LMI, Control design, Robustness

Convex Optimization, Lecture 10

by Sergei Savin

Spring 2025

Content

- LMI
- Control design
- Robustness check
- S-procedure
- Robust Control Design
- Quadratic stability
- Appendix A

Linear matrix inequalities (LMI)

A linear matrix inequality (LMI) is a semidefinite constraint placed on a matrix:

$$\mathbf{S} \succ 0 \tag{1}$$

We assume (and this is true!) that there exist *solvers* that can solve problems with such constraints.

Example

Given $\mathbf{A}$, find such $\mathbf{S} \succ 0$ that $\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} \prec 0$.

Notice that the last example is continious-time Lyapunov eq. for LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, and if such $\mathbf{S}$ exists the system is stable.

Consider a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, control $\mathbf{u} = \mathbf{K}\mathbf{x}$ and a Lyapunov function $V = \mathbf{x}^\top \mathbf{S} \mathbf{x}$, $\mathbf{S} \succ 0$.

Closed-form of the system is $\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}$, and full derivative of the Lyapunov function:

$$\dot{V} = \mathbf{x}^\top (\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x} < 0 \tag{2}$$

This can be re-written as an LMI:

$$(\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S} + \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K}) \prec 0 \tag{3}$$

This is *not linear* in decision variables ($\mathbf{S}$ and $\mathbf{K}$), and can't be solved directly using popular solvers.

Introducing new variable $\mathbf{P} = \mathbf{S}^{-1}$ and multiplying (3) by $\mathbf{P}$ on both sides (we can do it, as both $\mathbf{P}$ and $\mathbf{S}$ are full rank, and thus it is a congruence transformation which preserves definiteness, see appendix) we get:

$$\mathbf{P}(\mathbf{A} + \mathbf{B}\mathbf{K})^\top + (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{P} \prec 0 \tag{4}$$

Now we introduce one more variable $\mathbf{L} = \mathbf{K}\mathbf{P}$ and get an LMI constraint:

$$\mathbf{P}\mathbf{A}^\top + \mathbf{A}\mathbf{P} + \mathbf{L}^\top\mathbf{B}^\top + \mathbf{B}\mathbf{L} \prec 0 \tag{5}$$

Solving (5) gives us $\mathbf{P}$ and $\mathbf{L}$, from which we can compute $\mathbf{K} = \mathbf{L}\mathbf{P}^{-1}$ and $\mathbf{S} = \mathbf{P}^{-1}$, solving the original problem.

Discrete Lyapunov equation

A discrete dynamical system $\mathbf{x}_{i+1} = \mathbf{f}(\mathbf{x}_i)$ is stable if there exists a Lyapunov function $V(\mathbf{x}_i)$, such that:

- $V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) < 0$;
- $V(\mathbf{x}_i) > 0$ for all $\mathbf{x}_i \neq 0$.

Control design (discrete-time), 1

Consider a system $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i$, control $\mathbf{u}_i = \mathbf{K}\mathbf{x}_i$ and a Lyapunov function $V(\mathbf{x}_i) = \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i$, $\mathbf{S} \succ 0$.

Closed-form of the system is $\mathbf{x}_{i+1} = (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}_i$, and discrete dynamics of the Lyapunov function is:

$$V(\mathbf{x}_{i+1}) - V(\mathbf{x}_i) < 0 \tag{6}$$

$$\mathbf{x}_{i+1}^\top \mathbf{S}\mathbf{x}_{i+1} - \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i < 0 \tag{7}$$

$$\mathbf{x}_i^\top (\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}_i - \mathbf{x}_i^\top \mathbf{S}\mathbf{x}_i < 0 \tag{8}$$

$$\mathbf{x}_i^\top ((\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K}) - \mathbf{S})\mathbf{x}_i < 0 \tag{9}$$

The last equation is equivalent to the following semidefinite inequality:

$$(\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K}) - \mathbf{S} \prec 0 \tag{10}$$

Semidefinite inequality $(\mathbf{A}+\mathbf{B}\mathbf{K})^\top \mathbf{S}(\mathbf{A}+\mathbf{B}\mathbf{K}) - \mathbf{S} \prec 0$ can be re-written:

$$-\mathbf{S} - (\mathbf{A}+\mathbf{B}\mathbf{K})^\top(-\mathbf{S})(\mathbf{A}+\mathbf{B}\mathbf{K}) \prec 0 \tag{11}$$

