Robot Dynamics

Fundamentals of Robotics, Lecture 8

by Sergei Savin

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Lagrange equations

Remember Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} = \tau \tag{1}$$

Let us remember that in the general case of point masses, kinetic energy is:

$$T = \sum 0.5 m_i \dot{\mathbf{r}}_i^\top \dot{\mathbf{r}}_i, \tag{2}$$

and in the general case of rigid bodies, it is:

$$T = \sum 0.5 m_i \dot{\mathbf{r}}_i^\top \dot{\mathbf{r}}_i + \sum 0.5 \mathbf{w}_i^\top \mathbf{I}_i \mathbf{w}_i, \tag{3}$$

Where $\dot{\mathbf{r}}_i$ is the velocity of the center of mass of the i-th body, and $\mathbf{w}$ is the angular velocity of that body.

Using chain rule we can describe the velocity of the center of mass:

$$\dot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}_i^v \dot{\mathbf{q}} \tag{4}$$

This establishes the connection between $\dot{\mathbf{r}}_i$ and generalized velocities.

For the rotations, it is not as simple. We start by using a Poisson formula to connect rotation matrix $\mathbf{T}(\mathbf{q})$ of a body to angular velocity of a body:

$$\mathbf{W}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{T}\dot{\mathbf{T}}, \quad \dot{\mathbf{T}}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial \mathbf{T}}{\partial \mathbf{q}} \dot{\mathbf{q}} \tag{5}$$

where $\mathbf{W}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$.

We can create notation:

$$\mathbf{w}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -\mathbf{W}_{2,3} \\ \mathbf{W}_{1,3} \\ -\mathbf{W}_{1,2} \end{bmatrix} \tag{6}$$

Homework 1

Prove that $\mathbf{w}(\mathbf{q}, \dot{\mathbf{q}})$ is linear with respect to $\dot{\mathbf{q}}$.

Now we can *find angular velocity Jacobian* of (6) w.r.t. $\dot{\mathbf{q}}$:

$$\mathbf{J}_i^w = \frac{\partial \mathbf{w}_i}{\partial \dot{\mathbf{q}}} \tag{7}$$

Since $\mathbf{w}_i$ is linear w.r.t. $\dot{\mathbf{q}}$, we can represent it as:

$$\mathbf{w}_i = \mathbf{J}_i^w \dot{\mathbf{q}} \tag{8}$$

Therefor me can rewrite the kinetic energy in terms of generalized velocity:

$$T = \sum 0.5\dot{\mathbf{q}}^\top (\mathbf{J}_i^v)^\top m_i \mathbf{J}_i^v \dot{\mathbf{q}} + \sum 0.5\dot{\mathbf{q}}^\top (\mathbf{J}_i^w)^\top \mathbf{I}_i \mathbf{J}_i^w \dot{\mathbf{q}} \tag{9}$$

Kinetic energy is a *quadratic form* of the generalized velocities. We can define the matrix of the quadratic form:

$$\mathbf{H} = \sum (\mathbf{J}_i^v)^\top m_i \mathbf{J}_i^v + \sum (\mathbf{J}_i^w)^\top \mathbf{I}_i \mathbf{J}_i^w \tag{10}$$

And therefor: $T = 0.5\dot{\mathbf{q}}^\top \mathbf{H} \dot{\mathbf{q}}$

We can find derivatives of the kinetic energy (remembering that $T = T^\top$, and therefore $\mathbf{H} = \mathbf{H}^\top$):

$$\frac{\partial T}{\partial \dot{\mathbf{q}}} = \mathbf{H}\dot{\mathbf{q}} \tag{11}$$

$$\frac{\partial T}{\partial \mathbf{q}} = 0.5\dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} \tag{12}$$

Notice that it is very tempting to say that $0.5\dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} = 0.5\dot{\mathbf{q}}^\top \dot{\mathbf{H}}$ but it is *not* the case. $\frac{\partial \mathbf{H}}{\partial \mathbf{q}}$ is a three dimensional tensor, symmetric along the first and second dimension (so, transposing along these two dimensions doesn't change the products of the tensor with matrices or vectors). Multiplication by $\dot{\mathbf{q}}$ happens along the first and second dimensions, while the partial differentiation happens along the third dimension, therefore the result is not necessarily equals to $0.5\dot{\mathbf{q}}^\top \dot{\mathbf{H}}$.

Left-hand side of the Lagrange equations can be re-written as:

$$\frac{d}{dt}\left(\mathbf{H}\dot{\mathbf{q}}\right) - 0.5\dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} = \tau \tag{13}$$

We can expand the derivative of a product:

$$\mathbf{H}\ddot{\mathbf{q}} + \dot{\mathbf{H}}\dot{\mathbf{q}} - 0.5\dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} = \tau \tag{14}$$

Expression $\dot{\mathbf{H}}\dot{\mathbf{q}} - 0.5\dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}}$ is often cast as a linear form. $\mathbf{C}\dot{\mathbf{q}}$ The classic formula for calculating $\mathbf{C}\dot{\mathbf{q}}$ uses Christoffel symbols.

