DC motor

Mechatronics, Lecture 4

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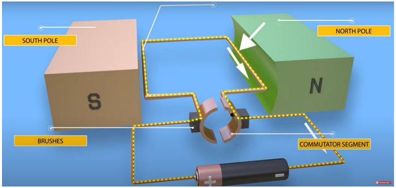
The basic idea of DC motor's operation is the use of Lorentz force: we run a current through a wire in magnetic field created by permanent magnets. As the wire forms a loop, this creates a net torque, rotating the wire and the shaft attached to it. Once the wire rotates by a certain angle, the brushes that supply electrical connection switch the direction of the DC current, allowing rotation to continue (instead of reversing)



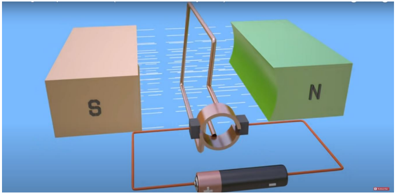


Figure 1: DC motor scheme. Images from `https://youtu.be/j_F4limaHYI`

DC motor mechanical model, 1

We can describe the dynamics of the motor shaft:

$$J\frac{d}{dt}\omega = \tau \tag{1}$$

where J is moment of inertia, ω is angular velocity and τ is torque. If we additionally consider linear viscous friction $F\omega$, we get:

$$J\frac{d}{dt}\omega + F\omega = \tau \tag{2}$$

The torque τ can be modeled as a linear function of current in the motor winding:

$$\tau = C_\tau I \tag{3}$$

where C_τ is torque coefficient.

Note that as long as the variable is angular velocity of the shaft ω, the dynamics is a first-order ODE. If we instead consider orientation of the shaft φ, the result is a second-order ODE.

$$J \frac{d^2}{dt^2} \varphi + F \frac{d}{dt} \varphi = \tau \tag{4}$$

$$\frac{d}{dt} \varphi = \omega \tag{5}$$

DC motor mechanical model, 3

The ODE $J\frac{d}{dt}\omega + F\omega = \tau$ is equivalent to the model of RL circuit. This means that the frequency response (considering torque as input and angular velocity as output) is:

$$|W(\alpha)| = \frac{1}{\sqrt{J^2\alpha^2 + F^2}} \tag{6}$$

where α is the input frequency.

- As the input frequency goes to infinity, the amplitude gain goes to zero.
- As the input frequency goes to zero, the amplitude gain goes to $\frac{1}{F}$.

DC motor winding is essentially a RL circuit. We can describe the electrodynamics of the motor winding as:

$$L\frac{d}{dt}I + RI + C_w\omega = u \tag{7}$$

where I is current in motor winding, L is induction coefficient of the motor winding, R is resistance of the motor winding, C_w is back-EMF coefficient and u is input voltage.

Note that this model also behaves like a RL circuit.

Full electro-mechanical model of a DC motor is given by the next system of ODEs:

$$\begin{cases} L\dot{I} + RI + C_w\omega = u \\ J\dot{\omega} + F\omega = C_\tau I \end{cases} \tag{8}$$

Laplace transform of this model is:

$$\begin{cases} LsI(s) + RI(s) + C_w\omega(s) = u(s) \\ Js\omega(s) + F\omega(s) = C_\tau I(s) \end{cases} \tag{9}$$

We can describe transfer functions:

$$\begin{bmatrix} Ls + R & C_w \\ -C_\tau & Js + F \end{bmatrix} \begin{bmatrix} I(s) \\ \omega(s) \end{bmatrix} = \begin{bmatrix} u(s) \\ 0 \end{bmatrix} \tag{10}$$

$$\det = (Ls + R)(Js + F) + C_w C_\tau \tag{11}$$

$$\mathrm{inv} = \frac{1}{\det} \begin{bmatrix} Js + F & -C_w \\ C_\tau & Ls + R \end{bmatrix} \tag{12}$$

Giving us transfer functions:

$$I(s) = \frac{Js + F}{(Ls + R)(Js + F) + C_w C_\tau} u(s), \tag{13}$$

$$\omega(s) = \frac{C_\tau}{(Ls + R)(Js + F) + C_w C_\tau} u(s) \tag{14}$$

We can open the brackets in the transfer function:

$$\omega(s) = \frac{C_\tau}{JLs^2 + (LF + JR)s + FR + C_w C_\tau} u(s) \tag{15}$$

With that, we can transform the model back to time domain, giving us a second-order ODE:

$$JL\ddot{\omega} + (LF + JR)\dot{\omega} + (FR + C_w C_\tau)\omega = C_\tau u \tag{16}$$

We can write a state-space model with state variables I and ω:

$$\begin{bmatrix} \dot{I} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -R/L & -C_w/L \\ C_\tau/J & -F/J \end{bmatrix} \begin{bmatrix} I \\ \omega \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u \tag{17}$$

Orientation of the motor is given by the angle φ, where $\dot{\varphi}(t) = \omega(t)$, or equivalently $s\varphi(s) = \omega(s)$. The transfer function from input voltage to angle φ is:

$$\varphi(s) = \frac{C_\tau}{JLs^3 + (LF + JR)s^2 + (FR + C_w C_\tau)s} u(s) \qquad (18)$$

The state-space model then becomes:

$$\begin{bmatrix} \dot{I} \\ \dot{\omega} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} -R/L & -C_w/L & 0 \\ C_\tau/J & -F/J & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I \\ \omega \\ \varphi \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix} u \qquad (19)$$

Payload

Let us consider a payload with moment of inertia J_p and torque τ_p. Then electro-mechanical dynamics of the motor is:

$$\begin{cases} L\dot{I} + RI + C_w\omega = u \\ (J + J_p)\dot{\omega} + F\omega = C_\tau I + \tau_p \end{cases} \tag{20}$$

For example, if the payload is a pendulum with mass m and length l, the dynamics becomes:

$$\begin{cases} L\dot{I} + RI + C_w\dot{\varphi} = u \\ (J + ml^2)\ddot{\varphi} + F\dot{\varphi} = C_\tau I + mgl\sin(\varphi) \end{cases} \tag{21}$$

Reducer (gearbox)

A reducer is a mechanism for reducing angular velocity of the output shaft of the motor, while increasing the output torque. Let us define angular velocity of the output shaft ω_o and torque of the output shaft τ_o. An ideal reducer is defined as:

$$\omega_o = \frac{1}{N}\omega \tag{22}$$

$$\tau_o = N\tau \tag{23}$$

where $N > 1$ is the reduction ratio.

The key idea is that an ideal reducer allows us to model orientation and velocity of the input shaft and the output shaft using the same coordinates.

Consider a payload with inertial J_p attached to the output shaft of the reducer, rotating at the angular velocity ω_o. The kinetic energy of the entire system is given by:

$$T = \frac{1}{2}J\omega^2 + \frac{1}{2}J_p\omega_o^2 = \tag{24}$$

$$= \frac{1}{2}J\omega^2 + \frac{1}{2}\frac{1}{N^2}J_p\omega^2 = \tag{25}$$

$$= \frac{1}{2}(J + J_p/N^2)\omega^2 \tag{26}$$

Thus, dynamic of the drive becomes:

$$\begin{cases} L\dot{I} + RI + C_w\omega = u \\ (J + J_p/N^2)\dot{\omega} + F\omega = C_\tau I + \tau_p/N \end{cases} \tag{27}$$

Read more

- University of Michigan. DC Motor Speed: System Modeling
- First-Order DC Electric Motor Model, Mark Drela, MIT Aero & Astro
- University of Alberta. DC Motor Speed Modeling

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Mechatronics-2023