

Laplace Transform and Transfer Functions

Control Theory, Lecture 3

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Spring 2022

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$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u \quad (1)$$

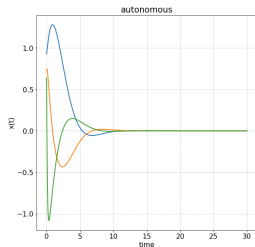


Figure 1: Autonomous ODE ($u = 0$)

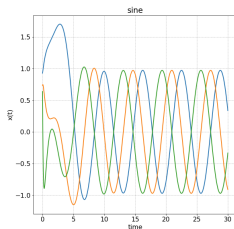


Figure 2: reaction to sine wave ($u = \sin(t)$)

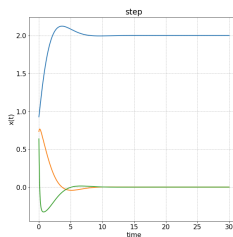


Figure 3: Reaction to step function ($u = 1$)

By definition, Laplace transform of a function $f(t)$ is given as:

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \quad (2)$$

where $F(s)$ is called an *image* of the function.

The study of Laplace transform is a separate mathematical field with applications in solving ODEs, which we won't cover.

However, we will consider transform of one case of interest - transform of a derivative.

LAPLACE TRANSFORM OF A DERIVATIVE

Consider a derivative $\frac{dx}{dt}$ and its transform:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = \int_0^{\infty} \frac{dx}{dt} e^{-st} dt \quad (3)$$

we will make use of the integration by parts formula:

Integration by parts

$$\int v \frac{du}{dt} dt = vu - \int \frac{dv}{dt} u dt \quad (4)$$

In our case, $\frac{du}{dt} = \frac{dx}{dt}$, $u = x$, $v = e^{-st}$, $\frac{dv}{dt} = -se^{-st}$:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = [xe^{-st}]_0^{\infty} - \int_0^{\infty} -se^{-st} x dt \quad (5)$$

$$\mathcal{L}\left(\frac{dx}{dt}\right) = -x(0) + s\mathcal{L}(x) \quad (6)$$

Thus, assuming that $x(0) = 0$, we can obtain a *derivative operator*:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = s\mathcal{L}(x) \quad (7)$$

Please notice that (7) is only true when $x(0) = 0$; it generally does not look very elegant either. Introducing a big-time abuse of notation, we can denote $x(s) = \mathcal{L}(x)$ and then drop the brackets, leaving us with:

$$\frac{dx}{dt} \longrightarrow sx \quad (8)$$

This form of a derivative operator has a very strange notation in terms of the Laplace transform theory, but is very simple to use in practice.

TRANSFER FUNCTION

Consider the following ODE, where u is an input (function of time that influences the solution of the ODE):

$$\ddot{x} + a\dot{x} + bx = u \quad (9)$$

We can rewrite it using the derivative operator:

$$s^2x + asx + bx = u \quad (10)$$

and then collect x on the left-hand-side:

$$x = \frac{1}{s^2 + as + b}u \quad (11)$$

At this point the mathematical meaning of this expression as an ODE is very vague, but it has a different direct use; this form is called a *transfer function*.

TRANSFER FUNCTION

Examples

Example

Given ODE: $2\ddot{x} + 5\dot{x} - 40x = 10u$

The transfer function for it looks: $x = \frac{10}{2s^3 + 5s - 40}u$

Example

Given ODE: $2\dot{x} - 4x = u$

The transfer function for it looks: $x = \frac{1}{2s - 4}u$

Example

Given ODE: $3\ddot{x} + 4x = u$

The transfer function for it looks: $x = \frac{1}{2s^3 + 4}u$

TRANSFER FUNCTION

Interesting things done easy

Consider the following (strange) ODE:

$$2\ddot{x} + 3\dot{x} + 2x = 10\dot{u} - u \quad (12)$$

Using the differential equation:

$$2s^2x + 3sx + 2x = 10su - u \quad (13)$$

...which is the same as:

$$(2s^2 + 3s + 2)x = (10s - 1)u \quad (14)$$

The transfer function for it looks:

$$x = \frac{10s - 1}{2s^2 + 3s + 2}u \quad (15)$$

STATE-SPACE TO TRANSFER FUNCTION CONVERSION

Transfer functions are being used to study the relation between the input and the output of the dynamical system.

Consider standard form state-space dynamical system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \end{cases} \quad (16)$$

We can rewrite it using the derivative operator:

$$\begin{cases} s\mathbf{Ix} - \mathbf{Ax} = \mathbf{Bu} \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \end{cases} \quad (17)$$

and then collect \mathbf{x} on the left-hand-side: $\mathbf{x} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{Bu}$
and finally, express \mathbf{y} out:

$$\mathbf{y} = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}) \mathbf{u} \quad (18)$$

Let the dynamic system be described as a transfer function:

$$y = G(s)x \quad (19)$$

We can try to modify the input based on how the output looks-like. Since we always do it in a linear way, we can write it as:

$$y = G(s)(x - H(s)y) \quad (20)$$

where $H(s)y$ is called *feedback*.

How would the transfer function from x to y look like?

From $y = G(s)(x - H(s)y)$ we go:

$$y = G(s)x - G(s)H(s)y \quad (21)$$

$$y + G(s)H(s)y = G(s)x \quad (22)$$

$$y = \frac{G(s)}{1 + G(s)H(s)}x \quad (23)$$

Thus, we found *closed-loop* transfer function:

$$W(s) = \frac{G(s)}{1 + G(s)H(s)} \quad (24)$$

- Chapter 6 Transfer Functions
- Control Systems Lectures - Transfer Functions, by Brian Douglas
- The Laplace Transform - A Graphical Approach, by Brian Douglas

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.

