

# Parameter estimation

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- Parameter estimation
  - Problem statement
  - Parameter uncertainty sources
  - Problem statement - more precise
  - System Linear in Parameters
  - Example
- Least Squares in parameter estimation
  - Problem statement
  - Solution
  - Multiple measurements

# Parameter estimation

## Problem statement

Assume we have a system, whose model is defined in terms of parameters  $\theta$ :

$$\dot{x} = f(x, u, \theta)$$

Task: knowing relations  $f(x, u) = f(x, u, \theta)$ , find a stabilizing control law.

In case of a linear system, the dynamics is given in the form:

$$\dot{x} = A(\theta)x + B(\theta)u$$

and the task is: knowing relations  $A = A(\theta)$  and  $B = B(\theta)$ , find a stabilizing control law.

# Parameter estimation

## Parameter uncertainty sources

There are a number of reasons parameters of the model can be unknown:

- Parameters that can't be measured.
- Parameters change with time.
- Unmodeled dynamics.
- Unknown forces.
- Dynamic changes in the system (picking up a load, etc).

# Parameter estimation

## Problem statement - Example 1

Unfortunately, in order to treat the problem we need to assume that we can, directly or indirectly, measure higher order derivatives.

### *Example 1*

Let us consider spring-damper system:  $m\ddot{q} + \mu\dot{q} + kq = u$ . Introducing notation  $y = u$  we arrive at:

$$y = m\ddot{q} + \mu\dot{q} + kq$$

# Parameter estimation

## Problem statement - Example 2

### *Example 2*

Consider a general mechanical system:

$$H\ddot{q} + c(\dot{q}, q) = u$$

This system is *linear* with respect to parameters such as  $m_i$ ,  $l_i/l_j$ ,  $l_i^2$ ,  $m_i l_i^2$ ,  $m_i g$ , etc., where  $l_i$  are lengths,  $m_i$  are masses,  $g$  is the gravitational acceleration. Same as before, we make  $y = u$ . Therefore we can rewrite it as follows:

$$y = Y(\ddot{q}, \dot{q}, q)\theta$$

where matrix  $Y$  can be a nonlinear function of the coordinates  $q$  and their derivatives. It is called a *regressor*.

# Parameter estimation

## System Linear in Parameters

From here on we assume that we measure  $y$  and we know that  $y$  is a function of state and it's derivatives, and parameters  $\theta$ .

Moreover, we assume that  $y$  is *linear with respect to parameters*  $\theta$ :

$$y = M(x, \dot{x})\theta$$

where  $M$  will be referred to as regressor matrix.

We introduce definitions:

- $\tilde{\theta}$  - estimated value of parameters  $\theta$ ;
- $\varepsilon_{\theta} = \theta - \tilde{\theta}$  - parameter estimation error;
- $e_{\theta} = y - M\tilde{\theta}$  - parameter estimation output error.

# Parameter estimation

## Example

Going back to the spring-damper system  $y = m\ddot{q} + \mu\dot{q} + kq$ , we can denote:

- $M = [\ddot{q} \ \dot{q} \ q]$  - regressor matrix;
- $\theta = [m \ \mu \ k]^\top$  - parameters;

Thus we get:

$$y = [\ddot{q} \ \dot{q} \ q][m \ \mu \ k]^\top = M\theta$$

Another example is a pendulum:  $y = ml^2\ddot{\varphi} + mgl\sin(\varphi)$ . We can denote:

- $M = [\ddot{\varphi} \ \sin(\varphi)]$  - regressor matrix;
- $\theta = [ml^2 \ mgl]^\top$  - parameters;

Thus we get:

$$y = [\ddot{\varphi} \ \sin(\varphi)][ml^2 \ mgl]^\top = M\theta$$



# Least Squares in parameter estimation

## Problem statement

We want to minimize parameter estimation error  $\varepsilon_\theta$ . However, we do not measure it directly. Let us instead minimize directly measured parameter estimation output error  $\mathbf{e}_\theta$ .

To this end we introduce a cost function  $J$ :

$$J = \frac{1}{2} \mathbf{e}_\theta^\top \mathbf{e}_\theta$$

Expanding the definition, we get:

$$J = \frac{1}{2} (\mathbf{y} - \mathbf{M}\tilde{\boldsymbol{\theta}})^\top (\mathbf{y} - \mathbf{M}\tilde{\boldsymbol{\theta}}) = \frac{1}{2} (\mathbf{y}^\top \mathbf{y} - 2\tilde{\boldsymbol{\theta}}^\top \mathbf{M}^\top \mathbf{y} + \tilde{\boldsymbol{\theta}}^\top \mathbf{M}^\top \mathbf{M} \tilde{\boldsymbol{\theta}})$$

Derivative of  $J$  with respect to parameters estimate is:

$$\frac{\partial J}{\partial \tilde{\boldsymbol{\theta}}} = -\mathbf{M}^\top \mathbf{y} + \mathbf{M}^\top \mathbf{M} \tilde{\boldsymbol{\theta}}$$

# Least Squares in parameter estimation

## Solution

We know that when the optimal estimation is found, the derivative  $\frac{\partial J}{\partial \tilde{\theta}}$  of the cost function will be equal to zero. Therefore:

$$\frac{\partial J}{\partial \tilde{\theta}} = -M^T y + M^T M \tilde{\theta} = 0$$

$$\tilde{\theta} = (M^T M)^{-1} M^T y$$

This presents the *least squares solution* for the estimation problem.

# Least Squares in parameter estimation

## Multiple measurements

If the parameters  $\theta$  are constant, we can use *multiple measurements* to find them. Let us denote the value of  $M$  matrix for  $i$ -th measurement as  $M_i$  and the corresponding value of output vector  $y$  as  $y_i$ .

Then we can introduce compound output and estimation matrices for  $n$  measurements:

$$\bar{M} = \begin{bmatrix} M_1 \\ \dots \\ M_n \end{bmatrix}$$

$$\bar{y} = \begin{bmatrix} y_1 \\ \dots \\ y_n \end{bmatrix}$$

Then we can use least squares to determine optimal parameter estimate same as before:

$$\tilde{\theta} = (\bar{M}^\top \bar{M})^{-1} \bar{M}^\top \bar{y}$$

Lecture slides are available via Moodle.

You can help improve these slides at:

<https://github.com/SergeiSa/Linear-Control-Slides-Spring-2020>

Check Moodle for additional links, videos, textbook suggestions.