Filters

Mechatronics, Lecture 2

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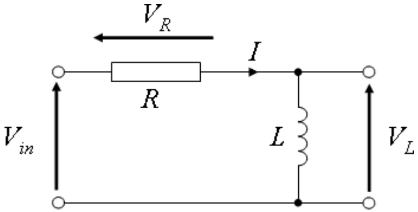
RL circuit as an ODE

(from the last lecture) RL circuit (resistor-inductor circuit) in the simplest case contains a power source, a resistor and an inductor.



We can model it as a first order differential equation (ODE):

$$L\frac{dI}{dt} + IR = V(t) \tag{1}$$

where L is inductance, I is current in the circuit, R is the resistance of the resistor and V is the voltage of source (input / battery, etc).

RL: short circuit, 1

Short circuit equation takes form:

$$L\frac{dI}{dt} + IR = 0 \tag{2}$$

Or equivalently:

$$\frac{d}{dt}I = -\frac{R}{L}I \tag{3}$$

We can easily solve it:

$$I(t) = I_0 e^{-\frac{R}{L}t} \tag{4}$$

where $I_0 = I(0)$.

RL: SHORT CIRCUIT, 2

We can make some observations:

- System's eigenvalue is $-\frac{R}{L}$. It is *always* negative (since $L > 0$ and $R > 0$), the system is asymptotically stable.
- Current $I(t)$ exponentially decays from the initial value to zero.
- The larger the inductance L the slower the exponential decay. The larger the resistance R the faster the decay.

RL: voltage input

Given a voltage input, the ODE describing an RL circuit becomes:

$$L\frac{dI}{dt} + IR = V(t) \tag{5}$$

Analytical solution for this equation is:

$$I(t) = I_0 e^{-\left(\frac{R}{L}t\right)} + e^{-\left(\frac{R}{L}t\right)} \int_0^t e^{\left(\frac{R}{L}\tau\right)} V(\tau) d\tau \tag{6}$$

RL: steady state, constant input

Given a constant input voltage $V = \text{const}$, the steady-state of the circuit ($\frac{dI}{dt} = 0$) is described as:

$$IR = V \tag{7}$$

Giving us a steady-state solution:

$$I = V/R \tag{8}$$

Notice that inductance L does not influence the steady-state solution.

Given a harmonic input voltage $V(t) = c\cos(\omega t) + d\sin(\omega t)$, we can attempt to find a steady state response of the system in the form:

$$I(t) = a\cos(\omega t) + b\sin(\omega t) \tag{9}$$

Relations between pairs (a, b) and (c, d) encode the change in the phase and amplitude of the input signal.

Differentiating $I(t)$ we get:

$$\frac{d}{dt}I(t) = -a\omega\sin(\omega t) + b\omega\cos(\omega t) \tag{10}$$

Substituting into the original ODE we obtain:

$$
\begin{aligned}
&L\omega(-a\sin(\omega t) + b\cos(\omega t)) + \\
&+R(a\cos(\omega t) + b\sin(\omega t)) = c\cos(\omega t) + d\sin(\omega t)
\end{aligned}
$$

We can re-write it as:

$$
(L\omega b + Ra - c)\cos(\omega t) + (-L\omega a + Rb - d)\sin(\omega t) = 0
$$

which would hold if:

$$
\begin{cases}
L\omega b + Ra - c = 0 \\
-L\omega a + Rb - d = 0
\end{cases}
\tag{11}
$$

RL: STEADY STATE, HARMONIC INPUT, 3

We can re-write the last expression as:

$$\begin{cases} L\omega b + Ra = c \\ -L\omega a + Rb = d \end{cases} \tag{12}$$

Or in a matrix form:

$$\begin{bmatrix} R & L\omega \\ -L\omega & R \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} \tag{13}$$

To find the expressions for (a, b) we find inverse of the matrix:

$$\begin{bmatrix} R & L\omega \\ -L\omega & R \end{bmatrix}^{-1} = \frac{1}{R^2 + L^2\omega^2} \begin{bmatrix} R & -L\omega \\ L\omega & R \end{bmatrix} \tag{14}$$

Now we can find (a, b):

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{R^2 + L^2\omega^2} \begin{bmatrix} R & -L\omega \\ L\omega & R \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{R^2 + L^2\omega^2} \begin{bmatrix} Rc - L\omega d \\ L\omega c + Rd \end{bmatrix}$$

With that, we can find how amplitude of the output sine wave $\mathbf{amp}(I(t)) = \sqrt{a^2 + b^2}$ depends on the amplitude of the input sine wave $\mathbf{amp}(V(t)) = \sqrt{c^2 + d^2}$:

$$\begin{aligned} \sqrt{a^2 + b^2} &= \frac{\sqrt{(Rc - L\omega d)^2 + (L\omega c + Rd)^2}}{R^2 + L^2\omega^2} = \\ &= \frac{\sqrt{R^2c^2 + L^2\omega^2 d^2 + L^2\omega^2 c^2 + R^2 d^2}}{R^2 + L^2\omega^2} = \\ &= \frac{\sqrt{(R^2 + L^2\omega^2)(c^2 + d^2)}}{R^2 + L^2\omega^2} = \frac{1}{\sqrt{R^2 + L^2\omega^2}} \sqrt{c^2 + d^2} \end{aligned}$$