Define $\mathbf{P}^{-1} = \mathbf{S}$. Note that $\mathbf{P} \succ 0$. We can multiply (11) by $\mathbf{P}$ on both sides (congruence transformation preserves definiteness, see Appendix):

$$-\mathbf{P} - \mathbf{P}(\mathbf{A}+\mathbf{B}\mathbf{K})^\top(-\mathbf{P}^{-1})(\mathbf{A}+\mathbf{B}\mathbf{K})\mathbf{P} \prec 0 \tag{12}$$

$$-\mathbf{P} - (\mathbf{A}\mathbf{P}+\mathbf{B}\mathbf{K}\mathbf{P})^\top(-\mathbf{P}^{-1})(\mathbf{A}\mathbf{P}+\mathbf{B}\mathbf{K}\mathbf{P}) \prec 0 \tag{13}$$

We define $\mathbf{L} = \mathbf{K}\mathbf{P}$:

$$-\mathbf{P} - (\mathbf{A}\mathbf{P}+\mathbf{B}\mathbf{L})^\top(-\mathbf{P}^{-1})(\mathbf{A}\mathbf{P}+\mathbf{B}\mathbf{L}) \prec 0 \tag{14}$$

Theorem (Schur complement)

Given matrix $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{bmatrix}$, *where* $\mathbf{C} \prec 0$, *the following statements are equivalent:*

1. $\mathbf{A} - \mathbf{B}^\top \mathbf{C}^{-1} \mathbf{B} \prec 0$
2. $\mathbf{M} \prec 0$

Semidefinite inequality
$-\mathbf{P} - (\mathbf{AP} + \mathbf{BL})^\top(-\mathbf{P}^{-1})(\mathbf{AP} + \mathbf{BL}) \prec 0$ can transformed with Schur inequality:

$$\begin{bmatrix} -\mathbf{P} & \mathbf{AP} + \mathbf{BL} \\ (\mathbf{AP} + \mathbf{BL})^\top & -\mathbf{P} \end{bmatrix} \prec 0 \tag{15}$$

This is a linear matrix inequality.

Robustness, 1

Consider a system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, but when you don't know $\mathbf{A}$ exactly. In other words, you don't know the model exactly. This is not to say that we know nothing about the model, but there is an uncertainty in our knowledge.

One way to model this model with *uncertainty* is given below:

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{F}\Delta\mathbf{E})\mathbf{x} \tag{16}$$

where $\mathbf{F}$ and $\mathbf{E}$ are arbitrary matrices, and Δ is a *norm-bounded* matrix: $||\Delta|| \leq 1$.

We can think of it this way: $\mathbf{A} + \mathbf{F}\Delta\mathbf{E}$ is the true but unknown model, and the range of all possible models we can expect is bounded by the possible values of Δ.

Lets write the Lyapunov equation for the system (16):

$$\dot{V} = \mathbf{x}^\top (\mathbf{A} + \mathbf{F}\Delta\mathbf{E})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}(\mathbf{A} + \mathbf{F}\Delta\mathbf{E})\mathbf{x} \leq 0 \tag{17}$$

Let us introduce a new variable $\mathbf{w} = \Delta\mathbf{E}\mathbf{x}$:

$$\dot{V} = \mathbf{x}^\top (\mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A})\mathbf{x} + \mathbf{w}^\top \mathbf{F}^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}\mathbf{F}\mathbf{w} \leq 0 \tag{18}$$

Let us consider $\mathbf{w}^\top \mathbf{w}$:

$$\mathbf{w}^\top \mathbf{w} = \mathbf{x}^\top \mathbf{E}^\top \Delta\Delta\mathbf{E}\mathbf{x} \leq \mathbf{x}^\top \mathbf{E}^\top \mathbf{E}\mathbf{x} \tag{19}$$

which is true because $||\Delta|| \leq 1$. In fact, the only property of the norm that we need here is that the delta inequality (19) holds.

