Task Prioritization

Fundamentals of Robotics, Lecture 7

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Consider a point K (position is given by vector $\mathbf{r}_K(\mathbf{q})$). We can formulate the velocity problem as such - find such $\dot{\mathbf{q}}^*$ that:

$$\mathbf{J}_K \dot{\mathbf{q}}^* = \mathbf{v}_K \tag{1}$$

We can solve velocity problem for the point K:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z} \tag{2}$$

Now, assume that we want to also find such velocity that one of the links acquires angular velocity ω:

$$\mathbf{J}_\omega \dot{\mathbf{q}}^* = \omega \tag{3}$$

Linear Velocity and Angular Velocity

How do we solve it? We can easily combine both problems together and then we get:

$$\begin{bmatrix} \mathbf{J}_K \\ \mathbf{J}_\omega \end{bmatrix} \dot{\mathbf{q}}^* = \begin{bmatrix} \mathbf{v}_K \\ \omega \end{bmatrix} \tag{4}$$

Which is solved as:

$$\dot{\mathbf{q}}^* = \begin{bmatrix} \mathbf{J}_K \\ \mathbf{J}_\omega \end{bmatrix}^+ \begin{bmatrix} \mathbf{v}_K \\ \omega \end{bmatrix} \tag{5}$$

If a solution exists, we will obtain it. But if a solution does not exit, we will find a least-residual solution, with errors distributed across both *tasks*. What if we want to make sure one task is achieved, while the second task can be failed?

As we noted before, all solutions to the velocity problem are given as:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z} \tag{6}$$

Then we can re-write the angular velocity problem $\mathbf{J}_\omega \dot{\mathbf{q}}^* = \omega$ as:

$$\mathbf{J}_\omega(\mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z}) = \omega \tag{7}$$

This can be solved:

$$\mathbf{J}_\omega \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{J}_\omega \mathbf{N}\mathbf{z} = \omega \tag{8}$$

$$\mathbf{z} = (\mathbf{J}_\omega \mathbf{N})^{-1}(\omega - \mathbf{J}_\omega \mathbf{J}_K^+ \mathbf{v}_K) \tag{9}$$

$$\dot{\mathbf{q}}^{**} = \mathbf{N}(\mathbf{J}_\omega \mathbf{N})^{-1}(\omega - \mathbf{J}_\omega \mathbf{J}_K^+ \mathbf{v}_K) + \mathbf{J}_K^+ \mathbf{v}_K \tag{10}$$

Considering the solution $\dot{\mathbf{q}}^{**} = \mathbf{N}(\mathbf{J}_\omega \mathbf{N})^{-1}(\omega - \mathbf{J}_\omega \mathbf{J}_K^+ \mathbf{v}_K) + \mathbf{J}_K^+ \mathbf{v}_K$ we can observe the following:

- The solution looks a little ugly.
- It looks like it will only get worse if we consider the third task.

We can do better.

By now we know how to solve a problem of the type $\mathbf{J}_1\dot{\mathbf{q}} = \mathbf{v}_1$. The solution is:

$$\dot{\mathbf{q}}_1 = \mathbf{J}_1^+\mathbf{v}_1 \tag{11}$$

We add second task $\mathbf{J}_2(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2$. Can we solve it, while keeping the solution to the first task $\mathbf{J}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_1$?

Our proposition: we solve an alternative task:

$$\mathbf{J}_2\mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2 \tag{12}$$

$$\mathbf{P}_1 = \mathbf{I} - \mathbf{J}_1^+\mathbf{J}_1 \tag{13}$$

We claim that the first and the second task will both be satisfied (if possible) even if we solve them *sequentially*.

Let us study the proposed solution. First, the joint velocities $\dot{\mathbf{q}}_1$ are found as $\dot{\mathbf{q}}_1 = \mathbf{J}_1^+ \mathbf{v}_1$.

Second, matrix $\mathbf{P}_1 = \mathbf{I} - \mathbf{J}_1^+ \mathbf{J}_1$ is a null space projector for the jacobian $\mathbf{J}_1$.

