3D linkages

Fundamentals of Robotics, Lecture 3

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Fall 2022

Consider a vector $\mathbf{v}$. As we noted in the previous lecture, we can express it in terms of its coordinates in a basis of of choice. Or - we can just leave it alone. Examples of vectors in Robotics are linear and angular velocities, as well as accelerations.

Consider a point P in Euclidean space $\mathcal{E}$. Given some other point O we can associate a vector $\mathbf{r}_{OP}$ with any point $P \in \mathcal{E}$ as difference in position of points P and O. We call point O origin. Origin and basis together form a *frame*.

Notice that to find coordinates of a vector we only need a basis, while for a point we also need an origin - in other words, we need a frame to express a point.

Consider the following example. We know coordinates of $\mathbf{r}_{OP}$ expressed in basis $\mathcal{V}$, and we know $\mathbf{r}_{OA}$, also expressed in $\mathcal{V}$. Can express $\mathbf{r}_{AP}$ in $\mathcal{V}$?

We know that $\mathbf{r}_{OP} = P - O$ and $\mathbf{r}_{OA} = A - O$ and $\mathbf{r}_{AP} = P - A$, hence $\mathbf{r}_{AP} = \mathbf{r}_{OP} - \mathbf{r}_{OA}$, which is true in any basis (of course, to sum coordinates in a meaningful way they should be associated with the same basis).

To denote that $\mathbf{r}_{AP}$ is expressed in $\mathcal{V}$ we write it as ${}^{\mathcal{V}}\mathbf{r}_{AP}$.

Another example. We have a point K and frames $\mathcal{F}_1$ and $\mathcal{F}_2$, as well as a *world frame*. Frame $\mathcal{F}_1$ given by its origin O_1 and basis $\mathcal{T}_1$. Frame $\mathcal{F}_2$ given by its origin O_2 and basis $\mathcal{T}_2$.

We know position of point O_1 in the world frame (its coordinates with respect to the origin of the world frame, expressed in the world frame basis) - ${}^{\mathcal{W}}\mathbf{r}_{O1}$. Basis $\mathcal{T}_1$ is given by a matrix ${}^{\mathcal{W}}\mathbf{T}_1$ whose columns are coordinates of the vectors forming $\mathcal{T}_1$ expressed in the world frame.

We know position of point O_2 in the $\mathcal{F}_1$ frame - ${}^{\mathcal{T}_1}\mathbf{r}_{O1O2}$. Basis $\mathcal{T}_2$ is given by a matrix ${}^{\mathcal{T}_1}\mathbf{T}_2$ whose columns are coordinates of the vectors forming $\mathcal{T}_2$ expressed in the world frame.

Task - express K in world frame given ${}^{\mathcal{T}_2}\mathbf{r}_{O2K}$ - position of K in the frame $\mathcal{F}_2$. Scary.

A good way to deal with these problems is to express every basis in terms of the world frame basis, and then do the same with all points.

First, we already have basis $\mathcal{T}_1$ in the world frame basis - ${}^{\mathcal{W}}\mathbf{T}_1$. Now to express $\mathcal{T}_2$.

Given coordinates ${}^{\mathcal{T}_1}\mathbf{T}_2$ in $\mathcal{T}_1$ we can find $\mathbf{T}_2 = \mathbf{T}_1 \, {}^{\mathcal{T}_1}\mathbf{T}_2$, hence ${}^{\mathcal{W}}\mathbf{T}_2 = {}^{\mathcal{W}}\mathbf{T}_1 \, {}^{\mathcal{T}_1}\mathbf{T}_2$.

