Manipulator equations, Linearization

Control Theory, Lecture 13

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Content

- Linearization, Taylor
- Variations
- Linearization of manipulator equations

In general, for a system of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, there is a simple way to linearize it around any given point. The process if based on Taylor expansion:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0) \tag{1}$$

Denoting $\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \mathbf{A}$ and $\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \mathbf{B}$, we get:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \mathbf{A}(\mathbf{x} - \mathbf{x}_0) + \mathbf{B}(\mathbf{u} - \mathbf{u}_0) \tag{2}$$

Finally, denoting $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{A}\mathbf{x}_0 - \mathbf{B}\mathbf{u}_0 = \mathbf{c}$ we achieve local linearization of the nonlinear dynamics:

$$\dot{\mathbf{x}} \sim \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \tag{3}$$

Let us define variation $\delta \mathbf{x} = \mathbf{x} - \mathbf{x}_0$ and $\delta \mathbf{u} = \mathbf{u} - \mathbf{u}_0$. Then we can compute $\delta \dot{\mathbf{x}}$:

$$\delta \dot{\mathbf{x}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_0 \tag{4}$$

where $\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0)$. Previously we computed: $\mathbf{f}(\mathbf{x}, \mathbf{u}) \sim \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0)$. we can rewrite it as:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) \sim \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0) \tag{5}$$

$$\delta \dot{\mathbf{x}} \sim \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \delta \mathbf{u} \tag{6}$$

$$\delta \dot{\mathbf{x}} \sim \mathbf{A} \delta \mathbf{x} + \mathbf{B} \delta \mathbf{u} \tag{7}$$

Notice that this expression is always linear. Also notice that this expression is sufficient for stabilizing control design.

Newton eq., Lagrange eq.

In classical mechanics, *Newton equations* are accepted as an axiom and are given (for a system of points):

$$m_i \ddot{\mathbf{r}}_i = \mathbf{f}_i, \quad i = 1, \; ... \; m \tag{8}$$

where m_i are masses of the particles, $\mathbf{r}_i$ are their positions, and $\mathbf{f}_i$ are forces acting on them.

From it, there is a way to derive *Lagrange equations*:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} = \tau \tag{9}$$

where kinetic energy T is defined as: $T = 0.5\dot{\mathbf{r}}^\top \mathbf{M}\dot{\mathbf{r}}$, $\mathbf{q}$ are generalized coordinates, and τ are generalized forces.

Lagrange equations are useful when modelling a system of rigid bodies connected via *joints*. Examples include robot arms, walking robots and others. However, there is an even more useful form of these equations, called *manipulator equations*:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau_n \tag{10}$$

where $\mathbf{H} = \mathbf{H}(\mathbf{q})$ is a generalized inertia matrix, $\mathbf{C} = \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is inertial force matrix, $\mathbf{g} = \mathbf{g}(\mathbf{q})$ is generalized gravity and other conservative forces, and τ_n are generalized non-conservative forces, such as control inputs (motor torques, etc.)

Consider the following problem: given $\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau_n$, design control law that would stabilize the system at the origin.

Linearization

One of the ways to think about this problem is to imagine that if there existed a linear system, that behaves exactly like the original system in the vicinity of the origin, we could stabilize the linear one and then use the found control law to stabilize the non-linear version.

We, on the other hand, have eq. in the form $\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau_n$. Let us assume $\tau_n = \mathbf{T}\mathbf{u}$, where $\mathbf{T} = \mathbf{T}(\mathbf{q})$. We can also define $\mathbf{x}$:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \tag{11}$$

Then our attempt to write system of first order ODE gives us:

$$\frac{d}{dt}\left(\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}\right) = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{H}^{-1}(\mathbf{T}\mathbf{u} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \end{bmatrix} \tag{12}$$

In this case, state matrix $\mathbf{A}$ becomes:

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ \frac{\partial(\mathbf{H}^{-1}(\mathbf{Tu} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}))}{\partial \mathbf{q}} & \frac{\partial(\mathbf{H}^{-1}(\mathbf{Tu} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}))}{\partial \dot{\mathbf{q}}} \end{bmatrix} \tag{13}$$

$$\frac{\partial}{\partial \mathbf{q}}(\mathbf{H}^{-1}(\mathbf{Tu} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g})) = \frac{\partial \mathbf{H}^{-1}}{\partial \mathbf{q}}(\mathbf{Tu} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \mathbf{H}^{-1}\frac{\partial}{\partial \mathbf{q}}(\mathbf{Tu} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}))$$

$$\frac{\partial(\mathbf{H}^{-1}(\mathbf{Tu} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}))}{\partial \dot{\mathbf{q}}} = -\mathbf{H}^{-1}\mathbf{C} - \mathbf{H}^{-1}\frac{\partial \mathbf{C}}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} \tag{14}$$

$$\frac{\partial \mathbf{H}^{-1}}{\partial \mathbf{q}} = -\mathbf{H}^{-1}\frac{\partial \mathbf{H}}{\partial \mathbf{q}}\mathbf{H}^{-1} \tag{15}$$

In the general case it is:
Let us make an assumption that our linearization point $\mathbf{q}_0$, $\dot{\mathbf{q}}_0$ and $\mathbf{u}_0$ is a node, meaning that $\ddot{\mathbf{q}}_0 = 0$, which implies:

$$\mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} \tag{16}$$

Then

$$\frac{\partial}{\partial \mathbf{q}}(\mathbf{H}^{-1}(\mathbf{T}\mathbf{u} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g})) = \frac{\partial \mathbf{H}^{-1}}{\partial \mathbf{q}} 0 + \mathbf{H}^{-1}\frac{\partial}{\partial \mathbf{q}}(\mathbf{T}\mathbf{u} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}))$$

Control matrix **B** becomes:

$$\mathbf{B} = \begin{bmatrix} 0 \\ \frac{\partial(\mathbf{H}^{-1}(\mathbf{T}\mathbf{u} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}))}{\partial \mathbf{u}} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{H}^{-1}\mathbf{T} \end{bmatrix} \tag{17}$$

...and this does not look very clean and nice to use. Indeed, it is not easy or nice in practice.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.