State Space Representation

Fadi Younes

a state space representation is a linear representation of a dynamic system either in a continuous or discrete form.

The most general time-continuous linear dynamic system has the following form:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y = C(t)x(t) + D(t)u(t)$$

where:
t denotes time
the first equation is called the *state equation* and the second is called the *output equation*
$x(t)$ is the state vector, $u(t)$ is the input vector, and $y(t)$ is the output vector

Note: $x(t)$ is called the state vector because:

- Future output depends **only** on current state and future input
- Future output depends on past input **only through** current state
- State summarizes effect of past inputs on future output (like the memory of the system)

A, B, C and D are matrices where:
$A(t)$ is the dynamics matrix
$B(t)$ is the input matrix
$C(t)$ is the output matrix
$D(t)$ is the feedthrough matrix

What are the state variables?

the minimum set of variables that fully describe (enough information to predict the future behavior) the system.

The first step of representing a system is to select a state vector, which needs to be chosen according to the following:
1- A minimum number of state variables must be selected as components of the state vector.

How do we know what is the **minimum number**?
The minimum number is the order of the differential equation describing the system.
If we have a TF (transfer function) the the minimum number is the order of the denominator of the transfer function after canceling common factors in the numerator and denominator

2- The minimum number of state variables must be **linearly independent**.

Converting from state-space to a transfer function

as the above representation of the state space:

$$\begin{gathered}\dot{x}(t)=A(t)x(t)+B(t)u(t)\\ y=C(t)x(t)+D(t)u(t)\end{gathered}$$

we take the Laplace transformation (integral transform that converts a function of a real variable (time here) to a function of a complex variable):

$$\begin{gathered}sX(s)=AX(s)+BU(s)\\ Y(s)=CX(s)+DU(s)\end{gathered}$$

then solving for $X(s)$:

$$X(s)=(sI-A)^{-1}BU(s)$$

where I is the identity matrix.

Then, substituting this equation into $Y(s)$ equation above we get:

$$Y(s)=[C(sI-A)^{-1}B+D]U(s)$$

and this is the transfer function we have for our state space representation.

How do we benefit from state-space representation?

1- Stability
the state space model is **stable** if all eigenvalues of the matrix A are negative real numbers or the real part of the complex eigenvalues are negative. If at least one **eigenvalue** has a positive real part, then the system is unstable.

2- Controllability

A continuous time-invariant linear state-space model is **controllable** if and only if:

$$\operatorname{rank}\begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}=n$$

where rank is the number of linearly independent rows in a matrix, and where n is the number of state variables.

3- Observability

A continuous time-invariant linear state-space model is **observable** if and only if:

$$\operatorname{rank}\begin{bmatrix}\mathbf{C}\\ \mathbf{C}\mathbf{A}\\ \vdots\\ \mathbf{C}\mathbf{A}^{n-1}\end{bmatrix}=n.$$

2- Can be applied to a non-linear system

3- Can be applied to time invariant systems

4- Can be applied to multiple input multiple output systems known as (MIMO) systems.

References

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