

Compliance control, Force control

Fundamentals of Robotics, Lecture 10

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Let us consider a task $\mathbf{r}_K = \mathbf{r}_K(\mathbf{q})$. We can differentiate it twice:

$$\dot{\mathbf{r}}_K = \mathbf{J}_K \dot{\mathbf{q}} \quad (1)$$

$$\ddot{\mathbf{r}}_K = \mathbf{J}_K \ddot{\mathbf{q}} + \dot{\mathbf{J}}_K \dot{\mathbf{q}} \quad (2)$$

But remember, we know what robot dynamics is:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \boldsymbol{\tau} \quad (3)$$

Expressing $\ddot{\mathbf{q}}$ and substituting it to $\ddot{\mathbf{r}}_K$ we get:

$$\ddot{\mathbf{r}}_K = \mathbf{J}_K \mathbf{H}^{-1}(\boldsymbol{\tau} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \dot{\mathbf{J}}_K \dot{\mathbf{q}} \quad (4)$$

If \mathbf{J}_K is full rank (which happens with some robot arms) we can rewrite the last equation further:

$$\ddot{\mathbf{r}}_K = \mathbf{J}_K \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \dot{\mathbf{J}}_K \dot{\mathbf{q}} \quad (5)$$

$$\mathbf{H}\mathbf{J}_K^{-1}\ddot{\mathbf{r}}_K = (\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \mathbf{H}\mathbf{J}_K^{-1}\dot{\mathbf{J}}_K\dot{\mathbf{q}} \quad (6)$$

Then we can multiply both sides by $\mathbf{J}_K^{-\top}$ and define $\mathbf{J}_K^{\top}\mathbf{f}_K = \tau$ we get:

$$\mathbf{J}_K^{-\top}\mathbf{H}\mathbf{J}_K^{-1}\ddot{\mathbf{r}}_K = \mathbf{J}_K^{-\top}\tau - \mathbf{J}_K^{-\top}\mathbf{C}\dot{\mathbf{q}} - \mathbf{J}_K^{-\top}\mathbf{g} + \mathbf{J}_K^{-\top}\mathbf{H}\mathbf{J}_K^{-1}\dot{\mathbf{J}}_K\dot{\mathbf{q}} \quad (7)$$

$$\mathbf{J}_K^{-\top}\mathbf{H}\mathbf{J}_K^{-1}\ddot{\mathbf{r}}_K = \mathbf{f}_K - \mathbf{J}_K^{-\top}\mathbf{C}\dot{\mathbf{q}} - \mathbf{J}_K^{-\top}\mathbf{g} + \mathbf{J}_K^{-\top}\mathbf{H}\mathbf{J}_K^{-1}\dot{\mathbf{J}}_K\dot{\mathbf{q}} \quad (8)$$

We define operation space inertial matrix $\mathbf{H}_K = \mathbf{J}_K^{-\top} \mathbf{H} \mathbf{J}_K^{-1}$:

$$\mathbf{H}_K \ddot{\mathbf{r}}_K = \mathbf{f}_K - \mathbf{J}_K^{-\top} \mathbf{C} \dot{\mathbf{q}} - \mathbf{J}_K^{-\top} \mathbf{g} + \mathbf{H}_K \dot{\mathbf{J}}_K \dot{\mathbf{q}} \quad (9)$$

$$\mathbf{H}_K \ddot{\mathbf{r}}_K + \mathbf{J}_K^{-\top} \mathbf{C} \dot{\mathbf{q}} - \mathbf{H}_K \dot{\mathbf{J}}_K \dot{\mathbf{q}} + \mathbf{J}_K^{-\top} \mathbf{g} = \mathbf{f}_K \quad (10)$$

We additionally define:

$$\mathbf{J}_K^{-\top} \mathbf{C} \dot{\mathbf{q}} - \mathbf{H}_K \dot{\mathbf{J}}_K \dot{\mathbf{q}} = \mathbf{C}_K \dot{\mathbf{r}}_K \quad (11)$$

$$\mathbf{J}_K^{-\top} \mathbf{g} = \mathbf{g}_K \quad (12)$$

Which gets us to the operation-space equations:

$$\mathbf{H}_K \ddot{\mathbf{r}}_K + \mathbf{C}_K \dot{\mathbf{r}}_K + \mathbf{g}_K = \mathbf{f}_K \quad (13)$$

Let us look at these equations closely:

$$\mathbf{H}_K \ddot{\mathbf{r}}_K + \mathbf{C}_K \dot{\mathbf{r}}_K + \mathbf{g}_K = \mathbf{f}_K \quad (14)$$

- They tell us how the task will evolve under the influence of input forces \mathbf{f}_K
- Operation-space inertia matrix is full rank as long as \mathbf{J}_K is full rank.
- As long as \mathbf{J}_K is full rank, there is one-to-one correspondence between the operational space and joint space dynamics.

