Controllability

Ahmad Hamdan

May 2020

1 Introduction

A natural question arises in linear control theory: To what extent can closed-loop feedback $u = -Kx$ manipulate the behavior of the system $\dot{x} = Ax + Bu$. In this lecture we will provide concrete conditions on when the system can be controllable and we will provide some examples.

2 Controllability

The ability to design the eigenvalues of the closed-loop system with the choice of $\mathbf{K}$ relies on the system in $\dot{x} = Ax+Bu$ being *controllable*. The controllability of a linear system is determined entirely by the column space of the **controllability** matrix $\mathbf{C}$:

$$\mathbf{C} = \begin{bmatrix} B & AB & A^2B & ... & A^{n-1}B \end{bmatrix} \tag{1}$$

If the matrix $\mathbf{C}$ has n linearly independent columns, so that it spans all of R^n then the system $\dot{x} = Ax + Bu$ is controllable. The span of the columns of the controllability matrix $\mathbf{C}$ forms a Krylov subspace that determines which state vector directions in R^n may be manipulated with control. Thus, in addition to controllability implying arbitrary eigenvalue placement, it also implies that any state $\xi \in R^n$ is reachable in a finite time with some actuation signal $u(t)$. The following three conditions are equivalent:

1. *Controllability.* The span of $\mathbf{C}$ is R^n. The matrix $\mathbf{C}$ may be generated by

   ```
   ctrb(A,B)
   ```

   and and the rank may be tested to see if it is equal to n, by

   ```
   rank(ctrb(A,B))
   ```

2. *Arbitrary eigenvalue placement.* It is possible to design the eigenvalues of the closed-loop system through choice of feedback $u = -Kx$:

$$\frac{d}{dt}x = Ax + Bu = (A - BK)x \tag{2}$$

Given a set of desired eigenvalues, the gain K can be determined by

```
K=place(A,B,neweig);
```

3. *Reachability of R^n*. It is possible to steer the system to any arbitrary state $x(t) = \xi \in R^n$ in a finite time with some actuation signal $u(t)$.

Note that reachability also applies to open-loop systems. In particular, if a direction ξ is not in the span of $\mathbf{C}$, then it is impossible for control to push in this direction in either open-loop or closed-loop.

Examples. The notion of controllability is more easily understood by investigating a few simple examples. First, consider the following system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \Rightarrow \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \tag{3}$$

This system is not controllable, because the controllability matrix $\mathbf{C}$ consists of two linearly dependent vectors and does not span R^n. Even before checking the rank of the controllability matrix, it is easy to see that the system won't be controllable since the states x_1 and x_2 are completely decoupled and the actuation input u only effects the second state.
Modifying this example to include two actuation inputs makes the system controllable by increasing the control authority:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \tag{4}$$

This *fully actuated* system is clearly controllable because x_1 and x_2 may be independently controlled with u_1 and u_2 The controllability of this system is confirmed by checking that the columns of $\mathbf{C}$ do span R^n.
The most interesting cases are less obvious than these two examples. Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \Rightarrow \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \tag{5}$$

This two-state system is controllable with a single actuation input because the states x_1 and x_2 are now coupled through the dynamics. Similarly,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \Rightarrow \mathbf{C} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \tag{6}$$

is controllable even though the dynamics of x_1 and x_2 are decoupled, because the actuator $\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is able to simultaneously affect both states and they have different timescales.
Further you will see that controllability is intimately related to the alignment of the columns of $\mathbf{B}$ with the eigenvector directions of $\mathbf{A}$.

3 References

1. Data-Driven Science and Engineering: Machine Learning, Dynamical Systems, and Control by J. Nathan Kutz and Steven L. Brunton

2. https://www.youtube.com/watch?v=u5Sv7YKAkt4list=PLMrJAkhIeNNR20Mz-VpzgfQs5zrYi085mindex=5

3. Control Engineering Fifth Edition by Katsuhiko Ogata