School of Mathematics and Physics – UNIVERSITY OF PORTSMOUTH

TOPICS IN GEOMETRY

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Part II of module TOPICS IN ALGEBRA AND GEOMETRY
(M31448, year 2023/2024)

FURTHER READING

Coxeter, H.S.M., *Introduction to Geometry*
Coxeter, H.S.M. and Greitzer., S. L., *Geometry Revisited*
Shively, L. S., *An introduction to modern geometry*
Simon, S., ALGEBRA and TOPOLOGY Lecture Notes (Moodle)

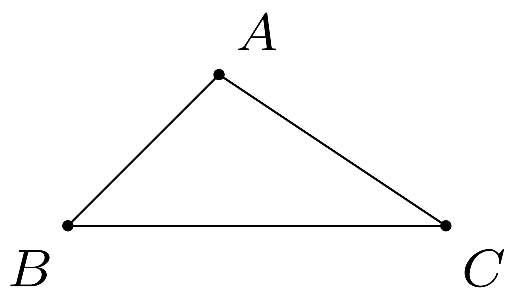
Portsmouth, November 5, 2024

I Geometry of the Euclidean plane

1. The triangle

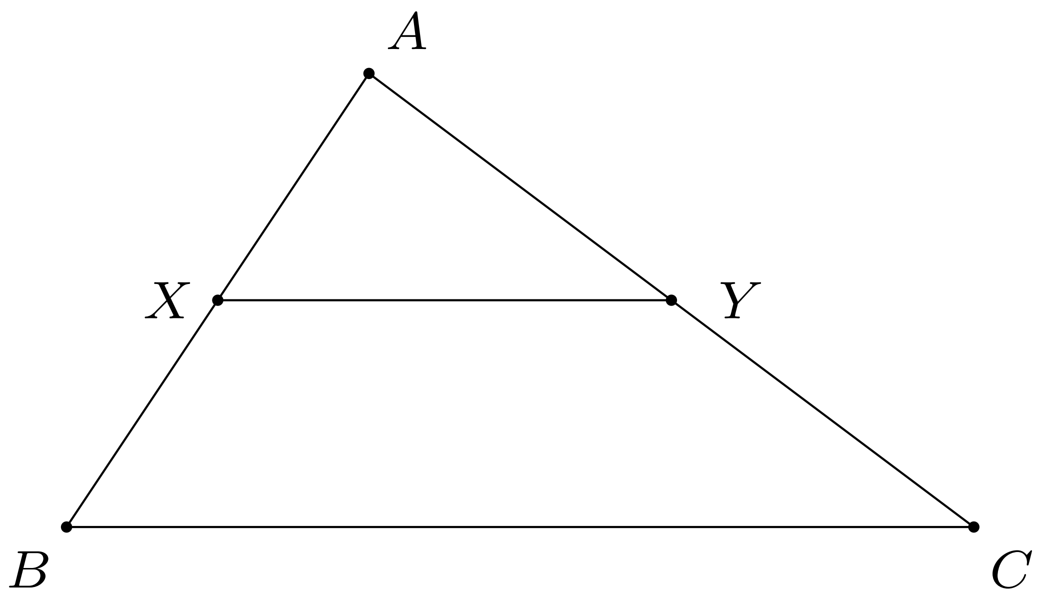
- Let us consider the Euclidean plane $\mathbb{R}^2$, i.e. $\mathbb{R} \times \mathbb{R}$ endowed with the Euclidean Topology (see §1-2 in the TOPOLOGY NOTES).
- **Orthogonal transformations** are those preserving distance in a manner that is entirely transitive: given two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$, there will always be an orthogonal transformation relating (read: mapping) one to the other.
- In a context where measuring *proximity* in a plausible way is possible (and the Euclidean plane is one such context), we can also define *angles* and this leads to the following:

Definition 1. *A* ***triangle*** *is a triple of points on the plane. We call the triangle* ***degenerate*** *if all three points are aligned, and will henceforth assume this not to be the case unless stated otherwise.*



Properties

- **Thales' Theorem** can be formulated as follows: given the following situation,



if $\overline{XY} \parallel \overline{BC}$, then the following segment ratios are equal:

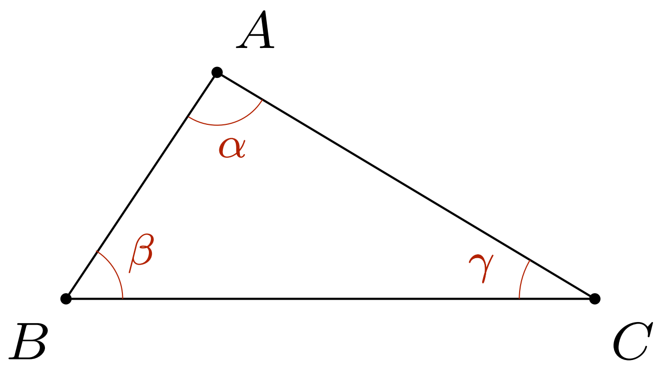
$$\frac{AX}{AB} = \frac{AY}{AC} = \frac{XY}{BC};$$

The converse of the Theorem is also true: if the above ratios are equal, then $\overline{XY} \parallel \overline{BC}$.

Proof: Exercise.

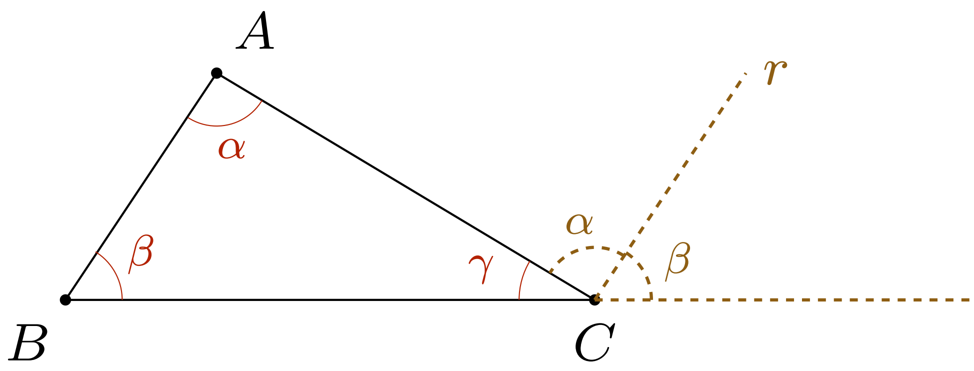
- The sum of the angles in a triangle is π.

  PROOF:



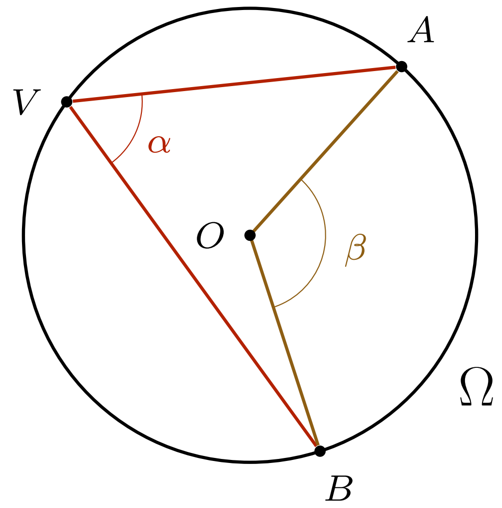
$\alpha + \beta + \gamma = \pi$?

