Second-order cone programming

Convex Optimization, Lecture 9

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2-norm, 1

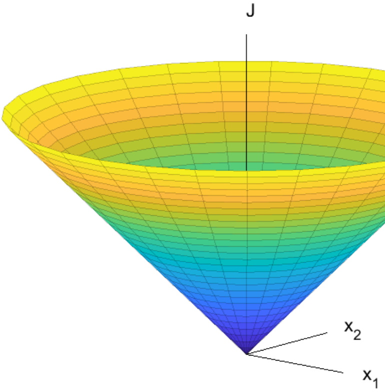
Let us consider a 2-norm as a function $f(\mathbf{x}): \mathbb{R}^n \to \mathbb{R}$:

$$f(\mathbf{x}) = ||\mathbf{x}||_2 \tag{1}$$

$$f(\mathbf{x}) = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{2}$$

We can describe 2-norm as a surface in the $\mathcal{S} \subset \mathbb{R}^{n+1}$ space:

$$\mathcal{S} = \{(J, \mathbf{x}) : \; J = ||\mathbf{x}||_2\} \tag{3}$$



Cone

The shape of the surface $\mathcal{S} = \{(J, \mathbf{x}) : \; J = ||\mathbf{x}||_2\}$ is a *cone*. We observe the following properties of a cone:

- There is a single tip point τ and a normal direction.
- Slicing cone with planes orthogonal to the normal direction, we produce ellipsoids (we can call it tangent sets).
- For any point p on the cone, the half-line from the tip point τ through p lies on the cone. The angle between this line and the normal is called *vertex angle*.

A second-order cone constraint has the following form:

$$||\mathbf{A}\mathbf{x} + \mathbf{b}|| \leq \mathbf{c}^\top \mathbf{x} + d \tag{4}$$

where $\mathbf{A} \in \mathbb{R}^{n,n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

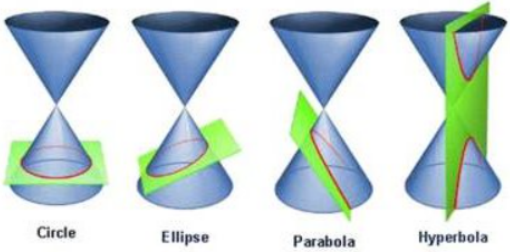
This constraint describes interior of a cone. The surface of the cone is an intersection of two surfaces:

$$J = ||\mathbf{A}\mathbf{x} + \mathbf{b}|| \tag{5}$$

$$P = \mathbf{c}^\top \mathbf{x} + d \tag{6}$$

First is a cone and second is a plane. Their intersection is called a *conic section.*

Typical conic sections are shown below:



As we can see, they represent ellipsoid and parabola. In order for them to represent a cone, the plane S needs to pass through the tip of the cone J. This can be achieved with the appropriate choice of constant d, which shifts S up or down.

The role of the free constant, 2

The surface of a second-order cone (SOC) is:

$$||\mathbf{A}\mathbf{x} + \mathbf{b}|| = \mathbf{c}^\top \mathbf{x} + d \tag{7}$$

we can find a tip point; it corresponds to both right-hand side and left-hand side becoming zero:

$$\begin{cases} \mathbf{A}\mathbf{x} + \mathbf{b} = 0 \\ \mathbf{c}^\top \mathbf{x} + d = 0 \end{cases} \tag{8}$$

Given full rank matrix $\mathbf{A}$, the solution is $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$. The system would hold if:

$$-\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} + d = 0 \tag{9}$$

The general form of a Second-order cone program (SOCP) is:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{f}^\top \mathbf{x}, \\ &\text{subject to} \quad \begin{cases} ||\mathbf{A}_i \mathbf{x} + \mathbf{b}_i||_2 \leq \mathbf{c}_i^\top \mathbf{x} + d_i, \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{10}$$

LP, QP and QCQP are subsets of SOCP.

We can write problem where our domain is a ball as SOCP:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & ||\mathbf{x}||_2 \leq d_i \end{aligned} \tag{11}$$

Same for ellipsoidal constraints:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} \quad & ||\mathbf{A}_i\mathbf{x}||_2 \leq d_i \end{aligned} \tag{12}$$

Consider a path planning problem: find a sequence of points $\mathbf{x}_i$ starting at $\mathbf{x}_0$, making a shortest path to the goal point $\mathbf{x}_g$, such that neightbouring points are no further than h from one another.

