

# Dynamically-consistent pseudoinverse and task prioritization

Fundamentals of Robotics, Lecture 13

by Sergei Savin

Fall 2022

A task is an equation that we wish robot trajectory to abide to. For example, let us consider a task  $\mathbf{r}_t(t, \mathbf{q})$ :

$$\mathbf{r}_t(t, \mathbf{q}) = \mathbf{r}_e(\mathbf{q}) - \mathbf{r}_e^*(t) = 0 \quad (1)$$

where  $\mathbf{r}_e(\mathbf{q})$  is the position of the end effector subject to generalized coordinates, and  $\mathbf{r}_e^*(t)$  is the desired position of the end effector.

Tasks are very similar to constraints. A task of the form  $\mathbf{r}_t(\mathbf{q}) = \mathbf{r}_t^*(t)$  can be differentiated twice:

$$\mathbf{J}\dot{\mathbf{q}} = \dot{\mathbf{r}}_t^* \quad (2)$$

$$\mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} = \ddot{\mathbf{r}}_t^* \quad (3)$$

However, there are no naturally occurring forces that implement the tasks (like reaction forces did for constraints). It is the control implemented by us that acts as a reaction force pushing the system to follow the task.

How would the robot behave if a task is obeyed? On the most basic level, it would look like this:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \boldsymbol{\tau} \\ \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}} = \ddot{\mathbf{r}}_t^* \end{cases} \quad (4)$$

Assuming that  $\boldsymbol{\tau} = \boldsymbol{\tau}^* + \mathbf{J}^\top \boldsymbol{\lambda}$ , we can rewrite it as:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} = \boldsymbol{\tau}^* + \mathbf{J}^\top \boldsymbol{\lambda} - (\mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) \\ \mathbf{J}\ddot{\mathbf{q}} = \ddot{\mathbf{r}}_t^* - \dot{\mathbf{J}}\dot{\mathbf{q}} \end{cases} \quad (5)$$

Finally, we define  $\boldsymbol{\tau}^* = \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}$  and  $\mathbf{y} = \ddot{\mathbf{r}}_t^* - \dot{\mathbf{J}}\dot{\mathbf{q}}$ , giving us:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} = \mathbf{J}^\top \boldsymbol{\lambda} \\ \mathbf{J}\ddot{\mathbf{q}} = \mathbf{y} \end{cases} \quad (6)$$

# SOLVING FOR A SINGLE TASK

The equation  $\begin{cases} \mathbf{H}\ddot{\mathbf{q}} = \mathbf{J}^\top \lambda \\ \mathbf{J}\ddot{\mathbf{q}} = \mathbf{y} \end{cases}$  can be solved:

$$\ddot{\mathbf{q}} = \mathbf{H}^{-1} \mathbf{J}^\top \lambda \quad (7)$$

$$\mathbf{J} \mathbf{H}^{-1} \mathbf{J}^\top \lambda = \mathbf{y} \quad (8)$$

$$\lambda = (\mathbf{J} \mathbf{H}^{-1} \mathbf{J}^\top)^{-1} \mathbf{y} \quad (9)$$

$$\ddot{\mathbf{q}} = \mathbf{H}^{-1} \mathbf{J}^\top (\mathbf{J} \mathbf{H}^{-1} \mathbf{J}^\top)^{-1} \mathbf{y} \quad (10)$$

$$\ddot{\mathbf{q}} = \mathbf{H}^{-1} \mathbf{J}^\top (\mathbf{J} \mathbf{H}^{-1} \mathbf{J}^\top)^{-1} (\ddot{\mathbf{r}}_t^* - \dot{\mathbf{J}}\dot{\mathbf{q}}) \quad (11)$$

So, the control law is:

$$\ddot{\mathbf{q}} = \mathbf{J}_\mathbf{H}^\# \mathbf{y} \quad (12)$$

$$\mathbf{J}_\mathbf{H}^\# = \mathbf{H}^{-1} \mathbf{J}^\top (\mathbf{J} \mathbf{H}^{-1} \mathbf{J}^\top)^{-1} \quad (13)$$

where  $\mathbf{J}_\mathbf{H}^\#$  is a *dynamically-consistent pseudoinverse*.

