Parameter estimation

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Parameter estimation
Problem statement

Assume we have a system, whose model is defined in terms of parameters θ:

$$\dot{x} = f(x, u, \theta)$$

Task: knowing relations $f(x, u) = f(x, u, \theta)$, find a stabilizing control law.

In case of a linear system, the dynamics is given in the form:

$$\dot{x} = A(\theta)x + B(\theta)u$$

and the task is: knowing relations $A = A(\theta)$ and $B = B(\theta)$, find a stabilizing control law.

Parameter estimation
Parameter uncertainty sources

There are a number of reasons parameters of the model can be unknown:

- Parameters that can't be measured.
- Parameters change with time.
- Unmodeled dynamics.
- Unknown forces.
- Dynamic changes in the system (picking up a load, etc).

Unfortunately, in order to treat the problem we need to assume that we can, directly or indirectly, measure higher order derivatives.

Example 1

Let us consider spring-damper system: $m\ddot{q} + \mu\dot{q} + kq = u$. Introducing notation $y = u$ we arrive at:

$$y = m\ddot{q} + \mu\dot{q} + kq$$

Example 2
Consider a general mechanical system:

$$H\ddot{q} + c(\dot{q}, q) = u$$

This system is *linear* with respect to parameters such as m_i, $l_i l_j$, l_i^2, $m_i l_i^2$, $m_i g$, etc., where l_i are lengths, m_i are masses, g is the gravitational acceleration. Same as before, we make $y = u$. Therefore we can rewrite it as follows:

$$y = Y(\ddot{q}, \dot{q}, q)\theta$$

where matrix Y can be a nonlinear function of the coordinates q and their derivatives. It is called a *regressor*.

Parameter estimation
System Linear in Parameters

From here on we assume that we measure y and we know that y is a function of state and it's derivatives, and parameters θ.

Moreover, we assume that y is *linear with respect to parameters θ*:

$$y = M(x, \dot{x})\theta$$

where M will be referred to as regressor matrix.

We introduce definitions:

- $\tilde{\theta}$ - estimated value of parameters θ;
- $\varepsilon_\theta = \theta - \tilde{\theta}$ - parameter estimation error;
- $e_\theta = y - M\tilde{\theta}$ - parameter estimation output error.

Parameter estimation
Example

Going back to the spring-damper system $y = m\ddot{q} + \mu\dot{q} + kq$, we can denote:

- $\mathsf{M} = [\ddot{q}\ \dot{q}\ q]$ - regressor matrix;
- $\theta = [m\ \mu\ k]^\top$ - parameters;

Thus we get:

$$y = [\ddot{q}\ \dot{q}\ q][m\ \mu\ k]^\top = \mathsf{M}\theta$$

Another example is a pendulum: $y = ml^2\ddot{\varphi} + mgl\sin(\varphi)$. We can denote:

- $\mathsf{M} = [\ddot{\varphi}\ \sin(\varphi)]$ - regressor matrix;
- $\theta = [ml^2\ mgl]^\top$ - parameters;

Thus we get:

$$y = [\ddot{\varphi}\ \sin(\varphi)][ml^2\ mgl]^\top = \mathsf{M}\theta$$

Least Squares in parameter estimation

We want to minimize parameter estimation error ε_θ. However, we do not measure it directly. Let us instead minimize directly measured parameter estimation output error e_θ.

To this end we introduce a cost function J:

$$J = \frac{1}{2} e_\theta^\top e_\theta$$

Expanding the definition, we get:

$$J = \frac{1}{2}(y - M\tilde{\theta})^\top (y - M\tilde{\theta}) = \frac{1}{2}(y^\top y - 2\tilde{\theta}^\top M^\top y + \tilde{\theta}^\top M^\top M\tilde{\theta})$$

Derivative of J with respect to parameters estimate is:

$$\frac{\partial J}{\partial \tilde{\theta}} = -M^\top y + M^\top M\tilde{\theta}$$

We know that when the optimal estimation is found, the derivative $\frac{\partial J}{\partial \tilde{\theta}}$ of the cost function will be equal to zero. Therefore:

$$\frac{\partial J}{\partial \tilde{\theta}} = -\mathsf{M}^\top \mathsf{y} + \mathsf{M}^\top \mathsf{M} \tilde{\theta} = 0$$

$$\tilde{\theta} = (\mathsf{M}^\top \mathsf{M})^{-1} \mathsf{M}^\top \mathsf{y}$$

This presents the *least squares solution* for the estimation problem.

Least Squares in parameter estimation
Multiple measurements

If the parameters θ are constant, we can use *multiple measurements* to find them. Let us denote the value of M matrix for i-th measurement as M_i and the corresponding value of output vector y as y_i. Then we can introduce compound output and estimation matrices for n measurements:

$$\bar{M} = \begin{bmatrix} M_1 \\ ... \\ M_n \end{bmatrix}$$

$$\bar{y} = \begin{bmatrix} y_1 \\ ... \\ y_n \end{bmatrix}$$

Then we can use least squares to determine optimal parameter estimate same as before:

$$\tilde{\theta} = (\bar{M}^\top \bar{M})^{-1} \bar{M}^\top \bar{y}$$

Lecture slides are available via Moodle.

You can help improve these slides at:
https://github.com/SergeiSa/Linear-Control-Slides-Spring-2020

Check Moodle for additional links, videos, textbook suggestions.