Euclid's Parallel Axiom entails: given a point off a line, there is one line through the point parallel to the line. This postulate provides the existence of line $r \parallel \overline{AB}$ containing C thus:



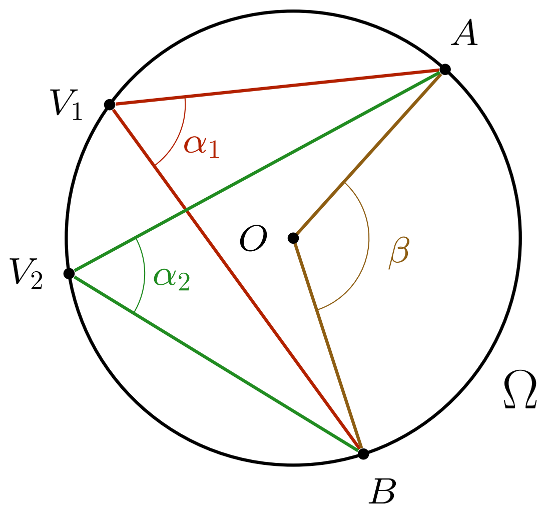
$\boxed{\alpha + \beta + \gamma = \pi}$

- Let Ω be a circumference and α an angle circumscribed in it.



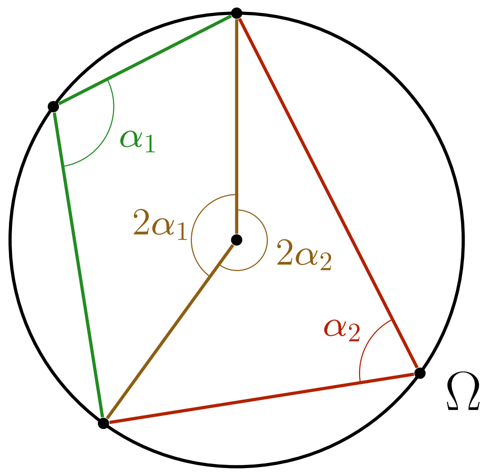
then the angle $\widehat{AOB}$ is twice the circumscribed angle $\widehat{AVB}$: $\beta = 2\alpha$. Exercise.

- Consequence: if two inscribed angles comprise equal arc segments, then both angles are equal.

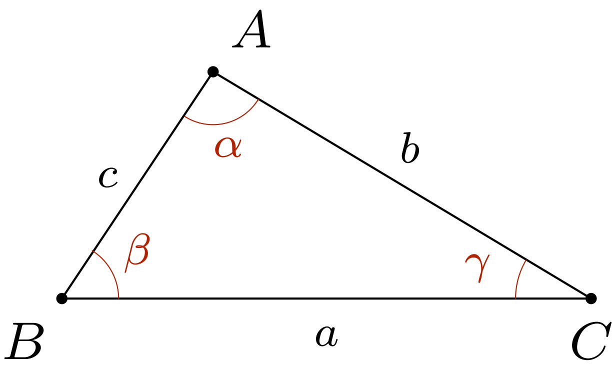


Proof: the central angle is the same for α_1, α_2, thus $2\alpha_1 = 2\alpha_2$ implies $\alpha_1 = \alpha_2$.

- More consequences:
  - if the visible arc of α equals half the circumference, then $\alpha = \pi/2$.
  - two inscribed angles with complementary visible arcs are supplementary angles:



    Proof: $2\alpha_1 + 2\alpha_2 = 2\pi$ implies $\alpha_1 + \alpha_2 = \pi$.
  - the **Sine rule**: given a triangle ABC,



$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R}$$

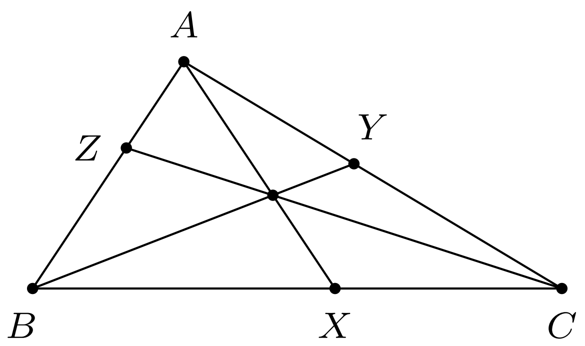
    where R is the radius of the circumscribed circle of the triangle.

    Proof: Exercise.

Ceva's Theorem

- Let ABC be a triangle. A **cevian** is any segment joining a vertex to a point of its opposite side.

Theorem 2 (Ceva's Theorem). *Given triangle ABC, let X, Y, Z be points such that the cevian segments AX, BY, CZ are concurrent, i.e. intersect:*



Then

$$\boxed{\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1,}$$

where segment lengths are signed (i.e. positive or negative depending on their orientation with respect to X, Y, Z).

Proof: Exercise.

Theorem 3 (Converse of Ceva's Theorem). *Given triangle ABC, if X, Y, Z are points of their respective opposing sides such that*

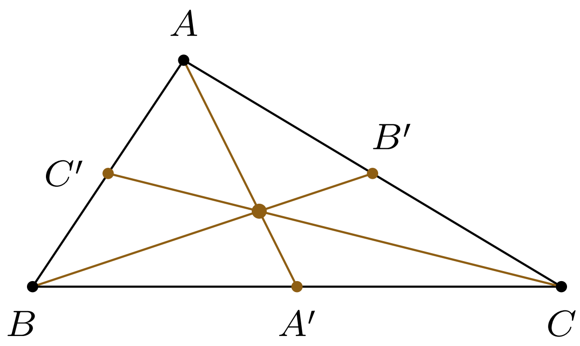
$$\boxed{\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1,}$$

then cevian segments AX, BY, CZ intersect.

PROOF: EXERCISE.

- **Medians** of a triangle are the cevians joining every vertex to the midpoint of the opposite side.

Corollary 4. *Medians always intersect:*



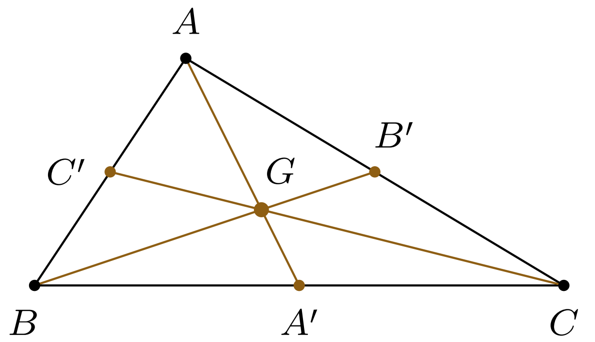
- PROOF: $\frac{AC'}{C'B} = \frac{BA'}{A'C} = \frac{CB'}{B'A} = 1$, thus so is their product and the converse of Ceva applies.

