

Inverse Kinematics

Fundamentals of Robotics, Lecture 6

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VELOCITY PROBLEM

Given a point (e.g. end effector) K with position given by vector $\mathbf{r}_K(\mathbf{q})$, we can find its velocity:

$$\dot{\mathbf{r}}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \dot{\mathbf{q}} \quad (1)$$

Let us introduce notation:

$$\mathbf{J}_K = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \quad (2)$$

$$\mathbf{v}_K = \dot{\mathbf{r}}_K(\mathbf{q}) \quad (3)$$

Thus we get:

$$\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}} \quad (4)$$

Given $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$, can we find least-residual solution to this problem? Yes!

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K \quad (5)$$

Is this the only solution? No. All solutions are:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N} \mathbf{z} \quad (6)$$

where $\mathbf{N} = \text{null}(\mathbf{J}_K)$ and \mathbf{z} are null space coordinates.

Alternatively, we can use a projector to do the same thing with less new notation:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + (\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K) \dot{\mathbf{q}} \quad (7)$$

where $\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$ is a null space projector.

Now let us find closest joint velocity to $\dot{\mathbf{q}}_0$ that solves the velocity problem $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$:

$$\begin{aligned} & \underset{\dot{\mathbf{q}}}{\text{minimize}} \quad \|\dot{\mathbf{q}} - \dot{\mathbf{q}}_0\|, \\ & \text{subject to} \quad \mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}} \end{aligned} \tag{8}$$

We can solve it by first finding all solutions:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N} \mathbf{z} \tag{9}$$

Then we minimize cost function in terms of the null space variable \mathbf{z} :

$$\|\mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N} \mathbf{z} - \dot{\mathbf{q}}_0\| \rightarrow \min \tag{10}$$

First we simplify $||\mathbf{J}_K^+ \mathbf{v}_K + \mathbf{Nz} - \dot{\mathbf{q}}_0|| \rightarrow \min$ with notation

$$\mathbf{c} = \mathbf{J}_K^+ \mathbf{v}_K - \dot{\mathbf{q}}_0 \quad (11)$$

Then we square $||\mathbf{Nz} + \mathbf{c}||$, and consider its derivative:

$$(\mathbf{Nz} + \mathbf{c})^\top (\mathbf{Nz} + \mathbf{c}) \rightarrow \min \quad (12)$$

$$\frac{\partial}{\partial \mathbf{z}} (\mathbf{Nz} + \mathbf{c})^\top (\mathbf{Nz} + \mathbf{c}) = 0 \quad (13)$$

$$2\mathbf{z}^\top \mathbf{N}^\top \mathbf{N} + 2\mathbf{c}^\top \mathbf{N} = 0 \quad (14)$$

$$\mathbf{z} = -(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \mathbf{c} \quad (15)$$

Knowing that $\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{Nz}$ we get:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \mathbf{c} \quad (16)$$

Let us examine the solution we obtained:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top (\mathbf{J}_K^+ \mathbf{v}_K - \dot{\mathbf{q}}_0) \quad (17)$$

$$\dot{\mathbf{q}} = (\mathbf{I} - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top) \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \dot{\mathbf{q}}_0 \quad (18)$$

Let us examine the matrices:

$$\mathbf{P}_N = \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \quad (19)$$

$$\mathbf{P}_R = \mathbf{I} - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top = \mathbf{I} - \mathbf{P}_N \quad (20)$$

$$\dot{\mathbf{q}} = \mathbf{P}_R \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{P}_N \dot{\mathbf{q}}_0 \quad (21)$$

where \mathbf{P}_N is column space projector for \mathbf{N} , hence it is a null space projector for the jacobian \mathbf{J}_K . And $\mathbf{I} - \mathbf{P}_N$ is a projector to the orthogonal complement, hence it is row space projector.

So, we have eq. $\dot{\mathbf{q}} = \mathbf{P}_R \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{P}_N \dot{\mathbf{q}}_0$ and we have null space projector \mathbf{P}_N and row space projector \mathbf{P}_R .

We know that pseudoinverse lies in the column space, so:

$$\mathbf{P}_R \mathbf{J}_K^+ = \mathbf{J}_K^+ \quad (22)$$

Also we know that null space projector can be found as $\mathbf{P}_N = \mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + (\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K) \dot{\mathbf{q}}_0 \quad (23)$$

Notice, this is almost exactly the same as what we found before. We can interpret it as "the solution is given by row-space least squares solution, plus null space projection of $\dot{\mathbf{q}}_0$ ".

