# Inverse Kinematics
## Fundamentals of Robotics, Lecture 6

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Given a point (e.g. end effector) $K$ with position given by vector $\mathbf{r}_K(\mathbf{q})$, we can find its velocity:

$$\dot{\mathbf{r}}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \dot{\mathbf{q}} \tag{1}$$

Let us introduce notation:

$$\mathbf{J}_K = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \tag{2}$$

$$\mathbf{v}_K = \dot{\mathbf{r}}_K(\mathbf{q}) \tag{3}$$

Thus we get:

$$\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}} \tag{4}$$

Given $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$, can we find least-residual solution to this problem? Yes!

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K \tag{5}$$

Is this the only solution? No. All solutions are:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z} \tag{6}$$

where $\mathbf{N} = \text{null}(\mathbf{J}_K)$ and $\mathbf{z}$ are null space coordinates.
Alternatively, we can use a projector to do the same thing with less new notation:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + (\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K)\dot{\mathbf{q}} \tag{7}$$

where $\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$ is a null space projector.

Now let us find closest joint velocity to $\dot{\mathbf{q}}_0$ that solves the velocity problem $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$:

$$
\begin{aligned}
\underset{\dot{\mathbf{q}}}{\text{minimize}} \quad & ||\dot{\mathbf{q}} - \dot{\mathbf{q}}_0||, \\
\text{subject to} \quad & \mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}
\end{aligned}
\tag{8}
$$

We can solve it by first finding all solutions:

$$
\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z}
\tag{9}
$$

Then we minimize cost function in terms of the null space variable $\mathbf{z}$:

$$
||\mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z} - \dot{\mathbf{q}}_0|| \to min
\tag{10}
$$

# Optimal velocity problem

First we simplify $||\mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z} - \dot{\mathbf{q}}_0|| \to min$ with notation

$$\mathbf{c} = \mathbf{J}_K^+ \mathbf{v}_K - \dot{\mathbf{q}}_0 \qquad (11)$$

Then we square $||\mathbf{N}\mathbf{z} + \mathbf{c}||$, and consider its derivative:

$$(\mathbf{N}\mathbf{z} + \mathbf{c})^\top (\mathbf{N}\mathbf{z} + \mathbf{c}) \to min \qquad (12)$$

$$\frac{\partial}{\partial \mathbf{z}} (\mathbf{N}\mathbf{z} + \mathbf{c})^\top (\mathbf{N}\mathbf{z} + \mathbf{c}) = 0 \qquad (13)$$

$$2\mathbf{z}^\top \mathbf{N}^\top \mathbf{N} + 2\mathbf{c}^\top \mathbf{N} = 0 \qquad (14)$$

$$\mathbf{z} = -(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \mathbf{c} \qquad (15)$$

Knowing that $\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z}$ we get:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \mathbf{c} \qquad (16)$$

Let us examine the solution we obtained:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1}\mathbf{N}^\top (\mathbf{J}_K^+ \mathbf{v}_K - \dot{\mathbf{q}}_0) \qquad (17)$$

$$\dot{\mathbf{q}} = (\mathbf{I} - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1}\mathbf{N}^\top)\mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1}\mathbf{N}^\top \dot{\mathbf{q}}_0 \qquad (18)$$

Let us examine the matrices:

$$\mathbf{P}_N = \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1}\mathbf{N}^\top \qquad (19)$$

$$\mathbf{P}_R = \mathbf{I} - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1}\mathbf{N}^\top = \mathbf{I} - \mathbf{P}_N \qquad (20)$$

$$\dot{\mathbf{q}} = \mathbf{P}_R \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{P}_N \dot{\mathbf{q}}_0 \qquad (21)$$

where $\mathbf{P}_N$ is column space projector for $\mathbf{N}$, hence it is a null space projector for the jacobian $\mathbf{J}_K$. And $\mathbf{I} - \mathbf{P}_N$ is a projector to the orthogonal compliment, hence it is row space projector.

So, we have eq. $\dot{\mathbf{q}} = \mathbf{P}_R \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{P}_N \dot{\mathbf{q}}_0$ and we have null space projector $\mathbf{P}_N$ and row space projector $\mathbf{P}_R$.

We know that pseudoinverse lies in the column space, so:

$$\mathbf{P}_R \mathbf{J}_K^+ = \mathbf{J}_K^+ \tag{22}$$

Also we know that null space projector can be found as $\mathbf{P}_N = \mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + (\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K)\dot{\mathbf{q}}_0 \tag{23}$$

Notice, this is almost exactly the same as what we found before. We can interpret it as "the solution is given by row-space least squares solution, plus null space projection of $\dot{\mathbf{q}}_0$".

