

LMI: Control design and robustness

Control Theory, Lecture 11

by Sergei Savin

Spring 2022

- LMI
- Control design
- Robustness
- S-procedure
- Appendix A

A linear matrix inequality (LMI) is a semidefinite constraint placed on a matrix:

$$\mathbf{S} \succ 0 \quad (1)$$

We assume (and this is true!) that there exist *solvers* that can solve problems with such constraints.

Example

Given \mathbf{A} , find such $\mathbf{S} \succ 0$ that $\mathbf{A}^\top \mathbf{S} + \mathbf{S} \mathbf{A} \prec 0$.

Notice that the last example is continuous-time Lyapunov eq. for LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, and if such \mathbf{S} exists the system is stable.

Consider a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, control $\mathbf{u} = \mathbf{K}\mathbf{x}$ and a Lyapunov function $V = \mathbf{x}^\top \mathbf{S}\mathbf{x}$, $\mathbf{S} \succ 0$.

Closed-form of the system is $\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}$, and full derivative of the Lyapunov function:

$$\dot{V} = \mathbf{x}^\top (\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x} \leq 0 \quad (2)$$

This can be re-written as an LMI:

$$(\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S} + \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K}) \prec 0 \quad (3)$$

This is *not linear* in decision variables (\mathbf{S} and \mathbf{K}), and can't be solved directly using popular solvers.

Introducing new variable $\mathbf{P} = \mathbf{S}^{-1}$ and multiplying (3) by \mathbf{P} on both sides (we can do it, as both \mathbf{P} and \mathbf{S} are full rank, and thus it is a congruence transformation which preserves definiteness, see appendix) we get:

$$\mathbf{P}(\mathbf{A} + \mathbf{BK})^\top + (\mathbf{A} + \mathbf{BK})\mathbf{P} \prec 0 \quad (4)$$

Now we introduce one more variable $\mathbf{L} = \mathbf{KP}$ and get an LMI constraint:

$$\mathbf{PA}^\top + \mathbf{AP} + \mathbf{L}^\top \mathbf{B}^\top + \mathbf{BL} \prec 0 \quad (5)$$

Solving (5) gives us \mathbf{P} and \mathbf{L} , from which we can compute $\mathbf{K} = \mathbf{LP}^{-1}$ and $\mathbf{S} = \mathbf{P}^{-1}$, solving the original problem.

Consider a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, but when you don't know \mathbf{A} exactly. In other words, you don't know the model exactly. This is not to say that we know nothing about the model, but there is an uncertainty in our knowledge.

A good way to model is lack of model knowledge, this *uncertainty*, is this:

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{F}\Delta\mathbf{E})\mathbf{x} \tag{6}$$

where \mathbf{F} and \mathbf{E} are arbitrary matrices, and Δ is a *norm-bounded* matrix: $\Delta \leq 1$.

We can think of it this way: $\mathbf{A} + \mathbf{F}\Delta\mathbf{E}$ is the true but unknown model, and the range of all possible models we can expect is bounded by the possible values of Δ .

Lets write the Lyapunov equation for the system (6):

$$\dot{V} = \mathbf{x}^\top (\mathbf{A} + \mathbf{F}\Delta\mathbf{E})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}(\mathbf{A} + \mathbf{F}\Delta\mathbf{E})\mathbf{x} \leq 0 \quad (7)$$

Let us introduce a new variable $\mathbf{w} = \Delta\mathbf{E}\mathbf{x}$:

$$\dot{V} = \mathbf{x}^\top (\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A})\mathbf{x} + \mathbf{w}^\top \mathbf{F}^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}\mathbf{F}\mathbf{w} \leq 0 \quad (8)$$

Let us consider $\mathbf{w}^\top \mathbf{w}$:

$$\mathbf{w}^\top \mathbf{w} = \mathbf{x}^\top \mathbf{E}^\top \Delta \Delta \mathbf{E} \mathbf{x} \leq \mathbf{x}^\top \mathbf{E}^\top \mathbf{E} \mathbf{x} \quad (9)$$

which is true because $\|\Delta\| \leq 1$. In fact, the only property of the norm that we need here is that the delta inequality (9) holds.

With $\mathbf{w}^\top \mathbf{w} \leq \mathbf{x}^\top \mathbf{E}^\top \mathbf{E} \mathbf{x}$ we can write:

$$\mathbf{x}^\top \mathbf{E}^\top \mathbf{E} \mathbf{x} - \mathbf{w}^\top \mathbf{w} \geq 0 \quad (10)$$

Which is the same as:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} \mathbf{E}^\top \mathbf{E} & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \geq 0 \quad (11)$$

The same way we can rewrite (8):

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} \mathbf{A}^\top \mathbf{S} + \mathbf{S} \mathbf{A} & \mathbf{S} \mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \leq 0 \quad (12)$$

which only need to hold while (11) holds.

There is a way to enforce constraint $\mathbf{z}^\top \mathbf{M} \mathbf{z} \leq 0$ for such \mathbf{z} that $\mathbf{z}^\top \mathbf{N} \mathbf{z} \geq 0$. This is called *s-procedure*.

