# Geometrical analysis of polynomial lens distortion models

**José I. Ronda, Antonio Valdés**

**Abstract** Polynomial functions are a usual choice to model the nonlinearity of lenses. Typically, these models are obtained through physical analysis of the lens system or on purely empirical grounds. The aim of this work is to facilitate an alternative approach to the selection or design of these models based on establishing a priori the desired geometrical properties of the distortion functions. With this purpose we obtain all the possible isotropic linear models and also those that are formed by functions with symmetry with respect to some axis. In this way, the classical models (decentering, thin prism distortion) are found to be particular instances of the family of models found by geometric considerations. These results allow to find generalizations of the most usually employed models while preserving the desired geometrical properties. Our results also provide a better understanding of the geometric properties of the models employed in the most usual computer vision software libraries.

José I. Ronda
Grupo de Tratamiento de Imágenes
Universidad Politécnica de Madrid
E-mail: jir@gti.ssr.upm.es
Antonio Valdés
Departamento de Álgebra, Geometría y Topología
Universidad Complutense de Madrid
E-mail: avaldes@ucm.es

## 1 Introduction

The correction of lens distortion is a relevant problem in computer vision and photogrammetry [8]. Lens distortion models the departure of the image capturing device from the theoretical pin-hole model and consists essentially in an image warping process.

Most of the proposed lens distortion models are given by an analytical expression of the space variables and the model parameters, although some efforts have also being made in order to depart from concrete analytical expressions [7]. These closed-form expressions usually express the position of the distorted points as a function of the ideal undistorted points given by the pinhole assumption, although in some cases it is the inverse of this function what is given by the model functions [4].

Lens distortion models can either result from the analysis of the physical problem or from a pragmatic approach led by the empirical capacity of the model to fit the observed data and the existence of practical algorithms to compute the model parameters. The concrete parameters of the distortion function are frequently computed within the bundle-adjustment process of a 3D scene reconstruction [4,14,9], but it is often possible to obtain these parameters from a single image that contains an element of known geometry, such as a calibration grid or a set of lines [1,11,15,6].

The first and probably most employed analytical form of lens distortion models is given by polynomials [5,3,14]. A natural generalization is that of rational functions [4], although some empirical studies [12]

attribute a similar modeling capabilities to both approaches.

A large part of the literature on these models assumes a radial rotationally invariant (RRI) distortion function [8, p. 191]. This strong geometrical requirement stems from the assumption that the capturing system is a rotationally symmetric structure. While these models suffice for some applications, those requiring higher precision must also account for such phenomenons as the non-alignment of the axes of the lens surfaces or the lack of paralellism of the lens and the imaging surface. The first is usually addressed by the decentering lens distortion model [5] and the second by means of the thin prism model [3]. The model employed in the computer vision software library OpenCV [2] integrates a rational term to model radial rotationally invariant distortion with polynomial terms accounting for thin prism and decentering distortion.

Radial rotationally invariant distortion, decentering distortion and thin-prism distortion are examples of models with interesting geometrical properties. They are linear, in the sense that the models constitute a vector space, they are isotropic, i.e., invariant to plane coordinate rotation and, from physical considerations, are formed of functions that are reflection-symmetric with respect to some axis. Some questions arise naturally:

– Are decentering and thin-prism distortion the only quadratic models with the three properties mentioned above? Or do they belong to a larger family of models from which we can select a better choice?
– How can we combine these models or extend them while keeping all these properties?
– Is it necessary to sacrifice some of these properties in order to obtain models with larger number of parameters?

In this work we intend to complement the physical approach to the analysis of lens distortion models with a geometrical perspective. To this purpose we formalize the relevant geometric properties of the models and obtain those that comply with these properties. In this way, we are in conditions to check to what extent the most employed models enjoy these properties and propose extensions that preserve them.

The paper is organized as follows. In section 2 we formalize the concept of lens distortion model and the main geometric properties of interest. In section 3 we study the basic properties of polynomial models introducing their complex representation that will be essential in the later analysis. Section 4 includes the first result of this work, which is the specification of all the possible polynomial linear isotropic lens distortion

models. Section 5 elaborates on this result, providing all the models that enjoy the previous properties and at the same time are formed of functions with reflection symmetry. Section 6 analyzes the properties of the most popular polynomial lens distortion models, placing them in the framework introduced by the theoretical results of the previous sections. Some extensions of these models are considered in section 7, that also includes the corresponding experiments. The conclusions are provided in section 8. An appendix at the end gathers the proofs of the theorems.

## 2 Lens distortion models

### 2.1 Distortion functions

We will term *lens distortion function* with *distortion center* $\mathbf{p}_0$ a smooth mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ that keeps fixed $\mathbf{p}_0$ and has identity Jacobian $J(F)$ at this point. To simplify the formulation we will assume that $\mathbf{p}_0$ is at the origin of coordinates. This is not restrictive in most practical situations, since the center of distortion is usually assumed to coincide with the principal point of the projection. Then the distortion function can be written as a mapping of the form

$$F(\mathbf{p}) = \mathbf{p} + G(\mathbf{p})$$

where $G(\mathbf{0}) = \mathbf{0}$ and $JG(\mathbf{0}) = \mathbf{0}$. Function $G$ will be termed *displacement function*. With this definition we are separating the linear and non-linear parts of the imaging process, the linear part being associated to the intrinsic parameter matrix. Two interesting analytical properties of lens distortion functions are easy to check:

– Each distortion function has a local inverse that is also of the same form.
– The composition of two distortions functions is another function of the same form.

Some physical properties of the imaging system have a correspondence with geometric properties of the displacement function. If the lens has perfect rotational symmetry and the image plane is perfectly orthogonal to the lens symmetry axis, the displacement function must be rotationally invariant. Formally, if $R_\theta$ represents the planar rotation of angle $\theta$, given by

$$\mathbf{p} = (x, y)^\top \mapsto R_\theta(\mathbf{p}) = \mathtt{R}_\theta \mathbf{p}, \ \mathtt{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \ (1)$$

a displacement function $G$ is *rotationally invariant* if it satisfies

$$G = R_{-\theta} \circ G \circ R_\theta$$

where $\circ$ denotes function composition.

Lack of parallelism between lens and image plane results in an image formation system that is no longer rotationally symmetric, but is symmetric with respect to the plane through the optical axis orthogonal to both lens and image planes. Displacement functions corresponding to this situation should exhibit *reflection symmetry* with respect to some line through the distortion center (symmetry axis). Formally, if $T_{\mathbf{u}}$ is the reflection leaving invariant the line through the origin with director vector $\mathbf{u}$, we have

$$G = T_{\mathbf{u}} \circ G \circ T_{\mathbf{u}}.$$

The displacement function $G(x,y)$ of a lens distortion model can be seen as a vector field on $\mathbb{R}^2$ that vanishes at the origin. An orthogonal basis for such vector fields is given by $\mathbf{u}(x,y) = (x,y)^{\mathsf{T}}$, $\mathbf{v}(x,y) = (-y,x)^{\mathsf{T}}$. Therefore, each displacement function can be written univoquely as the sum of a *radial* and a *tangential* displacement functions:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} g_r(x,y) + \begin{pmatrix} -y \\ x \end{pmatrix} g_t(x,y). \qquad (2)$$

### 2.2 Distortion models

We define a *lens distortion model* $\mathcal{M}$ as a set of set of displacement functions. A model will be termed *linear* if it is a vector space under the natural operations of sum and multiplication by scalars. Linear models are of practical importance because they greatly simplify the computational processes of obtainment of camera parameters.

A model is *isotropic* if it is invariant, as a set of functions, with respect to coordinate rotations. It is natural to consider in practice only models having this property because otherwise the characteristics of the model would vary with a rotation of the data. Formally, if $G$ is any function of the model $\mathcal{M}$, the model is *isotropic* if there is a $\tilde{G} \in \mathcal{M}$ such that

$$\tilde{G} = R_{-\theta} \circ G \circ R_{\theta}. \qquad (3)$$

We will also pay special attention to those models including only functions that are reflection symmetric with respect to some axis.

## 3 Polynomial models

### 3.1 Polynomial lens displacement functions

The $n$th-degree *polynomial lens distortion model* is the set of displacement functions of the form

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} X(x,y) \\ Y(x,y) \end{pmatrix}, \qquad (4)$$

where $X$ and $Y$ are polynomials of degree $\leq n$ without linear terms, so its Jacobian vanishes. We will also consider *homogeneous* $n$th-degree polynomial models in which $X$ and $Y$ are homogeneous polynomials of degree $n$.

For an arbitrary degree $n$ we define the vector mapping

$$v_n(x,y) = (x^n, x^{n-1}y, \ldots, y^n)^{\mathsf{T}}, \qquad (5)$$

so that we can express homogeneous displacement functions as

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \boldsymbol{w}_0^{\mathsf{T}} \\ \boldsymbol{w}_1^{\mathsf{T}} \end{pmatrix} v_n(x,y) = \mathtt{M}v_n(x,y), \ \mathbf{w}_i \in \mathbb{R}^{n+1}.$$

General (i.e., non-homogeneous displacement functions) can be expressed as sum of homogeneous displacement functions, and, consequently, can be represented by sets of matrices.

