$$P_T(t) = \underbrace{\int_t^T \lambda(s)^2 e^{-2\theta(s-t)} \underbrace{\left(\int_s^T \lambda(r)^2 e^{-\theta(r-s)} \, dr \right)}_{(\mathrm{I})} ds}_{(\mathrm{II})}$$

(I) $\quad I(s) = \min \{ i = 1, \ldots, N \mid s \leq T_i \}$

$$\int_s^T \lambda(r)^2 e^{-\theta(r-s)} \, dr$$

$$= \int_s^{T_{I(s)}} \lambda_{I(s)}^2 e^{-\theta(r-s)} \, dr + \sum_{i=I(s)+1}^{N} \lambda_i^2 \int_{T_{i-1}}^{T_i} e^{-\theta(r-s)} \, dr$$

$$= \frac{\lambda_{I(s)}^2}{\theta} \left[1 - e^{-\theta(T_{I(s)} - s)} \right] + \sum_{i=I(s)+1}^{N} \frac{\lambda_i^2}{\theta} \left[e^{-\theta(T_{i-1}-s)} - e^{-\theta(T_i - s)} \right]$$

(II) $$\int_t^T (\ldots) \, ds = \underbrace{\int_t^{I(t)} (\ldots) \, ds}_{(\mathrm{II}b)} + \sum_{i=I(t)+1}^{N} \underbrace{\int_{T_{i-1}}^{T_i} (\ldots) \, ds}_{(\mathrm{II}a)}$$

(IIa) $$\int_{T_{i-1}}^{T_i} \lambda_i^2 e^{-2\theta(s-t)} \cdot \left\{ \frac{\lambda_i^2}{\theta} \left[1 - e^{-\theta(T_i - s)} \right] + \sum_{j=i+1}^{N} \frac{\lambda_j^2}{\theta} \left[e^{-\theta(T_{j-1}-s)} - e^{-\theta(T_j - s)} \right] \right\}$$

$$= \ldots = \frac{\lambda_i^4}{2\theta^2} \left[e^{-\theta(T_{i-1}-t)} - e^{-\theta(T_i - t)} \right]^2 +$$

$$\sum_{j=i+1}^{N} \frac{\lambda_i^2 \lambda_j^2}{\theta^2} \left[e^{-\theta(T_{i-1}-t)} - e^{-\theta(T_i - t)} \right] \cdot \left[e^{-\theta(T_{j-1}-t)} - e^{-\theta(T_j - t)} \right]$$

$$\sum_{i=I(t)+1}^{N} \int_{T_{i-1}}^{T_i} (\dots)\, ds =$$

$$\frac{1}{2} \sum_{i=I(t)+1}^{N} \frac{\lambda_i^4}{\theta^2} \cdot \left[e^{-\theta(T_{i-1}-t)} - e^{-\theta(T_i - t)} \right]^2 +$$

$$\sum_{i=I(t)+1}^{N} \cdot \sum_{j=i+1}^{N} \frac{\lambda_i^2 \lambda_j^2}{\theta^2} \left[e^{-\theta(T_{i-1}-t)} - e^{-\theta(T_i - t)} \right] \cdot \left[e^{-\theta(T_{j-1}-t)} - e^{-\theta(T_j - t)} \right]$$

$$= \frac{1}{2} \cdot \left\{ \sum_{i=I(t)+1}^{N} \frac{\lambda_i^2}{\theta} \left[e^{-\theta(T_{i-1}-t)} - e^{-\theta(T_i - t)} \right] \right\}^2$$

(IIb) $$\int_t^{I(t)} (\dots)\, ds$$

$$= \frac{\lambda_{I(t)}^4}{2\theta^2} \left[1 - e^{-\theta(T_{I(t)}-t)} \right]^2$$

$$+ \sum_{j=I(t)+1} \frac{\lambda_{I(t)}^2 \lambda_j^2}{\theta^2} \left[1 - e^{-\theta(T_{I(t)}-t)} \right] \cdot \left[e^{-\theta(T_{j-1}-t)} - e^{-\theta(T_j - t)} \right]$$

$$= \frac{\lambda_{I(t)}^4}{2\theta^2} \left[1 - e^{-\theta(T_{I(t)}-t)} \right]^2$$

$$+ \frac{\lambda_{I(t)}^2}{\theta} \left[1 - e^{-\theta(T_{I(t)}-t)} \right] \cdot \left(\sum_{j=I(t)+1}^{N} \frac{\lambda_j^2}{\theta} \left[e^{-\theta(T_{j-1}-t)} - e^{-\theta(T_j - t)} \right] \right)$$

