Stability

Control Theory, Lecture 2

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Content

Critical point (node)

Consider the following ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

Let $\mathbf{x}_0$ be such a state that:

$$\mathbf{f}(\mathbf{x}_0, t) = 0 \tag{2}$$

Then such state $\mathbf{x}_0$ is called a *node* or a *critical point*.

Node $\mathbf{x}_0$ is called *stable* iff for any constant δ there exists constant ε such that:

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \ \longrightarrow \ ||\mathbf{x}(t) - \mathbf{x}_0|| < \varepsilon \tag{3}$$

Think of it as "for any initial point that lies at most δ away from $\mathbf{x}_0$, the rest of the trajectory $\mathbf{x}(t)$ will be at most ε away from $\mathbf{x}_0$".

Or, more picturesque, think of it as "the solutions with different initial conditions do not diverge from the node"

Node $\mathbf{x}_0$ is called *asymptotically stable* iff for any constant δ it is true that:

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \longrightarrow \lim_{t \to \infty} \mathbf{x}(t) = \mathbf{x}_0 \tag{4}$$

Think of it as "for any initial point that lies at most δ away from $\mathbf{x}_0$, the trajectory $\mathbf{x}(t)$ will asymptotically approach the point $\mathbf{x}_0$".

Or, more picturesque, think of it as "the solutions with different initial conditions converge to the node"

Stability vs Asymptotic stability

Example

Consider dynamical system $\dot{x} = 0$, and solution $x = 7$. This solution is stable, but not asymptotically stable (other solutions do not diverge from $x = 7$, but do not converge to it either).

Example

Consider dynamical system $\dot{x} = -x$, and solution $x = 0$. This solution is stable and asymptotically stable (other solutions converge to $x = 0$).

Example

Consider dynamical system $\dot{x} = x$, and solution $x = 0$. This solution is unstable (other solutions diverge from $x = 0$).

LTI and autonomous LTI

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5}$$

This is called a *linear time-invariant system*, or *LTI*.

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

This is also an LTI, but it is also called an *autonomous system*, since its evolution depends only on the state of the system.

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{7}$$

where $\mathbf{A}$ can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

$$\dot{\mathbf{x}} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x} \tag{8}$$

Multiply it by $\mathbf{V}^{-1} \longrightarrow \mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x}$.
Define $\mathbf{z} = \mathbf{V}^{-1}\mathbf{x} \longrightarrow \dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$.

Since elements of $\mathbf{D}$ are real, we can clearly see, that iff they are *all negative* will the system be asymptotically stable. If they are non-positive, the system is stable. And those elements are eigenvalues of $\mathbf{A}$.

Complex eigenvalues, 2-dimensional case (1)

Let us consider the following system:

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{9}$$

The eigenvalues of the system are $\alpha \pm i\beta$. We denote $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \mathbf{x}$.

We start by claiming that the system will be stable iff the $\dot{\mathbf{x}}^\top \mathbf{x} < 0$. Indeed, vector $\dot{\mathbf{x}}$ can always be decomposed into two components, $\dot{\mathbf{x}}_{||}$ parallel to $\mathbf{x}$, and $\dot{\mathbf{x}}_{\perp}$ perpendicular to $\mathbf{x}$. By definition $\dot{\mathbf{x}}_{\perp}^\top \mathbf{x} = 0$, and is responsible for the change in orientation of $\mathbf{x}$. The value of $\dot{\mathbf{x}}_{||}$ is responsible for the change in the length of $\mathbf{x}$; the length would shrink iff $\dot{\mathbf{x}}_{||}$ is of opposite direction to $\mathbf{x}$, giving negative value of the dot product $\dot{\mathbf{x}}^\top \mathbf{x}$.

Complex eigenvalues, 2-dimensional case (2)

Let us compute $\dot{\mathbf{x}}^\top \mathbf{x}$:

$$\dot{\mathbf{x}}^\top \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{10}$$

$$\dot{\mathbf{x}}^\top \mathbf{x} = \alpha(\mathbf{x}_1^2 + \mathbf{x}_2^2) \tag{11}$$

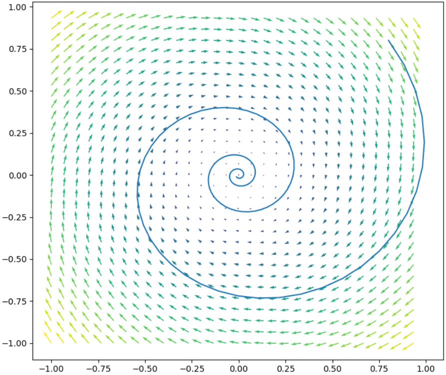
From this it is clear that the product $\dot{\mathbf{x}}^\top \mathbf{x} < 0$ is negative iff $\alpha < 0$.

