Physical systems

Math and modeling for high school

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Free fall

A unit mass is flying up with speed to v, such that $v(0) = 9.81$. Assuming that gravitational acceleration is $g = 9.81$, and initial position of the unit mass is $s(0) = 0$, when will it hit the ground?

Remembering Newton's law we get:

$$ma = -mg$$

We know *acceleration is the derivative of velocity* and *velocity is the derivative of position*:

$$\dot{v} = a \tag{1}$$

$$\dot{s} = v \tag{2}$$

Free fall

For our problem we have:

$$\dot{v} = -g \tag{3}$$

$$\dot{s} = v \tag{4}$$

From that, we can find that $v(t) = -gt + C_1$, and since $v(0) = 9.81$, we get:

$$v(0) = 0 + C_1 = 9.81 \tag{5}$$

Finally we get: $v(t) = -9.81t + 9.81$. From that we get:

$$\dot{s} = -9.81t + 9.81 \tag{6}$$

$$s(t) = -\frac{1}{2}9.81t^2 + 9.81t + C_2 \tag{7}$$

From $s(0) = 0$ we get $C_2 = 0$, so:

$$s(t) = -\frac{1}{2}9.81t^2 + 9.81t \tag{8}$$

Free fall

Given $s(t) = -\frac{1}{2}9.81t^2 + 9.81t$, we can try to find when the unit mass will hit the ground, i.e. when $s(t) = 0$:

$$-\frac{1}{2}9.81t^2 + 9.81t = 0 \tag{9}$$

$$t_1 = 0 \tag{10}$$

$$t_2 = 2 \tag{11}$$

And that is the answer!

Let us remember that this analysis will work just as well with numerical solution for the differential equation. Meaning, we are not limited to simple tasks with simple solution here.

Sinking

Consider a unit mass sinking in viscous liquid. The viscosity force is modeled as $f_v = \mu v = 4v$. The buoyancy force is $f_b = 1$. The mass of the unit is $m = 0.5$, the gravitational acceleration $g = 10$. What is the maximum velocity that the unit mass can achieve?

We can model the system as

$$ma = f_b - mg - \mu v \tag{12}$$

As before, we know that $\dot{v} = a$, so $\dot{v} = f_b - mg - \mu v$. When the unit mass achieves its maximal velocity, it means that $\dot{v} = 0$:

$$f_b - mg - \mu v_{max} = 0 \tag{13}$$

$$v_{max} = \frac{f_b - mg}{\mu} = \frac{1 - 5}{4} = -1 \tag{14}$$

Sinking

Notice that, given $\dot{v} = f_b - mg - \mu v$ we can also answer questions such as

- When will $v > 0.9 v_{max}$?
- How far will the unit mass sink from point $t = 1$ to the point $t = 3$, given zero initial velocity?

These questions can be easily answered if you solve the equation numerically.

Flying

Consider a body with mass $m = 2$, flying horizontally with propulsive force $f = 10$ from point $t = 0$ until point $t = 2$. After $t = 2$, the force ceases. Initial conditions $v(0) = 1$, $s(0) = 0$. When will the body pass 100 meter mark?

We can model the system as:

$$\begin{cases} 2a = 10 \\ 2a = 0 \end{cases} \tag{15}$$

We know that $\dot{v} = a$, so $2\dot{v} = 10$, so:

$$2\dot{v} = 10 \tag{16}$$

$$v = 5t + C_1 \tag{17}$$

Flying

Since $v(0) = 0 + C_1 = 1$, we get:

$$v(t) = \dot{s} = 5t + 1 \tag{18}$$

$$s = 2.5t^2 + t + C_2 \tag{19}$$

Since $s(0) = 0$, so $C_2 = 0$:

$$s = 2.5t^2 + t \tag{20}$$

What is the position when $t = 2$:

$$s(2) = 10 + 2 = 12 \tag{21}$$

$$v(2) = 10 + 1 = 11 \tag{22}$$

Flying

Now we can go back and consider what happens after $t = 2$:

$$\dot{v} = 0 \tag{23}$$
$$v = C_3 \tag{24}$$
$$v(2) = C_3 = 11 \tag{25}$$

And finally, the position:

$$\dot{s} = 11 \tag{26}$$
$$s = 11t + C_4 \tag{27}$$
$$s(2) = 22 + C_4 = 12 \tag{28}$$
$$C_4 = -10 \tag{29}$$
$$s(t) = 11t - 10 \tag{30}$$

So, when will $s(t) = 11t - 10 = 100$? The answer is $t = 110/11 = 10$

Sliding down a slope

Let us consider a body with mass $m = 1$, sliding down a slope with angle $\alpha = \pi/4$ and friction coefficient $\mu = 0.5$ along first $\sqrt{2}$ meters, and 0.75 later. We use parameters: gravitational acceleration $g = 10$, $x(0) = 0$, $v(0) = 0$.

We can find acceleration as:

$$ma = mg\sin(\alpha) - \mu N \tag{31}$$

where N is normal reaction force. We can find the reaction force as $N = mg\cos(\alpha)$. So we get:

$$ma = mg\sin(\alpha) - \mu mg\cos(\alpha) \tag{32}$$

$$a = g(\sin(\alpha) - \mu\cos(\alpha)) = \begin{cases} \frac{10}{4}\sqrt{2} & \text{if} \quad s \leq \sqrt{2} \\ \frac{10}{8}\sqrt{2} & \text{if} \quad s > \sqrt{2} \end{cases} \tag{33}$$

Sliding down a slope

Considering that derivative of the velocity is acceleration, we get:

$$\dot{v} = \begin{cases} \frac{10}{4}\sqrt{2} & \text{if} \quad s \leq \sqrt{2} \\ \frac{10}{8}\sqrt{2} & \text{if} \quad s > \sqrt{2} \end{cases}$$

The first segment is $\dot{v} = \frac{10}{4}\sqrt{2}$ and $v = \frac{10}{4}\sqrt{2}t + C_1$. We know that $v(0) = 0$, so $C_1 = 0$. Thus:

$$\dot{s} = \frac{10}{4}\sqrt{2}t \tag{34}$$

$$s = \frac{10}{8}\sqrt{2}t^2 + C_2 \tag{35}$$

$$s(0) = 0 \tag{36}$$

$$C_2 = 0 \tag{37}$$

$$s(t) = \frac{10}{8}\sqrt{2}t^2 \tag{38}$$

Sliding down a slope

Given $s(t) = \frac{10}{8}\sqrt{2}t^2$, what is the time when $s = \sqrt{2}$?

$$\sqrt{2} = \frac{10}{8}\sqrt{2}t^2 \tag{39}$$

$$t_1 = \sqrt{0.8} \tag{40}$$

With that, we can consider equation $\dot{v} = \frac{10}{8}\sqrt{2}$ with initial conditions $v(\sqrt{0.8}) = \frac{10}{4}\sqrt{1.8}$ and $s(\sqrt{0.8}) = \sqrt{2}$.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Extra-math-for-high-school

Check Moodle for additional links, videos, textbook suggestions.