- The **barycenter** (or **centroid**, **center of mass** or **centroid**) G of a triangle is the point of intersection of its medians.

Theorem 5. *The medians of a triangle divide it into six subtriangles of equal area:*

$$(BA'G) = (A'CG) = (CGB') = (B'GA) = (AGC') = (C'GB)$$

Corollary 6. *The barycenter divides the medians into segments of length proportions* $1/3 : 2/3$,



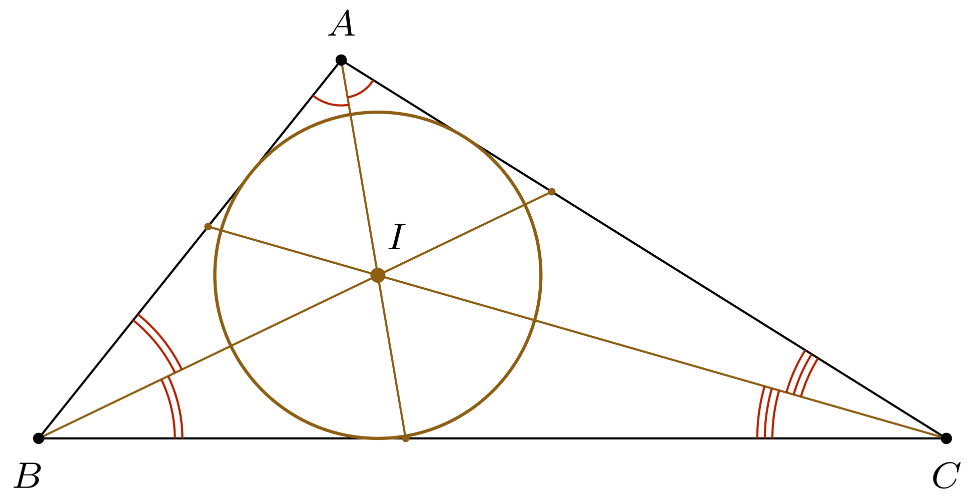
$$GA = 2GA'$$
$$GB = 2GB'$$
$$GC = 2GC'$$

PROOF OF BOTH RESULTS: EXERCISE.

Angle bisectors

- A **bisectrix** or **angle bisector** of ABC is a cevian dividing one of its angles into equal halves.

Theorem 7. *The three angle bisectors of a triangle intersect. The point where they meet, which we label I, is called the* ***incenter*** *and is the center of its inscribed circle:*

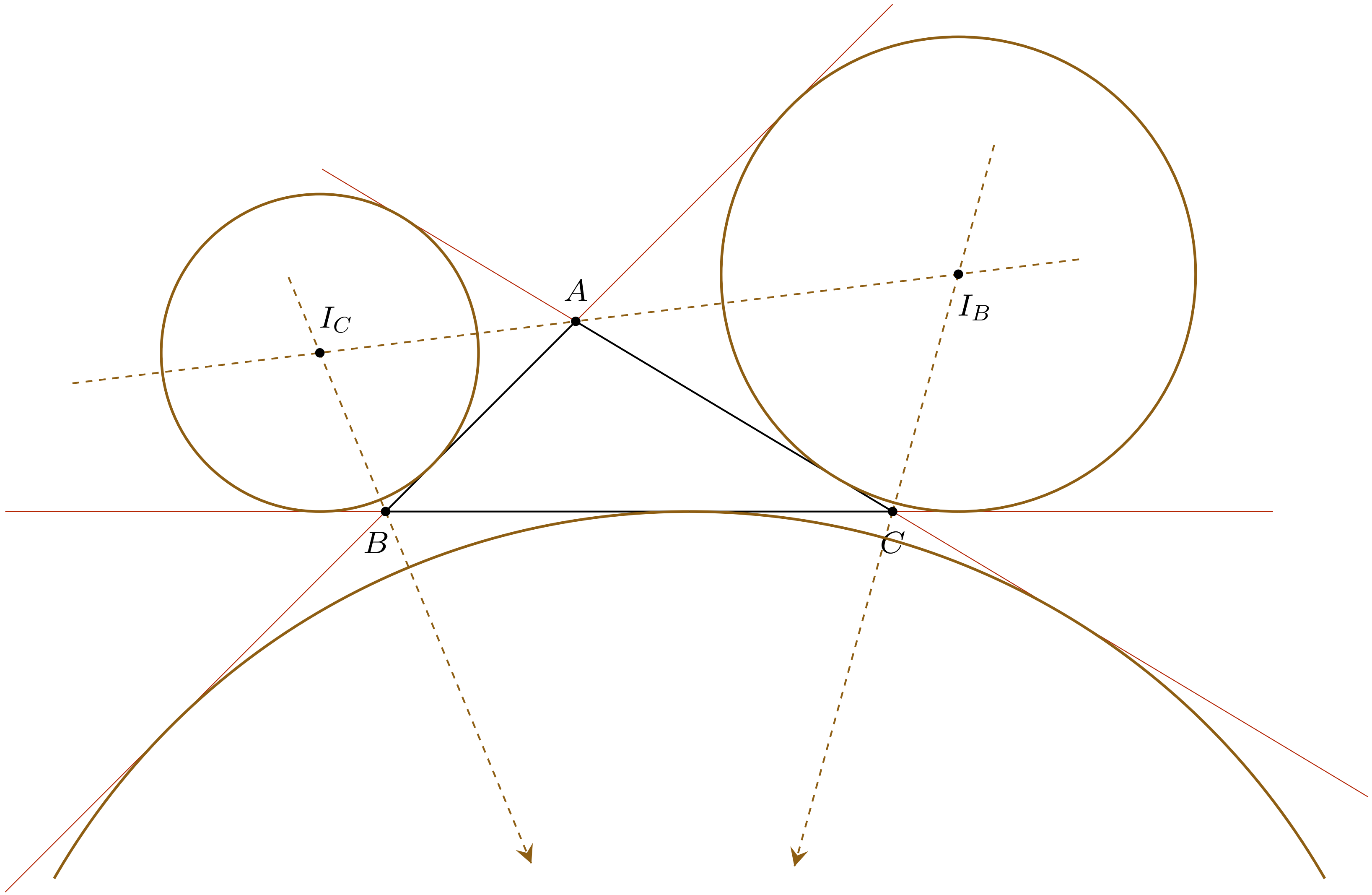


and the radius of said circle equals $\frac{1}{s}(ABC)$, where $s = \frac{1}{2}(a+b+c)$ is the ***semi-perimeter***.