Third, we consider equation $\mathbf{J}_2 \mathbf{P}_1 (\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2$:

$$\mathbf{J}_2 \mathbf{P}_1 (\mathbf{J}_1^+ \mathbf{v}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2 \tag{14}$$

Since $\mathbf{J}_1^+ \in \mathrm{row}(\mathbf{J}_1)$ we can conclude that $\mathbf{P}_1 \mathbf{J}_1^+ = 0$. Finally, we have the equation in the form:

$$\mathbf{J}_2 \mathbf{P}_1 \dot{\mathbf{q}}_2 = \mathbf{v}_2 \tag{15}$$

Given $\mathbf{J}_2\mathbf{P}_1\dot{\mathbf{q}}_2 = \mathbf{v}_2$ we can solve it:

$$\mathbf{P}_1\dot{\mathbf{q}}_2 = \mathbf{J}_2^+\mathbf{v}_2 \tag{16}$$

$$\mathbf{P}_1 = \mathbf{C}_1\mathbf{C}_1^\top \tag{17}$$

$$\mathbf{C}_1\mathbf{C}_1^\top\dot{\mathbf{q}}_2 = \mathbf{J}_2^+\mathbf{v}_2 \tag{18}$$

$$\mathbf{C}_1^\top\dot{\mathbf{q}}_2 = \mathbf{C}_1^\top\mathbf{J}_2^+\mathbf{v}_2 \tag{19}$$

$$\dot{\mathbf{q}}_2 = \mathbf{C}_1\mathbf{C}_1^\top\mathbf{J}_2^+\mathbf{v}_2 \tag{20}$$

$$\dot{\mathbf{q}}_2 = \mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 \tag{21}$$

We can summarize it as $(\mathbf{J}_2\mathbf{P}_1)^+ = \mathbf{P}_1\mathbf{J}_2^+$.

So, The solution is:

$$\dot{\mathbf{q}}_1 = \mathbf{J}_1^+ \mathbf{v}_1 \tag{22}$$

$$\dot{\mathbf{q}}_2 = \mathbf{P}_1 \mathbf{J}_2^+ \mathbf{v}_2 \tag{23}$$

Can we prove that the following holds?

$$\begin{cases} \mathbf{J}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_1 \\ \mathbf{J}_2 \mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2 \end{cases} \tag{24}$$

We study equation $\mathbf{J}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_1$:

$$\mathbf{J}_1\dot{\mathbf{q}}_1 + \mathbf{J}_1\mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 = \mathbf{v}_1 \tag{25}$$

Assuming that $\mathbf{J}_1\dot{\mathbf{q}}_1 = \mathbf{v}_1$ (the residual is zero), we get:

$$\mathbf{J}_1\mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 = 0 \tag{26}$$

Since $\mathbf{P}_1$ is null space projector it means that its columns lie in the null space of $\mathbf{J}_1$ - meaning that $\mathbf{J}_1\mathbf{P}_1 = 0$, q.e.d.

We study equation $\mathbf{J}_2\mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2$:

$$\mathbf{J}_2\mathbf{P}_1\mathbf{J}_1^+\mathbf{v}_1 + \mathbf{J}_2\mathbf{P}_1\dot{\mathbf{q}}_2 = \mathbf{v}_2 \tag{27}$$

If $\mathbf{J}_2\mathbf{P}_1\dot{\mathbf{q}}_2 = \mathbf{v}_2$, meaning that we found a zero-residual solution, then:

$$\mathbf{J}_2\mathbf{P}_1\mathbf{J}_1^+\mathbf{v}_1 = 0 \tag{28}$$

Since $\mathbf{J}_1^+$ is in the row space of $\mathbf{J}_1$, so $\mathbf{P}_1\mathbf{J}_1^+ = 0$.

$$\mathbf{J}_2\mathbf{0}\mathbf{v}_1 = 0 \tag{29}$$

$$0 = 0, \quad \text{q.e.d.} \tag{30}$$

Thus, we proved that the proposed solution works.