Assume the original dynamics includes external force \mathbf{f}_e with jacobian \mathbf{J}_e :

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \boldsymbol{\tau} + \mathbf{J}_e^\top \mathbf{f}_e \quad (15)$$

Then we can write the corresponding operational space dynamics:

$$\mathbf{H}_K \ddot{\mathbf{r}}_K + \mathbf{C}_K \dot{\mathbf{r}}_K + \mathbf{g}_K = \mathbf{f}_K + \mathbf{T}_e \mathbf{f}_e \quad (16)$$

where $\mathbf{T}_e = \mathbf{J}_K^{-\top} \mathbf{J}_e^\top$.

If external force is elastic, proportional to the displacement in \mathbf{r}_K , it can be re-written as:

$$\mathbf{f}_e = \mathbf{K}_e(\mathbf{r}_K^* - \mathbf{r}_K) = -\mathbf{K}_e\mathbf{e}_K \quad (17)$$

where $\mathbf{e}_K = \mathbf{r}_K - \mathbf{r}_K^*$ - displacement in \mathbf{r}_K and \mathbf{K}_e is a full-rank stiffness matrix.

If $\mathbf{r}_K^* = \text{const}$, then $\dot{\mathbf{e}}_K = \dot{\mathbf{r}}_K$ and $\ddot{\mathbf{e}}_K = \ddot{\mathbf{r}}_K$, and the dynamics assumes the form:

$$\mathbf{H}_K\ddot{\mathbf{e}}_K + \mathbf{C}_K\dot{\mathbf{e}}_K + \mathbf{g}_K = \mathbf{f}_K - \mathbf{T}_e\mathbf{K}_e\mathbf{e}_K \quad (18)$$

$$\mathbf{H}_K\ddot{\mathbf{e}}_K + \mathbf{C}_K\dot{\mathbf{e}}_K + \mathbf{T}_e\mathbf{K}_e\mathbf{e}_K + \mathbf{g}_K = \mathbf{f}_K \quad (19)$$

We propose Lyapunov function:

$$V = \frac{1}{2} \dot{\mathbf{e}}_K^\top \mathbf{H}_K \dot{\mathbf{e}}_K + \frac{1}{2} \mathbf{e}_K^\top \mathbf{K}_p \mathbf{e}_K \quad (20)$$

Let us find its time-derivative:

$$\begin{aligned} \dot{V} &= \dot{\mathbf{e}}_K^\top \mathbf{H}_K \ddot{\mathbf{e}}_K + \frac{1}{2} \dot{\mathbf{e}}_K^\top \dot{\mathbf{H}}_K \dot{\mathbf{e}}_K + \dot{\mathbf{e}}_K^\top \mathbf{K}_p \mathbf{e}_K \\ \dot{V} &= \dot{\mathbf{e}}_K^\top \mathbf{f}_K - \dot{\mathbf{e}}_K^\top (\mathbf{C}_K \dot{\mathbf{e}}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K + \mathbf{g}_K) + \\ &\quad + \frac{1}{2} \dot{\mathbf{e}}_K^\top \dot{\mathbf{H}}_K \dot{\mathbf{e}}_K + \dot{\mathbf{e}}_K^\top \mathbf{K}_p \mathbf{e}_K \\ \dot{V} &= \dot{\mathbf{e}}_K^\top \mathbf{f}_K - \dot{\mathbf{e}}_K^\top \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K - \dot{\mathbf{e}}_K^\top \mathbf{g}_K + \dot{\mathbf{e}}_K^\top \mathbf{K}_p \mathbf{e}_K \end{aligned}$$