Theorem 8 (Steiner-Lehmus, 1840)**.** *A triangle with two equal angle bisectors is an isosceles triangle, i.e. a triangle with two equal sides.*

Proof of both results: Exercise

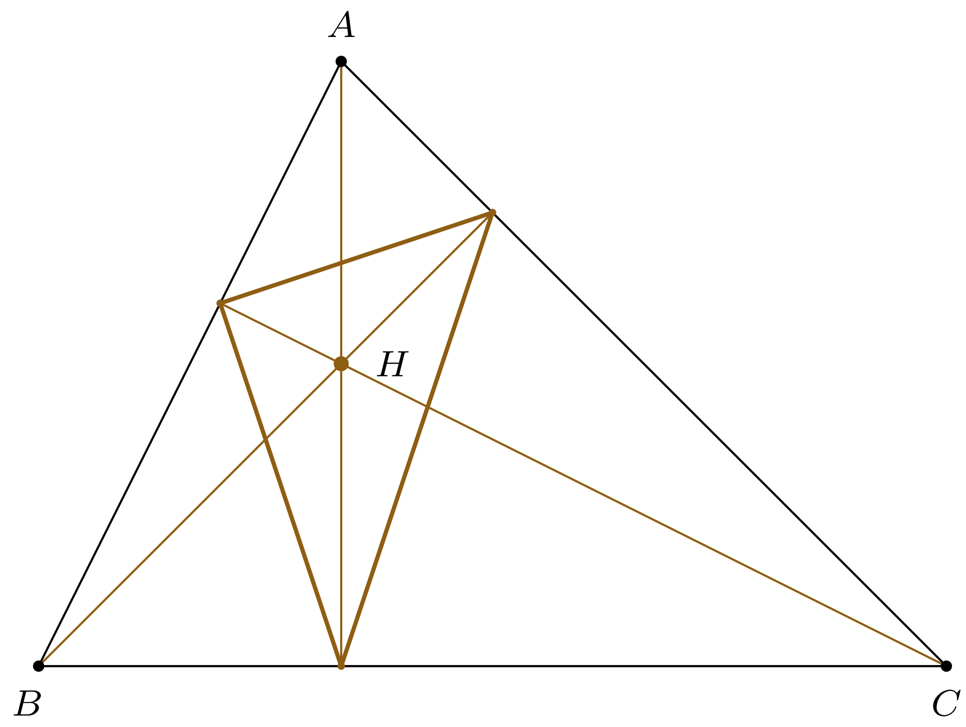
- **Exterior bisectrices** arise from considering the sides of a triangle ABC as lines instead of line segments. They determine the three centers I_A, I_B, I_C of the exterior circumferences tangent to the sides of the triangle, in a very straightforward way (EXERCISE: prove it):



The orthic triangle

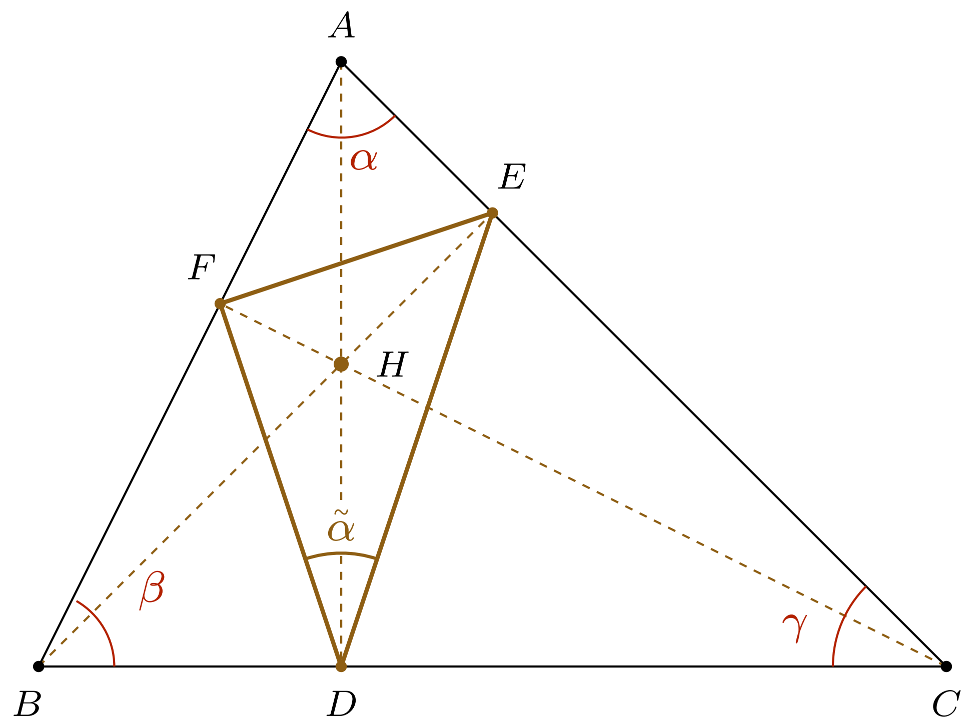
- An **altitude** of a triangle $\triangle ABC$ is a cevian segment orthogonal (i.e. perpendicular) to the opposite side. Exercise: prove the following,

Proposition 9. *The three altitudes of $\triangle ABC$ intersect in a point, called the* ***orthocenter*** *H of the triangle. The triangle determined by the endpoints of the three altitudes is called the* ***orthic triangle****.*



Proposition 10. *If $\triangle ABC$ is an* ***acute triangle*** *(i.e. all of its angles are $< \pi/2$) then the three altitudes of $\triangle ABC$ are the angle bisectors of the orthic triangle.*

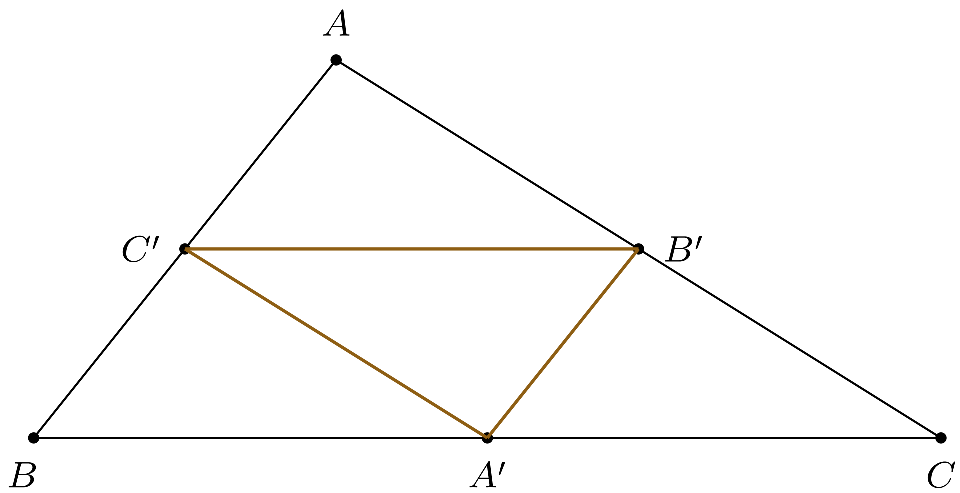
Corollary 11. *The angles of orthic triangle $\triangle DEF$ of $\triangle ABC$ are $\tilde{\alpha} = \widehat{FDE} = \pi - 2\alpha$, etc.*



Exercise: prove the above and complete the "etc" in the Corollary.

