Introduction, ODE and State Space

Control Theory, Lecture 1

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What is control?

The first obvious question is, what is control theory? The easiest strategy to answer this question is to bring examples of systems that you can *learn how to control*:

Figure 1: Drones

Figure 2: Industrial robot arms

But beware, this is not the whole answer!

Why Control?

The second most natural question to ask is - why do we need to study Control Theory?

The easy answer is:

Control is one of the fundamental aspects of Robotics, together with Mechanical and Electrical Engineering, Sensing, Software development, etc.

A lot of typical practical issues occurring in robotics (such as lag and delays, lack of sensory data, external disturbances, robot having dynamics slightly different from the expected, etc.) are addressed in control theory; you need to understand the topic top be able to use solutions that control theory gives us.

Enough for the motivation

Now that we know (kinda) why we do it:

Let's start with the content of the course!

Ordinary differential equations

1st order

Let us remember the normal form of first-order *ordinary differential equations (ODEs):*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where $\mathbf{x} = \mathbf{x}(t)$ is the solution of the equation and t is a free variable (usually - time).

Definition

We can call this equation (same as any other ODEs) a *dynamical system*, and $\mathbf{x}$ is called the *state* of the dynamical system.

Example

$$\dot{x} = -3x^3 - 7 \tag{2}$$

State

State of a dynamical system is a minimal set of variables that describe the system, in the sense that knowing current state and all future inputs you can predict the behavior of the system.

Example

For a spring-damper system, the state variables could be position and velocity of the mass.

Example

For a double pendulum, the state variables could be joint angles and joint velocities.

Ordinary differential equations, n-th order

The normal form of an *n-th order* ordinary differential equation is:

$$x^{(n)} = f(x^{(n-1)}, x^{(n-2)}, \dots, \ddot{x}, \dot{x}, x, t) \tag{3}$$

where $x = x(t)$ is the solution of the equation. Same as before, it is a *dynamical system*, but this time we need more variables to describe the state of this system, for example we can use the set $\{x,\ \dot{x}, \dots, x^{(n-1)}\}$.

Example

$$\ddot{x} = \cos(2\dot{x}) - 10x + 7 \tag{4}$$

Example

$$\begin{cases} \dddot{x}_1 = \dot{x}_1 + x_1 + x_2^2 - 4 \\ \dddot{x}_2 = 10x_1^3 + \ddot{x}_2 \end{cases} \tag{5}$$

Linear differential equations

1st order

Linear ODEs of the first order have normal form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

Example

$$\begin{cases} \dot{x}_1 = -20x_1 + 7x_2 \\ \dot{x}_2 = 10.5x_1 - 3x_2 \end{cases} \tag{7}$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -8 & 5 & 2 \\ 0.5 & -10 & -2 \\ 1 & -1 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{8}$$

Linear differential equations, n-th order

A single linear ODE of the n-th order are often written in the form:

$$a_n x^{(n)} + ... + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0 \tag{9}$$

Example

$$12 \dddot{x} - 3\ddot{x} + 5.5\dot{x} + 2x = 0 \tag{10}$$

Example

$$5\ddot{x} - 2\dot{x} + 10x = 0 \tag{11}$$

Equations with an input

Sometimes it is convenient to write an ODE in the form with an *input*, for example:

$$a_2\ddot{x} + a_1\dot{x} + a_0 x = u(t) \tag{12}$$

In this equation $u(t)$ is a function of time. This form offers us many uses:

- We can use $u(t)$ to model *control input*, (e.g. voltage, motor torque) that we directly control.
- We can use $u(t)$ to model external forces acting on the system.
- We can substitute particular function instead of $u(t)$, e.g. sine wave or step function, to study how the system behaves with such an input.

