

Controllability, Observability

Control Theory, Lecture 11

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- Controllability of Discrete LTI
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- Limited control

CONTROLLABILITY

CONTROLLABILITY OF DISCRETE LTI

Consider discrete LTI:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \quad (1)$$

Assume the initial state is \mathbf{x}_1 . Then we can deduce that:

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1$$

$$\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2$$

$$\mathbf{x}_4 = \mathbf{A}\mathbf{x}_3 + \mathbf{B}\mathbf{u}_3 = \mathbf{A}(\mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2) + \mathbf{B}\mathbf{u}_3$$

...

$$\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + \dots + \mathbf{A}^{n-k} \mathbf{B}\mathbf{u}_k + \dots + \mathbf{B}\mathbf{u}_n$$

CONTROLLABILITY OF DISCRETE LTI

Controllability matrix

Equation $\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + \dots + \mathbf{A}^{n-k} \mathbf{B} \mathbf{u}_k + \dots \mathbf{B} \mathbf{u}_n$ can be re-written as:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \mathbf{u}_{k-2} \\ \dots \\ \mathbf{u}_1 \end{bmatrix} \quad (2)$$

Notice that in order for the system to go from \mathbf{x}_1 to \mathbf{x}_{n+1} , vector $\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1$ needs to be in the column space of $\mathcal{C} = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$.

Since \mathbf{x}_{n+1} can be anything, and \mathbf{x}_1 might be equal to zero (among other possibilities), we should require that all vectors in \mathbb{R}^n need to be in the column space of \mathcal{C} , meaning \mathcal{C} needs to be full rank.

CONTROLLABILITY OF DISCRETE LTI

Controllability criterion

Controllability

For a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$, where $\mathbf{x} \in \mathbb{R}^n$, if the matrix $\mathcal{C} = [\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$ is full row rank (i.e. $\text{rank}(\mathcal{C}) = n$), any state can be reached, which means that *the system is controllable*.

If you are interested why the controllability matrix for not include more columns, like \mathbf{A}^n , see Appendix A.

OBSERVABILITY OF DISCRETE LTI

Consider discrete LTI:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i \end{cases} \quad (3)$$

And an observer:

$$\hat{\mathbf{x}}_{i+1} = \mathbf{A}\hat{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i + \mathbf{L}(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_i) \quad (4)$$

Remember that we can define observation error $\mathbf{e}_i = \hat{\mathbf{x}}_i - \mathbf{x}_i$ and write its dynamics:

$$\mathbf{e}_{i+1} = \mathbf{A}\mathbf{e}_i - \mathbf{L}\mathbf{C}\mathbf{e}_i \quad (5)$$

Dual system (which is stable if and only if the original is stable), has form:

$$\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i \quad (6)$$

OBSERVABILITY OF DISCRETE LTI

Dual system

Dynamical system $\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i$, we can be represented as:

$$\begin{cases} \varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i + \mathbf{C}^\top \mathbf{v}_i \\ \mathbf{v}_i = -\mathbf{L}^\top \varepsilon_i \end{cases} \quad (7)$$

Controllability matrix of this system is:

$$\mathcal{O}^\top = [\mathbf{C}^\top \quad (\mathbf{A}^\top)\mathbf{C}^\top \quad \dots \quad (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top] \quad (8)$$

It is easier to represent this matrix in its transposed form:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix} \quad (9)$$

OBSERVABILITY OF DISCRETE LTI

Observability criterion

Observability

For a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$ and $\mathbf{y}_i = \mathbf{C}\mathbf{x}_i$, where $\mathbf{x} \in \mathbb{R}^n$, if

the matrix $\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix}$ is full column rank (i.e.

$\text{rank}(\mathcal{O}) = n$), observation error can go to zero from any initial position, which means that *the system is observable*.

Let's look at this equation one more time:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \mathbf{u}_{k-2} \\ \dots \\ \mathbf{u}_1 \end{bmatrix} \quad (10)$$

If the system is controllable, it means *every state can be reached from any other space in only n steps*. This seem to disagree with our real-world experience.

Let's look at an even simpler equation $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$. Let's rewrite the equation as follows:

$$\mathbf{x}_f - \mathbf{A}\mathbf{x}_1 = \mathbf{B}\mathbf{u}_1 \quad (11)$$

As long as $\mathbf{x}_f - \mathbf{A}\mathbf{x}_1$ lies in the column space of \mathbf{B} , it *can be achieved in a single step*, using control:

$$\mathbf{u}_1 = \mathbf{B}^+(\mathbf{x}_f - \mathbf{A}\mathbf{x}_1) \quad (12)$$

This as well, seem to disagree with our real-world experience.

In the actual engineering reality we often have to deal with equations, that look closer to:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ ||\mathbf{D}\mathbf{u}_i||_r \leq 1 \end{cases} \quad (13)$$

... which is a *second-order cone program*. Or:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{D}\mathbf{u}_i \leq \mathbf{d} \end{cases} \quad (14)$$

... which is a *quadratic program*. Notice, those equations *can't be solved analytically*.

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.



Why does controllability matrix $\mathcal{C} = [\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$ includes only column blocks up to $\mathbf{A}^{n-1}\mathbf{B}$ and not, for example, $\mathbf{A}^n\mathbf{B}$? We start with:

Theorem (Cayley–Hamilton)

A matrix $\mathbf{M} \in \mathbb{R}^{n,n}$ satisfies its own characteristic equation.

A characteristic equation can be written as $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$, meaning that we can write:

$$\mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + \dots + a_0\mathbf{I} = 0 \quad (15)$$

Meaning that \mathbf{M}^n is a linear combination of \mathbf{M}^{n-1} , \mathbf{M}^{n-2} , ..., \mathbf{I} .

The controllability matrix can be written as

$$\mathcal{C} = [\mathbf{I} \quad \mathbf{A} \quad \dots \quad \mathbf{A}^{n-1}] \begin{bmatrix} \mathbf{B} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{B} \end{bmatrix} \quad (16)$$

meaning that the rank of \mathcal{C} depends only on matrix $[\mathbf{I} \quad \mathbf{A} \quad \dots \quad \mathbf{A}^{n-1}]$. Adding to it columns \mathbf{A}^n does not change the rank, as \mathbf{A}^n is a linear combination of the other columns, as we proved in the previous slide.