*Example 1* The simplest case is the quadratic model, corresponding to $n = 2$, for which the general and the homogeneous cases coincide. The displacement functions are of the form:

$$\begin{aligned} \Delta x &= a_0 x^2 + a_1 xy + a_2 y^2 \\ \Delta y &= b_0 x^2 + b_1 xy + b_2 y^2, \\ & \quad a_i, b_j \in \mathbb{R}, \end{aligned} \qquad (6)$$

that can be expressed in matrix form as

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}. \qquad (7)$$

A *polynomial radial displacement* is of the form

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} p(x,y),$$

where $p$ is a polynomial. As an example we have the well known $n$-coefficient radial rotationally invariant (RRI) model, given by functions of the form

$$\begin{aligned} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} (\alpha_1 r^2 + \cdots + \alpha_n r^{2n}) \\ r^2 &= x^2 + y^2. \end{aligned} \qquad (8)$$

It is easy to check that all the polynomial radial distortions that are invariant with respect to rotations are of this form.

We define analogously the *polynomial tangential displacement* functions as those of the form

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} q(x, y),$$

where $q$ is a polynomial.

In the homogeneous case radial displacement functions can be expressed as

$$\begin{aligned} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} \mathbf{w}^{\top} v_{n-1}(x, y) \\ &= \begin{pmatrix} w_1 & \cdots & w_n & 0 \\ 0 & w_1 & \cdots & w_n \end{pmatrix} v_n(x, y), \end{aligned} \tag{9}$$

and tangential distortion functions as

$$\begin{aligned} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} &= \begin{pmatrix} -y \\ x \end{pmatrix} \mathbf{w}^{\top} v_{n-1}(x, y) \\ &= \begin{pmatrix} 0 & -w_1 & \cdots & -w_n \\ w_1 & \cdots & w_n & 0 \end{pmatrix} v_n(x, y). \end{aligned} \tag{10}$$

Therefore radial and tangential displacement functions constitute linear subspaces of dimension $n$ of the matrix space $\mathbb{R}^{2 \times (n+1)}$, that intersect trivially. Since the dimension of the matrix space is $2(n+1) > 2n$, the functions $g_r$ and $g_t$ in the decomposition (2) are not in general polynomial for a polynomial displacement function. So we have the following proposition.

**Proposition 1** *The sets of $n$th-degree homogeneous radial or tangential displacements constitute isotropic subspaces of dimension $n$ of the matrix space $\mathbb{R}^{2 \times (n+1)}$, that intersect trivially.*

*Example 2* In the quadratic case the radial displacements are those of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} (t_1 x + t_2 y) = \begin{pmatrix} t_1 & t_2 & 0 \\ 0 & t_1 & t_2 \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}, \tag{11}$$

and the tangential displacements are those of the form

$$\begin{pmatrix} -y \\ x \end{pmatrix} (u_1 x + u_2 y) = \begin{pmatrix} 0 & -u_1 & -u_2 \\ u_1 & u_2 & 0 \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}. \tag{12}$$

The direct sum of the corresponding linear models is a vector subspace of dimension four of $\mathbb{R}^{2 \times 3}$, with which we can identify the set of quadratic distortion functions. Any quadratic displacement function outside this four-dimensional subspace has non-polynomial radial or tangential components.

## 3.2 Complex polynomial formulation of displacement functions

Polynomial displacement functions (4) can be expressed equivalently as a single complex polynomial in the complex variables $z$ and $\bar{z}$,

$$f(z, \bar{z}) = \Delta z = \sum_{(k,l) \in I}^{n} \gamma_{kl} z^k \bar{z}^l, \ \gamma_{kl} \in \mathbb{C}, \tag{13}$$

where $I$ is any finite set of index pairs $(k, l)$ such that $k \geq 0$, $l \geq 0$, $k + l \geq 2$. These polynomials have not been so far, to the authors knowledge, employed to express lens distortion functions, and we will see that they facilitate enormously the geometrical analysis of models.

The real polynomial (4) and the complex polynomial formulations (13) are indeed equivalent, since, if we write $P(x, y) = X(x, y) + iY(x, y)$, we have that

$$P(x, y) = P\left( \frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z}) \right) = f(z, \bar{z}).$$

Conversely, since $z = x + iy$, we recover $P = X + iY$ from $f$.

*Example 3* In the quadratic case a general complex polynomial is given by

$$\Delta z = \gamma_{20} z^2 + \gamma_{11} z \bar{z} + \gamma_{02} \bar{z}^2.$$

Let us write $\gamma_{kl} = \alpha_{kl} + i\beta_{kl}$. The corresponding real polynomial expression will be of the form

$$\Delta \mathbf{p} = \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}.$$

If we denote $\mathbf{a} = (a_0, a_1, a_2)^{\top}$, $\mathbf{b} = (b_0, b_1, b_2)^{\top}$, $\boldsymbol{\alpha} = (\alpha_{20}, \alpha_{11}, \alpha_{02})^{\top}$, $\boldsymbol{\beta} = (\beta_{20}, \beta_{11}, \beta_{02})^{\top}$ and $\mathbf{c} = \mathbf{a} + i\mathbf{b}$, $\boldsymbol{\gamma} = \boldsymbol{\alpha} + i\boldsymbol{\beta}$, it is easy to check that the correspondence between both sets of parameters is given by

$$\mathbf{c} = \mathtt{C}\boldsymbol{\gamma},$$

where

$$\mathtt{C} = \begin{pmatrix} 1 & 1 & 1 \\ 2i & 0 & -2i \\ -1 & 1 & -1 \end{pmatrix}.$$

The matrix $\mathtt{C}$ is invertible as a consequence of the equivalence between both kinds of parameterizations.

Radial and tangential displacement functions are also easily expressed in complex polynomial notation. Since $z$ corresponds to the radial vector $(x, y)$ and $iz$ to the tangential vector $(-y, x)$, radial and tangential displacements are given respectively by expressions of the form

$$z p(z, \bar{z}), \quad i z q(z, \bar{z}),$$

where $p(z, \bar{z})$ and $q(z, \bar{z})$ are *real-valued complex polynomials*, i.e., such that for any $z \in \mathbb{C}$ their evaluation is real. It is easy to check that this is equivalent to having coefficients satisfying $\gamma_{kl} = \bar{\gamma}_{lk}$.

Therefore the complex polynomials that are multiples of $z$ represent displacement functions that lie in the space generated by radial and tangential displacement functions. The only monomials that do not lie in this space are those of the form $\bar{z}^n$, thus providing a natural complement of that space (see proposition 1).

## 4 Linear isotropic models

In this section we aim at obtaining the polynomial models that enjoy at the same time the properties of being linear and rotationally invariant. To this purpose we will make use of the theory of group representations.

4.1 Group representations on polynomial spaces

Given a group $G$, a representation of $G$ on a vector space $V$ is a group homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(V),$$

where $\mathrm{Aut}(V)$ stands for the group of automorphisms of $V$, i.e., the set of invertible linear mappings $f : V \to V$. Hence, a representation is just a group action on the vector space $V$ such that the transformations defined by the elements of $G$ are linear mappings $V \to V$.

As an example that will be useful for our purposes, let us consider the group $G = SO(2)$ of plane rotations and the vector space $V = \mathcal{H}^n$ of homogeneous polynomials $P : \mathbb{R}^2 \to \mathbb{R}$ of degree $n$ in the variables $(x, y)$. The group representation

$$\rho : SO(2) \longrightarrow \mathrm{Aut}(\mathcal{H}^n)$$

is simply given by $\rho(\mathrm{R}_\theta)(P) = P'$ where

$$P'(\mathbf{p}) = P(\mathrm{R}_\theta \mathbf{p}),$$

where $\mathbf{p} = (x, y)^\mathsf{T}$. It is immediate to check that $\rho(\mathrm{R}_\theta)$ is a linear mapping whose inverse is $\rho(\mathrm{R}_{-\theta})$.

Since $\rho(\mathrm{R}_\theta)$ is an automorphism of $\mathcal{H}^n$, the elements of the basis of $\mathcal{H}^n$ given by the components of $v_n(\mathbf{p})$ (defined in (5)) are transformed into the basis

$$\left( \rho(\mathrm{R}_\theta)(x^n), \rho(\mathrm{R}_\theta)(x^{n-1}y), \ldots, \rho(\mathrm{R}_\theta)(y^n) \right)^\mathsf{T}$$
$$= \rho(\mathrm{R}_\theta)(v_n(\mathbf{p})) = v_n(\mathrm{R}_\theta \mathbf{p}),$$

and so there exists a regular matrix $V_n(\mathrm{R}_\theta)$ of order $n + 1$ such that

$$v_n(\mathrm{R}_\theta \mathbf{p}) = \mathrm{V}_n(\mathrm{R}_\theta) v_n(\mathbf{p}). \tag{14}$$

For instance, for $n = 2$ we have

$$\mathrm{V}_2(\mathrm{R}_\theta) = \begin{pmatrix} \cos^2 \theta & -\sin 2\theta & \sin^2 \theta \\ \frac{1}{2}\sin 2\theta & \cos 2\theta & -\frac{1}{2}\sin 2\theta \\ \sin \theta^2 & \sin 2\theta & \cos^2 \theta \end{pmatrix}.$$

A vector subspace $W \subset V$ is called $G$-invariant if $\rho(g)(W) \subset W$ for every $g \in G$. A representation $\rho : G \to \mathrm{Aut}(V)$ is said to be irreducible if there exist no $G$-invariant subspace but the trivial ones, i.e., the null-subspace and $V$ itself.

