Filters Mechatronics, Lecture 2

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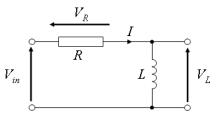
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CONTENT

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- RC circuit
 - ▶ Laplace transform, Transfer function
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RL CIRCUIT AS AN ODE

(from the last lecture) RL circuit (resistor-inductor circuit) in the simplest case contains a power source, a resistor and an inductor.



We can model it as a first order differential equation (ODE):

$$L\frac{dI}{dt} + IR = V(t) \tag{1}$$

where L is inductance, I is current in the circuit, R is the resistance of the resistor and V is the voltage of source (input / battery, etc).

RL: SHORT CIRCUIT, 1

Short circuit equation takes form:

$$L\frac{dI}{dt} + IR = 0 (2)$$

Or equivalently:

$$\frac{d}{dt}I = -\frac{R}{L}I\tag{3}$$

We can easily solve it:

$$I(t) = I_0 e^{-\frac{R}{L}t} \tag{4}$$

where $I_0 = I(0)$.

RL: SHORT CIRCUIT, 2

We can make some observations:

- System's eigenvalue is $-\frac{R}{L}$. It is always negative (since L > 0 and R > 0), the system is asymptotically stable.
- Current I(t) exponentially decays from the initial value to zero.
- The larger the inductance L the slower the exponential decay. The larger the resistance R the faster the decay.

RL: VOLTAGE INPUT

Given a voltage input, the ODE describing an RL circuit becomes:

$$L\frac{dI}{dt} + IR = V(t) \tag{5}$$

Analytical solution for this equation is:

$$I(t) = I_0 e^{-(\frac{R}{L}t)} + e^{-(\frac{R}{L}t)} \int_0^t e^{(\frac{R}{L}\tau)} V(\tau) d\tau$$
 (6)

RL: STEADY STATE, CONSTANT INPUT

Given a constant input voltage V = const, the steady-state of the circuit $(\frac{dI}{dt} = 0)$ is described as:

$$IR = V \tag{7}$$

Giving us a steady-state solution:

$$I = V/R \tag{8}$$

Notice that inductance L does not influence the steady-state solution.

Given a harmonic input voltage $V(t) = c\cos(\omega t) + d\sin(\omega t)$, we can attempt to find a steady state response of the system in the form:

$$I(t) = a\cos(\omega t) + b\sin(\omega t) \tag{9}$$

Relations between pairs (a, b) and (c, d) encode the change in the phase and amplitude of the input signal.

Differentiating I(t) we get:

$$\frac{d}{dt}I(t) = -a\omega\sin(\omega t) + b\omega\cos(\omega t) \tag{10}$$

Substituting into the original ODE we obtain:

$$L\omega(-a\sin(\omega t) + b\cos(\omega t)) + \\ + R(a\cos(\omega t) + b\sin(\omega t)) = c\cos(\omega t) + d\sin(\omega t)$$

We can re-write it as:

$$(L\omega b + Ra - c)\cos(\omega t) + (-L\omega a + Rb - d)\sin(\omega t) = 0$$

which would hold if:

$$\begin{cases}
L\omega b + Ra - c = 0 \\
-L\omega a + Rb - d = 0
\end{cases}$$
(11)

We can re-write the last expression as:

$$\begin{cases} L\omega b + Ra = c \\ -L\omega a + Rb = d \end{cases}$$
 (12)

Or in a matrix form:

$$\begin{bmatrix} R & L\omega \\ -L\omega & R \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$
 (13)

To find the expressions for (a, b) we find inverse of the matrix:

$$\begin{bmatrix} R & L\omega \\ -L\omega & R \end{bmatrix}^{-1} = \frac{1}{R^2 + L^2\omega^2} \begin{bmatrix} R & -L\omega \\ L\omega & R \end{bmatrix}$$
(14)

Now we can find (a, b):

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{R^2 + L^2 \omega^2} \begin{bmatrix} R & -L\omega \\ L\omega & R \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{R^2 + L^2 \omega^2} \begin{bmatrix} Rc - L\omega d \\ L\omega c + Rd \end{bmatrix}$$

With that, we can find how amplitude of the output sine wave $\operatorname{amp}(I(t)) = \sqrt{a^2 + b^2}$ depends on the amplitude of the input sine wave $\operatorname{amp}(V(t)) = \sqrt{c^2 + d^2}$:

$$\sqrt{a^2 + b^2} = \frac{\sqrt{(Rc - L\omega d)^2 + (L\omega c + Rd)^2}}{R^2 + L^2\omega^2} =
= \frac{\sqrt{R^2c^2 + L^2\omega^2d^2 + L^2\omega^2c^2 + R^2d^2}}{R^2 + L^2\omega^2} =
= \frac{\sqrt{(R^2 + L^2\omega^2)(c^2 + d^2)}}{R^2 + L^2\omega^2} = \frac{1}{\sqrt{R^2 + L^2\omega^2}} \sqrt{c^2 + d^2}$$

Thus, we have the amplitude gain $G(\omega)$:

$$G(\omega) = \frac{1}{\sqrt{R^2 + L^2 \omega^2}} \tag{15}$$

$$\mathbf{amp}(I(t)) = \frac{1}{\sqrt{R^2 + L^2 \omega^2}} \mathbf{amp}(V(t))$$
 (16)

This is called frequency response. We can plot the amplitude gain $G(\omega)$ as a function of frequency ω .

