

# Jacobians

## Fundamentals of Robotics, Lecture 4

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# FORWARD KINEMATICS WITH SPECIAL EUCLIDEAN GROUP

Consider a two-link robot with frames  $\mathcal{F}_i$ , bases  $\mathcal{T}_i$ , joints and origins  $O_i$ . We know  ${}^{\mathcal{W}}\mathbf{T}_1$  and  ${}^{\mathcal{T}_1}\mathbf{T}_2$ , as well as  ${}^{\mathcal{W}}\mathbf{r}_{O_1}$  and  ${}^{\mathcal{T}_1}\mathbf{r}_{O_1O_2}$  (see last lecture for details).

To express everything in terms of the world frame we do the familiar steps:

$${}^{\mathcal{W}}\mathbf{T}_2 = {}^{\mathcal{W}}\mathbf{T}_1 {}^{\mathcal{T}_1}\mathbf{T}_2 \tag{1}$$

$${}^{\mathcal{W}}\mathbf{r}_{O_1O_2} = {}^{\mathcal{W}}\mathbf{T}_1 {}^{\mathcal{T}_1}\mathbf{r}_{O_1O_2} \tag{2}$$

$${}^{\mathcal{W}}\mathbf{r}_{O_2} = {}^{\mathcal{W}}\mathbf{r}_{O_1} + {}^{\mathcal{W}}\mathbf{r}_{O_1O_2} \tag{3}$$

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We could do the same thing in the following fashion:

$${}^{\mathcal{W}}\mathbf{E}_1 = \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_1 & {}^{\mathcal{W}}\mathbf{r}_{O_1} \\ \mathbf{0} & 1 \end{bmatrix}, \quad {}^{\varepsilon_1}\mathbf{E}_2 = \begin{bmatrix} \tau_1\mathbf{T}_2 & \tau_1\mathbf{r}_{O_1O_2} \\ \mathbf{0} & 1 \end{bmatrix} \quad (4)$$

$${}^{\mathcal{W}}\mathbf{E}_2 = \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_2 & {}^{\mathcal{W}}\mathbf{r}_{O_2} \\ \mathbf{0} & 1 \end{bmatrix} \quad (5)$$

$${}^{\mathcal{W}}\mathbf{E}_2 = {}^{\mathcal{W}}\mathbf{E}_1 {}^{\varepsilon_1}\mathbf{E}_2 \quad (6)$$

$$\begin{aligned} {}^{\mathcal{W}}\mathbf{E}_2 &= \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_1 & {}^{\mathcal{W}}\mathbf{r}_{O_1} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \tau_1\mathbf{T}_2 & \tau_1\mathbf{r}_{O_1O_2} \\ \mathbf{0} & 1 \end{bmatrix} = \\ &= \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_1 \tau_1\mathbf{T}_2 & ({}^{\mathcal{W}}\mathbf{T}_1 \tau_1\mathbf{r}_{O_1O_2} + {}^{\mathcal{W}}\mathbf{r}_{O_1}) \\ \mathbf{0} & 1 \end{bmatrix} \end{aligned}$$

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In general, we can define a transformation from frame  $i$  to the frame  $i + 1$

$$\mathcal{E}_i \mathbf{E}_{i+1} = \begin{bmatrix} \mathbf{T}_i & \mathbf{p}_i \\ \mathbf{0} & 1 \end{bmatrix} \quad (7)$$

Where  $\mathbf{T}_i$  are coordinates of the basis of the frame  $i + 1$  in terms of frame  $i$ , and  $\mathbf{p}_i$  is the vector pointing from the origin of the frame  $i$  to the origin of the frame  $i + 1$  expressed in the basis of the frame  $i$ .

To get transformation from the world frame to the  $n$ -th frame we get:

$${}^{\mathcal{W}}\mathbf{E}_n = \prod_{i=1}^n \mathcal{E}_{i-1} \mathbf{E}_i \quad (8)$$

Last lecture we focused on how to find expressions of radius-vectors (vectors describing positions of points) in world frame, given relative positions and orientations of frames.

Today, we focus on derivatives of these expressions.

Consider vector  $\mathbf{r}_K$  describing position of the point  $K$ . What is its derivative?

In order to answer this question we need to understand, which parameters appearing in the expression for  $\mathbf{r}_K$  are changing with time, and which do not.

Consider an example:

$$\mathbf{r}_K(\mathbf{q}) = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \\ 0 \end{bmatrix} \quad (9)$$

In this example, we know that it is joint coordinate  $q_1$  that is going to change with time.

