Controllability, Observability

Control Theory, Lecture 11

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Content

Theorem (Cayley–Hamilton)

A matrix $\mathbf{M} \in \mathbb{R}^{n,n}$ satisfies its own characteristic equation.

A characteristic equation can be written as $\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_0 = 0$, meaning that we can write:

$$\mathbf{M}^n + a_{n-1}\mathbf{M}^{n-1} + ... + a_0\mathbf{I} = 0 \tag{1}$$

Meaning that $\mathbf{M}^n$ is a linear combination of $\mathbf{M}^{n-1}$, $\mathbf{M}^{n-2}$, ..., $\mathbf{I}$.

Consider discrete LTI:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \tag{2}$$

Assume the initial state is $\mathbf{x}_1$. Then we can deduce that:

$$\begin{aligned}
\mathbf{x}_2 &= \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1 \\
\mathbf{x}_3 &= \mathbf{A}\mathbf{x}_2 + \mathbf{B}\mathbf{u}_2 = \mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2 \\
\mathbf{x}_4 &= \mathbf{A}\mathbf{x}_3 + \mathbf{B}\mathbf{u}_3 = \mathbf{A}(\mathbf{A}(\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u}_1) + \mathbf{B}\mathbf{u}_2) + \mathbf{B}\mathbf{u}_3 \\
&... \\
\mathbf{x}_{n+1} &= \mathbf{A}^n\mathbf{x}_1 + ... + \mathbf{A}^{n-k}\mathbf{B}\mathbf{u}_k + ... + \mathbf{B}\mathbf{u}_n
\end{aligned}$$

Equation $\mathbf{x}_{n+1} = \mathbf{A}^n \mathbf{x}_1 + ... + \mathbf{A}^{n-k}\mathbf{B}\mathbf{u}_k + ...\mathbf{B}\mathbf{u}_n$ can be re-written as:

$$\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1 = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \mathbf{u}_{k-2} \\ ... \\ \mathbf{u}_1 \end{bmatrix} \tag{3}$$

Notice that in order for the system to go from $\mathbf{x}_1$ to $\mathbf{x}_{n+1}$, vector $\mathbf{x}_{n+1} - \mathbf{A}^n \mathbf{x}_1$ needs be in the column space of $\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$.
Since $\mathbf{x}_{n+1}$ can be anything, and $\mathbf{x}_1$ might be equal to zero (among other possibilities), we should require that all vectors in $\mathbb{R}^n$ need to be in the column space of $\mathcal{C}$, meaning $\mathcal{C}$ needs to be full rank.

Controllability criterion

Controllability

For a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$, where $\mathbf{x} \in \mathbb{R}^n$, if the matrix $\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & ... & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$ is full row rank (i.e. $\text{rank}(\mathcal{C}) = n$), any state can be reached, which means that *the system is controllable.*

If you are interested why the controllability matrix for not include more columns, like $\mathbf{A}^n$, Consider the following:
The controllability matrix can be written as

$$\mathcal{C} = \begin{bmatrix} \mathbf{I} & \mathbf{A} & ... & \mathbf{A}^{n-1} \end{bmatrix} \begin{bmatrix} \mathbf{B} & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & ... & \mathbf{0} \\ ... & ... & ... & ... \\ \mathbf{0} & \mathbf{0} & ... & \mathbf{B} \end{bmatrix} \tag{4}$$

meaning that the rank of $\mathcal{C}$ depends only on matrix $\begin{bmatrix} \mathbf{I} & \mathbf{A} & ... & \mathbf{A}^{n-1} \end{bmatrix}$. Adding to it columns $\mathbf{A}^n$ does not change the rank, as $\mathbf{A}^n$ is a linear combination of the other columns, as we proved in the previous slide.

Consider discrete LTI:

$$\begin{cases} \mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \\ \mathbf{y}_i = \mathbf{C}\mathbf{x}_i \end{cases} \tag{5}$$

And an observer:

$$\hat{\mathbf{x}}_{i+1} = \mathbf{A}\hat{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i + \mathbf{L}(\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_i) \tag{6}$$

Remember that we can define observation error $\mathbf{e}_i = \hat{\mathbf{x}}_i - \mathbf{x}_i$ and write its dynamics:

$$\mathbf{e}_{i+1} = \mathbf{A}\mathbf{e}_i - \mathbf{L}\mathbf{C}\mathbf{e}_i \tag{7}$$

Dual system (which is stable if and only if the original is stable), has form:

$$\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i \tag{8}$$

Dynamical system $\varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i - \mathbf{C}^\top \mathbf{L}^\top \varepsilon_i$, we can be represented as:

$$\begin{cases} \varepsilon_{i+1} = \mathbf{A}^\top \varepsilon_i + \mathbf{C}^\top \mathbf{v}_i \\ \mathbf{v}_i = -\mathbf{L}^\top \varepsilon_i \end{cases} \tag{9}$$

Controllability matrix of this system is:

$$\mathcal{O}^\top = \begin{bmatrix} \mathbf{C}^\top & (\mathbf{A}^\top)\mathbf{C}^\top & ... & (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top \end{bmatrix} \tag{10}$$

It is easier to represent this matrix in its transposed form:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ ... \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{11}$$

Observability

For a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$ and $\mathbf{y}_i = \mathbf{C}\mathbf{x}_i$, where $\mathbf{x} \in \mathbb{R}^n$, if the matrix $\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ ... \\ \mathbf{CA}^{n-1} \end{bmatrix}$ is full column rank (i.e. $\mathrm{rank}(\mathcal{O}) = n$), observation error can go to zero from any initial position, which means that *the system is observable.*

