Frequency response, Bode

Control Theory, Lecture 4

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- *Fourier series* can be seen as representing a periodic function as a sum of harmonics (sines and cosines). These sines and cosines can be thought of as forming a basis in a linear space. The coefficients of the series can be thought of as a discrete spectrum of the function.
- *Fourier transform* gives a continuous spectrum of the function. The "basis" is still made of harmonic functions.
- *Laplace transform* also gives a continuous spectrum of the function, but in a different basis: the basis is given by complex exponentials. I like to think of this basis as solutions of second order ODEs.

Laplace and Fourier transforms

Let's compare. Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi j t \omega} dt, \quad \omega \in \mathbb{R} \tag{1}$$

Laplace transform:

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt, \quad s \in \mathbb{C} \tag{2}$$

We can see that Fourier looks like Laplace with purely imaginary number in the exponent.

From analysing solutions of linear ODEs we know that, given harmonic input (sine, cosine, their combination) "after the transient process is over, the solution approaches a harmonic with the same frequency", but possibly different amplitude and phase.

Intuitively we can think of the imaginary part of s as having to do with this frequency response.

The kernel function of the Laplace transform is e^{-st} with $s = \sigma + j\omega$ being a complex variable. If $\sigma = 0$, the kernel becomes $e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$. You can see the similarity with Fourier transform kernel.

The first key idea of a Bode plot is substitution of purely complex variable $j\omega$ in place of Laplace variable s, which can have non-zero real part.

Given a transfer function $W(s)$, $s = \sigma + j\omega$ we can analyse its behaviour when $\sigma = 0$. We can plot its amplitude $a(\omega) = |W(j\omega)|$ and its phase $\varphi(\omega) = \text{atan2}(\text{im}(W(j\omega)),\ \text{real}(W(j\omega)))$.

Bode plot is actually two plots, 1) $20 \cdot \log(a(\omega))$ and 2) $\frac{180}{\pi}\varphi(\omega)$. The 20 and log has to do with the vertical axis being in decibels.

Consider $W(s) = \frac{1}{1+s}$. Then $W(j\omega) = \frac{1}{1+j\omega}$. We can transform it as:

$$W(j\omega) = \frac{1 - j\omega}{(1 + j\omega)(1 - j\omega)} = \frac{1 - j\omega}{1 + \omega^2} \tag{3}$$

We that we know that $\mathrm{real}(W(j\omega)) = \frac{1}{1+\omega^2}$ and $\mathrm{im}(W(j\omega)) = -\frac{\omega}{1+\omega^2}$.

Bode plot is then given as:

$$a(\omega) = \sqrt{\frac{1 - \omega^2}{(1 + \omega^2)^2}} \tag{4}$$

$$\varphi(\omega) = \mathrm{atan2}\left(-\frac{\omega}{1 + \omega^2},\ \frac{1}{1 + \omega^2}\right) \tag{5}$$

Before we discuss the use of Bode plot, let us remember that closed-loop transfer function has form (when simple feedback is used):

$$W(s) = \frac{G(s)}{1 + G(s)} \tag{6}$$

Substituting $s \longrightarrow j\omega$ we get:

$$W(\omega) = \frac{G(j\omega)}{1 + G(j\omega)} \tag{7}$$

From this we can see that $W(\omega)$ becomes ill-defined if $G(j\omega) = -1$. Meaning, we want to avoid two things happening simultaneously: the amplitude of $G(j\omega)$ being equal to 1, and its phase (argument) being equal to $180°$ (remember, phase of $0°$ is pure positive real number, phase of $90°$ is pure positive imaginary number, $180°$ is pure negative real number, etc.).

Let's check an illustration:

Check the colab notebook based on the example above for an illustration of how the Bode plot can be made by hand or via scipy signal library.

Read more

- Control System Lectures - Bode Plots, Introduction

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.