(II)

$$p_T(t) = \frac{1}{2}\cdot\left\{\left(1-e^{-\theta(T_{I(t)}-t)}\right)\frac{\lambda_{I(t)}^2}{\theta} + \sum_{i=I(t)+1}^{N}\frac{\lambda_i^2}{\theta}\left[e^{-\theta(T_{i-1}-t)} - e^{-\theta(T_i-t)}\right]\right\}^2$$

$$= \frac{1}{2}\left\{\frac{\lambda_{I(t)}^2}{\theta}\left(1-e^{-\theta(T_{I(t)}-t)}\right) + e^{\theta t}\cdot\sum_{i=I(t)+1}^{N}\frac{\lambda_i^2}{\theta}\left[e^{-\theta T_{i-1}} - e^{-\theta T_i}\right]\right\}^2$$

Weight: $(T_K \le T_N)$

$$\int_{T_{K-1}}^{T_K} p_T(t)\,dt = \frac{1}{2}\int_{T_{K-1}}^{T_K} f_K(t)^2\,dt,$$

$$f(t) = f_{d_i,K}(t) = \frac{\lambda_K^2}{\theta}\left(1-e^{-\theta(T_K-t)}\right) + \sum_{i=K+1}^{N}\frac{\lambda_i^2}{\theta}\left(e^{-\theta(T_{i-1}-t)} - e^{-\theta(T_i-t)}\right)$$

$$g(t) = f(t)^2, \quad f'(t) = \theta f(t) - \lambda_K^2$$

$$G(t) = \frac{1}{2\theta} f(t)^2 + \frac{\lambda_K^2}{\theta^2} f(t) + \frac{\lambda_K^4}{\theta^2} t$$

$$G'(t) = \frac{1}{\theta} f(t)\cdot f'(t) + \frac{\lambda_K^2}{\theta^2} f'(t) + \frac{\lambda_K^4}{\theta^2}$$

$$= \frac{1}{\theta} f(t)\left(\theta f(t) - \lambda_K^2\right) + \frac{\lambda_K^2}{\theta^2}\left(\theta f(t) - \lambda_K^2\right) + \frac{\lambda_K^4}{\theta^2}$$

$$= f(t)^2$$

$$\int_{T_{k-1}}^{T_k} P_1(t)\,dt = \frac{1}{2}\left[g(T_k) - g(T_{k-1}) \right]$$

$$= \frac{1}{2}\left[\frac{1}{2\Theta}\left(f(T_k)^2 - f(T_{k-1})^2 \right) + \frac{\lambda_k^2}{\Theta^2}\left(f(T_k) - f(T_{k-1}) \right) + \frac{\lambda_k^4}{\Theta^2}\left(T_k - T_{k-1} \right) \right]$$

$$f(T_k) = \sum_{i=k+1}^{N} \frac{\lambda_i^2}{\Theta}\left(e^{-\Theta(T_{i-1} - T_k)} - e^{-\Theta(T_i - T_k)} \right)$$

$$f(T_{k-1}) = \frac{\lambda_k^2}{\Theta}\left(1 - e^{-\Theta(T_k - T_{k-1})} \right) + \sum_{i=k+1}^{N} \frac{\lambda_i^2}{\Theta}\left(e^{-\Theta(T_{i-1} - T_{k-1})} - e^{-\Theta(T_i - T_{k-1})} \right)$$

$$= \frac{\lambda_k^2}{\Theta}\left(1 - e^{-\Theta(T_k - T_{k-1})} \right) +$$

$$e^{-\Theta(T_k - T_{k-1})} \cdot f(T_k)$$

$$f(T_N) = 0$$

$$f(T_{k-1}) = \frac{\lambda_k^2}{\Theta}\left(1 - e^{-\Theta(T_k - T_{k-1})} \right) + e^{-\Theta(T_k - T_{k-1})} f(T_k)$$

