Inverse Kinematics

Fundamentals of Robotics, Lecture 6

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Velocity problem

Given a point (e.g. end effector) K with position given by vector $\mathbf{r}_K(\mathbf{q})$, we can find its velocity:

$$\dot{\mathbf{r}}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \dot{\mathbf{q}} \tag{1}$$

Let us introduce notation:

$$\mathbf{J}_K = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \tag{2}$$

$$\mathbf{v}_K = \dot{\mathbf{r}}_K(\mathbf{q}) \tag{3}$$

Thus we get:

$$\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}} \tag{4}$$

Given $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$, can we find least-residual solution to this problem? Yes!

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K \tag{5}$$

Is this the only solution? No. All solutions are:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z} \tag{6}$$

where $\mathbf{N} = \mathrm{null}(\mathbf{J}_K)$ and $\mathbf{z}$ are null space coordinates. Alternatively, we can use a projector to do the same thing with less new notation:

$$\dot{\mathbf{q}}^* = \mathbf{J}_K^+ \mathbf{v}_K + (\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K)\dot{\mathbf{q}} \tag{7}$$

where $\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$ is a null space projector.

Now let us find closest joint velocity to $\dot{\mathbf{q}}_0$ that solves the velocity problem $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$:

$$\begin{aligned} \underset{\dot{\mathbf{q}}}{\text{minimize}} \quad & ||\dot{\mathbf{q}} - \dot{\mathbf{q}}_0||, \\ \text{subject to} \quad & \mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}} \end{aligned} \tag{8}$$

We can solve it by first finding all solutions:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z} \tag{9}$$

Then we minimize cost function in terms of the null space variable $\mathbf{z}$:

$$||\mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}\mathbf{z} - \dot{\mathbf{q}}_0|| \to min \tag{10}$$

First we simplify $||\mathbf{J}_K^+\mathbf{v}_K + \mathbf{N}\mathbf{z} - \dot{\mathbf{q}}_0|| \rightarrow min$ with notation

$$\mathbf{c} = \mathbf{J}_K^+\mathbf{v}_K - \dot{\mathbf{q}}_0 \tag{11}$$

Then we square $||\mathbf{N}\mathbf{z} + \mathbf{c}||$, and consider its derivative:

$$(\mathbf{N}\mathbf{z} + \mathbf{c})^\top(\mathbf{N}\mathbf{z} + \mathbf{c}) \rightarrow min \tag{12}$$

$$\frac{\partial}{\partial \mathbf{z}}(\mathbf{N}\mathbf{z} + \mathbf{c})^\top(\mathbf{N}\mathbf{z} + \mathbf{c}) = 0 \tag{13}$$

$$2\mathbf{z}^\top\mathbf{N}^\top\mathbf{N} + 2\mathbf{c}^\top\mathbf{N} = 0 \tag{14}$$

$$\mathbf{z} = -(\mathbf{N}^\top\mathbf{N})^{-1}\mathbf{N}^\top\mathbf{c} \tag{15}$$

Knowing that $\dot{\mathbf{q}} = \mathbf{J}_K^+\mathbf{v}_K + \mathbf{N}\mathbf{z}$ we get:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+\mathbf{v}_K - \mathbf{N}(\mathbf{N}^\top\mathbf{N})^{-1}\mathbf{N}^\top\mathbf{c} \tag{16}$$

Let us examine the solution we obtained:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top (\mathbf{J}_K^+ \mathbf{v}_K - \dot{\mathbf{q}}_0) \tag{17}$$

$$\dot{\mathbf{q}} = (\mathbf{I} - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top) \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \dot{\mathbf{q}}_0 \tag{18}$$

Let us examine the matrices:

$$\mathbf{P}_N = \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top \tag{19}$$

$$\mathbf{P}_R = \mathbf{I} - \mathbf{N}(\mathbf{N}^\top \mathbf{N})^{-1} \mathbf{N}^\top = \mathbf{I} - \mathbf{P}_N \tag{20}$$

$$\dot{\mathbf{q}} = \mathbf{P}_R \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{P}_N \dot{\mathbf{q}}_0 \tag{21}$$

where $\mathbf{P}_N$ is column space projector for $\mathbf{N}$, hence it is a null space projector for the jacobian $\mathbf{J}_K$. And $\mathbf{I} - \mathbf{P}_N$ is a projector to the orthogonal compliment, hence it is row space projector.

So, we have eq. $\dot{\mathbf{q}} = \mathbf{P}_R \mathbf{J}_K^+ \mathbf{v}_K + \mathbf{P}_N \dot{\mathbf{q}}_0$ and we have null space projector $\mathbf{P}_N$ and row space projector $\mathbf{P}_R$.

We know that pseudoinverse lies in the column space, so:

$$\mathbf{P}_R \mathbf{J}_K^+ = \mathbf{J}_K^+ \tag{22}$$

Also we know that null space projector can be found as $\mathbf{P}_N = \mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K$:

$$\dot{\mathbf{q}} = \mathbf{J}_K^+ \mathbf{v}_K + (\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K) \dot{\mathbf{q}}_0 \tag{23}$$

Notice, this is almost exactly the same as what we found before. We can interpret it as "the solution is given by row-space least squares solution, plus null space projection of $\dot{\mathbf{q}}_0$".

