Controllability, Observability

Control Theory, Lecture 10

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Consider discrete LTI:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \tag{1}$$

Assume the initial state is $\mathbf{x}_1$. Then we can deduce that:

$$\begin{aligned}
\mathbf{x}_2 &= \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1 \\
\mathbf{x}_3 &= \mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2 \\
\mathbf{x}_4 &= \mathbf{A}\mathbf{x}_3 + \mathbf{B}\mathbf{u}_3 = \mathbf{A}(\mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2) + \mathbf{B}\mathbf{u}_3 \\
&\dots \\
\mathbf{x}_{n+1} &= \mathbf{A}^n\mathbf{x}_1 + \dots + \mathbf{A}^{n-k}\mathbf{B}\mathbf{u}_k + \dots + \mathbf{B}\mathbf{u}_n
\end{aligned}$$

Equation $\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + ... + \mathbf{A}^{n-k} \mathbf{B} \mathbf{u}_k + ... \mathbf{B} \mathbf{u}_n$ can be re-written as:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \mathbf{u}_{k-2} \\ ... \\ \mathbf{u}_1 \end{bmatrix} \tag{2}$$

Notice that in order for the system to go from $\mathbf{x}_1$ to $\mathbf{x}_{n+1}$, vector $\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1$ needs be in the column space of $\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$.
Since $\mathbf{x}_{n+1}$ can be anything, and $\mathbf{x}_1$ might be equal to zero (among other possibilities), we should require that all vectors in $\mathbb{R}^n$ need to be in the column space of $\mathcal{C}$, meaning $\mathcal{C}$ needs to be full rank.

Controllability

For a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$, where $\mathbf{x} \in \mathbb{R}^n$, if the matrix $\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$ is full row rank (i.e. $\mathrm{rank}(\mathcal{C}) = n$), any state can be reached, which means that *the system is controllable.*

If you are interested why the controllability matrix for not include more columns, like $\mathbf{A}^n$, see Appendix A.

Observability of Discrete LTI

Consider discrete LTI:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i \end{cases} \tag{3}$$

And an observer:

$$\hat{\mathbf{x}}_{i+1} = \mathbf{A}\hat{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i + \mathbf{L}(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_i) \tag{4}$$

Remember that we can define observation error $\mathbf{e}_i = \hat{\mathbf{x}}_i - \mathbf{x}_i$ and write its dynamics:

$$\mathbf{e}_{i+1} = \mathbf{A}\mathbf{e}_i - \mathbf{L}\mathbf{C}\mathbf{e}_i \tag{5}$$

Dual system (which is stable if and only if the original is stable), has form:

$$\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i \tag{6}$$

Dynamical system $\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i$, we can be represented as:

$$\begin{cases} \varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i + \mathbf{C}^\top \mathbf{v}_i \\ \mathbf{v}_i = -\mathbf{L}^\top \varepsilon_i \end{cases} \tag{7}$$

Controllability matrix of this system is:

$$\mathcal{O}^\top = \begin{bmatrix} \mathbf{C}^\top & (\mathbf{A}^\top)\mathbf{C}^\top & ... & (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top \end{bmatrix} \tag{8}$$

It is easier to represent this matrix in its transposed form:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ ... \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{9}$$

Observability

For a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$ and $\mathbf{y}_i = \mathbf{C}\mathbf{x}_i$, where $\mathbf{x} \in \mathbb{R}^n$, if the matrix $\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ ... \\ \mathbf{CA}^{n-1} \end{bmatrix}$ is full column rank (i.e. $\text{rank}(\mathcal{O}) = n$), observation error can go to zero from any initial position, which means that *the system is observable*.

Let's look at this equation one more time:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \mathbf{u}_{k-2} \\ ... \\ \mathbf{u}_1 \end{bmatrix} \tag{10}$$

If the system is controllable, it means *every state can be reached from any other space in only n steps*. This seem to disagree with our real-world experience.

Let's look at an even simpler equation $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$. Let's rewrite the equation as follows:

$$\mathbf{x}_f - \mathbf{A}\mathbf{x}_1 = \mathbf{B}\mathbf{u}_1 \tag{11}$$

As long as $\mathbf{x}_f - \mathbf{A}\mathbf{x}_1$ lies in the column space of $\mathbf{B}$, it *can be achieved in a single step*, using control:

$$\mathbf{u}_1 = \mathbf{B}^+(\mathbf{x}_f - \mathbf{A}\mathbf{x}_1) \tag{12}$$

This as well, seem to disagree with our real-world experience.

Limited control

In the actual engineering reality we often have to deal with equations, that look closer to:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ ||\mathbf{D}\mathbf{u}_i||_r \leq 1 \end{cases} \tag{13}$$

... which is a *second-order cone program.* Or:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{D}\mathbf{u}_i \leq \mathbf{d} \end{cases} \tag{14}$$

... which is a *quadratic program.* Notice, those equations *can't be solved analytically.*

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.

Why does controllability matrix $\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$ includes only column blocks up to $\mathbf{A}^{n-1}\mathbf{B}$ and not, for example, $\mathbf{A}^{n}\mathbf{B}$? We start with:

Theorem (Cayley–Hamilton)

A matrix $\mathbf{M} \in \mathbb{R}^{n,n}$ satisfies its own characteristic equation.

A characteristic equation can be written as $\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_0 = 0$, meaning that we can write:

$$\mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + ... + a_0\mathbf{I} = 0 \tag{15}$$

Meaning that $\mathbf{M}^n$ is a linear combination of $\mathbf{M}^{n-1}$, $\mathbf{M}^{n-2}$, ..., $\mathbf{I}$.

The controllability matrix can be written as

$$\mathcal{C} = \begin{bmatrix} \mathbf{I} & \mathbf{A} & ... & \mathbf{A}^{n-1} \end{bmatrix} \begin{bmatrix} \mathbf{B} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & ... & \mathbf{0} \\ ... & ... & ... & ... \\ \mathbf{0} & \mathbf{0} & ... & \mathbf{B} \end{bmatrix} \tag{16}$$

meaning that the rank of $\mathcal{C}$ depends only on matrix $\begin{bmatrix} \mathbf{I} & \mathbf{A} & ... & \mathbf{A}^{n-1} \end{bmatrix}$. Adding to it columns $\mathbf{A}^n$ does not change the rank, as $\mathbf{A}^n$ is a linear combination of the other columns, as we proved in the previous slide.