$$\omega_K = \int_{T_{K-1}}^{T_K} v(t)^2 \lambda(t)^2 \, dt = \lambda_K^2 \cdot \underbrace{\int_{T_{K-1}}^{T_K} v(t)^2 \, dt}_{V_K}$$

$$V_K = \int_{T_{K-1}}^{T_K} v(t)^2 \, dt$$

$$= \underbrace{\int_{T_{K-1}}^{T_K} z_0^2 \int_0^t \lambda^2(s) \, ds \, dt}_{V_K^1} + \underbrace{\int_{T_{K-1}}^{T_K} z_0 e^{-\theta t} \int_0^t \lambda^2(s) e^{-\theta s} \int_0^s \eta^2(u) e^{2\theta u} \, du \, ds \, dt}_{V_K^2}$$

$$V_K^1 = z_0^2 \int_{T_{K-1}}^{T_K} \left[\int_0^{T_{K-1}} \lambda^2(s) \, ds + \int_{T_{K-1}}^t \lambda^2(s) \, ds \right] dt$$

$$= z_0^2 \int_{T_{K-1}}^{T_K} \left[\sum_{i=1}^{K-1} \lambda_i^2 \left(T_i - T_{i-1} \right) + \lambda_K^2 \left(t - T_{K-1} \right) \right] dt$$

$$= z_0^2 \left[\left(T_K - T_{K-1} \right) \cdot \sum_{i=1}^{K-1} \lambda_i^2 \left(T_i - T_{i-1} \right) + \lambda_K^2 \cdot \frac{1}{2} \left(T_K - T_{K-1} \right)^2 \right]$$

$$= z_0^2 \left[\frac{\lambda_K^2}{2} \left(T_K - T_{K-1} \right)^2 + \left(T_K - T_{K-1} \right) \sum_{i=1}^{K-1} \lambda_i^2 \left(T_i - T_{i-1} \right) \right]$$

V_K^2:
$$\int_0^s \eta^2(u) e^{2\theta u} \, du = \underbrace{\sum_{i=1}^{I(s)-1} \eta_i^2 \frac{1}{2\theta} \left(e^{2\theta T_i} - e^{2\theta T_{i-1}} \right)}_{S_1(I(s)-1)} + \frac{\eta_{I(s)}^2}{2\theta} \left(e^{2\theta s} - e^{2\theta T_{I(s)-1}} \right)$$

$$= \frac{\eta_{I(s)}^2}{2\theta} \left(e^{2\theta s} - e^{2\theta T_{I(s)-1}} \right) + S_1 \left(I(s) - 1 \right),$$

$$S_1(k) = S_1(k-1) + \frac{\eta_k^2}{2\theta} \left(e^{2\theta T_k} - e^{2\theta T_{k-1}} \right)$$

$$\int_0^t \lambda^2(s)\, e^{-\theta s} \cdot \left[\frac{\eta_{I(s)}^2}{2\theta} \left(e^{2\theta s} - e^{2\theta T_{I(s)-1}} \right) + S_1\left(I(s)-1\right) \right] ds$$

$$= \sum_{i=1}^{I(t)-1} \int_{T_{i-1}}^{T_i} \lambda_i^2\, e^{-\theta s} \cdot \left[\frac{\eta_i^2}{2\theta} \left(e^{2\theta s} - e^{2\theta T_{i-1}} \right) + S_1(i-1) \right] ds$$

$$+ \int_{T_{I(t)-1}}^{t} \lambda_{I(t)}^2\, e^{-\theta s} \left[\frac{\eta_{I(t)}^2}{2\theta} \left(e^{2\theta s} - e^{2\theta T_{I(t)-1}} \right) + S_1\left(I(t)-1\right) \right] ds$$

$$= \sum_{i=1}^{I(t)-1} \int_{T_{i-1}}^{T_i} \lambda_i^2 \frac{\eta_i^2}{2\theta} \left(e^{\theta s} - e^{-\theta s + 2\theta T_{i-1}} \right) ds + \sum_{i=1}^{I(t)-1} \lambda_i^2\, S_1(i-1) \cdot \int_{T_{i-1}}^{T_i} e^{-\theta s} ds$$

