

# DC motor

## Mechatronics, Lecture 4

by Sergei Savin

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- Working principle
- Mechanical model
- Electrical model
- Electro-mechanical second-order model (ODE, transfer function, state-space)
- Electro-mechanical third-order model
- Reducer (gearbox)

# DC MOTOR WORKING PRINCIPLE

The basic idea of DC motor's operation is the use of Lorentz force: we run a current through a wire in magnetic field created by permanent magnets. As the wire forms a loop, this creates a net torque, rotating the wire and the shaft attached to it. Once the wire rotates by a certain angle, the brushes that supply electrical connection switch the direction of the DC current, allowing rotation to continue (instead of reversing)

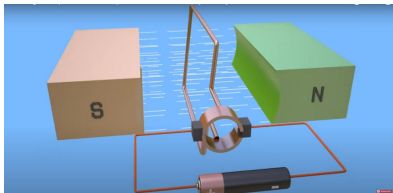
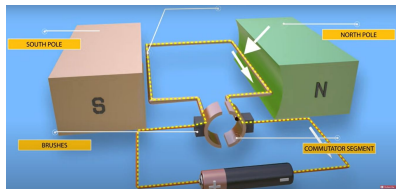


Figure 1: DC motor scheme. Images from [https://youtu.be/j\\_F4limaHYI](https://youtu.be/j_F4limaHYI)

We can describe the dynamics of the motor shaft:

$$J \frac{d}{dt} \omega = \tau \quad (1)$$

where  $J$  is moment of inertia,  $\omega$  is angular velocity and  $\tau$  is torque. If we additionally consider linear viscous friction  $F\omega$ , we get:

$$J \frac{d}{dt} \omega + F\omega = \tau \quad (2)$$

The torque  $\tau$  can be modeled as a linear function of current in the motor winding:

$$\tau = C_\tau I \quad (3)$$

where  $C_\tau$  is torque coefficient.

Note that as long as the variable is angular velocity of the shaft  $\omega$ , the dynamics is a first-order ODE. If we instead consider orientation of the shaft  $\varphi$ , the result is a second-order ODE.

$$J \frac{d^2}{dt^2} \varphi + F \frac{d}{dt} \varphi = \tau \quad (4)$$

$$\frac{d}{dt} \varphi = \omega \quad (5)$$

The ODE  $J \frac{d}{dt} \omega + F \omega = \tau$  is equivalent to the model of RL circuit. This means that the frequency response (considering torque as input and angular velocity as output) is:

$$|W(\alpha)| = \frac{1}{\sqrt{J^2 \alpha^2 + F^2}} \quad (6)$$

where  $\alpha$  is the input frequency.

- As the input frequency goes to infinity, the amplitude gain goes to zero.
- As the input frequency goes to zero, the amplitude gain goes to  $\frac{1}{F}$ .

DC motor winding is essentially a RL circuit. We can describe the electrodynamics of the motor winding as:

$$L \frac{d}{dt} I + RI + C_w \omega = u \quad (7)$$

where  $I$  is current in motor winding,  $L$  is induction coefficient of the motor winding,  $R$  is resistance of the motor winding,  $C_w$  is back-EMF coefficient and  $u$  is input voltage.

Note that this model also behaves like a RL circuit.

Full electro-mechanical model of a DC motor is given by the next system of ODEs:

$$\begin{cases} L\dot{I} + RI + C_w\omega = u \\ J\dot{\omega} + F\omega = C_\tau I \end{cases} \quad (8)$$

Laplace transform of this model is:

$$\begin{cases} LsI(s) + RI(s) + C_w\omega(s) = u(s) \\ Js\omega(s) + F\omega(s) = C_\tau I(s) \end{cases} \quad (9)$$



# DC MOTOR, TRANSFER FUNCTIONS, 1

We can describe transfer functions:

$$\begin{bmatrix} Ls + R & C_w \\ -C_\tau & Js + F \end{bmatrix} \begin{bmatrix} I(s) \\ \omega(s) \end{bmatrix} = \begin{bmatrix} u(s) \\ 0 \end{bmatrix} \quad (10)$$

$$\det = (Ls + R)(Js + F) + C_w C_\tau \quad (11)$$

$$\text{inv} = \frac{1}{\det} \begin{bmatrix} Js + F & -C_w \\ C_\tau & Ls + R \end{bmatrix} \quad (12)$$

