Laplace Transform and Transfer Functions

Control Theory, Lecture 3

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Spring 2022

Content

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u \tag{1}$$

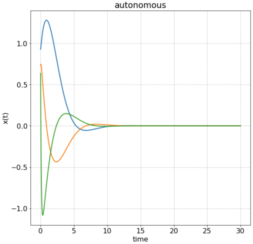

Figure 1: Autonomous ODE ($u = 0$)

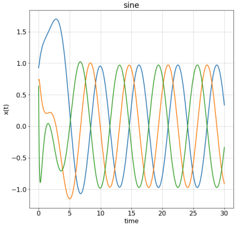

Figure 2: reaction to sine wave ($u = sin(t)$)

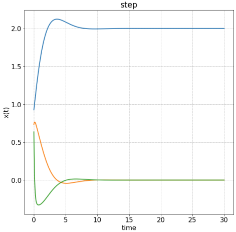

Figure 3: Reaction to step function ($u = 1$)

Laplace Transform

By definition, Laplace transform of a function $f(t)$ is given as:

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{2}$$

where $F(s)$ is called an *image* of the function.

The study of Laplace transform is a separate mathematical field with applications in solving ODEs, which we won't cover. However, we will consider transform of one case of interest - transform of a derivative.

Consider a derivative $\frac{dx}{dt}$ and its transform:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = \int_0^\infty \frac{dx}{dt} e^{-st} dt \tag{3}$$

we will make use of the integration by parts formula:

Integration by parts

$$\int v \frac{du}{dt} dt = vu - \int \frac{dv}{dt} u dt \tag{4}$$

In our case, $\frac{du}{dt} = \frac{dx}{dt}$, $u = x$, $v = e^{-st}$, $\frac{dv}{dt} = -se^{-st}$:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = \left[xe^{-st}\right]_0^\infty - \int_0^\infty -se^{-st} x dt \tag{5}$$

$$\mathcal{L}\left(\frac{dx}{dt}\right) = -x(0) + s\mathcal{L}(x) \tag{6}$$

Thus, assuming that $x(0) = 0$, we can obtain a *derivative operator*:

$$\mathcal{L}\left(\frac{dx}{dt}\right) = s\mathcal{L}\left(x\right) \tag{7}$$

Please notice that (7) is only true when $x(0) = 0$; it generally does not look very elegant either. Introducing a big-time abuse of notation, we can denote $x(s) = \mathcal{L}\left(x\right)$ and then drop the brackets, leaving us with:

$$\frac{dx}{dt} \longrightarrow sx \tag{8}$$

This form of a derivative operator has a very strange notation in terms of the Laplace transform theory, but is very simple to use in practice.

Consider the following ODE, where u is an input (function of time that influences the solution of the ODE):

$$\ddot{x} + a\dot{x} + bx = u \tag{9}$$

We can rewrite it using the derivative operator:

$$s^2 x + asx + bx = u \tag{10}$$

and then collect x on the left-hand-side:

$$x = \frac{1}{s^2 + as + b} u \tag{11}$$

At this point the mathematical meaning of this expression as an ODE is very vague, but it has a different direct use; this form is called a *transfer function*.

Example

Given ODE: $2\dddot{x} + 5\dot{x} - 40x = 10u$

The transfer function for it looks: $x = \frac{10}{2s^3+5s-40}u$

Example

Given ODE: $2\dot{x} - 4x = u$

The transfer function for it looks: $x = \frac{1}{2s-4}u$

Example

Given ODE: $3\dddot{x} + 4x = u$

The transfer function for it looks: $x = \frac{1}{2s^3+4}u$

Consider the following (strange) ODE:

$$2\ddot{x} + 3\dot{x} + 2x = 10\dot{u} - u \tag{12}$$

Using the differential equation:

$$2s^2x + 3sx + 2x = 10su - u \tag{13}$$

...which is the same as:

$$(2s^2 + 3s + 2)x = (10s - 1)u \tag{14}$$

The transfer function for it looks:

$$x = \frac{10s - 1}{2s^2 + 3s + 2}u \tag{15}$$

Transfer functions are being used to study the relation between the input and the output of the dynamical system.

Consider standard form state-space dynamical system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \tag{16}$$

We can rewrite it using the derivative operator:

$$\begin{cases} s\mathbf{I}\mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \tag{17}$$

and then collect $\mathbf{x}$ on the left-hand-side: $\mathbf{x} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}$ and finally, express $\mathbf{y}$ out:

$$\mathbf{y} = \left(\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{u} \tag{18}$$

Let the dynamic system be described as a transfer function:

$$y = G(s)x \tag{19}$$

We can try to modify the input based on how the output looks-like. Since we always do it in a linear way, we can write it as:

$$y = G(s)(x - H(s)y) \tag{20}$$

where $H(s)y$ is called *feedback*.

How would the transfer function from x to y look like?

From $y = G(s)(x - H(s)y)$ we go:

$$y = G(s)x - G(s)H(s)y \tag{21}$$

$$y + G(s)H(s)y = G(s)x \tag{22}$$

$$y = \frac{G(s)}{1 + G(s)H(s)}x \tag{23}$$

Thus, we found *closed-loop* transfer function:

$$W(s) = \frac{G(s)}{1 + G(s)H(s)} \tag{24}$$

Read more

- Chapter 6 Transfer Functions
- Control Systems Lectures - Transfer Functions, by Brian Douglas
- The Laplace Transform - A Graphical Approach, by Brian Douglas

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.