$$+ \frac{\lambda_{I(t)}^2 \eta_{I(t)}^2}{2\theta} \cdot \int_{T_{I(t)-1}}^{t} \left(e^{\theta s} - e^{-\theta s + 2\theta T_{I(t)-1}} \right) ds + \lambda_{I(t)}^2 \cdot S_1(I(t)-1) \cdot \int_{T_{I(t)-1}}^{t} e^{-\theta s} ds$$

$$= \sum_{i=1}^{I(t)-1} \frac{\lambda_i^2 \eta_i^2}{2\theta^2} \left[e^{\theta s} + e^{-\theta s + 2\theta T_{i-1}} \right]_{T_{i-1}}^{T_i} - \sum_{i=1}^{I(t)-1} \frac{\lambda_i^2}{\theta} \cdot S_1(i-1) \cdot \left[e^{-\theta s} \right]_{T_{i-1}}^{T_i}$$

$$+ \frac{\lambda_{I(t)}^2 \eta_{I(t)}^2}{2\theta^2} \left[e^{\theta s} + e^{-\theta s + 2\theta T_{I(t)-1}} \right]_{T_{I(t)-1}}^{t} - \frac{\lambda_{I(t)}^2}{\theta} \cdot S_1(I(t)-1) \cdot \left[e^{-\theta s} \right]_{T_{I(t)-1}}^{t}$$

$$= \sum_{i=1}^{I(t)-1} \frac{\lambda_i^2 \eta_i^2}{2\theta^2} e^{\theta T_i} \left(1 - e^{-\theta(T_i - T_{i-1})} \right)^2 + \sum_{i=1}^{I(t)-1} \frac{\lambda_i^2 \cdot S_1(i-1)}{\theta} \cdot e^{-\theta T_{i-1}} \cdot \left(1 - e^{-\theta(T_i - T_{i-1})} \right)$$

$$+ \frac{\lambda_{I(t)}^2 \eta_{I(t)}^2}{2\theta^2} e^{\theta t} \left(1 - e^{-\theta(t - T_{I(t)-1})} \right)^2 + \frac{\lambda_{I(t)}^2 \cdot S_1(I(t)-1)}{\theta} e^{-\theta T_{I(t)-1}} \cdot \left(1 - e^{-\theta(t - T_{I(t)-1})} \right)$$

$$\int_0^t \lambda^2(t)\, e^{-\theta s} \cdot [\ldots]\, ds$$

$$= S_2(I(t)-1) + S_3(I(t)-1)$$

$$+ \frac{\lambda_{I(t)}^2 \eta_{I(t)}^2}{2\theta^2} e^{\theta t} \cdot \left(1 - e^{-\theta(t - T_{I(t)-1})}\right)^2$$

$$\cancel{+ \lambda_{I(t)}^2 \eta_{I(t)}^2} + \frac{\lambda_{I(t)}^2 \cdot S_1(I(t)-1)}{\theta} e^{-\theta T_{I(t)-1}} \left(1 - e^{-\theta(t - T_{I(t)-1})}\right)$$

$$S_2(k) = S_2(k-1) + \frac{\lambda_k^2 \eta_k^2}{2\theta^2} e^{\theta T_k}\left(1 - e^{-\theta(T_k - T_{k-1})}\right)^2$$

$$S_3(k) = S_3(k-1) + \frac{\lambda_k^2 \cdot S_1(k-1)}{\theta} e^{-\theta T_{k-1}} \left(1 - e^{-\theta(T_k - T_{k-1})}\right)$$

$$V_k^2 = \int_{T_{k-1}}^{T_k} z_0\, e^{-\theta t} \cdot \left[S_2(k-1) + S_3(k-1)\right] dt$$

$$+ \int_{T_{k-1}}^{T_k} z_0 \frac{\lambda_k^2 \eta_k^2}{2\theta^2} \left(1 - e^{-\theta(t - T_{k-1})}\right)^2 dt$$

