

# Filters

## Mechatronics, Lecture 2

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Fall 2023

## ■ RL circuit

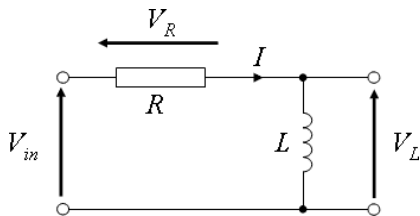
- ▶ ODE
- ▶ short circuit
- ▶ voltage input
- ▶ steady state, constant input
- ▶ steady state, harmonic input, frequency response
- ▶ Laplace transform, Transfer function, frequency response

## ■ RC circuit

- ▶ Laplace transform, Transfer function
- ▶ Frequency response

# RL CIRCUIT AS AN ODE

(from the last lecture) RL circuit (resistor-inductor circuit) in the simplest case contains a power source, a resistor and an inductor.



We can model it as a first order differential equation (ODE):

$$L \frac{dI}{dt} + IR = V(t) \quad (1)$$

where  $L$  is inductance,  $I$  is current in the circuit,  $R$  is the resistance of the resistor and  $V$  is the voltage of source (input / battery, etc).

Short circuit equation takes form:

$$L \frac{dI}{dt} + IR = 0 \quad (2)$$

Or equivalently:

$$\frac{d}{dt}I = -\frac{R}{L}I \quad (3)$$

We can easily solve it:

$$I(t) = I_0 e^{-\frac{R}{L}t} \quad (4)$$

where  $I_0 = I(0)$ .

We can make some observations:

- System's eigenvalue is  $-\frac{R}{L}$ . It is *always* negative (since  $L > 0$  and  $R > 0$ ), the system is asymptotically stable.
- Current  $I(t)$  exponentially decays from the initial value to zero.
- The larger the inductance  $L$  the slower the exponential decay. The larger the resistance  $R$  the faster the decay.

Given a voltage input, the ODE describing an RL circuit becomes:

$$L \frac{dI}{dt} + IR = V(t) \quad (5)$$

Analytical solution for this equation is:

$$I(t) = I_0 e^{-(\frac{R}{L}t)} + e^{-(\frac{R}{L}t)} \int_0^t e^{(\frac{R}{L}\tau)} V(\tau) d\tau \quad (6)$$

## RL: STEADY STATE, CONSTANT INPUT

Given a constant input voltage  $V = \text{const}$ , the steady-state of the circuit ( $\frac{dI}{dt} = 0$ ) is described as:

$$IR = V \quad (7)$$

Giving us a steady-state solution:

$$I = V/R \quad (8)$$

Notice that inductance  $L$  does not influence the steady-state solution.

Given a harmonic input voltage  $V(t) = c \cos(\omega t) + d \sin(\omega t)$ , we can attempt to find a steady state response of the system in the form:

$$I(t) = a \cos(\omega t) + b \sin(\omega t) \quad (9)$$

Relations between pairs  $(a, b)$  and  $(c, d)$  encode the change in the phase and amplitude of the input signal.

Differentiating  $I(t)$  we get:

$$\frac{d}{dt}I(t) = -a\omega \sin(\omega t) + b\omega \cos(\omega t) \quad (10)$$



Substituting into the original ODE we obtain:

$$L\omega(-a \sin(\omega t) + b \cos(\omega t)) + \\ + R(a \cos(\omega t) + b \sin(\omega t)) = c \cos(\omega t) + d \sin(\omega t)$$

We can re-write it as:

$$(L\omega b + Ra - c) \cos(\omega t) + (-L\omega a + Rb - d) \sin(\omega t) = 0$$

which would hold if:

$$\begin{cases} L\omega b + Ra - c = 0 \\ -L\omega a + Rb - d = 0 \end{cases} \quad (11)$$

We can re-write the last expression as:

$$\begin{cases} L\omega b + Ra = c \\ -L\omega a + Rb = d \end{cases} \quad (12)$$

