Control for systems with explicit constraints

Control Theory, Lecture 13

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- Explicit and implicit constraints
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LTI systems we studied before have the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state of the system. This form has no explicit constraints.

Let us introduce one of the forms of a linear dynamical system with explicit constraints:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\lambda \\ \mathbf{G}\dot{\mathbf{x}} = \mathbf{0} \end{cases}$$

where $\mathbf{G}$ is a constraint matrix, $\lambda \in \mathbb{R}^k$ is the constraints reaction forces, and $\mathbf{F}$ is the reaction force Jacobian.

Consider a two mass system with a spring:

$$\begin{cases} \ddot{x}_1 + \mu \dot{x}_1 + k(x_1 - x_2) = 0 \\ \ddot{x}_2 + \mu \dot{x}_2 + k(x_2 - x_1) = 0 \end{cases}$$

We can add a constraint $x_2 = 10$. This implies that $\ddot{x}_2 = 0$. Corresponding system of equations is:

$$\begin{cases} \ddot{x}_1 + \mu \dot{x}_1 + k(x_1 - x_2) = 0 \\ \ddot{x}_2 + \mu \dot{x}_2 + k(x_2 - x_1) = \lambda \\ \ddot{x}_2 = 0 \end{cases}$$

But that is the same as:

$$\ddot{x}_1 + \mu \dot{x}_1 + k(x_1 - 10) = 0$$

Thus we transformed the system with *explicit constraints* into a system with *implicit constraints*

Figure 1: Walking robots

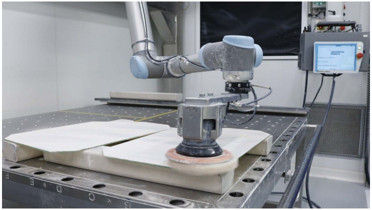

Figure 2: Polishing with industrial arms

Typical reasons why explicit constraints arise

Explicit constraints are usually not a necessity and not a physical property of the problem. However, they are often encountered in practice. Typical situations when they are encountered as as follows:

- Systems with contact interactions.
- Hybrid systems (two or more different dynamics which switch between one-another).
- Nonholonomic constraints in the dynamics (dynamics of a unicycle, bicycle, etc.).
- Dynamics is more clear and easy to work which when non-minimal representation is used.

Ways to control systems with explicit constraints

There are basic ways to deal with such systems:

- Reduce to a system with implicit constraints and control that system instead.
- Treat reaction forces as a yet another external force.
- Design control law based on the explicit representation of constraints.

Consider equations in the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \tag{1}$$

where $\mathbf{A}$ is the state matrix, $\mathbf{B}$ is the control matrix and $\mathbf{c}$ is the affine term of the affine dynamics model.

For systems with constraints the same linearization takes form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{S}\lambda + \mathbf{c} \\ \mathbf{G}\dot{\mathbf{x}} = 0 \end{cases} \tag{2}$$

where $\mathbf{S}$ is linearized constraint Jacobian and $\mathbf{G} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \dot{\mathbf{F}} & \mathbf{F} \end{bmatrix}$.

We can observe that constraint $\mathbf{G}\dot{\mathbf{x}} = 0$ implies that all feasible state velocities $\dot{\mathbf{x}}$ lie in the null space of $\mathbf{G}$. This means that we can introduce a new lower dimensional variable $\mathbf{z}$ to describe $\mathbf{x}$ (assuming initial value of $\mathbf{x}$ lies in the column space of $\mathbf{N}$):

$$\mathbf{N}\mathbf{z} = \mathbf{x} \tag{3}$$

where $\mathbf{N} = \text{null}(\mathbf{G})$ - orthonormal basis in the null space of $\mathbf{G}$.

Let us re-express dynamics (2) in terms of $\mathbf{z}$ by multiplying it by $\mathbf{N}^\top$ on the left:

$$\mathbf{N}^\top \dot{\mathbf{x}} = \mathbf{N}^\top \mathbf{A}\mathbf{x} + \mathbf{N}^\top \mathbf{B}\mathbf{u} + \mathbf{N}^\top \mathbf{S}\lambda + \mathbf{N}^\top \mathbf{c} \tag{4}$$

We can prove that $\mathbf{N}^\top \mathbf{S} = 0$ for all mechanical systems (for example, by observing that mechanical constrains do not do work) or check that our particular $\mathbf{S}$ lies in the row space of our $\mathbf{G}$.