With $\mathbf{w}^\top \mathbf{w} \leq \mathbf{x}^\top \mathbf{E}^\top \mathbf{E} \mathbf{x}$ we can write:

$$\mathbf{x}^\top \mathbf{E}^\top \mathbf{E} \mathbf{x} - \mathbf{w}^\top \mathbf{w} \geq 0 \tag{20}$$

Which is the same as:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} \mathbf{E}^\top \mathbf{E} & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \geq 0 \tag{21}$$

The same way we can rewrite (18):

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} \mathbf{A}^\top \mathbf{S} + \mathbf{S} \mathbf{A} & \mathbf{S} \mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \leq 0 \tag{22}$$

which only need to hold while (21) holds.

There is a way to enforce constraint $\mathbf{z}^\top \mathbf{M} \mathbf{z} \leq 0$ for such $\mathbf{z}$ that $\mathbf{z}^\top \mathbf{N} \mathbf{z} \geq 0$. This is called *s-procedure*.

Theorem (S-procedure)

If $\gamma > 0$ *and* $\mathbf{M} + \gamma \mathbf{N} \prec 0$ *then* $\mathbf{z}^\top \mathbf{N} \mathbf{z} \geq 0 \implies \mathbf{z}^\top \mathbf{M} \mathbf{z} \leq 0$

Using s-procedure we enforce (22) when (21) holds:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} \mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} + \gamma \mathbf{E}^\top \mathbf{E} & \mathbf{S}\mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & -\gamma \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \leq 0 \tag{23}$$

In LMI form this is:

$$\begin{bmatrix} \mathbf{A}^\top \mathbf{S} + \mathbf{S}\mathbf{A} + \gamma \mathbf{E}^\top \mathbf{E} & \mathbf{S}\mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & -\gamma \mathbf{I} \end{bmatrix} \prec 0 \tag{24}$$

This is a condition that the system is stable for all values of Δ. The decision variables are $\mathbf{S}$ and γ.

Consider a system:

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{F}\Delta\mathbf{E})\mathbf{x} + \mathbf{B}\mathbf{u} \tag{25}$$

Let us design control for this system that guarantees stability.

We assume control to be linear $\mathbf{u} = \mathbf{K}\mathbf{x}$. The system takes the form:

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{F}\Delta\mathbf{E})\mathbf{x} \tag{26}$$

Lyapunov eq. then becomes:

$$\mathbf{x}^\top(\mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{F}\Delta\mathbf{E})^\top\mathbf{S}\mathbf{x} + \mathbf{x}^\top\mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{F}\Delta\mathbf{E})\mathbf{x} \leq 0 \tag{27}$$

Robust Control Design, 2

We introduce new variable $\mathbf{w} = \Delta \mathbf{E} \mathbf{x}$:

$$\mathbf{x}^\top (\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}\mathbf{x} + \mathbf{x}^\top \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{x}^\top \mathbf{S}\mathbf{F}\mathbf{w} + \mathbf{w}^\top \mathbf{F}^\top \mathbf{S}\mathbf{x} \leq 0$$

We re-write it as a matrix:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} (\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S} + \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K}) & \mathbf{S}\mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \leq 0 \qquad (28)$$

This equation needs to hold only for $\mathbf{w} = \Delta \mathbf{E} \mathbf{x}$.

Notice that $\mathbf{w}^\top \mathbf{w} \leq \mathbf{x}^\top \mathbf{E}^\top \mathbf{E} \mathbf{x}$, since $||\Delta|| \leq 1$. In matrix form it is:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}^\top \begin{bmatrix} \mathbf{E}^\top \mathbf{E} & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} \geq 0 \tag{29}$$

Using S-procedure we get:

$$\begin{bmatrix} (\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S} + \mathbf{S}(\mathbf{A} + \mathbf{B}\mathbf{K}) + \gamma \mathbf{E}^\top \mathbf{E} & \mathbf{S}\mathbf{F} \\ \mathbf{F}^\top \mathbf{S} & -\gamma \mathbf{I} \end{bmatrix} \prec 0 \tag{30}$$

where $\gamma > 0$. Dividing the resulting LMI by γ and defining $\mathbf{S}_\gamma = \frac{1}{\gamma}\mathbf{S}$ we get:

$$\begin{bmatrix} (\mathbf{A} + \mathbf{B}\mathbf{K})^\top \mathbf{S}_\gamma + \mathbf{S}_\gamma(\mathbf{A} + \mathbf{B}\mathbf{K}) + \mathbf{E}^\top \mathbf{E} & \mathbf{S}_\gamma \mathbf{F} \\ \mathbf{F}^\top \mathbf{S}_\gamma & -\mathbf{I} \end{bmatrix} \prec 0 \tag{31}$$

