Algebra in 3D: examples

Math and modeling for high school

by Sergei Savin

Fall 2022

Given a cube $ABCDA_1B_1C_1D_1$ with a side length 9. Point $K \in BB_1$, with $||KB|| = 7$ (meaning the distance from K to B is 7). Plane α passes through K and C_1, and is parallel to BD_1. P is a point of intersection of α with A_1B_1

- Prove that $||A_1P||/||PB_1|| = 2.5$.
- Find angle between α and BB_1C_1.

Our approach here can be straight-forward: we compute everything that can be computed exactly.

Later we can discard useless steps, but it can be counter-productive to start discarding them before we know how we will arrive at the solution.

For example, assuming that $C_1 = [0,\ 0,\ 0]$ (why? so α passes through the origin), and orienting the axes such that $D_1 = [9,\ 0,\ 0]$, $B_1 = [0,\ 0,\ 9]$, we get:

$$A_1 = [9,\ 0,\ 9] \tag{1}$$
$$A = [9,\ 9,\ 9] \tag{2}$$
$$B = [0,\ 9,\ 9] \tag{3}$$
$$C = [0,\ 9,\ 0] \tag{4}$$
$$D = [9,\ 9,\ 0] \tag{5}$$

Do we need all these points? No. Is it trivial to find their coordinates? Yes. Is it easier to think about the problem when you know all their coordinates? Yes.

Since $K \in BB_1$ and $||KB|| = 7$, and $B = [0,\ 9,\ 9]$, $B_1 = [0,\ 0,\ 9]$, we can find coordinates of K:

$$K = [0,\ 2,\ 9] \tag{6}$$

There are two ways to arrive there. 1) You know that K is $7/9$ of the way between B and B_1, so:
$K = B + \frac{7}{9}(B_1 - B) = \frac{2}{9}B + \frac{7}{9}B_1$.

Or, 2) The direction from B to B_1 is given as $\mathbf{b} = [0,\ -1,\ 0]$, and we know that the distance is 7, so $K = B + 7\mathbf{b}$.

Plane α passes through K and C_1, and is parallel to BD_1. The fact that it passes through K and C_1, it means that a vector from K to C_1 is tangent to the plane. So is a vector from B to D_1. Let us find those two vectors:

$$\mathbf{r}_{BD_1} = B - D_1 = [0,\ 9,\ 9] - [9,\ 0,\ 0] = [-9,\ 9,\ 9] \tag{7}$$

$$\mathbf{r}_{KC_1} = K - C_1 = [0,\ 2,\ 9] - [0,\ 0,\ 0] = [0,\ 2,\ 9] \tag{8}$$

We can find the norm to the plane by taking a cross product of $\mathbf{r}_{BD_1}$ and $\mathbf{r}_{KC_1}$:

$$\mathbf{n} = \mathbf{r}_{BD_1} \times \mathbf{r}_{KC_1} = \begin{bmatrix} 81 - 18 \\ 0 + 81 \\ -18 - 0 \end{bmatrix} = \begin{bmatrix} 63 \\ 81 \\ -18 \end{bmatrix} \tag{9}$$

Finding point P

P is a point of intersection of α with A_1B_1. Let us find a vector from A_1 to B_1:

$$\mathbf{r}_{A_1B_1} = A_1 - B_1 = [9,\ 0,\ 9] - [0,\ 0,\ 9] = [9,\ 0,\ 0] \qquad (10)$$

Let us denote coordinates of P as $\mathbf{p}$ and coordinates of B_1 as $\mathbf{b}_1$. We know that $\mathbf{p}^\top \mathbf{n} = 0$ and $\mathbf{p} = \lambda \mathbf{r}_{A_1B_1} + \mathbf{b}_1$. Therefore:

$$(\lambda \mathbf{r}_{A_1B_1} + \mathbf{b}_1)^\top \mathbf{n} = 0 \qquad (11)$$

$$\lambda = -\frac{\mathbf{b}_1^\top \mathbf{n}}{\mathbf{r}_{A_1B_1}^\top \mathbf{n}} \qquad (12)$$