An important property of compact groups as $SO(2)$ is that any representation is completely reducible, i.e., the associated vector space can be decomposed as $V = V_1 \oplus \cdots \oplus V_N$, the restriction of the representation $\rho$ to any $V_i$ being an irreducible representation [13].

4.2 Polynomial displacements and geometric transformations

The set of homogeneous displacement functions of degree $n$ $\mathbf{P} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{P}(x, y) = (X(x, y), Y(x, y))$ is a vector space $\mathcal{V}^n$ in which the plane rotation group $SO(2)$ acts according to equation (3). Specifically, a rotation transforms the mapping $\mathbf{P}$ into the mapping $\mathbf{P}'$ given by

$$\mathbf{P}'(\mathbf{x}) = \mathrm{R}_\theta^\mathsf{T} \mathbf{P} \left( \mathrm{R}_\theta \mathbf{x} \right),$$

where $\mathbf{x} = (x, y)^\mathsf{T}$ and $\mathrm{R}_\theta$ is defined in (1).

Let us consider in more detail the homogeneous case. The displacement function is then given by the equation

$$\Delta \mathbf{p} = \mathrm{M} v_n(\mathbf{p}), \ \Delta \mathbf{p} \equiv \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}, \ \mathbf{p} \equiv \begin{pmatrix} x \\ y \end{pmatrix}, \tag{15}$$

where $\mathrm{M}$ is a $2 \times (n + 1)$ matrix. In order to see how matrix $\mathrm{M}$ in (15) changes with coordinate rotation we substitute in this equation

$$\mathbf{p} = \mathrm{R}\bar{\mathbf{p}}, \ \Delta \mathbf{p} = \mathrm{R}\Delta\bar{\mathbf{p}},$$

obtaining

$$\Delta\bar{\mathbf{p}} = \mathtt{R}^\top \mathtt{M} v_n (\mathtt{R}\bar{\mathbf{p}})$$
$$= \mathtt{R}^\top \mathtt{M} V_n (\mathtt{R}) \, v_n (\bar{\mathbf{p}})$$
$$= \bar{\mathtt{M}} v_n (\bar{\mathbf{p}}),$$

where

$$\bar{\mathtt{M}} = \mathtt{R}^\top \mathtt{M} V_n(\mathtt{R}). \tag{16}$$

Thus a homogeneous distortion function transforms itself under the action of a coordinate rotation into another one given by the previous formula. And, in particular, we have that polynomial models, homogeneous or not, are isotropic.

The complex function formulation (13) allows for an easier treatment of coordinate rotation. Using complex numbers, a coordinate rotation of angle $\theta$ can be written as

$$z = e^{i\theta} w, \ \Delta z = e^{i\theta} \Delta w.$$

Let us see how these changes of variables induce a transformation in the complex polynomial. We have

$$e^{i\theta} \Delta w = \sum_{(k,l) \in I} \gamma_{kl} e^{i\theta(k-l)} w^k \bar{w}^l,$$

so that the new polynomial is

$$\Delta w = \sum_{(k,l) \in I} \gamma_{kl} e^{i\theta(k-l-1)} w^k \bar{w}^l. \tag{17}$$

In the case of monomials, the corresponding transformation is

$$z^k \bar{z}^l \mapsto e^{i\theta(k-l-1)} w^k \bar{w}^l. \tag{18}$$

We will call the number $m = k - l - 1$ the *winding number* of the monomial. Table 1 shows a classification of the monomials of degrees from two to five according to their associated winding number.

*Example 4* For degree two a coordinate rotation transforms the coefficients according to

$$(\gamma_{20}, \gamma_{11}, \gamma_{02}) \mapsto (e^{i\theta}\gamma_{20}, e^{-i\theta}\gamma_{11}, e^{-3i\theta}\gamma_{02}). \tag{19}$$
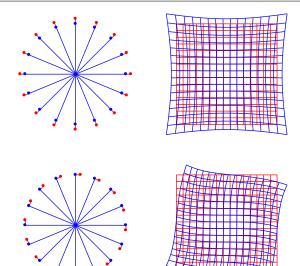


**Figure 1** Action on a circle and on a grid of the rotationally invariant cubic distortions corresponding to matrices (21). Top: radial invariant distortion, bottom: tangential invariant distortion.

### 4.3 Rotation-invariant distortion functions

We will call *invariant monomials* those of zero winding number, i.e., those that are invariant with respect to coordinate rotations (18). They are of the form

$$z^{k+1}\bar{z}^k, \ k > 0, \tag{20}$$

and therefore there are no invariant monomials of even degree. The displacement functions that do not change under coordinate rotations are those given by complex linear combinations of invariant monomials.

We can write the term corresponding to an invariant monomial $\gamma z^k \bar{z}^{k+1}$ as the sum of a radial and a tangential term as

$$\gamma z^k \bar{z}^{k+1} = z \left( a z^k \bar{z}^k \right) + (iz) \left( b z^k \bar{z}^k \right),$$

with $\gamma$ being $a + ib$.

In the case of degree three, the radial and tangential terms correspond respectively to the matrices

$$\begin{pmatrix} 1 \ 0 \ 1 \ 0 \\ 0 \ 1 \ 0 \ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \ 1 \ 0 \ 1 \\ -1 \ 0 \ -1 \ 0 \end{pmatrix}. \tag{21}$$

The first one corresponds to the cubic (one-parameter) invariant radial distortion of equation (8) and the other one to invariant tangential distortion. Figure 1 shows the action of the corresponding distortion functions on points of a circle and on a grid.

| $m$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $\bar z^2$ | | $z\bar z$ | | $z^2$ | | | |
| | | | $\bar z^3$ | | $z\bar z^2$ | | $z^2\bar z$ | | $z^3$ | | |
| | | $\bar z^4$ | | $z\bar z^3$ | | $z^2\bar z^2$ | | $z^3\bar z$ | | $z^4$ | |
| | $\bar z^5$ | | $z\bar z^4$ | | $z^2\bar z^3$ | | $z^3\bar z^2$ | | $z^4\bar z$ | | $z^5$ |

**Table 1** Classification of monomials up to degree five by their winding number.

### 4.4 Linear isotropic models

In this subsection we obtain all the linear isotropic polynomial models of functions of a given maximum degree. In the language of group representations, these are the invariant subspaces of the representation of the planar rotation group on the vector space of displacement functions. As we mentioned in section 4.1, these invariant subspaces are direct sum of irreducible invariant subspaces. Therefore the problem is that of finding these irreducible subspaces.

Some notation will be useful in the sequel. We will denote by $\mathcal{P}^{(n)}$ the complex vector space of polynomials $f(z,\bar z)$ spanned by the monomials $z^k\bar z^l$ of degree $k + l \in \{2,\ldots,n\}$, by $\mathcal{P}_m^{(n)}$ the subspace of $\mathcal{P}^{(n)}$ generated by the monomials with winding number $m$ and $\mathcal{W}^{(n)}$ the subspace generated by all the monomials with winding number $m \neq 0$, i.e., the non-invariant monomials. Therefore we have

$$\mathcal{P}^{(n)} = \mathcal{P}_0^{(n)} \oplus \mathcal{W}^{(n)},$$
$$\mathcal{W}^{(n)} = \bigoplus_{m\neq 0} \mathcal{P}_m^{(n)}.$$

Let us denote by $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{C}^*$ the complex projective line. Its points are equivalence classes

$$[(\mu,\nu)] = \{(\gamma\mu,\gamma\nu):\gamma \in \mathbb{C}^*\}.$$

We will denote $[(\mu,\nu)] = (\mu:\nu)$. Analogously, the real projective line $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R}^2 \setminus \{(0,0)\}/\mathbb{R}^* = \mathbb{C}^*/\mathbb{R}^*$ and its points will be denoted as $[\mu]$ for $\mu \in \mathbb{C}^*$.

Since $\mathcal{P}^{(n)} = \mathcal{P}_0^{(n)} \oplus \mathcal{W}^{(n)}$ and the elements of $\mathcal{P}_0^{(n)}$ are kept fixed by the representation, we just have to obtain the irreducible subspaces of $\mathcal{W}^{(n)}$. Albeit the set $\mathcal{P}^{(n)}$ has a natural structure of complex vector space, we are interested in $\mathcal{P}^{(n)}$ as a real vector space, since we are identifying it with pairs $(P(x,y),Q(x,y))$ of polynomials in two real variables. We will denote by $\mathcal{P}^{(n)}_{\mathbb{R}}$ this real vector space.