Medial triangle

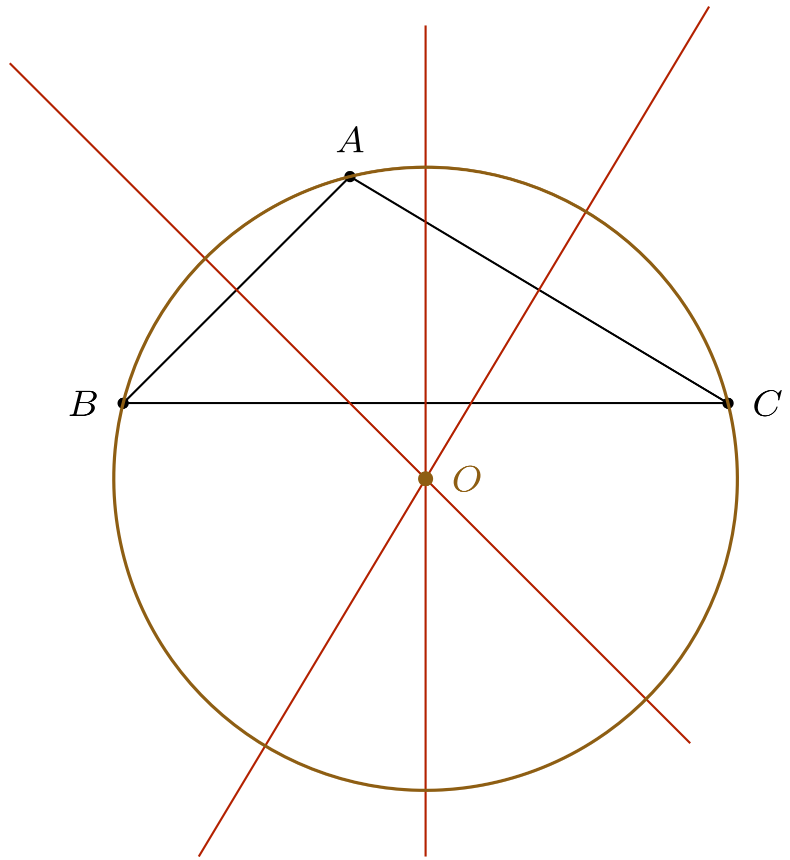
- The **medial** (or **midpoint**, or **complementary**) **triangle** of $\triangle ABC$ is the triangle whose vertices A', B', C' are the midpoints of sides a, b, c respectively.



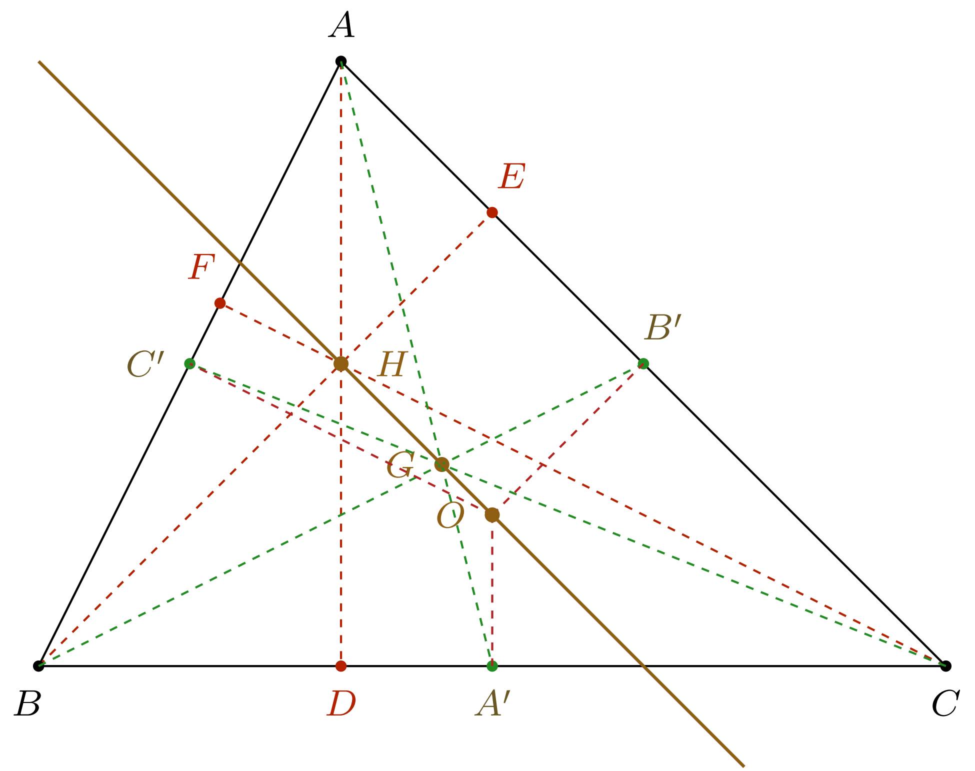
- C', B' are the midpoints of c, b respectively, thus $C'B' \parallel CB$ and Thales's Theorem implies $BC = 2B'C'$. Similarly, $AC = 2A'C'$ and $AB = 2A'B'$, Thus both triangles are **similar** or **homothetical** to one another, i.e. they have the same shapes (albeit different sizes).
- Proportionality of sides is 2, thus the areas of both triangles satisfy $(ABC) = 4\,(A'B'C')$.
- The medians of $\triangle ABC$ are also medians of its complementary triangle (all you have to do is draw said medians in the above figure).
- As a consequence, both triangles $\triangle ABC$ and $\triangle A'B'C'$ share the same barycenter: $G = G'$.