Transpose of the weighted inverse has form:

$$(\mathbf{J}_{\mathbf{H}}^{\#})^{\top} = (\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^{\top})^{-1}\mathbf{J}\mathbf{H}^{-1} \quad (14)$$

Consider matrix  $\mathbf{C}$  acting a little like column space projector:

$$\mathbf{C} = \mathbf{J}\mathbf{J}_{\mathbf{H}}^{\#} = \mathbf{J}\mathbf{H}^{-1}\mathbf{J}^{\top}(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^{\top})^{-1} = \mathbf{I} \quad (15)$$

Consider matrix  $\mathbf{N}$  acting a little like null space projector:

$$\mathbf{N} = \mathbf{I} - \mathbf{J}_H^\# \mathbf{J} = \mathbf{I} - \mathbf{H}^{-1} \mathbf{J}^\top (\mathbf{J} \mathbf{H}^{-1} \mathbf{J}^\top)^{-1} \mathbf{J} \quad (16)$$

Let us prove that  $\mathbf{N} \mathbf{J}_H^\# = 0$ .

$$e = (\mathbf{I} - \mathbf{J}_H^\# \mathbf{J}) \mathbf{J}_H^\# = \mathbf{J}_H^\# - \mathbf{J}_H^\# \mathbf{J} \mathbf{J}_H^\# \quad (17)$$

Since  $\mathbf{J} \mathbf{J}_H^\# = \mathbf{I}$ , we get  $e = \mathbf{J}_H^\# - \mathbf{J}_H^\# = 0$ .

Let us prove that  $\mathbf{J} \mathbf{N} = 0$ .

$$e = \mathbf{J} \mathbf{N} = \mathbf{J} (\mathbf{I} - \mathbf{J}_H^\# \mathbf{J}) = \quad (18)$$

$$= \mathbf{J} - \mathbf{J} \mathbf{J}_H^\# \mathbf{J} = \mathbf{J} - \mathbf{J} = 0 \quad (19)$$

## TWO TASKS

Consider a situation when we have two tasks. Assuming we can accommodate both, it would be great to solve for them sequentially.

We remember that the solution to the following system is:

$$\begin{cases} \mathbf{H}\mathbf{x} = \mathbf{J}^\top \lambda \\ \mathbf{J}\mathbf{x} = \mathbf{y} \end{cases} \quad (20)$$

is  $\mathbf{x} = \mathbf{J}_\mathbf{H}^\# \mathbf{y}$ . Now we fix  $\mathbf{x}$  and  $\lambda = (\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1}\mathbf{y}$  and introduce a new system:

$$\begin{cases} \mathbf{H}(\mathbf{x} + \chi) = \mathbf{J}^\top \lambda + (\mathbf{G}\mathbf{N})^\top \beta \\ \mathbf{G}\mathbf{N}(\mathbf{x} + \chi) = \mathbf{z} \\ \mathbf{J}(\mathbf{x} + \chi) = \mathbf{y} \end{cases} \quad (21)$$

where  $\mathbf{N} = \mathbf{I} - \mathbf{J}_\mathbf{H}^\# \mathbf{J} = \mathbf{I} - \mathbf{H}^{-1}\mathbf{J}^\top(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^\top)^{-1}\mathbf{J}$ . Let us find  $\chi$ .



Let us first study  $\mathbf{GNx}$ :

$$e = \mathbf{GNx} = \mathbf{GNJ}_H^{\#} \mathbf{y} = 0 \quad (22)$$

We found that  $\mathbf{GNx} = 0$ . Hence we can simplify:

$$\begin{cases} \mathbf{H}(\mathbf{x} + \chi) = \mathbf{J}^{\top} \lambda + (\mathbf{GN})^{\top} \beta \\ \mathbf{GN}\chi = \mathbf{z} \\ \mathbf{J}(\mathbf{x} + \chi) = \mathbf{y} \end{cases} \quad (23)$$

Additionally, we observe that  $\mathbf{Hx} = \mathbf{J}^{\top} \lambda$  so we simplify:

$$\begin{cases} \mathbf{H}\chi = (\mathbf{GN})^{\top} \beta \\ \mathbf{GN}\chi = \mathbf{z} \\ \mathbf{J}\chi = 0 \end{cases} \quad (24)$$

# TWO TASKS

So, we have:

$$\begin{cases} \mathbf{H}\chi = (\mathbf{GN})^\top \beta \\ \mathbf{GN}\chi = \mathbf{z} \\ \mathbf{J}\chi = 0 \end{cases} \quad (25)$$

