

# Task Prioritization

## Fundamentals of Robotics, Lecture 7

by Sergei Savin

Fall 2022

Consider a point  $K$  (position is given by vector  $\mathbf{r}_K(\mathbf{q})$ ). We can formulate the velocity problem as such - find such  $\dot{\mathbf{q}}^*$  that:

$$\mathbf{J}_K \dot{\mathbf{q}}^* = \mathbf{v}_K \quad (1)$$

We can solve velocity problem for the point  $K$ :

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N} \mathbf{z} \quad (2)$$

Now, assume that we want to also find such velocity that one of the links acquires angular velocity  $\omega$ :

$$\mathbf{J}_\omega \dot{\mathbf{q}}^* = \omega \quad (3)$$

How do we solve it? We can easily combine both problems together and then we get:

$$\begin{bmatrix} \mathbf{J}_K \\ \mathbf{J}_\omega \end{bmatrix} \dot{\mathbf{q}}^* = \begin{bmatrix} \mathbf{v}_K \\ \omega \end{bmatrix} \quad (4)$$

Which is solved as:

$$\dot{\mathbf{q}}^* = \begin{bmatrix} \mathbf{J}_K \\ \mathbf{J}_\omega \end{bmatrix}^+ \begin{bmatrix} \mathbf{v}_K \\ \omega \end{bmatrix} \quad (5)$$

If a solution exists, we will obtain it. But if a solution does not exist, we will find a least-residual solution, with errors distributed across both *tasks*. What if we want to make sure one task is achieved, while the second task can be failed?

# LINEAR VELOCITY AND ANGULAR VELOCITY

As we noted before, all solutions to the velocity problem are given as:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{Nz} \quad (6)$$

Then we can re-write the angular velocity problem  $\mathbf{J}_\omega \dot{\mathbf{q}}^* = \omega$  as:

$$\mathbf{J}_\omega (\mathbf{J}_K^+ \mathbf{v}_K + \mathbf{Nz}) = \omega \quad (7)$$

This can be solved:

$$\mathbf{J}_\omega \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{J}_\omega \mathbf{Nz} = \omega \quad (8)$$

$$\mathbf{z} = (\mathbf{J}_\omega \mathbf{N})^{-1} (\omega - \mathbf{J}_\omega \mathbf{J}_K^+ \mathbf{v}_K) \quad (9)$$

$$\dot{\mathbf{q}}^{**} = \mathbf{N} (\mathbf{J}_\omega \mathbf{N})^{-1} (\omega - \mathbf{J}_\omega \mathbf{J}_K^+ \mathbf{v}_K) + \mathbf{J}_K^+ \mathbf{v}_K \quad (10)$$

Considering the solution

$\dot{\mathbf{q}}^{**} = \mathbf{N}(\mathbf{J}_\omega \mathbf{N})^{-1}(\omega - \mathbf{J}_\omega \mathbf{J}_K^+ \mathbf{v}_K) + \mathbf{J}_K^+ \mathbf{v}_K$  we can observe the following:

- The solution looks a little ugly.
- It looks like it will only get worse if we consider the third task.

We can do better.

By now we know how to solve a problem of the type  $\mathbf{J}_1 \dot{\mathbf{q}} = \mathbf{v}_1$ .  
The solution is:

$$\dot{\mathbf{q}}_1 = \mathbf{J}_1^+ \mathbf{v}_1 \quad (11)$$

We add second task  $\mathbf{J}_2(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2$ . Can we solve it, while keeping the solution to the first task  $\mathbf{J}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_1$ ?

Our proposition: we solve an alternative task:

$$\mathbf{J}_2 \mathbf{P}_1 (\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2 \quad (12)$$

$$\mathbf{P}_1 = \mathbf{I} - \mathbf{J}_1^+ \mathbf{J}_1 \quad (13)$$

We claim that the first and the second task will both be satisfied (if possible) even if we solve them *sequentially*.

Let us study the proposed solution. First, the joint velocities  $\dot{\mathbf{q}}_1$  are found as  $\dot{\mathbf{q}}_1 = \mathbf{J}_1^+ \mathbf{v}_1$ .

Second, matrix  $\mathbf{P}_1 = \mathbf{I} - \mathbf{J}_1^+ \mathbf{J}_1$  is a null space projector for the jacobian  $\mathbf{J}_1$ .