The expression:

$$\begin{bmatrix} (\mathbf{A}+\mathbf{B}\mathbf{K})^\top \mathbf{S}_\gamma + \mathbf{S}_\gamma(\mathbf{A}+\mathbf{B}\mathbf{K}) + \mathbf{E}^\top\mathbf{E} & \mathbf{S}_\gamma\mathbf{F} \\ \mathbf{F}^\top\mathbf{S}_\gamma & -\mathbf{I} \end{bmatrix} \prec 0$$

can be re-written as:

$$\begin{bmatrix} (\mathbf{A}+\mathbf{B}\mathbf{K})^\top \mathbf{S}_\gamma + \mathbf{S}_\gamma(\mathbf{A}+\mathbf{B}\mathbf{K}) & \mathbf{S}_\gamma\mathbf{F} \\ \mathbf{F}^\top\mathbf{S}_\gamma & -\mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathbf{E}^\top \\ 0 \end{bmatrix} (-\mathbf{I})^{-1} \begin{bmatrix} \mathbf{E} & 0 \end{bmatrix} \prec 0$$

then using Schur complement we get:

$$\begin{bmatrix} (\mathbf{A}+\mathbf{B}\mathbf{K})^\top \mathbf{S}_\gamma + \mathbf{S}_\gamma(\mathbf{A}+\mathbf{B}\mathbf{K}) & \mathbf{S}_\gamma\mathbf{F} & \mathbf{E}^\top \\ \mathbf{F}^\top\mathbf{S}_\gamma & -\mathbf{I} & 0 \\ \mathbf{E} & 0 & -\mathbf{I} \end{bmatrix} \prec 0 \tag{32}$$

We can introduce a new variable $\mathbf{P} = \mathbf{S}_\gamma^{-1}$ and multiply both sides (congruence transformation) of the previously found inequality by $\begin{bmatrix} \mathbf{P} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix}$, to get:

$$\begin{bmatrix} \mathbf{P}(\mathbf{A}+\mathbf{B}\mathbf{K})^\top + (\mathbf{A}+\mathbf{B}\mathbf{K})\mathbf{P} & \mathbf{F} & \mathbf{P}\mathbf{E}^\top \\ \mathbf{F}^\top & -\mathbf{I} & 0 \\ \mathbf{E}\mathbf{P} & 0 & -\mathbf{I} \end{bmatrix} \prec 0 \tag{33}$$

Finally, introducing the last new variable $\mathbf{L} = \mathbf{K}\mathbf{P}$ we find an LMI that we can solve:

$$\begin{bmatrix} \mathbf{P}\mathbf{A}^\top + \mathbf{A}\mathbf{P} + \mathbf{B}\mathbf{L} + \mathbf{L}^\top\mathbf{B}^\top & \mathbf{F} & \mathbf{P}\mathbf{E}^\top \\ \mathbf{F}^\top & -\mathbf{I} & 0 \\ \mathbf{E}\mathbf{P} & 0 & -\mathbf{I} \end{bmatrix} \prec 0 \tag{34}$$

Quadratic stability, 1

Let us consider the following system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{35}$$

where $\mathbf{A} = \sum_{i=1}^{n} \alpha_i \mathbf{A}_i$, $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$ with known $\mathbf{A}_i$ but unknown coefficients α_i. Is it stable for all possible values of α_i? Note that we can't use eigenvalue analysis in this case.

Geometrically, this means $\mathbf{A}$ is in a polytope with vertices $\mathbf{A}_i$.