$$+ \int_{T_{k-1}}^{T_k} z_0 \frac{\lambda_k^2 S_1(k-1)}{\theta} e^{-\theta T_{k-1}} e^{-\theta t} \cdot \left(1 - e^{-\theta(t - T_{k-1})}\right) dt$$

$$= V_k^3 + V_k^4 + V_k^5$$

$$V_k^3 = \int_{T_{k-1}}^{T_k} z_0 e^{-\theta t} \left(S_2(k-1) + S_3(k-1) \right) dt$$

$$= z_0 \left(S_2(k-1) + S_3(k-1) \right) \cdot \frac{1}{\theta} \left(e^{-\theta T_{k-1}} - e^{-\theta T_k} \right)$$

$$= \frac{z_0}{\theta} \left(S_2(k-1) + S_3(k-1) \right) \cdot \left(e^{-\theta T_{k-1}} - e^{-\theta T_k} \right)$$

$$V_k^4 = z_0 \int_{T_{k-1}}^{T_k} \frac{\lambda_k^2 \eta_k^2}{2\theta^2} \left(1 - 2 e^{-\theta(t-T_{k-1})} + e^{-2\theta(t-T_{k-1})} \right) dt$$

$$= \frac{\lambda_k^2 \eta_k^2 \cdot z_0}{2\theta^2} \cdot \left[t + \frac{2}{\theta} e^{-\theta(t-T_{k-1})} - \frac{1}{2\theta} e^{-2\theta(t-T_{k-1})} \right]_{T_{k-1}}^{T_k}$$

$$= z_0 \frac{\lambda_k^2 \eta_k^2}{2\theta^2} \left[T_k - T_{k-1} + \frac{2}{\theta} e^{-\theta(T_k - T_{k-1})} - \frac{1}{2\theta} e^{-2\theta(T_k - T_{k-1})} - \frac{2}{\theta} + \frac{1}{2\theta} \right]$$

$$= z_0 \frac{\lambda_k^2 \eta_k^2}{2\theta^2} \left[\frac{2}{\theta} e^{-\theta(T_k - T_{k-1})} - \frac{1}{2\theta} e^{-2\theta(T_k - T_{k-1})} + (T_k - T_{k-1}) - \frac{3}{2\theta} \right]$$

$$= \frac{\lambda_k^2 \eta_k^2}{2\theta^2} \cdot z_0 \left[t - \frac{1}{\theta} \left(1 - e^{-\theta(t-T_{k-1})} \right) - \frac{1}{2\theta} \left(1 - e^{-\theta(t-T_{k-1})} \right)^2 \right]_{T_{k-1}}^{T_k}$$

$$= z_0 \frac{\lambda_k^2 \eta_k^2}{2\theta^2} \left[(T_k - T_{k-1}) - \frac{1}{\theta} \left(1 - e^{-\theta(T_k - T_{k-1})} \right) - \frac{1}{2\theta} \left(1 - e^{-\theta(T_k - T_{k-1})} \right)^2 \right]$$

$$V_k^5 = \int_{T_{k-1}}^{T_k} z_0 \frac{\lambda_k^2 S_1(k-1)}{\theta} \cdot e^{\theta t} \cdot \left(e^{-\theta T_{k-1}} - e^{-\theta t} \right) dt$$

$$= z_0 \frac{\lambda_k^2 S_1(k-1)}{\theta} \int_{T_{k-1}}^{T_k} \left[e^{-\theta t - \theta T_{k-1}} - e^{-2\theta t} \right] dt$$

$$V_k^5 = z_0 \frac{\lambda_k^2 S_1(k-1)}{\Theta} \cdot \left[\frac{1}{2\Theta} e^{-2\Theta t} - \frac{1}{\Theta} e^{-\Theta t - \Theta T_{k-1}} \right]_{T_{k-1}}^{T_k}$$

$$= z_0 \frac{\lambda_k^2 S_1(k-1)}{2 \cdot \Theta^2} \cdot \left[e^{-2\Theta T_k} - e^{-2\Theta T_{k-1}} - 2 e^{-\Theta T_k - \Theta T_{k-1}} + 2 e^{-\Theta T_{k-1} - \Theta T_{k-1}} \right]$$