Third, we consider equation  $\mathbf{J}_2 \mathbf{P}_1 (\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2$ :

$$\mathbf{J}_2 \mathbf{P}_1 (\mathbf{J}_1^+ \mathbf{v}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2 \quad (14)$$

Since  $\mathbf{J}_1^+ \in \text{row}(\mathbf{J}_1)$  we can conclude that  $\mathbf{P}_1 \mathbf{J}_1^+ = 0$ . Finally, we have the equation in the form:

$$\mathbf{J}_2 \mathbf{P}_1 \dot{\mathbf{q}}_2 = \mathbf{v}_2 \quad (15)$$

Given  $\mathbf{J}_2 \mathbf{P}_1 \dot{\mathbf{q}}_2 = \mathbf{v}_2$  we can solve it:

$$\mathbf{P}_1 \dot{\mathbf{q}}_2 = \mathbf{J}_2^+ \mathbf{v}_2 \quad (16)$$

$$\mathbf{P}_1 = \mathbf{C}_1 \mathbf{C}_1^\top \quad (17)$$

$$\mathbf{C}_1 \mathbf{C}_1^\top \dot{\mathbf{q}}_2 = \mathbf{J}_2^+ \mathbf{v}_2 \quad (18)$$

$$\mathbf{C}_1^\top \dot{\mathbf{q}}_2 = \mathbf{C}_1^\top \mathbf{J}_2^+ \mathbf{v}_2 \quad (19)$$

$$\dot{\mathbf{q}}_2 = \mathbf{C}_1 \mathbf{C}_1^\top \mathbf{J}_2^+ \mathbf{v}_2 \quad (20)$$

$$\dot{\mathbf{q}}_2 = \mathbf{P}_1 \mathbf{J}_2^+ \mathbf{v}_2 \quad (21)$$

We can summarize it as  $(\mathbf{J}_2 \mathbf{P}_1)^+ = \mathbf{P}_1 \mathbf{J}_2^+$ .



So, The solution is:

$$\dot{\mathbf{q}}_1 = \mathbf{J}_1^+ \mathbf{v}_1 \quad (22)$$

$$\dot{\mathbf{q}}_2 = \mathbf{P}_1 \mathbf{J}_2^+ \mathbf{v}_2 \quad (23)$$

Can we prove that the following holds?

$$\begin{cases} \mathbf{J}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_1 \\ \mathbf{J}_2 \mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2 \end{cases} \quad (24)$$

We study equation  $\mathbf{J}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_1$ :

$$\mathbf{J}_1\dot{\mathbf{q}}_1 + \mathbf{J}_1\mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 = \mathbf{v}_1 \quad (25)$$

Assuming that  $\mathbf{J}_1\dot{\mathbf{q}}_1 = \mathbf{v}_1$  (the residual is zero), we get:

$$\mathbf{J}_1\mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 = 0 \quad (26)$$

Since  $\mathbf{P}_1$  is null space projector it means that its columns lie in the null space of  $\mathbf{J}_1$  - meaning that  $\mathbf{J}_1\mathbf{P}_1 = 0$ , q.e.d.

We study equation  $\mathbf{J}_2\mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = \mathbf{v}_2$ :

$$\mathbf{J}_2\mathbf{P}_1\mathbf{J}_1^+\mathbf{v}_1 + \mathbf{J}_2\mathbf{P}_1\dot{\mathbf{q}}_2 = \mathbf{v}_2 \quad (27)$$

If  $\mathbf{J}_2\mathbf{P}_1\dot{\mathbf{q}}_2 = \mathbf{v}_2$ , meaning that we found a zero-residual solution, then:

$$\mathbf{J}_2\mathbf{P}_1\mathbf{J}_1^+\mathbf{v}_1 = 0 \quad (28)$$

Since  $\mathbf{J}_1^+$  is in the row space of  $\mathbf{J}_1$ , so  $\mathbf{P}_1\mathbf{J}_1^+ = 0$ .

$$\mathbf{J}_2\mathbf{0}\mathbf{v}_1 = 0 \quad (29)$$

$$0 = 0, \quad \text{q.e.d.} \quad (30)$$

Thus, we proved that the proposed solution works.

### 3 TASKS

Let us try to solve three tasks, one after another:

$$\mathbf{J}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_1 \quad (31)$$

$$\mathbf{J}_2\mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_2 \quad (32)$$

$$\mathbf{J}_3\mathbf{P}_2(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_3 \quad (33)$$

$$\mathbf{P}_1 = \mathbf{I} - \mathbf{J}_1^+\mathbf{J}_1 \quad (34)$$

$$\mathbf{P}_2 = \mathbf{P}_1(\mathbf{I} - \mathbf{J}_2^+\mathbf{J}_2) \quad (35)$$

Where  $\mathbf{P}_1$  is a projector onto the null space of  $\mathbf{J}_1$ , and  $\mathbf{P}_2$  is a projector onto the intersection of null spaces of  $\mathbf{J}_1$  and  $\mathbf{J}_2$ .

