectors that can't be produced as outputs of matrix  $\mathbf{A}$  is called *left null space*. Zero vector is included, as in linear spaces.

If we want to project vector  $\mathbf{y}$  onto the left null space of  $\mathbf{A}$ , we do it as:

$$\mathbf{y}_l = (\mathbf{I} - \mathbf{A}\mathbf{A}^+) \mathbf{y} \in \mathcal{C}^\perp(\mathbf{A}) \quad (4)$$



# FINDING FIXED POINTS

Given LTI system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ , where  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{u} \in \mathbb{R}^m$ , 1) find if there are states that can be made into fixed points, 2) find all states that can be made into fixed points with a constant control law.

Solution:

- 1 Yes, state  $\mathbf{x} = \mathbf{0}$  becomes a fixed point under control law  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{u} = -\mathbf{K}\mathbf{x}$ .
- 2 Let us find null space of the matrix  $[\mathbf{A} \ \mathbf{B}]$  as  $\mathbf{N} = \text{null}([\mathbf{A} \ \mathbf{B}])$ . We can find all  $\mathbf{x}, \mathbf{u}$  pairs that produce fixed points as follows:  $\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \mathbf{N}\mathbf{z}, \forall \mathbf{z}$ . Let  $\mathbf{N}_x$  be the first  $n$  rows of  $\mathbf{N}$ . Then all states that can be made into fixed points are given as  $\mathbf{x}^* = \mathbf{N}_x \mathbf{z}_x, \forall \mathbf{z}_x$

# CHECKING FIXED POINTS

Given LTI system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ , where  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{u} \in \mathbb{R}^m$ , 1) check if  $\mathbf{x}^*$  can be made into a fixed point, 2) find control constant  $\mathbf{u}^*$  that does it, given control law  $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$ .

Solution:

- 1 We can check that  $(\mathbf{A} - \mathbf{BK})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* = \mathbf{0}$  has a solution, in other words that  $-(\mathbf{A} - \mathbf{BK})\mathbf{x}^* \in \text{col}(\mathbf{B})$ . Resulting condition is given via projection into the left null space of  $\mathbf{B}$ :  $(\mathbf{I} - \mathbf{BB}^+)(\mathbf{A} - \mathbf{BK})\mathbf{x}^* = \mathbf{0}$
- 2 This means finding such  $\mathbf{u}^*$  that  $(\mathbf{A} - \mathbf{BK})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* = \mathbf{0}$ . This is done via pseudo-inverse, which provides exact solution, as long as it exists:  $\mathbf{u}^* = -\mathbf{B}^+(\mathbf{A} - \mathbf{BK})\mathbf{x}^*$ .

Given LTI system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ , where  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{u} \in \mathbb{R}^m$ , and a state  $\mathbf{x}^d$  which can not be made into a fixed point under constant control law, find the closest to it state  $\mathbf{x}^f$  which can be made into a fixed point.

As we know from the first example, for this LTI system all fixed points under constant control are given as  $\mathbf{x}^* = \mathbf{N}_x \mathbf{z}_x$ . To find the closest point to a given vector in a subspace, you project the vector into that subspace. In this case, we project  $\mathbf{x}^d$  to the column space (span) of  $\mathbf{N}_x$ :

$$\mathbf{x}^f = \mathbf{N}_x \mathbf{N}_x^+ \mathbf{x}^d \quad (5)$$

# FINDING FIXED POINTS FOR AFFINE SYSTEMS

Given LTI system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c}$ , where  $\mathbf{x} \in \mathbb{R}^n$ , and control law  $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$ , find all states that can be made fixed points by choosing appropriate  $\mathbf{u}^*$ .

We are required to find all solutions to the equation  $(\mathbf{A} - \mathbf{BK})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* + \mathbf{c} = \mathbf{0}$ . Let us define state-control pairs  $\mathbf{v} = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}$ .

We can easily find particular solution to this linear system:  
 $\mathbf{v}^p = -[(\mathbf{A} - \mathbf{BK}) \quad \mathbf{B}]^+ \mathbf{c}$ .

Finding null space basis  $\mathbf{N}$  for the matrix of this linear system:  $\mathbf{N} = \text{null}([\mathbf{A} - \mathbf{BK} \quad \mathbf{B}])$  we get the general solution as follows:  $\mathbf{v}^* = \mathbf{v}^p + \mathbf{N}\mathbf{z}$ . First  $n$  equations in the expression defining  $\mathbf{v}^*$  give us  $\mathbf{x}^*$ , the rest -  $\mathbf{u}^*$ .

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Control-Theory-Slides-Spring-2021](https://github.com/SergeiSa/Control-Theory-Slides-Spring-2021)

Check Moodle for additional links, videos, textbook suggestions.