# Robotics with Contact

## Fundamentals of Robotics, Lecture 12

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# Constraints

Assume the dynamics has to obey a certain equation, e.g.:

$$\mathbf{g}(\mathbf{q}) = 0 \tag{1}$$

This can happen, for example, if part of a robot is attached to the ground. Bolting the end-effector of a robot arm to the ground makes it into a parallel robot.

Equation (1) can be differentiated to find constraints on velocity or acceleration:

$$\mathbf{F}\dot{\mathbf{q}} = 0 \tag{2}$$

$$\mathbf{F}\ddot{\mathbf{q}} + \dot{\mathbf{F}}\dot{\mathbf{q}} = 0 \tag{3}$$

where $\mathbf{F} = \frac{\partial \mathbf{g}}{\partial \mathbf{q}}$.

Remember - without constraints the dynamics looks like $\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau$. Presence of constraints adds reaction forces $\lambda$:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \mathbf{F}^\top \lambda \tag{4}$$

Let us note that this equation is not solvable without the acceleration constraint equations. With constraints it has the following form:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \mathbf{F}^\top \lambda \\ \mathbf{F}\ddot{\mathbf{q}} + \dot{\mathbf{F}}\dot{\mathbf{q}} = 0 \end{cases} \tag{5}$$

As long as $\mathbf{F}$ has independent rows, it has only one solution.

We can re-write the equation in a block-matrix form:

$$\begin{bmatrix} \mathbf{H} & -\mathbf{F}^\top \\ -\mathbf{F} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g} \\ \dot{\mathbf{F}}\dot{\mathbf{q}} \end{bmatrix} \tag{6}$$

The block matrix on the left-hand side is invertible (as long as $\mathbf{F}$ has full row-rank).

## Constraints - dynamics

Dynamics equations can be solved:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \mathbf{F}^\top \lambda \tag{7}$$

$$\ddot{\mathbf{q}} = \mathbf{H}^{-1}\mathbf{F}^\top \lambda + \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \tag{8}$$

$$-\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top \lambda - \mathbf{F}\mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) = \dot{\mathbf{F}}\dot{\mathbf{q}} \tag{9}$$

$$-\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top \lambda = \mathbf{F}\mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \dot{\mathbf{F}}\dot{\mathbf{q}} \tag{10}$$

We define $\mathbf{f}_1 = \tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}$ and $\mathbf{f}_2 = \dot{\mathbf{F}}\dot{\mathbf{q}}$:

$$-\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top \lambda = \mathbf{F}\mathbf{H}^{-1}\mathbf{f}_1 + \mathbf{f}_2 \tag{11}$$

$$\lambda = -(\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top)^{-1}(\mathbf{F}\mathbf{H}^{-1}\mathbf{f}_1 - \mathbf{f}_2) \tag{12}$$

$$\ddot{\mathbf{q}} = -\mathbf{H}^{-1}\mathbf{F}^\top(\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top)^{-1}(\mathbf{F}\mathbf{H}^{-1}\mathbf{f}_1 - \mathbf{f}_2) + \mathbf{H}^{-1}\mathbf{f}_1 \tag{13}$$

$$\ddot{\mathbf{q}} = (\mathbf{I} - \mathbf{H}^{-1}\mathbf{F}^\top(\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top)^{-1}\mathbf{F}\mathbf{H}^{-1})\mathbf{f}_1 - \tag{14}$$

$$-\mathbf{H}^{-1}\mathbf{F}^\top(\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top)^{-1}\mathbf{F}\mathbf{H}^{-1}\mathbf{f}_2 \tag{15}$$

So, we have:

$$\begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{I} - \mathbf{H}^{-1}\mathbf{F}^{\top}\mathbf{M}\mathbf{F}\mathbf{H}^{-1} & \mathbf{H}^{-1}\mathbf{F}^{\top}\mathbf{M}\mathbf{F}\mathbf{H}^{-1} \\ \mathbf{M}\mathbf{F}\mathbf{H}^{-1} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \tag{16}$$

where $\mathbf{M} = (\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^{\top})^{-1}$.

# Friction cone

Note that reaction forces cannot assume arbitrary values. If $\mathbf{f}$ is a reaction force acting at a contact point, then constraints on possible values of reaction forces are given as:

$$||\mathbf{T}^\top \mathbf{f}|| \leq \mu \mathbf{n}^\top \mathbf{f} \tag{17}$$

where $\mathbf{T} \in \mathbb{R}^{3 \times 2}$ is a basis in a tangent plane on the supporting surface, $\mathbf{n}$ is a normal to the supporting surface and $\mu$ is a friction coefficient.

For a horizontal surface with friction coefficient $\mu = 0.5$ and $\mathbf{f} = [f_x \ f_y \ f_z]$ the equation loos like:

$$\sqrt{f_x^2 + f_y^2} \leq 0.5 f_z \tag{18}$$

Alternatively, a linear approximation can be used:

$$\mathbf{A}\mathbf{f} \leq \mathbf{b} \tag{19}$$

For a horizontal surface with friction coefficient $\mu = 1$ approximation of the cone by a pyramid with 4 faces can look like:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