Noting that $\dot{\mathbf{z}} = \mathbf{N}^\top \dot{\mathbf{x}}$ and $\mathbf{x} = \mathbf{N}\mathbf{z}$ we get:

$$\dot{\mathbf{z}} = \mathbf{N}^\top \mathbf{A}\mathbf{N}\mathbf{z} + \mathbf{N}^\top \mathbf{B}\mathbf{u} + \mathbf{N}^\top \mathbf{c} \tag{5}$$

Defining $\mathbf{A}_N = \mathbf{N}^\top \mathbf{A}\mathbf{N}$, $\mathbf{B}_N = \mathbf{N}^\top \mathbf{B}$ and $\mathbf{c}_N = \mathbf{N}^\top \mathbf{c}$ we get:

$$\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{B}_N \mathbf{u} + \mathbf{c}_N \tag{6}$$

Since we achieved that our constrained dynamics is written in the standard LTI form:

$$\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{B}_N \mathbf{u} + \mathbf{c}_N, \tag{7}$$

we can use standard LTI control methods on it, for example finding optimal feedback gains via pole placement or LQR:

$$\mathbf{K}_N = \mathrm{lqr}(\mathbf{A}_N, \mathbf{B}_N, \mathbf{Q}, \mathbf{R}) \tag{8}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are matrices defining cost function for the LQR problem.

For any LTI system, including the LTI form of a constrained system we saw previously, inverse dynamics can be solved precisely by a pseudo-inverse, as long as there exist a solution. The following condition verifies it:

$$(\mathbf{I} - \mathbf{B}\mathbf{B}^{+})(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c}) = 0, \tag{9}$$

The condition checks if vector $(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c})$ lies in the column space of $\mathbf{B}$. If it holds, precise solution to inverse kinematics can be found as:

$$\mathbf{u}_{ID} = \mathbf{B}^{+}(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c}). \tag{10}$$

For a constrained mechanical system we can solve inverse dynamics without the need for linearization. Consider the following dynamics:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{F}^\top \lambda \tag{11}$$

We can represent constraint Jacobian $\mathbf{F}^\top$ as its QR decomposition: $\mathbf{F}^\top = \mathbf{Q}\begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}$, where $\mathbf{Q}^\top\mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$ and $\mathbf{R}$ is convertible.

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{Q}\begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}\lambda \tag{12}$$

Let us multiply the equation by $\mathbf{Q}^\top$:

$$\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{Q}^\top\mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{Q}^\top\mathbf{T}\mathbf{u} + \begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}\lambda \tag{13}$$

Introducing switching variables (to divide upper and lower part of the equations) $\mathbf{S}_1 = \begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}$ and $\mathbf{S}_2 = \begin{bmatrix}\mathbf{0} & \mathbf{I}\end{bmatrix}$ and multiplying equations by one and the other we get two systems:

$$\begin{cases}\mathbf{S}_1\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{Q}^\top\mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{S}_1\mathbf{Q}^\top\mathbf{T}\mathbf{u} + \mathbf{R}\lambda \\ \mathbf{S}_2\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{Q}^\top\mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{S}_2\mathbf{Q}^\top\mathbf{T}\mathbf{u}\end{cases} \tag{14}$$

The main advantage we achieved is that now we can calculate both $\mathbf{u}$ and λ

Resulting expression for $\mathbf{u}$ is:

$$\mathbf{u} = (\mathbf{S}_2\mathbf{Q}^\top\mathbf{T})^+\mathbf{S}_2\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{Q}^\top\mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) \tag{15}$$

Expression for λ is:

$$\lambda = \mathbf{R}^{-1}\mathbf{S}_1\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{Q}^\top\mathbf{C}\dot{\mathbf{q}} + \mathbf{g} - \mathbf{T}\mathbf{u}) \tag{16}$$

We can notice a pseudo-inverse, implying that the no-residual solution does not have to exist.

We can easily write inverse dynamics as a QP:

$$
\begin{aligned}
\underset{\mathbf{u},\lambda}{\text{minimize}} \quad & ||\mathbf{u}||, \\
\text{subject to} \quad & \begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{F}^{\top}\lambda \\ \mathbf{F}\ddot{\mathbf{q}} + \dot{\mathbf{F}}\dot{\mathbf{q}} = 0 \end{cases}
\end{aligned}
\tag{17}
$$

If there are some constraints or limits on the control input (torque limits, for instance) or the reaction forces are restricted (by friction cones, for instance), those can be directly added.

Read more

- Mason, S., Righetti, L. and Schaal, S., 2014, November. Full dynamics LQR control of a humanoid robot: An experimental study on balancing and squatting. In 2014 IEEE-RAS International Conference on Humanoid Robots (pp. 374-379). IEEE.
- Mason, S., Rotella, N., Schaal, S. and Righetti, L., 2016, November. Balancing and walking using full dynamics LQR control with contact constraints. In 2016 IEEE-RAS 16th International Conference on Humanoid Robots (Humanoids) (pp. 63-68). IEEE. - arxiv.org/pdf/1701.08179
- Mistry, M., Buchli, J. and Schaal, S., 2010, May. Inverse dynamics control of floating base systems using orthogonal decomposition. In 2010 IEEE international conference on robotics and automation (pp. 3406-3412). IEEE. - citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.212.3601&rep=re

Thank you!

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