Jacobians

Fundamentals of Robotics, Lecture 4

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Forward kinematics with special Euclidean group

Consider a two-link robot with frames $\mathcal{F}_i$, bases $\mathcal{T}_i$, joints and origins O_i. We know ${}^{\mathcal{W}}\mathbf{T}_1$ and ${}^{\mathcal{T}_1}\mathbf{T}_2$, as well as ${}^{\mathcal{W}}\mathbf{r}_{O_1}$ and ${}^{\mathcal{T}_1}\mathbf{r}_{O_1 O_2}$ (see last lecture for details).

To express everything in terms of the world frame we do the familiar steps:

$$
{}^{\mathcal{W}}\mathbf{T}_2 = {}^{\mathcal{W}}\mathbf{T}_1 \, {}^{\mathcal{T}_1}\mathbf{T}_2 \tag{1}
$$

$$
{}^{\mathcal{W}}\mathbf{r}_{O_1 O_2} = {}^{\mathcal{W}}\mathbf{T}_1 \, {}^{\mathcal{T}_1}\mathbf{r}_{O_1 O_2} \tag{2}
$$

$$
{}^{\mathcal{W}}\mathbf{r}_{O_2} = {}^{\mathcal{W}}\mathbf{r}_{O_1} + {}^{\mathcal{W}}\mathbf{r}_{O_1 O_2} \tag{3}
$$

Forward kinematics with special Euclidean group

We could do the same thing in the following fashion:

$$
{}^{\mathcal{W}}\mathbf{E}_1 = \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_1 & {}^{\mathcal{W}}\mathbf{r}_{O_1} \\ \mathbf{0} & 1 \end{bmatrix}, \quad {}^{\mathcal{E}_1}\mathbf{E}_2 = \begin{bmatrix} {}^{\mathcal{T}_1}\mathbf{T}_2 & {}^{\mathcal{T}_1}\mathbf{r}_{O_1O_2} \\ \mathbf{0} & 1 \end{bmatrix} \tag{4}
$$

$$
{}^{\mathcal{W}}\mathbf{E}_2 = \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_2 & {}^{\mathcal{W}}\mathbf{r}_{O_2} \\ \mathbf{0} & 1 \end{bmatrix} \tag{5}
$$

$$
{}^{\mathcal{W}}\mathbf{E}_2 = {}^{\mathcal{W}}\mathbf{E}_1 \, {}^{\mathcal{E}_1}\mathbf{E}_2 \tag{6}
$$

$$
\begin{aligned}
{}^{\mathcal{W}}\mathbf{E}_2 &= \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_1 & {}^{\mathcal{W}}\mathbf{r}_{O_1} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} {}^{\mathcal{T}_1}\mathbf{T}_2 & {}^{\mathcal{T}_1}\mathbf{r}_{O_1O_2} \\ \mathbf{0} & 1 \end{bmatrix} = \\
&= \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_1 \, {}^{\mathcal{T}_1}\mathbf{T}_2 & ({}^{\mathcal{W}}\mathbf{T}_1 \, {}^{\mathcal{T}_1}\mathbf{r}_{O_1O_2} + {}^{\mathcal{W}}\mathbf{r}_{O_1}) \\ \mathbf{0} & 1 \end{bmatrix}
\end{aligned}
$$

Forward kinematics with special Euclidean group

In general, we can define a transformation from frame i to the frame $i+1$

$$^{\mathcal{E}_i}\mathbf{E}_{i+1} = \begin{bmatrix} \mathbf{T}_i & \mathbf{p}_i \\ \mathbf{0} & 1 \end{bmatrix} \tag{7}$$

Where $\mathbf{T}_i$ are coordinates of the basis of the frame $i+1$ in terms of frame i, and $\mathbf{p}_i$ is the vector pointing from the origin of the frame i to the origin of the frame $i+1$ expressed in the basis of the frame i.

To get transformation from the world frame to the n-th frame we get:

$$^{\mathcal{W}}\mathbf{E}_n = \prod_{i=1}^{n} {}^{\mathcal{E}_{i-1}}\mathbf{E}_i \tag{8}$$

Last lecture we focused on how to find expressions of radius-vectors (vectors describing positions of points) in world frame, given relative positions and orientations of frames.

Today, we focus on derivatives of these expressions.

Consider vector $\mathbf{r}_K$ describing position of the point K. What is its derivative?

In order to answer this question we need to understand, which parameters appearing in the expression for $\mathbf{r}_K$ are changing with time, and which do not.

Consider an example:

$$\mathbf{r}_K(\mathbf{q}) = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \\ 0 \end{bmatrix} \tag{9}$$

In this example, we know that it is joint coordinate q_1 that is going to change with time.

