

# Inverse Kinematics

## Fundamentals of Robotics, Lecture 6

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Fall 2022

Given a point (e.g. end effector)  $K$  with position given by vector  $\mathbf{r}_K(\mathbf{q})$ , we can find its velocity:

$$\dot{\mathbf{r}}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \dot{\mathbf{q}} \quad (1)$$

Let us introduce notation:

$$\mathbf{J}_K = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \quad (2)$$

$$\mathbf{v}_K = \dot{\mathbf{r}}_K(\mathbf{q}) \quad (3)$$

Thus we get:

$$\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}} \quad (4)$$

Given  $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$ , can we find least-residual solution to this problem? Yes!

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K \quad (5)$$

Is this the only solution? No. All solutions are:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N} \mathbf{z} \quad (6)$$

where  $\mathbf{N} = \text{null}(\mathbf{J}_K)$  and  $\mathbf{z}$  are null space coordinates.

Alternatively, we can use a projector to do the same thing with less new notation:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + (\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K) \dot{\mathbf{q}} \quad (7)$$

where  $\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$  is a null space projector.

Now let us find closest joint velocity to  $\dot{\mathbf{q}}_0$  that solves the velocity problem  $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$ :

$$\begin{aligned} & \underset{\dot{\mathbf{q}}}{\text{minimize}} \quad \|\dot{\mathbf{q}} - \dot{\mathbf{q}}_0\|, \\ & \text{subject to} \quad \mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}} \end{aligned} \tag{8}$$

We can solve it by first finding all solutions:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N} \mathbf{z} \tag{9}$$

Then we minimize cost function in terms of the null space variable  $\mathbf{z}$ :

$$\|\mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N} \mathbf{z} - \dot{\mathbf{q}}_0\| \rightarrow \min \tag{10}$$

First we simplify  $||\mathbf{J}_K^+ \mathbf{v}_K + \mathbf{Nz} - \dot{\mathbf{q}}_0|| \rightarrow \min$  with notation

$$\mathbf{c} = \mathbf{J}_K^+ \mathbf{v}_K - \dot{\mathbf{q}}_0 \quad (11)$$

Then we square  $||\mathbf{Nz} + \mathbf{c}||$ , and consider its derivative:

$$(\mathbf{Nz} + \mathbf{c})^\top (\mathbf{Nz} + \mathbf{c}) \rightarrow \min \quad (12)$$

$$\frac{\partial}{\partial \mathbf{z}} (\mathbf{Nz} + \mathbf{c})^\top (\mathbf{Nz} + \mathbf{c}) = 0 \quad (13)$$

$$2\mathbf{z}^\top \mathbf{N}^\top \mathbf{N} + 2\mathbf{c}^\top \mathbf{N} = 0 \quad (14)$$

$$\mathbf{z} = -(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \mathbf{c} \quad (15)$$

Knowing that  $\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{Nz}$  we get:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \mathbf{c} \quad (16)$$

Let us examine the solution we obtained:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top (\mathbf{J}_K^+ \mathbf{v}_K - \dot{\mathbf{q}}_0) \quad (17)$$

$$\dot{\mathbf{q}} = (\mathbf{I} - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top) \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \dot{\mathbf{q}}_0 \quad (18)$$

Let us examine the matrices:

$$\mathbf{P}_N = \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \quad (19)$$

$$\mathbf{P}_R = \mathbf{I} - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top = \mathbf{I} - \mathbf{P}_N \quad (20)$$

$$\dot{\mathbf{q}} = \mathbf{P}_R \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{P}_N \dot{\mathbf{q}}_0 \quad (21)$$

where  $\mathbf{P}_N$  is column space projector for  $\mathbf{N}$ , hence it is a null space projector for the jacobian  $\mathbf{J}_K$ . And  $\mathbf{I} - \mathbf{P}_N$  is a projector to the orthogonal complement, hence it is row space projector.

So, we have eq.  $\dot{\mathbf{q}} = \mathbf{P}_R \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{P}_N \dot{\mathbf{q}}_0$  and we have null space projector  $\mathbf{P}_N$  and row space projector  $\mathbf{P}_R$ .

We know that pseudoinverse lies in the column space, so:

$$\mathbf{P}_R \mathbf{J}_K^+ = \mathbf{J}_K^+ \quad (22)$$

Also we know that null space projector can be found as  $\mathbf{P}_N = \mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$ :

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + (\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K) \dot{\mathbf{q}}_0 \quad (23)$$

Notice, this is almost exactly the same as what we found before. We can interpret it as "the solution is given by row-space least squares solution, plus null space projection of  $\dot{\mathbf{q}}_0$ ".

