

# Controllability, Observability

## Control Theory, Lecture 11

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## Theorem (Cayley–Hamilton)

*A matrix  $\mathbf{M} \in \mathbb{R}^{n,n}$  satisfies its own characteristic equation.*

A characteristic equation can be written as

$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$ , meaning that we can write:

$$\mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + \dots + a_0\mathbf{I} = 0 \quad (1)$$

Meaning that  $\mathbf{M}^n$  is a linear combination of  $\mathbf{M}^{n-1}$ ,  $\mathbf{M}^{n-2}$ , ...,  $\mathbf{I}$ .

# CONTROLLABILITY OF DISCRETE LTI

Consider discrete LTI:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \quad (2)$$

Assume the initial state is  $\mathbf{x}_1$ . Then we can deduce that:

$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1$$

$$\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2$$

$$\mathbf{x}_4 = \mathbf{A}\mathbf{x}_3 + \mathbf{B}\mathbf{u}_3 = \mathbf{A}(\mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2) + \mathbf{B}\mathbf{u}_3$$

...

$$\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + \dots + \mathbf{A}^{n-k} \mathbf{B}\mathbf{u}_k + \dots + \mathbf{B}\mathbf{u}_n$$

# CONTROLLABILITY MATRIX

Equation  $\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + \dots + \mathbf{A}^{n-k} \mathbf{B} \mathbf{u}_k + \dots \mathbf{B} \mathbf{u}_n$  can be re-written as:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \mathbf{u}_{k-2} \\ \dots \\ \mathbf{u}_1 \end{bmatrix} \quad (3)$$

Notice that in order for the system to go from  $\mathbf{x}_1$  to  $\mathbf{x}_{n+1}$ , vector  $\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1$  needs to be in the column space of  $\mathcal{C} = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$ .

Since  $\mathbf{x}_{n+1}$  can be anything, and  $\mathbf{x}_1$  might be equal to zero (among other possibilities), we should require that all vectors in  $\mathbb{R}^n$  need to be in the column space of  $\mathcal{C}$ , meaning  $\mathcal{C}$  needs to be full rank.

## Controllability

For a system  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$ , where  $\mathbf{x} \in \mathbb{R}^n$ , if the matrix  $\mathcal{C} = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$  is full row rank (i.e.  $\text{rank}(\mathcal{C}) = n$ ), any state can be reached, which means that *the system is controllable*.

If you are interested why the controllability matrix for not include more columns, like  $\mathbf{A}^n$ , Consider the following:  
The controllability matrix can be written as

$$\mathcal{C} = [\mathbf{I} \quad \mathbf{A} \quad \dots \quad \mathbf{A}^{n-1}] \begin{bmatrix} \mathbf{B} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{B} \end{bmatrix} \quad (4)$$

meaning that the rank of  $\mathcal{C}$  depends only on matrix  $[\mathbf{I} \quad \mathbf{A} \quad \dots \quad \mathbf{A}^{n-1}]$ . Adding to it columns  $\mathbf{A}^n$  does not change the rank, as  $\mathbf{A}^n$  is a linear combination of the other columns, as we proved in the previous slide.

# OBSERVABILITY OF DISCRETE LTI

Consider discrete LTI:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i \end{cases} \quad (5)$$

And an observer:

$$\hat{\mathbf{x}}_{i+1} = \mathbf{A}\hat{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i + \mathbf{L}(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_i) \quad (6)$$

Remember that we can define observation error  $\mathbf{e}_i = \hat{\mathbf{x}}_i - \mathbf{x}_i$  and write its dynamics:

$$\mathbf{e}_{i+1} = \mathbf{A}\mathbf{e}_i - \mathbf{L}\mathbf{C}\mathbf{e}_i \quad (7)$$

Dual system (which is stable if and only if the original is stable), has form:

$$\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i \quad (8)$$



# OBSERVABILITY OF DISCRETE LTI

## Dual system

Dynamical system  $\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i$ , we can be represented as:

$$\begin{cases} \varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i + \mathbf{C}^\top \mathbf{v}_i \\ \mathbf{v}_i = -\mathbf{L}^\top \varepsilon_i \end{cases} \quad (9)$$

Controllability matrix of this system is:

$$\mathcal{O}^\top = [\mathbf{C}^\top \quad (\mathbf{A}^\top)\mathbf{C}^\top \quad \dots \quad (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top] \quad (10)$$

It is easier to represent this matrix in its transposed form:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix} \quad (11)$$

# OBSERVABILITY OF DISCRETE LTI

## Observability criterion

### Observability

For a system  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$  and  $\mathbf{y}_i = \mathbf{C}\mathbf{x}_i$ , where  $\mathbf{x} \in \mathbb{R}^n$ , if

the matrix  $\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \dots \\ \mathbf{CA}^{n-1} \end{bmatrix}$  is full column rank (i.e.

$\text{rank}(\mathcal{O}) = n$ ), observation error can go to zero from any initial position, which means that *the system is observable*.

Exponential  $e^a$  is defined as a series:

$$e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}a^n \quad (12)$$

Matrix exponential  $e^{\mathbf{A}}$  is defined as a series:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}\mathbf{A} + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n \quad (13)$$

# ANALYTICAL SOLUTION TO ODE

An ODE of the form  $\dot{x} = ax$  has analytical solution  $x(t) = e^{at}x(0)$ .

An ODE of the form  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$  has analytical solution  $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$ .