Another example:

$$\mathbf{r}_K(\mathbf{q}) = \begin{bmatrix} q_1 \\ q_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_3 & -\sin q_3 \\ 0 & \sin q_3 & \cos q_3 \end{bmatrix} \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 + l_1 \cos q_3 \\ l_1 \sin q_3 \end{bmatrix}$$

Now, q_1 and q_2 are translations and q_3 is a rotation, and $\mathbf{q} = [q_1,\ q_2,\ q_3]$ are changing joint coordinates.

So, what is a derivative of $\mathbf{r}_K(\mathbf{q})$ with respect to time?

$$\frac{d}{dt}\mathbf{r}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}}\dot{\mathbf{q}} \tag{10}$$

We can denote $\mathbf{J}_K = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}}$ and call it a jacobian matrix. We can also denote velocity of the point K as $\mathbf{v}_K(\mathbf{q}, \dot{\mathbf{q}}) = \frac{d}{dt}\mathbf{r}_K(\mathbf{q})$.

$$\mathbf{v}_K = \mathbf{J}_K\dot{\mathbf{q}} \tag{11}$$

Notice that velocity $\mathbf{v}_K$ is linear with respect to the joint velocities $\dot{\mathbf{q}}$, and the linear relation is given by the jacobian $\mathbf{J}_K$.

Consider $\mathbf{r}_K = \mathbf{r}_{OO_1} + \mathbf{r}_{O_1O_2} + ... + \mathbf{r}_{O_nK}$. What is a jacobian of $\mathbf{r}_K$?

$$\mathbf{J}_K = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} = \frac{\partial \mathbf{r}_{OO_1}}{\partial \mathbf{q}} + \frac{\partial \mathbf{r}_{O_1O_2}}{\partial \mathbf{q}} + ... + \frac{\partial \mathbf{r}_{O_nK}}{\partial \mathbf{q}} =$$
$$= \mathbf{J}_{OO_1} + \mathbf{J}_{O_1O_2} + ... + \mathbf{J}_{O_nK}$$

Jacobians have an additive structure inherited from the additive structure of the position vectors.

Notice also that if your vectors and jacobians are expressed via coordinates in different bases - you have to express them in a single basis, and then do the additions, as usual.

Let us consider a rigid body rotating with angular velocity ω with basis $\mathcal{T}$, given by a matrix $\mathbf{T}$ (whose coordinates in the world frame basis are ${}^{\mathcal{W}}\mathbf{T}$), stationary with respect to the rigid body. Consider a point K on the link, defined by a vector $\mathbf{r}$. We know coordinates of $\mathbf{r}$ in terms of $\mathcal{T}$, which we denote as ${}^{\mathcal{T}}\mathbf{r}$.

What is velocity of K? By definition of angular velocity, $\mathbf{v} = \omega \times \mathbf{r}$. The same can be represented as a vector-matrix multiplication:

$$\mathbf{v} = \omega \times \mathbf{r} = \Omega \mathbf{r} \tag{12}$$

$$\Omega = [\omega]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{13}$$

Note that expression $\mathbf{v} = \omega \times \mathbf{r}$ works as long as we are talking about vectors themselves, or their coordinate representation in the same basis:

$$\mathbf{v} = \omega \times \mathbf{r} = \Omega \mathbf{r} \tag{14}$$

$$^{\mathcal{W}}\mathbf{v} = ^{\mathcal{W}}\omega \times \ ^{\mathcal{W}}\mathbf{r} = ^{\mathcal{W}}\Omega \ ^{\mathcal{W}}\mathbf{r} \tag{15}$$

$$^{\mathcal{T}}\mathbf{v} = ^{\mathcal{T}}\omega \times \ ^{\mathcal{T}}\mathbf{r} = ^{\mathcal{T}}\Omega \ ^{\mathcal{T}}\mathbf{r} \tag{16}$$

At the same time, we can find position of K as $\mathbf{r} = \mathbf{T}\ {}^{\mathcal{T}}\mathbf{r}$ and ${}^{\mathcal{W}}\mathbf{r} = {}^{\mathcal{W}}\ \mathbf{T}\ {}^{\mathcal{T}}\mathbf{r}$, where $\frac{d}{dt}{}^{\mathcal{T}}\mathbf{r} = 0$. We can find its time derivative as:

$$\mathbf{v} = \frac{d}{dt}\mathbf{r} = \frac{d}{dt}\mathbf{T}\ {}^{\mathcal{T}}\mathbf{r} = \dot{\mathbf{T}}\ {}^{\mathcal{T}}\mathbf{r} \tag{17}$$

$${}^{\mathcal{W}}\mathbf{v} = {}^{\mathcal{W}}\ \dot{\mathbf{T}}\ {}^{\mathcal{T}}\mathbf{r} \tag{18}$$

