

Jacobians

Fundamentals of Robotics, Lecture 4

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Fall 2022

FORWARD KINEMATICS WITH SPECIAL EUCLIDEAN GROUP

Consider a two-link robot with frames \mathcal{F}_i , bases \mathcal{T}_i , joints and origins O_i . We know ${}^{\mathcal{W}}\mathbf{T}_1$ and ${}^{\mathcal{T}_1}\mathbf{T}_2$, as well as ${}^{\mathcal{W}}\mathbf{r}_{O_1}$ and ${}^{\mathcal{T}_1}\mathbf{r}_{O_1O_2}$ (see last lecture for details).

To express everything in terms of the world frame we do the familiar steps:

$${}^{\mathcal{W}}\mathbf{T}_2 = {}^{\mathcal{W}}\mathbf{T}_1 {}^{\mathcal{T}_1}\mathbf{T}_2 \tag{1}$$

$${}^{\mathcal{W}}\mathbf{r}_{O_1O_2} = {}^{\mathcal{W}}\mathbf{T}_1 {}^{\mathcal{T}_1}\mathbf{r}_{O_1O_2} \tag{2}$$

$${}^{\mathcal{W}}\mathbf{r}_{O_2} = {}^{\mathcal{W}}\mathbf{r}_{O_1} + {}^{\mathcal{W}}\mathbf{r}_{O_1O_2} \tag{3}$$

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We could do the same thing in the following fashion:

$${}^{\mathcal{W}}\mathbf{E}_1 = \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_1 & {}^{\mathcal{W}}\mathbf{r}_{O_1} \\ \mathbf{0} & 1 \end{bmatrix}, \quad {}^{\varepsilon_1}\mathbf{E}_2 = \begin{bmatrix} \tau_1\mathbf{T}_2 & \tau_1\mathbf{r}_{O_1O_2} \\ \mathbf{0} & 1 \end{bmatrix} \quad (4)$$

$${}^{\mathcal{W}}\mathbf{E}_2 = \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_2 & {}^{\mathcal{W}}\mathbf{r}_{O_2} \\ \mathbf{0} & 1 \end{bmatrix} \quad (5)$$

$${}^{\mathcal{W}}\mathbf{E}_2 = {}^{\mathcal{W}}\mathbf{E}_1 {}^{\varepsilon_1}\mathbf{E}_2 \quad (6)$$

$$\begin{aligned} {}^{\mathcal{W}}\mathbf{E}_2 &= \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_1 & {}^{\mathcal{W}}\mathbf{r}_{O_1} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \tau_1\mathbf{T}_2 & \tau_1\mathbf{r}_{O_1O_2} \\ \mathbf{0} & 1 \end{bmatrix} = \\ &= \begin{bmatrix} {}^{\mathcal{W}}\mathbf{T}_1 \tau_1\mathbf{T}_2 & ({}^{\mathcal{W}}\mathbf{T}_1 \tau_1\mathbf{r}_{O_1O_2} + {}^{\mathcal{W}}\mathbf{r}_{O_1}) \\ \mathbf{0} & 1 \end{bmatrix} \end{aligned}$$

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In general, we can define a transformation from frame i to the frame $i + 1$

$$\mathcal{E}_i \mathbf{E}_{i+1} = \begin{bmatrix} \mathbf{T}_i & \mathbf{p}_i \\ \mathbf{0} & 1 \end{bmatrix} \quad (7)$$

Where \mathbf{T}_i are coordinates of the basis of the frame $i + 1$ in terms of frame i , and \mathbf{p}_i is the vector pointing from the origin of the frame i to the origin of the frame $i + 1$ expressed in the basis of the frame i .

To get transformation from the world frame to the n -th frame we get:

$${}^{\mathcal{W}}\mathbf{E}_n = \prod_{i=1}^n \mathcal{E}_{i-1} \mathbf{E}_i \quad (8)$$

Last lecture we focused on how to find expressions of radius-vectors (vectors describing positions of points) in world frame, given relative positions and orientations of frames.

Today, we focus on derivatives of these expressions.

Consider vector \mathbf{r}_K describing position of the point K . What is its derivative?

In order to answer this question we need to understand, which parameters appearing in the expression for \mathbf{r}_K are changing with time, and which do not.

Consider an example:

$$\mathbf{r}_K(\mathbf{q}) = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \\ 0 \end{bmatrix} \quad (9)$$

In this example, we know that it is joint coordinate q_1 that is going to change with time.