We propose to solve it as:

$$\dot{\mathbf{q}}_1 = \mathbf{J}_1^+\mathbf{v}_1 \quad (36)$$

$$\dot{\mathbf{q}}_2 = \mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 \quad (37)$$

$$\dot{\mathbf{q}}_3 = \mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3 \quad (38)$$

First equation:  $\mathbf{J}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_1$ .

If the first task has a zero-residual  $\mathbf{J}_1\dot{\mathbf{q}}_1 = \mathbf{v}_1$ , we obtain:

$$\text{(what we want to prove)} \quad \mathbf{J}_1(\dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = 0 \quad (39)$$

$$\mathbf{J}_1(\mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 + \mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3) = 0 \quad (40)$$

We can observe  $\mathbf{J}_1\mathbf{P}_1 = 0$  and  $\mathbf{J}_1\mathbf{P}_2 = \mathbf{J}_1\mathbf{P}_1(\mathbf{I} - \mathbf{J}_2^+\mathbf{J}_2) = 0$ :

$$\mathbf{J}_1(\mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 + \mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3) = \mathbf{J}_1(0 + 0) = 0 \quad (41)$$

Second equation:  $\mathbf{J}_2\mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_2$ .

Given that  $\mathbf{J}_2\mathbf{P}_1\dot{\mathbf{q}}_2 = \mathbf{v}_2$ , we obtain:

$$\text{(what we want to prove)} \quad \mathbf{J}_2\mathbf{P}_1(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_3) = 0 \quad (42)$$

$$\mathbf{J}_2\mathbf{P}_1(\mathbf{J}_1^+\mathbf{v}_1 + \mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3) = 0 \quad (43)$$

$$\mathbf{J}_2\mathbf{P}_1\mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3 = 0 \quad (44)$$

$$\mathbf{J}_2\mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3 = 0 \quad (45)$$

Matrix  $\mathbf{P}_2$  is null space projector for the  $\mathbf{J}_2$ , further projected onto the null space of  $\mathbf{J}_1$ ; hence  $\mathbf{J}_2\mathbf{P}_2 = 0$ :

$$\mathbf{J}_2\mathbf{P}_2\mathbf{J}_3^+\mathbf{v}_3 = \mathbf{0}\mathbf{J}_3^+\mathbf{v}_3 = 0 = 0 \quad (46)$$

Third equation:  $\mathbf{J}_3\mathbf{P}_2(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2 + \dot{\mathbf{q}}_3) = \mathbf{v}_3$ .

Given that  $\mathbf{J}_3\mathbf{P}_2\dot{\mathbf{q}}_3 = \mathbf{v}_3$ , we obtain:

$$\text{(what we want to prove)} \quad \mathbf{J}_3\mathbf{P}_2(\dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) = 0 \quad (47)$$

$$\mathbf{J}_3\mathbf{P}_2(\mathbf{J}_1^+\mathbf{v}_1 + \mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2) = 0 \quad (48)$$

We can observe that  $\mathbf{P}_2\mathbf{J}_1^+ = 0$  and  $\mathbf{P}_2\mathbf{P}_1\mathbf{J}_2^+ = 0$ , so

$$\mathbf{J}_3\mathbf{P}_2\mathbf{J}_1^+\mathbf{v}_1 + \mathbf{J}_3\mathbf{P}_2\mathbf{P}_1\mathbf{J}_2^+\mathbf{v}_2 = \mathbf{J}_3\mathbf{0}\mathbf{v}_1 + \mathbf{J}_3\mathbf{0}\mathbf{v}_2 = 0 = 0 \quad (49)$$

In general, we have the following method for sequential tasks:

$$\dot{\mathbf{q}}_i = \mathbf{P}_{i-1} \mathbf{J}_i^+ \mathbf{v}_i \quad (50)$$

$$\mathbf{P}_i = \mathbf{P}_{i-1} (\mathbf{I} - \mathbf{J}_i^+ \mathbf{J}_i) \quad (51)$$

We can see advantages of the approach:

- Complexity does not increase with the number of tasks
- We only need to invert jacobians once



# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Fundamentals-of-robotics-2022](https://github.com/SergeiSa/Fundamentals-of-robotics-2022)

Check Moodle for additional links, videos, textbook suggestions.