At the same time, ${}^{\mathcal{W}}\mathbf{v} = {}^{\mathcal{W}}\ \Omega\ {}^{\mathcal{W}}\mathbf{r}$. Hence:

$${}^{\mathcal{W}}\mathbf{v} = {}^{\mathcal{W}}\ \Omega\ {}^{\mathcal{W}}\mathbf{T}\ {}^{\mathcal{T}}\mathbf{r} \tag{19}$$

With that, we know that:

$$\Omega\mathbf{T} = \dot{\mathbf{T}} \tag{20}$$

$$\Omega = \dot{\mathbf{T}}\mathbf{T}^{\top} \tag{21}$$

Notice that $\mathbf{T}^\top\mathbf{T} = \mathbf{I}$, so $\frac{d}{dt}(\mathbf{T}^\top\mathbf{T}) = 0$, and thus $\dot{\mathbf{T}}^\top\mathbf{T} = -\mathbf{T}^\top\dot{\mathbf{T}}$:

$$\Omega = \dot{\mathbf{T}}\mathbf{T}^\top \tag{22}$$

$$\Omega\mathbf{T} = \dot{\mathbf{T}} \tag{23}$$

$$\mathbf{T}^\top\Omega\mathbf{T} = \mathbf{T}^\top\dot{\mathbf{T}} \tag{24}$$

$$\mathbf{T}^\top\Omega\mathbf{T} = -\dot{\mathbf{T}}^\top\mathbf{T} \tag{25}$$

$$\mathbf{T}^\top\Omega = -\dot{\mathbf{T}}^\top \tag{26}$$

$$\Omega = -\mathbf{T}\dot{\mathbf{T}}^\top \tag{27}$$

Note, in these formulas Ω, $\mathbf{T}$ and $\dot{\mathbf{T}}$ are expressed in the same coordinates.

Given Ω, we can find ω:

$$\omega = \text{skew2vec}(\Omega) \tag{28}$$

$$\text{skew2vec}\left(\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \right) = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{29}$$

You can notice that both ω and Ω are linear with respect to $\dot{\mathbf{T}}$, which in turn is linear with respect to $\dot{\mathbf{q}}$.

Angular velocity jacobian

As we mentioned, $\omega = \omega(\mathbf{q}, \dot{\mathbf{q}})$ is linear with respect to $\dot{\mathbf{q}}$:

$$\omega = \frac{\partial \omega}{\partial \dot{\mathbf{q}}} \dot{\mathbf{q}} = \mathbf{J}_\omega \dot{\mathbf{q}} \tag{30}$$

Note that there are big differences in both computation and definition between translation jacobian $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$ and rotational jacobian $\mathbf{J}_\omega \dot{\mathbf{q}}$.

To define translation jacobian we need to define a link and a point on the link, whose jacobian we discuss; for rotation jacobian, we only need to identify the link whose rotation we describe.

Assume we have a vector ${}^{\mathcal{W}}\mathbf{r}(\mathbf{q})$ expressed in the basis $\mathcal{W}$. We can find its jacobian:

$$ {}^{\mathcal{W}}\mathbf{J}(\mathbf{q}) = \frac{\partial\, {}^{\mathcal{W}}\mathbf{r}}{\partial \mathbf{q}} \tag{31} $$

Now, given basis $\mathcal{T}$, expressed by matrix ${}^{\mathcal{W}}\mathbf{T}$, we can represent $\mathbf{r}$ in $\mathcal{T} = const$:

$$ {}^{\mathcal{T}}\mathbf{r}(\mathbf{q}) = {}^{\mathcal{W}}\mathbf{T}^{\top}\, {}^{\mathcal{W}}\mathbf{r}(\mathbf{q}) \tag{32} $$

And of course we can find jacobian of ${}^{\mathcal{T}}\mathbf{r}(\mathbf{q})$:

$$ {}^{\mathcal{T}}\mathbf{J}(\mathbf{q}) = \frac{\partial\, {}^{\mathcal{T}}\mathbf{r}}{\partial \mathbf{q}} = {}^{\mathcal{W}}\mathbf{T}^{\top}\, {}^{\mathcal{W}}\mathbf{J}(\mathbf{q}) \tag{33} $$

We can play the same game with angular velocities ω. Jacobians depend on the bases same as vectors do.

Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Fundamentals-of-robotics-2022

Check Moodle for additional links, videos, textbook suggestions.