Controllability, Observability

Control Theory, Lecture 11

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Content

- Controllability of Discrete LTI
  - Controllability matrix
  - Controllability criterion
- Observability of Discrete LTI
  - Dual system
  - Observability criterion
- "Unlimited control"
- Limited control

Controllability of Discrete LTI

Consider discrete LTI:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \tag{1}$$

Assume the initial state is $\mathbf{x}_1$. Then we can deduce that:

$$\begin{aligned}
\mathbf{x}_2 &= \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1 \\
\mathbf{x}_3 &= \mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2 \\
\mathbf{x}_4 &= \mathbf{A}\mathbf{x}_3 + \mathbf{B}\mathbf{u}_3 = \mathbf{A}(\mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2) + \mathbf{B}\mathbf{u}_3 \\
&\ldots \\
\mathbf{x}_{n+1} &= \mathbf{A}^n\mathbf{x}_1 + \ldots + \mathbf{A}^{n-k}\mathbf{B}\mathbf{u}_k + \ldots + \mathbf{B}\mathbf{u}_n
\end{aligned}$$

Equation $\mathbf{x}_{n+1} = \mathbf{A}^n\mathbf{x}_1 + ... + \mathbf{A}^{n-k}\mathbf{B}\mathbf{u}_k + ...\mathbf{B}\mathbf{u}_n$ can be re-written as:

$$\mathbf{x}_{n+1} - \mathbf{A}^n\mathbf{x}_1 = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & ... & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \mathbf{u}_{k-2} \\ ... \\ \mathbf{u}_1\end{bmatrix} \tag{2}$$

Notice that in order for the system to go from $\mathbf{x}_1$ to $\mathbf{x}_{n+1}$, it needs be in the column space of $\begin{bmatrix}\mathbf{B} & \mathbf{AB} & ... & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$.

Controllability

For a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$, where $\mathbf{x} \in \mathbb{R}^n$, if the matrix $\mathcal{B} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$ is full row rank (i.e. $\text{rank}(\mathcal{B}) = n$), any state can be reached, which means that *the system is controllable.*

Observability of Discrete LTI

Consider discrete LTI:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i \end{cases} \tag{3}$$

And an observer:

$$\hat{\mathbf{x}}_{i+1} = \mathbf{A}\hat{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i + \mathbf{L}(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_i) \tag{4}$$

Remember that we can define observation error $\mathbf{e}_i = \hat{\mathbf{x}}_i - \mathbf{x}_i$ and white its dynamics:

$$\mathbf{e}_{i+1} = \mathbf{A}\mathbf{e}_i - \mathbf{L}\mathbf{C}\mathbf{e}_i \tag{5}$$

Dual system (which is stable if and only if the original is stable), has form:

$$\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i \tag{6}$$

Observability of Discrete LTI

Dual system

Dynamical system $\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i$, we can be represented as:

$$\begin{cases} \varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i + \mathbf{C}^\top \mathbf{v}_i \\ \mathbf{v}_i = -\mathbf{L}^\top \varepsilon_i \end{cases} \tag{7}$$

Controllability matrix of this system is:

$$\mathcal{C}^\top = \begin{bmatrix} \mathbf{C}^\top & (\mathbf{A}^\top)\mathbf{C}^\top & \dots & (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top \end{bmatrix} \tag{8}$$

It is easier to represent this matrix in its transposed form:

$$\mathcal{C} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{9}$$

Observability of Discrete LTI

Observability criterion

Observability

For a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$ and $\mathbf{y}_i = \mathbf{C}\mathbf{x}_i$, where $\mathbf{x} \in \mathbb{R}^n$, if the matrix $\mathcal{C} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix}$ is full column rank (i.e. $\text{rank}(\mathcal{C}) = n$), observation error can go to zero from any initial position, which means that *the system is observable*.

Let's look at this equation one more time:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \mathbf{u}_{k-2} \\ \dots \\ \mathbf{u}_1 \end{bmatrix} \tag{10}$$

If the system is controllable, it means *every state can be reached from any other space in only n steps*. This seem to disagree with our real-world experience.

Let's look at an even simpler equation $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$. Let's rewrite the equation as follows:

$$\mathbf{x}_f - \mathbf{A}\mathbf{x}_1 = \mathbf{B}\mathbf{u}_1 \tag{11}$$

As long as $\mathbf{x}_f - \mathbf{A}\mathbf{x}_1$ lies in the column space of $\mathbf{B}$, it *can be achieved in a single step*, using control:

$$\mathbf{u}_1 = \mathbf{B}^+(\mathbf{x}_f - \mathbf{A}\mathbf{x}_1) \tag{12}$$

This as well, seem to disagree with our real-world experience.

Limited control

In the actual engineering reality we often have to deal with equations, that look closer to:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ ||\mathbf{D}\mathbf{u}_i||_r \leq 1 \end{cases} \tag{13}$$

... which is a *second-order cone program*. Or:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{D}\mathbf{u}_i \leq \mathbf{d} \end{cases} \tag{14}$$

... which is a *quadratic program*. Notice, those equations *can't be solved analytically*.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.