Voltage V_R across the resistor is given as $V_R = IR$. Therefore, the amplitude gain $G_R(\omega) = \mathbf{amp}(V_R(t))/\mathbf{amp}(V(t))$ of the voltage V_R is:

$$G_R(\omega) = \frac{R}{\sqrt{R^2 + L^2 \omega^2}} \tag{17}$$

Voltage V_L across the inducer is given as $V_L = L \frac{d}{dt} I(t)$. Its amplitude is given as $\mathbf{amp}(V_L(t)) = L\omega \sqrt{a^2 + b^2}$:

$$G_R(\omega) = \frac{L\omega}{\sqrt{R^2 + L^2\omega^2}} \tag{18}$$

Analyzing the amplitude gain $G(\omega) = \frac{1}{\sqrt{R^2 + L^2 \omega^2}}$ we can see that:

- As frequency tends to infinity, the gain goes to zero.
- As frequency tends to zero, the gain goes to 1/R.

This makes RL circuit a *low-pass filter*: high-frequency signals are attenuated (suppressed) by the circuit.

LAPLACE TRANSFORM

By definition, Laplace transform of a function f(t) is given as:

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{19}$$

In particular, Laplace transform of a derivative $\frac{d}{dt}x(t)$ is sX(s).

RL: LAPLACE TRANSFORM

Time domain ODE for the RL circuit is:

$$L\frac{dI}{dt} + IR = V(t) \tag{20}$$

Laplace transform of this equation is:

$$LsI(s) + I(s)R = V(s)$$
(21)

where I(s) and V(s) are Laplace transform of current I(t) and voltage V(t).

RL: Transfer functions

Laplace domain ODE is LsI(s) + I(s)R = V(s). The transfer function from V(s) to I(s) is $I(s) = W_I(s)V(s)$:

$$W_I(s) = \frac{1}{Ls + R} \tag{22}$$

Voltage across the resistor $V_R = IR$ can be computed via its transfer function $V_R(s) = W_R(s)V(s)$:

$$W_R(s) = \frac{R}{Ls + R} \tag{23}$$

Voltage across the inducer $V_L(s) = LsI(s)$ can be computed via its transfer function $V_L(s) = W_L(s)V(s)$:

$$W_L(s) = \frac{Ls}{Ls + R} \tag{24}$$

RL: Frequency response, 1

Knowing transfer function W(s), we can compute frequency response by substituting ωj for s into the transfer function and finding its absolute value. Doing it for the transfer function $W_I(s) = \frac{1}{Ls+R}$ we get:

$$W_I(\omega) = \frac{1}{L\omega j + R} = \frac{L\omega j - R}{L^2\omega^2 + R^2};$$
 (25)

$$|W_I(\omega)| = \frac{\sqrt{L^2 \omega^2 + R^2}}{L^2 \omega^2 + R^2} = \frac{1}{\sqrt{L^2 \omega^2 + R^2}}$$
 (26)

RL: Frequency response, 2

Same for the transfer function $W_R(s) = \frac{R}{Ls+R} = RW_I(s)$ yields:

$$|W_R(\omega)| = \frac{R}{\sqrt{L^2 \omega^2 + R^2}} \tag{27}$$

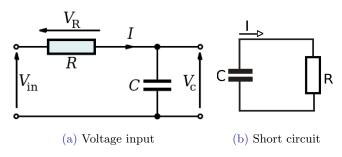
Considering transfer function $W_L(s) = \frac{Ls}{Ls+R}$ yields:

$$W_{L}(\omega) = \frac{L\omega j}{L\omega j + R} = \frac{-L^{2}\omega^{2} - RL\omega j}{L^{2}\omega^{2} + R^{2}};$$
$$|W_{L}(\omega)| = \frac{\sqrt{L^{4}\omega^{4} + R^{2}L^{2}\omega^{2}}}{L^{2}\omega^{2} + R^{2}} = L\omega \frac{\sqrt{L^{2}\omega^{2} + R^{2}}}{L^{2}\omega^{2} + R^{2}} = \frac{L\omega}{\sqrt{L^{2}\omega^{2} + R^{2}}}$$

Thus, we found frequency responses from earlier without solving the ODEs.