Consider second derivative of the position of the point $K$:

$$\ddot{\mathbf{r}}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \ddot{\mathbf{q}} + \frac{d}{dt}\left(\frac{\partial \mathbf{r}_K}{\partial \mathbf{q}}\right)\dot{\mathbf{q}} \tag{24}$$

Defining $\mathbf{a}_K = \ddot{\mathbf{r}}_K$ we get:

$$\mathbf{a}_K = \mathbf{J}_K \ddot{\mathbf{q}} + \dot{\mathbf{J}}_K \dot{\mathbf{q}} \tag{25}$$

The least residual solution is easily found:

$$\ddot{\mathbf{q}} = \mathbf{J}_K^+ (\mathbf{a}_K - \dot{\mathbf{J}}_K \dot{\mathbf{q}}) \tag{26}$$

Given $\mathbf{a}_K = \mathbf{J}_K \ddot{\mathbf{q}} + \dot{\mathbf{J}}_K \dot{\mathbf{q}}$ let us find acceleration closest to $\ddot{\mathbf{q}}_0$ that solves the acceleration problem:

$$\ddot{\mathbf{q}} = \mathbf{J}_K^+ (\mathbf{a}_K - \dot{\mathbf{J}}_K \dot{\mathbf{q}}) + \mathbf{P}_N \ddot{\mathbf{q}}_0 \qquad (27)$$

where $\mathbf{P}_N = \mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$.

What if we want to find such $\mathbf{q}^*$ that $\mathbf{r}_K(\mathbf{q}^*) = \mathbf{r}^*$. Can we do it?

Unlike previous, this is not a linear problem. It often involves trigonometric functions, and other nonlinear ones.

Our approach will be to linearize the expression $\mathbf{r}_K(\mathbf{q})$ and find its solution via an iterative procedure.

Given exact solution $\mathbf{q}_0$ for problem $\mathbf{r}_K(\mathbf{q}_0) = \mathbf{r}_K(0)$. Then, knowing velocity $\dot{\mathbf{q}}_0$, then we can find an approximation of the position in the next moment of time:

$$\frac{\mathbf{q}_1 - \mathbf{q}_0}{\Delta t} \approx \dot{\mathbf{q}}_0 \tag{28}$$

$$\mathbf{q}_1 \approx \mathbf{q}_0 + \dot{\mathbf{q}}_0 \Delta t \tag{29}$$

This works tolerably well, for improvements we can look to other schemes of solving ODEs.

But what if we don't have an exact solution $\mathbf{q}_0$? After all, it was that which allowed us to use local linearization.

Given initial guess $\mathbf{q}_0$ we will try to solve the problem $\mathbf{r}_K(\mathbf{q}) = \mathbf{r}_K^*$. First let us define discrepancy:

$$\mathbf{e}(\mathbf{q}) = \mathbf{r}_K(\mathbf{q}) - \mathbf{r}_K^* \qquad (30)$$

We define cost function $f = \mathbf{e}^\top \mathbf{e}$, initial position $\mathbf{r}_{K,0} = \mathbf{r}_K(\mathbf{q}_0)$, initial discrepancy $\mathbf{e}_0 = \mathbf{r}_{K,0} - \mathbf{r}_K^*$ and gen. coordinates displacement $\delta = \mathbf{q} - \mathbf{q}_0$ and produce Taylor expansion of the cost:

$$f \approx \mathbf{e}_0^\top \mathbf{e}_0 + \mathbf{e}_0^\top \mathbf{J}_K \delta + \delta^\top \mathbf{J}_K^\top \mathbf{e}_0 + \delta^\top \mathbf{J}_K^\top \mathbf{J}_K \delta \qquad (31)$$

Now we take derivative and set it to zero:

$$2\mathbf{e}_0^\top \mathbf{J}_K + 2\delta^\top \mathbf{J}_K^\top \mathbf{J}_K = 0 \qquad (32)$$

We obtained expression:

$$\delta = -(\mathbf{J}_K^\top \mathbf{J}_K)^{-1}\mathbf{J}_K^\top \mathbf{e}_0 \tag{33}$$

And remembering the substitutions we made we get:

$$\mathbf{q} - \mathbf{q}_0 = -(\mathbf{J}_K^\top \mathbf{J}_K)^{-1}\mathbf{J}_K^\top (\mathbf{r}_{K,0} - \mathbf{r}_K^*) \tag{34}$$

$$\mathbf{q} = \mathbf{q}_0 - \mathbf{J}_K^+ (\mathbf{r}_{K,0} - \mathbf{r}_K^*) \tag{35}$$

We can use the final expression to update our initial guess, then the-linearize the problem at the new position and repeat the process, until we converge.

How can we check if a velocity $\mathbf{v}_K$ can be achieved, given $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$?

If $\mathbf{v}_K$ lies in the column space of $\mathbf{J}_K$, it is achievable:

$$(\mathbf{I} - \mathbf{J}_K \mathbf{J}_K^+)\mathbf{v}_K = 0 \tag{36}$$

How can we check if a solution $\dot{\mathbf{q}}$ is minimal-norm, given $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$?

If $\dot{\mathbf{q}}$ lies in the row space of $\mathbf{J}_K$, it is minimal:

$$(\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K)\dot{\mathbf{q}} = 0 \qquad (37)$$

# Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Fundamentals-of-robotics-2022

Check Moodle for additional links, videos, textbook suggestions.