Nonlinear Control, PD, Gravity compensation, Lyapunov

Fundamentals of Robotics, Lecture 9

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Asymptotic stability criteria

Autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is assymptotically stable, if there exists a scalar function $V = V(\mathbf{x}) > 0$, whose time derivative is negative $\dot{V}(\mathbf{x}) < 0$, except $V(\mathbf{0}) = 0$, $\dot{V}(\mathbf{0}) = 0$.

Marginal stability criteria

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is stable in the sense of Lyapunov, $\exists V(\mathbf{x}) > 0$, $\dot{V}(\mathbf{x}) \leq 0$.

Definition

Function $V(\mathbf{x}) > 0$ in this case is called *Lyapunov function.*

This is not the only type of stability as you remember, you are invited to study criteria for other stability types on your own.

Basic second-order system

Let us consider the following basic system:

$$\ddot{\mathbf{q}} + \mathbf{K}_d \dot{\mathbf{q}} + \mathbf{K}_p \mathbf{q} = 0 \tag{1}$$

where $\mathbf{K}_d > 0$ and $\mathbf{K}_p > 0$ are semidefinite matrices. Can we analyze its stability?

We propose Lyapunov function

$$V = \frac{1}{2}\dot{\mathbf{q}}^\top \dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^\top \mathbf{K}_p \mathbf{q} \tag{2}$$

Derivative of the Lyapunov function is:

$$\dot{V} = \dot{\mathbf{q}}^\top \ddot{\mathbf{q}} + \mathbf{q}^\top \mathbf{K}_p \dot{\mathbf{q}} \tag{3}$$

But we know that $\ddot{\mathbf{q}} = -\mathbf{K}_d \dot{\mathbf{q}} - \mathbf{K}_p \mathbf{q}$, so:

$$\dot{V} = -\dot{\mathbf{q}}^\top (\mathbf{K}_d \dot{\mathbf{q}} + \mathbf{K}_p \mathbf{q}) + \mathbf{q}^\top \mathbf{K}_p \dot{\mathbf{q}} \tag{4}$$

$$\dot{V} = -\dot{\mathbf{q}}^\top \mathbf{K}_d \dot{\mathbf{q}} \tag{5}$$

This means that the system is at least marginally stable: $\dot{V} \leq 0$, namely $\dot{V} = 0$ for $\dot{\mathbf{q}} = 0$ and $\forall \mathbf{q}$.

Now, we want to prove that the only trajectory in the fixed point ($\dot{\mathbf{q}} = 0$, $\ddot{\mathbf{q}} = 0$) is $\mathbf{q} = 0$.

Consider the fixed point $\dot{\mathbf{q}} = 0$, $\ddot{\mathbf{q}} = 0$:

$$\ddot{\mathbf{q}} + \mathbf{K}_d \dot{\mathbf{q}} + \mathbf{K}_p \mathbf{q} = 0 \quad \rightarrow \quad \mathbf{K}_p \mathbf{q} = 0 \tag{6}$$

Since $\mathbf{K}_p > 0$, it is full rank and has a trivial null space, so $\mathbf{q} = 0$ is the only trajectory in the fixed point space. So, the system is asymptotically stable by LaSalle's invariance principle.

Now, we can consider manipulator equations:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau \tag{7}$$

We propose a Lyapunov function:

$$V = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{H}\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^\top \mathbf{K}_p \mathbf{q} \tag{8}$$

$$\dot{V} = \dot{\mathbf{q}}^\top \mathbf{H}\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^\top \dot{\mathbf{H}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^\top \mathbf{K}_p \mathbf{q} \tag{9}$$

$$\dot{V} = \dot{\mathbf{q}}^\top (\tau - \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) + \frac{1}{2}\dot{\mathbf{q}}^\top \dot{\mathbf{H}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^\top \mathbf{K}_p \mathbf{q} \tag{10}$$

$$\dot{V} = \frac{1}{2}\dot{\mathbf{q}}^\top (\dot{\mathbf{H}} - 2\mathbf{C})\dot{\mathbf{q}} + \dot{\mathbf{q}}^\top (\tau - \mathbf{g}) + \dot{\mathbf{q}}^\top \mathbf{K}_p \mathbf{q} \tag{11}$$

Looking at the expression:

$$\dot{V} = \frac{1}{2}\dot{\mathbf{q}}^\top(\dot{\mathbf{H}} - 2\mathbf{C})\dot{\mathbf{q}} + \dot{\mathbf{q}}^\top(\tau - \mathbf{g}) + \dot{\mathbf{q}}^\top\mathbf{K}_p\mathbf{q}$$

we can observe that $\dot{\mathbf{H}} - 2\mathbf{C}$ is skew-symmetric, and therefore $\dot{\mathbf{q}}^\top(\dot{\mathbf{H}} - 2\mathbf{C})\dot{\mathbf{q}} = 0$
We can propose the following τ:

$$\tau = \mathbf{g} - \mathbf{K}_d\dot{\mathbf{q}} - \mathbf{K}_p\mathbf{q} \tag{12}$$

Then the Lyapunov function derivative takes form:

$$\dot{V} = -\dot{\mathbf{q}}^\top\mathbf{K}_d\dot{\mathbf{q}} - \dot{\mathbf{q}}^\top\mathbf{K}_p\mathbf{q} + \dot{\mathbf{q}}^\top\mathbf{K}_p\mathbf{q} \tag{13}$$

$$\dot{V} = -\dot{\mathbf{q}}^\top\mathbf{K}_d\dot{\mathbf{q}} \leq 0 \tag{14}$$

So, $\dot{V} = -\dot{\mathbf{q}}^\top \mathbf{K}_d \dot{\mathbf{q}} \leq 0$. More precisely $\dot{V} = 0$ for $\dot{\mathbf{q}} = 0$ and $\forall \mathbf{q}$, giving us marginal stability.

