Discrete Dynamics

Control Theory, Lecture 6

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The following dynamical system is called *discrete*:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \tag{1}$$

Note that those:

- have no derivatives in the equation;
- are easily simulated.

The affine control for this system can be given as:

$$\mathbf{u}_i = -\mathbf{K}\mathbf{x}_i + \mathbf{u}_i^* \tag{2}$$

Let us consider stability of the discrete dynamical system where matrix $\mathbf{A}$ has purely real eigenvalues:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i \tag{3}$$

With eigendecomposition $\mathbf{A} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}$ (where $\mathbf{D}$ is a diagonal matrix with eigenvalues λ_j of $\mathbf{A}$ on its diagonal) and introducing notation $\mathbf{z}_i = \mathbf{V}\mathbf{x}_i$ we get:

$$\mathbf{x}_{i+1} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}\mathbf{x}_i \tag{4}$$

$$\mathbf{z}_{i+1} = \mathbf{D}\mathbf{z}_i \tag{5}$$

Meaning that the dynamics became a system of independent scalar equations $z_{j,i+1} = \lambda_j z_{j,i}$.

Thus, with $z_{j,i+1} = \lambda_j z_{j,i}$ we can find now the absolute value of the scalars z_j will dwindle with time iff $|\lambda_j| < 1$:

$$\left| \frac{z_{j,i+1}}{z_{j,i}} \right| = |\lambda_j| \tag{6}$$

Let us consider stability of the discrete dynamical system with a 2-by-2 matrix $\mathbf{A}$:

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \tag{7}$$

Let us find norms of $\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}$ and $\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}$:

$$\left|\left|\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}\right|\right|^2 = x_{1,i}^2 + x_{2,i}^2 \tag{8}$$

$$\left|\left|\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}\right|\right|^2 = (\alpha^2 + \beta^2)(x_{1,i}^2 + x_{2,i}^2) \tag{9}$$

We can find the ratio of the norms of $\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix}$ and $\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}$:

$$\left|\left| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \right|\right|^2 / \left|\left| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \right|\right|^2 = \alpha^2 + \beta^2 \tag{10}$$

Remembering that eigenvalues of the system are $\lambda = \alpha \pm j\beta$, we can rewrite teh expression above as:

$$\left|\left| \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} \right|\right|^2 / \left|\left| \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} \right|\right|^2 = |\lambda| \tag{11}$$

We can see that the norm of the variable $\mathbf{x}$ will dwindle with time iff $|\lambda| < 1$.

General stability criterion is given below:

Stability criterion

In general, discrete systems $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$ are stable as long as the eigenvalues of $\mathbf{A}$ are smaller than 1 by absolute value: $|\lambda_i(\mathbf{A})| \leq 1,\ \forall i.$ This is true for complex eigenvalues as well.

Consider linear time-invariant autonomous system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{12}$$

The time derivative $\dot{\mathbf{x}}$ can be replaces with a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + \Delta t) - \mathbf{x}(t)) \tag{13}$$

Note that we could have also used other definitions of a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + 0.5\Delta t) - \mathbf{x}(t - 0.5\Delta t)) \tag{14}$$

or

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t) - \mathbf{x}(t - \Delta t)) \tag{15}$$

We can introduce notation:

$$\begin{cases} \mathbf{x}_0 = \mathbf{x}(0) \\ \mathbf{x}_1 = \mathbf{x}(\Delta t) \\ \mathbf{x}_2 = \mathbf{x}(2\Delta t) \\ ... \\ \mathbf{x}_n = \mathbf{x}(n\Delta t) \end{cases} \tag{16}$$

We say that $\mathbf{x}_i$ is the value of $\mathbf{x}$ at the time step i. Then the finite difference can be written, for example, as follows:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) \tag{17}$$

We can rewrite our original autonomous LTI as follows:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_i \tag{18}$$

Isolating $\mathbf{x}_{i+1}$ on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{A}\Delta t + \mathbf{I})\mathbf{x}_i \tag{19}$$

Or alternatively:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_{i+1} \tag{20}$$

Isolating $\mathbf{x}_{i+1}$ on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{I} - \mathbf{A}\Delta t)^{-1}\mathbf{x}_i \tag{21}$$

Defining *discrete state space matrix* $\bar{\mathbf{A}}$ and *discrete control matrix* $\bar{\mathbf{B}}$ as follows:

$$\bar{\mathbf{A}} = \mathbf{A}\Delta t + \mathbf{I} \tag{22}$$

$$\bar{\mathbf{B}} = \mathbf{B}\Delta t \tag{23}$$

We get discrete dynamics:

$$\mathbf{x}_{i+1} = \bar{\mathbf{A}}\mathbf{x}_i + \bar{\mathbf{B}}\mathbf{u}_i \tag{24}$$

This way of defining discrete dynamics is called *zero order hold (ZOH)*.

Graphically, we can understand what zero order hold is, by comparing it to the first order hold:

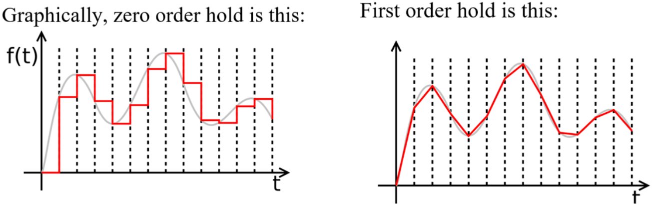

Figure 1: Different types of discretization

Let the discrete state $\mathbf{x}_i$ correspond to continuous state $\mathbf{x}$ at the moment of time t_i. Then, we can say that the discretization is *exact* the following holds for any solution $\mathbf{x}(t)$

$$\mathbf{x}_0 = \mathbf{x}(t_0) \rightarrow \mathbf{x}_i = \mathbf{x}(t_i),\ \forall i \tag{25}$$

We can compute the exact discretization as follows:

$$\bar{\mathbf{A}} = e^{\mathbf{A}\Delta t} \tag{26}$$

$$\bar{\mathbf{B}} = \mathbf{B} \int_{t_0}^{t_0+\Delta t} e^{\mathbf{A}s} ds \tag{27}$$

Read more

- Automatic Control 1 Discrete-time linear systems, Prof. Alberto Bemporad, University of Trento

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.