

Robot Dynamics

Fundamentals of Robotics, Lecture 8

by Sergei Savin

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Remember Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \tau \quad (1)$$

Let us remember that in the general case of point masses, kinetic energy is:

$$T = \sum 0.5 m_i \dot{\mathbf{r}}_i^\top \dot{\mathbf{r}}_i, \quad (2)$$

and in the general case of rigid bodies, it is:

$$T = \sum 0.5 m_i \dot{\mathbf{r}}_i^\top \dot{\mathbf{r}}_i + \sum 0.5 \mathbf{w}_i^\top \mathbf{I}_i \mathbf{w}_i, \quad (3)$$

Where $\dot{\mathbf{r}}_i$ is the velocity of the center of mass of the i -th body, and \mathbf{w} is the angular velocity of that body.

Using chain rule we can describe the velocity of the center of mass:

$$\dot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J}_i^v \dot{\mathbf{q}} \quad (4)$$

This establishes the connection between $\dot{\mathbf{r}}_i$ and generalized velocities.

For the rotations, it is not as simple. We start by using a Poisson formula to connect rotation matrix $\mathbf{T}(\mathbf{q})$ of a body to angular velocity of a body:

$$\mathbf{W}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{T}\dot{\mathbf{T}}, \quad \dot{\mathbf{T}}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial \mathbf{T}}{\partial \mathbf{q}} \dot{\mathbf{q}} \quad (5)$$

where $\mathbf{W}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$

We can create notation:

$$\mathbf{w}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -\mathbf{W}_{2,3} \\ \mathbf{W}_{1,3} \\ -\mathbf{W}_{1,2} \end{bmatrix} \quad (6)$$

Homework 1

Prove that $\mathbf{w}(\mathbf{q}, \dot{\mathbf{q}})$ is linear with respect to $\dot{\mathbf{q}}$.

Now we can *find angular velocity Jacobian* of (6) w.r.t. $\dot{\mathbf{q}}$:

$$\mathbf{J}_i^w = \frac{\partial \mathbf{w}_i}{\partial \dot{\mathbf{q}}} \quad (7)$$

Since \mathbf{w}_i is linear w.r.t. $\dot{\mathbf{q}}$, we can represent it as:

$$\mathbf{w}_i = \mathbf{J}_i^w \dot{\mathbf{q}} \quad (8)$$

Therefor me can rewrite the kinetic energy in terms of generalized velocity:

$$T = \sum 0.5 \dot{\mathbf{q}}^\top (\mathbf{J}_i^v)^\top m_i \mathbf{J}_i^v \dot{\mathbf{q}} + \sum 0.5 \dot{\mathbf{q}}^\top (\mathbf{J}_i^w)^\top \mathbf{I}_i \mathbf{J}_i^w \dot{\mathbf{q}} \quad (9)$$

Kinetic energy is a *quadratic form* of the generalized velocities. We can define the matrix of the quadratic form:

$$\mathbf{H} = \sum (\mathbf{J}_i^v)^\top m_i \mathbf{J}_i^v + \sum (\mathbf{J}_i^w)^\top \mathbf{I}_i \mathbf{J}_i^w \quad (10)$$

And therefor: $T = 0.5 \dot{\mathbf{q}}^\top \mathbf{H} \dot{\mathbf{q}}$

We can find derivatives of the kinetic energy (remembering that $T = T^\top$, and therefore $\mathbf{H} = \mathbf{H}^\top$):

$$\frac{\partial T}{\partial \dot{\mathbf{q}}} = \mathbf{H} \dot{\mathbf{q}} \quad (11)$$

$$\frac{\partial T}{\partial \mathbf{q}} = 0.5 \dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} \quad (12)$$

Notice that it is very tempting to say that $0.5 \dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} = 0.5 \dot{\mathbf{q}}^\top \dot{\mathbf{H}}$ but it is *not* the case. $\frac{\partial \mathbf{H}}{\partial \mathbf{q}}$ is a three dimensional tensor, symmetric along the first and second dimension (so, transposing along these two dimensions doesn't change the products of the tensor with matrices or vectors). Multiplication by $\dot{\mathbf{q}}$ happens along the first and second dimensions, while the partial differentiation happens along the third dimension, therefore the result is not necessarily equals to $0.5 \dot{\mathbf{q}}^\top \dot{\mathbf{H}}$.

Left-hand side of the Lagrange equations can be re-written as:

$$\frac{d}{dt} \left(\mathbf{H} \dot{\mathbf{q}} \right) - 0.5 \dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} = \tau \quad (13)$$

We can expand the derivative of a product:

$$\mathbf{H} \ddot{\mathbf{q}} + \dot{\mathbf{H}} \dot{\mathbf{q}} - 0.5 \dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}} = \tau \quad (14)$$

Expression $\dot{\mathbf{H}} \dot{\mathbf{q}} - 0.5 \dot{\mathbf{q}}^\top \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \dot{\mathbf{q}}$ is often cast as a linear form. $\mathbf{C} \dot{\mathbf{q}}$
The classic formula for calculating $\mathbf{C} \dot{\mathbf{q}}$ uses Christoffel symbols.

