From linear ODE to State Space

Given an ODE:

$$a_k y^{(k)} + a_{k-1} y^{(k-1)} + \ldots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_0$$

find its state space representation:

$$\dot{x} = Ax + b$$

Process

The first step is to express higher derivatives

Step 1.1:

$$y^{(k)} + \frac{a_{k-1}}{a_k} y^{(k-1)} + \ldots + \frac{a_2}{a_k} \ddot{y} + \frac{a_1}{a_k} \dot{y} + \frac{a_0}{a_k} y = \frac{b_0}{a_k}$$

Step 1.2:

$$y^{(k)} = -\frac{a_{k-1}}{a_k} y^{(k-1)} - \ldots - \frac{a_2}{a_k} \ddot{y} - \frac{a_1}{a_k} \dot{y} - \frac{a_0}{a_k} y + \frac{b_0}{a_k}$$

Second step s introduction of new variables x:

Step 2.1:

$$\begin{gathered} x_k = y^{(k-1)} \\ x_{k-1} = y^{(k-2)} \\ \ldots \\ x_1 = y \end{gathered}$$

Step 2.2:

$$\begin{gathered} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \ldots \\ \dot{x}_k = -\frac{a_{k-1}}{a_k} x_k - \ldots - \frac{a_2}{a_k} x_3 - \frac{a_1}{a_k} x_2 - \frac{a_0}{a_k} x_1 + \frac{b_0}{a_k} \end{gathered}$$

Finally, we write it in a matrix form.

Tasks 1.1: ODE to State Space conversion

Convert to State Space represantation and to a transfer function representation

- $10y^{(4)} - 7y^{(3)} + 2\ddot{y} + 0.5\dot{y} + 4y = 15u$
- $5y^{(4)} - 17y^{(3)} - 3\ddot{y} + 1.5\dot{y} + 2y = 25u$
- $-3y^{(4)} + 22y^{(3)} + 4\ddot{y} + 1.5\dot{y} + 1y = 15u$

- $5y^{(4)} - 17y^{(3)} - 1.5\ddot{y} + 100\dot{y} + 1.1y = 45u$
- $1.5y^{(4)} - 23y^{(3)} - 2.5\ddot{y} + 0.1\dot{y} + 100y = -10u$

Task 1.2 (higher difficulty)

Convert the following to a second order ODE and to a transfer function representation:

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ -5 & -10 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$\dot{x} = \begin{pmatrix} 0 & 8 \\ 1 & 3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$\dot{x} = \begin{pmatrix} 0 & 8 \\ 6 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

For all of the above,

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Solve ODE

Below is an example of how one can solve and ODE in Python

```
import numpy as np
from scipy.integrate import odeint

n = 4
A = np.array([[0, 1, 0], [0, 0, 1], [-10, -5, -2]])

# x_dot from state space
def StateSpace(x, t):
    return A.dot(x)# + B*np.sin(t)

time = np.linspace(0, 1, 1000)
x0 = np.random.rand(n-1)  # initial state

solution = {"SS": odeint(StateSpace, x0, time)}

import matplotlib.pyplot as plt

plt.subplot(121)
plt.plot(time, solution["SS"])
plt.xlabel('time')
plt.ylabel('x(t)')

plt.show()
```

Task 1.3 Implement Euler Integration or Runge-Kutta Integration scheme, solve the equation from the Task 1 using it.

Task 2.1, convert to state space and simulate

- $10y^{(5)} + 10y^{(4)} - 7y^{(3)} + 2\ddot{y} + 0.5\dot{y} + 4y = 0$
- $1y^{(5)} + 5y^{(4)} - 17y^{(3)} - 3\ddot{y} + 1.5\dot{y} + 2y = 0$
- $6y^{(5)} - 3y^{(4)} + 22y^{(3)} + 4\ddot{y} + 1.5\dot{y} + 1y = 0$
- $22y^{(5)} + 5y^{(4)} - 17y^{(3)} - 1.5\ddot{y} + 100\dot{y} + 1.1y = 0$
- $-10y^{(5)} + 1.5y^{(4)} - 23y^{(3)} - 2.5\ddot{y} + 0.1\dot{y} + 100y = 0$

Task 2.2, convert to state space and simulate

- $10y^{(5)} + 10y^{(4)} - 7y^{(3)} + 2\ddot{y} + 0.5\dot{y} + 4y = \sin(t)$
- $1y^{(5)} + 5y^{(4)} - 17y^{(3)} - 3\ddot{y} + 1.5\dot{y} + 2y = \sin(t)$
- $6y^{(5)} - 3y^{(4)} + 22y^{(3)} + 4\ddot{y} + 1.5\dot{y} + 1y = \sin(t)$
- $22y^{(5)} + 5y^{(4)} - 17y^{(3)} - 1.5\ddot{y} + 100\dot{y} + 1.1y = \sin(t)$
- $-10y^{(5)} + 1.5y^{(4)} - 23y^{(3)} - 2.5\ddot{y} + 0.1\dot{y} + 100y = \sin(t)$

Subtask 2.3 Mass-spring-damper system

Find or derive equations for a mass-spring-damper system with mass 10kg, spting stiffness of 1000 N / m and damping coefficient 1 N s / m, write them in state-space and second order ODE forms, and simulate them.

Task 3.1, Convert to transfer functions

- $\begin{cases} \ddot{x} + 0.5\dot{x} + 4y = u \\ y = 1.5\dot{x} + 6x \end{cases}$
- $\begin{cases} 10\ddot{x} + 1.5\dot{x} + 8y = 0.5u \\ y = 15\dot{x} + 16x \end{cases}$
- $\begin{cases} \ddot{x} + 2\dot{x} - 5y = u \\ y = 2.5\dot{x} - 7x \end{cases}$
- $\begin{cases} \ddot{x} + 22\dot{x} + 10y = 10u \\ y = 10.5\dot{x} + 11x \end{cases}$

4. Stability of an autonomous linear system

Autonomous linear system is *stable*, iff the eigenvalues of its matrix have negative real parts. In other words, their should lie on the left half of the complex plane.

Consider the system:

$$\dot{x} = \begin{pmatrix} -1 & 0.4 \\ -20 & -16 \end{pmatrix} x$$

Let us find its eigenvalues:

```
import numpy as np
from numpy.linalg import eig

A = np.array([[-1, 0.4], [-20, -16]]) # state matrix
e, v = eig(A)
print("eigenvalues of A:", e)
```

```
eigenvalues of A: [ -1.55377801 -15.44622199]
```

The eigenvalues are $\lambda_1 = -1.55$ and $\lambda_1 = -15.44$, both real and negative. Let us test those and show that the system's state converges:

```
from scipy.integrate import odeint
import matplotlib.pyplot as plt

def LTI(x, t):
    return A.dot(x)

time = np.linspace(0, 10, 1000)    # interval from 0 to 10
x0 = np.random.rand(2)            # initial state

solution = odeint(LTI, x0, time)

plt.plot(time, solution)
plt.xlabel('time')
plt.ylabel('x(t)')
plt.show()
```

Task 4.1. Find if the following autonomous linear systems are stable

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ -5 & -10 \end{pmatrix} x$$
$$\dot{x} = \begin{pmatrix} 0 & 8 \\ 1 & 3 \end{pmatrix} x$$
$$\dot{x} = \begin{pmatrix} 0 & 8 \\ 6 & 0 \end{pmatrix} x$$
$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} x$$

Task 4.2 Simulate all of them, to show convergence.

Task 4.3 Add a constant term to the equation and show via simulation how the point where the system converges changes (two examples are sufficient).