Thus, we have the *amplitude gain* $G(\omega)$:

$$G(\omega) = \frac{1}{\sqrt{R^2 + L^2\omega^2}} \tag{15}$$

$$\mathbf{amp}(I(t)) = \frac{1}{\sqrt{R^2 + L^2\omega^2}}\mathbf{amp}(V(t)) \tag{16}$$

This is called frequency response. We can plot the amplitude gain $G(\omega)$ as a function of frequency ω.

Voltage V_R across the resistor is given as $V_R = IR$. Therefore, the amplitude gain $G_R(\omega) = \mathbf{amp}(V_R(t))/\mathbf{amp}(V(t))$ of the voltage V_R is:

$$G_R(\omega) = \frac{R}{\sqrt{R^2 + L^2\omega^2}} \tag{17}$$

Voltage V_L across the inducer is given as $V_L = L\frac{d}{dt}I(t)$. Its amplitude is given as $\mathbf{amp}(V_L(t)) = L\omega\sqrt{a^2 + b^2}$:

$$G_R(\omega) = \frac{L\omega}{\sqrt{R^2 + L^2\omega^2}} \tag{18}$$

Analyzing the amplitude gain $G(\omega) = \frac{1}{\sqrt{R^2+L^2\omega^2}}$ we can see that:

- As frequency tends to infinity, the gain goes to zero.
- As frequency tends to zero, the gain goes to $1/R$.

This makes RL circuit a *low-pass filter*: high-frequency signals are attenuated (suppressed) by the circuit.

By definition, Laplace transform of a function $f(t)$ is given as:

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{19}$$

In particular, Laplace transform of a derivative $\frac{d}{dt}x(t)$ is $sX(s)$.

RL: Laplace transform

Time domain ODE for the RL circuit is:

$$L\frac{dI}{dt} + IR = V(t) \tag{20}$$

Laplace transform of this equation is:

$$LsI(s) + I(s)R = V(s) \tag{21}$$

where $I(s)$ and $V(s)$ are Laplace transform of current $I(t)$ and voltage $V(t)$.

Laplace domain ODE is $LsI(s) + I(s)R = V(s)$. The transfer function from $V(s)$ to $I(s)$ is $I(s) = W_I(s)V(s)$:

$$W_I(s) = \frac{1}{Ls + R} \tag{22}$$

Voltage across the resistor $V_R = IR$ can be computed via its transfer function $V_R(s) = W_R(s)V(s)$:

$$W_R(s) = \frac{R}{Ls + R} \tag{23}$$

Voltage across the inducer $V_L(s) = LsI(s)$ can be computed via its transfer function $V_L(s) = W_L(s)V(s)$:

$$W_L(s) = \frac{Ls}{Ls + R} \tag{24}$$

RL: Frequency response, 1

Knowing transfer function $W(s)$, we can compute frequency response by substituting ωj for s into the transfer function and finding its absolute value. Doing it for the transfer function $W_I(s) = \frac{1}{Ls+R}$ we get:

$$W_I(\omega) = \frac{1}{L\omega j + R} = \frac{-L\omega j + R}{L^2\omega^2 + R^2}; \tag{25}$$

$$|W_I(\omega)| = \frac{\sqrt{L^2\omega^2 + R^2}}{L^2\omega^2 + R^2} = \frac{1}{\sqrt{L^2\omega^2 + R^2}} \tag{26}$$

RL: Frequency response, 2

Same for the transfer function $W_R(s) = \frac{R}{Ls+R} = RW_I(s)$ yields:

$$|W_R(\omega)| = \frac{R}{\sqrt{L^2\omega^2 + R^2}} \tag{27}$$

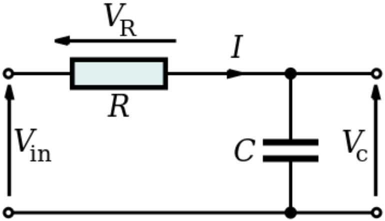
Considering transfer function $W_L(s) = \frac{Ls}{Ls+R}$ yields:

$$W_L(\omega) = \frac{L\omega j}{L\omega j + R} = \frac{L^2\omega^2 + RL\omega j}{L^2\omega^2 + R^2};$$

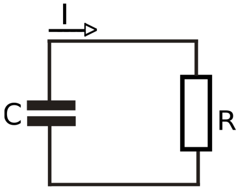
$$|W_L(\omega)| = \frac{\sqrt{L^4\omega^4 + R^2L^2\omega^2}}{L^2\omega^2 + R^2} = L\omega\frac{\sqrt{L^2\omega^2 + R^2}}{L^2\omega^2 + R^2} = \frac{L\omega}{\sqrt{L^2\omega^2 + R^2}}$$

Thus, we found frequency responses from earlier without solving the ODEs.

