Compliance control, Force control

Fundamentals of Robotics, Lecture 10

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Let us consider a task $\mathbf{r}_K = \mathbf{r}_K(\mathbf{q})$. We can differentiate it twice:

$$\dot{\mathbf{r}}_K = \mathbf{J}_K \dot{\mathbf{q}} \tag{1}$$

$$\ddot{\mathbf{r}}_K = \mathbf{J}_K \ddot{\mathbf{q}} + \dot{\mathbf{J}}_K \dot{\mathbf{q}} \tag{2}$$

But remember, we know what robot dynamics is:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau \tag{3}$$

Expressing $\ddot{\mathbf{q}}$ and substituting it to $\ddot{\mathbf{r}}_K$ we get:

$$\ddot{\mathbf{r}}_K = \mathbf{J}_K \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \dot{\mathbf{J}}_K \dot{\mathbf{q}} \tag{4}$$

If $\mathbf{J}_K$ is full rank (which happens with some robot arms) we can rewrite the last equation further:

$$\ddot{\mathbf{r}}_K = \mathbf{J}_K \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \dot{\mathbf{J}}_K \dot{\mathbf{q}} \tag{5}$$

$$\mathbf{H}\mathbf{J}_K^{-1}\ddot{\mathbf{r}}_K = (\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \mathbf{H}\mathbf{J}_K^{-1}\dot{\mathbf{J}}_K \dot{\mathbf{q}} \tag{6}$$

Then we can multiply both sides by $\mathbf{J}_K^{-\top}$ and define $\mathbf{J}_K^{\top}\mathbf{f}_K = \tau$ we get:

$$\mathbf{J}_K^{-\top}\mathbf{H}\mathbf{J}_K^{-1}\ddot{\mathbf{r}}_K = \mathbf{J}_K^{-\top}\tau - \mathbf{J}_K^{-\top}\mathbf{C}\dot{\mathbf{q}} - \mathbf{J}_K^{-\top}\mathbf{g} + \mathbf{J}_K^{-\top}\mathbf{H}\mathbf{J}_K^{-1}\dot{\mathbf{J}}_K \dot{\mathbf{q}} \tag{7}$$

$$\mathbf{J}_K^{-\top}\mathbf{H}\mathbf{J}_K^{-1}\ddot{\mathbf{r}}_K = \mathbf{f}_K - \mathbf{J}_K^{-\top}\mathbf{C}\dot{\mathbf{q}} - \mathbf{J}_K^{-\top}\mathbf{g} + \mathbf{J}_K^{-\top}\mathbf{H}\mathbf{J}_K^{-1}\dot{\mathbf{J}}_K \dot{\mathbf{q}} \tag{8}$$

We define operation space inertial matrix $\mathbf{H}_K = \mathbf{J}_K^{-\top} \mathbf{H} \mathbf{J}_K^{-1}$:

$$\mathbf{H}_K \ddot{\mathbf{r}}_K = \mathbf{f}_K - \mathbf{J}_K^{-\top} \mathbf{C} \dot{\mathbf{q}} - \mathbf{J}_K^{-\top} \mathbf{g} + \mathbf{H}_K \dot{\mathbf{J}}_K \dot{\mathbf{q}} \tag{9}$$

$$\mathbf{H}_K \ddot{\mathbf{r}}_K + \mathbf{J}_K^{-\top} \mathbf{C} \dot{\mathbf{q}} - \mathbf{H}_K \dot{\mathbf{J}}_K \dot{\mathbf{q}} + \mathbf{J}_K^{-\top} \mathbf{g} = \mathbf{f}_K \tag{10}$$

We additionally define:

$$\mathbf{J}_K^{-\top} \mathbf{C} \dot{\mathbf{q}} - \mathbf{H}_K \dot{\mathbf{J}}_K \dot{\mathbf{q}} = \mathbf{C}_K \dot{\mathbf{r}}_K \tag{11}$$

$$\mathbf{J}_K^{-\top} \mathbf{g} = \mathbf{g}_K \tag{12}$$

Which gets us to the operation-space equations:

$$\mathbf{H}_K \ddot{\mathbf{r}}_K + \mathbf{C}_K \dot{\mathbf{r}}_K + \mathbf{g}_K = \mathbf{f}_K \tag{13}$$

Let us look at these equations closely:

$$\mathbf{H}_K \ddot{\mathbf{r}}_K + \mathbf{C}_K \dot{\mathbf{r}}_K + \mathbf{g}_K = \mathbf{f}_K \tag{14}$$

- They tell as how the task will evolve under the influence of input forces $\mathbf{f}_K$
- Operation-space inertia matrix is full rank as long as $\mathbf{J}_K$ is full rank.
- As long as $\mathbf{J}_K$ is full rank, there is one-to-one correspondence between the operational space and joint space dynamics.