$$\mathbf{b}_1^\top \mathbf{n} = 0 + 0 + 9 \cdot 2 = -162 \qquad (13)$$

$$\mathbf{r}_{A_1B_1}^\top \mathbf{n} = -9 \cdot 11 + 0 + 0 = 567 \qquad (14)$$

$$\lambda = -\frac{-162}{567} = 2/7 \qquad (15)$$

Task 1

We found that $\lambda = 2/7$ and $\mathbf{p} = \lambda \mathbf{r}_{A_1 B_1} + \mathbf{b}_1$. Then:

$$\mathbf{p} = 2/7[9,\ 0,\ 0] + [0,\ 0,\ 9] = [18/7,\ 0,\ 9] \tag{16}$$

We can find $||A_1 P||$ and $||B_1 P||$; These vectors are $A_1 P = [45/7,\ 0,\ 0]$ and $B_1 P = [18/7,\ 0,\ 0]$. With that we know:

$$||A_1 P|| = 45/7 \tag{17}$$
$$||B_1 P|| = 18/7 \tag{18}$$
$$||A_1 P|| / ||B_1 P|| = 45\ 18 = 5/2 \tag{19}$$

Find angle between α and BB_1C_1. This is the same as angle between the normals to these planes. First we find the norm to the plane BB_1C_1:

$$B - B_1 = [0,\ 9,\ 0] \tag{20}$$

$$B_1 - C_1 = [0,\ 0,\ 9] \tag{21}$$

$$\mathbf{m} = (B - B_1) \times (B_1 - C_1) = [81,\ 0,\ 0] \tag{22}$$

With that we know normals to both α and BB_1C_1, and can find the angle between them using dot product.

Task 2

We know that:

$$\mathbf{m} \cdot \mathbf{n} = ||\mathbf{n}|| \; ||\mathbf{m}|| \cos(\varphi), \tag{23}$$

where φ is the angle we seek. We need to find $\mathbf{m} \cdot \mathbf{n}$, as well as $||\mathbf{m}||$ and $||\mathbf{n}||$:

$$\mathbf{m} \cdot \mathbf{n} = 63 \cdot 81 + 0 + 0 = 5103 \tag{24}$$

$$||\mathbf{n}|| = \sqrt{10854} \tag{25}$$

$$||\mathbf{m}|| = 81 \tag{26}$$

$$\cos(\varphi) = \frac{\mathbf{m} \cdot \mathbf{n}}{||\mathbf{n}|| \; ||\mathbf{m}||} = 5103/(81\sqrt{10854}) \approx 0.6047 \tag{27}$$

$$\varphi = 52.79^\circ \tag{28}$$

The code in Matlab for this problem is:

```
dx = [9;0;0]; dy = [0;9;0]; dz = [0;0;9];
C1 = sym([0;0;0]);
D1 = C1 + dx; A1 = C1 + dx + dz; B1 = C1 + dz;
B = C1 + dy + dz;

K = (2/9) * B + (7/9) * B1;
BD1 = B - D1;
KC1 = K - C1;
n = cross(BD1, KC1);
lambda = -dot(B1, n) / dot(dx, n);
P = lambda*dx + B1;
A1P = A1 - P;
B1P = B1 - P;
disp( norm(A1P) / norm(B1P) ) %task 1

m = cross(dy, dz);
cos_phi = dot(n, m) / (norm(n) * norm(m));
%task 2:
phi = round(double(acos(cos_phi))*180/pi, 3)
```

Homework

Consider a cube $ABCDA_1B_1C_1D_1$ with a side length 3. Point $S_1 \in DB_1$, with $||DS_1|| = 1$ (meaning the distance from D to S_1 is 1). Point $S_2 \in AB_1$, with $||AS_2|| = 2$ (meaning the distance from A to S_2 is 2). Plane α_1 passes through S_1, S_2 and D. Plane α_2 is orthogonal to C_1S_1 through C. Point P is an intersection between the plane α_1 and the line passing through points D_1, B.

- Find distance between P and A.
- Find angle between α_1 and α_2.
- Find distance between α_1 and all vertices of the cube.
- Prove that P lies on B_1S_1.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Extra-math-for-high-school

Check Moodle for additional links, videos, textbook suggestions.