# Nonlinear Control, PD, Gravity compensation, Lyapunov

## Fundamentals of Robotics, Lecture 9

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### Asymptotic stability criteria

Autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is assymptotically stable, if there exists a scalar function $V = V(\mathbf{x}) > 0$, whose time derivative is negative $\dot{V}(\mathbf{x}) < 0$, except $V(\mathbf{0}) = 0$, $\dot{V}(\mathbf{0}) = 0$.

### Marginal stability criteria

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is stable in the sense of Lyapunov, $\exists V(\mathbf{x}) > 0$, $\dot{V}(\mathbf{x}) \leq 0$.

### Definition

Function $V(\mathbf{x}) > 0$ in this case is called *Lyapunov function*.

This is not the only type of stability as you remember, you are invited to study criteria for other stability types on your own.

Let us consider the following basic system:

$$\ddot{\mathbf{q}} + \mathbf{K}_d\dot{\mathbf{q}} + \mathbf{K}_p\mathbf{q} = 0 \tag{1}$$

where $\mathbf{K}_d > 0$ and $\mathbf{K}_p > 0$ are semidefinite matrices. Can we analyze its stability?

We propose Lyapunov function

$$V = \frac{1}{2}\dot{\mathbf{q}}^\top\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^\top\mathbf{K}_p\mathbf{q} \tag{2}$$

Derivative of the Lyapunov function is:

$$\dot{V} = \dot{\mathbf{q}}^\top \ddot{\mathbf{q}} + \mathbf{q}^\top \mathbf{K}_p \dot{\mathbf{q}} \tag{3}$$

But we know that $\ddot{\mathbf{q}} = -\mathbf{K}_d \dot{\mathbf{q}} - \mathbf{K}_p \mathbf{q}$, so:

$$\dot{V} = -\dot{\mathbf{q}}^\top (\mathbf{K}_d \dot{\mathbf{q}} + \mathbf{K}_p \mathbf{q}) + \mathbf{q}^\top \mathbf{K}_p \dot{\mathbf{q}} \tag{4}$$

$$\dot{V} = -\dot{\mathbf{q}}^\top \mathbf{K}_d \dot{\mathbf{q}} \tag{5}$$

This means that the system is at least marginally stable: $\dot{V} \leq 0$, namely $\dot{V} = 0$ for $\dot{\mathbf{q}} = 0$ and $\forall \mathbf{q}$.

Now, we want to prove that the only trajectory in the fixed point ($\dot{\mathbf{q}} = 0$, $\ddot{\mathbf{q}} = 0$) is $\mathbf{q} = 0$.

Consider the fixed point $\dot{\mathbf{q}} = 0$, $\ddot{\mathbf{q}} = 0$:

$$\ddot{\mathbf{q}} + \mathbf{K}_d \dot{\mathbf{q}} + \mathbf{K}_p \mathbf{q} = 0 \quad \rightarrow \quad \mathbf{K}_p \mathbf{q} = 0 \tag{6}$$

Since $\mathbf{K}_p > 0$, it is full rank and has a trivial null space, so $\mathbf{q} = 0$ is the only trajectory in the fixed point space. So, the system is asymptotically stable by LaSalle's invariance principle.

Now, we can consider manipulator equations:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau \tag{7}$$

We propose a Lyapunov function:

$$V = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{H}\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^\top \mathbf{K}_p \mathbf{q} \tag{8}$$

$$\dot{V} = \dot{\mathbf{q}}^\top \mathbf{H}\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^\top \dot{\mathbf{H}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^\top \mathbf{K}_p \mathbf{q} \tag{9}$$

$$\dot{V} = \dot{\mathbf{q}}^\top (\tau - \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) + \frac{1}{2}\dot{\mathbf{q}}^\top \dot{\mathbf{H}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^\top \mathbf{K}_p \mathbf{q} \tag{10}$$

$$\dot{V} = \frac{1}{2}\dot{\mathbf{q}}^\top (\dot{\mathbf{H}} - 2\mathbf{C})\dot{\mathbf{q}} + \dot{\mathbf{q}}^\top (\tau - \mathbf{g}) + \dot{\mathbf{q}}^\top \mathbf{K}_p \mathbf{q} \tag{11}$$

Looking at the expression:

$$\dot{V} = \frac{1}{2}\dot{\mathbf{q}}^\top(\dot{\mathbf{H}} - 2\mathbf{C})\dot{\mathbf{q}} + \dot{\mathbf{q}}^\top(\tau - \mathbf{g}) + \dot{\mathbf{q}}^\top\mathbf{K}_p\mathbf{q}$$

we can observe that $\dot{\mathbf{H}} - 2\mathbf{C}$ is skew-symmetric, and therefore $\dot{\mathbf{q}}^\top(\dot{\mathbf{H}} - 2\mathbf{C})\dot{\mathbf{q}} = 0$
We can propose the following $\tau$:

$$\tau = \mathbf{g} - \mathbf{K}_d\dot{\mathbf{q}} - \mathbf{K}_p\mathbf{q} \tag{12}$$

Then the Lyapunov function derivative takes form:

$$\dot{V} = -\dot{\mathbf{q}}^\top\mathbf{K}_d\dot{\mathbf{q}} - \dot{\mathbf{q}}^\top\mathbf{K}_p\mathbf{q} + \dot{\mathbf{q}}^\top\mathbf{K}_p\mathbf{q} \tag{13}$$

$$\dot{V} = -\dot{\mathbf{q}}^\top\mathbf{K}_d\dot{\mathbf{q}} \leq 0 \tag{14}$$

# Gravity compensation

So, $\dot{V} = -\dot{\mathbf{q}}^\top \mathbf{K}_d \dot{\mathbf{q}} \leq 0$. More precisely $\dot{V} = 0$ for $\dot{\mathbf{q}} = 0$ and $\forall \mathbf{q}$, giving us marginal stability.

