Vector ODEs
Math and modeling for high school

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Consider two differential equations:

$$\begin{cases} \dot{x} = -10x - 3t \\ \dot{y} = -2y + t \end{cases} \tag{1}$$

Introducing a vector variable $\mathbf{v} = [x, \ y]$ we can re-write the system as:

$$\dot{\mathbf{v}} = \begin{bmatrix} -10 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{v} + \begin{bmatrix} -3t \\ t \end{bmatrix} \tag{2}$$

Notice that it is still two independent equations. But it doesn't have to be. Here is an example of a system of equations which are not interdependent:

$$\dot{\mathbf{v}} = \begin{bmatrix} -10 & 2 \\ 0 & -2 \end{bmatrix} \mathbf{v} + \begin{bmatrix} -3t \\ t \end{bmatrix} \tag{3}$$

In general, consider equation

$$\dot{\mathbf{v}} = \mathbf{A}\mathbf{v} + \mathbf{c} \tag{4}$$

As before, we can discuss how the derivative of a variable in a ratio between the increase in its value and the time interval over which the increase has taken place:

$$\dot{\mathbf{v}} \approx \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} \tag{5}$$

So, we can make it into an *integration scheme*:

$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + (\mathbf{A}\mathbf{v}(t) + \mathbf{c})\Delta t \tag{6}$$

When do system of ODEs become important?

Consider a problem: given a satellite orbiting Earth at the velocity $\mathbf{v}$, at the height h. Find its orbit.

Sounds quite intimidating, doesn't it?

First, let us remember the Newton's law:

$$m\dot{\mathbf{v}} = \mathbf{f} \tag{7}$$

where m is the mass of the satellite, $\mathbf{v}$ is its velocity and $\mathbf{f}$ is the sum of all external forces acting on the satellite.

Second, we assume that the only external force acting on the satellite is gravitational force.

For us it will be insufficient to assume that the gravity force is given by mg directed towards the center of the Earth. Instead, we want to use a more precise model of gravitational force:

$$f = G\frac{mM}{h^2} \tag{8}$$

where f is the magnitude of the gravitational force, m is the mass of the satellite, M is the mass of the Earth and h is the distance between the center of the Earth and the satellite.

Here are some of these constants:
- $M = 5.972 \cdot 10^{24}$ kg
- $G = 6.673 \cdot 10^{-11}$ N m^2/kg^2
- radius of Earth is $6.371 \cdot 10^6$ m

Let us assume that the position of the satellite relative to the center of the Earth, is given by the vector $\mathbf{r}$. How can we use this information to compute the magnitude of the gravity force?

Our strategy is to realize that $h = ||\mathbf{r}||$, and use it in the formula:

$$f = G\frac{mM}{||\mathbf{r}||^2} \tag{9}$$

But how do we give this force a direction? It always points towards Earth, so in the direction $-\mathbf{r}$. The gravity force should be pointing in the same direction.

The way to ensure it, is to multiply the magnitude $f = G\frac{mM}{||\mathbf{r}||^2}$ by the direction unit vector $-\mathbf{r}/||\mathbf{r}||$:

$$f = -G\frac{mM}{||\mathbf{r}||^3}\mathbf{r} \tag{10}$$

Now we know the gravity force, and we can write the equations of motion for the satellite:

$$\begin{cases} \dot{\mathbf{r}} = \mathbf{v} \\ m\dot{\mathbf{v}} = -G\frac{mM}{\|\mathbf{r}\|^3}\mathbf{r} \end{cases} \tag{11}$$

Clearly we can divide the first equation by m, ridding us from this parameter - meaning the mass of the satellite does not play a role in this problem. This is true because we do not consider the effect of other forces than gravity, and we do not consider the effects the satellite has on the Earth.

We can solve this system in MATLAB using the following code:

```matlab
G = 6.673 * 10^(-11);
M = 5.972 * 10^24;
r0 = randn(3, 1) * 7*10^6;
v0 = randn(3, 1) * 1000;
tspan = linspace(0, 20^4, 10^5);
x0 = [r0; v0];
odefun = @(t, x) [x(4:6); -G*M*x(1:3) / ...
((norm(x(1:3)))^3)];
opts = odeset('Reltol',1e-13,'AbsTol',1e-14,'Stats','on');
[t, x] = ode45(odefun,tspan,x0, opts);
plot3(x(:, 1), x(:, 2), x(:, 3));
```

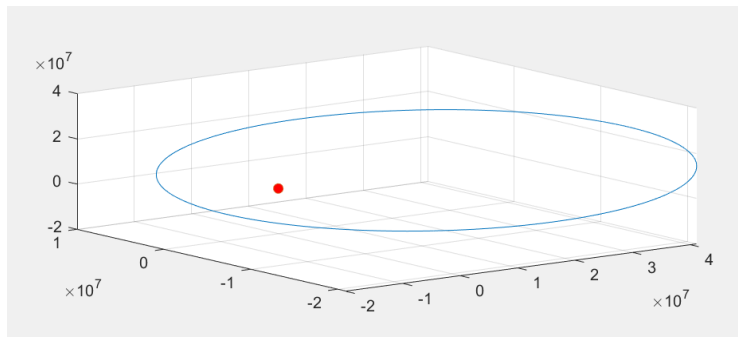We plot the solution, and the result is shown below:

Figure 1: Orbit of a satellite, found by ode45; center of the Earth is given by a red dot

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Extra-math-for-high-school

Check Moodle for additional links, videos, textbook suggestions.