The Euler line and the nine-point circle

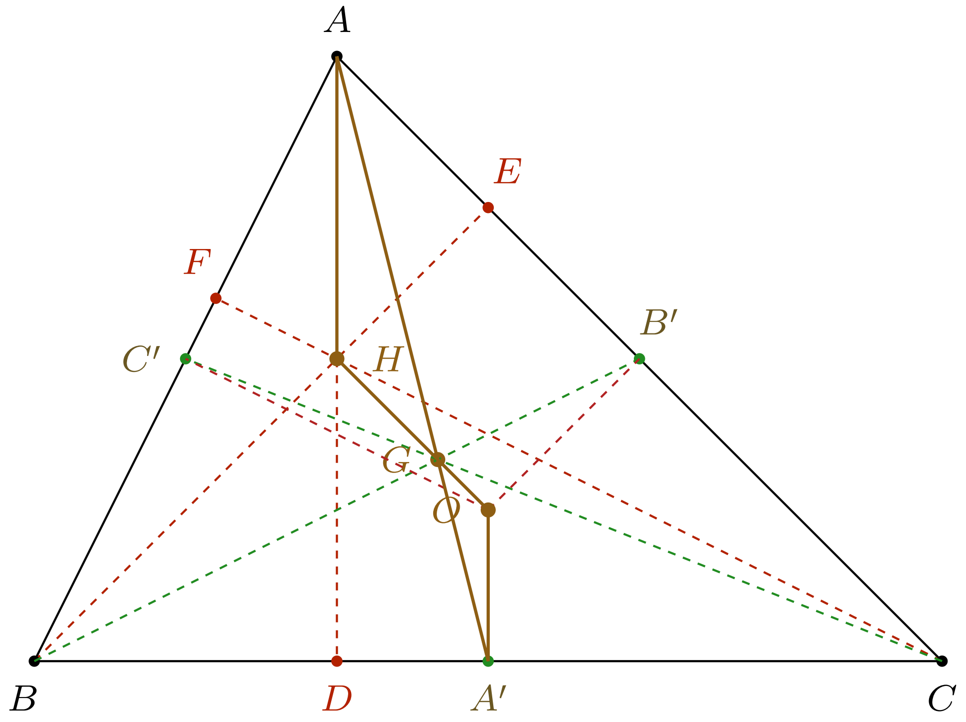
- Given $\triangle ABC$, its **circumcenter** is the center O of its circumscribed circle (i.e. the circle containing A, B, C). It is the intersection of the perpendicular bisectors of sides a, b, c
  EXERCISE: prove that such intersection exists and serves the aforementioned function.



Theorem 12 (Euler's Theorem). *Given $\triangle ABC$, let H be its orthocenter, G its barycenter and O its circumcenter. There is a straight line containing all three points O, H, G.*

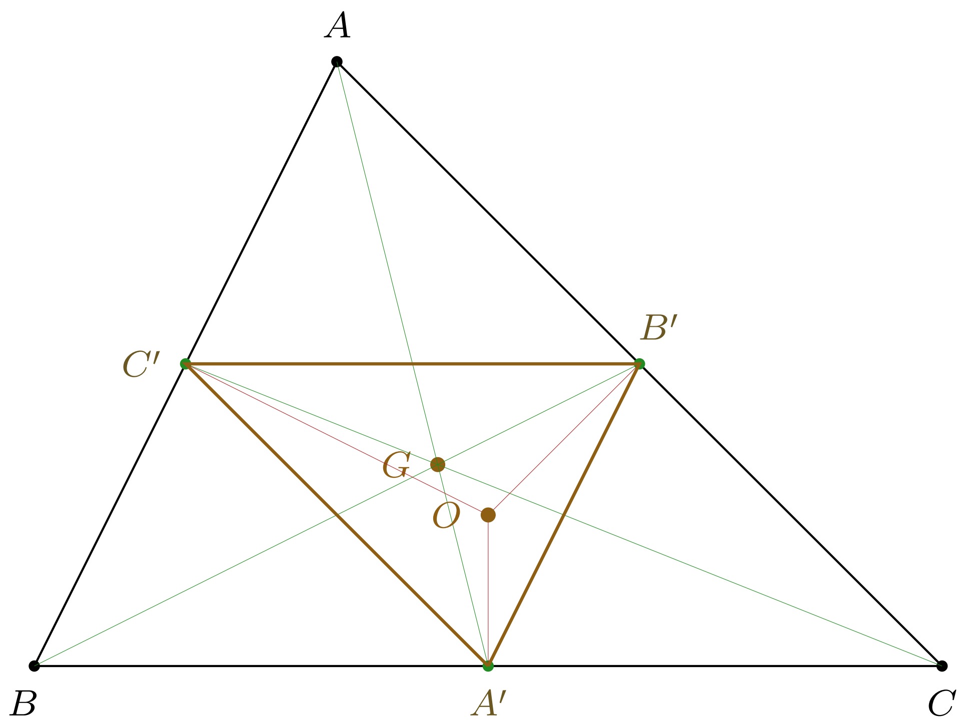


- PROOF:
  - $A'O \parallel AD$ because they are both perpendicular to BC.
  - let us prove that triangles AHG and $A'OG$ are **similar**, i.e. $\triangle AHG \sim \triangle A'OG$ (i.e. they have the same angles):



once this is proven, the Theorem will be proved.

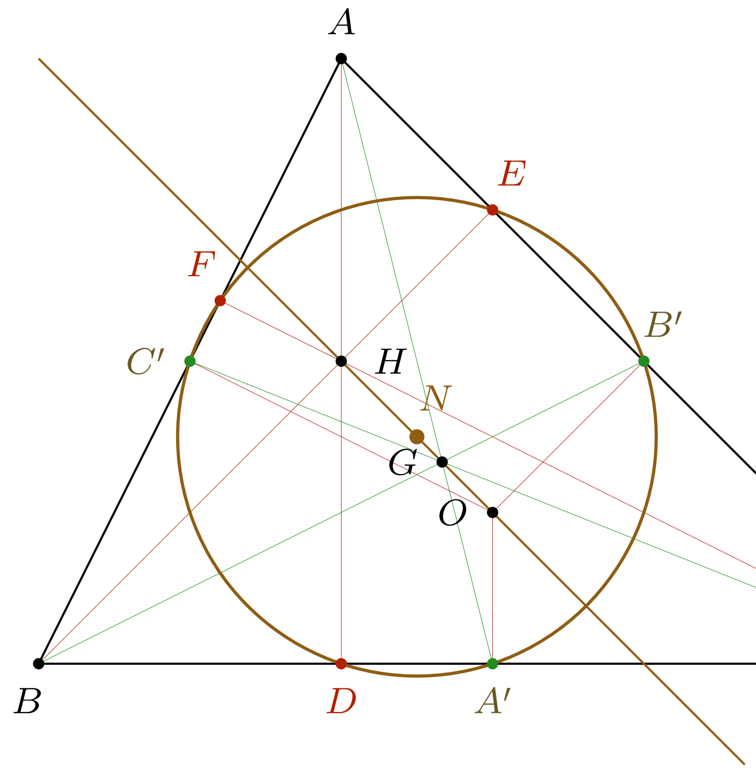
- Medial triangle $\triangle A'B'C'$ shares its barycenter with that of $\triangle ABC$ ($G' = G$) and its orthocenter is the circumcenter of original triangle, $H' = O$:



and triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar (Exercise), thus so are AHG and $A'H'G' = A'OG$ highlighted in the previous slide.

