Control for systems with explicit constraints

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Explicit and no constraints

LTI systems we studied before have the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state of the system. This form has no explicit constraints.

Let us introduce one form of a linear dynamical system with explicit constraints:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\lambda \\ \mathbf{G}\dot{\mathbf{x}} = \mathbf{0} \end{cases}$$

where $\mathbf{G}$ is a constraint matrix, $\lambda \in \mathbb{R}^k$ is the constraints reaction forces, and $\mathbf{F}$ is the reaction force Jacobian.

Explicit and implicit constraints
Example

Consider a two mass system with a spring:

$$\begin{cases} \ddot{x}_1 + \mu \dot{x}_1 + k(x_1 - x_2) = 0 \\ \ddot{x}_2 + \mu \dot{x}_2 + k(x_2 - x_1) = 0 \end{cases}$$

We can add a constraint $x_2 = 10$. This implies that $\ddot{x}_2 = 0$. Corresponding system of equations is:

$$\begin{cases} \ddot{x}_1 + \mu \dot{x}_1 + k(x_1 - x_2) = 0 \\ \ddot{x}_2 + \mu \dot{x}_2 + k(x_2 - x_1) = \lambda \\ \ddot{x}_2 = 0 \end{cases}$$

But that is the same as:

$$\ddot{x}_1 + \mu \dot{x}_1 + k(x_1 - 10) = 0$$

Thus we transformed the system with *explicit constraints* into a system with *implicit constraints*

Examples of systems with constraints

Figure 1: Walking robots

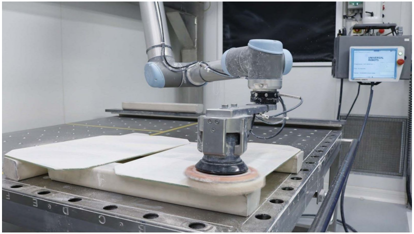

Figure 2: Polishing with industrial arms

Typical reasons why explicit constraints arise

Explicit constraints are usually not a necessity and not a physical property of the problem. However, they are often encountered in practice. Typical situations when they are encountered as as follows:

- Systems with contact interactions.
- Hybrid systems (two or more different dynamics which switch between one-another).
- Nonholonomic constraints in the dynamics (dynamics of a unicycle, bicycle, etc.).
- Dynamics is more clear and easy to work which when non-minimal representation is used.

Ways to control systems with explicit constraints

There are basic ways to deal with such systems:

- Reduce to a system with implicit constraints and control that system instead.
- Treat reaction forces as a yet another external force.
- Design control law based on the explicit representation of constraints.

Geometry of the constrained LTI system

Let's consider the system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{F}\lambda \\ \mathbf{G}\dot{\mathbf{x}} = \mathbf{0} \end{cases}$$

We can deduce that:

- $\dot{\mathbf{x}} \in \text{null}(\mathbf{G})$ - from the second eq.
- $\mathbf{F}\lambda \in (\text{null}(\mathbf{G}))^{\perp}$ - otherwise the reaction force actually influences the motion allowed by constraint.

Let $\mathbf{n}_1$, $\mathbf{n}_2$, ..., $\mathbf{n}_{n-k}$ to be an orthonormal basis in the null space of $\mathbf{G}$), and $\mathbf{N} = [\mathbf{n}_1, \ \dots \ \mathbf{n}_{n-k}]$. Then all possible accelerations $\dot{\mathbf{x}}$ can be represented as:

$$\dot{\mathbf{x}} = \mathbf{N}\dot{\mathbf{z}}$$

where $\mathbf{z} \in \mathbb{R}^{n-k}$ are reduced coordinates in the reduced dynamics of the system, or coordinates in the zero space of the constraints.

Reducing LTI to a system with implicit constraints

A simple way to reduce an LTI system with explicit constraints to a system with implicit constraints is to multiply it by $\mathbf{N}^\top$:

$$\mathbf{N}^\top \dot{\mathbf{x}} = \mathbf{N}^\top \mathbf{A}\mathbf{x} + \mathbf{N}^\top \mathbf{B}\mathbf{u} + \mathbf{N}^\top \mathbf{F}\lambda$$

Remembering that $\mathbf{N}$ is representing an orthonotmal basis, and $\mathbf{N}\mathbf{N}^\top = \mathbf{I}$:

$$\mathbf{N}^\top \dot{\mathbf{x}} = \mathbf{N}^\top \mathbf{A}\mathbf{N}\mathbf{N}^\top \mathbf{x} + \mathbf{N}^\top \mathbf{B}\mathbf{u}$$

$$\dot{\mathbf{z}} = \mathbf{N}^\top \mathbf{A}\mathbf{N}\mathbf{z} + \mathbf{N}^\top \mathbf{B}\mathbf{u}$$

That is a system with implicit constraints.

Constrained LQR

Assume we had a cost function:

$$J = \int_0^\infty \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{u}^\top \mathbf{R} \mathbf{u} \, dt$$

Introducing change of variables $\mathbf{z} = \mathbf{N}^\top \mathbf{x}$ we obtain:

$$J = \int_0^\infty \mathbf{z}^\top \mathbf{N}^\top \mathbf{Q} \mathbf{N} \mathbf{z} + \mathbf{u}^\top \mathbf{R} \mathbf{u} \, dt$$

Together with previously considered dynamics we obtain all values necessary for the formulation of the optimal control problem, which will constitute a Riccati eq. in new variables. It can be solved numerically:
`K = lqr(`$(\mathbf{N}^\top \mathbf{A} \mathbf{N})$, $(\mathbf{N}^\top \mathbf{B})$, $(\mathbf{N}^\top \mathbf{Q} \mathbf{N})$, $\mathbf{R}$ `)`

Lecture slides are available via Moodle.

You can help improve these slides at:
`https://github.com/SergeiSa/Linear-Control-Slides-Spring-2020`

Check Moodle for additional links, videos, textbook suggestions.