Sequence of frames

We already know ${}^{\mathcal{W}}\mathbf{r}_{O1}$ and we know ${}^{\mathcal{T}_1}\mathbf{r}_{O1O2}$. Vector $\mathbf{r}_{O1O2}$ is therefore:

$$\mathbf{r}_{O1O2} = \mathbf{T}_1 \ {}^{\mathcal{T}_1}\mathbf{r}_{O1O2} \tag{1}$$

$${}^{\mathcal{W}}\mathbf{r}_{O1O2} = {}^{\mathcal{W}} \mathbf{T}_1 \ {}^{\mathcal{T}_1}\mathbf{r}_{O1O2} \tag{2}$$

We remember that $\mathbf{r}_{O1O2} = O_2 - O_1$ and $\mathbf{r}_{O1} = O_1 - O$ (where O is the world frame origin), and we want to find $\mathbf{r}_{O2} = O_2 - O$, so:

$$\mathbf{r}_{O2} = \mathbf{r}_{O1} + \mathbf{r}_{O1O2} \tag{3}$$

$${}^{\mathcal{W}}\mathbf{r}_{O2} = {}^{\mathcal{W}} \mathbf{r}_{O1} + {}^{\mathcal{W}} \mathbf{r}_{O1O2} \tag{4}$$

$${}^{\mathcal{W}}\mathbf{r}_{O2} = {}^{\mathcal{W}} \mathbf{r}_{O1} + {}^{\mathcal{W}} \mathbf{T}_1 \ {}^{\mathcal{T}_1}\mathbf{r}_{O1O2} \tag{5}$$

We know ${}^{\mathcal{T}_2}\mathbf{r}_{O2K}$, so vector $\mathbf{r}_{O2K}$ can be found as:

$$\mathbf{r}_{O2K} = \mathbf{T}_2\ {}^{\mathcal{T}_2}\mathbf{r}_{O2K} \tag{6}$$

$${}^{\mathcal{W}}\mathbf{r}_{O2K} = {}^{\mathcal{W}}\ \mathbf{T}_2\ {}^{\mathcal{T}_2}\mathbf{r}_{O2K} \tag{7}$$

$${}^{\mathcal{W}}\mathbf{r}_{O2K} = {}^{\mathcal{W}}\ \mathbf{T}_1\ {}^{\mathcal{T}_1}\mathbf{T}_2\ {}^{\mathcal{T}_2}\mathbf{r}_{O2K} \tag{8}$$

Finally, we observe that $\mathbf{r}_{O2K} = K - O_2$, $\mathbf{r}_{O2} = O_2 - O$ and $\mathbf{r}_K = K - O$, so:

$$\mathbf{r}_K = \mathbf{r}_{O2} + \mathbf{r}_{O2K} \tag{9}$$

$${}^{\mathcal{W}}\mathbf{r}_K = {}^{\mathcal{W}}\ \mathbf{r}_{O2} + {}^{\mathcal{W}}\ \mathbf{r}_{O2K} \tag{10}$$

$${}^{\mathcal{W}}\mathbf{r}_K = {}^{\mathcal{W}}\ \mathbf{r}_{O1} + {}^{\mathcal{W}}\ \mathbf{T}_1\ {}^{\mathcal{T}_1}\mathbf{r}_{O1O2} + {}^{\mathcal{W}}\ \mathbf{T}_1\ {}^{\mathcal{T}_1}\mathbf{T}_2\ {}^{\mathcal{T}_2}\mathbf{r}_{O2K} \tag{11}$$

Consider a serial non-branching linkage. The i-th link has frame $\mathcal{F}_i$ attached to it, meaning the coordinates of any point on the link with respect to the frame do not change.

For simplicity, let us assume that the joint (i-th) by which the i-th link is attached (to its *parent link*) has coordinates $[0,\ 0,\ 0]$ in the frame $\mathcal{F}_i$. The point O_i is i-th joint and the origin of the frame $\mathcal{F}_i$; the basis associated with that frame is $\mathcal{T}_i$. We define vector $\mathbf{r}_{O_i O_{i+1}}$ pointing from the current joint to the next one. For the last n-th link we define $\mathbf{r}_{O_n K}$ vector pointing from the joint to the end-effector

We know ${}^{\mathcal{T}_i}\mathbf{r}_{O_i O_{i+1}}$ and ${}^{\mathcal{T}_i}\mathbf{T}_{i+1}$. We need to find everything in terms of world frame.

The process is rather simple. First consider rotations. We are given ${}^{\mathcal{W}}\mathbf{T}_1$, ${}^{\mathcal{T}_1}\mathbf{T}_2$, ${}^{\mathcal{T}_2}\mathbf{T}_3$, etc.