Christoffel symbols-based formula for the $\mathbf{C}\dot{\mathbf{q}}$ is:

$$\mathbf{C}\dot{\mathbf{q}} = \begin{bmatrix} \sum_{j,k}^{n} \Gamma_{1,j,k}\dot{q}_j\dot{q}_k \\ \dots \\ \sum_{j,k}^{n} \Gamma_{n,j,k}\dot{q}_j\dot{q}_k \end{bmatrix} \tag{15}$$

where Christoffel symbols $\Gamma_{i,j,k}$ are given as:

$$\Gamma_{i,j,k} = \frac{1}{2}\left(\frac{\partial H_{i,j}}{\partial q_k} + \frac{\partial H_{i,k}}{\partial q_j} - \frac{\partial H_{k,j}}{\partial q_i} \right) \tag{16}$$

My apologies for not providing a derivation

Sometimes we need to find matrix $\mathbf{C}$ specifically, rather than linear form $\mathbf{C}\dot{\mathbf{q}}$. This can be achieved using Christoffel symbols as well.

$$C_{i,j} = \sum_{k}^{n} \Gamma_{i,j,k} \dot{q}_k \tag{17}$$

If you use auto-differentiation, you can consider directly using expression (14) to find $\mathbf{C}\dot{\mathbf{q}}$:

$$\mathbf{C}\dot{\mathbf{q}} = \dot{\mathbf{H}}\dot{\mathbf{q}} - 0.5\frac{\partial \dot{\mathbf{q}}^\top \mathbf{H}\dot{\mathbf{q}}}{\partial \mathbf{q}} \tag{18}$$

In Matlab code it looks like:

```
C_times_v = reshape(dHdt*v, length(v), 1) - reshape(jacobian((0.5*v' * H * v), q), length(v), 1);
```

Generalized inertia

Part 6, Alternative formulation

Alternatively, you can use the following formula:

$$\mathbf{C}\dot{\mathbf{q}} = \dot{\mathbf{H}}\dot{\mathbf{q}} - 0.5\frac{\partial \mathrm{vec}(\mathbf{H})}{\partial \mathbf{q}}(\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}) \tag{19}$$

where $\otimes$ is a Kronecker product, and vec() is vectorization of matrix (representing all its elements as a vector.

In Matlab code it looks like:

```
C_times_v = reshape(dHdt*v, [], 1) - 0.5*( jacobian(
    reshape(H, [], 1), q)'*kron(v, v) );
```

You can express generalized forces of all kinds using discussed previously multiplication by the Jacobian:

$$\tau = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^{\top} \mathbf{f} \tag{20}$$

where $\mathbf{f}$ is an external force, and $\mathbf{r}$ is the radius-vector giving position of that force.

For a torque ξ applied to a rigid body, the corresponding generalized force is:

$$\tau = \mathbf{J}_w^{\top} \xi \tag{21}$$

where $\mathbf{J}_w$ is the angular velocity Jacobian of that body. Note that both $\mathbf{J}_w$ and ξ need to be expressed in the same basis.

If the force is conservative, it is often easy to describe potential energy U associated with it. Then you can find the relevant generalized forces as:

$$\mathbf{g} = -\frac{\partial U}{\partial \mathbf{q}} \tag{22}$$

Typically this is useful for gravitational forces and elastic forces.

Finally we can write the form of manipulator equations:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} = \tau \tag{23}$$

Another popular form specifically points out conservative forces $\mathbf{g}$:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \tau \tag{24}$$

The most concise and useful for this class form is:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{c} = \tau \tag{25}$$

where $\mathbf{c} = \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}$.

Importantly, manipulator equations can always be re-written in an explicit form (where the highest-order derivative is isolated on the left-hand-side):

$$\ddot{\mathbf{q}} = \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \tag{26}$$

This is possible because $\mathbf{H}$ is full-rank. This in turn is possible, because generalized coordinates are independent.

You can read more at:

- Chapter 4. Robot Dynamics and Control - part 3.2, an interesting derivation.
- Robot Dynamics Lecture Notes. Robotic Systems Lab, ETH Zurich HS 2017:
  - 2.5 Angular Velocity
  - Chapter 3. Dynamics
  - 3.4.2 Kinetic Energy
  - 3.4.3 Potential Energy
  - 3.4.5 Additional Constraints (note some notational differences)
  - 3.5.2 Deriving Generalized Equations of Motion

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Fundamentals-of-robotics-2022

Check Moodle for additional links, videos, textbook suggestions.