Christoffel symbols-based formula for the $\mathbf{C}\dot{\mathbf{q}}$ is:

$$\mathbf{C}\dot{\mathbf{q}} = \begin{bmatrix} \sum_{j,k}^n \Gamma_{1,j,k} \dot{q}_j \dot{q}_k \\ \dots \\ \sum_{j,k}^n \Gamma_{n,j,k} \dot{q}_j \dot{q}_k \end{bmatrix} \quad (15)$$

where Christoffel symbols $\Gamma_{i,j,k}$ are given as:

$$\Gamma_{i,j,k} = \frac{1}{2} \left(\frac{\partial H_{i,j}}{\partial q_k} + \frac{\partial H_{i,k}}{\partial q_j} - \frac{\partial H_{k,j}}{\partial q_i} \right) \quad (16)$$

My apologies for not providing a derivation

GENERALIZED INERTIA

Part 4, Christoffel symbols

Sometimes we need to find matrix \mathbf{C} specifically, rather than linear form $\mathbf{C}\dot{\mathbf{q}}$. This can be achieved using Christoffel symbols as well.

$$C_{i,j} = \sum_k^n \Gamma_{i,j,k} \dot{q}_k \quad (17)$$

If you use auto-differentiation, you can consider directly using expression (14) to find $\mathbf{C}\dot{\mathbf{q}}$:

$$\mathbf{C}\dot{\mathbf{q}} = \dot{\mathbf{H}}\dot{\mathbf{q}} - 0.5 \frac{\partial \dot{\mathbf{q}}^\top \mathbf{H} \dot{\mathbf{q}}}{\partial \mathbf{q}} \quad (18)$$

In Matlab code it looks like:

```
0 C_times_v = reshape(dHdt*v, length(v), 1) - reshape  
jacobian((0.5*v' * H * v), q), length(v), 1);
```

Alternatively, you can use the following formula:

$$\mathbf{C}\dot{\mathbf{q}} = \dot{\mathbf{H}}\dot{\mathbf{q}} - 0.5 \frac{\partial \text{vec}(\mathbf{H})}{\partial \mathbf{q}} (\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}) \quad (19)$$

where \otimes is a Kronecker product, and $\text{vec}()$ is vectorization of matrix (representing all its elements as a vector).

In Matlab code it looks like:

```
0 C_times_v = reshape(dHdt*v, [], 1) - 0.5*( jacobian(  
    reshape(H, [], 1), q)'*kron(v, v) );
```

GENERALIZED FORCES

Part 1, General case

You can express generalized forces of all kinds using discussed previously multiplication by the Jacobian:

$$\tau = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^\top \mathbf{f} \quad (20)$$

where \mathbf{f} is an external force, and \mathbf{r} is the radius-vector giving position of that force.

For a torque ξ applied to a rigid body, the corresponding generalized force is:

$$\tau = \mathbf{J}_w^\top \xi \quad (21)$$

where \mathbf{J}_w is the angular velocity Jacobian of that body. Note that both \mathbf{J}_w and ξ need to be expressed in the same basis.

GENERALIZED FORCES

Part 2, Conservative forces

If the force is conservative, it is often easy to describe potential energy U associated with it. Then you can find the relevant generalized forces as:

$$\mathbf{g} = -\frac{\partial U}{\partial \mathbf{q}} \quad (22)$$

Typically this is useful for gravitational forces and elastic forces.

Finally we can write the form of manipulator equations:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} = \boldsymbol{\tau} \quad (23)$$

Another popular form specifically points out conservative forces \mathbf{g} :

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \boldsymbol{\tau} \quad (24)$$

The most concise and useful for this class form is:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{c} = \boldsymbol{\tau} \quad (25)$$

where $\mathbf{c} = \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}$.

Importantly, manipulator equations can always be re-written in an explicit form (where the highest-order derivative is isolated on the left-hand-side):

$$\ddot{\mathbf{q}} = \mathbf{H}^{-1}(\tau - \mathbf{C}\dot{\mathbf{q}} - \mathbf{g}) \quad (26)$$

This is possible because \mathbf{H} is full-rank. This in turn is possible, because generalized coordinates are independent.

You can read more at:

- [Chapter 4. Robot Dynamics and Control](#) - part 3.2, an interesting derivation.
- [Robot Dynamics Lecture Notes. Robotic Systems Lab, ETH Zurich HS 2017:](#)
 - ▶ 2.5 Angular Velocity
 - ▶ Chapter 3. Dynamics
 - ▶ 3.4.2 Kinetic Energy
 - ▶ 3.4.3 Potential Energy
 - ▶ 3.4.5 Additional Constraints (note some notational differences)
 - ▶ 3.5.2 Deriving Generalized Equations of Motion

THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

github.com/SergeiSa/Fundamentals-of-robotics-2022

Check Moodle for additional links, videos, textbook suggestions.