We can use our familiar arguments about coordinates to show that:

$$
\begin{aligned}
{}^{\mathcal{W}}\mathbf{T}_2 &= {}^{\mathcal{W}}\mathbf{T}_1 \; {}^{\mathcal{T}_1}\mathbf{T}_2 && (12)\\
{}^{\mathcal{W}}\mathbf{T}_3 &= {}^{\mathcal{W}}\mathbf{T}_2 \; {}^{\mathcal{T}_2}\mathbf{T}_3 && (13)\\
{}^{\mathcal{W}}\mathbf{T}_{i+1} &= {}^{\mathcal{W}}\mathbf{T}_i \; {}^{\mathcal{T}_i}\mathbf{T}_{i+1} && (14)
\end{aligned}
$$

We find *absolute orientation* of a link ${}^{\mathcal{W}}\mathbf{T}_i$ and then we can find absolute orientation of the next link just by using its local basis ${}^{\mathcal{T}_i}\mathbf{T}_{i+1}$.
You can think about it as "we know orientation of the previous link - next one needs just one more rotation, given by ${}^{\mathcal{T}_i}\mathbf{T}_{i+1}$".

Next, we take "translations". All vectors ${}^{\mathcal{T}_i}\mathbf{r}_{O_iO_{i+1}}$ are known to us in local coordinates. We can transfer then to world coordinates:

$$ {}^{\mathcal{W}}\mathbf{r}_{O_iO_{i+1}} = {}^{\mathcal{W}}\mathbf{T}_i \, {}^{\mathcal{T}_i}\mathbf{r}_{O_iO_{i+1}} \tag{15} $$

With that, we can easily find position of any point on the robot, end-effector for instance:

$$ {}^{\mathcal{W}}\mathbf{r}_K = {}^{\mathcal{W}}\mathbf{r}_{O_1} + {}^{\mathcal{W}}\mathbf{r}_{O_1O_2} + {}^{\mathcal{W}}\mathbf{r}_{O_2O_3} + ... + {}^{\mathcal{W}}\mathbf{r}_{O_nK} \tag{16} $$

How do we integrate joints into this discussion?

Remember rotation matrices:

$$\mathbf{R}_x(\varphi_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_x & -\sin\varphi_x \\ 0 & \sin\varphi_x & \cos\varphi_x \end{bmatrix} \tag{17}$$

$$\mathbf{R}_y(\varphi_y) = \begin{bmatrix} \cos\varphi_y & 0 & \sin\varphi_y \\ 0 & 1 & 0 \\ -\sin\varphi_y & 0 & \cos\varphi_y \end{bmatrix} \tag{18}$$

$$\mathbf{R}_z(\varphi_z) = \begin{bmatrix} \cos\varphi_z & -\sin\varphi_z & 0 \\ \sin\varphi_z & \cos\varphi_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

If your joint aligns with one of the principle axes of your parent basis, the transformation from the patent basis to the child basis can be defined by one of those matrices. That is quite common in practice.

Assume the i-th link is connected to the $i-1$ link via a pin joint aligned with the x-axes of the frame $\mathcal{F}_{i-1}$. Then, using our previous notation we get ${}^{\mathcal{T}_{i-1}}\mathbf{T}_i = \mathbf{R}_x(\varphi_i)$. We call it *parametrization*. In this example φ_i is a joint angle.

It is also easy to find examples where ${}^{\mathcal{T}_{i-1}}\mathbf{T}_i = {}^{\mathcal{T}_{i-1}}\mathbf{C}_i\,\mathbf{R}_x(\varphi_i)$ where ${}^{\mathcal{T}_{i-1}}\mathbf{C}_i = const$ that depicts how the child link is oriented when $\varphi_i = 0$. We can use this constant orientation offset to change pose of the robot corresponding to zero joint angles, as well as to account for geometric quirks of the linkage.

Let's draw examples!

Thank you!

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github.com/SergeiSa/Fundumentals-of-robotics-2022

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