Solution. The problem becomes:

$$\begin{aligned} \underset{\mathbf{x}_1,\ldots,\mathbf{x}_n}{\text{minimize}} \quad & \sum_{i=1}^{n} (\mathbf{x}_i - \mathbf{x}_g)^\top (\mathbf{x}_i - \mathbf{x}_g), \\ \text{subject to} \quad & ||\mathbf{x}_i - \mathbf{x}_{i+1}|| < h, \quad i \in 0, 1, 2, ..., n \end{aligned} \tag{13}$$

Set $\mathbf{c}_i = 0$ and recognize that $||\mathbf{A}_i\mathbf{x} + \mathbf{b}_i||_2 \leq d_i$ is the same as $(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i)^\top(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i) \leq d_i^2$ (since the first implies that d_i is non-negative).

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{f}^\top\mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{x}^\top\mathbf{A}_i^\top\mathbf{A}_i\mathbf{x} + 2\mathbf{b}_i^\top\mathbf{A}_i\mathbf{x} + \mathbf{b}_i^\top\mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{14}$$

Now to make the cost quadratic:

$$\begin{aligned} & \underset{\mathbf{x},t}{\text{minimize}} \quad t, \\ & \text{subject to} \quad \begin{cases} \mathbf{x}^\top \mathbf{A}_0^\top \mathbf{A}_0 \mathbf{x} + 2\mathbf{b}_0^\top \mathbf{A}_0 \mathbf{x} + \mathbf{b}_0^\top \mathbf{b}_0 \leq t \\ \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{15}$$

Which is the same as:

$$\begin{aligned} & \underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{x}^\top \mathbf{H} \mathbf{x} + \mathbf{f}^\top \mathbf{x}, \\ & \text{subject to} \quad \begin{cases} \mathbf{x}^\top \mathbf{A}_i^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{A}_i \mathbf{x} + \mathbf{b}_i^\top \mathbf{b}_i \leq d_i^2 \\ \mathbf{F}\mathbf{x} = \mathbf{g}. \end{cases} \end{aligned} \tag{16}$$

As long as $\mathbf{A}_0 = \sqrt{\mathbf{H}}$, and $\mathbf{b}_0 = 0.5\mathbf{A}_0^{-1}\mathbf{f}$.

Friction cone as an SOC

Friction in 3D, 1

Friction force $\mathbf{f}_\tau$ together with normal reaction force $\mathbf{f}_n$ together form contact reaction force $\mathbf{f}_R$:

$$\mathbf{f}_R = \mathbf{f}_n + \mathbf{f}_\tau \tag{17}$$

We can choose to represent the reaction force in a basis $\mathbf{B}$ formed by concatenating normal direction $\mathbf{n}$ and two tangent directions $\mathbf{t}_1$, $\mathbf{t}_2$.

$$\mathbf{f}_R = \mathbf{B} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} \tag{18}$$

We can prove that $f_n = \mathbf{n}^\top \mathbf{f}_R$:

$$\mathbf{n}^\top \mathbf{f}_R = \mathbf{n}^\top \begin{bmatrix} \mathbf{n} & \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = f_n \qquad (19)$$

We can prove that $\begin{bmatrix} f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}^\top \mathbf{f}_R$:

$$\begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}^\top \mathbf{f}_R = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}^\top \begin{bmatrix} \mathbf{n} & \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \qquad (20)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_n \\ f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} = \begin{bmatrix} f_{\tau,1} \\ f_{\tau,2} \end{bmatrix} \qquad (21)$$

We can write friction cone constraint as follows:

$$\sqrt{f_{\tau,1}^2 + f_{\tau,2}^2} \leq \mu f_n \tag{22}$$

where μ is friction coefficient, f_τ is the magnitude of the friction force and f_n is the magnitude of the normal reaction force.

We can describe it as *element-wise description*. The simplicity of this description makes it quite attractive.

Friction cone representations, 2

It is possible to re-write the same constraint as:

$$\left|\left|\begin{bmatrix}\mathbf{t}_1 & \mathbf{t}_2\end{bmatrix}^\top \mathbf{f}_R\right|\right| \leq \mu \mathbf{n}^\top \mathbf{f}_R \tag{23}$$

We can call it a *vector description*. The advantage of this description is the use of a single vector variable $\mathbf{f}_R$. It takes the form of a second-order cone (SOC) constraint.

Note that $\mathbf{t}_1$ and $\mathbf{t}_2$ are usually not given, and can be chosen arbitrarily, up to rotation. We can find them as a left null space of the normal vector: $\mathbf{T} = \begin{bmatrix}\mathbf{t}_1 & \mathbf{t}_2\end{bmatrix} = \text{null}(\mathbf{n}^\top)$:

$$||\mathbf{T}^\top \mathbf{f}_R|| \leq \mu \mathbf{n}^\top \mathbf{f}_R \tag{24}$$

We can do the same with projectors:

$$||(\mathbf{I} - \mathbf{n}\mathbf{n}^\top)\mathbf{f}_R|| \leq \mu \mathbf{n}^\top \mathbf{f}_R \tag{25}$$

Homework

Plot a cone from a given direction and a given vertex angle.