Quadratic stability, 2

Theorem (Quadratic stability)

$\mathbf{A}_i^\top \mathbf{S} + \mathbf{S}\mathbf{A}_i \leq 0$ *implies* $\dot{\mathbf{x}} = \sum_{i=1}^{n} \alpha_i \mathbf{A}_i \mathbf{x}$ *is stable, where* $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$

Proof: $\dot{V} = \left(\sum_{i=1}^{n} \alpha_i \mathbf{A}_i\right)^\top \mathbf{S} + \mathbf{S}\left(\sum_{i=1}^{n} \alpha_i \mathbf{A}_i\right) \leq 0$ can be re-written as: $\dot{V} = \sum_{i=1}^{n} \left(\alpha_i(\mathbf{A}_i^\top \mathbf{S} + \mathbf{S}\mathbf{A}_i)\right)$ and since $\mathbf{A}_i^\top \mathbf{S} + \mathbf{S}\mathbf{A}_i \leq 0$ and $\alpha_i \geq 0$, then $\dot{V} \leq 0$. $\square$

Let us consider the following system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x} \tag{36}$$

where $\mathbf{A} = \sum_{i=1}^{n} \alpha_i \mathbf{A}_i$, $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$ with known $\mathbf{A}_i$ but unknown coefficients α_i. How to design control law $\mathbf{u} = \mathbf{K}\mathbf{x}$ making the system stable for all possible values of α_i?

The closed-loop form of the system is:

$$\dot{\mathbf{x}} = (\sum_{i=1}^{n} \alpha_i \mathbf{A}_i + \mathbf{B}\mathbf{K})\mathbf{x} \tag{37}$$

Let us write Lyapunov eq. for the system:

$$\left(\sum_{i=1}^{n} \alpha_i (\mathbf{A}_i + \mathbf{B}\mathbf{K})\right)^{\top} \mathbf{S} + \mathbf{S} \left(\sum_{i=1}^{n} \alpha_i (\mathbf{A}_i + \mathbf{B}\mathbf{K})\right) \prec 0 \qquad (38)$$

We can re-write it as:

$$\sum_{i=1}^{n} \alpha_i \left((\mathbf{A}_i + \mathbf{B}\mathbf{K})^{\top} \mathbf{S} + \mathbf{S}(\mathbf{A}_i + \mathbf{B}\mathbf{K})\right) \prec 0 \qquad (39)$$

Hence if $(\mathbf{A}_i + \mathbf{B}\mathbf{K})^{\top} \mathbf{S} + \mathbf{S}(\mathbf{A}_i + \mathbf{B}\mathbf{K}) \prec 0$, the original system is stable.

From $(\mathbf{A}_i + \mathbf{BK})^\top \mathbf{S} + \mathbf{S}(\mathbf{A}_i + \mathbf{BK}) \prec 0$, we can go on to do control design. Introducing $\mathbf{P} = \mathbf{S}^{-1}$, we use congruence transformation multiplying by $\mathbf{P}$ on both sides:

$$\mathbf{P}(\mathbf{A}_i + \mathbf{BK})^\top + (\mathbf{A}_i + \mathbf{BK})\mathbf{P} \prec 0 \tag{40}$$

Introducing new variable $\mathbf{L} = \mathbf{KP}$ we get a problem linear in decision variables:

$$\mathbf{P}\mathbf{A}_i^\top + \mathbf{A}_i\mathbf{P} + \mathbf{L}^\top\mathbf{B}^\top + \mathbf{BL} \prec 0 \tag{41}$$

where the decision variables are $\mathbf{P}$ and $\mathbf{L}$. The control gain matrix is found as $\mathbf{K} = \mathbf{LP}^{-1}$.

Consider matrices $\mathbf{P} \succ 0$, and $\mathbf{V} \in \mathbb{R}^{n,n}$ is full rank. We can prove that:

$$\mathbf{P} \succ 0 \implies \mathbf{V}^\top \mathbf{P} \mathbf{V} \succ 0 \tag{42}$$

Proof: $\mathbf{x}^\top \mathbf{V}^\top \mathbf{P} \mathbf{V} \mathbf{x} = \mathbf{z}^\top \mathbf{P} \mathbf{z}$, where $\mathbf{z} = \mathbf{V}\mathbf{x}$. Since $\mathbf{P} \succ 0$, $\mathbf{z}^\top \mathbf{P} \mathbf{z} \geq 0$, hence $\mathbf{x}^\top \mathbf{V}^\top \mathbf{P} \mathbf{V} \mathbf{x} \geq 0$.

Definition

Congruence transformation preserves semi-definiteness: $\det(\mathbf{V}) \neq 0, \ \mathbf{P} \succ 0 \implies \mathbf{V}^\top \mathbf{P} \mathbf{V} \succ 0$

Lecture slides are available via Github, links are on Moodle:

github.com/SergeiSa/Convex-Optimization