Consider second derivative of the position of the point  $K$ :

$$\ddot{\mathbf{r}}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \ddot{\mathbf{q}} + \frac{d}{dt} \left( \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \right) \dot{\mathbf{q}} \quad (24)$$

Defining  $\mathbf{a}_K = \ddot{\mathbf{r}}_K$  we get:

$$\mathbf{a}_K = \mathbf{J}_K \ddot{\mathbf{q}} + \dot{\mathbf{J}}_K \dot{\mathbf{q}} \quad (25)$$

The least residual solution is easily found:

$$\ddot{\mathbf{q}} = \mathbf{J}_K^+ (\mathbf{a}_K - \dot{\mathbf{J}}_K \dot{\mathbf{q}}) \quad (26)$$



Given  $\mathbf{a}_K = \mathbf{J}_K \ddot{\mathbf{q}} + \dot{\mathbf{J}}_K \dot{\mathbf{q}}$  let us find acceleration closest to  $\ddot{\mathbf{q}}_0$  that solves the acceleration problem:

$$\ddot{\mathbf{q}} = \mathbf{J}_K^+ (\mathbf{a}_K - \dot{\mathbf{J}}_K \dot{\mathbf{q}}) + \mathbf{P}_N \ddot{\mathbf{q}}_0 \quad (27)$$

where  $\mathbf{P}_N = \mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$ .

What if we want to find such  $\mathbf{q}^*$  that  $\mathbf{r}_K(\mathbf{q}^*) = \mathbf{r}^*$ . Can we do it?

Unlike previous, this is not a linear problem. It often involves trigonometric functions, and other nonlinear ones.

Our approach will be to linearize the expression  $\mathbf{r}_K(\mathbf{q})$  and find its solution via an iterative procedure.

Given exact solution  $\mathbf{q}_0$  for problem  $\mathbf{r}_K(\mathbf{q}_0) = \mathbf{r}_K(0)$ . Then, knowing velocity  $\dot{\mathbf{q}}_0$ , then we can find an approximation of the position in the next moment of time:

$$\frac{\mathbf{q}_1 - \mathbf{q}_0}{\Delta t} \approx \dot{\mathbf{q}}_0 \quad (28)$$

$$\mathbf{q}_1 \approx \mathbf{q}_0 + \dot{\mathbf{q}}_0 \Delta t \quad (29)$$

This works tolerably well, for improvements we can look to other schemes of solving ODEs.

But what if we don't have an exact solution  $\mathbf{q}_0$ ? After all, it was that which allowed us to use local linearization.

# POSITION PROBLEM - GENERAL

Given initial guess  $\mathbf{q}_0$  we will try to solve the problem  $\mathbf{r}_K(\mathbf{q}) = \mathbf{r}_K^*$ . First let us define discrepancy:

$$\mathbf{e}(\mathbf{q}) = \mathbf{r}_K(\mathbf{q}) - \mathbf{r}_K^* \quad (30)$$

We define cost function  $f = \mathbf{e}^\top \mathbf{e}$ , initial position  $\mathbf{r}_{K,0} = \mathbf{r}_K(\mathbf{q}_0)$ , initial discrepancy  $\mathbf{e}_0 = \mathbf{r}_{K,0} - \mathbf{r}_K^*$  and gen. coordinates displacement  $\delta = \mathbf{q} - \mathbf{q}_0$  and produce Taylor expansion of the cost:

$$f \approx \mathbf{e}_0^\top \mathbf{e}_0 + \mathbf{e}_0^\top \mathbf{J}_K \delta + \delta^\top \mathbf{J}_K^\top \mathbf{e}_0 + \delta^\top \mathbf{J}_K^\top \mathbf{J}_K \delta \quad (31)$$

Now we take derivative and set it to zero:

$$2\mathbf{e}_0^\top \mathbf{J}_K + 2\delta^\top \mathbf{J}_K^\top \mathbf{J}_K = 0 \quad (32)$$

We obtained expression:

$$\delta = -(\mathbf{J}_K^\top \mathbf{J}_K)^{-1} \mathbf{J}_K^\top \mathbf{e}_0 \quad (33)$$

And remembering the substitutions we made we get:

$$\mathbf{q} - \mathbf{q}_0 = -(\mathbf{J}_K^\top \mathbf{J}_K)^{-1} \mathbf{J}_K^\top (\mathbf{r}_{K,0} - \mathbf{r}_K^*) \quad (34)$$

$$\mathbf{q} = \mathbf{q}_0 - \mathbf{J}_K^+ (\mathbf{r}_{K,0} - \mathbf{r}_K^*) \quad (35)$$

We can use the final expression to update our initial guess, then the-linearize the problem at the new position and repeat the process, until we converge.

How can we check if a velocity  $\mathbf{v}_K$  can be achieved, given  $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$ ?

If  $\mathbf{v}_K$  lies in the column space of  $\mathbf{J}_K$ , it is achievable:

$$(\mathbf{I} - \mathbf{J}_K \mathbf{J}_K^+) \mathbf{v}_K = 0 \quad (36)$$

How can we check if a solution  $\dot{\mathbf{q}}$  is minimal-norm, given  $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$ ?

If  $\dot{\mathbf{q}}$  lies in the row space of  $\mathbf{J}_K$ , it is minimal:

$$(\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K) \dot{\mathbf{q}} = 0 \quad (37)$$

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Fundamentals-of-robotics-2022](https://github.com/SergeiSa/Fundamentals-of-robotics-2022)

Check Moodle for additional links, videos, textbook suggestions.