Let us try to solve three tasks, one after another:

$$\mathbf{J}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_1 \tag{31}$$

$$\mathbf{J}_2\mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_2 \tag{32}$$

$$\mathbf{J}_3\mathbf{P}_2(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_3 \tag{33}$$

$$\mathbf{P}_1 = \mathbf{I} - \mathbf{J}_1^+\mathbf{J}_1 \tag{34}$$

$$\mathbf{P}_2 = \mathbf{P}_1(\mathbf{I} - \mathbf{J}_2^+\mathbf{J}_2) \tag{35}$$

Where $\mathbf{P}_1$ is a projector onto the null space of $\mathbf{J}_1$, and $\mathbf{P}_2$ is a projector onto the intersection of null spaces of $\mathbf{J}_1$ and $\mathbf{J}_2$.
We propose to solve it as:

$$\dot{\mathbf{q}}_1 = \mathbf{J}_1^+\mathbf{v}_1 \tag{36}$$

$$\dot{\mathbf{q}}_2 = \mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 \tag{37}$$

$$\dot{\mathbf{q}}_3 = \mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3 \tag{38}$$

First equation: $\mathbf{J}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_1$.

If the first task has a zero-residual $\mathbf{J}_1\dot{\mathbf{q}}_1 = \mathbf{v}_1$, we obtain:

$$\text{(what we want to prove)} \quad \mathbf{J}_1(\dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = 0 \tag{39}$$

$$\mathbf{J}_1(\mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 + \mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3) = 0 \tag{40}$$

We can observe $\mathbf{J}_1\mathbf{P}_1 = 0$ and $\mathbf{J}_1\mathbf{P}_2 = \mathbf{J}_1\mathbf{P}_1(\mathbf{I} - \mathbf{J}_2^+\mathbf{J}_2) = 0$:

$$\mathbf{J}_1(\mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 + \mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3) = \mathbf{J}_1(0 + 0) = 0 \tag{41}$$

Second equation: $\mathbf{J}_2\mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_2$.

Given that $\mathbf{J}_2\mathbf{P}_1\dot{\mathbf{q}}_2 = \mathbf{v}_2$, we obtain:

$$\text{(what we want to prove)} \quad \mathbf{J}_2\mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_3) = 0 \tag{42}$$

$$\mathbf{J}_2\mathbf{P}_1(\mathbf{J}_1^+\mathbf{v}_1 + \mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3) = 0 \tag{43}$$

$$\mathbf{J}_2\mathbf{P}_1\mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3 = 0 \tag{44}$$

$$\mathbf{J}_2\mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3 = 0 \tag{45}$$

Matrix $\mathbf{P}_2$ is null space projector for the $\mathbf{J}_2$, further projected onto the null space of $\mathbf{J}_1$; hence $\mathbf{J}_2\mathbf{P}_2 = 0$:

$$\mathbf{J}_2\mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3 = \mathbf{0}\mathbf{J}_3^+\mathbf{v}_3 = 0 = 0 \tag{46}$$

Third equation: $\mathbf{J}_3\mathbf{P}_2(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_3$.

Given that $\mathbf{J}_3\mathbf{P}_2\dot{\mathbf{q}}_3 = \mathbf{v}_3$, we obtain:

$$\text{(what we want to prove)} \quad \mathbf{J}_3\mathbf{P}_2(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = 0 \tag{47}$$

$$\mathbf{J}_3\mathbf{P}_2(\mathbf{J}_1^+\mathbf{v}_1 + \mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2) = 0 \tag{48}$$

We can observe that $\mathbf{P}_2\mathbf{J}_1^+ = 0$ and $\mathbf{P}_2\mathbf{P}_1\mathbf{J}_2^+ = 0$, so

$$\mathbf{J}_3\mathbf{P}_2\mathbf{J}_1^+\mathbf{v}_1 + \mathbf{J}_3\mathbf{P}_2\mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 = \mathbf{J}_3\mathbf{0}\mathbf{v}_1 + \mathbf{J}_3\mathbf{0}\mathbf{v}_2 = 0 = 0 \tag{49}$$

In general, we have the following method for sequential tasks:

$$\dot{\mathbf{q}}_i = \mathbf{P}_{i-1}\mathbf{J}_i^{+}\mathbf{v}_i \tag{50}$$

$$\mathbf{P}_i = \mathbf{P}_{i-1}(\mathbf{I} - \mathbf{J}_i^{+}\mathbf{J}_i) \tag{51}$$

We can see advantages of the approach:

- Complexity does not increase with the number of tasks
- We only need to invert jacobians once

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Fundamentals-of-robotics-2022

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