Second order systems

Mechatronics, Lecture 6

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Content

- Eigenvalues
- Characteristic Polynomial
- Poles
- Natural Frequency
- Frequency response

Some of the second-order dynamical systems we have seen before:

- Spring-mass-damper dynamics.
- RLC circuit.
- DC motor angular velocity dynamics.
- Linearized model of a pendulum.

In a few lectures we will see that PID controller also acts in a way similar to a second-order dynamical system.

Since any polynomial can be represented as a product of linear polynomials and quadratic polynomials, we can also re-write any transfer function as a product of transfer functions with quadratic polynomials in the denominator. We can see second-order dynamical systems as fundamental building blocks of dynamical systems.

Observing eq. $m\ddot{y} + \mu_0\dot{y} + c_0 y = 0$ we can tell that it is stable if (sufficient but not necessary condition) $m > 0$, $\mu > 0$, and $c > 0$ - this follows from the physics of the system.

A more principled approach is to find eigenvalues of the linear system. We start by dividing the equation by m:

$$\ddot{y} + \mu\dot{y} + cy = 0 \tag{1}$$

where $\mu = \mu_0/m$ and $c = c_0/m$. Defining $x_1 = y$ and $x_2 = \dot{y}$, the system can be equivalently represented as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -\mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

With linear system $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -\mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we need to find its eigenvalues. We know that there is a formula for eigenvalues based on trace and determinant:

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \tag{3}$$

where T is trace and D is the determinant.

In our case $T = -\mu$ and $D = c$, and eigenvalues are:

$$\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4c}}{2} \tag{4}$$

Lets analyze eigenvalues $\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4c}}{2}$. We can see that if $\mu \geq 0$ and $c \geq 0$, there are only two scenarios:

1. $\mu^2 - 4c \geq 0$, in which case $\sqrt{\mu^2 - 4c} \leq \mu$, the eigenvalues are purely real and negative.
2. $\mu^2 - 4c < 0$, in which case $\sqrt{\mu^2 - 4c}$ is a purely imaginary number, the eigenvalues are complex with negative real parts.

If $\mu \geq 0$ and $c = 0$, $\lambda_1 = -\mu$, $\lambda_2 = 0$, hence the system is marginally stable.

If $\mu \geq 0$ and $c < 0$, then $\sqrt{\mu^2 - 4c} \geq \mu$, and eigenvalues are purely real and one of them is positive, the system is unstable. If $\mu < 0$ and $c < 0$ at least one of the eigenvalues is still positive.

If $\mu < 0$ and $c \geq 0$, then again there are only two scenarios:

1. $\mu^2 - 4c \geq 0$, in which case $\sqrt{\mu^2 - 4c} \leq \mu$, the eigenvalues are purely real and positive.
2. $\mu^2 - 4c < 0$, in which case $\sqrt{\mu^2 - 4c}$ is a purely imaginary number, the eigenvalues are complex with positive real parts.

Definition

If $\mu \geq 0$ and $c \geq 0$ the system is stable, if $\mu < 0$ or $c < 0$ it is unstable.

Going back to the eq. $\ddot{y} + \mu\dot{y} + cy = 0$ we can write characteristic eq. for it:

$$k^2 y + \mu k + c = 0 \tag{5}$$

Its roots are given by the formula:

$$k = \frac{-\mu \pm \sqrt{\mu^2 - 4c}}{2} \tag{6}$$

As we can see, it is exactly the same as the determinant-trace formula.

Transfer function poles

Now, lets consider the transfer function:

$$W(s) = \frac{1}{s^2 + \mu s + c} \tag{7}$$

Poles of the transfer functions are solutions of the polynomial in its denominator; notice that this polynomial is exactly the same as the previously discussed characteristic polynomial. Meaning the poles of the transfer function are eigenvalues of the linear system.

Another popular form of writing a second-order ODE is the following:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0 \tag{8}$$

where ω_n is a *natural frequency* and ζ is *damping factor*. The quantity $\zeta\omega_n$ is called *damping attenuation*. The relation between ζ, ω_n and μ, c is:

$$\omega_n = \sqrt{c}, \qquad \zeta = \frac{\mu}{2\sqrt{c}} \tag{9}$$

With this, the expression for eigenvalues is:

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{10}$$

With the representation $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0$ we can tell:

- the system is overdamped (has pure real negative eigenvalues) is $\zeta > 1$, critically damped if $\zeta = 1$ and underdamped (produces oscillations) if $0 \leq \zeta < 1$;
- the system has purely imaginary eigenvalues (non-decaying oscillations) iff $\zeta = 0$ (since $\omega_n = 0$ corresponds to the trivial solution).

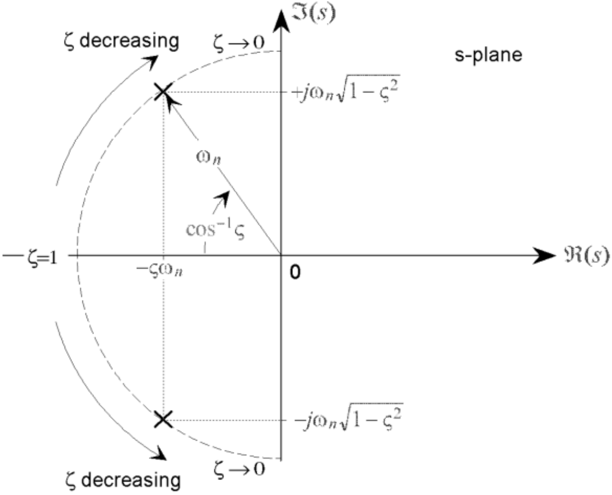

Figure 1: Poles and ζ, ω_n. Credit: MIT.

Given $\zeta = 0$ (implying $\mu = 0$) we have pure imaginary poles / eigenvalues $\lambda = \pm i\omega_n$. Moreover, the dynamics equations become:

$$\ddot{y} + \omega_n^2 y = 0 \tag{11}$$

Proposing solution $y = A\sin(\omega_n t + \varphi)$, noting that $\dot{y} = \omega_n A\cos(\omega_n t + \varphi)$ and $\ddot{y} = -\omega_n^2 A\sin(\omega_n t + \varphi)$, we see:

$$-\omega_n^2 A\sin(\omega_n t + \varphi) + \omega_n^2 A\sin(\omega_n t + \varphi) = 0 \tag{12}$$

which is an equality.

Previous slide showed that the system without damping $\ddot{y} + \omega_n^2 y = 0$ will oscillate with angular frequency ω_n; this motivates the name "natural frequency".

Also it shows that, if we pass a harmonic signal with angular frequency ω_n through a transfer function $W(s) = s^2 + \omega_n^2$ we will get a zero output.

Alternately, passing a harmonic signal with angular frequency ω_n through a transfer function $W(s) = \frac{1}{s^2+\omega_n^2}$ we observe an infinite gain. Same can be observed by substituting $i\omega$ for s and evaluating the absolute value of the transfer function at $\omega = \omega_n$. This effect is called *resonance*.

Given a transfer function $W(s) = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$ we can find frequency response gain by computing absolute value of $W(i\omega)$:

$$G = |W(i\omega)| = \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \tag{13}$$

- As $\omega \to 0$, $G \to \frac{1}{\omega_n^2}$.
- As $\omega \to \omega_n$, $G \to \frac{1}{\sqrt{2\zeta\omega_n^2}}$.
- As $\omega \to \infty$, $G \to 0$.

Read more

- MIT. 2.14 Analysis and Design of Feedback Control Systems. Understanding Poles and Zeros

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Mechatronics-2023