Equations with an input

General form of an n-th order linear ODE with an input can be presented as follows:

$$a_n x^{(n)} + ... + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = u(t) \tag{13}$$

State-space representation of a linear system with an input is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{14}$$

Equations with an output

Equations can also have an output. The meaning of what is an output of an equation depends on the particular use-case - it is not a mathematical issue, it is a question of interpretation. For example, an output can mean:

- What we measure (height of a quadrotor, angular velocity of motor's rotor, etc.).
- We care about and/or what we can to control (position and orientation of a quadrotor, velocity of a car, etc.)
- etc.

We often denote output as y, and it depends on the state of the system: $y = g(\mathbf{x})$

Equations with an output

State-space representation of a linear system with an input and an output is:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{15}$$

If $\mathbf{u} \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}$ (i.e. if they are scalars) and you want to represent the system with an output as a single ODE, it is typical to treat the output as ODE the variable:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{16}$$

Linear differential equations

In this course we will focus entirely on linear dynamical systems. In particular, we will take a good use of the following two forms:

$$a_n y^{(n)} + ... + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{17}$$

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \tag{18}$$

the last one is called *state-space representation*. In general both $\mathbf{u}$ and $\mathbf{y}$ can be vectors.

Good news:

Both of those can be used to express any linear system, hence we can change one into the other.

n-th order ODE to State-Space

Consider eq. $\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = u$.

Make a substitution: $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$. Therefore:

$$\begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = x_3 \\ \dot{x}_3 = u - a_2\ddot{y} - a_1\dot{y} - a_0 y = u - a_2 x_3 - a_1 x_2 - a_0 x_1 \end{cases} \tag{19}$$

Which can be directly put in the state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \tag{20}$$

State Space to ODE

An example of how linear algebra serves
to solve a seemingly difficult problem

(advanced, not going to be on the test)

State Space to ODE

Consider a system in state-space form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases} \iff \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \\ y = \mathbf{C}\mathbf{x} \end{cases} \tag{21}$$

We want to rewrite it as a linear ODE:

$$\ddot{y} + b_2\dot{y} + b_1 y = 0 \tag{22}$$

Note that initial conditions of both equation need to agree.

State Space to ODE

Since $y = \mathbf{Cx}$, its derivative is $\dot{y} = \mathbf{C}\dot{\mathbf{x}}$:

$$\dot{y} = \mathbf{CAx} \tag{23}$$

$$\dot{y} = \begin{bmatrix} (a_{11}c_1 + a_{21}c_2) & (a_{12}c_1 + a_{22}c_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{24}$$

Analogous for $\ddot{y}$:

$$\ddot{y} = \mathbf{CAAx} \tag{25}$$

State Space to ODE

Combining our results we find the linear transformation between the variables x_1, x_2 and y, $\dot{y}$:

$$\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ (a_{11}c_1 + a_{21}c_2) & (a_{12}c_1 + a_{22}c_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{26}$$

Resulting transformation matrix is:

$$\mathbf{T} = \begin{bmatrix} c_1 & c_2 \\ (a_{11}c_1 + a_{21}c_2) & (a_{12}c_1 + a_{22}c_2) \end{bmatrix} \tag{27}$$

$$\mathbf{x} = \mathbf{T}^{-1} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \tag{28}$$

State Space to ODE

Remember that:

$$\ddot{y} = \mathbf{CAAx} \tag{29}$$

$$\ddot{y} = \mathbf{CAAT}^{-1} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \tag{30}$$

So, we obtained $\ddot{y}$ as a linear function of y, $\dot{y}$. From this it is clear how the same can be generalized to higher dimensions.

State Space to ODE

Check out the code implementation.

Read more

- 2.14 Analysis and Design of Feedback Control Systems:
  - State-Space Representation of LTI Systems
  - Time-Domain Solution of LTI State Equations
- Linear Physical Systems Analysis:
  - State Space Representations of Linear Physical Systems
    lpsa.swarthmore.edu/Representations/SysRepSS.html
  - Transformation: Differential Equation to State Space
    lpsa.swarthmore.edu/.../DE2SS.html

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2023