$$= z_0 \frac{\lambda_k^2 \cdot S_1(k-1)}{2\Theta^2} \left[e^{-2\Theta T_k} - e^{-2\Theta T_{k-1}} - 2 e^{-\Theta(T_k + T_{k-1})} + 2 e^{-2\Theta T_{k-1}} \right]$$

$$= z_0 \frac{\lambda_k^2 \cdot S_1(k-1)}{2\Theta^2} \left[e^{-2\Theta T_k} + e^{-2\Theta T_{k-1}} - 2 e^{-\Theta(T_k + T_{k-1})} \right]$$

$$= z_0 \frac{\lambda_k^2 S_1(k-1)}{2\Theta^2} \cdot \left(e^{-\Theta T_k} - e^{-\Theta T_{k-1}} \right)^2$$

Skew:

$$\omega_k = \lambda_k^2 \cdot \left\{ V_k^1 + V_k^3 + V_k^4 + V_k^5 \right\}$$

$$V_k^1 = z_0^2 \left[\frac{\lambda_k^2}{2} (T_k - T_{k-1})^2 + (T_k - T_{k-1}) \cdot \sum_{i=1}^{k-1} \lambda_i^2 (T_i - T_{i-1}) \right]$$

$$S_1(k) = S_1(k-1) + \frac{\eta_k^2}{2\Theta} \left(e^{2\Theta T_k} - e^{2\Theta T_{k-1}} \right)$$

$$S_2(k) = S_2(k-1) + \frac{\lambda_k^2 \eta_k^2}{2\Theta^2} e^{\Theta T_k} \left(1 - e^{-\Theta(T_k - T_{k-1})} \right)^2$$

$$S_3(k) = S_3(k-1) + \frac{\lambda_k^2 S_1(k-1)}{\Theta} e^{-\Theta T_{k-1}} \left(1 - e^{-\Theta(T_k - T_{k-1})} \right)$$

$$V_k^3 = \frac{z_0}{\Theta} \left[S_2(k-1) + S_3(k-1) \right] \left(e^{-\Theta T_{k-1}} - e^{-\Theta T_k} \right)$$

$$V_k^4 = z_0 \frac{\lambda_k^2 \eta_k^2}{2\Theta^2} \left[(T_k - T_{k-1}) - \frac{1}{\Theta} \left(1 - e^{-\Theta(T_k - T_{k-1})} \right) - \frac{1}{2\Theta} \left(1 - e^{-\Theta(T_k - T_{k-1})} \right)^2 \right]$$

$$V_k^5 = z_0 \frac{\lambda_k^2 S_1(k-1)}{2\Theta^2} \cdot \left(e^{-\Theta T_k} - e^{-\Theta T_{k-1}} \right)^2$$

Transformation

$$\tilde{S}_1(k) = e^{-2\theta T_k} S_1(k)$$

$$= e^{-2\theta(T_k - T_{k-1})} \cdot \tilde{S}_1(k-1) + \frac{\lambda_k^2}{2\theta}\left(1 - e^{-2\theta(T_k - T_{k-1})}\right)$$

$$\tilde{S}_2(k) = e^{-\theta T_k} S_2(k)$$

$$= e^{-\theta(T_k - T_{k-1})} \cdot \tilde{S}(k-1) + \frac{\lambda_k^2 \lambda_k^2}{2\theta^2}\left(1 - e^{-\theta(T_k - T_{k-1})}\right)$$

$$\tilde{S}_3(k) = e^{-\theta(T_k - T_{k-1})} \cdot \left[\tilde{S}_3(k-1) + \frac{\lambda_k^2 \tilde{S}_1(k-1)}{\theta}\left(1 - e^{-\theta(T_k - T_{k-1})}\right)^{*}\right]$$

$$V_k^3 = \frac{z_0}{\theta}\left[\tilde{S}_2(k-1) + \tilde{S}_3(k-1)\right]\left(1 - e^{-\theta(T_k - T_{k-1})}\right)$$

$$V_k^5 = z_0 \cdot \frac{\lambda_k^2}{2\theta^2} \cdot \tilde{S}(k-1) \cdot \left(1 - e^{-\theta(T_k - T_{k-1})}\right)$$