**Theorem 1** *The irreducible real subspaces of the representation $\rho : SO(2) \to \mathrm{Aut}(\mathcal{P}^{(n)})$ are the one-dimensional real subspaces of $\mathcal{P}_0^{(n)}$ together with the bidimensional subspaces of the form*

$$\mathcal{M}_m^{(n)}[f,g] = \{\gamma f(z,\bar z) + \bar\gamma g(z,\bar z):\gamma \in \mathbb{C}\}, \qquad (22)$$

*where $f \in \mathcal{P}_m^{(n)}$, $g \in \mathcal{P}_{-m}^{(n)}$.*

*Proof* Consider the basis of $\mathcal{P}_{\mathbb{R}}^{(n)}$

$$\mathcal{B} = \left\{z^k\bar z^l, iz^k\bar z^l\right\}_{k,l\geq 0,\, 2\leq k+l\leq n},$$

where we suppose that the monomials are ordered by their winding number $m = k - l - 1$. Since

$$\rho(e^{i\theta})(z^k\bar z^l) = e^{im\theta}z^k\bar z^l,$$

the matrix $\mathtt{M}$ of the automorphism $\rho(e^{i\theta})$ with respect to $\mathcal{B}$ is built with diagonal blocks

$$\mathtt{M}_m = \begin{pmatrix} \cos m\theta & -\sin m\theta \\ \sin m\theta & \cos m\theta \end{pmatrix}.$$

An irreducible invariant real subspace $W$ of $\mathtt{M}$ must be associated to a pair of complex conjugate eigenvalues, which necessarily are of the form $e^{im\theta}, e^{-im\theta}$. Therefore $W$ must be an irreducible invariant subspace of

$$\mathcal{P}_m^{(n)} \oplus \mathcal{P}_{-m}^{(n)}.$$

Such subspaces are obtained in lemma 1 and are of the form $\{\gamma f(z,\bar z) + \bar\gamma g(z,\bar z):\gamma \in \mathbb{C}\}$, $f \in \mathcal{P}_m^{(n)}$, $g \in \mathcal{P}_{-m}^{(n)}$, as stated. $\square$

*Remark 1* Observe that $\mathcal{M}_m^{(n)}[f,g]$ and $\mathcal{M}_m^{(n)}\left[\tilde f,\tilde g\right]$ are the same space if and only if $\tilde f = \alpha f$, $\tilde g = \bar\alpha g$ for some $\alpha \in \mathbb{C}^*$. Otherwise the spaces have trivial intersection.

*Example 5* In degree $n = 2$ we have only three monomials, each of them with a different winding number: $z^2$ ($m = 1$), $z\bar z$ ($m = -1$) and $\bar z^2$ ($m = -3$). Therefore there are no invariant monomials. Thus a generic polynomial of $\mathcal{P}_1^{(2)}$ is of the form $f = \mu z^2$, $\mu \in \mathbb{C}$, and a generic polynomial of $\mathcal{P}_{-1}^{(2)}$ is of the form $g = \bar\nu z\bar z$. Therefore, we can parameterize the set of irreducible invariant subspaces $\mathcal{M}_m^{(n)}[f,g]$ by the pair of coefficients $(\mu,\nu)$, and since, by remark 1, $(\mu,\nu)$ and $(\alpha\mu,\alpha\nu)$ produce the same space, we have that the irreducible subspaces of $\mathcal{P}_1^{(2)} \oplus \mathcal{P}_{-1}^{(2)}$ can be adequately parameterized by the projective points $(\mu:\nu) \in \mathbb{P}^1_{\mathbb{C}}$. These subspaces are thus given by

$$\mathcal{M}_1^{(2)}(\mu:\nu) = \left\{\gamma\mu z^2 + \bar\gamma\bar\nu z\bar z:\gamma \in \mathbb{C}\right\}, \quad (\mu:\nu) \in \mathbb{P}^1_{\mathbb{C}}. \tag{23}$$

Observe that

$$\mathcal{M}_1^{(2)}(1:1) = \left\{z\left(\gamma z + \bar\gamma\bar z\right):\gamma \in \mathbb{C}\right\},$$

with $\gamma\bar{z} + \bar{\gamma}z$ being real-valued, is the space of radial displacements and

$$\mathcal{M}_1^{(2)}(1:-1) = \{z\,(\gamma z - \bar{\gamma}\bar{z}) : \gamma \in \mathbb{C}\},$$

is the space of tangential displacements, as $\gamma z - \bar{\gamma}\bar{z}$ takes only pure imaginary values. Since different irreducible subspaces intersect trivially, we have that the direct sum of any two different subspaces of the form (23) is the whole four-dimensional space

$$\mathcal{P}_1^{(2)} \oplus \mathcal{P}_{-1}^{(2)} = \{\gamma_1 z^2 + \gamma_2 z\bar{z} : \gamma_1, \gamma_2 \in \mathbb{C}\} \qquad (24)$$
$$= \mathcal{M}_1^{(2)}(1:1) \oplus \mathcal{M}_1^{(2)}(1:-1).$$

In section 6 we will see another interesting decomposition of this space (see equation (37)).

In the case of winding number $m = -3$ the subspace generated by the only associated monomial,

$$\mathcal{P}_{-3}^{(2)} = \{\gamma\bar{z}^2 : \gamma \in \mathbb{C}\}$$

already coincides with the irreducible invariant subspace $\mathcal{M}_3^{(2)}[\bar{z}^2, 0]$.

## 5 Reflection-symmetric distortion functions

As we have mentioned before, distortion functions that have reflection symmetry with respect to some axis are important in order to model some optical phenomenons. In this section we obtain all the polynomial models that enjoy at the same time the three properties of being linear, isotropic, and being formed by functions with reflection symmetry. We will see that this triple requirement happens to limit severely the dimensionality of the possible models, thus pointing towards the need of relaxing some of the constraints in order to gain flexibility.

### 5.1 Equations and parameterizations of the variety

The following theorem describes the polynomial displacement functions with reflection symmetry.

**Proposition 2** *A polynomial displacement function*

$$f(z, \bar{z}) = \sum_{(k,l)\in I} \gamma_{kl} z^k \bar{z}^l$$

*is reflection-symmetric with respect to the axis $\langle e^{i\theta} \rangle = \{ae^{i\theta} : a \in \mathbb{R}\}$ if and only if it satisfies*

$$e^{2i\theta}\overline{f(z, \bar{z})} = f(e^{2i\theta}\bar{z}, e^{-2i\theta}z),$$

*which is equivalent to have coefficients of the form*

$$\gamma_{kl} = a_{kl}e^{im\theta},$$
$$a_{kl}, \theta \in \mathbb{R}, \, m = k - l - 1, \qquad (25)$$

*and therefore the coefficients satisfy the relation*

$$\mathrm{Im}\left[\gamma_{kl}^{m'}\bar{\gamma}_{k'l'}^m\right] = 0. \qquad (26)$$

*Proof* A reflection with respect to the axis $\langle e^{i\theta} \rangle = \{ae^{i\theta} : a \in \mathbb{R}\}$ is expressed in terms of complex numbers by the mapping

$$z \mapsto e^{2i\theta}\bar{z}.$$

Therefore a displacement

$$\Delta z = f(z, \bar{z})$$

is reflection-symmetric with respect to this axis if

$$e^{2i\theta}\overline{\Delta z} = f(e^{2i\theta}\bar{z}, e^{-2i\theta}z),$$

i.e., if

$$e^{2i\theta}\overline{f(z, \bar{z})} = f(e^{2i\theta}\bar{z}, e^{-2i\theta}z).$$

A straightforward computation shows that this is equivalent to have coefficients satisfying

$$\gamma_{kl} = e^{-2i\theta m}\overline{\gamma}_{kl}, \, m = k - l - 1. \qquad (27)$$

Writing $\gamma_{kl} = \rho_{kl}e^{i\phi_{kl}}$, with $\rho_{kl} \geq 0$, the equation above implies

$$e^{2i\phi_{kl}} = e^{-2i\theta m},$$

i.e.,

$$2\phi_{kl} = -2\theta m + 2k\pi, k \in \mathbb{Z}$$
$$\Leftrightarrow \phi_{kl} = -\theta m + k\pi$$
$$\Leftrightarrow \gamma_{kl} = \rho_{kl}e^{-i\theta m}e^{ik\pi} = \pm\rho_{kl}e^{-i\theta m}.$$

From (27), for $(k,l) \neq (k',l')$, denoting $m' = k' - l' - 1$, we must have

$$\left(\frac{\gamma_{kl}}{\bar{\gamma}_{kl}}\right)^{m'} = \left(\frac{\gamma_{k'l'}}{\bar{\gamma}_{k'l'}}\right)^m. \qquad (28)$$

i.e.,

$$\gamma_{kl}^{m'}\bar{\gamma}_{k'l'}^m = \bar{\gamma}_{kl}^{m'}\gamma_{k'l'}^m$$

or equivalently

$$\mathrm{Im}\left[\gamma_{kl}^{m'}\bar{\gamma}_{k'l'}^m\right] = 0.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

*Remark 2* The equations (26) are sufficient conditions if there exists a monomial with winding number $m = 1$, as it is easy to check. However, in the general case they are not sufficient conditions as the polynomial

$$f(z, \bar{z}) = z^3 + iz\bar{z}^2$$

shows.

