

# Discrete Dynamics

## Control Theory, Lecture 6

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- Discrete Dynamics
- Stability of the Discrete Dynamics
- Discretization
  - ▶ Finite difference
  - ▶ Finite difference in an autonomous LTI
- Zero order hold
- ZOH and other types of discretization
  - ▶ Zero order hold vs First order hold
  - ▶ Exact discretization
- Read more

The following dynamical system is called *discrete*:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \quad (1)$$

Note that those:

- have no derivatives in the equation;
- are easily simulated.

The affine control for this system can be given as:

$$\mathbf{u}_i = -\mathbf{K}\mathbf{x}_i + \mathbf{u}_i^* \quad (2)$$

# STABILITY OF THE DISCRETE DYNAMICS

## Real eigenvalues

Let us consider stability of the discrete dynamical system where matrix  $\mathbf{A}$  has purely real eigenvalues:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i \quad (3)$$

With eigendecomposition  $\mathbf{A} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}$  (where  $\mathbf{D}$  is a diagonal matrix with eigenvalues  $\lambda_j$  of  $\mathbf{A}$  on its diagonal) and introducing notation  $\mathbf{z}_i = \mathbf{V}\mathbf{x}_i$  we get:

$$\mathbf{x}_{i+1} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}\mathbf{x}_i \quad (4)$$

$$\mathbf{z}_{i+1} = \mathbf{D}\mathbf{z}_i \quad (5)$$

Meaning that the dynamics became a system of independent scalar equations  $z_{j,i+1} = \lambda_j z_{j,i}$ .

# STABILITY OF THE DISCRETE DYNAMICS

## Real eigenvalues

Thus, with  $z_{j,i+1} = \lambda_j z_{j,i}$  we can find now the absolute value of the scalars  $z_j$  will dwindle with time iff  $|\lambda_j| < 1$ :

$$\left| \frac{z_{j,i+1}}{z_{j,i}} \right| = |\lambda_j| \quad (6)$$

# STABILITY OF THE DISCRETE DYNAMICS

## 2x2 system

Let us consider stability of the discrete dynamical system with a 2-by-2 matrix  $\mathbf{A}$ :

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \quad (7)$$

Let us find norms of  $\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}$  and  $\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}$ :

$$\left\| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \right\|^2 = x_{1,i}^2 + x_{2,i}^2 \quad (8)$$

$$\left\| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \right\|^2 = (\alpha^2 + \beta^2)(x_{1,i}^2 + x_{2,i}^2) \quad (9)$$

# STABILITY OF THE DISCRETE DYNAMICS

## 2x2 system

We can find the ratio of the norms of  $\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}$  and  $\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}$ :

$$\left\| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \right\|^2 / \left\| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \right\|^2 = \alpha^2 + \beta^2 \quad (10)$$

Remembering that eigenvalues of the system are  $\lambda = \alpha \pm j\beta$ , we can rewrite the expression above as:

$$\left\| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \right\|^2 / \left\| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \right\|^2 = |\lambda| \quad (11)$$

We can see that the norm of the variable  $\mathbf{x}$  will dwindle with time iff  $|\lambda| < 1$ .

# STABILITY OF THE DISCRETE DYNAMICS

General stability criterion is given below:

## Stability criterion

In general, discrete systems  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$  are stable as long as the eigenvalues of  $\mathbf{A}$  are smaller than 1 by absolute value:  $|\lambda_i(\mathbf{A})| \leq 1, \forall i$ . This is true for complex eigenvalues as well.



# DISCRETIZATION

## Finite difference

Consider linear time-invariant autonomous system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (12)$$

The time derivative  $\dot{\mathbf{x}}$  can be replaced with a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + \Delta t) - \mathbf{x}(t)) \quad (13)$$

Note that we could have also used other definitions of a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + 0.5\Delta t) - \mathbf{x}(t - 0.5\Delta t)) \quad (14)$$

or

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t) - \mathbf{x}(t - \Delta t)) \quad (15)$$

# DISCRETIZATION

## Finite difference notation

We can introduce notation:

$$\begin{cases} \mathbf{x}_0 = \mathbf{x}(0) \\ \mathbf{x}_1 = \mathbf{x}(\Delta t) \\ \mathbf{x}_2 = \mathbf{x}(2\Delta t) \\ \dots \\ \mathbf{x}_n = \mathbf{x}(n\Delta t) \end{cases} \quad (16)$$

We say that  $\mathbf{x}_i$  is the value of  $\mathbf{x}$  at the time step  $i$ . Then the finite difference can be written, for example, as follows:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t} (\mathbf{x}_{i+1} - \mathbf{x}_i) \quad (17)$$

# DISCRETIZATION

## Finite difference in an autonomous LTI

We can rewrite our original autonomous LTI as follows:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_i \quad (18)$$

Isolating  $\mathbf{x}_{i+1}$  on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{A}\Delta t + \mathbf{I})\mathbf{x}_i \quad (19)$$

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Or alternatively:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_{i+1} \quad (20)$$

Isolating  $\mathbf{x}_{i+1}$  on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{I} - \mathbf{A}\Delta t)^{-1}\mathbf{x}_i \quad (21)$$

# DISCRETIZATION

## Zero order hold

Defining *discrete state space matrix*  $\bar{\mathbf{A}}$  and *discrete control matrix*  $\bar{\mathbf{B}}$  as follows:

$$\bar{\mathbf{A}} = \mathbf{A}\Delta t + \mathbf{I} \quad (22)$$

$$\bar{\mathbf{B}} = \mathbf{B}\Delta t \quad (23)$$

We get discrete dynamics:

$$\mathbf{x}_{i+1} = \bar{\mathbf{A}}\mathbf{x}_i + \bar{\mathbf{B}}\mathbf{u}_i \quad (24)$$

This way of defining discrete dynamics is called *zero order hold (ZOH)*.

# ZOH AND OTHER TYPES OF DISCRETIZATION

## Zero order hold vs First order hold

Graphically, we can understand what zero order hold is, by comparing it to the first order hold:

Graphically, zero order hold is this:



First order hold is this:

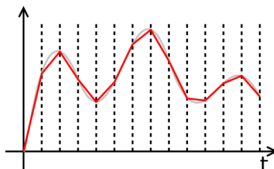


Figure 1: Different types of discretization

# ZOH AND OTHER TYPES OF DISCRETIZATION

## Exact discretization

Let the discrete state  $\mathbf{x}_i$  correspond to continuous state  $\mathbf{x}$  at the moment of time  $t_i$ . Then, we can say that the discretization is *exact* the following holds for any solution  $\mathbf{x}(t)$

$$\mathbf{x}_0 = \mathbf{x}(t_0) \rightarrow \mathbf{x}_i = \mathbf{x}(t_i), \quad \forall i \quad (25)$$

We can compute the exact discretization as follows:

$$\bar{\mathbf{A}} = e^{\mathbf{A}\Delta t} \quad (26)$$

$$\bar{\mathbf{B}} = \mathbf{B} \int_{t_0}^{t_0+\Delta t} e^{\mathbf{A}s} ds \quad (27)$$

- [Automatic Control 1 Discrete-time linear systems](#), Prof. Alberto Bemporad, University of Trento

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Control-Theory-Slides-Spring-2022](https://github.com/SergeiSa/Control-Theory-Slides-Spring-2022)

Check Moodle for additional links, videos, textbook suggestions.

