Adaptive Control

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Content

- Feed-forward Adaptive Control
  - Parameter estimation
  - FF control of mechanical systems
- Feedback Adaptive Control
  - LQR
- Real-time Parameter Estimation
  - Linear feedback form
  - Parameter estimation error dynamics
  - Parameter estimation error dynamics stability

Feed-forward Adaptive Control

Parameter estimation

Last time we learned that we can represent a system in the form:

$$\mathbf{y} = \mathbf{M}(\mathbf{x}, \dot{\mathbf{x}})\theta$$

and then collect data and solve it for θ via Linear Least Squares or, equivalently, we pseudoinverse.

Let us demonstrate how it can be used to generate feedback control.

Feed-forward Adaptive Control

FF control of mechanical systems

One example of a system for which we can do it is the following:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q}) = \mathbf{u} \quad \Leftrightarrow \quad \begin{cases} \mathbf{y} = \mathbf{Y}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q})\theta \\ \mathbf{u} \equiv \mathbf{y} \end{cases}$$

(see previous lecture for notation and derivation)

Assume we have desired trajectory $\mathbf{q}^* = \mathbf{q}^*(t)$ and we found precise values of θ. Then we can define control as follows:

$$\mathbf{u} = \mathbf{K}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{K}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}}) + \mathbf{Y}(\ddot{\mathbf{q}}^*, \dot{\mathbf{q}}, \mathbf{q})\theta$$

where

$$\mathbf{Y}(\ddot{\mathbf{q}}^*, \dot{\mathbf{q}}, \mathbf{q})\theta = \mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q})$$

Feed-forward Adaptive Control

FF control of mechanical systems

Let's prove that the proposed control is stable. We know that:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q}) = \mathbf{u} \\ \mathbf{u} = \mathbf{K}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{K}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}}) + \mathbf{H}\ddot{\mathbf{q}}^* + \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q}) \end{cases}$$

then, introducing $\mathbf{e} = \mathbf{q}^* - \mathbf{q}$, and choosing $\mathbf{K}_p = \mathbf{H}\mathbf{D}_p$ and $\mathbf{K}_d = \mathbf{H}\mathbf{D}_d$, where $\mathbf{D}_p > 0$ and $\mathbf{D}_d > 0$ are positive-definite diagonal matrices:

$$\mathbf{H}\ddot{\mathbf{q}}^* - \mathbf{H}\ddot{\mathbf{q}} + \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q}) - \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q}) + \mathbf{K}_p(\mathbf{q}^* - \mathbf{q}) + \mathbf{K}_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}}) = \mathbf{0}$$

$$\mathbf{H}\ddot{\mathbf{e}} + \mathbf{K}_d\dot{\mathbf{e}} + \mathbf{K}_p\mathbf{e} = \mathbf{0}$$

$$\ddot{\mathbf{e}} + \mathbf{D}_d\dot{\mathbf{e}} + \mathbf{D}_p\mathbf{e} = \mathbf{0}$$

And thus we got a system of independent second order linear differential equations with strictly positive coefficients, which are stable:

$$\ddot{e}_i + d_{d,i}\dot{e}_i + d_{p,i}e_i = 0$$

Feedback Adaptive Control

LQR

Assume we had a linear system, whose model is a function of the estimated parameters θ:

$$\mathbf{A} = \mathbf{A}(\theta), \;\; \mathbf{B} = \mathbf{B}(\theta)$$

Then if we know a good estimate of the parameters $\tilde{\theta}$, we can use matrices $\mathbf{A}$ and $\mathbf{B}$ to solve Riccati eq.

$$\begin{cases} \mathbf{Q} - \mathbf{S}\mathbf{B}(\tilde{\theta})\mathbf{R}^{-1}\mathbf{B}^{\top}(\tilde{\theta})\mathbf{S} + 2\mathbf{S}\mathbf{A}(\tilde{\theta}) = 0 \\ \mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^{\top}(\tilde{\theta})\mathbf{S}\mathbf{x} \end{cases}$$

However, this guarantees neither optimality, nor stability.

Real-time Parameter Estimation

Linear feedback form

In order to use control methods discussed previously, we need to be able to estimate parameters on the fly. This is especially true if the parameters are changing in time.

We can estimate parameters in real time same as we did with state estimation, using a linear feedback law:

$$\frac{d}{dt}\tilde{\theta} = \Gamma \mathbf{M}^\top \mathbf{e}_\theta$$

where $\mathbf{M}$ is a regressor matrix.

Real-time Parameter Estimation

Parameter estimation error dynamics

Since $\mathbf{e}_\theta = \mathbf{y} - \mathbf{M}\tilde{\theta}$, we have:

$$\frac{d}{dt}\tilde{\theta} = \Gamma \mathbf{M}^\top (\mathbf{y} - \mathbf{M}\tilde{\theta})$$

By definition $\varepsilon_\theta = \theta - \tilde{\theta}$ and $\frac{d}{dt}\varepsilon_\theta = \frac{d}{dt}\theta - \frac{d}{dt}\tilde{\theta}$, and if we assume that $\frac{d}{dt}\theta = 0$, we get:

$$\frac{d}{dt}\varepsilon_\theta = \Gamma \mathbf{M}^\top (\mathbf{M}(\theta - \varepsilon_\theta) - \mathbf{y}) = \Gamma \mathbf{M}^\top (\mathbf{M}\theta - \mathbf{M}\varepsilon_\theta - \mathbf{M}\theta)$$

where we used the fact that by definition $\mathbf{M}\theta = \mathbf{y}$. Finally, we get the parameter estimation error dynamics equation:

$$\frac{d}{dt}\varepsilon_\theta = -\Gamma \mathbf{M}^\top \mathbf{M}\varepsilon_\theta$$

Real-time Parameter Estimation

Parameter estimation error dynamics stability

In order to achieve correct estimates of parameters, parameter estimation error dynamics needs to be stable:

$$-\Gamma \mathbf{M}^\top \mathbf{M} < 0$$

This means that we need to find such Γ that $\Gamma \mathbf{M}^\top \mathbf{M} > 0$. This is a non-trivial task. If $\text{null}(\mathbf{M}^\top \mathbf{M}) \neq 0$, then it is generally impossible.

If $\mathbf{M}^\top \mathbf{M}$ is invertible, we can choose Γ as:

$$\Gamma = \mathbf{D}(\mathbf{M}^\top \mathbf{M})^{-1}$$

where $\mathbf{D} > 0$. Then $-\Gamma \mathbf{M}^\top \mathbf{M} = -\mathbf{D} < 0$, and the estimator will be stable.

Lecture slides are available via Moodle.

You can help improve these slides at:
`https://github.com/SergeiSa/Linear-Control-Slides-Spring-2020`

Check Moodle for additional links, videos, textbook suggestions.