*Remark 3* In particular, for the invariant monomials ($m = 0$) this implies

$$\hat{\gamma}_{kl} = a_{kl} \in \mathbb{R}.$$

*Example 6* For degree two, the functions symmetric with respect to the horizontal axis are

$$f(z, \bar{z}) = a_0 z^2 + a_1 z\bar{z} + a_2 \bar{z}^2, \ a_i \in \mathbb{R},$$

and after coordinate rotation we obtain

$$\hat{f}(z, \bar{z}) = a_0 e^{i\theta} z^2 + a_1 e^{-i\theta} z\bar{z} + a_2 e^{-3i\theta} \bar{z}^2. \quad (29)$$

Let us see that the first two terms can be written as the sum of a radial term and a tangential term. Writing $a = a_0 + a_1$, $b = a_0 - a_1$, we have

$$a_0 e^{i\theta} z^2 + a_1 e^{-i\theta} z\bar{z} = az\frac{1}{2}(e^{i\theta}z + e^{-i\theta}\bar{z}) + biz\frac{1}{2i}(e^{i\theta}z - e^{-i\theta}\bar{z}),$$

so that in real polynomial form the first two terms of $\hat{f}(z, \bar{z})$ are

$$a \begin{pmatrix} x \\ y \end{pmatrix} (x\cos\theta - y\sin\theta) + b \begin{pmatrix} -y \\ x \end{pmatrix} (x\sin\theta + y\cos\theta),$$

and in real matrix form, including the three terms, we obtain

$$
a \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ 0 & \cos\theta & -\sin\theta \end{pmatrix}
$$
$$
+ b \begin{pmatrix} 0 & \sin\theta & \cos\theta \\ -\sin\theta & -\cos\theta & 0 \end{pmatrix} \quad (30)
$$
$$
+ c \begin{pmatrix} \cos 3\theta & -2\sin 3\theta & -\cos 3\theta \\ -\sin 3\theta & -2\cos 3\theta & \sin 3\theta \end{pmatrix}.
$$

Figure 2 shows the action of each of these terms on points on a circle and on a grid oriented according to the symmetry axis.

If we consider functions of degree $n = 3$ an analogous process leads to the parameterization

$$
d \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}
$$
$$
+ e \begin{pmatrix} \cos 2\theta & -2\sin 2\theta & -\cos 2\theta & 0 \\ 0 & \cos 2\theta & -2\sin 2\theta & -\cos 2\theta \end{pmatrix}
$$
$$
\quad (31)
$$
$$
+ f \begin{pmatrix} 0 & \sin 2\theta & 2\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -2\cos 2\theta & \sin 2\theta & 0 \end{pmatrix}
$$
$$
+ g \begin{pmatrix} \cos 4\theta & -3\sin 4\theta & -3\cos 4\theta & \sin 4\theta \\ -\sin 4\theta & -3\cos 4\theta & 3\sin 4\theta & \cos 4\theta \end{pmatrix},
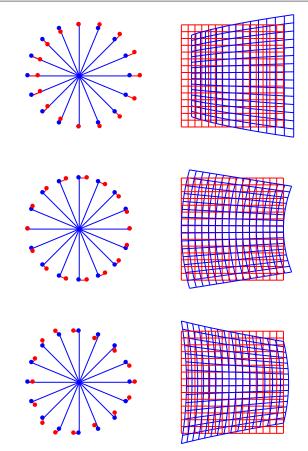$$



**Figure 2** Quadratic distortions given by each of the matrices in (30), ordered from top to bottom and symmetric with respect to the horizontal axis. Action on points a circle and on a grid.

where the first term is radial rotationally invariant, the second is radial, the third tangential, and the fourth is of none of these types. Figure 3 shows the action of each of these terms on points on a circle and on a grid oriented according to the symmetry axis.

Although for a given value of parameter $\theta$ the function sets given by (30) or by (31) are linear subspaces, when we consider the union of the sets corresponding to all the possible values of $\theta$ we do not obtain a linear subspace. For example, the polynomials

$$f_1(z, \bar{z}) = z^2, \ \ f_2(z, \bar{z}) = iz\bar{z}$$

are of the form (29) but their sum is not. The obtainment of isotropic linear models constituted by displacement functions with reflection symmetry is addressed in the following section.

## 5.2 Linear isotropic reflection-symmetric models

The previous results can be employed to obtain a practical description of linear isotropic quadratic models of
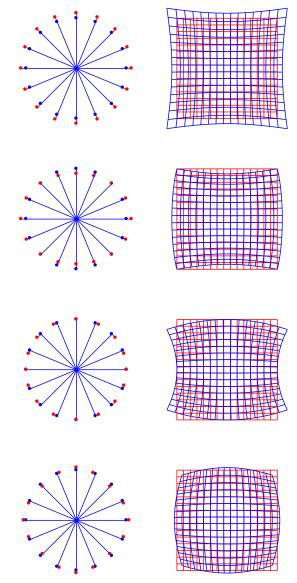
**Figure 3** Cubic distortions given by each of the matrices in (31), ordered from top to bottom and symmetric with respect to the horizontal axis. Action on points a circle and on a grid.

reflection symmetric functions, given by the following theorem, whose proof is included in the 9.2, in the appendix.

**Theorem 2** *The linear isotropic distortion models with monomials of degree at most n constituted by functions with reflection symmetry are those of the form*

$$\mathcal{M}_m^{(n)}[f,g] \oplus \mathcal{F}, \qquad (32)$$

*where the spaces $\mathcal{M}_m^{(n)}[f,g]$ are defined in theorem 1, $f, g$ are polynomials with real coefficients, and $\mathcal{F}$ is a subspace generated by invariant monomials (20) with real coefficients.*[1]

---

[1] Note that if $f = g = 0$ then $\mathcal{M}_m^{(n)}[f,g] = \{0\}$ and that $\mathcal{F}$ can also be the null vector subspace.

*Example 7* As we saw in example 5, the irreducible subspaces in $\mathcal{P}^{(2)}$ are the spaces

$$\mathcal{M}_1^{(2)}(\mu:\nu) = \left\{\gamma\mu z^2 + \bar{\gamma}\bar{\nu}z\bar{z} \colon \gamma \in \mathbb{C}\right\}, \ (\mu:\nu) \in \mathbb{P}_{\mathbb{C}}^1$$

and the space

$$\mathcal{P}_{-3}^{(2)} = \mathcal{M}_3^{(2)}\left[\bar{z}^2, 0\right] = \left\{\gamma\bar{z}^2 \colon \gamma \in \mathbb{C}\right\}$$

and there are not invariant monomials. Therefore the linear isotropic quadratic distortion models constituted by functions with reflection symmetry are the spaces $\mathcal{M}_1^{(2)}(\mu:\nu)$ with $\mu, \nu \in \mathbb{R}$ and $\mathcal{P}_3^{(2)}$. In the first case we have, noting $\mu = r$, $\nu = s$, $r, s \in \mathbb{R}$, and $\gamma = ae^{i\phi}$, $a, \phi \in \mathbb{R}$,

$$\mathcal{M}_1^{(2)}(r:s) = \left\{a\left(re^{i\phi}z^2 + se^{-i\phi}z\bar{z}\right) : a, \phi \in \mathbb{R}\right\}.$$

Noting $p = r + s$, $q = s - r$, $t_1 = a\cos\phi$, $t_2 = a\sin\phi$, it is easy to check that the real matrix form for these models is

$$p\begin{pmatrix} t_1 & -t_2 & 0 \\ 0 & t_1 & -t_2 \end{pmatrix} + q\begin{pmatrix} 0 & t_2 & t_1 \\ -t_2 & -t_1 & 0 \end{pmatrix}, \ t_1, t_2 \in \mathbb{R}, \qquad (33)$$

where the first term corresponds to radial distortion and the second to tangential distortion. Therefore the different models of this family are specified by the ratio between these two displacement terms.

The functions of the space $\mathcal{P}_{-3}^{(2)}$ are those of the form

$$f(z, \bar{z}) = ae^{i\phi}\bar{z}^2, \ \alpha, \phi \in \mathbb{R},$$

and with the identification $t_1 = a\cos\phi$, $t_2 = a\sin\phi$, have matrix form

$$\begin{pmatrix} t_1 & 2t_2 & -t_1 \\ t_2 & -2t_1 & -t_2 \end{pmatrix}, \ t_1, t_2 \in \mathbb{R}. \qquad (34)$$

Therefore the set of linear isotropic quadratic distortion models with functions with reflection symmetry consists in a one-parameter family (parametrized by the ratio $(p:q)$) and an additional model. All these models are two-dimensional and the ratio of their parameters, $t_2/t_1$ determines the symmetry axis according to the relation $t_2/t_1 = \tan\phi$ for the models of the one-parameter family and $t_2/t_1 = \tan 3\phi$ for the additional model.

Figure 4 provides a topology-preserving representation of the parameter space of the irreducible isotropic linear models of degree two. Each point of the sphere corresponds to a bidimensional isotropic linear model $\mathcal{M}_1^{(2)}(\mu:\nu)$ (see equation (23)) within the four-dimensional radial-tangential space. The parameter space $\mathbb{P}_{\mathbb{C}}^1$ is represented as a sphere through the stereographic projection $\mathbb{P}_{\mathbb{C}}^1 \ni (\mu:\nu) \mapsto (2\mu\bar{\nu}, |\mu|^2 - |\nu|^2) \in \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3$. The blue circle on the sphere corresponds to those of these models that are constituted by functions with reflection symmetry with respect to some axis (i.e., those
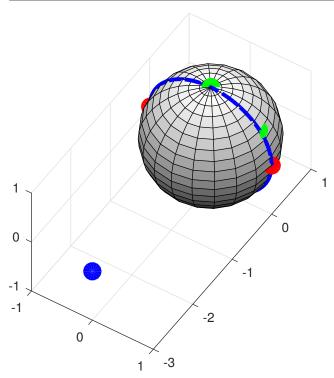
**Figure 4** Topology-preserving representation of the parameter space of the irreducible isotropic linear models of degree two (see example 7).

given by (33)), the red dots on this circle correspond to the radial and tangential models and the green dots correspond to the thin prism and lens decentering models as we will see in the next section. The isolated point corresponds to the space $\mathcal{P}_{-3}^{(2)}$ (34), also constituted by functions with reflection symmetry.

