Linear Algebra for LTI systems

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Which states are possible?

Consider discrete autonomous LTI system:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state of the system, expressed in the basis $\mathbf{O}$.
What possible values can $\mathbf{x}_{i+1}$ attain?

From the (1) follows that all possible $\mathbf{x}_{i+1}$ are in the *column space* of $\mathbf{A}$.

$$\mathbf{x}_{i+1} \in \mathcal{X} = \mathrm{col}(\mathbf{A}) \subseteq \mathbb{R}^n$$

How to check if the state is possible?

Part 1

How can we tell if a particular value of $\mathbf{x}_{i+1}$ is possible? More general, having an the equation $\mathbf{h} = \mathbf{A}\mathbf{x}$, how do check that for a particular $\mathbf{h}$, $\exists \mathbf{x}$, s.t. the equality is satisfied?

Let $\mathbf{P}$ be an orthonormal basis in the column space of $\mathbf{A}$:

$$\mathbf{P} = \text{orth}(\mathbf{A})$$

Columns of $\mathbf{P}$ span the column space of $\mathbf{A}$, or in other words, the column spaces of $\mathbf{A}$ and $\mathbf{P}$ and the same: $\mathcal{X} = \text{col}(\mathbf{A}) = \text{col}(\mathbf{P})$.

How to check if the state is possible?

Part 2

Vector $\mathbf{h}$ is expressed in the basis $\mathbf{O}$, which we can denote as $\mathbf{h}^{\mathbf{O}}$. Then, we can find coordinates of the projection of $\mathbf{h}$ onto $\mathcal{X}$ in the basis $\mathbf{P}$:

$$\mathbf{h}_p^{\mathbf{P}} = \mathbf{P}^\top \mathbf{h}^{\mathbf{O}}$$

In the basis $\mathbf{O}$, vector $\mathbf{h}_P$ is given by the equation $\mathbf{h}_p^{\mathbf{O}} = \mathbf{P}\mathbf{h}_p^{\mathbf{P}}$:

$$\mathbf{h}_p^{\mathbf{O}} = \mathbf{P}\mathbf{P}^\top \mathbf{h}^{\mathbf{O}}$$

Notice, that if vector $\mathbf{h}$ lies in the column space $\mathcal{X}$, its projection onto $\mathcal{X}$, namely $\mathbf{h}_p^{\mathbf{O}}$, should be equal to $\mathbf{h}$. Let us define projection residual $\mathbf{e}$:

$$\mathbf{e}^{\mathbf{O}} = \mathbf{h}_p^{\mathbf{O}} - \mathbf{h}^{\mathbf{O}}$$

Therefore we con formulate that $\mathbf{h} \in \mathcal{X}$ iff $\mathbf{e} = 0$.

How to check if the state is possible?

Part 3

Now we can formulate the condition for $\mathbf{h} \in \mathcal{X} = \mathrm{col}(\mathbf{A})$:

$$\mathbf{P}\mathbf{P}^\top \mathbf{h} - \mathbf{h} = 0$$

or:

$$(\mathbf{P}\mathbf{P}^\top - \mathbf{I})\mathbf{h} = 0$$

where $\mathbf{P} = \mathrm{orth}(\mathbf{A})$

Check if affine system is stabilizable

Part 1

Consider affine linear system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \tag{2}$$

Can it be stabilized? Let the control law be give affine:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} - \mathbf{u}_0$$

We want to find such $\mathbf{K}$ and $\mathbf{u}_0$ that:

- dynamics of the system becomes $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$.
- $\mathbf{A} - \mathbf{B}\mathbf{K} < 0$.

Check if affine system is stabilizable

Part 2

In order for the $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \\ \mathbf{u} = -\mathbf{K}\mathbf{x} - \mathbf{u}_0 \end{cases}$ to become $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$, the following equality needs to hold:

$$\mathbf{B}\mathbf{u}_0 = \mathbf{c}$$

In other words, $\mathbf{c}$ needs to be in the column space of $\mathbf{B}$. And we know the conditions we need to check:

$$(\mathbf{P}\mathbf{P}^\top - \mathbf{I})\mathbf{c} = 0,$$

where $\mathbf{P} = \text{orth}(\mathbf{B})$

Criteria without orthonormal basis

Finding an orthonormal basis might be excessive for this task, and there is a more direct way of checking if for a given $\mathbf{h}$, $\exists \mathbf{x}$, s.t. $\mathbf{h} = \mathbf{A}\mathbf{x}$. If there exists $\mathbf{x}^*$ such that $\mathbf{h} = \mathbf{A}\mathbf{x}^*$, then it can be found as:

$$\mathbf{x}^* = \text{argmin } ||\mathbf{h} - \mathbf{A}\mathbf{x}||$$

Solution to this *linear least squares problem* is a psuedo inverse:

$$\mathbf{x}^* = \mathbf{A}^+\mathbf{h}$$

Therefore, projection residual equation can be written as:

$$\mathbf{e} = \mathbf{A}\mathbf{x}^* - \mathbf{h}$$

Same as before, original question then comes down to proving that $\mathbf{e} = 0$:

$$\mathbf{A}\mathbf{A}^+\mathbf{h} - \mathbf{h} = 0$$

or

$$(\mathbf{A}\mathbf{A}^+ - \mathbf{I})\mathbf{h} = 0$$

Checking if an arbitrary point can be stabilized

Consider a linear system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}_0) - \mathbf{u}_0 \end{cases} \tag{3}$$

Is point $\mathbf{x}_0$ stabilizable? What this question means, is can we find such $\mathbf{K}$ and $\mathbf{u}_0$ that the system converges to $\mathbf{x}_0$?

Assume we are at the point $\mathbf{x}_0$. Then $\mathbf{K}(\mathbf{x} - \mathbf{x}_0) = 0$. Can we make sure we stay at this point? If not, it is not stabilized.
This comes down to solving equation:

$$\mathbf{A}\mathbf{x}_0 - \mathbf{B}\mathbf{u}_0 = 0$$

In other words, we need to find if $\mathbf{A}\mathbf{x}_0$ is in the column space of $\mathbf{B}$.
Here is the criterion:

$$(\mathbf{B}\mathbf{B}^+ - \mathbf{I})(\mathbf{A}\mathbf{x}_0) = 0$$

All stabilizable points

For the system from the previous example we can easily find all points that are stabilizable. For that we consider equation:

$$\mathbf{A}\mathbf{x}_0 - \mathbf{B}\mathbf{u}_0 = 0$$

but this time we make both $\mathbf{x}_0$ and $\mathbf{u}_0$ our variables. This system has a nontrivial solution if matrix $[\mathbf{A},\ -\mathbf{B}]$ has a nontrivial *null space*.

Let $\mathbf{z} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix}$. And let $\mathbf{N}$ be a basis in the null space of $[\mathbf{A},\ -\mathbf{B}]$:

$$\mathbf{N} = \text{null}([\mathbf{A},\ -\mathbf{B}])$$

Then any combination of its columns produces a vector $\mathbf{z}$, which is stabilizable:

$$\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix} = \mathbf{z} = \mathbf{N}\mathbf{r}$$

where $\mathbf{r}$ is a random vector.

Lecture slides are available via Moodle.

You can help improve these slides at:
`https://github.com/SergeiSa/Linear-Control-Slides-Spring-2020`

Check Moodle for additional links, videos, textbook suggestions.