Another example:

$$\mathbf{r}_K(\mathbf{q}) = \begin{bmatrix} q_1 \\ q_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_3 & -\sin q_3 \\ 0 & \sin q_3 & \cos q_3 \end{bmatrix} \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 + l_1 \cos q_3 \\ l_1 \sin q_3 \end{bmatrix}$$

Now,  $q_1$  and  $q_2$  are translations and  $q_3$  is a rotation, and  $\mathbf{q} = [q_1, q_2, q_3]$  are changing joint coordinates.

So, what is a derivative of  $\mathbf{r}_K(\mathbf{q})$  with respect to time?

$$\frac{d}{dt}\mathbf{r}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}}\dot{\mathbf{q}} \quad (10)$$

We can denote  $\mathbf{J}_K = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}}$  and call it a jacobian matrix. We can also denote velocity of the point  $K$  as  $\mathbf{v}_K(\mathbf{q}, \dot{\mathbf{q}}) = \frac{d}{dt}\mathbf{r}_K(\mathbf{q})$ .

$$\mathbf{v}_K = \mathbf{J}_K\dot{\mathbf{q}} \quad (11)$$

Notice that velocity  $\mathbf{v}_K$  is linear with respect to the joint velocities  $\dot{\mathbf{q}}$ , and the linear relation is given by the jacobian  $\mathbf{J}_K$ .



Consider  $\mathbf{r}_K = \mathbf{r}_{OO_1} + \mathbf{r}_{O_1O_2} + \dots + \mathbf{r}_{O_nK}$ . What is a jacobian of  $\mathbf{r}_K$ ?

$$\begin{aligned}\mathbf{J}_K &= \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} = \frac{\partial \mathbf{r}_{OO_1}}{\partial \mathbf{q}} + \frac{\partial \mathbf{r}_{O_1O_2}}{\partial \mathbf{q}} + \dots + \frac{\partial \mathbf{r}_{O_nK}}{\partial \mathbf{q}} = \\ &= \mathbf{J}_{OO_1} + \mathbf{J}_{O_1O_2} + \dots + \mathbf{J}_{O_nK}\end{aligned}$$

Jacobians have an additive structure inherited from the additive structure of the position vectors.

Notice also that if your vectors and jacobians are expressed via coordinates in different bases - you have to express them in a single basis, and then do the additions, as usual.

Let us consider a rigid body rotating with angular velocity  $\omega$  with basis  $\mathcal{T}$ , given by a matrix  $\mathbf{T}$  (whose coordinates in the world frame basis are  ${}^W\mathbf{T}$ ), stationary with respect to the rigid body. Consider a point  $K$  on the link, defined by a vector  $\mathbf{r}$ . We know coordinates of  $\mathbf{r}$  in terms of  $\mathcal{T}$ , which we denote as  ${}^{\mathcal{T}}\mathbf{r}$ .

What is velocity of  $K$ ? By definition of angular velocity,  $\mathbf{v} = \omega \times \mathbf{r}$ . The same can be represented as a vector-matrix multiplication:

$$\mathbf{v} = \omega \times \mathbf{r} = \mathbf{\Omega} \mathbf{r} \quad (12)$$

$$\mathbf{\Omega} = [\omega]_{\times} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \quad (13)$$

Note that expression  $\mathbf{v} = \omega \times \mathbf{r}$  works as long as we are talking about vectors themselves, or their coordinate representation in the same basis:

$$\mathbf{v} = \omega \times \mathbf{r} = \Omega \mathbf{r} \quad (14)$$

$$\mathcal{W}_{\mathbf{v}} = \mathcal{W}_{\omega} \times \mathcal{W}_{\mathbf{r}} = \mathcal{W}_{\Omega} \mathcal{W}_{\mathbf{r}} \quad (15)$$

$$\mathcal{T}_{\mathbf{v}} = \mathcal{T}_{\omega} \times \mathcal{T}_{\mathbf{r}} = \mathcal{T}_{\Omega} \mathcal{T}_{\mathbf{r}} \quad (16)$$

At the same time, we can find position of  $K$  as  $\mathbf{r} = \mathbf{T}^\top \mathbf{r}$  and  ${}^{\mathcal{W}}_{\mathbf{r}} = {}^{\mathcal{W}}_{\mathbf{T}^\top \mathbf{r}}$ , where  $\frac{d}{dt} \mathbf{T}^\top \mathbf{r} = 0$ . We can find its time derivative as:

$$\mathbf{v} = \frac{d}{dt} \mathbf{r} = \frac{d}{dt} \mathbf{T}^\top \mathbf{r} = \dot{\mathbf{T}}^\top \mathbf{r} \quad (17)$$

$${}^{\mathcal{W}}_{\mathbf{v}} = {}^{\mathcal{W}}_{\dot{\mathbf{T}}^\top \mathbf{r}} \quad (18)$$

