PID Control

Mechatronics, Lecture 7

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Content

- Steady State

One of the most standard and widely used control laws is proportional-integral-derivative (PID) control. It is especially suitable for:

- SISO control, decentralized control, low-level control loop
- stabilizing control,
- shaping frequency response,
- shaping performance / step response,
- robust control (rejecting constant additive disturbances).

Proportional-derivative (PD) control is quite similar (sans robustness), and is widely used in theoretical research.

DC motor model

Dynamics of the DC motor can be represented as:

$$J L \ddot{\omega} + (LF + JR)\dot{\omega} + (FR + C_w C_\tau)\omega = C_\tau u \tag{1}$$

We can re-write the model in new variables:

$$\ddot{\omega} + a\dot{\omega} + c\omega = bu \tag{2}$$

where $a = \frac{LF+JR}{JL}$, $c = \frac{FR+C_w C_\tau}{JL}$, and $b = \frac{C_\tau}{JL}$.

If we want to control angular velocity with a *PD control*, the control law will take form:

$$u = -K_p \omega - K_d \dot{\omega} \tag{3}$$

Substituting the control law into the dynamics equations gives us *closed-loop dynamics*:

$$\ddot{\omega} + a\dot{\omega} + c\omega = -bK_p\omega - bK_d\dot{\omega} \tag{4}$$

Grouping the terms we get:

$$\ddot{\omega} + (a + bK_d)\dot{\omega} + (c + bK_p)\omega = 0 \tag{5}$$

We can manipulate coefficients K_p and K_d to achieve desired behavior of the system.

PD Control example

Consider the following dynamical system:

$$\ddot{\omega} + 2\dot{\omega} + 5\omega = 0.5u \tag{6}$$

We will attempt to find PD control law that turns it into a system $\ddot{\omega} + 5\dot{\omega} + 10\omega = 0$.

We need to solve the following linear equations:

$$2 + 0.5K_d = 5 \tag{7}$$

$$5 + 0.5K_p = 10 \tag{8}$$

giving us $K_d = 6$ and $K_p = 10$ and PD control law:

$$u = -10\omega - 6\dot{\omega} \tag{9}$$

Often we use control to follow a *reference signal* $\omega_r(t)$. Control law in that case takes form:

$$u = K_p(\omega_r(t) - \omega) + K_d(\dot{\omega}_r(t) - \dot{\omega}) \tag{10}$$

Substituting it into dynamics equation $\ddot{\omega} + a\dot{\omega} + c\omega = bu$ we get:

$$\ddot{\omega} + a\dot{\omega} + c\omega = bK_p(\omega_r(t) - \omega) + bK_d(\dot{\omega}_r(t) - \dot{\omega}) \tag{11}$$

$$\ddot{\omega} + (a + bK_d)\dot{\omega} + (c + bK_p)\omega = bK_p\omega_r(t) + bK_d\dot{\omega}_r(t) \tag{12}$$

We can transform the last equation into Laplace domain:

$$(s^2 + (a + bK_d)s + (c + bK_p))\omega(s) = (bK_p + bK_d s)\omega_r(s) \quad (13)$$

We find transfer function from the reference signal to the angular velocity $\omega(s)$:

$$W_r(s) = \frac{bK_d s + bK_p}{s^2 + (a + bK_d)s + (c + bK_p)} \quad (14)$$

$$\omega(s) = W_r(s)\omega_r(s) \quad (15)$$

Sometimes it is hard to model the motor exactly; In particular, this can be expressed by considering additive disturbance:

$$JL\ddot{\omega} + (LF + JR)\dot{\omega} + (FR + C_w C_\tau)\omega = C_\tau u + d \tag{16}$$

where d is the additive disturbance. We can re-write the model in new variables:

$$\ddot{\omega} + a\dot{\omega} + c\omega = bu + d. \tag{17}$$

If we want to control angular velocity with a *PID control*, the control law will take form:

$$u(t) = -K_d \dot{\omega}(t) - K_p \omega(t) - K_i \int_0^t \omega(\tau) d\tau \tag{18}$$

Defining φ such that $\dot{\varphi} = \omega$ we get:

$$u(t) = -K_d \ddot{\varphi}(t) - K_p \dot{\varphi}(t) - K_i \varphi(t) \tag{19}$$

Substituting the control law into the dynamics equations gives us *closed-loop dynamics*:

$$\dddot{\varphi} + a\ddot{\varphi} + c\dot{\varphi} = d - bK_d\ddot{\varphi} - bK_p\dot{\varphi} - bK_i\varphi \tag{20}$$

Grouping the terms we get:

$$\dddot{\varphi} + (a + bK_d)\ddot{\varphi} + (c + bK_p)\dot{\varphi} + bK_i\varphi = d \tag{21}$$

Considering the steady state of the equation
$\dddot{\varphi} + (a + bK_d)\ddot{\varphi} + (c + bK_p)\dot{\varphi} + bK_i\varphi = d$, we get:

$$bK_i\varphi = d \tag{22}$$

With that, we can find steady-state value of $\varphi = d/(bK_i)$. Notice that it allows steady state solution with $\omega = 0$; the steady state value of φ is irrelevant for the angular velocity control. This is the idea behind the integral part of PID control.

PID control in Laplace domain looks like:

$$u(s) = -(K_d s + K_p + \frac{K_i}{s})\omega(s) \tag{23}$$

With reference signal, PID control leads to the transfer function (from reference to angular velocity):

$$W_r(s) = b\frac{K_d s^2 + K_p s + K_i}{s^3 + (a + bK_d)s^2 + (c + bK_p)s + bK_i} \tag{24}$$

If we want to control orientation of motor shaft, we have to re-write the dynamics in terms of φ:

$$\varphi(s) = \frac{b}{s^3 + as^2 + cs} u(s) \tag{25}$$

The PID control will take the usual form:

$$u(s) = (K_d s + K_p + \frac{K_i}{s})(\varphi_r(s) - \varphi(s)) \tag{26}$$

$$\varphi(s) = \frac{1}{s^3 + as^2 + cs} u(s) \tag{27}$$

The closed-loop transfer function will be:

$$\varphi(s) = b \frac{K_d s^2 + K_p s + K_i}{s^4 + as^3 + (c + bK_d)s^2 + bK_p s + bK_i} \varphi_r(s) \tag{28}$$

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Mechatronics-2023