Constraints simulating physical contact can be written as inequalities:

$$\mathbf{g}(\mathbf{q}) \geq 0 \tag{21}$$

When $\mathbf{g}(\mathbf{q}) > 0$, the reaction force $\lambda = 0$, and conversely - if $\lambda > 0$, then $\mathbf{g}(\mathbf{q}) = 0$. This can be described via complimentary constraint:

$$\begin{cases} \mathbf{g}(\mathbf{q})^\top \lambda = 0 \\ \mathbf{g}(\mathbf{q}) \geq 0 \\ \lambda \geq 0 \end{cases} \tag{22}$$

Given nominal acceleration $\ddot{\mathbf{q}}^*$ and nominal reaction forces $\lambda^*$, we can formulate reactive control

$$\begin{aligned}
\underset{\ddot{\mathbf{q}},\ \lambda}{\text{minimize}} \quad & ||\ddot{\mathbf{q}}||_{\mathbf{Q}_1} + ||\lambda||_{\mathbf{Q}_2} && (23) \\
\text{subject to:} \quad & \mathbf{H}(\ddot{\mathbf{q}}^* + \ddot{\mathbf{q}}) + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \mathbf{F}^\top(\lambda^* + \lambda) && (24) \\
& \mathbf{F}(\ddot{\mathbf{q}}^* + \ddot{\mathbf{q}}) + \dot{\mathbf{F}}\dot{\mathbf{q}} = 0 && (25)
\end{aligned}$$

where $|| \cdot ||_{\mathbf{Q}_i}$ is a weighted norm with weight matrix $\mathbf{Q}_i$. This control law does not guarantee stability and requires solving a quadratic problem on each iteration.

For a constrained mechanical system we can solve inverse dynamics without the need for linearization. Consider the following dynamics:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{B}\mathbf{u} + \mathbf{F}^\top \lambda \tag{26}$$

We can represent constraint Jacobian $\mathbf{F}^\top$ as its QR decomposition: $\mathbf{F}^\top = \mathbf{Q}\begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}$, where $\mathbf{Q}^\top\mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$ and $\mathbf{R}$ is convertible.

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{B}\mathbf{u} + \mathbf{Q}\begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}\lambda \tag{27}$$

Let us multiply the equation by $\mathbf{Q}^\top$:

$$\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{Q}^\top\mathbf{B}\mathbf{u} + \begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}\lambda \tag{28}$$

Introducing switching variables (to divide upper and lower part of the equations) $\mathbf{S}_1 = \begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}$ and $\mathbf{S}_2 = \begin{bmatrix}\mathbf{0} & \mathbf{I}\end{bmatrix}$ and multiplying equations by one and the other we get two systems:

$$\begin{cases}\mathbf{S}_1\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{S}_1\mathbf{Q}^\top\mathbf{B}\mathbf{u} + \mathbf{R}\lambda \\ \mathbf{S}_2\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{S}_2\mathbf{Q}^\top\mathbf{B}\mathbf{u}\end{cases} \tag{29}$$

The main advantage we achieved is that now we can calculate both $\mathbf{u}$ and $\lambda$

Resulting expression for $\mathbf{u}$ is:

$$\mathbf{u} = (\mathbf{S}_2 \mathbf{Q}^\top \mathbf{B})^+ \mathbf{S}_2 \mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) \tag{30}$$

Expression for $\lambda$ is:

$$\lambda = \mathbf{R}^{-1} \mathbf{S}_1 \mathbf{Q}^\top (\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} - \mathbf{B}\mathbf{u}) \tag{31}$$

We can notice a pseudo-inverse, implying that the no-residual solution does not have to exist.

# Read more


- Righetti, L., Buchli, J., Mistry, M. and Schaal, S., 2011, May. Inverse dynamics control of floating-base robots with external constraints: A unified view. In 2011 IEEE international conference on robotics and automation (pp. 1085-1090). IEEE.
- Mistry, M., Buchli, J. and Schaal, S., 2010, May. Inverse dynamics control of floating base systems using orthogonal decomposition. In 2010 IEEE international conference on robotics and automation (pp. 3406-3412). IEEE.
- Righetti, L., Buchli, J., Mistry, M., Kalakrishnan, M. and Schaal, S., 2013. Optimal distribution of contact forces with inverse-dynamics control. The International Journal of Robotics Research, 32(3), pp.280-298.
- Nakanishi, J., Mistry, M. and Schaal, S., 2007, April. Inverse dynamics control with floating base and constraints. In Proceedings 2007 IEEE International Conference on Robotics and Automation (pp. 1942-1947). IEEE.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Fundamentals-of-robotics-2022

Check Moodle for additional links, videos, textbook suggestions.

Prove that $\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top$ is full rank if $\mathbf{F}$ is full row-rank and $\mathbf{H} > 0$ - positive-definite.

We can prove that $\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top$ is positive-definite:

$$J = \mathbf{x}^\top \mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top \mathbf{x} \tag{32}$$

Defining $\mathbf{y} = \mathbf{F}^\top \mathbf{x}$. Since $\mathbf{F}$ is full row-rank, then $||\mathbf{x}|| \neq 0$ implies $||\mathbf{y}|| \neq 0$ because $\mathbf{F}^\top$ has trivial null space. We get: $J = \mathbf{y}^\top \mathbf{H}^{-1} \mathbf{y}$ which is positive for any non-zero $\mathbf{y}$ (because $\mathbf{H}$ is positive-definite), meaning $J > 0$ for any non-zero $\mathbf{x}$. Since $\mathbf{F}\mathbf{H}^{-1}\mathbf{F}^\top > 0$, it implies full-rank.

This is called *congruent transformation*, which leaves positive-definiteness in-tact.