Assume the original dynamics includes external force $\mathbf{f}_e$ with jacobian $\mathbf{J}_e$:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau + \mathbf{J}_e^\top \mathbf{f}_e \tag{15}$$

Then we can write the corresponding operational space dynamics:

$$\mathbf{H}_K \ddot{\mathbf{r}}_K + \mathbf{C}_K \dot{\mathbf{r}}_K + \mathbf{g}_K = \mathbf{f}_K + \mathbf{T}_e \mathbf{f}_e \tag{16}$$

where $\mathbf{T}_e = \mathbf{J}_K^{-\top} \mathbf{J}_e^\top$.

If external force is elastic, proportional to the displacement in $\mathbf{r}_K$, it can be re-written as:

$$\mathbf{f}_e = \mathbf{K}_e(\mathbf{r}_K^* - \mathbf{r}_K) = -\mathbf{K}_e \mathbf{e}_K \tag{17}$$

where $\mathbf{e}_K = \mathbf{r}_K - \mathbf{r}_K^*$ - displacement in $\mathbf{r}_K$ and $\mathbf{K}_e$ is a full-rank stiffness matrix.

If $\mathbf{r}_K^* = const$, then $\dot{\mathbf{e}}_K = \dot{\mathbf{r}}_K$ and $\ddot{\mathbf{e}}_K = \ddot{\mathbf{r}}_K$, and the dynamics assumes the form:

$$\mathbf{H}_K \ddot{\mathbf{e}}_K + \mathbf{C}_K \dot{\mathbf{e}}_K + \mathbf{g}_K = \mathbf{f}_K - \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K \tag{18}$$

$$\mathbf{H}_K \ddot{\mathbf{e}}_K + \mathbf{C}_K \dot{\mathbf{e}}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K + \mathbf{g}_K = \mathbf{f}_K \tag{19}$$

We propose Lyapunov function:

$$V = \frac{1}{2}\dot{\mathbf{e}}_K^\top \mathbf{H}_K \dot{\mathbf{e}}_K + \frac{1}{2}\mathbf{e}_K^\top \mathbf{K}_p \mathbf{e}_K \tag{20}$$

Let us find its time-derivative:

$$\begin{aligned}
\dot{V} &= \dot{\mathbf{e}}_K^\top \mathbf{H}_K \ddot{\mathbf{e}}_K + \frac{1}{2}\dot{\mathbf{e}}_K^\top \dot{\mathbf{H}}_K \dot{\mathbf{e}}_K + \dot{\mathbf{e}}_K^\top \mathbf{K}_p \mathbf{e}_K \\
\dot{V} &= \dot{\mathbf{e}}_K^\top \mathbf{f}_K - \dot{\mathbf{e}}_K^\top (\mathbf{C}_K \dot{\mathbf{e}}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K + \mathbf{g}_K) + \\
&\quad + \frac{1}{2}\dot{\mathbf{e}}_K^\top \dot{\mathbf{H}}_K \dot{\mathbf{e}}_K + \dot{\mathbf{e}}_K^\top \mathbf{K}_p \mathbf{e}_K \\
\dot{V} &= \dot{\mathbf{e}}_K^\top \mathbf{f}_K - \dot{\mathbf{e}}_K^\top \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K - \dot{\mathbf{e}}_K^\top \mathbf{g}_K + \dot{\mathbf{e}}_K^\top \mathbf{K}_p \mathbf{e}_K
\end{aligned}$$

We can propose the following control law

$$\mathbf{f}_K = \mathbf{g}_K - \mathbf{K}_p \mathbf{e}_K - \mathbf{K}_d \dot{\mathbf{e}}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K \tag{21}$$

where $\mathbf{K}_p$, $\mathbf{K}_d$ are positive-definite matrices. Then, derivative of the Lyapunov function is:

$$\begin{aligned}\dot{V} &= \dot{\mathbf{e}}_K^\top (\mathbf{g}_K - \mathbf{K}_p \mathbf{e}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K - \mathbf{K}_d \dot{\mathbf{e}}_K) - \\ &- \dot{\mathbf{e}}_K^\top \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K - \dot{\mathbf{e}}_K^\top \mathbf{g}_K + \dot{\mathbf{e}}_K^\top \mathbf{K}_p \mathbf{e}_K \\ \dot{V} &= -\dot{\mathbf{e}}_K^\top \mathbf{K}_d \dot{\mathbf{e}}_K\end{aligned}$$

We can see that $\dot{V} \leq 0$.