Another example:

$$\mathbf{r}_K(\mathbf{q}) = \begin{bmatrix} q_1 \\ q_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q_3 & -\sin q_3 \\ 0 & \sin q_3 & \cos q_3 \end{bmatrix} \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 + l_1 \cos q_3 \\ l_1 \sin q_3 \end{bmatrix}$$

Now, q_1 and q_2 are translations and q_3 is a rotation, and $\mathbf{q} = [q_1, q_2, q_3]$ are changing joint coordinates.

So, what is a derivative of $\mathbf{r}_K(\mathbf{q})$ with respect to time?

$$\frac{d}{dt}\mathbf{r}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}}\dot{\mathbf{q}} \quad (10)$$

We can denote $\mathbf{J}_K = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}}$ and call it a jacobian matrix. We can also denote velocity of the point K as $\mathbf{v}_K(\mathbf{q}, \dot{\mathbf{q}}) = \frac{d}{dt}\mathbf{r}_K(\mathbf{q})$.

$$\mathbf{v}_K = \mathbf{J}_K\dot{\mathbf{q}} \quad (11)$$

Notice that velocity \mathbf{v}_K is linear with respect to the joint velocities $\dot{\mathbf{q}}$, and the linear relation is given by the jacobian \mathbf{J}_K .

Consider $\mathbf{r}_K = \mathbf{r}_{OO_1} + \mathbf{r}_{O_1O_2} + \dots + \mathbf{r}_{O_nK}$. What is a jacobian of \mathbf{r}_K ?

$$\begin{aligned}\mathbf{J}_K &= \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} = \frac{\partial \mathbf{r}_{OO_1}}{\partial \mathbf{q}} + \frac{\partial \mathbf{r}_{O_1O_2}}{\partial \mathbf{q}} + \dots + \frac{\partial \mathbf{r}_{O_nK}}{\partial \mathbf{q}} = \\ &= \mathbf{J}_{OO_1} + \mathbf{J}_{O_1O_2} + \dots + \mathbf{J}_{O_nK}\end{aligned}$$

Jacobians have an additive structure inherited from the additive structure of the position vectors.

Notice also that if your vectors and jacobians are expressed via coordinates in different bases - you have to express them in a single basis, and then do the additions, as usual.

Let us consider a rigid body rotating with angular velocity ω with basis \mathcal{T} , given by a matrix \mathbf{T} (whose coordinates in the world frame basis are ${}^W\mathbf{T}$), stationary with respect to the rigid body. Consider a point K on the link, defined by a vector \mathbf{r} . We know coordinates of \mathbf{r} in terms of \mathcal{T} , which we denote as ${}^{\mathcal{T}}\mathbf{r}$.

What is velocity of K ? By definition of angular velocity, $\mathbf{v} = \omega \times \mathbf{r}$. The same can be represented as a vector-matrix multiplication:

$$\mathbf{v} = \omega \times \mathbf{r} = \mathbf{\Omega} \mathbf{r} \quad (12)$$

$$\mathbf{\Omega} = [\omega]_{\times} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \quad (13)$$

Note that expression $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ works as long as we are talking about vectors themselves, or their coordinate representation in the same basis:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\Omega} \mathbf{r} \quad (14)$$

$$\mathcal{W}_{\mathbf{v}} = \mathcal{W}_{\boldsymbol{\omega}} \times \mathcal{W}_{\mathbf{r}} = \mathcal{W}_{\boldsymbol{\Omega}} \mathcal{W}_{\mathbf{r}} \quad (15)$$

$$\mathcal{T}_{\mathbf{v}} = \mathcal{T}_{\boldsymbol{\omega}} \times \mathcal{T}_{\mathbf{r}} = \mathcal{T}_{\boldsymbol{\Omega}} \mathcal{T}_{\mathbf{r}} \quad (16)$$

At the same time, we can find position of K as $\mathbf{r} = \mathbf{T}^\top \mathbf{r}$ and ${}^{\mathcal{W}}_{\mathbf{r}} = {}^{\mathcal{W}}_{\mathbf{T}^\top \mathbf{r}}$, where $\frac{d}{dt} \mathbf{r} = 0$. We can find its time derivative as:

$$\mathbf{v} = \frac{d}{dt} \mathbf{r} = \frac{d}{dt} \mathbf{T}^\top \mathbf{r} = \dot{\mathbf{T}}^\top \mathbf{r} \quad (17)$$

$${}^{\mathcal{W}}_{\mathbf{v}} = {}^{\mathcal{W}}_{\dot{\mathbf{T}}^\top \mathbf{r}} \quad (18)$$