At the same time,  ${}^{\mathcal{W}}_{\mathbf{v}} = {}^{\mathcal{W}}_{\Omega} {}^{\mathcal{W}}_{\mathbf{r}}$ . Hence:

$${}^{\mathcal{W}}_{\mathbf{v}} = {}^{\mathcal{W}}_{\Omega} {}^{\mathcal{W}}_{\mathbf{T}^\top \mathbf{r}} \quad (19)$$

With that, we know that:

$$\Omega \mathbf{T} = \dot{\mathbf{T}} \quad (20)$$

$$\Omega = \dot{\mathbf{T}} \mathbf{T}^\top \quad (21)$$

Notice that  $\mathbf{T}^\top \mathbf{T} = \mathbf{I}$ , so  $\frac{d}{dt}(\mathbf{T}^\top \mathbf{T}) = 0$ , and thus  $\dot{\mathbf{T}}^\top \mathbf{T} = -\mathbf{T}^\top \dot{\mathbf{T}}$ :

$$\Omega = \dot{\mathbf{T}} \mathbf{T}^\top \quad (22)$$

$$\Omega \mathbf{T} = \dot{\mathbf{T}} \quad (23)$$

$$\mathbf{T}^\top \Omega \mathbf{T} = \mathbf{T}^\top \dot{\mathbf{T}} \quad (24)$$

$$\mathbf{T}^\top \Omega \mathbf{T} = -\dot{\mathbf{T}}^\top \mathbf{T} \quad (25)$$

$$\mathbf{T}^\top \Omega = -\dot{\mathbf{T}}^\top \quad (26)$$

$$\Omega = -\mathbf{T} \dot{\mathbf{T}}^\top \quad (27)$$

Note, in these formulas  $\Omega$ ,  $\mathbf{T}$  and  $\dot{\mathbf{T}}$  are expressed in the same coordinates.

Given  $\Omega$ , we can find  $\omega$ :

$$\omega = \text{skew2vec}(\Omega) \quad (28)$$

$$\text{skew2vec} \left( \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \right) = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad (29)$$

You can notice that both  $\omega$  and  $\Omega$  are linear with respect to  $\dot{\mathbf{T}}$ , which in turn is linear with respect to  $\dot{\mathbf{q}}$ .

As we mentioned,  $\omega = \omega(\mathbf{q}, \dot{\mathbf{q}})$  is linear with respect to  $\dot{\mathbf{q}}$ :

$$\omega = \frac{\partial \omega}{\partial \dot{\mathbf{q}}} \dot{\mathbf{q}} = \mathbf{J}_\omega \dot{\mathbf{q}} \quad (30)$$

Note that there are big differences in both computation and definition between translation jacobian  $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$  and rotational jacobian  $\mathbf{J}_\omega \dot{\mathbf{q}}$ .

To define translation jacobian we need to define a link and a point on the link, whose jacobian we discuss; for rotation jacobian, we only need to identify the link whose rotation we describe.

Assume we have a vector  ${}^{\mathcal{W}}\mathbf{r}(\mathbf{q})$  expressed in the basis  $\mathcal{W}$ . We can find its jacobian:

$${}^{\mathcal{W}}\mathbf{J}(\mathbf{q}) = \frac{\partial {}^{\mathcal{W}}\mathbf{r}}{\partial \mathbf{q}} \quad (31)$$

Now, given basis  $\mathcal{T}$ , expressed by matrix  ${}^{\mathcal{W}}\mathbf{T}$ , we can represent  $\mathbf{r}$  in  $\mathcal{T} = const$ :

$${}^{\mathcal{T}}\mathbf{r}(\mathbf{q}) = {}^{\mathcal{W}}\mathbf{T}^{\top} {}^{\mathcal{W}}\mathbf{r}(\mathbf{q}) \quad (32)$$

And of course we can find jacobian of  ${}^{\mathcal{T}}\mathbf{r}(\mathbf{q})$ :

$${}^{\mathcal{T}}\mathbf{J}(\mathbf{q}) = \frac{\partial {}^{\mathcal{T}}\mathbf{r}}{\partial \mathbf{q}} = {}^{\mathcal{W}}\mathbf{T}^{\top} {}^{\mathcal{W}}\mathbf{J}(\mathbf{q}) \quad (33)$$

We can play the same game with angular velocities  $\omega$ . Jacobians depend on the bases same as vectors do.



# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Fundamentals-of-robotics-2022](https://github.com/SergeiSa/Fundamentals-of-robotics-2022)

Check Moodle for additional links, videos, textbook suggestions.

