

# Controllability, Observability

## Control Theory, Lecture 11

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# CONTROLLABILITY

# CONTROLLABILITY OF DISCRETE LTI

Consider discrete LTI:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \quad (1)$$

Assume the initial state is  $\mathbf{x}_1$ . Then we can deduce that:

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1$$

$$\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2$$

$$\mathbf{x}_4 = \mathbf{A}\mathbf{x}_3 + \mathbf{B}\mathbf{u}_3 = \mathbf{A}(\mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2) + \mathbf{B}\mathbf{u}_3$$

...

$$\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + \dots + \mathbf{A}^{n-k} \mathbf{B}\mathbf{u}_k + \dots + \mathbf{B}\mathbf{u}_n$$

# CONTROLLABILITY OF DISCRETE LTI

## Controllability matrix

Equation  $\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + \dots + \mathbf{A}^{n-k} \mathbf{B} \mathbf{u}_k + \dots \mathbf{B} \mathbf{u}_n$  can be re-written as:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \mathbf{u}_{k-2} \\ \dots \\ \mathbf{u}_1 \end{bmatrix} \quad (2)$$

Notice that in order for the system to go from  $\mathbf{x}_1$  to  $\mathbf{x}_{n+1}$ , vector  $\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1$  needs to be in the column space of  $\mathcal{C} = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$ .

Since  $\mathbf{x}_{n+1}$  can be anything, and  $\mathbf{x}_1$  might be equal to zero (among other possibilities), we should require that all vectors in  $\mathbb{R}^n$  need to be in the column space of  $\mathcal{C}$ , meaning  $\mathcal{C}$  needs to be full rank.

# CONTROLLABILITY OF DISCRETE LTI

## Controllability criterion

### Controllability

For a system  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$ , where  $\mathbf{x} \in \mathbb{R}^n$ , if the matrix  $\mathcal{C} = [\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$  is full row rank (i.e.  $\text{rank}(\mathcal{C}) = n$ ), any state can be reached, which means that *the system is controllable*.

If you are interested why the controllability matrix for not include more columns, like  $\mathbf{A}^n$ , see Appendix A.

# OBSERVABILITY OF DISCRETE LTI

Consider discrete LTI:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i \end{cases} \quad (3)$$

And an observer:

$$\hat{\mathbf{x}}_{i+1} = \mathbf{A}\hat{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i + \mathbf{L}(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_i) \quad (4)$$

Remember that we can define observation error  $\mathbf{e}_i = \hat{\mathbf{x}}_i - \mathbf{x}_i$  and write its dynamics:

$$\mathbf{e}_{i+1} = \mathbf{A}\mathbf{e}_i - \mathbf{L}\mathbf{C}\mathbf{e}_i \quad (5)$$

Dual system (which is stable if and only if the original is stable), has form:

$$\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i \quad (6)$$

# OBSERVABILITY OF DISCRETE LTI

## Dual system

Dynamical system  $\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i$ , we can be represented as:

$$\begin{cases} \varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i + \mathbf{C}^\top \mathbf{v}_i \\ \mathbf{v}_i = -\mathbf{L}^\top \varepsilon_i \end{cases} \quad (7)$$

Controllability matrix of this system is:

$$\mathcal{O}^\top = [\mathbf{C}^\top \quad (\mathbf{A}^\top)\mathbf{C}^\top \quad \dots \quad (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top] \quad (8)$$

It is easier to represent this matrix in its transposed form:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix} \quad (9)$$



# OBSERVABILITY OF DISCRETE LTI

## Observability criterion

### Observability

For a system  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$  and  $\mathbf{y}_i = \mathbf{C}\mathbf{x}_i$ , where  $\mathbf{x} \in \mathbb{R}^n$ , if

the matrix  $\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix}$  is full column rank (i.e.

$\text{rank}(\mathcal{O}) = n$ ), observation error can go to zero from any initial position, which means that *the system is observable*.

Let's look at this equation one more time:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \mathbf{u}_{k-2} \\ \dots \\ \mathbf{u}_1 \end{bmatrix} \quad (10)$$

If the system is controllable, it means *every state can be reached from any other space in only  $n$  steps*. This seem to disagree with our real-world experience.

Let's look at an even simpler equation  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$ . Let's rewrite the equation as follows:

$$\mathbf{x}_f - \mathbf{A}\mathbf{x}_1 = \mathbf{B}\mathbf{u}_1 \quad (11)$$

As long as  $\mathbf{x}_f - \mathbf{A}\mathbf{x}_1$  lies in the column space of  $\mathbf{B}$ , it *can be achieved in a single step*, using control:

$$\mathbf{u}_1 = \mathbf{B}^+(\mathbf{x}_f - \mathbf{A}\mathbf{x}_1) \quad (12)$$

This as well, seem to disagree with our real-world experience.

In the actual engineering reality we often have to deal with equations, that look closer to:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ ||\mathbf{D}\mathbf{u}_i||_r \leq 1 \end{cases} \quad (13)$$

... which is a *second-order cone program*. Or:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{D}\mathbf{u}_i \leq \mathbf{d} \end{cases} \quad (14)$$

... which is a *quadratic program*. Notice, those equations *can't be solved analytically*.

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Control-Theory-Slides-Spring-2022](https://github.com/SergeiSa/Control-Theory-Slides-Spring-2022)

Check Moodle for additional links, videos, textbook suggestions.



Why does controllability matrix  $\mathcal{C} = [\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$  includes only column blocks up to  $\mathbf{A}^{n-1}\mathbf{B}$  and not, for example,  $\mathbf{A}^n\mathbf{B}$ ? We start with:

**Theorem (Cayley–Hamilton)**

*A matrix  $\mathbf{M} \in \mathbb{R}^{n,n}$  satisfies its own characteristic equation.*

A characteristic equation can be written as  $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$ , meaning that we can write:

$$\mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + \dots + a_0\mathbf{I} = 0 \quad (15)$$

Meaning that  $\mathbf{M}^n$  is a linear combination of  $\mathbf{M}^{n-1}$ ,  $\mathbf{M}^{n-2}$ , ...,  $\mathbf{I}$ .

The controllability matrix can be written as

$$\mathcal{C} = [\mathbf{I} \quad \mathbf{A} \quad \dots \quad \mathbf{A}^{n-1}] \begin{bmatrix} \mathbf{B} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{B} \end{bmatrix} \quad (16)$$

meaning that the rank of  $\mathcal{C}$  depends only on matrix  $[\mathbf{I} \quad \mathbf{A} \quad \dots \quad \mathbf{A}^{n-1}]$ . Adding to it columns  $\mathbf{A}^n$  does not change the rank, as  $\mathbf{A}^n$  is a linear combination of the other columns, as we proved in the previous slide.