Observers

Control Theory, Lecture 8

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Spring 2023

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Before we considered systems and control laws of the following type:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{u} = \mathbf{K}\mathbf{x} \end{cases} \tag{1}$$

But when we implement that control law, how do we know the current value of $\mathbf{x}$?

In practice, we can *estimate* it using *measurement*.

Why information is imperfect?

There are a number of reasons why we can not directly measure the state of the system. Here are some:

- Digital measurements are done in discrete time intervals.
- Unpredicted events (faults, collisions, etc.).
- Un-modelled kinematics or dynamics (links bending, gear box backlash, friction, etc.) making the very definition of the state disconnected from reality.
- Lack of sensors.
- Imprecise, nonlinear and biased sensors.
- Other physical effects.

Let us introduce new notation. We have an LTI system of the following form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \hat{\mathbf{x}}(t) = \text{estimate}\left(\mathbf{y}(t)\right) \\ \mathbf{u} = -\mathbf{K}\hat{\mathbf{x}} \end{cases} \tag{2}$$

Then:

- $\mathbf{x}$ and $\mathbf{y}$ are the state and output (actual or true)
- $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}$ are the estimated (observed) state and output.

Notice that we never know true state $\mathbf{x}$, and therefore for the control purposes we have to use the estimated state $\hat{\mathbf{x}}$.

Estimation error

How can we quantify the error in our estimation? We can do it directly as state estimation error:

$$\varepsilon = \hat{\mathbf{x}} - \mathbf{x} \tag{3}$$

But this is impossible to compute, since we do not know $\mathbf{x}$. Alternatively, we can compare measured output $\mathbf{y}$ with estimated output $\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}$:

$$\tilde{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} - \mathbf{y} \tag{4}$$

This can always be computed.

Let us consider autonomous dynamical system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{5}$$

with measurements $\mathbf{y}$. We want to get as good an estimate of the state $\hat{\mathbf{x}}$ as we can.

First note: dynamics should also hold for our observed state:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} \tag{6}$$

Therefore if we know the initial conditions of our system exactly, and we know our model exactly, we can find exact state of the system without using measurement $\mathbf{y}$. We can call it an open loop observation. Unfortunately, we know neither the model nor the initial conditions precisely.

We propose *observer* that takes into account measurements in a linear way; analogues with linear control $-\mathbf{K}\mathbf{x}$, here we propose a linear law $-\mathbf{L}\tilde{\mathbf{y}}$. Remembering that $\tilde{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} - \mathbf{y}$ we get:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{7}$$

With this observer, we want to get as good estimate of the state $\hat{\mathbf{x}}$ as we can.

We can subtract (5) from (7), to get *observer error dynamics*:

$$\dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{A}\mathbf{x} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{8}$$

$$\dot{\varepsilon} = (\mathbf{A} - \mathbf{L}\mathbf{C})\varepsilon \tag{9}$$

Observer gains

The observer $\dot{\varepsilon} = (\mathbf{A} - \mathbf{LC})\varepsilon$ is *stable* (i.e., the state estimation error tends to zero), as long as the following matrix has eigenvalues with negative real parts:

$$\mathbf{A} - \mathbf{LC} \in \mathbb{H}$$

We need to find $\mathbf{L}$. Let us observe the key difference between observer design and controller design:

- Controller design: find such $\mathbf{K}$ that $\mathbf{A} - \mathbf{BK} \in \mathbb{H}$.
- Observer design: find such $\mathbf{L}$ that: $\mathbf{A} - \mathbf{LC} \in \mathbb{H}$

We have instruments for finding $\mathbf{K}$, what about $\mathbf{L}$?

In general, we can observe that if $\mathbf{A} - \mathbf{LC} \in \mathbb{H}$, then $(\mathbf{A} - \mathbf{LC})^\top \in \mathbb{H}$ (eigenvalues of a matrix and its transpose are the same, see Appendix).

Therefore, we can solve the following *dual problem*:

- find such $\mathbf{L}$ that $\mathbf{A}^\top - \mathbf{C}^\top \mathbf{L}^\top \in \mathbb{H}$.

The dual problem is *equivalent* to the control design problem. We can solve it by producing and solving algebraic Riccati equation, as in the LQR formulation. In pseudo-code it can be represented the following way:

$\mathbf{L}^\top$ `= lqr`$(\mathbf{A}^\top, \mathbf{C}^\top, \mathbf{Q}, \mathbf{R})$.
where $\mathbf{Q}$ and $\mathbf{R}$ are weight matrices, determining the "sensitivity" or "aggressiveness" of the observer.

