Hamilton-Jacobi-Bellman eq., Riccati eq., Linear Quadratic Regulator

Control Theory, Lecture 9

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Let us define dynamics:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{1}$$

with initial conditions $\mathbf{x}(0)$.

Additionally we define *control policy* as:

$$\mathbf{u} = \pi(\mathbf{x}, t) \tag{2}$$

To connect with the previous ways we talked about control, we can say that choosing different control gains and different feed-forward control amounts to choosing a different control policy.

Let J be an additive cost function:

$$J(\mathbf{x}_0, \pi(\mathbf{x}, t)) = \int_0^\infty g(\mathbf{x}, \mathbf{u}) dt \tag{3}$$

where $g(\mathbf{x}, \mathbf{u})$ is instantaneous cost and $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial conditions. Notice that J depends on $\mathbf{x}_0$ rather than $\mathbf{x}(t)$, since initial conditions and control policy completely define the trajectory of the system $\mathbf{x}(t)$.

Let J^* be the optimal (lowest possible) cost. In other words:

$$J^*(\mathbf{x}_0) = \inf_\pi J(\mathbf{x}_0, \pi(\mathbf{x}, t)) \tag{4}$$

Optimal cost is attained when optimal policy is attained: $\pi = \pi^*(\mathbf{x}, t)$

With this, we can formulate *Hamilton-Jacobi-Bellman equation* (HJB):

$$\min_{\mathbf{u}} \left[g(\mathbf{x}, \mathbf{u}) + \frac{\partial J^*}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) \right] = 0 \tag{5}$$

This can be loosely interpreted as follows: the value in square brackets is $\dot{J}(\mathbf{x}_0, \pi)$, which is equal to 0 when $\pi = \pi^*(\mathbf{x}, t)$, and is positive otherwise (in the small vicinity of π^*), as $J(\mathbf{x}_0, \pi^*)$ is smaller than any $J(\mathbf{x}_0, \pi)$, $\pi^* \neq \pi$.

We can find control that delivers minimum to the function (5):

$$u^* = \arg\min_{\mathbf{u}} \left[g(\mathbf{x}, \mathbf{u}) + \frac{\partial J^*}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) \right] \tag{6}$$

For LTI, dynamics is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{7}$$

We can choose quadratic cost:

$$g(\mathbf{x}, \mathbf{u}) = \mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{u}^\top \mathbf{R}\mathbf{u} \tag{8}$$

Then HJB becomes:

$$\min_{\mathbf{u}} \left[\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{u}^\top \mathbf{R}\mathbf{u} + \frac{\partial J^*}{\partial \mathbf{x}}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})\right] = 0 \tag{9}$$

where $\mathbf{Q} = \mathbf{Q}^\top \geq 0$ and $\mathbf{R} = \mathbf{R}^\top > 0$.

There is a theorem that says that for LTI with quadratic cost, J^* has the form:

$$J^* = \mathbf{x}^\top \mathbf{S} \mathbf{x} \tag{10}$$

where $\mathbf{S} = \mathbf{S}^\top \geq 0$.

Then HJB becomes:

$$\min_{\mathbf{u}} \left[\mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{u}^\top \mathbf{R} \mathbf{u} + \mathbf{x}^\top \mathbf{S} (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) + (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})^\top \mathbf{S} \mathbf{x} \right] = 0$$

Simplifying, we get:

$$\min_{\mathbf{u}} \left[\mathbf{u}^\top \mathbf{R} \mathbf{u} + \mathbf{x}^\top (\mathbf{Q} + \mathbf{S}\mathbf{A} + \mathbf{A}^\top \mathbf{S}) \mathbf{x} + \mathbf{x}^\top \mathbf{S} \mathbf{B} \mathbf{u} + \mathbf{u}^\top \mathbf{B}^\top \mathbf{S} \mathbf{x} \right] = 0$$

Finding partial derivative of the HJB with respect to $\mathbf{u}$ and setting it to zero (as it is an extreme point) we get:

$$2\mathbf{u}^\top \mathbf{R} + 2\mathbf{x}^\top \mathbf{S}\mathbf{B} = 0 \tag{11}$$

This expression can be transposed and $\mathbf{u}$ separated:

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{S}\mathbf{x} \tag{12}$$

This is the desired control law. We can see that it is *proportional*. We can re-write it as:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \tag{13}$$

where $\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\top \mathbf{S}$ is the controller gain. This control law is called Linear Quadratic Regulator (LQR).

Substituting found control law into the HJB, we find:

$$\min_{\mathbf{u}} [\mathbf{x}^\top(\mathbf{Q} + \mathbf{S}\mathbf{A} + \mathbf{A}^\top\mathbf{S})\mathbf{x} + \mathbf{x}^\top\mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{R}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{S}\mathbf{x} - \\ - \mathbf{x}^\top\mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{S}\mathbf{x} - \mathbf{x}^\top\mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{S}\mathbf{x}] = 0 \tag{14}$$

Simplifying, we get:

$$\mathbf{x}^\top(\mathbf{Q} + \mathbf{S}\mathbf{A} + \mathbf{A}^\top\mathbf{S} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{S})\mathbf{x} = 0 \tag{15}$$

which would hold for all $\mathbf{x}$ iff:

$$\mathbf{Q} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{A}^\top\mathbf{S} = 0 \tag{16}$$

This is the *Algebraic Riccati equation*.

There are a number of ways to solve LQR:

- In MATLAB there is a function `[K,S,P] = lqr(A,B,Q,R)`, where `P=eig(A-B*K)`
- In Python, there is `S = scipy.linalg.solve_continuous_are(A,B,Q,R)`
- In Drake (by MIT and Toyota Research) there is a function `(K,S) = LinearQuadraticRegulator(A,B,Q,R)`

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2022

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