We can consider the fixed points $\dot{\mathbf{e}}_K = 0$, $\ddot{\mathbf{e}}_K = 0$:

$$\begin{aligned}
\mathbf{H}_K\ddot{\mathbf{e}}_K + \mathbf{C}_K\dot{\mathbf{e}}_K + \mathbf{T}_e\mathbf{K}_e\mathbf{e}_K + \mathbf{g}_K &= \mathbf{g}_K - \mathbf{K}_p\mathbf{e}_K - \mathbf{K}_d\dot{\mathbf{e}}_K + \mathbf{T}_e\mathbf{K}_e\mathbf{e}_K \\
\mathbf{T}_e\mathbf{K}_e\mathbf{e}_K + \mathbf{g}_K &= \mathbf{g}_K - \mathbf{K}_p\mathbf{e}_K + \mathbf{T}_e\mathbf{K}_e\mathbf{e}_K \\
0 &= -\mathbf{K}_p\mathbf{e}_K
\end{aligned}$$

Since $\mathbf{K}_p$ is full rank, therefore $\mathbf{e}_K = 0$. So, the system is asymptotically stable.

$$\mathbf{f}_K = \mathbf{g}_K - \mathbf{K}_p \mathbf{e}_K - \mathbf{K}_d \dot{\mathbf{e}}_K + \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K \tag{22}$$

$$\tau = \mathbf{J}_K^\top \mathbf{g}_K - \mathbf{J}_K^\top \mathbf{K}_p \mathbf{e}_K - \mathbf{J}_K^\top \mathbf{K}_d \dot{\mathbf{e}}_K + \mathbf{J}_K^\top \mathbf{T}_e \mathbf{K}_e \mathbf{e}_K \tag{23}$$

$$\tau = \mathbf{J}_K^\top \mathbf{g}_K + (\mathbf{J}_K^\top \mathbf{T}_e \mathbf{K}_e - \mathbf{J}_K^\top \mathbf{K}_p) \mathbf{e}_K - \mathbf{J}_K^\top \mathbf{K}_d \mathbf{J}_e \dot{\mathbf{q}} \tag{24}$$

$$\tau = \mathbf{J}_K^\top \mathbf{J}_K^{-\top} \mathbf{g} + (\mathbf{J}_K^\top \mathbf{J}_K^{-\top} \mathbf{J}_e^\top \mathbf{K}_e - \mathbf{J}_K^\top \mathbf{K}_p) \mathbf{e}_K - \mathbf{J}_K^\top \mathbf{K}_d \mathbf{J}_e \dot{\mathbf{q}} \tag{25}$$

$$\tau = \mathbf{g} + (\mathbf{J}_e^\top \mathbf{K}_e - \mathbf{J}_K^\top \mathbf{K}_p) \mathbf{e}_K - \mathbf{J}_K^\top \mathbf{K}_d \mathbf{J}_e \dot{\mathbf{q}} \tag{26}$$

In a case when $\mathbf{J}_e = \mathbf{J}_K = \mathbf{J}$, we get:

$$\tau = \mathbf{g} + \mathbf{J}^\top (\mathbf{K}_e - \mathbf{K}_p) \mathbf{e}_K - \mathbf{J}^\top \mathbf{K}_d \mathbf{J} \dot{\mathbf{q}} \tag{27}$$

Resulting control law:

$$\tau = \mathbf{g} + \mathbf{J}^\top(\mathbf{K}_e - \mathbf{K}_p)\mathbf{e}_K - \mathbf{J}^\top \mathbf{K}_d \mathbf{J}\dot{\mathbf{q}} \tag{28}$$

results in a robot acting as if the contact interaction is governed by an elastic force with stiffness matrix $\mathbf{K} = \mathbf{K}_e - \mathbf{K}_p$.

For a system with no natural elasticity $\mathbf{K}_e$ we can still achieve elastic-like behavior by choosing $\mathbf{K}_p$.

Read more

You can read more at *Siciliano, B., Sciavicco, L., Villani, L. and Oriolo, G., 2009. Robotics. Advanced textbooks in control and signal processing*, Chapter 9.2

Thank you!

Lecture slides are available via Moodle.

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