## 6 Application: analysis of some well-known polynomial models

In this section we discuss how the most commonly used lens distortion models fit in the framework presented above.

*Decentering distortion* [5] is an analytical model of the effect of imperfect alignment of the revolution axes of the lens surfaces. The displacement functions of the model are given by the quadratic functions

$$
\begin{aligned}
\Delta x &= s_1\left(3x^2 + y^2\right) + 2s_2 xy \\
\Delta y &= 2s_1 xy + s_2\left(x^2 + 3y^2\right).
\end{aligned}
\tag{35}
$$

In our matrix notation, the model is given by the matrices

$$
\begin{pmatrix} 3s_1 & 2s_2 & s_1 \\ s_2 & 2s_1 & 3s_2 \end{pmatrix}, \ s_1, s_2 \in \mathbb{R}.
$$

This model is obviously linear and, as is known from physical considerations, it is isotropic and formed by

functions with reflection symmetry. Therefore it must be an instance of the models (33) or (34). It is easy to check that we are in the first case, with coefficients

$$
(p : q) = (3 : 1)
$$

and taking $t_1 = s_1$ and $t_2 = -s_2$ in (33).

*Thin prism distortion* [3] models the effect of imperfection in the lens manufacturing process and is given by the expression

$$
\begin{aligned}
\Delta x &= u_1\left(x^2 + y^2\right) \\
\Delta y &= u_2\left(x^2 + y^2\right),
\end{aligned}
\tag{36}
$$

so that its matrix is

$$
\begin{pmatrix} u_1 & 0 & u_1 \\ u_2 & 0 & u_2 \end{pmatrix}, \ u_1, u_2 \in \mathbb{R}.
$$

Observe that the displacement is always proportional to $(u_1, u_2)$. We see again that this is a particular case of (33), now corresponding to the coefficients

$$
(p : q) = (1 : 1)
$$

and taking $t_1 = s_1$ and $t_2 = -s_2$. Therefore these two models correspond to two points in the one-parameter family of models defined by equation (33) as a consequence of theorem 2, represented as the green dots in figure 4.

Let us see how these models are combined in practice. The model employed in the Matlab Computer Vision Toolbox [10] is the direct sum of three-coefficient RRI distortion (8) and quadratic decentering distortion (35) (named in the documentation "tangential distortion"), i.e., the model is a particular case of (32), given by

$$
\mathcal{M}_1^{(2)}(1 : 1) \oplus \mathcal{G},
$$

where

$$
\mathcal{G} = \left\{ z\left(a_1 z\bar{z} + a_2 z^2\bar{z}^2 + a_3 z^4\bar{z}^4\right) : a_1, a_2, a_3 \in \mathbb{R} \right\}.
$$

Therefore, the model is composed of reflection symmetric functions.

In [14] a four parameter model consisting in the sum of models given by (35) and (36) is introduced. Such a model coincides with the sum of the polynomial radial and polynomial tangential models $\mathcal{P}_1^{(2)} \oplus \mathcal{P}_{-1}^{(2)}$ (see equation (24)) which is then written as

$$
\mathcal{P}_1^{(2)} \oplus \mathcal{P}_{-1}^{(2)} = \mathcal{M}_1^{(2)}(3 : 1) \oplus \mathcal{M}_1^{(2)}(1 : 1).
\tag{37}
$$

Finally we consider a more complex model employed in OpenCV 3.3 [2]. The OpenCV model substitute the polynomial RRI distortion found in the [14] model just considered by a rational RRI distortion and the quadratic

thin prism distortion is substituted by a quartic expression

$$\Delta x = s_1 r^2 + s_2 r^4$$
$$\Delta y = s_3 r^2 + s_4 r^4. \tag{38}$$

In order to analyze this part of the model, we observe first that it corresponds to the complex polynomials

$$f(z, \bar{z}) = \gamma_{11} z\bar{z} + \gamma_{22} z^2 \bar{z}^2$$
$$\gamma_{11} = s_1 + i s_3 = \rho_1 e^{i\theta_1}$$
$$\gamma_{22} = s_2 + i s_4 = \rho_2 e^{i\theta_2}.$$

Since this model has real dimension 4 and does not include invariant monomials, it does not have the reflection symmetric property, according to theorem 2. To see this directly, just observe that both monomials share the winding number $m = -1$ (see table 1), but according to equations 28, the function will be reflection symmetric if and only if

$$e^{2i\theta_1} = e^{2i\theta_2},$$

i.e., if $\theta_1 = \pm\theta_2$, that requires $s_3/s_1 = \pm s_4/s_2$. Therefore the model given by (38) does not preserve the property of being formed of reflection symmetric functions as one might expect for thin prism distortion.

## 7 Application: extending known models

In this section we apply our results by proposing some extensions of the usual lens distortion models and doing some preliminary testing of them.

In order to compare different models with real images we obtain images of a board in different positions with a GoPro camera. We first obtain a 3D reconstruction and initial values of the distortion parameters. For this we use the Matlab camera calibration toolbox and its model consisting of rotationally-symmetric radial distortion of two coefficients and quadratic decentering distortion. The distortion center is assumed to coincide with the image principal point. Then we perform a re-optimization of the 3D reconstruction using a different lens distortion model and compute the residual error. Figure 5 shows some original images and their corrected versions with the best algorithm. Table 2 shows the reprojection errors obtained with different models.

The first set of tests is performed with RRI distortion (8) with different number of coefficients. The improvement stops at three coefficients. The corresponding model, which is the one generated by the invariant monomials of degrees 3, 5, and 7, is kept as an integrating part of the models considered in the remaining experiments.
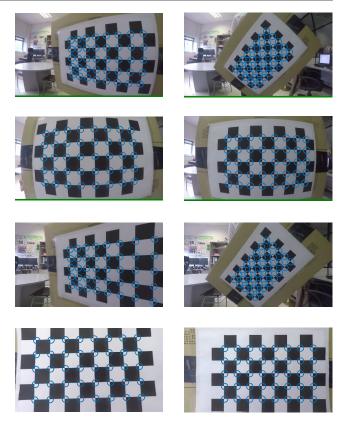


**Figure 5** Original (four top images) and corrected (four bottom) images with the model that minimizes reprojection error.

In the second set of experiments we consider different models of the form

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \Delta_1 x \\ \Delta_1 y \end{pmatrix} + \begin{pmatrix} \Delta_2 x \\ \Delta_2 y \end{pmatrix}, \tag{39}$$

where the first term, introduced in equation (33), generalizes decentering and thin prism distortion and is given by

$$\begin{pmatrix} \Delta_1 x \\ \Delta_1 y \end{pmatrix} = \left[ p \begin{pmatrix} t_1 & -t_2 & 0 \\ 0 & t_1 & -t_2 \end{pmatrix} + q \begin{pmatrix} 0 & t_2 & t_1 \\ -t_2 & -t_1 & 0 \end{pmatrix} \right] \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix},$$

while the second term is the three-parameter RRI distortion (8)

$$\begin{pmatrix} \Delta_2 x \\ \Delta_2 y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \left( \alpha_1 r^2 + \alpha_2 r^4 + \alpha_3 r^6 \right).$$

Figure 6 shows the residual error as a function of the parameter $\phi$, where $(p : q) = (\cos\phi : \sin\phi)$. We observe that the best results are achieved by models for which radial distortion is the dominant term, i.e., for $\phi$ close to 0 or $\pi$.

Then we consider models in which either linearity or reflection-symmetry of the model functions is lost.
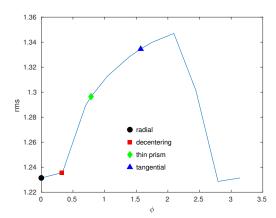
**Figure 6** Residual errors (rms) for the model (39) with different parameters $(p : q) = (\cos \phi : \sin \phi)$.

First we consider linear models not ensuring reflection-symmetry:

- Direct sum of decentering and thin prism distortion plus three coefficient RRI.
- Full quadratic and cubic distortions with two additional coefficients of RRI, so that the RRI term also has three coefficients in total.

Finally a nonlinear model is tested consisting in monomials of degrees two and three ensuring reflection-symmetry (equations (30) and (31)), plus two additional RRI terms in order to include three-coefficient RRI.

In table 2 we see that the model resulting in the minimum reprojection error is the one with largest number of parameters, but it is closely followed by the proposed non-linear model, that has nearly half of the parameters and enjoys the property of being formed by reflection-symmetric functions. Therefore it seems that for the calibration of the considered lens system the use of models ensuring the adequate geometric properties is effective in terms of obtaining good performance with a reduced number of parameters.