Definition

As long as the *real parts of the eigenvalues* of the system are *strictly negative*, the system is *asymptotically stable*. If the real parts of the eigenvalues of the system are zero, the system is *marginally stable*.

Complex eigenvalues, 2-dimensional case (3)

Vector field of $\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$ is shown below:



General case (1)

Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{U}\mathbf{C}\mathbf{U}^{-1}$, where $\mathbf{C}$ is a complex-valued diagonal matrix and $\mathbf{U}$ is a complex-valued inevitable matrix.

We multiply both sides by $\mathbf{U}^{-1}$, then define $\mathbf{z} = \mathbf{U}^{-1}\mathbf{x}$ to arrive at:

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} \tag{12}$$

which falls into a set of independent equations, with complex coefficients c_j:

$$\dot{z}_j = c_j z_j \tag{13}$$

General case (2)

Expanding $c_j = \alpha + i\beta$, and $z_j = u + iv$ (we dismiss subscripts for clarity), we find that $\dot{z}_j = c_j z_j$ can be expanded as:

$$\dot{u} + i\dot{v} = \dot{z}_j = c_j z_j = (\alpha + i\beta)(u + iv) \tag{14}$$

$$\dot{u} + i\dot{v} = \alpha u + i\beta u + i\alpha v - \beta v \tag{15}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{16}$$

As we can see, $\dot{z}_j = c_j z_j$ is asymptotically stable iff $\mathrm{Re}(c_j) < 0$, and marginally stable if $\alpha = \mathrm{Re}(c_j) = 0$. Same is true for $\dot{\mathbf{z}} = \mathbf{C}\mathbf{z}$ and hence, for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, as $\mathbf{U}$ is invertible.

Consider an autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{17}$$

Definition

Eq. (17) is stable iff real parts of eigenvalues of $\mathbf{A}$ are non-positive.

Definition

Eq. (17) is asymptotically stable iff real parts of eigenvalues of $\mathbf{A}$ are negative.

Illustration

Here is an illustration of *phase portraits* of two-dimensional LTIs with different types of stability:

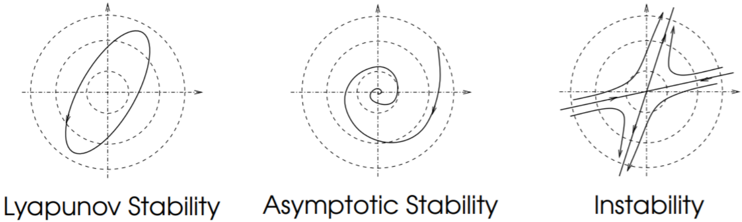
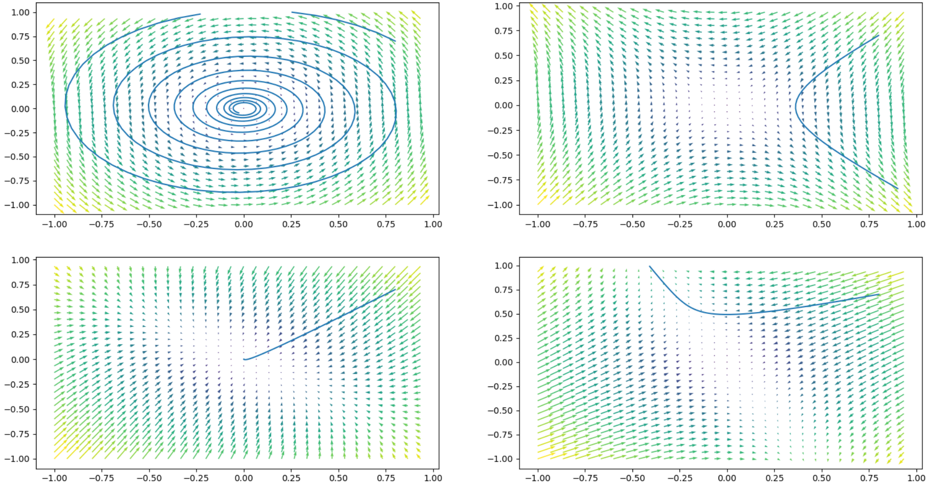


Figure 1: phase portraits for different types of stability

Credit: staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf


1.00 0.75 0.50 0.25 0.00 −0.25 −0.50 −0.75 −1.00
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00

- Control Systems Design, by Julio H. Braslavsky
  staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.