Thus we get dynamics+observer combination:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \\ \mathbf{y} = \mathbf{C}\mathbf{x} \\ \mathbf{u} = -\mathbf{K}\hat{\mathbf{x}} \end{cases} \tag{10}$$

where $\mathbf{A} - \mathbf{B}\mathbf{K} \in \mathbb{H}$ and $\mathbf{A}^\top - \mathbf{C}^\top \mathbf{L}^\top \in \mathbb{H}$.

Let us re-write the dynamics

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}\mathbf{C}\mathbf{x} - \mathbf{L}\mathbf{C}\hat{\mathbf{x}} \end{cases} \tag{11}$$

in a matrix form:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{C} & (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} \tag{12}$$

We can't directly reason about eigenvalues of this matrix. Next slide will show a way to do it with a change of variables.

Let us use the following substitution: $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, which implies $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{e}$:
Our system had form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}\mathbf{C}\mathbf{x} - \mathbf{L}\mathbf{C}\hat{\mathbf{x}} \end{cases} \tag{13}$$

Since $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}$, we get:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} - (\mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} + \mathbf{L}\mathbf{C}\mathbf{x} - \mathbf{L}\mathbf{C}\hat{\mathbf{x}})$$

$$\dot{\mathbf{e}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) - \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}})$$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}$$

Equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}$ takes form:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e}$$

Collecting $\dot{\mathbf{x}}$ and $\dot{\mathbf{e}}$ we get:

$$\begin{cases} \dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BKe} \\ \dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC})\mathbf{e} \end{cases} \tag{14}$$

In matrix form it becomes:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} (\mathbf{A} - \mathbf{BK}) & \mathbf{BK} \\ 0 & (\mathbf{A} - \mathbf{LC}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{15}$$

Eigenvalues of a upper block-triangular matrices equal to the union of the eigenvalues of the blocks on the main diagonal (see Appendix B). Hence here, the eigenvalues of the system are equal to the union of eigenvalues of $(\mathbf{A} - \mathbf{BK})$ and $(\mathbf{A} - \mathbf{LC})$.

Since the eigenvalues of the system are equal to the union of eigenvalues of $(\mathbf{A} - \mathbf{BK})$ and $(\mathbf{A} - \mathbf{LC})$, we can make the following observation:

Separation principle

As long as the observer and the controller are stable independently, the overall system is stable too. This is called *separation principle*.

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2023

Given matrix $\mathbf{M}$ and its eigenvalue λ and eigenvector $\mathbf{v}$. we can prove that λ is an eigenvector of $\mathbf{M}^\top$:

$$\mathbf{M}\mathbf{v} = \lambda\mathbf{v} \tag{16}$$

$$\det\left(\mathbf{M} - \mathbf{I}\lambda\right) = 0 \tag{17}$$

$$\det\left(\mathbf{M}^\top - \mathbf{I}\lambda\right) = 0 \tag{18}$$

$$\mathbf{M}^\top\mathbf{u} = \lambda\mathbf{u} \tag{19}$$

We used the fact that determinant of a matrix is equal to the determinant of its transpose: $\det\left(\mathbf{A}\right) = \det\left(\mathbf{A}^\top\right)$.

Eig. values of block-diagonal matrices

Given matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \tag{20}$$

Let λ, $\mathbf{v}$ be an eigenvalue and eigenvector of $\mathbf{A}$ and μ, $\mathbf{u}$ be an eigenvalue and eigenvector of $\mathbf{C}$. We can prove that λ, $\mathbf{v}_M = \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix}$ are eigenvalue and eigenvector of $\mathbf{M}$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{v} \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix}. \tag{21}$$

If μ is not an eigenvalue of $\mathbf{A}$, we can prove that μ, $\mathbf{u}_M = \begin{bmatrix} (\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{u} \end{bmatrix}$ are eigenvalue and eigenvector of $\mathbf{M}$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} (\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} + \mathbf{B}\mathbf{u} \\ \mathbf{C}\mathbf{u} \end{bmatrix} =$$

$$= \begin{bmatrix} (\mathbf{I} + \mathbf{A}(\mathbf{I}\mu - \mathbf{A})^{-1})\mathbf{B}\mathbf{u} \\ \mu\mathbf{u} \end{bmatrix} = \begin{bmatrix} (\mathbf{I}\mu - \mathbf{A} + \mathbf{A})(\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mu\mathbf{u} \end{bmatrix} =$$

$$= \begin{bmatrix} \mu(\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mu\mathbf{u} \end{bmatrix} = \mu \begin{bmatrix} (\mathbf{I}\mu - \mathbf{A})^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{u} \end{bmatrix}.$$

Counting the number of eigenvalues we observe that eigenvalues of $\mathbf{M}$ include only eigenvalues of $\mathbf{A}$ and $\mathbf{B}$.