# DC motor

## Mechatronics, Lecture 4

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Fall 2023

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# DC motor working principle

The basic idea of DC motor's operation is the use of Lorentz force: we run a current through a wire in magnetic field created by permanent magnets. As the wire forms a loop, this creates a net torque, rotating the wire and the shaft attached to it. Once the wire rotates by a certain angle, the brushes that supply electrical connection switch the direction of the DC current, allowing rotation to continue (instead of reversing)

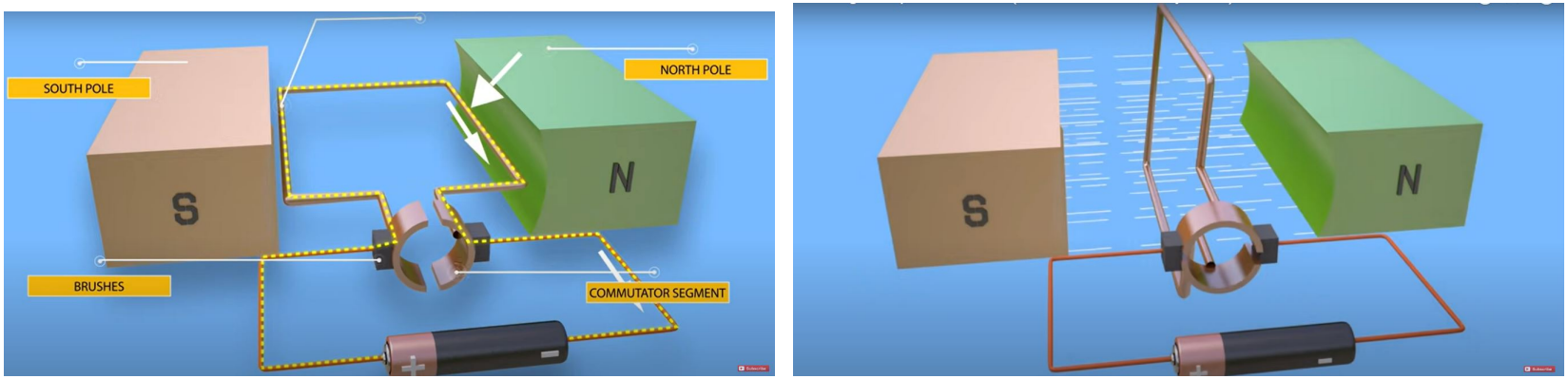


Figure 1: DC motor scheme. Images from `https://youtu.be/j_F4limaHYI`

# DC motor mechanical model, 1

We can describe the dynamics of the motor shaft:

$$J\frac{d}{dt}\omega = \tau \tag{1}$$

where $J$ is moment of inertia, $\omega$ is angular velocity and $\tau$ is torque. If we additionally consider linear viscous friction $F\omega$, we get:

$$J\frac{d}{dt}\omega + F\omega = \tau \tag{2}$$

The torque $\tau$ can be modeled as a linear function of current in the motor winding:

$$\tau = C_\tau I \tag{3}$$

where $C_\tau$ is torque coefficient.

# DC motor mechanical model, 2

Note that as long as the variable is angular velocity of the shaft $\omega$, the dynamics is a first-order ODE. If we instead consider orientation of the shaft $\varphi$, the result is a second-order ODE.

$$J\frac{d^2}{dt^2}\varphi + F\frac{d}{dt}\varphi = \tau \tag{4}$$

$$\frac{d}{dt}\varphi = \omega \tag{5}$$

# DC motor mechanical model, 3

The ODE $J\frac{d}{dt}\omega + F\omega = \tau$ is equivalent to the model of RL circuit. This means that the frequency response (considering torque as input and angular velocity as output) is:

$$|W(\alpha)| = \frac{1}{\sqrt{J^2\alpha^2 + F^2}} \tag{6}$$

where $\alpha$ is the input frequency.

- As the input frequency goes to infinity, the amplitude gain goes to zero.
- As the input frequency goes to zero, the amplitude gain goes to $\frac{1}{F}$.

# DC motor electrical model

DC motor winding is essentially a RL circuit. We can describe the electrodynamics of the motor winding as:

$$L\frac{d}{dt}I + RI + C_w\omega = u \tag{7}$$

where $I$ is current in motor winding, $L$ is induction coefficient of the motor winding, $R$ is resistance of the motor winding, $C_w$ is back-EMF coefficient and $u$ is input voltage.

Note that this model also behaves like a RL circuit.