At the same time, ${}^{\mathcal{W}}_{\mathbf{v}} = {}^{\mathcal{W}}_{\Omega} {}^{\mathcal{W}}_{\mathbf{r}}$. Hence:

$${}^{\mathcal{W}}_{\mathbf{v}} = {}^{\mathcal{W}}_{\Omega} {}^{\mathcal{W}}_{\mathbf{T}^\top \mathbf{r}} \quad (19)$$

With that, we know that:

$$\Omega \mathbf{T} = \dot{\mathbf{T}} \quad (20)$$

$$\Omega = \dot{\mathbf{T}} \mathbf{T}^\top \quad (21)$$

Notice that $\mathbf{T}^\top \mathbf{T} = \mathbf{I}$, so $\frac{d}{dt}(\mathbf{T}^\top \mathbf{T}) = 0$, and thus $\dot{\mathbf{T}}^\top \mathbf{T} = -\mathbf{T}^\top \dot{\mathbf{T}}$:

$$\Omega = \dot{\mathbf{T}} \mathbf{T}^\top \quad (22)$$

$$\Omega \mathbf{T} = \dot{\mathbf{T}} \quad (23)$$

$$\mathbf{T}^\top \Omega \mathbf{T} = \mathbf{T}^\top \dot{\mathbf{T}} \quad (24)$$

$$\mathbf{T}^\top \Omega \mathbf{T} = -\dot{\mathbf{T}}^\top \mathbf{T} \quad (25)$$

$$\mathbf{T}^\top \Omega = -\dot{\mathbf{T}}^\top \quad (26)$$

$$\Omega = -\mathbf{T} \dot{\mathbf{T}}^\top \quad (27)$$

Note, in these formulas Ω , \mathbf{T} and $\dot{\mathbf{T}}$ are expressed in the same coordinates.

Given Ω , we can find ω :

$$\omega = \text{skew2vec}(\Omega) \quad (28)$$

$$\text{skew2vec} \left(\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \right) = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad (29)$$

You can notice that both ω and Ω are linear with respect to $\dot{\mathbf{T}}$, which in turn is linear with respect to $\dot{\mathbf{q}}$.

As we mentioned, $\omega = \omega(\mathbf{q}, \dot{\mathbf{q}})$ is linear with respect to $\dot{\mathbf{q}}$:

$$\omega = \frac{\partial \omega}{\partial \dot{\mathbf{q}}} \dot{\mathbf{q}} = \mathbf{J}_\omega \dot{\mathbf{q}} \quad (30)$$

Note that there are big differences in both computation and definition between translation jacobian $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$ and rotational jacobian $\mathbf{J}_\omega \dot{\mathbf{q}}$.

To define translation jacobian we need to define a link and a point on the link, whose jacobian we discuss; for rotation jacobian, we only need to identify the link whose rotation we describe.

Assume we have a vector ${}^{\mathcal{W}}\mathbf{r}(\mathbf{q})$ expressed in the basis \mathcal{W} . We can find its jacobian:

$${}^{\mathcal{W}}\mathbf{J}(\mathbf{q}) = \frac{\partial {}^{\mathcal{W}}\mathbf{r}}{\partial \mathbf{q}} \quad (31)$$

Now, given basis \mathcal{T} , expressed by matrix ${}^{\mathcal{W}}\mathbf{T}$, we can represent \mathbf{r} in $\mathcal{T} = \text{const}$:

$${}^{\mathcal{T}}\mathbf{r}(\mathbf{q}) = {}^{\mathcal{W}}\mathbf{T}^{\top} {}^{\mathcal{W}}\mathbf{r}(\mathbf{q}) \quad (32)$$

And of course we can find jacobian of ${}^{\mathcal{T}}\mathbf{r}(\mathbf{q})$:

$${}^{\mathcal{T}}\mathbf{J}(\mathbf{q}) = \frac{\partial {}^{\mathcal{T}}\mathbf{r}}{\partial \mathbf{q}} = {}^{\mathcal{W}}\mathbf{T}^{\top} {}^{\mathcal{W}}\mathbf{J}(\mathbf{q}) \quad (33)$$

We can play the same game with angular velocities ω . Jacobians depend on the bases same as vectors do.

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Fundamentals-of-robotics-2022

Check Moodle for additional links, videos, textbook suggestions.