Lecture slides are available via Github, links are on Moodle:

github.com/SergeiSa/Convex-Optimization

Appendix A - canonical form

Consider the following SOC constraint:

$$||\mathbf{A}\mathbf{x} + \mathbf{b}||_2 \leq \mathbf{c}^\top \mathbf{x} + d \tag{26}$$

Let us consider a special case when $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{(n-1)\times n}$ and $\text{rank}\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix}\right) = n$. Then we can introduce the following substitution:

$$\xi = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} \mathbf{E} \\ \mathbf{e}^\top \end{bmatrix} \tag{27}$$

where $\mathbf{I} \in \mathbb{R}^{n,n}$ is an identity matrix. Then constraint (26) becomes:

$$||\mathbf{E}\xi||_2 \leq \mathbf{e}^\top \xi \tag{28}$$

Notice that $||\mathbf{E}\xi||_2 \leq \mathbf{e}^\top \xi$ is equivalent to:

$$\sum_{i=1}^{n-1} \xi_i^2 \leq \xi_n^2 \tag{29}$$

which is a standard form of a cone. A map back from ξ to $\mathbf{x}$ is given as:

$$\mathbf{x} = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^\top \end{bmatrix}^{-1} \left(\xi - \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix} \right) \tag{30}$$

Appendix B - plotting cones

Plotting level sets, 1

To plot a cone it is convenient to first use change of coordinates $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$, meaning $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b})$, giving us SOC:

$$||\mathbf{y}|| = \mathbf{c}^\top \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b}) + d \tag{31}$$

Note that $d - \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{b} = 0$ for a cone with a tip; so SOC becomes:

$$||\mathbf{y}|| = \mathbf{c}^\top \mathbf{A}^{-1}\mathbf{y} \tag{32}$$

To plot level sets of this cone we choose height of the level set h and pick point $\mathbf{y}_h = h\frac{\mathbf{A}^{-T}\mathbf{c}}{\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{A}^{-T}\mathbf{c}}$; we note that $\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{y}_h = h$. Then we consider points on the plane $\mathcal{P}$ orthogonal to $\mathbf{c}^\top \mathbf{A}^{-1}$ and passing through $\mathbf{y}_h$:

$$\mathcal{P} = \mathbf{y}_h + \mathbf{T}\mathbf{z} : \ \forall \mathbf{z} \tag{33}$$

where $\mathbf{T} = \text{null}(\mathbf{c}^\top \mathbf{A}^{-1})$, so $\mathbf{c}^\top \mathbf{A}^{-1}\mathbf{T} = 0$.

Since SOC becomes:

$$||\mathbf{y}_h + \mathbf{T}\mathbf{z}|| = h \tag{34}$$

Since $\mathbf{y}_h$ and $\mathbf{T}\mathbf{z}$ are orthogonal, it is equivalent to:

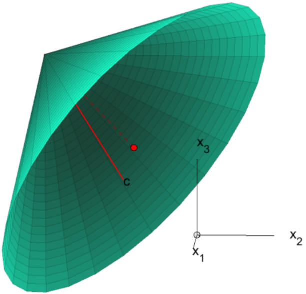
$$||\mathbf{T}\mathbf{z}|| = g \tag{35}$$

where $g = \sqrt{h^2 - \mathbf{y}_h^\top \mathbf{y}_h}$. In the 3D case, this is a circle with radius g. We can find N consecutive evenly spaced points of this circle, resulting in the next sequence of $\mathbf{y}_l$:

$$\mathbf{y}_l = \mathbf{y}_h + \mathbf{T}\begin{bmatrix} g\cos(\varphi) \\ -g\sin(\varphi) \end{bmatrix}, \quad \varphi = 0,\ \frac{2\pi}{N},\ 2\frac{2\pi}{N},\ ...,\ 2\pi \tag{36}$$

$$\mathbf{x}_l = \mathbf{A}^{-1}(\mathbf{y}_l - \mathbf{b}) \tag{37}$$

The center of the ellipsoid representing this level set lies at the point $\mathbf{x} = \mathbf{A}^{-1}(\mathbf{y}_h - \mathbf{b})$.



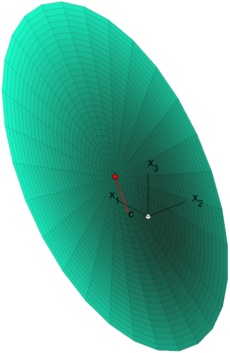


Figure 1: Cone. Dashed line - centers of level-sets.