Or in a matrix form:

$$\begin{bmatrix} R & L\omega \\ -L\omega & R \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} \quad (13)$$

To find the expressions for  $(a, b)$  we find inverse of the matrix:

$$\begin{bmatrix} R & L\omega \\ -L\omega & R \end{bmatrix}^{-1} = \frac{1}{R^2 + L^2\omega^2} \begin{bmatrix} R & -L\omega \\ L\omega & R \end{bmatrix} \quad (14)$$

Now we can find  $(a, b)$ :

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{R^2 + L^2\omega^2} \begin{bmatrix} R & -L\omega \\ L\omega & R \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{R^2 + L^2\omega^2} \begin{bmatrix} Rc - L\omega d \\ L\omega c + Rd \end{bmatrix}$$

With that, we can find how amplitude of the output sine wave  $\mathbf{amp}(I(t)) = \sqrt{a^2 + b^2}$  depends on the amplitude of the input sine wave  $\mathbf{amp}(V(t)) = \sqrt{c^2 + d^2}$ :

$$\begin{aligned} \sqrt{a^2 + b^2} &= \frac{\sqrt{(Rc - L\omega d)^2 + (L\omega c + Rd)^2}}{R^2 + L^2\omega^2} = \\ &= \frac{\sqrt{R^2c^2 + L^2\omega^2d^2 + L^2\omega^2c^2 + R^2d^2}}{R^2 + L^2\omega^2} = \\ &= \frac{\sqrt{(R^2 + L^2\omega^2)(c^2 + d^2)}}{R^2 + L^2\omega^2} = \frac{1}{\sqrt{R^2 + L^2\omega^2}} \sqrt{c^2 + d^2} \end{aligned}$$

Thus, we have the *amplitude gain*  $G(\omega)$ :

$$G(\omega) = \frac{1}{\sqrt{R^2 + L^2\omega^2}} \quad (15)$$

$$\mathbf{amp}(I(t)) = \frac{1}{\sqrt{R^2 + L^2\omega^2}} \mathbf{amp}(V(t)) \quad (16)$$

This is called frequency response. We can plot the amplitude gain  $G(\omega)$  as a function of frequency  $\omega$ .

Voltage  $V_R$  across the resistor is given as  $V_R = IR$ . Therefore, the amplitude gain  $G_R(\omega) = \mathbf{amp}(V_R(t))/\mathbf{amp}(V(t))$  of the voltage  $V_R$  is:

$$G_R(\omega) = \frac{R}{\sqrt{R^2 + L^2\omega^2}} \quad (17)$$

Voltage  $V_L$  across the inducer is given as  $V_L = L \frac{d}{dt} I(t)$ . Its amplitude is given as  $\mathbf{amp}(V_L(t)) = L\omega\sqrt{a^2 + b^2}$ :

$$G_R(\omega) = \frac{L\omega}{\sqrt{R^2 + L^2\omega^2}} \quad (18)$$

Analyzing the amplitude gain  $G(\omega) = \frac{1}{\sqrt{R^2 + L^2\omega^2}}$  we can see that:

- As frequency tends to infinity, the gain goes to zero.
- As frequency tends to zero, the gain goes to  $1/R$ .

This makes RL circuit a *low-pass filter*: high-frequency signals are attenuated (suppressed) by the circuit.

By definition, Laplace transform of a function  $f(t)$  is given as:

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \quad (19)$$

In particular, Laplace transform of a derivative  $\frac{d}{dt}x(t)$  is  $sX(s)$ .

Time domain ODE for the RL circuit is:

$$L \frac{dI}{dt} + IR = V(t) \quad (20)$$

Laplace transform of this equation is:

$$LsI(s) + I(s)R = V(s) \quad (21)$$

where  $I(s)$  and  $V(s)$  are Laplace transform of current  $I(t)$  and voltage  $V(t)$ .