Let us check that this is a solution:

$$\mathbf{x}(t) = \left( \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + \dots \right) \mathbf{x}(0) \quad (14)$$

$$\dot{\mathbf{x}}(t) = \left( \mathbf{A} + \mathbf{A}\mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}\mathbf{A}t^2 + \dots \right) \mathbf{x}(0) \quad (15)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A} \left( \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \dots \right) \mathbf{x}(0) \quad (16)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) \quad (17)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad \square \quad (18)$$

# FORCED STATE RESPONSE (LTI) (1)

An ODE of the form  $\dot{x} = ax + bu(t)$  also has analytical solution.  
To find it, we first find the following derivative:

$$\frac{d}{dt} (e^{-at}x(t)) = e^{-at}\dot{x}(t) - ae^{-at}x(t) \quad (19)$$

Multiplying  $\dot{x} = ax + bu(t)$  by  $e^{-at}$  we see:

$$e^{-at}\dot{x} = e^{-at}ax + e^{-at}bu(t) \quad (20)$$

$$e^{-at}\dot{x} - e^{-at}ax = e^{-at}bu(t) \quad (21)$$

$$\frac{d}{dt} (e^{-at}x(t)) = e^{-at}bu(t) \quad (22)$$

$$\int_0^t \frac{d}{d\tau} (e^{-a\tau}x(\tau)) d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \quad (23)$$

# FORCED STATE RESPONSE (LTI) (2)

Continuing the derivation:

$$\int_0^t \frac{d}{d\tau} (e^{-a\tau} x(\tau)) d\tau = \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (24)$$

$$e^{-at} x(t) - x(0) = \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (25)$$

$$x(t) = e^{at} x(0) + e^{at} \int_0^t e^{-a\tau} bu(\tau) d\tau \quad (26)$$

$$x(t) = e^{at} x(0) + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \quad (27)$$

## FORCED STATE RESPONSE (LTI) (3)

State-space equation  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t)$  also has an analytical solution:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \quad (28)$$

The same can be re-written as:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \quad (29)$$

# CONTROLLABILITY, CONTINUOUS-TIME (1)

Let us consider matrix exponential  $e^{\mathbf{A}t}$  is defined as a series:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + \dots \quad (30)$$

Using Cayley–Hamilton we can observe that any powers of  $\mathbf{A}$  higher than  $n$  can be represented as a linear combination of lower powers. This gives us the following expression:

$$e^{\mathbf{A}t} = \phi_0(t)\mathbf{I} + \phi_1(t)\mathbf{A} + \phi_2(t)\mathbf{A}^2 + \dots + \phi_{n-1}(t)\mathbf{A}^{n-1} \quad (31)$$

This allows us to re-write the forced state response:

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau \\ \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t (\phi_0(t-\tau)\mathbf{I} + \phi_1(t-\tau)\mathbf{A} + \dots \\ &\quad + \phi_{n-1}(t-\tau)\mathbf{A}^{n-1})\mathbf{b}u(\tau) d\tau \end{aligned}$$



## CONTROLLABILITY, CONTINUOUS-TIME (2)

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \phi_0(t-\tau)\mathbf{b}u(\tau)d\tau \\ &\quad + \int_0^t \phi_1(t-\tau)\mathbf{A}\mathbf{b}u(\tau)d\tau + \dots \int_0^t \phi_{n-1}(t-\tau)\mathbf{A}^{n-1}\mathbf{b}u(\tau)d\tau\end{aligned}$$

$$\mathbf{x}(t) - e^{\mathbf{A}t}\mathbf{x}(0) = [\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{b}] \begin{bmatrix} \int_0^t \phi_0(t-\tau)u(\tau)d\tau \\ \int_0^t \phi_1(t-\tau)u(\tau)d\tau \\ \dots \\ \int_0^t \phi_{n-1}(t-\tau)u(\tau)d\tau \end{bmatrix}$$

This shows that if controllability matrix is rank-deficient, it would not be possible to achieve some state from some initial condition.

# PBH CONTROLLABILITY CRITERION

There is an alternative way to test if pair  $(\mathbf{A}, \mathbf{B})$  is controllable:

## PBH controllability criterion

If for any  $\lambda \in \mathbb{C}$ , the the matrix  $[(\mathbf{A} - \lambda\mathbf{I}), \mathbf{B}]$  has full row rank, then the pair  $(\mathbf{A}, \mathbf{B})$  is controllable.

- If  $\lambda$  is not an eigenvalue of  $\mathbf{A}$ , then  $\det(\mathbf{A} - \lambda\mathbf{I}) \neq 0$  and the matrix has full row rank.
- If  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$  and  $\mathcal{V} = \text{null}(\mathbf{A} - \lambda\mathbf{I})$ ,  $\dim(\mathcal{V}) = 1$  and  $\mathbf{v} \in \mathcal{V}$ , meaning  $\lambda, \mathbf{v}$  are eigenvalue and eigenvector of  $\mathbf{A}$ , then in order for the criterion to hold the columns fo  $\mathbf{B}$  should not all be orthogonal to  $\mathbf{v}$ :  $\mathbf{v}^\top \mathbf{B} \neq 0$ .
- If eigenspace  $\mathcal{V}$  is  $k$ -dimensional, the projection of  $\mathbf{B}$  onto that eigenspace should also be  $k$ -dimensional.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:

[github.com/SergeiSa/Control-Theory-Slides-Spring-2023](https://github.com/SergeiSa/Control-Theory-Slides-Spring-2023)