## 8 Conclusions and future work

In this work we have studied polynomial lens distortion models from a geometrical point of view. After identifying the key geometrical properties of lens distortion models, we have:

- provided a complete description of the models enjoying this properties,
- placed the most commonly employed polynomial models in the resulting picture,
- proposed some extensions to these models enjoying the desired properties and tested them for the calibration of a camera.

In our study we have employed the framework provided by the theory of group representations and, to the authors knowledge, a novel representation of polynomial models in terms of complex functions that greatly facilitates this geometrical analysis.

Our first result has been the identification of isotropic linear models. Then we have obtained a parameterization of the polynomial lens distortion functions that are symmetric with respect to some axis and also the linear isotropic models formed by functions with this property. As an application of this result we have described all the linear quadratic lens distortion models that are composed of reflection-symmetric functions and found that they constitute a one-parameter family plus one particular additional model. We have then observed that the decentering distortion model and the thin prism model are two instances of this one parameter family.

Our analysis facilitates the design of polynomial models, linear or not, enjoying the desired geometrical properties. As a practical application of the results, some extensions of known lens distortion models have been proposed and tested for the calibration of a camera.

A natural development of this work would be its extension to the case of rational models.

## 9 Appendix: Proofs of theorems

### 9.1 Lemma to prove theorem *1*

**Lemma 1** *Let $V$ be a complex vector space with a basis $\{u_1, \ldots, u_p, v_1, \ldots, v_q\}$ and a complex endomorphism $f : V \to V$ given by*

$$f(u_i) = \lambda u_i, \, i = 1, \ldots, p$$
$$f(v_j) = \bar{\lambda} v_j, \, j = 1, \ldots, q$$
$$\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C} \setminus \mathbb{R}.$$

*Then the irreducible invariant subspaces of $f$ with respect to the realification $V_\mathbb{R}$ of $V$ (i.e., the consideration of $V$ as a real vector space by restricting the scalars to the real numbers) are of the form*

$$\mathcal{S}_{(\alpha:\beta)} = \left\{ \gamma \sum_{i=1}^{p} \alpha_i u_i + \bar{\gamma} \sum_{j=1}^{q} \bar{\beta}_j v_j : \gamma \in \mathbb{C} \right\},$$

*where $(\alpha : \beta)$ is an abbreviation for*

$$(\alpha_1 : \ldots : \alpha_p : \beta_1 : \ldots : \beta_q) \in \mathbb{P}^{p+q-1}.$$

*Besides, if $(\alpha : \beta) \neq (\alpha' : \beta')$ then*

$$\mathcal{S}_{(\alpha:\beta)} \cap \mathcal{S}_{(\alpha':\beta')} = \{0\}.$$

| Method | NP | Linear | RRI | RSF | Rep. error |
|---|---|---|---|---|---|
| 1 coef. RRI | 1 | Y | Y | Y | 2.71 |
| 2 coefs. RRI | 2 | Y | Y | Y | 1.48 |
| 3 coefs. RRI | 3 | Y | Y | Y | 1.35 |
| 4 coefs. RRI | 4 | Y | Y | Y | 1.36 |
| 5 coefs. RRI | 5 | Y | Y | Y | 1.36 |
| Decentering + 3 coefs. RRI | 5 | Y | N | Y | 1.24 |
| Thin prism + 3 coefs. RRI | 5 | Y | N | Y | 1.30 |
| Radial quadratic + 3 coefs. RRI | 5 | Y | N | Y | 1.23 |
| Decentering + thin prism + 3 coefs. RRI | 7 | Y | N | N | 1.20 |
| Nonlinear quadratic and cubic + 2 extra coefs. RRI | 9 | N | N | Y | 0.95 |
| Full quadratic and cubic + 2 extra coefs. RRI | 16 | Y | N | N | 0.85 |

**Table 2** Reprojection errors (rms) obtained after bundle adjustment with different lens distortion models. For each model we also indicate its number of parameters, whether it is linear, radially rotationally invariant (RRI) and formed by reflection-symmetric functions (RSF).

*Proof* A basis for $V_{\mathbb{R}}$ is given by

$$\{u_1, iu_1, \ldots, u_p, iu_p, v_1, iv_1, \ldots, v_q, iv_q\},$$

so we can identify $V \approx \mathbb{C}^{p+q}$ and $V_{\mathbb{R}} \approx \mathbb{R}^{2(p+q)}$. With this identification, the matrix $\mathtt{M}$ of $f$ as an endomorphism of $\mathbb{R}^{2(p+q)}$ is block-diagonal with $p$ blocks

$$\mathtt{B} = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$$

and $q$ blocks $\mathtt{B}^{\top}$. From the diagonalization

$$\mathtt{B} = \mathtt{U} \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} \bar{\mathtt{U}}^{\top}, \text{ where}$$

$$\mathtt{U} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

we easily obtain a diagonalization of $\mathtt{M}$ and from it we see that the eigenvectors of this matrix associated to the eigenvalue $\lambda$ are of the form

$$\mathbf{w} = (\alpha_1, -i\alpha_1, \ldots, \alpha_p, -i\alpha_p, \beta_1, i\beta_1, \ldots, \beta_q, i\beta_q)^{\top}$$
$$\alpha_i, \beta_j \in \mathbb{C}, \tag{40}$$

and those associated to the eigenvalue $\bar{\lambda}$ are their conjugates. Given a non null vector $\mathbf{w} = \mathbf{w}_1 + i\mathbf{w}_2$ of this form, $\mathbf{w}$ and $\bar{\mathbf{w}}$ span an invariant subspace of $\mathtt{M}$ whose realification admits the basis $\{\mathbf{w}_1, \mathbf{w}_2\}$. Denoting $\alpha_i = a_i + ib_i$, $\beta_j = c_j + id_j$, we have

$$\mathbf{w}_1 = (a_1, b_1, \ldots, a_p, b_p, c_1, -d_1, \ldots, c_q, -d_q)^{\top}$$
$$\mathbf{w}_2 = (b_1, -a_1, \ldots, b_p, -a_p, d_1, c_1, \ldots, d_q, c_q)^{\top}.$$

The elements of this subspace have coordinates of the form

$$r_1 \mathbf{w}_1 + r_2 \mathbf{w}_2, \ r_1, r_2 \in \mathbb{R},$$

that correspond to the elements of $V$

$$r_1 \left( \sum_{i=1}^{p} \underbrace{(a_i + ib_i)}_{\alpha_i} u_i + \sum_{j=1}^{q} \underbrace{(c_j - id_j)}_{\bar{\beta}_j} v_j \right)$$

$$+ r_2 \left( \sum_{i=1}^{p} \underbrace{(b_i - ia_i)}_{-i\alpha_i} u_i + \sum_{j=1}^{q} \underbrace{(d_j + ic_j)}_{i\bar{\beta}_i} v_j \right)$$

$$= \underbrace{(r_1 - ir_2)}_{\gamma} \sum_{i=1}^{p} \alpha_i u_i + \underbrace{(r_1 + ir_2)}_{\bar{\gamma}} \sum_{j=1}^{q} \bar{\beta}_j v_j,$$

and so the subspace generated by $\mathbf{w}_1$ and $\mathbf{w}_2$ is of the form $\mathcal{S}_{(\alpha:\beta)}$, as required. Finally, let us see that all the irreducible subspaces are of this form. Since $\mathtt{M}$ is real and without real eigenvectors, its irreducible invariant subspaces are bidimensional. Therefore, let us consider an invariant bidimensional real subspace $W \subset \mathbb{R}^{2(p+q)} \subset \mathbb{C}^{2(p+q)}$. Let $W^{\mathbb{C}} = W \oplus iW$ be the associated complex vector subspace. The eigenvalues of the restriction to $W^{\mathbb{C}}$ of the endomorphism given by $\mathtt{M}$ must be complex conjugated and so they are $\{\lambda, \bar{\lambda}\}$. The eigenvector $\mathbf{x} = \mathbf{x}_1 + i\mathbf{x}_2$ associated to the first eigenvalue must be of the form (40). The endomorphism being real, the conjugate vector $\bar{\mathbf{x}}$ must belong to the invariant subspace $W^{\mathbb{C}}$ and so the real vectors $\mathbf{x}_1, \mathbf{x}_2 \in W$ and therefore $W$ is of the form $\mathcal{S}_{(\alpha:\beta)}$ as required.

As for the last assertion, just observe that if

$$\gamma \sum_{i=1}^{m} \alpha_i \mathbf{u}_i + \bar{\gamma} \sum_{j=1}^{n} \bar{\beta}_j \mathbf{v}_j = \gamma' \sum_{i=1}^{m} \alpha_i' \mathbf{u}_i + \bar{\gamma}' \sum_{j=1}^{n} \bar{\beta}_j' \mathbf{v}_j$$

then, the vectors being a base, we have that $\gamma \alpha_i = \gamma' \alpha_i'$ and $\bar{\gamma} \bar{\beta}_j = \bar{\gamma}' \bar{\beta}_j'$ and so $(\alpha_1 : \ldots : \alpha_m : \beta_1 : \ldots : \beta_n) = (\alpha_1' : \ldots : \alpha_m' : \beta_1' : \ldots : \beta_n')$. $\qquad\square$

9.2 Proof of theorem 2

If $\mathcal{S}$ is a subspace of $\mathcal{P}^{(n)}$ generated by some set of monomials and $f \in \mathcal{P}^{(n)}$, we define the *projection* $P_{\mathcal{S}}(f)$ as the polynomial obtained by keeping in $f$ only the monomials in $\mathcal{S}$. Therefore, we have a linear mapping

$$P_{\mathcal{S}} : \mathcal{P}^{(n)} \longrightarrow \mathcal{S}.$$

Now we can proceed to the proof of theorem 2.