RC CIRCUIT

Consider RC circuit (resistor-capacitor circuit):



RC CIRCUIT ODE

We can define electric charge of a capacitor as q, relating to the current as:

$$q(t) = \int_0^t I(\tau)d\tau \tag{28}$$

This means that $\frac{dq(t)}{dt} = I(t)$.

Voltage across the capacitor is $V(t) = \frac{q(t)}{C}$. This gives us ODE of the circuit:

$$\frac{q(t)}{C} + I(t)R = V(t) \tag{29}$$

Which is the same as:

$$\frac{q(t)}{C} + \frac{dq(t)}{dt}R = V(t) \tag{30}$$

RC CIRCUIT LAPLACE TRANSFORM

Note that $\frac{dq(t)}{dt} = I(t)$ in Laplace domain is sq(s) = I(s). IN other words:

$$q(s) = -\frac{1}{s}I(s) \tag{31}$$

That means the time domain equation $\frac{q(t)}{C} + I(t)R = V(t)$ becomes:

$$\frac{1}{Cs}I(s) + I(s)R = V(s) \tag{32}$$

Giving us a transfer function:

$$I(s) = \frac{sC}{1 + sCR}V(s) \tag{33}$$

RC CIRCUIT, TRANSFER FUNCTIONS

The transfer function from input voltage V(s) to current I(s) is $I(s) = W_I(s)V(s)$

$$W_I(s) = \frac{sC}{1 + sCR} \tag{34}$$

The transfer function from input voltage V(s) to voltage across the resistor $V_R(s) = I(s)R$ is:

$$W_R(s) = \frac{sCR}{1 + sCR} \tag{35}$$

The transfer function from input voltage V(s) to voltage across the capacitor $V_C(s) = \frac{1}{Cs}I(s)$ is:

$$W_C(s) = \frac{1}{1 + sCR} \tag{36}$$

RC CIRCUIT, FREQUENCY RESPONSE

The frequency response of the voltage across the capacitor is found by substituting ωj into $W_C(s)$

$$W_C(\omega) = \frac{1}{1 + CR\omega j} = \frac{1 - CR\omega j}{1 + C^2 R^2 \omega^2}$$
 (37)

$$|W_C(\omega)| = \frac{\sqrt{1 + C^2 R^2 \omega^2}}{1 + C^2 R^2 \omega^2} = \frac{1}{\sqrt{1 + C^2 R^2 \omega^2}}$$
(38)

The frequency response of the voltage across the resistor is found by substituting ωj into $W_R(s)$

$$W_R(\omega) = \frac{CR\omega j}{1 + CR\omega j} = \frac{CR\omega j + C^2R^2\omega^2}{1 + C^2R^2\omega^2} = CR\omega \frac{j + CR\omega}{1 + C^2R^2\omega^2}$$
$$|W_R(\omega)| = \frac{CR\omega}{\sqrt{1 + C^2R^2\omega^2}}$$

RC: FILTER

Analyzing the amplitude gain $|W_R(\omega)| = \frac{CR\omega}{\sqrt{1+C^2R^2\omega^2}}$ we can see that:

- As frequency tends to infinity, the gain goes to 1.
- \blacksquare As frequency tends to zero, the gain goes to 0.

This makes RL circuit a *high-pass filter*: low-frequency signals are attenuated (suppressed) by the circuit.

RLC CIRCUIT ODE, 1

Remember that voltage across the capacitor is $V_C(t) = \frac{q(t)}{C}$, and voltage across an inducer is $V_L(t) = L \frac{d}{dt} I(t)$. This gives us ODE of the circuit:

$$L\frac{d}{dt}I(t) + \frac{q(t)}{C} + I(t)R = V(t)$$
(39)

To find Laplace transform of this equation we remember that $V_C(s) = \frac{1}{sC}I(s)$ and $V_L(s) = sLI(s)$:

$$sLI(s) + RI(s) + \frac{1}{sC}I(s) = V(s)$$
(40)

$$I(s) = \frac{sC}{s^2LC + sRC + 1}V(s) \tag{41}$$

RLC CIRCUIT ODE, 2

Alternative ways to describe RLC circuit dynamics is:

$$L\frac{d}{dt}I(t) + I(t)R + \int_0^t I(\tau)d\tau = V(t)$$
(42)

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)I(s) = s^2V(s) \tag{43}$$

The voltage across the conductor is $V_C(s) = \frac{1}{sC}I(s)$:

$$V_C(s) = \frac{1}{s^2 LC + sRC + 1} V(s)$$
 (44)

The voltage across the inducer is $V_L(s) = sLI(s)$:

$$V_L(s) = \frac{s^2 LC}{s^2 LC + sRC + 1} V(s)$$

$$\tag{45}$$

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at: github.com/SergeiSa/Mechatronics-2023