Let us study the fixed points $\dot{\mathbf{q}} = 0$ and $\ddot{\mathbf{q}} = 0$ for the manipulator eq.:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{g} - \mathbf{K}_d\dot{\mathbf{q}} - \mathbf{K}_p\mathbf{q} \tag{15}$$

$$\mathbf{g} = \mathbf{g} - \mathbf{K}_p\mathbf{q} \tag{16}$$

$$\mathbf{K}_p\mathbf{q} = 0 \tag{17}$$

$$\mathbf{q} = 0 \tag{18}$$

We get that the only trajectory in the fixed space is $\mathbf{q} = 0$. So, the system is asymptotically stable by LaSalle's invariance principle.

The control that we proposed:

$$\tau = \mathbf{g} - \mathbf{K}_d \dot{\mathbf{q}} - \mathbf{K}_p \mathbf{q} \tag{19}$$

is gravity compensation and PD control. We have proven that it is stable.

If the system has linear dissipation force (viscous force) $\mathbf{F}\dot{\mathbf{q}}$:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{F}\dot{\mathbf{q}} + \mathbf{g} = \tau \tag{20}$$

...we can use proportional control without derivative component:

$$\tau = \mathbf{g} - \mathbf{K}_p\mathbf{q} \tag{21}$$

Where the same Lyapunov function will obtain derivative:

$$\dot{V} = -\dot{\mathbf{q}}^\top \mathbf{F}_d \dot{\mathbf{q}} \leq 0 \tag{22}$$

With the same conclusions as before. An example of system for which we can use it is industrial robot arms with gear box reducers.

Let us introduce *desired position* $\mathbf{q}^*$ and *position error* $\mathbf{e}$:

$$\mathbf{e} = \mathbf{q}^* - \mathbf{q} \tag{23}$$

If we are lucky, the *error dynamics* is evolving in accord with the following equation:

$$\ddot{\mathbf{e}} + \mathbf{K}_d \dot{\mathbf{e}} + \mathbf{K}_p \mathbf{e} = 0 \tag{24}$$

where $\mathbf{K}_d > 0$ and $\mathbf{K}_p > 0$. This system is asymptotically stable, meaning that $\mathbf{e} \to 0$ and $\mathbf{q} \to \mathbf{q}^*$.

Let us find what $\ddot{\mathbf{e}}$ is:

$$\ddot{\mathbf{e}} = \ddot{\mathbf{q}}^* - \ddot{\mathbf{q}} \tag{25}$$

$$\ddot{\mathbf{e}} = \ddot{\mathbf{q}}^* - \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \tag{26}$$

Then error dynamics is:

$$\ddot{\mathbf{q}}^* - \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \mathbf{K}_d\dot{\mathbf{e}} + \mathbf{K}_p\mathbf{e} = 0 \tag{27}$$

$$\mathbf{H}\ddot{\mathbf{q}}^* - (\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) + \mathbf{H}(\mathbf{K}_d\dot{\mathbf{e}} + \mathbf{K}_p\mathbf{e}) = 0 \tag{28}$$

$$\tau = \mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} + \mathbf{H}(\mathbf{K}_d\dot{\mathbf{e}} + \mathbf{K}_p\mathbf{e}) \tag{29}$$

So, we found a control law that makes error dynamics stable, meaning $\mathbf{q} \to \mathbf{q}^*$.

Often, it does not make sense to try to compute $\mathbf{H}$ and $\mathbf{C}\dot{\mathbf{q}} + \mathbf{g}$ based on current position of the robot. Instead, we can compute it based on the desired trajectory:

$$\mathbf{H}^* = \mathbf{H}(\mathbf{q}^*) \tag{30}$$

$$\mathbf{C}^* = \mathbf{C}(\dot{\mathbf{q}}^*, \mathbf{q}^*) \tag{31}$$

$$\mathbf{g}^* = \mathbf{g}(\mathbf{q}^*) \tag{32}$$

We can define inverse dynamics $\tau^*$ as:

$$\tau^* = \mathbf{H}^*\ddot{\mathbf{q}}^* + \mathbf{C}^*\dot{\mathbf{q}}^* + \mathbf{g}^* \tag{33}$$

Then, the control law can be re-written as:

$$\tau = \tau^* + \mathbf{H}^*(\mathbf{K}_d\dot{\mathbf{e}} + \mathbf{K}_p\mathbf{e}) \tag{34}$$

But, this way we do not have any stability guarantee.

You can read more at *Siciliano, B., Sciavicco, L., Villani, L. and Oriolo, G., 2009. Robotics. Advanced textbooks in control and signal processing*, Chapter 8.5

# Thank you!

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Fundamentals-of-robotics-2022

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