*Proof* We consider displacement functions expressed as complex polynomials in the variables $z$ and $\bar{z}$,

$$f(z, \bar{z}) = \sum_{(k,l) \in G^{(n)}} \gamma_{kl} z^k \bar{z}^l \in \mathcal{P}^{(n)}$$

with reflection symmetry with respect to some axis. Therefore the coefficients can be obtained through the parameterization (25).

Let us suppose that we have a real vector space $L$ of functions of this form which, at the same time, is invariant under the action of the unitary group $SO(2)$ according to (19), i.e.,

$$\gamma_{kl} \mapsto e^{i\theta m} \gamma_{kl}.$$

Given an element $f$ of $L$ there must exist an element $f_0$ of its orbit under the action of $SO(2)$ with reflection symmetry with respect to the horizontal axis, i.e., with real coefficients $\gamma_{kl} = a_{kl} \in \mathbb{R}$. Therefore, $L$ is determined by its subset $L_{\mathbb{R}}$ of its elements with real coefficients.

Denoting $m = k + l - 1$ and $m' = k' + l' - 1$, let us consider two pairs $(k, l)$ and $(k', l')$ such that

$$mm' \neq 0, \text{ and } |m| \neq |m'|.$$

Let us see that $L_{\mathbb{R}}$ cannot contain a polynomial with both coefficients $a_{kl} \neq 0$ and $a_{k'l'} \neq 0$. We denote by $\mathcal{S}$ the set of polynomials only with monomials $z^k \bar{z}^l$, $z^{k'} \bar{z}^{l'}$. Since $L$ is a linear subspace, so is its image by the linear mapping $P_{\mathcal{S}}$, that cancels all monomials but $z^k \bar{z}^l$ and $z^{k'} \bar{z}^{l'}$. If such a polynomial existed, both

$$c \left( a_{kl} z^k \bar{z}^l + a_{k'l'} z^{k'} \bar{z}^{l'} \right)$$

and

$$a_{kl} e^{i\theta m} z^k \bar{z}^l + a_{k'l'} e^{i\theta m'} z^{k'} \bar{z}^{l'}$$

would belong to this image for any $c, \theta \in \mathbb{R}$, so that its sum

$$a_{kl} \left( c + e^{i\theta m} \right) z^k \bar{z}^l + a_{k'l'} \left( c + e^{i\theta m'} \right) z^{k'} \bar{z}^{l'}$$

must also be in the image, and therefore satisfy (28), so that

$$\left( \frac{c + e^{i\theta m}}{c + e^{-i\theta m}} \right)^{2m'} = \left( \frac{c + e^{i\theta m'}}{c + e^{-i\theta m'}} \right)^{2m}$$

for any $c, \theta \in \mathbb{R}$. If this were true we would have that

$$F(z) = \left( \frac{c + z^m}{c + z^{-m}} \right)^{2m'} = \left( \frac{c + z^{m'}}{c + z^{-m'}} \right)^{2m} = G(z),$$

(41)

but

$$\left( \frac{d^3 F}{dz^3} - \frac{d^3 G}{dz^3} \right)(1) = -\frac{4\,c(c-1)}{(c+1)^3}(m'^2 - m^2)mm' \neq 0$$

unless $|m| = |m'|$ or $mm' = 0$, and therefore we have found a contradiction.

Let us see now that the image of $L_{\mathbb{R}}$ by the mapping $P_{\mathcal{W}}$, that only keeps the non-invariant monomials of each polynomial cannot be of dimension larger than one. It is easy to check that a vector space is of dimension larger than one if and only if some projection onto a coordinate plane has dimension larger than one. In our case, this means that there are two different monomials $z^k \bar{z}^{n-k}$, $z^{k'} \bar{z}^{n'-k'}$ such that $L_{\mathbb{R}}$ contains polynomials

$$\ldots + 1 z^k \bar{z}^l + 0 z^{k'} \bar{z}^{l'} + \ldots$$

and

$$\ldots + 0 z^k \bar{z}^l + 1 z^{k'} \bar{z}^{l'} + \ldots$$

with $m = k + l - 1$, $m' = k + l - 1$, $mm' \neq 0$, and using first the isotropy of $L$ and then its linearity, we see that $L$ must contain a polynomial

$$\ldots + e^{i\theta m} z^k \bar{z}^l + e^{i\theta' m'} z^{k'} \bar{z}^{l'} + \ldots$$

for any $\theta, \theta'$. And applying (28) to the coefficients of these monomials we would have for all $\theta, \theta' \in \mathbb{R}$,

$$\left( \frac{e^{i\theta m}}{e^{-i\theta m}} \right)^{2m'} = \left( \frac{e^{i\theta' m'}}{e^{-i\theta' m'}} \right)^{2m}$$

$$\Leftrightarrow e^{4i\theta mm'} = e^{4i\theta' mm'},$$

which is not true unless $mm' = 0$.

Therefore, if $P_{\mathcal{W}}(L_{\mathbb{R}})$ contains polynomials with some monomial $z^k \bar{z}^l$ with $m = k - l - 1 \neq 0$, $L_{\mathbb{R}}$ must be one-dimensional and, since it can only contain polynomials with monomials with $k - l - 1 \in \{-m, m\}$, it must be of the form

$$P_{\mathcal{W}}(L_{\mathbb{R}}) = \{ \alpha(f + g) : \alpha \in \mathbb{R} \},$$

where $f \in \mathcal{P}_m^{(n)}$, $g \in \mathcal{P}_{-m}^{(n)}$ are polynomials with real coefficients, so that the projection of $L$ onto the space of non-invariant monomials is

$$P_{\mathcal{W}}(L) = \left\{ \alpha \left( e^{im\theta} f + e^{-im\theta} g \right) e^{im\theta} : \alpha \in \mathbb{R} \right\}, \qquad (42)$$

that corresponds to $\mathcal{M}_m^{(n)}[f, g]$ in (22) with $\gamma = \alpha e^{im\theta}$.

So we have the following possibilities:

(a) If $L$ does not contain polynomials with invariant monomials, it must of the form (42),

(b) If $L$ only contains polynomials with invariant monomials, $L$ can be any linear subspace of invariant polynomials with real coefficients.

(c) Finally, if $L$ contains polynomials with invariant monomials and polynomials with non-invariant monomials, since $L$ is an invariant subspace it must contain an irreducible subspace of noninvariant monomials that must be of the form (42), and only one. Therefore $L$ must also contain its projection onto the space of invariant polynomials, and consequently $L$ is the direct sum of a space of the form (42) and a linear space of invariant polynomials with real coefficients. $\qquad \square$

## References

1. Luis Álvarez, Luis Gómez, and J. Rafael Sendra. An algebraic approach to lens distortion by line rectification. *Journal of Mathematical Imaging and Vision*, 35:36–50, 2009.
2. Gary Bradski and Adrian Kaehler. Opencv. *Dr. Dobb's journal of software tools*, 3, 2000.
3. D. C. Brown. Decentering distortion of lenses. 32(3):444–462, 1966.
4. D. Claus and A. W. Fitzgibbon. A rational function lens distortion model for general cameras. In *2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR'05)*, volume 1, pages 213–219 vol. 1, June 2005.
5. AE Conrady. Lens-systems, decentered. *Monthly notices of the royal astronomical society*, 79:384–390, 1919.
6. Frédéric Devernay and Olivier Faugeras. Straight lines have to be straight. *Machine Vision and Applications*, 13(1):14–24, Aug 2001.
7. R. Hartley and S. B. Kang. Parameter-free radial distortion correction with center of distortion estimation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 29(8):1309–1321, Aug 2007.
8. Richard Hartley and Andrew Zisserman. *Multiple View Geometry in Computer Vision*. Cambridge University Press, 2 edition, 2004.
9. Hongdong Li and Richard Hartley. A non-iterative method for correcting lens distortion from nine point correspondences. *OMNIVIS 2005*, 2:7, 2005.
10. Matlab computer vision toolbox, 2017. The MathWorks, Natick, MA, USA.
11. Rickard Strand and Eric Hayman. Correcting radial distortion by circle fitting. In *BMVC*, 2005.
12. Zhongwei Tang, Rafael Grompone von Gioi, Pascal Monasse, and Jean-Michel Morel. A precision analysis of camera distortion models. *IEEE Transactions on Image Processing*, 26(6):2694–2704, 2017.
13. Ernest B Vinberg. *Linear Representations of Groups*. Birkhäuser, 2012.
14. Juyang Weng, Paul Cohen, Marc Herniou, et al. Camera calibration with distortion models and accuracy evaluation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 14(10):965–980, 1992.
15. Fanlu Wu, Hong Wei, and Xiangjun Wang. Correction of image radial distortion based on division model. *Optical Engineering*, 56(1):013108, 2017.