Giving us transfer functions:

$$I(s) = \frac{Js + F}{(Ls + R)(Js + F) + C_w C_\tau} u(s), \quad (13)$$

$$\omega(s) = \frac{C_\tau}{(Ls + R)(Js + F) + C_w C_\tau} u(s) \quad (14)$$

We can open the brackets in the transfer function:

$$\omega(s) = \frac{C_\tau}{JLs^2 + (LF + JR)s + FR + C_w C_\tau} u(s) \quad (15)$$

With that, we can transform the model back to time domain, giving us a second-order ODE:

$$JL\ddot{\omega} + (LF + JR)\dot{\omega} + (FR + C_w C_\tau)\omega = C_\tau u \quad (16)$$

We can write a state-space model with state variables  $I$  and  $\omega$ :

$$\begin{bmatrix} \dot{I} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -R/L & -C_w/L \\ C_\tau/J & -F/J \end{bmatrix} \begin{bmatrix} I \\ \omega \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u \quad (17)$$

Orientation of the motor is given by the angle  $\varphi$ , where  $\dot{\varphi}(t) = \omega(t)$ , or equivalently  $s\varphi(s) = \omega(s)$ . The transfer function from input voltage to angle  $\varphi$  is:

$$\varphi(s) = \frac{C_\tau}{JLs^3 + (LF + JR)s^2 + (FR + C_w C_\tau)s} u(s) \quad (18)$$

The state-space model then becomes:

$$\begin{bmatrix} \dot{I} \\ \dot{\omega} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} -R/L & -C_w/L & 0 \\ C_\tau/J & -F/J & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I \\ \omega \\ \varphi \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix} u \quad (19)$$

Let us consider a payload with moment of inertia  $J_p$  and torque  $\tau_p$ . Then electro-mechanical dynamics of the motor is:

$$\begin{cases} L\dot{I} + RI + C_w\omega = u \\ (J + J_p)\dot{\omega} + F\omega = C_\tau I + \tau_p \end{cases} \quad (20)$$

For example, if the payload is a pendulum with mass  $m$  and length  $l$ , the dynamics becomes:

$$\begin{cases} L\dot{I} + RI + C_w\dot{\varphi} = u \\ (J + ml^2)\ddot{\varphi} + F\dot{\varphi} = C_\tau I + mgl \sin(\varphi) \end{cases} \quad (21)$$

# REDUCER (GEARBOX)

A reducer is a mechanism for reducing angular velocity of the output shaft of the motor, while increasing the output torque. Let us define angular velocity of the output shaft  $\omega_o$  and torque of the output shaft  $\tau_o$ . An ideal reducer is defined as:

$$\omega_o = \frac{1}{N}\omega \quad (22)$$

$$\tau_o = N\tau \quad (23)$$

where  $N > 1$  is the reduction ratio.

The key idea is that an ideal reducer allows us to model orientation and velocity of the input shaft and the output shaft using the same coordinates.

# MOMENT OF INERTIA WITH A REDUCER

Consider a payload with inertial  $J_p$  attached to the output shaft of the reducer, rotating at the angular velocity  $\omega_o$ . The kinetic energy of the entire system is given by:

$$T = \frac{1}{2}J\omega^2 + \frac{1}{2}J_p\omega_o^2 = \quad (24)$$

$$= \frac{1}{2}J\omega^2 + \frac{1}{2}\frac{1}{N^2}J_p\omega^2 = \quad (25)$$

$$= \frac{1}{2}(J + J_p/N^2)\omega^2 \quad (26)$$

Thus, dynamic of the drive becomes:

$$\begin{cases} L\dot{I} + RI + C_w\omega = u \\ (J + J_p/N^2)\dot{\omega} + F\omega = C_\tau I + \tau_p/N \end{cases} \quad (27)$$

- University of Michigan. DC Motor Speed: System Modeling
- First-Order DC Electric Motor Model, Mark Drela, MIT Aero & Astro
- University of Alberta. DC Motor Speed Modeling



Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:  
[github.com/SergeiSa/Mechatronics-2023](https://github.com/SergeiSa/Mechatronics-2023)

