

# Mechanics, Dynamics

## Mechatronics, Lecture 3

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- Introduction
- Newton - Euler laws
- Moment of inertia
- Mechanical Energy, Work, Power

# INTRODUCTION

What is Mechatronics? The term *Mechatronics* is a combination of words *mechanics* and *electronics*.

## Definition

Mechatronics is a synergistic integration of mechanical and electrical engineering, computer control in design and manufacturing.

Examples of mechatronic design include:

- Motors with built-in gears and sensors.
- Quadrotor brushless motors.
- Motor-wheel.

Modern robots, from robot-arm to Boston Dynamics's ATLAS are also examples of mechatronic design.

In this course we will focus on:

- Motors: working principles, control, sensing;
- Gears, reducers, transmissions: models, friction and other physical effects;
- Single input, single output (SISO) control, as applicable to simple electro-mechanical systems;
- Sensors, using sensor data.

Second Newton's law in 2D case:

$$\begin{cases} m\ddot{x} = \sum f_{i,x} \\ m\ddot{y} = \sum f_{i,y} \end{cases} \quad (1)$$

where  $m$  is mass of a body,  $f_{i,x}$ ,  $f_{i,y}$  - Cartesian components of forces acting on the body.

Euler law of motion in 2D case:

$$J\ddot{\varphi} = \sum \tau_i \quad (2)$$

where  $J$  is a moment of inertia of a body,  $\varphi$  is orientation of the body,  $\tau_i$  is torque. All of these entities are defined assuming rotation around the axis normal to the plane of motion, with positive direction of rotation defined as clock-wise.

We can see similarity of these equation with Newton laws. Note that this is only true for a 2D case, not for a general 3D case, where the Euler dynamics equations have the following form:

$$\mathbf{I}\dot{\omega} = \omega \times (\mathbf{I}\omega) + \tau \quad (3)$$

where  $\mathbf{I}$  is matrix of inertia,  $\omega$  is angular velocity and  $\tau$  is external torque (all in body frame).

A pendulum can be described by the following dynamical equations:

$$J\ddot{\varphi} = lmg \sin(\varphi) \quad (4)$$

where  $J$ ,  $l$ ,  $m$  and  $g$  are moment of inertia, length, mass and gravitational acceleration.

If there is motor torque  $\tau$  acting on the pendulum, the dynamics takes form:

$$J\ddot{\varphi} = lmg \sin(\varphi) + \tau \quad (5)$$



# TORQUE GENERATED BY A FORCE

A force  $\mathbf{f}$  applied to a rigid body at a point  $\mathbf{r}$  generates a torque  $\tau$ :

$$\tau = \mathbf{r} \times \mathbf{f} \quad (6)$$

In 2D case, a force with magnitude  $f$  acting along a line with a distance  $r$  from the axis (around which torque is computed) generates torque according to the formula:

$$\tau = rf \quad (7)$$

It is tempting to think of moment of inertia in 2D case as an analog of mass; moment of inertia determines the rate of change of **angular velocity** for a given **torque**, same as mass determines the rate of change of **linear velocity** for a given **force**.

$$\begin{cases} J\dot{\omega} = \tau \\ m\dot{v} = f \end{cases}$$

Another way to think about moment of inertia is through kinetic energy. Kinetic energy of a rigid body can be described as:

$$T = \frac{1}{2}m\mathbf{v}^\top\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}^\top\mathbf{I}\boldsymbol{\omega} \quad (8)$$

In 2D case it is simplified:

$$T = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}J\omega^2 \quad (9)$$

Moment of inertia determines how energy will the body acquire by the increase in its angular velocity.

Moment of inertia of a point mass  $m$  meters away from the axis of rotation is given as:

$$J = mr^2 \quad (10)$$

The same formula give moment of inertia for a rim of a disk with radius  $r$ . For a solid disk with uniform mass distribution the formula becomes:

$$J = \frac{1}{2}mr^2 \quad (11)$$

For a centrally mounted rod the moment of inertia is  $J = \frac{1}{12}mr^2$  and for a rod mounted like a pendulum it is  $J = \frac{1}{3}mr^2$ .

# MECHANICAL ENERGY

*Mechanical energy* is a sum of kinetic and potential energy. We already saw the form of kinetic energy in 2D case:

$$T = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}J\omega^2.$$

Potential energy is more complex: it depends on what type of potential (conservative) forces that act on the system. For example:

- gravitational potential energy:  $\Pi = mg(y - y_0)$ , where  $y_0$  is the value of vertical coordinate  $y$  for which potential energy is defined as zero;
- linear spring potential energy:  $\Pi = \frac{1}{2}k(r - r_0)^2$ , where  $k$  is the stiffness and  $r_0$  is the length of the spring in the relaxed state.

# MECHANICAL WORK, 1

The change of the total energy of the system is equal to the sum of *mechanical work* performed by the active forces:

$$\Delta T + \Delta \Pi = \sum A_i \quad (12)$$

where  $A_i$  is the work performed by the active force  $f_i$  or a torque  $\tau_i$ .

We could compute the work performed by a force (applied to a point mass) on the interval of time from  $t_0$  to  $t_f$ :

$$A = \int_{t_0}^{t_f} \mathbf{v}^\top \mathbf{f} \, dt \quad (13)$$

where  $\mathbf{v} = \mathbf{v}(t)$  is the velocity of the point mass and  $\mathbf{f} = \mathbf{f}(t)$  is the force.

## MECHANICAL WORK, 2

If a force  $\mathbf{f}$  is collinear with velocity  $\mathbf{v}$ , we re-write the formula for mechanical work using the absolute values of these vectors  $f$  and  $v = \dot{s}$ :

$$A = \int_{t_0}^{t_f} v f \, dt \quad (14)$$

As long as  $f = \text{const}$ , the expression becomes trivial:

$$A = (s(t_1) - s(t_0))f \quad (15)$$

We can derive the same formulas for mechanical work of a torque  $\tau$ :

$$A = \int_{t_0}^{t_f} \omega \tau \, dt \quad (16)$$

$$A = (\varphi(t_1) - \varphi(t_0))\tau \quad (\tau = \text{const}) \quad (17)$$

# MECHANICAL POWER

The entity  $\mathbf{v}^\top \mathbf{f}$  we saw before is called *mechanical power*. It can also be defined as:

$$P = \mathbf{v}^\top \mathbf{f} \quad (18)$$

$$P = \omega \tau \quad (19)$$

$$P = \frac{d}{dt} A \quad (20)$$

Note that where as mechanical work is an integral entity, mechanical power is instantaneous. Also, using the definition of mechanical energy we observe that:

$$\frac{d}{dt}(T + \Pi) = \sum P_i \quad (21)$$



Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:  
[github.com/SergeiSa/Mechatronics-2023](https://github.com/SergeiSa/Mechatronics-2023)

