Second order systems

Mechatronics, Lecture 6

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Second-order systems

Some of the second-order dynamical systems we have seen before:

- Spring-mass-damper dynamics.
- RLC circuit.
- DC motor angular velocity dynamics.
- Linearized model of a pendulum.

In a few lectures we will see that PID controller also acts in a way similar to a second-order dynamical system.

Since any polynomial can be represented as a product of linear polynomials and quadratic polynomials, we can also re-write any transfer function as a product of transfer functions with quadratic polynomials in the denominator. We can see second-order dynamical systems as fundamental building blocks of dynamical systems.

Eigenvalues, 1

Observing eq. $m\ddot{y} + \mu_0\dot{y} + c_0 y = 0$ we can tell that it is stable if (sufficient but not necessary condition) $m > 0$, $\mu > 0$, and $c > 0$ - this follows from the physics of the system.

A more principled approach is to find eigenvalues of the linear system. We start by dividing the equation by m:

$$\ddot{y} + \mu\dot{y} + cy = 0 \tag{1}$$

where $\mu = \mu_0/m$ and $c = c_0/m$. Defining $x_1 = y$ and $x_2 = \dot{y}$, the system can be equivalently represented as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -\mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

Eigenvalues, 2

With linear system $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -\mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we need to find its eigenvalues. We can use a formula for eigenvalues based on trace and determinant:

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \tag{3}$$

where T is trace and D is the determinant.

In our case $T = -\mu$ and $D = c$, and eigenvalues are:

$$\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4c}}{2} \tag{4}$$

Eigenvalues, 3

Let us analyze eigenvalues $\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4c}}{2}$. We can see that if $\mu \geq 0$ and $c \geq 0$, there are only two scenarios:

1. $\mu^2 - 4c \geq 0$, in which case $\sqrt{\mu^2 - 4c} \leq \mu$, the eigenvalues are purely real and negative.
2. $\mu^2 - 4c < 0$, in which case $\sqrt{\mu^2 - 4c}$ is a purely imaginary number, the eigenvalues are complex with negative real parts.

If $\mu \geq 0$ and $c = 0$, $\lambda_1 = -\mu$, $\lambda_2 = 0$, hence the system is marginally stable.

Eigenvalues, 4

If $\mu \geq 0$ and $c < 0$, then $\sqrt{\mu^2 - 4c} \geq \mu$, and eigenvalues are purely real and one of them is positive, the system is unstable. If $\mu < 0$ and $c < 0$ at least one of the eigenvalues is still positive.

If $\mu < 0$ and $c \geq 0$, then again there are only two scenarios:

1. $\mu^2 - 4c \geq 0$, in which case $\sqrt{\mu^2 - 4c} \leq \mu$, the eigenvalues are purely real and positive.
2. $\mu^2 - 4c < 0$, in which case $\sqrt{\mu^2 - 4c}$ is a purely imaginary number, the eigenvalues are complex with positive real parts.

Definition

If $\mu \geq 0$ and $c \geq 0$ the system is stable, if $\mu < 0$ or $c < 0$ it is unstable.

Characteristic polynomial

Going back to the eq. $\ddot{y} + \mu\dot{y} + cy = 0$ we can write characteristic eq. for it:

$$k^2 y + \mu k + c = 0 \tag{5}$$

Its roots are given by the formula:

$$k = \frac{-\mu \pm \sqrt{\mu^2 - 4c}}{2} \tag{6}$$

As we can see, it is exactly the same as the determinant-trace formula.

Transfer function poles

Now, let us consider the transfer function:

$$W(s) = \frac{1}{s^2 + \mu s + c} \tag{7}$$

Poles of the transfer functions are solutions of the polynomial in its denominator; notice that this polynomial is exactly the same as the previously discussed characteristic polynomial. Meaning the poles of the transfer function are eigenvalues of the linear system.

Natural frequency representation, 1

Another popular form of writing a second-order ODE is the following:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0 \tag{8}$$

where ω_n is a *natural frequency* and ζ is *damping factor*. The quantity $\zeta\omega_n$ is called *damping attenuation*. The relation between ζ, ω_n and μ, c is:

$$\omega_n = \sqrt{c}, \qquad \zeta = \frac{\mu}{2\sqrt{c}} \tag{9}$$

With this, the expression for eigenvalues is:

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{10}$$

Natural frequency representation, 2

With the representation $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0$ we can tell:

- the system is overdamped (has pure real negative eigenvalues) is $\zeta > 1$, critically damped if $\zeta = 1$ and underdamped (produces oscillations) if $0 \leq \zeta < 1$;
- the system has purely imaginary eigenvalues (non-decaying oscillations) iff $\zeta = 0$ (since $\omega_n = 0$ corresponds to the trivial solution).

