

Duality, Sensitivity, KKT

Convex Optimization, Lecture 13

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- Lagrange dual function
- Duality gap, strong and weak duality
- KKT conditions
- Sensitivity

Consider an optimization problem:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & f_0(\mathbf{x}), \\ \text{subject to} & \begin{cases} f_i(\mathbf{x}) \leq 0, \\ h_j(\mathbf{x}) = 0. \end{cases}\end{array}\quad (1)$$

It's *Lagrangian* is given as:

$$L(\mathbf{x}, \lambda_i, \nu_j) = f_0(\mathbf{x}) + \sum_i \lambda_i f_i(\mathbf{x}) + \sum_j \nu_j h_j(\mathbf{x}) \quad (2)$$

where $\lambda_i \geq 0$ and ν_j are Lagrange multipliers; they are sometimes called *dual variables*.

Given *Lagrangian* $L(\mathbf{x}, \lambda_i, \nu_j) = f_0(\mathbf{x}) + \sum_i \lambda_i f_i(\mathbf{x}) + \sum_j \nu_j h_j(\mathbf{x})$,
the associated *Lagrange dual function* is given as:

$$g(\lambda_i, \nu_j) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda_i, \nu_j). \quad (3)$$

Lagrange dual function is **always concave**. If p^* is the optimal value of the cost function of the original problem, then $g(\lambda_i, \nu_j)$ gives as a *lower bound* on its possible values.

In fact, substituting any ν_j and $\lambda_i > 0$ gives us a valid lower bound on the cost. Maximum of $g(\lambda_i, \nu_j)$ over the domain given by $\lambda_i > 0$ provides us optimal (largest) lower bound of the problem, denoted as g^* .

DUALITY GAP, STRONG AND WEAK DUALITY

If p^* is the optimal value of the cost function of the original problem and g^* is the optimal lower bound of the problem, then $p^* - g^*$ is called optimal *duality gap*.

If optimal duality gap is zero, the problem is said to have *strong duality*. If optimal duality gap greater than zero, the problem is said to have *weak duality*.

LAGRANGE DUAL FUNCTION FOR A QP, 1

Consider the following QP:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} && \mathbf{x}^\top \mathbf{H} \mathbf{x}, \\ & \text{subject to} && \mathbf{A} \mathbf{x} \leq \mathbf{b}. \end{aligned} \tag{4}$$

Its Lagrangian is:

$$L(\mathbf{x}, \lambda) = \mathbf{x}^\top \mathbf{H} \mathbf{x} + \lambda^\top (\mathbf{A} \mathbf{x} - \mathbf{b}) \tag{5}$$

In order to minimize the Lagrangian with respect to \mathbf{x} we find the gradient and set it to zero:

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = 2\mathbf{x}^\top \mathbf{H} + \lambda^\top \mathbf{A} = 0 \tag{6}$$

With that we can compute \mathbf{x} as a function of λ :

$$\mathbf{x} = -0.5\mathbf{H}^{-1}\mathbf{A}^\top \lambda \tag{7}$$

Knowing that $\mathbf{x} = -0.5\mathbf{H}^{-1}\mathbf{A}^\top\lambda$ we can compute $g(\lambda)$ by substituting the \mathbf{x} we found into the Lagrangian:

$$g(\lambda) = \frac{1}{4}\lambda^\top \mathbf{A}\mathbf{H}^{-1}\mathbf{H}\mathbf{H}^{-1}\mathbf{A}^\top\lambda - \frac{1}{2}\lambda^\top \mathbf{A}\mathbf{H}^{-1}\mathbf{A}^\top\lambda - \lambda^\top \mathbf{b} \quad (8)$$

$$g(\lambda) = -\frac{1}{4}\lambda^\top \mathbf{A}\mathbf{H}^{-1}\mathbf{A}^\top\lambda - \lambda^\top \mathbf{b} \quad (9)$$

In order to find the optimal lower bound we solve the following problem:

$$\begin{aligned} &\underset{\lambda}{\text{maximize}} && -\frac{1}{4}\lambda^\top \mathbf{A}\mathbf{H}^{-1}\mathbf{A}^\top\lambda - \lambda^\top \mathbf{b}, \\ &\text{subject to} && \lambda \geq 0. \end{aligned} \quad (10)$$

Solution of this problem is the solution of the original problem.

Karush-Kuhn-Tucker (KKT) conditions certify optimality of an optimization problem:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & f_0(\mathbf{x}), \\ \text{subject to} & \begin{cases} f_i(\mathbf{x}) \leq 0, \\ h_j(\mathbf{x}) = 0. \end{cases}\end{array}\quad (11)$$

- ❶ Primal feasibility: $f_i(\mathbf{x}) \leq 0$ and $h_j(\mathbf{x}) = 0$.
- ❷ Dual feasibility: $\lambda_i \geq 0$.
- ❸ Complementarity slackness: $\lambda_i f_i(\mathbf{x}) = 0$.
- ❹ Lagrangian stationarity:

$$\frac{\partial}{\partial \mathbf{x}} \left(f_0(\mathbf{x}) + \sum_i \lambda_i f_i(\mathbf{x}) + \sum_j \nu_j h_j(\mathbf{x}) \right) = 0.$$

Optimal values of Lagrange variables λ determine local sensitivity of the system with respect to small perturbations of constraints.

Consider perturbed optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & f_0(\mathbf{x}), \\ \text{subject to} & \begin{cases} f_i(\mathbf{x}) \leq u_i, \\ h_j(\mathbf{x}) = v_j. \end{cases} \end{array} \quad (12)$$

where u_i and v_j - perturbations of the constraints. Let $p(\mathbf{u}, \mathbf{v})$ be optimal value of the cost function for given values of u_i and v_j . Then $p(0, 0)$ is the optimal value of the unperturbed problem.

Sensitivity of the optimal value of the cost function to the constraint perturbation is given as:

$$\left. \frac{\partial p(\mathbf{u}, \mathbf{v})}{\partial \mathbf{u}} \right|_{\mathbf{u}=0, \mathbf{v}=0} = -\lambda^*; \quad (13)$$

$$\left. \frac{\partial p(\mathbf{u}, \mathbf{v})}{\partial \mathbf{v}} \right|_{\mathbf{u}=0, \mathbf{v}=0} = -\nu^*. \quad (14)$$

Thus, analysing values of lagrange variables allows us to assess local sensitivity to constraint perturbation.

- Convex Optimization, Lecture 12: KKT conditions, Ryan Tibshirani.
- EE 227A: Convex Optimization and Applications, Lecture 13: Optimality Conditions for Convex Problems, Laurent El Ghaoui.
- The Karush–Kuhn–Tucker (KKT) Conditions (video). Visually Explained.

Lecture slides are available via Github, links are on Moodle:

github.com/SergeiSa/Convex-Optimization