Consider second derivative of the position of the point K:

$$\ddot{\mathbf{r}}_K(\mathbf{q}) = \frac{\partial \mathbf{r}_K}{\partial \mathbf{q}} \ddot{\mathbf{q}} + \frac{d}{dt}\left(\frac{\partial \mathbf{r}_K}{\partial \mathbf{q}}\right) \dot{\mathbf{q}} \tag{24}$$

Defining $\mathbf{a}_K = \ddot{\mathbf{r}}_K$ we get:

$$\mathbf{a}_K = \mathbf{J}_K \ddot{\mathbf{q}} + \dot{\mathbf{J}}_K \dot{\mathbf{q}} \tag{25}$$

The least residual solution is easily found:

$$\ddot{\mathbf{q}} = \mathbf{J}_K^+ (\mathbf{a}_K - \dot{\mathbf{J}}_K \dot{\mathbf{q}}) \tag{26}$$

Given $\mathbf{a}_K = \mathbf{J}_K \ddot{\mathbf{q}} + \dot{\mathbf{J}}_K \dot{\mathbf{q}}$ let us find acceleration closest to $\ddot{\mathbf{q}}_0$ that solves the acceleration problem:

$$\ddot{\mathbf{q}} = \mathbf{J}_K^{+}(\mathbf{a}_K - \dot{\mathbf{J}}_K \dot{\mathbf{q}}) + \mathbf{P}_N \ddot{\mathbf{q}}_0 \tag{27}$$

where $\mathbf{P}_N = \mathbf{I} - \mathbf{J}_K^{+} \mathbf{J}_K$.

Position problem

What if we want to find such $\mathbf{q}^*$ that $\mathbf{r}_K(\mathbf{q}^*) = \mathbf{r}^*$. Can we do it?

Unlike previous, this is not a linear problem. It often involves trigonometric functions, and other nonlinear ones.

Our approach will be to linearize the expression $\mathbf{r}_K(\mathbf{q})$ and find its solution via an iterative procedure.

Given exact solution $\mathbf{q}_0$ for problem $\mathbf{r}_K(\mathbf{q}_0) = \mathbf{r}_K(0)$. Then, knowing velocity $\dot{\mathbf{q}}_0$, then we can find an approximation of the position in the next moment of time:

$$\frac{\mathbf{q}_1 - \mathbf{q}_0}{\Delta t} \approx \dot{\mathbf{q}}_0 \tag{28}$$

$$\mathbf{q}_1 \approx \mathbf{q}_0 + \dot{\mathbf{q}}_0 \Delta t \tag{29}$$

This works tolerably well, for improvements we can look to other schemes of solving ODEs.

But what if we don't have an exact solution $\mathbf{q}_0$? After all, it was that which allowed us to use local linearization.

Position problem - general

Given initial guess $\mathbf{q}_0$ we will try to solve the problem $\mathbf{r}_K(\mathbf{q}) = \mathbf{r}_K^*$. First let us define discrepancy:

$$\mathbf{e}(\mathbf{q}) = \mathbf{r}_K(\mathbf{q}) - \mathbf{r}_K^* \tag{30}$$

We define cost function $f = \mathbf{e}^\top \mathbf{e}$, initial position $\mathbf{r}_{K,0} = \mathbf{r}_K(\mathbf{q}_0)$, initial discrepancy $\mathbf{e}_0 = \mathbf{r}_{K,0} - \mathbf{r}_K^*$ and gen. coordinates displacement $\delta = \mathbf{q} - \mathbf{q}_0$ and produce Taylor expansion of the cost:

$$f \approx \mathbf{e}_0^\top \mathbf{e}_0 + \mathbf{e}_0^\top \mathbf{J}_K \delta + \delta^\top \mathbf{J}_K^\top \mathbf{e}_0 + \delta^\top \mathbf{J}_K^\top \mathbf{J}_K \delta \tag{31}$$

Now we take derivative and set it to zero:

$$2\mathbf{e}_0^\top \mathbf{J}_K + 2\delta^\top \mathbf{J}_K^\top \mathbf{J}_K = 0 \tag{32}$$

We obtained expression:

$$\delta = -(\mathbf{J}_K^\top \mathbf{J}_K)^{-1} \mathbf{J}_K^\top \mathbf{e}_0 \tag{33}$$

And remembering the substitutions we made we get:

$$\mathbf{q} - \mathbf{q}_0 = -(\mathbf{J}_K^\top \mathbf{J}_K)^{-1} \mathbf{J}_K^\top (\mathbf{r}_{K,0} - \mathbf{r}_K^*) \tag{34}$$

$$\mathbf{q} = \mathbf{q}_0 - \mathbf{J}_K^+ (\mathbf{r}_{K,0} - \mathbf{r}_K^*) \tag{35}$$

We can use the final expression to update our initial guess, then the-linearize the problem at the new position and repeat the process, until we converge.

How can we check if a velocity $\mathbf{v}_K$ can be achieved, given $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$?

If $\mathbf{v}_K$ lies in the column space of $\mathbf{J}_K$, it is achievable:

$$(\mathbf{I} - \mathbf{J}_K \mathbf{J}_K^+) \mathbf{v}_K = 0 \tag{36}$$

How can we check if a solution $\dot{\mathbf{q}}$ is minimal-norm, given $\mathbf{v}_K = \mathbf{J}_K \dot{\mathbf{q}}$?

If $\dot{\mathbf{q}}$ lies in the row space of $\mathbf{J}_K$, it is minimal:

$$(\mathbf{I} - \mathbf{J}_K^+ \mathbf{J}_K)\dot{\mathbf{q}} = 0 \tag{37}$$

Thank you!

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github.com/SergeiSa/Fundamentals-of-robotics-2022

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