Let us consider matrix exponential $e^{\mathbf{A}t}$ is defined as a series:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + ... \tag{12}$$

Using Cayley–Hamilton we can observe that any powers of $\mathbf{A}$ higher than n can be represented as a linear combination of lower powers. This gives us the following expression:

$$e^{\mathbf{A}t} = \phi_0(t)\mathbf{I} + \phi_1(t)\mathbf{A} + \phi_2(t)\mathbf{A}^2 + ... + \phi_{n-1}(t)\mathbf{A}^{n-1} \tag{13}$$

This allows us to re-write the forced state response:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t (\phi_0(t-\tau)\mathbf{I} + \phi_1(t-\tau)\mathbf{A} + ... + \phi_{n-1}(t-\tau)\mathbf{A}^{n-1})\mathbf{b}u(\tau)\, d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \phi_0(t-\tau)\mathbf{b}u(\tau)d\tau$$
$$+ \int_0^t \phi_1(t-\tau)\mathbf{A}\mathbf{b}u(\tau)d\tau + ... \int_0^t \phi_{n-1}(t-\tau)\mathbf{A}^{n-1}\mathbf{b}u(\tau)d\tau$$

$$\mathbf{x}(t) - e^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix}\mathbf{b} & \mathbf{A}\mathbf{b} & ... & \mathbf{A}^{n-1}\mathbf{b}\end{bmatrix} \begin{bmatrix} \int_0^t \phi_0(t-\tau)u(\tau)d\tau \\ \int_0^t \phi_1(t-\tau)u(\tau)d\tau \\ ... \\ \int_0^t \phi_{n-1}(t-\tau)u(\tau)d\tau \end{bmatrix}$$

This shows that if controllability matrix is rank-deficient, it would not be possible to achieve some state from some initial condition.

PBH controllability criterion

There is an alternative way to test if pair $(\mathbf{A},\ \mathbf{B})$ is controllable:

PBH controllability criterion

If for any $\lambda \in \mathbb{C}$, the the matrix $[(\mathbf{A} - \lambda \mathbf{I}), \mathbf{B}]$ has full row rank, then the pair $(\mathbf{A},\ \mathbf{B})$ is controllable.

- If λ is not an eigenvalue of $\mathbf{A}$, then $\det(\mathbf{A} - \lambda \mathbf{I}) \neq 0$ and the matrix has full row rank.
- If $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ and $\mathcal{V} = \mathrm{null}(\mathbf{A} - \lambda \mathbf{I})$, $\dim(\mathcal{V}) = 1$ and $\mathbf{v} \in \mathcal{V}$, meaning $\lambda,\ \mathbf{v}$ are eigenvalue and eigenvector of $\mathbf{A}$, then in order for the criterion to hold the columns fo $\mathbf{B}$ should not all be orthogonal to $\mathbf{v}$: $\mathbf{v}^\top \mathbf{B} \neq 0$.
- If eigenspace $\mathcal{V}$ is k-dimensional, the projection of $\mathbf{B}$ onto that eigenspace should also be k-dimensional.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2023

Appendix A: Analytical solution (recap)

Exponential e^a is defined as a series:

$$e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + ... = \sum_{n=0}^{\infty} \frac{1}{n!}a^n \tag{14}$$

Matrix exponential $e^{\mathbf{A}}$ is defined as a series:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}\mathbf{A} + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A} + ... = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n \tag{15}$$

Analytical solution to ODE

An ODE of the form $\dot{x} = ax$ has analytical solution $x(t) = e^{at}x(0)$.

An ODE of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ has analytical solution $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$.

Let us check that this is a solution:

$$\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + \frac{1}{6}\mathbf{A}\mathbf{A}\mathbf{A}t^3 + ...\right)\mathbf{x}(0) \tag{16}$$

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} + \mathbf{A}\mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}\mathbf{A}t^2 + ...\right)\mathbf{x}(0) \tag{17}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}\mathbf{A}t^2 + ...\right)\mathbf{x}(0) \tag{18}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) \tag{19}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \qquad \square \tag{20}$$

An ODE of the form $\dot{x} = ax + bu(t)$ also has analytical solution. To find it, we first find the following derivative:

$$\frac{d}{dt}\left(e^{-at}x(t)\right) = e^{-at}\dot{x}(t) - ae^{-at}x(t) \tag{21}$$

Multiplying $\dot{x} = ax + bu(t)$ by e^{-at} we see:

$$e^{-at}\dot{x} = e^{-at}ax + e^{-at}bu(t) \tag{22}$$

$$e^{-at}\dot{x} - e^{-at}ax = e^{-at}bu(t) \tag{23}$$

$$\frac{d}{dt}\left(e^{-at}x(t)\right) = e^{-at}bu(t) \tag{24}$$

$$\int_0^t \frac{d}{d\tau}\left(e^{-a\tau}x(\tau)\right)d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \tag{25}$$

Continuing the derivation:

$$\int_0^t \frac{d}{d\tau}\left(e^{-a\tau}x(\tau)\right)d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \tag{26}$$

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau \tag{27}$$

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(\tau)d\tau \tag{28}$$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \tag{29}$$

Forced State Response (LTI) (3)

State-space equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t)$ also has an analytical solution:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{30}$$

The same can be re-written as:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{31}$$