Mechanics, Dynamics

Mechatronics, Lecture 3

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Content

What is Mechatronics? The term *Mechatronics* is a combination of words *mechanics* and *electronics*.

Definition

Mechatronics is a synergistic integration of mechanical and electrical engineering, computer control in design and manufacturing.

Examples of mechatronic design include:

- Motors with built-in gears and sensors.
- Quadrotor brushless motors.
- Motor-wheel.

Modern robots, from robot-arm to Boston Dynamics's ATLAS are also examples of mechatronic design.

In this course we will focus on:

- Motors: working principles, control, sensing;
- Gears, reducers, transmissions: models, friction and other physical effects;
- Single input, single output (SISO) control, as applicable to simple electro-mechanical systems;
- Sensors, using sensor data.

Second Newton's law in 2D case:

$$\begin{cases} m\ddot{x} = \sum f_{i,x} \\ m\ddot{y} = \sum f_{i,y} \end{cases} \tag{1}$$

where m is mass of a body, $f_{i,x}$, $f_{i,y}$ - Cartesian components of forces acting on the body.

Euler law of motion in 2D case:

$$J\ddot{\varphi} = \sum \tau_i \tag{2}$$

where J is a moment of inertia of a body, φ is orientation of the body, τ_i is torque. All of these entities are defined assuming rotation around the axis normal to the plane of motion, with positive direction of rotation defined as clock-wise.

We can see similarity of these equation with Newton laws. Note that this is only true for a 2D case, not for a general 3D case, where the Euler dynamics equations have the following form:

$$\mathbf{I}\dot{\omega} = \omega \times (\mathbf{I}\omega) + \tau \tag{3}$$

where $\mathbf{I}$ is matrix of inertia, ω is angular velocity and τ is external torque (all in body frame).

A pendulum can be described by the following dynamical equations:

$$J\ddot{\varphi} = lmg\sin(\varphi) \tag{4}$$

where J, l, m and g are moment of inertia, length, mass and gravitational acceleration.

If there is motor torque τ acting on the pendulum, the dynamics takes form:

$$J\ddot{\varphi} = lmg\sin(\varphi) + \tau \tag{5}$$

Torque generated by a force

A force $\mathbf{f}$ applied to a rigid body at a point $\mathbf{r}$ generates a torque τ:

$$\tau = \mathbf{r} \times \mathbf{f} \tag{6}$$

In 2D case, a force with magnitude f acting along a line with a distance r from the axis (around which torque is computed) generates torque according to the formula:

$$\tau = rf \tag{7}$$

It is tempting to think of moment of inertia in 2D case as an analog of mass; moment of inertia determines the rate of change of angular velocity for a given torque, same as mass determines the rate of change of linear velocity for a given force.

$$\begin{cases} J\dot{\omega} = \tau \\ m\dot{v} = f \end{cases}$$

Moment of inertia, 2

Another way to think about moment of inertia is through kinetic energy. Kinetic energy of a rigid body can be described as:

$$T = \frac{1}{2} m \mathbf{v}^\top \mathbf{v} + \frac{1}{2} \omega^\top \mathbf{I} \omega \tag{8}$$

In 2D case it is simplified:

$$T = \frac{1}{2} m (v_x^2 + v_y^2) + \frac{1}{2} J \omega^2 \tag{9}$$

Moment of inertia determines how energy will the body acquire by the increase in its angular velocity.

Moment of inertia of a point mass r meters away from the axis of rotation is given as:

$$J = mr^2 \tag{10}$$

The same formula give moment of inertia for a rim of a disk with radius r. For a solid disk with uniform mass distribution the formula becomes:

$$J = \frac{1}{2}mr^2 \tag{11}$$

For a centrally mounted rod the moment of inertia is $J = \frac{1}{12}mr^2$ and for a rod mounted like a pendulum it is $J = \frac{1}{3}mr^2$.

Mechanical energy is a sum of kinetic and potential energy. We already saw the form of kinetic energy in 2D case:
$T = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}J\omega^2$.

Potential energy is more complex: it depends on what type of potential (conservative) forces that act on the system. For example:

- gravitational potential energy: $\Pi = mg(y - y_0)$, where y_0 is the value of vertical coordinate y for which potential energy is defined as zero;
- linear spring potential energy: $\Pi = \frac{1}{2}k(r - r_0)^2$, where k is the stiffness and r_0 is the length of the spring in the relaxed state.

The change of the total energy of the system is equal to the sum of *mechanical work* performed by the active forces:

$$\Delta T + \Delta \Pi = \sum A_i \tag{12}$$

where A_i is the work performed by the active force f_i or a torque τ_i.

We could compute the work performed by a force (applied to a point mass) on the interval of time from t_0 to t_f:

$$A = \int_{t_0}^{t_f} \mathbf{v}^\top \mathbf{f} \; dt \tag{13}$$

where $\mathbf{v} = \mathbf{v}(t)$ is the velocity of the point mass and $\mathbf{f} = \mathbf{f}(t)$ is the force.

Mechanical Work, 2

If a force $\mathbf{f}$ is collinear with velocity $\mathbf{v}$, we re-write the formula for mechanical work using the absolute values of these vectors f and $v = \dot{s}$:

$$A = \int_{t_0}^{t_f} vf \; dt \tag{14}$$

As long as $f = \text{const}$, the expression becomes trivial:

$$A = (s(t_1) - s(t_0))f \tag{15}$$

We can derive the same formulas for mechanical work of a torque τ:

$$A = \int_{t_0}^{t_f} \omega\tau \; dt \tag{16}$$

$$A = (\varphi(t_1) - \varphi(t_0))\tau \quad (\tau = \text{const}) \tag{17}$$

The entity $\mathbf{v}^\top \mathbf{f}$ we saw before is called *mechanical power*. It can also be defined as:

$$P = \mathbf{v}^\top \mathbf{f} \tag{18}$$

$$P = \omega \tau \tag{19}$$

$$P = \frac{d}{dt} A \tag{20}$$

Note that where as mechanical work is an integral entity, mechanical power is instantaneous. Also, using the definition of mechanical energy we observe that:

$$\frac{d}{dt}(T + \Pi) = \sum P_i \tag{21}$$

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Mechatronics-2023