

Stabilizing Control

Control Theory, Lecture 5

by Sergei Savin

Spring 2022

- Changing stability
- Stabilizing control
- Linear control: Closed-loop system,
- Affine control
- Error dynamics
- Affine trajectory tracking
- Point-to-point control
- Pure state feedback
- Read more

CHANGING STABILITY

Here are two LTIs:

$$\dot{x} = 2x \quad (1)$$

$$\dot{x} = 2x + u \quad (2)$$

First one is autonomous and unstable. Second one is not autonomous, and we won't know whether or not it is stable, until we know what u is.

If we pick $u = 0$, the result is an unstable equation. But we can also pick u such that the resulting dynamics is stable, such as $u = -3x$:

$$\dot{x} = 2x + u = 2x - 3x = -x \quad (3)$$

So, we can use *control input* u to change stability of the system!

Definition

The problem of finding control law \mathbf{u} that make a certain solution \mathbf{x}^* of dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ stable is called *stabilizing control problem*

This is true for both linear and non-linear systems. But for linear systems we can get a lot more details about this problem, if we restrict our choice of control law.

LINEAR CONTROL

Closed-loop system

Consider an LTI system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (4)$$

and let us chose *control as a linear function of the state x* :

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \quad (5)$$

Thus, we know how the system is going to look when the control is applied:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} \quad (6)$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \quad (7)$$

Note that (7) is an autonomous system. We call this a *closed loop* system.

Observing the system $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$ we obtained, we can notice that we already have the tools to analyse its stability:

Stability condition for LTI closed-loop system

The real parts of the eigenvalues of the matrix $(\mathbf{A} - \mathbf{BK})$ should be negative for asymptotic stability, or non-positive for stability in the sense of Lyapunov.

Hurwitz matrix

If square matrix \mathbf{M} has eigenvalues with strictly negative real parts, it is called Hurwitz. We will denote it as $\mathbf{M} \in \mathcal{H}$.

So, all you need to do is to find such \mathbf{K} that $(\mathbf{A} - \mathbf{BK})$ is Hurwitz, and you made a an asymptotically stable closed-loop system!

AFFINE CONTROL

Part 1

We don't have to limit ourselves to just this $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{u} = -\mathbf{K}\mathbf{x}$ pair.

In fact, this pair mostly works for the simple case when the solution we want to stabilize is trivial $\mathbf{x}^*(t) = 0$.

Let us consider a slightly more complicated system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \quad (8)$$

This is called *affine system*, because of the constant term \mathbf{c} .

What is the control that stabilizes this system? Let us propose an *affine control law*:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^* \quad (9)$$

where \mathbf{u}^* is a constant term.

Thus, from $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c}$ and $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$ we get the following closed-loop system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} + \mathbf{B}\mathbf{u}^* + \mathbf{c} \quad (10)$$

And as long as we can choose such \mathbf{u}^* that $\mathbf{B}\mathbf{u}^* = -\mathbf{c}$, we will get back to the previously seen form $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$.

Existence of the stabilizing control

Same as it is possible that there exists no such \mathbf{K} that $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$ is stable, there might exist no such \mathbf{u}^* that $\mathbf{B}\mathbf{u}^* = -\mathbf{c}$

Let us now consider an arbitrary solution $\mathbf{x}^* = \mathbf{x}^*(t)$ for the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (11)$$

and try to find a stabilizing control for it.

Our first step is to notice that, if $\mathbf{x}^* = \mathbf{x}^*(t)$ is a solution, that means that it satisfies the ODE (11):

$$\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \quad (12)$$

where $\mathbf{u}^* = \mathbf{u}^*(t)$ is some control law, for which the solution $\mathbf{x}^* = \mathbf{x}^*(t)$ is obtained.

If we are not given $\mathbf{u}^* = \mathbf{u}^*(t)$, we can compute it as:

$$\mathbf{u}^* = \mathbf{B}^+(\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^*) \quad (13)$$

where \mathbf{B}^+ is a pseudo-inverse, and the solution to this least-squared problem will have to have no residual (since $\mathbf{x}^* = \mathbf{x}^*(t)$ is a solution).

$$\|\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^* - \mathbf{B}\mathbf{B}^+(\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^*)\| = 0 \quad (14)$$

Now, let us introduce the concept of *control error* \mathbf{e} :

$$\mathbf{e} = \mathbf{x} - \mathbf{x}^* \quad (15)$$

Control error and stability

If control error goes to zero asymptotically, every solution goes to \mathbf{x}^* .

Remember that we have two simultaneous equations:

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^*$. We can now subtract one from the other to get:

$$\dot{\mathbf{x}} - \dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u} - \mathbf{B}\mathbf{u}^* \quad (16)$$

in other words:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \quad (17)$$

where $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$

We arrived at a new dynamical system $\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v}$, which is an LTI, and we are interested in stabilizing the solution $\mathbf{e}^* = 0$. We know how to do it with a linear control law:

$$\mathbf{v} = -\mathbf{K}\mathbf{e} \quad (18)$$

Now remember that $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$ and $\mathbf{e} = \mathbf{x} - \mathbf{x}^*$, this will become:

$$\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{u}^* \quad (19)$$

This control law $\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{u}^*$ can be thought of as consisting of two parts:

- Feedback control $\mathbf{u}_{FB} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*)$, which depends on the control error (which requires a feedback about the current state of your system)
- Feed-forward control $\mathbf{u}_{FF} = \mathbf{u}^*$, which depends only on the trajectory and the equations of dynamics of your system, but not on your current state

AFFINE TRAJECTORY TRACKING

Part 1

What we just did - stabilization of the arbitrary trajectory $\mathbf{x}^* = \mathbf{x}^*(t)$ - is also called *trajectory tracking control*, or *trajectory stabilization*. The solution we stabilized is called *trajectory*.

Just for completeness, let's consider the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \quad (20)$$

and stabilize trajectory $\mathbf{x}^* = \mathbf{x}^*(t)$.

AFFINE TRAJECTORY TRACKING

Part 2

We start by observing that, as before, our solution gives us equality:

$$\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* + \mathbf{c} \quad (21)$$

and after introducing control error and subtracting (21) from the original dynamics (20), we get:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \quad (22)$$

where $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$, which we already saw before. The only difference is that now \mathbf{u}^* is found as:

$$\mathbf{u}^* = \mathbf{B}^+(\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^* - \mathbf{c}) \quad (23)$$

What if we want to move our system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ from initial condition to some desired state \mathbf{x}^* . This can be called *point-to-point control*.

This is the same as trajectory control with $\mathbf{x}^* = \text{const}$ and $\dot{\mathbf{x}}^* = 0$.

- Richard M. Murray Control and Dynamical Systems
California Institute of Technology [Optimization-Based Control](#)
- [Dynamic Simulation in Python](#)

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.