# DC motor electro-mechanical model

Full electro-mechanical model of a DC motor is given by the next system of ODEs:

$$\begin{cases} L\dot{I} + RI + C_w\omega = u \\ J\dot{\omega} + F\omega = C_\tau I \end{cases} \tag{8}$$

Laplace transform of this model is:

$$\begin{cases} LsI(s) + RI(s) + C_w\omega(s) = u(s) \\ Js\omega(s) + F\omega(s) = C_\tau I(s) \end{cases} \tag{9}$$

# DC motor, transfer functions, 1

We can describe transfer functions:

$$\begin{bmatrix} Ls + R & C_w \\ -C_\tau & Js + F \end{bmatrix} \begin{bmatrix} I(s) \\ \omega(s) \end{bmatrix} = \begin{bmatrix} u(s) \\ 0 \end{bmatrix} \tag{10}$$

$$\det = (Ls + R)(Js + F) + C_w C_\tau \tag{11}$$

$$\text{inv} = \frac{1}{\det} \begin{bmatrix} Js + F & -C_w \\ C_\tau & Ls + R \end{bmatrix} \tag{12}$$

Giving us transfer functions:

$$I(s) = \frac{Js + F}{(Ls + R)(Js + F) + C_w C_\tau} u(s), \tag{13}$$

$$\omega(s) = \frac{C_\tau}{(Ls + R)(Js + F) + C_w C_\tau} u(s) \tag{14}$$

# DC motor, transfer functions, 2

We can open the brackets in the transfer function:

$$\omega(s) = \frac{C_\tau}{JLs^2 + (LF + JR)s + FR + C_w C_\tau} u(s) \tag{15}$$

With that, we can transform the model back to time domain, giving us a second-order ODE:

$$JL\ddot{\omega} + (LF + JR)\dot{\omega} + (FR + C_w C_\tau)\omega = C_\tau u \tag{16}$$

# DC motor, state-space

We can write a state-space model with state variables $I$ and $\omega$:

$$\begin{bmatrix} \dot{I} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -R/L & -C_w/L \\ C_\tau/J & -F/J \end{bmatrix} \begin{bmatrix} I \\ \omega \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u \tag{17}$$

# Angle control

Orientation of the motor is given by the angle $\varphi$, where $\dot{\varphi}(t) = \omega(t)$, or equivalently $s\varphi(s) = \omega(s)$. The transfer function from input voltage to angle $\varphi$ is:

$$\varphi(s) = \frac{C_\tau}{JLs^3 + (LF + JR)s^2 + (FR + C_w C_\tau)s} u(s) \tag{18}$$

The state-space model then becomes:

$$\begin{bmatrix} \dot{I} \\ \dot{\omega} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} -R/L & -C_w/L & 0 \\ C_\tau/J & -F/J & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I \\ \omega \\ \varphi \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix} u \tag{19}$$

# Payload

Let us consider a payload with moment of inertia $J_p$ and torque $\tau_p$. Then electro-mechanical dynamics of the motor is:

$$\begin{cases} L\dot{I} + RI + C_w\omega = u \\ (J + J_p)\dot{\omega} + F\omega = C_\tau I + \tau_p \end{cases} \tag{20}$$

For example, if the payload is a pendulum with mass $m$ and length $l$, the dynamics becomes:

$$\begin{cases} L\dot{I} + RI + C_w\dot{\varphi} = u \\ (J + ml^2)\ddot{\varphi} + F\dot{\varphi} = C_\tau I + mgl\sin(\varphi) \end{cases} \tag{21}$$

# Reducer (gearbox)

A reducer is a mechanism for reducing angular velocity of the output shaft of the motor, while increasing the output torque. Let us define angular velocity of the output shaft $\omega_o$ and torque of the output shaft $\tau_o$. An ideal reducer is defined as:

$$\omega_o = \frac{1}{N}\omega \tag{22}$$

$$\tau_o = N\tau \tag{23}$$

where $N > 1$ is the reduction ratio.

The key idea is that an ideal reducer allows us to model orientation and velocity of the input shaft and the output shaft using the same coordinates.

# Moment of inertia with a reducer

Consider a payload with inertial $J_p$ attached to the output shaft of the reducer, rotating at the angular velocity $\omega_o$. The kinetic energy of the entire system is given by:

$$T = \frac{1}{2}J\omega^2 + \frac{1}{2}J_p\omega_o^2 = \tag{24}$$

$$= \frac{1}{2}J\omega^2 + \frac{1}{2}\frac{1}{N^2}J_p\omega^2 = \tag{25}$$

$$= \frac{1}{2}(J + J_p/N^2)\omega^2 \tag{26}$$

Thus, dynamic of the drive becomes:

$$\begin{cases} L\dot{I} + RI + C_w\omega = u \\ (J + J_p/N^2)\dot{\omega} + F\omega = C_\tau I + \tau_p/N \end{cases} \tag{27}$$

# Read more

- University of Michigan. DC Motor Speed: System Modeling
- First-Order DC Electric Motor Model, Mark Drela, MIT Aero & Astro
- University of Alberta. DC Motor Speed Modeling

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Mechatronics-2023