Lyapunov Theory, Lyapunov equations

Control Theory, Lecture 7

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Content

Asymptotic stability criteria

Autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is assymptotically stable, if there exists a scalar function $V = V(\mathbf{x}) > 0$, whose time derivative is negative $\dot{V}(\mathbf{x}) < 0$, except $V(\mathbf{0}) = 0$, $\dot{V}(\mathbf{0}) = 0$.

Marginal stability criteria

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is stable in the sense of Lyapunov, $\exists V(\mathbf{x}) > 0$, $\dot{V}(\mathbf{x}) \leq 0$.

Definition

Function $V(\mathbf{x}) > 0$ in this case is called *Lyapunov function*.

This is not the only type of stability as you remember, you are invited to study criteria for other stability types on your own.

Take dynamical system $\dot{x} = -x$.

We propose a *Lyapunov function candidate* $V(x) = x^2 \geq 0$. Let's find its derivative:

$$\dot{V}(x) = \frac{\partial V}{\partial x}(-x) = 2x(-x) = -x^2 \leq 0 \tag{1}$$

This satisfies the Lyapunov criteria, so the system is stable. It is in fact asymptotically stable, because $\dot{V}(x) \neq 0$ if $x \neq 0$.

Consider oscillator $\ddot{q} = f(q, \dot{q}) = -\dot{q}$.

We propose a *Lyapunov function candidate* $V(q, \dot{q}) = T(q, \dot{q}) = \frac{1}{2}\dot{q}^2 \geq 0$, where $T(q, \dot{q})$ is kinetic energy of the system. Let's find its derivative:

$$\dot{V}(q, \dot{q}) = \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}} f(q, \dot{q}) = \dot{q}(-\dot{q}) = -\dot{q}^2 \leq 0 \qquad (2)$$

This satisfies the Lyapunov criteria, so the system is stable. But it is not proven to be asymptotically stable, because $\dot{V}(q, \dot{q}) = 0$ for any q as long as $\dot{q} = 0$.

Consider pendulum $\ddot{q} = f(q, \dot{q}) = -\dot{q} - sin(q)$.

We propose a *Lyapunov function candidate* $V(q, \dot{q}) = E(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + 1 - cos(q) \geq 0$, where $E(q, \dot{q})$ is total energy of the system. Let's find its derivative:

$$\dot{V}(q, \dot{q}) = \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}} f(q, \dot{q}) = \dot{q} sin(q) + \dot{q}(-\dot{q} - sin(q)) = -\dot{q}^2 \leq 0 \tag{3}$$

This satisfies the Lyapunov criteria, so the system is stable. It is not proven to be asymptotically stable, because $\dot{V}(q, \dot{q}) = 0$ for any q, as long as $\dot{q} = 0$.

As you saw, Lyapunov method allows you to deal with nonlinear systems, as well as linear ones. But for linear ones there are additional properties we can use.

Observation 1

For a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ we can always pick Lyapunov function candidate in the form $V = \mathbf{x}^\top \mathbf{S} \mathbf{x} \geq 0$, where $\mathbf{S}$ is a positive semidefinite matrix.

Next slides will shows where this leads us.

Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and $V = \mathbf{x}^\top \mathbf{S}\mathbf{x} \geq 0$, let's find its derivative:

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}\dot{\mathbf{x}} \tag{4}$$

$$\dot{V}(\mathbf{x}) = (\mathbf{A}\mathbf{x})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}\mathbf{A}\mathbf{x} = \mathbf{x}^\top (\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A})\mathbf{x} \tag{5}$$

Notice that $\dot{V}(x)$ should be negative for all $\mathbf{x}$ for the system to be stable, meaning that $\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A}$ should be negative semidefinite. A more strict form of this requirement is *Lyapunov equation*:

$$\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} = -\mathbf{Q} \tag{6}$$

where $\mathbf{Q}$ is a positive-definite matrix.

Marginal stability criteria, discrete case

Given $\mathbf{x}_{i+1} = \mathbf{f}(\mathbf{x}_i)$, if $V(\mathbf{x}_i) > 0$, and $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) \leq 0$, the system is stable.

Same as before, for linear systems we will be choosing *positive semidefinite quadratic forms* as Lyapunov function candidates.

Consider dynamics $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$ and $V = \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i \geq 0$, let's find $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i)$:

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) = (\mathbf{A}\mathbf{x}_i)^\top \mathbf{S}\mathbf{A}\mathbf{x}_i - \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i \tag{7}$$

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) = \mathbf{x}_i^\top (\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S})\mathbf{x}_i \tag{8}$$

Notice that $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i)$ should be negative for all $\mathbf{x}_i$ for the system to be stable, meaning that $\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S}$ should be negative semidefinite. A more strict form of this requirement is *Discrete Lyapunov equation*:

$$\mathbf{A}^\top \mathbf{S}\mathbf{A} - \mathbf{S} = -\mathbf{Q} \tag{9}$$

where $\mathbf{Q}$ is a positive-definite matrix.

In practice, you can easily use Lyapunov equations for stability verification. Python and MATLAB have built-in functionality to solve it:

- scipy: `linalg.solve_continuous_lyapunov(A, Q)`
- MATLAB: `lyap(A,Q)`

Read more

- 3.9 Liapunov's direct method
- Universita degli studi di Padova Dipartimento di Ingegneria dell'Informazione, Nicoletta Bof, Ruggero Carli, Luca Schenato, Technical Report, Lyapunov Theory for Discrete Time Systems

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2022

Check Moodle for additional links, videos, textbook suggestions.