We can propose the following control law

$$\mathbf{f}_K = \mathbf{g}_K - \mathbf{K}_p \mathbf{e}_K - \mathbf{K}_d \dot{\mathbf{e}}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K \quad (21)$$

where \mathbf{K}_p , \mathbf{K}_d are positive-definite matrices. Then, derivative of the Lyapunov function is:

$$\begin{aligned} \dot{V} &= \dot{\mathbf{e}}_K^\top (\mathbf{g}_K - \mathbf{K}_p \mathbf{e}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K - \mathbf{K}_d \dot{\mathbf{e}}_K) - \\ &\quad - \dot{\mathbf{e}}_K^\top \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K - \dot{\mathbf{e}}_K^\top \mathbf{g}_K + \dot{\mathbf{e}}_K^\top \mathbf{K}_p \mathbf{e}_K \\ \dot{V} &= -\dot{\mathbf{e}}_K^\top \mathbf{K}_d \dot{\mathbf{e}}_K \end{aligned}$$

We can see that $\dot{V} \leq 0$.

We can consider the fixed points $\dot{\mathbf{e}}_K = 0$, $\ddot{\mathbf{e}}_K = 0$:

$$\begin{aligned}\mathbf{H}_K \ddot{\mathbf{e}}_K + \mathbf{C}_K \dot{\mathbf{e}}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K + \mathbf{g}_K &= \mathbf{g}_K - \mathbf{K}_p \mathbf{e}_K - \mathbf{K}_d \dot{\mathbf{e}}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K \\ \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K + \mathbf{g}_K &= \mathbf{g}_K - \mathbf{K}_p \mathbf{e}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K \\ 0 &= -\mathbf{K}_p \mathbf{e}_K\end{aligned}$$

Since \mathbf{K}_p is full rank, therefore $\mathbf{e}_K = 0$. So, the system is asymptotically stable.

$$\mathbf{f}_K = \mathbf{g}_K - \mathbf{K}_p \mathbf{e}_K - \mathbf{K}_d \dot{\mathbf{e}}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K \quad (22)$$

$$\tau = \mathbf{J}_K^\top \mathbf{g}_K - \mathbf{J}_K^\top \mathbf{K}_p \mathbf{e}_K - \mathbf{J}_K^\top \mathbf{K}_d \dot{\mathbf{e}}_K + \mathbf{J}_K^\top \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K \quad (23)$$

$$\tau = \mathbf{J}_K^\top \mathbf{g}_K + (\mathbf{J}_K^\top \mathbf{T}_e \mathbf{K}_e - \mathbf{J}_K^\top \mathbf{K}_p) \mathbf{e}_K - \mathbf{J}_K^\top \mathbf{K}_d \mathbf{J}_e \dot{\mathbf{q}} \quad (24)$$

$$\tau = \mathbf{J}_K^\top \mathbf{J}_K^{-\top} \mathbf{g} + (\mathbf{J}_K^\top \mathbf{J}_K^{-\top} \mathbf{J}_e^\top \mathbf{K}_e - \mathbf{J}_K^\top \mathbf{K}_p) \mathbf{e}_K - \mathbf{J}_K^\top \mathbf{K}_d \mathbf{J}_e \dot{\mathbf{q}} \quad (25)$$

$$\tau = \mathbf{g} + (\mathbf{J}_e^\top \mathbf{K}_e - \mathbf{J}_K^\top \mathbf{K}_p) \mathbf{e}_K - \mathbf{J}_K^\top \mathbf{K}_d \mathbf{J}_e \dot{\mathbf{q}} \quad (26)$$

In a case when $\mathbf{J}_e = \mathbf{J}_K = \mathbf{J}$, we get:

$$\tau = \mathbf{g} + \mathbf{J}^\top (\mathbf{K}_e - \mathbf{K}_p) \mathbf{e}_K - \mathbf{J}^\top \mathbf{K}_d \mathbf{J} \dot{\mathbf{q}} \quad (27)$$

Resulting control law:

$$\tau = \mathbf{g} + \mathbf{J}^\top (\mathbf{K}_e - \mathbf{K}_p) \mathbf{e}_K - \mathbf{J}^\top \mathbf{K}_d \mathbf{J} \dot{\mathbf{q}} \quad (28)$$

results in a robot acting as if the contact interaction is governed by an elastic force with stiffness matrix $\mathbf{K} = \mathbf{K}_e - \mathbf{K}_p$.

For a system with no natural elasticity \mathbf{K}_e we can still achieve elastic-like behavior by choosing \mathbf{K}_p .

You can read more at *Siciliano, B., Sciavicco, L., Villani, L. and Oriolo, G., 2009. Robotics. Advanced textbooks in control and signal processing*, Chapter 9.2

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Fundamentals-of-robotics-2022

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