We solve for  $\chi$ :

$$\chi = \mathbf{H}^{-1}(\mathbf{GN})^\top \beta \quad (26)$$

$$\mathbf{GNH}^{-1}(\mathbf{GN})^\top \beta = \mathbf{z} \quad (27)$$

$$\beta = (\mathbf{GNH}^{-1}(\mathbf{GN})^\top)^{-1} \mathbf{z} \quad (28)$$

$$\chi = \mathbf{H}^{-1}(\mathbf{GN})^\top (\mathbf{GNH}^{-1}(\mathbf{GN})^\top)^{-1} \mathbf{z} \quad (29)$$

$$\chi = (\mathbf{GN})_{\mathbf{H}}^{\#} \mathbf{z} \quad (30)$$

Thus, we get a rather straight-forward way to solve for multiple tasks:

- first task:  $\mathbf{x} = \mathbf{J}_{\mathbf{H}}^{\#} \mathbf{y}$
- second task:  $\chi = (\mathbf{GN})_{\mathbf{H}}^{\#} \mathbf{z}$
- solution:  $\ddot{\mathbf{q}} = \mathbf{x} + \chi$

So, adding a new task, we simply multiply the jacobian by the null space "projector" and use the same dynamically consistent pseudoinverse. The resulting generalized acceleration is sum of the two components we found independently.

We can prove that the solution for the second task we found does not violate the original solution. I.e., does it honor the last constraint  $\mathbf{J}\chi = 0$ ?

$$e = \mathbf{J}(\mathbf{GN})_{\mathbf{H}}^{\#} \mathbf{z} \quad (31)$$

$$e = \mathbf{JH}^{-1}(\mathbf{GN})^{\top}(\mathbf{GNH}^{-1}(\mathbf{GN})^{\top})^{-1} \mathbf{z} \quad (32)$$

$$e = \mathbf{JH}^{-1} \mathbf{N}^{\top} \mathbf{G}^{\top}(\mathbf{GNH}^{-1} \mathbf{N}^{\top} \mathbf{G}^{\top})^{-1} \mathbf{z} \quad (33)$$

$$e = 0 \quad (34)$$

Let us prove that  $\mathbf{JH}^{-1}\mathbf{N}^\top = 0$ .

$$e = \mathbf{JH}^{-1}\mathbf{N}^\top \quad (35)$$

$$e = \mathbf{JH}^{-1}(\mathbf{I} - \mathbf{J}_H^\# \mathbf{J})^\top \quad (36)$$

$$e = \mathbf{JH}^{-1}(\mathbf{I} - \mathbf{J}^\top (\mathbf{J}_H^\#)^\top) \quad (37)$$

$$e = \mathbf{JH}^{-1} - \mathbf{JH}^{-1}\mathbf{J}^\top (\mathbf{J}_H^\#)^\top \quad (38)$$

$$e = \mathbf{JH}^{-1} - \mathbf{JH}^{-1}\mathbf{J}^\top (\mathbf{JH}^{-1}\mathbf{J}^\top)^{-1}\mathbf{JH}^{-1} \quad (39)$$

$$e = (\mathbf{I} - \mathbf{JH}^{-1}\mathbf{J}^\top (\mathbf{JH}^{-1}\mathbf{J}^\top)^{-1})\mathbf{JH}^{-1} \quad (40)$$

$$e = (\mathbf{I} - \mathbf{I})\mathbf{JH}^{-1} \quad (41)$$

$$e = 0 \quad (42)$$

We can put it this way. Sequential weighted inverse allows us to simplify:

$$\begin{cases} \mathbf{H}(\mathbf{x} + \chi) = \mathbf{J}^\top \lambda + (\mathbf{GN})^\top \beta \\ \mathbf{GN}(\mathbf{x} + \chi) = \mathbf{z} \\ \mathbf{J}(\mathbf{x} + \chi) = \mathbf{y} \end{cases} \longrightarrow \begin{cases} \mathbf{H}\chi = (\mathbf{GN})^\top \beta \\ \mathbf{GN}\chi = \mathbf{z} \end{cases}$$

And this was possible because we used Jacobian matrix multiplied by generalized null space projector on the right:  $\mathbf{GN}$ .

- Kim, D., Di Carlo, J., Katz, B., Bledt, G. and Kim, S., 2019. Highly dynamic quadruped locomotion via whole-body impulse control and model predictive control. arXiv preprint arXiv:1909.06586.

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Fundamentals-of-robotics-2022](https://github.com/SergeiSa/Fundamentals-of-robotics-2022)

Check Moodle for additional links, videos, textbook suggestions.