Consider second derivative of the position of the point K :

$$\ddot{\mathbf{r}}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \ddot{\mathbf{q}} + \frac{d}{dt} \left(\frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \right) \dot{\mathbf{q}} \quad (24)$$

Defining $\mathbf{a}_K = \ddot{\mathbf{r}}_K$ we get:

$$\mathbf{a}_K = \mathbf{J}_K \ddot{\mathbf{q}} + \dot{\mathbf{J}}_K \dot{\mathbf{q}} \quad (25)$$

The least residual solution is easily found:

$$\ddot{\mathbf{q}} = \mathbf{J}_K^+ (\mathbf{a}_K - \dot{\mathbf{J}}_K \dot{\mathbf{q}}) \quad (26)$$

Given $\mathbf{a}_K = \mathbf{J}_K \ddot{\mathbf{q}} + \dot{\mathbf{J}}_K \dot{\mathbf{q}}$ let us find acceleration closest to $\ddot{\mathbf{q}}_0$ that solves the acceleration problem:

$$\ddot{\mathbf{q}} = \mathbf{J}_K^+ (\mathbf{a}_K - \dot{\mathbf{J}}_K \dot{\mathbf{q}}) + \mathbf{P}_N \ddot{\mathbf{q}}_0 \quad (27)$$

where $\mathbf{P}_N = \mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$.

What if we want to find such \mathbf{q}^* that $\mathbf{r}_K(\mathbf{q}^*) = \mathbf{r}^*$. Can we do it?

Unlike previous, this is not a linear problem. It often involves trigonometric functions, and other nonlinear ones.

Our approach will be to linearize the expression $\mathbf{r}_K(\mathbf{q})$ and find its solution via an iterative procedure.

Given exact solution \mathbf{q}_0 for problem $\mathbf{r}_K(\mathbf{q}_0) = \mathbf{r}_K(0)$. Then, knowing velocity $\dot{\mathbf{q}}_0$, then we can find an approximation of the position in the next moment of time:

$$\frac{\mathbf{q}_1 - \mathbf{q}_0}{\Delta t} \approx \dot{\mathbf{q}}_0 \quad (28)$$

$$\mathbf{q}_1 \approx \mathbf{q}_0 + \dot{\mathbf{q}}_0 \Delta t \quad (29)$$

This works tolerably well, for improvements we can look to other schemes of solving ODEs.

But what if we don't have an exact solution \mathbf{q}_0 ? After all, it was that which allowed us to use local linearization.

Given initial guess \mathbf{q}_0 we will try to solve the problem $\mathbf{r}_K(\mathbf{q}) = \mathbf{r}_K^*$. First let us define discrepancy:

$$\mathbf{e}(\mathbf{q}) = \mathbf{r}_K(\mathbf{q}) - \mathbf{r}_K^* \quad (30)$$

We define cost function $f = \mathbf{e}^\top \mathbf{e}$, initial position $\mathbf{r}_{K,0} = \mathbf{r}_K(\mathbf{q}_0)$, initial discrepancy $\mathbf{e}_0 = \mathbf{r}_{K,0} - \mathbf{r}_K^*$ and gen. coordinates displacement $\delta = \mathbf{q} - \mathbf{q}_0$ and produce Taylor expansion of the cost:

$$f \approx \mathbf{e}_0^\top \mathbf{e}_0 + \mathbf{e}_0^\top \mathbf{J}_K \delta + \delta^\top \mathbf{J}_K^\top \mathbf{e}_0 + \delta^\top \mathbf{J}_K^\top \mathbf{J}_K \delta \quad (31)$$

Now we take derivative and set it to zero:

$$2\mathbf{e}_0^\top \mathbf{J}_K + 2\delta^\top \mathbf{J}_K^\top \mathbf{J}_K = 0 \quad (32)$$

We obtained expression:

$$\delta = -(\mathbf{J}_K^\top \mathbf{J}_K)^{-1} \mathbf{J}_K^\top \mathbf{e}_0 \quad (33)$$

And remembering the substitutions we made we get:

$$\mathbf{q} - \mathbf{q}_0 = -(\mathbf{J}_K^\top \mathbf{J}_K)^{-1} \mathbf{J}_K^\top (\mathbf{r}_{K,0} - \mathbf{r}_K^*) \quad (34)$$

$$\mathbf{q} = \mathbf{q}_0 - \mathbf{J}_K^+ (\mathbf{r}_{K,0} - \mathbf{r}_K^*) \quad (35)$$

We can use the final expression to update our initial guess, then the-linearize the problem at the new position and repeat the process, until we converge.

How can we check if a velocity \mathbf{v}_K can be achieved, given $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$?

If \mathbf{v}_K lies in the column space of \mathbf{J}_K , it is achievable:

$$(\mathbf{I} - \mathbf{J}_K \mathbf{J}_K^+) \mathbf{v}_K = \mathbf{0} \quad (36)$$

How can we check if a solution $\dot{\mathbf{q}}$ is minimal-norm, given $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$?

If $\dot{\mathbf{q}}$ lies in the row space of \mathbf{J}_K , it is minimal:

$$(\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K) \dot{\mathbf{q}} = 0 \quad (37)$$

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Fundamentals-of-robotics-2022

Check Moodle for additional links, videos, textbook suggestions.