Natural frequency representation, 3

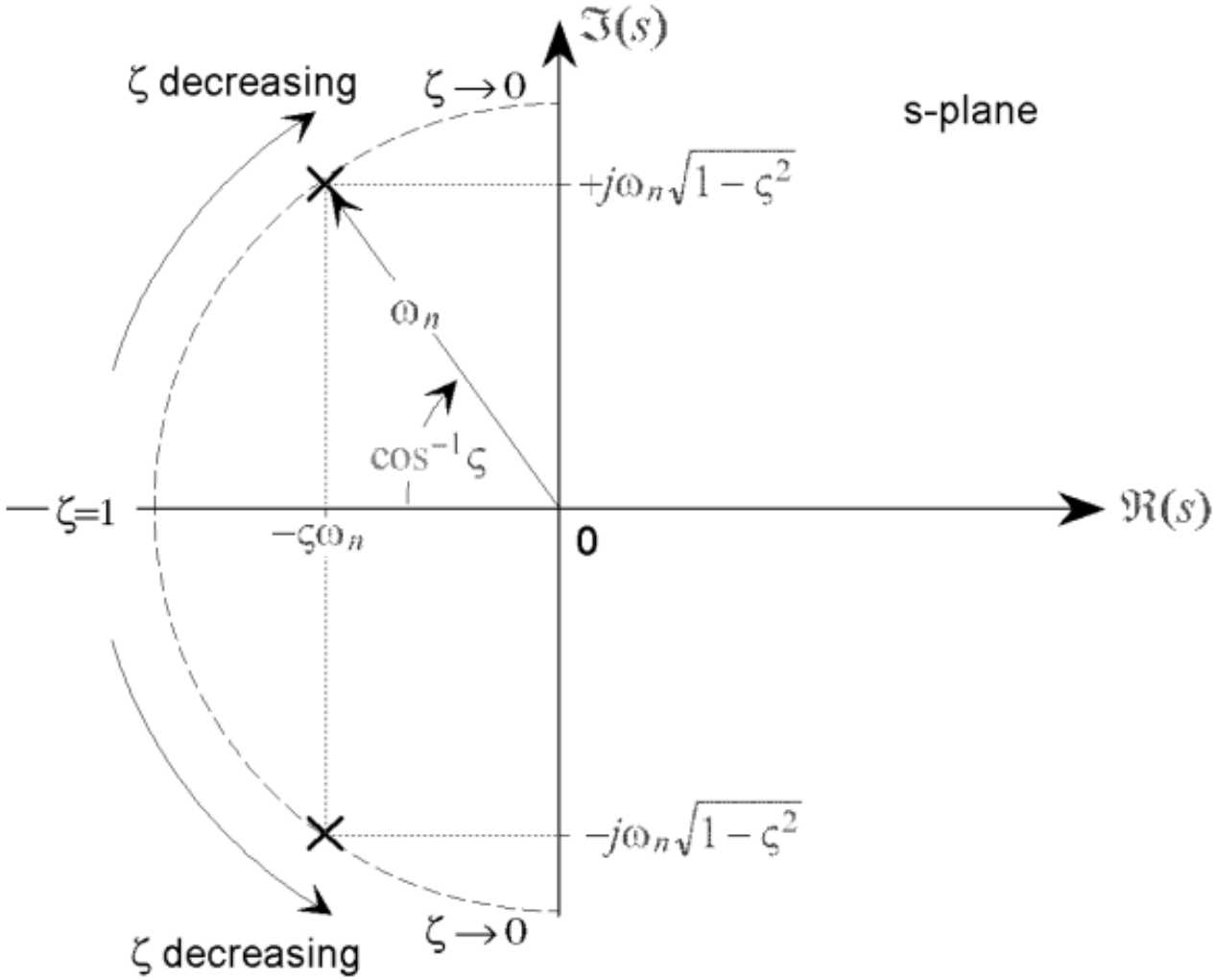

Figure 1: Poles and ζ, ω_n. Credit: MIT.

Pure imaginary poles, 1

Given $\zeta = 0$ (implying $\mu = 0$) we have pure imaginary poles / eigenvalues $\lambda = \pm i\omega_n$. Moreover, the dynamics equations become:

$$\ddot{y} + \omega_n^2 y = 0 \tag{11}$$

Proposing solution $y = A\sin(\omega_n t + \varphi)$, noting that $\dot{y} = \omega_n A\cos(\omega_n t + \varphi)$ and $\ddot{y} = -\omega_n^2 A\sin(\omega_n t + \varphi)$, we see:

$$-\omega_n^2 A\sin(\omega_n t + \varphi) + \omega_n^2 A\sin(\omega_n t + \varphi) = 0 \tag{12}$$

which is an equality.

Pure imaginary poles, 2

Previous slide showed that the system without damping $\ddot{y} + \omega_n^2 y = 0$ will oscillate with angular frequency ω_n; this motivates the name "natural frequency".

Also it shows that, if we pass a harmonic signal with angular frequency ω_n through a transfer function $W(s) = s^2 + \omega_n^2$ we will get a zero output.

Alternately, passing a harmonic signal with angular frequency ω_n through a transfer function $W(s) = \frac{1}{s^2+\omega_n^2}$ we observe an infinite gain. Same can be observed by substituting $i\omega$ for s and evaluating the absolute value of the transfer function at $\omega = \omega_n$. This effect is called *resonance*.

Frequency response

Given a transfer function $W(s) = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$ we can find frequency response gain by computing absolute value of $W(i\omega)$:

$$G = |W(i\omega)| = \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \tag{13}$$

- As $\omega \to 0$, $G \to \frac{1}{\omega_n^2}$.
- As $\omega \to \omega_n$, $G \to \frac{1}{\sqrt{2\zeta}\omega_n^2}$.
- As $\omega \to \infty$, $G \to 0$.

Read more

- MIT. 2.14 Analysis and Design of Feedback Control Systems. Understanding Poles and Zeros

Lecture slides are available via Github, links are on Moodle

You can help improve these slides at:
github.com/SergeiSa/Mechatronics-2023