Consider RC circuit (resistor-capacitor circuit):



(a) Voltage input



(b) Short circuit

RC circuit ODE

We can define electric charge of a capacitor as q, relating to the current as:

$$q(t) = q(0) + \int_0^t I(\tau) d\tau \tag{28}$$

This means that $\frac{dq(t)}{dt} = I(t)$.

Voltage across the capacitor is $V(t) = \frac{q(t)}{C}$. This gives us ODE of the circuit:

$$\frac{q(t)}{C} + I(t)R = V(t) \tag{29}$$

Which is the same as:

$$\frac{q(t)}{C} + \frac{dq(t)}{dt}R = V(t) \tag{30}$$

RC circuit Laplace Transform

Expression $\frac{dq(t)}{dt} = I(t)$ in Laplace domain becomes $sq(s) = I(s)$. Dividing by s we get:

$$q(s) = \frac{1}{s} I(s) \tag{31}$$

Note that $1/s$ is a representation of an *integrator* in Laplace domain. So, the time domain equation $\frac{q(t)}{C} + I(t)R = V(t)$ becomes:

$$\frac{1}{sC} I(s) + I(s)R = V(s) \tag{32}$$

Giving us a transfer function:

$$I(s) = \frac{sC}{1 + sCR} V(s) \tag{33}$$

The transfer function from input voltage $V(s)$ to current $I(s)$ is $I(s) = W_I(s)V(s)$

$$W_I(s) = \frac{sC}{1 + sCR} \tag{34}$$

The transfer function from input voltage $V(s)$ to voltage across the resistor $V_R(s) = I(s)R$ is:

$$W_R(s) = \frac{sCR}{1 + sCR} \tag{35}$$

The transfer function from input voltage $V(s)$ to voltage across the capacitor $V_C(s) = \frac{1}{sC}I(s)$ is:

$$W_C(s) = \frac{1}{1 + sCR} \tag{36}$$

The frequency response of the voltage across the capacitor is found by substituting ωj into $W_C(s)$

$$W_C(\omega) = \frac{1}{1 + CR\omega j} = \frac{1 - CR\omega j}{1 + C^2 R^2 \omega^2} \tag{37}$$

$$|W_C(\omega)| = \frac{\sqrt{1 + C^2 R^2 \omega^2}}{1 + C^2 R^2 \omega^2} = \frac{1}{\sqrt{1 + C^2 R^2 \omega^2}} \tag{38}$$

The frequency response of the voltage across the resistor is found by substituting ωj into $W_R(s)$

$$W_R(\omega) = \frac{CR\omega j}{1 + CR\omega j} = \frac{CR\omega j + C^2 R^2 \omega^2}{1 + C^2 R^2 \omega^2} = CR\omega \frac{j + CR\omega}{1 + C^2 R^2 \omega^2}$$

$$|W_R(\omega)| = \frac{CR\omega}{\sqrt{1 + C^2 R^2 \omega^2}}$$

Analyzing the amplitude gain $|W_R(\omega)| = \frac{CR\omega}{\sqrt{1+C^2R^2\omega^2}}$ we can see that:

- As frequency tends to infinity, the gain goes to 1.
- As frequency tends to zero, the gain goes to 0.

This makes RL circuit a *high-pass filter*: low-frequency signals are attenuated (suppressed) by the circuit.

Remember that voltage across the capacitor is $V_C(t) = \frac{q(t)}{C}$, and voltage across an inducer is $V_L(t) = L\frac{d}{dt}I(t)$. This gives us ODE of the circuit:

$$L\frac{d}{dt}I(t) + \frac{q(t)}{C} + I(t)R = V(t) \tag{39}$$

In Laplace domain these equalities are $V_C(s) = \frac{1}{sC}I(s)$ and $V_L(s) = sLI(s)$:

$$sLI(s) + RI(s) + \frac{1}{sC}I(s) = V(s) \tag{40}$$

$$\frac{s^2LC}{sC}I(s) + \frac{sCR}{sC}I(s) + \frac{1}{sC}I(s) = V(s) \tag{41}$$

$$I(s) = \frac{sC}{s^2LC + sRC + 1}V(s) \tag{42}$$

RLC circuit ODE, 2

Alternative ways to describe RLC circuit dynamics is:

$$L\frac{d}{dt}I(t) + I(t)R + \frac{1}{C}\int_0^t I(\tau)d\tau = V(t) \tag{43}$$

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)I(s) = sV(s) \tag{44}$$

The voltage across the conductor is $V_C(s) = \frac{1}{sC}I(s)$:

$$V_C(s) = \frac{1}{s^2LC + sRC + 1}V(s) \tag{45}$$

The voltage across the inducer is $V_L(s) = sLI(s)$:

$$V_L(s) = \frac{s^2LC}{s^2LC + sRC + 1}V(s) \tag{46}$$

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Mechatronics-2023