Theorem

If $\gamma > 0$ and $\mathbf{M} + \gamma \mathbf{N} \prec 0$ then $\mathbf{z}^\top \mathbf{N} \mathbf{z} \geq 0 \implies \mathbf{z}^\top \mathbf{M} \mathbf{z} \leq 0$

Using s-procedure we enforce (12) when (11) holds:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} \mathbf{A}^\top \mathbf{S} + \mathbf{S} \mathbf{A} + \gamma \mathbf{E}^\top \mathbf{E} & \mathbf{S} \mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & -\gamma \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \leq 0 \quad (13)$$

In LMI form this is:

$$\begin{bmatrix} \mathbf{A}^\top \mathbf{S} + \mathbf{S} \mathbf{A} + \gamma \mathbf{E}^\top \mathbf{E} & \mathbf{S} \mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & -\gamma \mathbf{I} \end{bmatrix} \prec 0 \quad (14)$$

This is a condition that the system is stable for all values of Δ .
The decision variables are \mathbf{S} and γ .

Let us consider the following system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (15)$$

where $\mathbf{A} = \sum_{i=1}^n \alpha_i \mathbf{A}_i$, $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$ with known \mathbf{A}_i but unknown coefficients α_i . Is it stable for all possible values of α_i ? Note that we can't use eigenvalue analysis in this case.

Geometrically, this means \mathbf{A} is in a polytope with vertices \mathbf{A}_i .

Theorem (Quadratic stability)

$\mathbf{A}_i^\top \mathbf{S} + \mathbf{S} \mathbf{A}_i \leq 0$ implies $\dot{\mathbf{x}} = \sum_{i=1}^n \alpha_i \mathbf{A}_i \mathbf{x}$ is stable, where $\alpha_i \geq 0$,
 $\sum_{i=1}^n \alpha_i = 1$

Proof: $\dot{V} = \left(\sum_{i=1}^n \alpha_i \mathbf{A}_i \right)^\top \mathbf{S} + \mathbf{S} \left(\sum_{i=1}^n \alpha_i \mathbf{A}_i \right) \leq 0$ can be

re-written as: $\dot{V} = \sum_{i=1}^n (\alpha_i (\mathbf{A}_i^\top \mathbf{S} + \mathbf{S} \mathbf{A}_i))$ and since

$\mathbf{A}_i^\top \mathbf{S} + \mathbf{S} \mathbf{A}_i \leq 0$ and $\alpha_i \geq 0$, then $\dot{V} \leq 0$. □

Let us consider the following system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (16)$$

where $\mathbf{A} = \sum_{i=1}^n \alpha_i \mathbf{A}_i$, $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$ with known \mathbf{A}_i but unknown coefficients α_i . How to design control law $\mathbf{u} = \mathbf{K}\mathbf{x}$ making the system stable for all possible values of α_i ?

The closed-loop form of the system is:

$$\dot{\mathbf{x}} = \left(\sum_{i=1}^n \alpha_i \mathbf{A}_i + \mathbf{B}\mathbf{K} \right) \mathbf{x} \quad (17)$$

Let us write Lyapunov eq. for the system:

$$\left(\sum_{i=1}^n \alpha_i (\mathbf{A}_i + \mathbf{BK}) \right)^{\top} \mathbf{S} + \mathbf{S} \left(\sum_{i=1}^n \alpha_i (\mathbf{A}_i + \mathbf{BK}) \right) \prec 0 \quad (18)$$

We can re-write it as:

$$\sum_{i=1}^n \alpha_i \left((\mathbf{A}_i + \mathbf{BK})^{\top} \mathbf{S} + \mathbf{S} (\mathbf{A}_i + \mathbf{BK}) \right) \prec 0 \quad (19)$$

Hence if $(\mathbf{A}_i + \mathbf{BK})^{\top} \mathbf{S} + \mathbf{S} (\mathbf{A}_i + \mathbf{BK}) \prec 0$, the original system is stable.

From $(\mathbf{A}_i + \mathbf{BK})^\top \mathbf{S} + \mathbf{S}(\mathbf{A}_i + \mathbf{BK}) \prec 0$, we can go on to do control design. Introducing $\mathbf{P} = \mathbf{S}^{-1}$, we use congruence transformation multiplying by \mathbf{P} on both sides:

$$\mathbf{P}(\mathbf{A}_i + \mathbf{BK})^\top + (\mathbf{A}_i + \mathbf{BK})\mathbf{P} \prec 0 \quad (20)$$

Introducing new variable $\mathbf{L} = \mathbf{KP}$ we get a problem linear in decision variables:

$$\mathbf{PA}_i^\top + \mathbf{A}_i\mathbf{P} + \mathbf{L}^\top \mathbf{B}^\top + \mathbf{BL} \prec 0 \quad (21)$$

where the decision variables are \mathbf{P} and \mathbf{L} . The control gain matrix is found as $\mathbf{K} = \mathbf{LP}^{-1}$.

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.



APPENDIX A

Congruence transformation and definiteness

Consider matrices $\mathbf{P} \succ 0$, and $\mathbf{V} \in \mathbb{R}^{n,n}$ is full rank. We can prove that:

$$\mathbf{P} \succ 0 \implies \mathbf{V}^\top \mathbf{P} \mathbf{V} \succ 0 \quad (22)$$

Proof: $\mathbf{x}^\top \mathbf{V}^\top \mathbf{P} \mathbf{V} \mathbf{x} = \mathbf{z}^\top \mathbf{P} \mathbf{z}$, where $\mathbf{z} = \mathbf{V} \mathbf{x}$. Since $\mathbf{P} \succ 0$, $\mathbf{z}^\top \mathbf{P} \mathbf{z} \geq 0$, hence $\mathbf{x}^\top \mathbf{V}^\top \mathbf{P} \mathbf{V} \mathbf{x} \geq 0$.

Definition

Congruence transformation preserves semi-definiteness:

$$\det(\mathbf{V}) \neq 0, \mathbf{P} \succ 0 \implies \mathbf{V}^\top \mathbf{P} \mathbf{V} \succ 0$$