- Given that both triangles $\triangle AHG \sim \triangle A'OG$ share a common vertex G and have parallel sides $A'O$ and AD, all three points H, G, O are aligned. $\square$

Theorem 13 (The Nine Point Circle). *Given $\triangle ABC$, let H be its orthocenter, G its barycenter and O its circumcenter. Let $\mathfrak{e}$ be the Euler line containing all three points O, H, G. If N is the mid-point of segment OH, then N is the center of a circle containing points A', B', C' and D, E, F.*

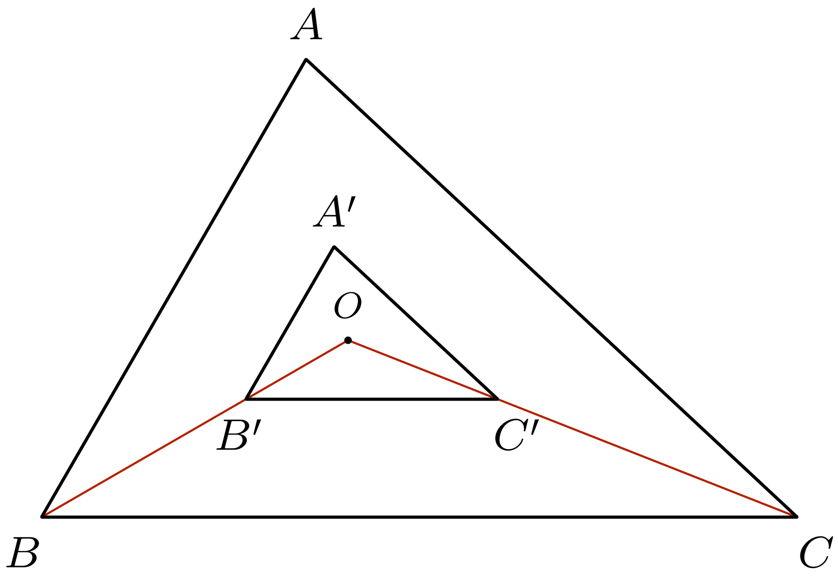


Exercise: find the remaining three notable points justifying the title, and prove the Theorem.

Exercises

- Prove everything marked as an EXERCISE in the preceding slides.
- Prove that the converse to one of the first statements of the chapter holds: if the opposite angles of a quadrilateral are supplementary, (i.e. their sum equals π) then the vertices of the quadrilateral are concyclic (i.e. they belong to the same circle).
- Prove that a converse to another one of those first statements holds as well: if two non-consecutive angles of a quadrilateral are right angles, then all four points lie on a circle.
- Let $\triangle ABC$ with respective angles α, β, γ and R be the radius of the circumference circumscribing it.
  - Prove that, even if β or γ are obtuse angles, you still have $a = b\cos\gamma + c\cos\beta$.
  - Use the Law of Sines to prove that $\sin(\beta+\gamma) = \sin\beta\cos\gamma + \sin\gamma\cos\beta$.
- Prove that for every triangle ABC,
  - $a(\sin\beta - \sin\gamma) + b(\sin\gamma - \sin\alpha) + c(\sin\alpha - \sin\beta) = 0$.
  - Let p, q be the radii of the two circumferences passing through point A and intersecting BC in B and C respectively. Prove that $pq = R^2$.
  - Prove that $\text{area}(\triangle ABC) = \frac{abc}{4R}$.
- Let X, Y, Z be the midpoints of a triangle ABC. Prove that the corresponding cevians intersect.
- Prove that the three cevians perpendicular to the opposite sides of a triangle intersect.

- Let ABC and $A'B'C'$ be the following two triangles having parallel sides:



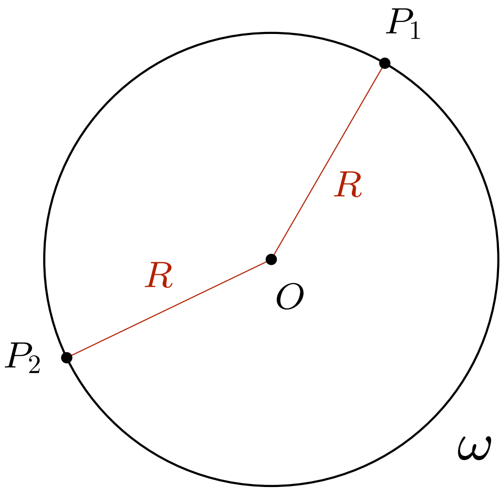
Prove that the three lines AA', BB', CC' intersect.

- Let AX be a cevian having length p, dividng BC into segments of length $|BX| = m$ and $|XC| = n$. Prove that $a\left(p^2 + mn\right) = b^2m + c^2n$.
- Prove that the orthocenter and the circumcenter of an obtuse triangle lie in its exterior.
- Find the ratio between the area of a triangle and that of a triangle whose sides have the same length as the medians of the former.
- Prove that every triangle with two equal medians is isosceles.
- Prove that every triangle with two equal heights is isosceles.

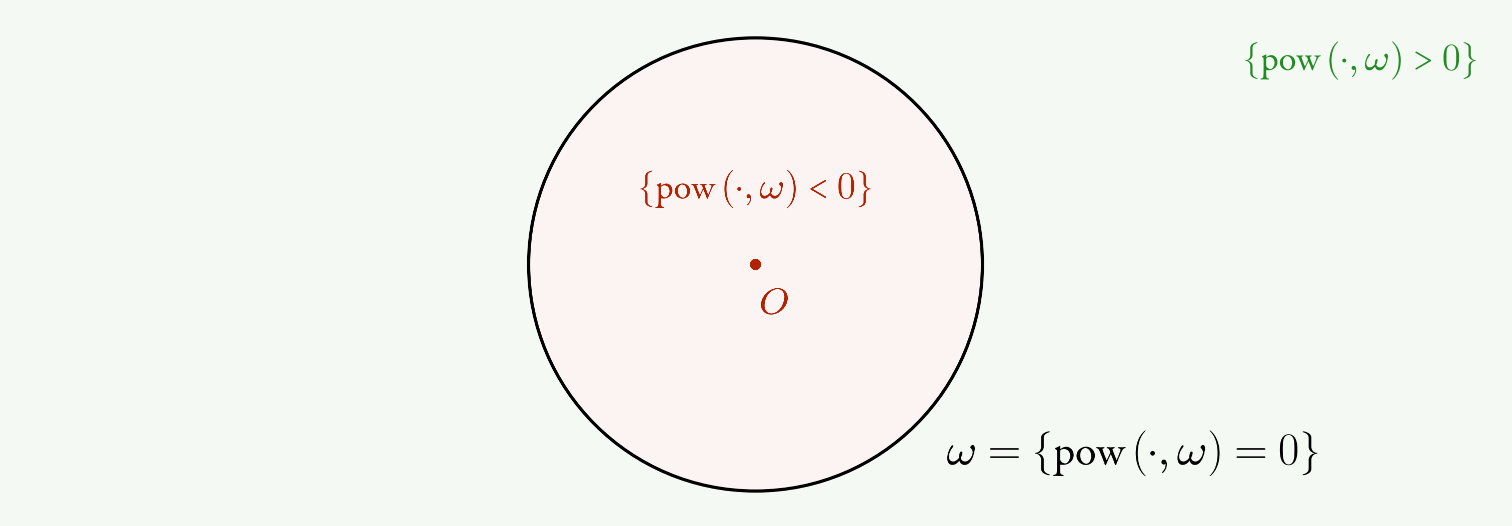
- Prove that the product of two sides of a triangle equals $2Rh$, where h is the length of the altitude of the third side.
- Prove that if three circles of centers A, B, C intersect one another externally, then their radii are $s-a, s-b, s-c$ where s equals the semi-perimeter.
- Given $\triangle ABC$, consider $\triangle I_aI_bI_c$ whose sides are the exterior angle bisectors having angles α, β, γ. Prove that $\triangle ABC$ is the orthic triangle of $\triangle I_aI_bI_c$.
- Let X, Y, Z be the tangency points of the inscribed circle of $\triangle ABC$ with the corresponding sides. Prove that the cevians AX, BY, CZ intersect.