Let us study the fixed points $\dot{\mathbf{q}} = 0$ and $\ddot{\mathbf{q}} = 0$ for the manipulator eq.:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{g} - \mathbf{K}_d \dot{\mathbf{q}} - \mathbf{K}_p \mathbf{q} \tag{15}$$

$$\mathbf{g} = \mathbf{g} - \mathbf{K}_p \mathbf{q} \tag{16}$$

$$\mathbf{K}_p \mathbf{q} = 0 \tag{17}$$

$$\mathbf{q} = 0 \tag{18}$$

We get that the only trajectory in the fixed space is $\mathbf{q} = 0$. So, the system is asymptotically stable by LaSalle's invariance principle.

The control that we proposed:

$$\tau = \mathbf{g} - \mathbf{K}_d \dot{\mathbf{q}} - \mathbf{K}_p \mathbf{q} \tag{19}$$

is gravity compensation and PD control. We have proven that it is stable.

If the system has linear dissipation force (viscous force) $\mathbf{F}\dot{\mathbf{q}}$:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{F}\dot{\mathbf{q}} + \mathbf{g} = \tau \tag{20}$$

...we can use proportional control without derivative component:

$$\tau = \mathbf{g} - \mathbf{K}_p \mathbf{q} \tag{21}$$

Where the same Lyapunov function will obtain derivative:

$$\dot{V} = -\dot{\mathbf{q}}^\top \mathbf{F}_d \dot{\mathbf{q}} \leq 0 \tag{22}$$

With the same conclusions as before. An example of system for which we can use it is industrial robot arms with gear box reducers.

Let us introduce *desired position* $\mathbf{q}^*$ and *position error* $\mathbf{e}$:

$$\mathbf{e} = \mathbf{q}^* - \mathbf{q} \tag{23}$$

If we are lucky, the *error dynamics* is evolving in accord with the following equation:

$$\ddot{\mathbf{e}} + \mathbf{K}_d \dot{\mathbf{e}} + \mathbf{K}_p \mathbf{e} = 0 \tag{24}$$

where $\mathbf{K}_d > 0$ and $\mathbf{K}_p > 0$. This system is asymptotically stable, meaning that $\mathbf{e} \to 0$ and $\mathbf{q} \to \mathbf{q}^*$.

Let us find what $\ddot{\mathbf{e}}$ is:

$$\ddot{\mathbf{e}} = \ddot{\mathbf{q}}^* - \ddot{\mathbf{q}} \tag{25}$$

$$\ddot{\mathbf{e}} = \ddot{\mathbf{q}}^* - \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \tag{26}$$

Then error dynamics is:

$$\ddot{\mathbf{q}}^* - \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \mathbf{K}_d\dot{\mathbf{e}} + \mathbf{K}_p\mathbf{e} = 0 \tag{27}$$

$$\mathbf{H}\ddot{\mathbf{q}}^* - (\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \mathbf{H}(\mathbf{K}_d\dot{\mathbf{e}} + \mathbf{K}_p\mathbf{e}) = 0 \tag{28}$$

$$\tau = \mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} + \mathbf{H}(\mathbf{K}_d\dot{\mathbf{e}} + \mathbf{K}_p\mathbf{e}) \tag{29}$$

So, we found a control law that makes error dynamics stable, meaning $\mathbf{q} \to \mathbf{q}^*$.

Often, it does not make sense to try to compute $\mathbf{H}$ and $\mathbf{C}\dot{\mathbf{q}} + \mathbf{g}$ based on current position of the robot. Instead, we can compute it based on the desired trajectory:

$$\mathbf{H}^* = \mathbf{H}(\mathbf{q}^*) \tag{30}$$

$$\mathbf{C}^* = \mathbf{C}(\dot{\mathbf{q}}^*, \mathbf{q}^*) \tag{31}$$

$$\mathbf{g}^* = \mathbf{g}(\mathbf{q}^*) \tag{32}$$

We can define inverse dynamics τ^* as:

$$\tau^* = \mathbf{H}^*\ddot{\mathbf{q}}^* + \mathbf{C}^*\dot{\mathbf{q}}^* + \mathbf{g}^* \tag{33}$$

Then, the control law can be re-written as:

$$\tau = \tau^* + \mathbf{H}^*(\mathbf{K}_d\dot{\mathbf{e}} + \mathbf{K}_p\mathbf{e}) \tag{34}$$

But, this way we do not have any stability guarantee.

Read more

You can read more at *Siciliano, B., Sciavicco, L., Villani, L. and Oriolo, G., 2009. Robotics. Advanced textbooks in control and signal processing*, Chapter 8.5

Thank you!

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