Laplace domain ODE is  $LsI(s) + I(s)R = V(s)$ . The transfer function from  $V(s)$  to  $I(s)$  is  $I(s) = W_I(s)V(s)$ :

$$W_I(s) = \frac{1}{Ls + R} \quad (22)$$

Voltage across the resistor  $V_R = IR$  can be computed via its transfer function  $V_R(s) = W_R(s)V(s)$ :

$$W_R(s) = \frac{R}{Ls + R} \quad (23)$$

Voltage across the inducer  $V_L(s) = LsI(s)$  can be computed via its transfer function  $V_L(s) = W_L(s)V(s)$ :

$$W_L(s) = \frac{Ls}{Ls + R} \quad (24)$$

Knowing transfer function  $W(s)$ , we can compute frequency response by substituting  $\omega j$  for  $s$  into the transfer function and finding its absolute value. Doing it for the transfer function  $W_I(s) = \frac{1}{Ls+R}$  we get:

$$W_I(\omega) = \frac{1}{L\omega j + R} = \frac{-L\omega j + R}{L^2\omega^2 + R^2}; \quad (25)$$

$$|W_I(\omega)| = \frac{\sqrt{L^2\omega^2 + R^2}}{L^2\omega^2 + R^2} = \frac{1}{\sqrt{L^2\omega^2 + R^2}} \quad (26)$$

Same for the transfer function  $W_R(s) = \frac{R}{Ls+R} = RW_I(s)$  yields:

$$|W_R(\omega)| = \frac{R}{\sqrt{L^2\omega^2 + R^2}} \quad (27)$$

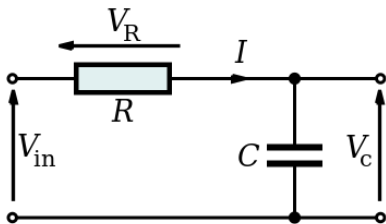
Considering transfer function  $W_L(s) = \frac{Ls}{Ls+R}$  yields:

$$\begin{aligned} W_L(\omega) &= \frac{L\omega j}{L\omega j + R} = \frac{L^2\omega^2 + RL\omega j}{L^2\omega^2 + R^2}; \\ |W_L(\omega)| &= \frac{\sqrt{L^4\omega^4 + R^2L^2\omega^2}}{L^2\omega^2 + R^2} = L\omega \frac{\sqrt{L^2\omega^2 + R^2}}{L^2\omega^2 + R^2} = \frac{L\omega}{\sqrt{L^2\omega^2 + R^2}} \end{aligned}$$

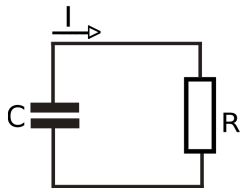
Thus, we found frequency responses from earlier without solving the ODEs.

# RC CIRCUIT

Consider RC circuit (resistor-capacitor circuit):



(a) Voltage input



(b) Short circuit

We can define electric charge of a capacitor as  $q$ , relating to the current as:

$$q(t) = q(0) + \int_0^t I(\tau) d\tau \quad (28)$$

This means that  $\frac{dq(t)}{dt} = I(t)$ .

Voltage across the capacitor is  $V(t) = \frac{q(t)}{C}$ . This gives us ODE of the circuit:

$$\frac{q(t)}{C} + I(t)R = V(t) \quad (29)$$

Which is the same as:

$$\frac{q(t)}{C} + \frac{dq(t)}{dt}R = V(t) \quad (30)$$

# RC CIRCUIT LAPLACE TRANSFORM

Expression  $\frac{dq(t)}{dt} = I(t)$  in Laplace domain becomes  $sq(s) = I(s)$ . Dividing by  $s$  we get:

$$q(s) = \frac{1}{s}I(s) \quad (31)$$

Note that  $1/s$  is a representation of an *integrator* in Laplace domain. So, the time domain equation  $\frac{q(t)}{C} + I(t)R = V(t)$  becomes:

$$\frac{1}{sC}I(s) + I(s)R = V(s) \quad (32)$$

Giving us a transfer function:

$$I(s) = \frac{sC}{1 + sCR}V(s) \quad (33)$$

# RC CIRCUIT, TRANSFER FUNCTIONS

The transfer function from input voltage  $V(s)$  to current  $I(s)$  is  $I(s) = W_I(s)V(s)$

$$W_I(s) = \frac{sC}{1 + sCR} \quad (34)$$

The transfer function from input voltage  $V(s)$  to voltage across the resistor  $V_R(s) = I(s)R$  is:

$$W_R(s) = \frac{sCR}{1 + sCR} \quad (35)$$

The transfer function from input voltage  $V(s)$  to voltage across the capacitor  $V_C(s) = \frac{1}{sC}I(s)$  is:

$$W_C(s) = \frac{1}{1 + sCR} \quad (36)$$

## RC CIRCUIT, FREQUENCY RESPONSE

The frequency response of the voltage across the capacitor is found by substituting  $\omega j$  into  $W_C(s)$

$$W_C(\omega) = \frac{1}{1 + CR\omega j} = \frac{1 - CR\omega j}{1 + C^2 R^2 \omega^2} \quad (37)$$

$$|W_C(\omega)| = \frac{\sqrt{1 + C^2 R^2 \omega^2}}{1 + C^2 R^2 \omega^2} = \frac{1}{\sqrt{1 + C^2 R^2 \omega^2}} \quad (38)$$

The frequency response of the voltage across the resistor is found by substituting  $\omega j$  into  $W_R(s)$

$$W_R(\omega) = \frac{CR\omega j}{1 + CR\omega j} = \frac{CR\omega j + C^2 R^2 \omega^2}{1 + C^2 R^2 \omega^2} = CR\omega \frac{j + CR\omega}{1 + C^2 R^2 \omega^2}$$
$$|W_R(\omega)| = \frac{CR\omega}{\sqrt{1 + C^2 R^2 \omega^2}}$$



Analyzing the amplitude gain  $|W_R(\omega)| = \frac{CR\omega}{\sqrt{1+C^2R^2\omega^2}}$  we can see that:

- As frequency tends to infinity, the gain goes to 1.
- As frequency tends to zero, the gain goes to 0.

This makes RL circuit a *high-pass filter*: low-frequency signals are attenuated (suppressed) by the circuit.

# RLC CIRCUIT ODE, 1

Remember that voltage across the capacitor is  $V_C(t) = \frac{q(t)}{C}$ , and voltage across an inducer is  $V_L(t) = L \frac{d}{dt} I(t)$ . This gives us ODE of the circuit:

$$L \frac{d}{dt} I(t) + \frac{q(t)}{C} + I(t)R = V(t) \quad (39)$$

In Laplace domain these equalities are  $V_C(s) = \frac{1}{sC} I(s)$  and  $V_L(s) = sLI(s)$ :

$$sLI(s) + RI(s) + \frac{1}{sC} I(s) = V(s) \quad (40)$$

$$\frac{s^2 LC}{sC} I(s) + \frac{sCR}{sC} I(s) + \frac{1}{sC} I(s) = V(s) \quad (41)$$

$$I(s) = \frac{sC}{s^2 LC + sRC + 1} V(s) \quad (42)$$

Alternative ways to describe RLC circuit dynamics is:

$$L \frac{d}{dt} I(t) + I(t)R + \frac{1}{C} \int_0^t I(\tau) d\tau = V(t) \quad (43)$$

$$\left( s^2 + \frac{R}{L}s + \frac{1}{LC} \right) I(s) = sV(s) \quad (44)$$

The voltage across the conductor is  $V_C(s) = \frac{1}{sC}I(s)$ :

$$V_C(s) = \frac{1}{s^2LC + sRC + 1} V(s) \quad (45)$$

The voltage across the inducer is  $V_L(s) = sLI(s)$ :

$$V_L(s) = \frac{s^2LC}{s^2LC + sRC + 1} V(s) \quad (46)$$

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:  
[github.com/SergeiSa/Mechatronics-2023](https://github.com/SergeiSa/Mechatronics-2023)