2. The circle

- A **circle ω of center O and radius R** is defined as the geometric locus of points P in the Euclidean plane such that $d(P, O) = R$:

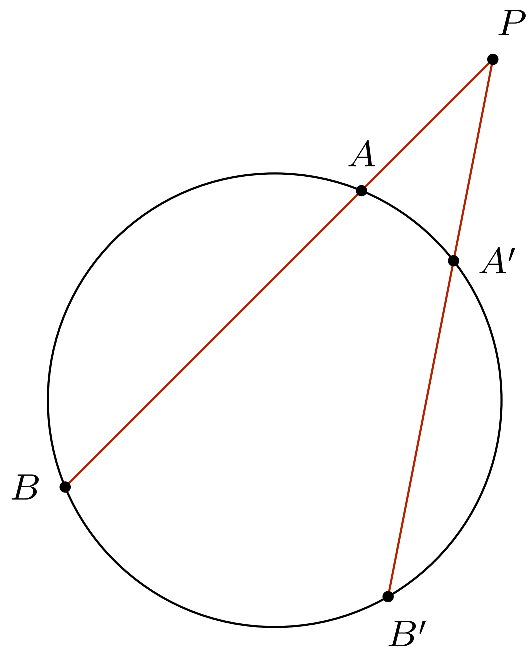


- The **power of a point P with respect to ω** as $\operatorname{pow}(P, \omega) := \operatorname{pow}_\omega(P) := d(O, P)^2 - R^2$.



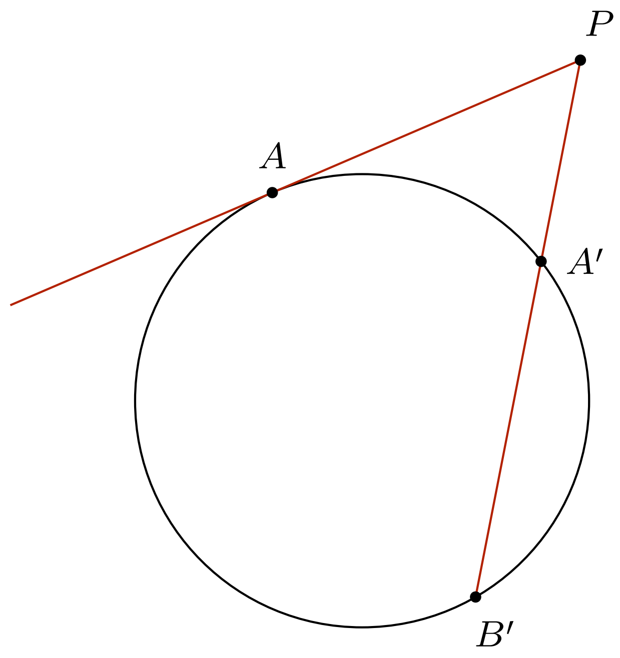
- A useful geometrical interpretation:

Proposition 14. *If P is a point exterior to ω and ℓ, ℓ' two lines containing P and secant to ω through points A, B and A', B' respectively. Then $PA \cdot PB = PA' \cdot PB' = \operatorname{pow}(P, \omega)$.*



PROOF: EXERCISE.

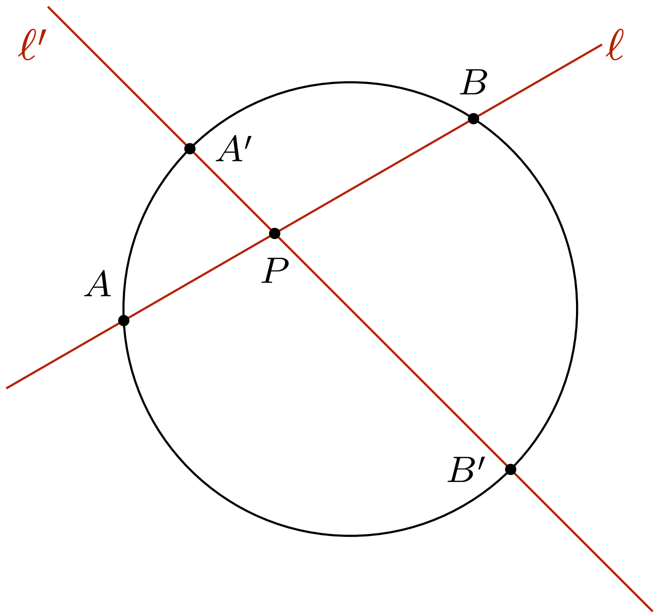
- Needless to say, if the secant becomes a tangent, i.e. $A = B$,



$$\boxed{PA \cdot PB = (PA)^2}$$

and a quick application of Pythagoras Theorem implies that this square $(PA)^2$ is indeed $\mathrm{pow}_\omega(P)$ (think why).

Proposition 15. *If P is a point interior to ω and ℓ, ℓ' two lines containing P and intersecting ω at points A, B and A', B' respectively, then $PA \cdot PB = PA' \cdot PB' = -\text{pow}(P, \omega)$.*



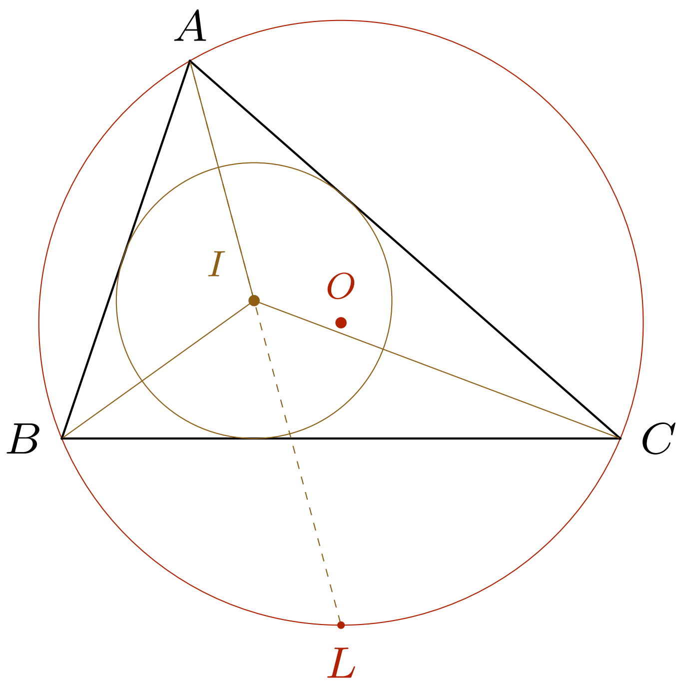
Proof: Exercise.

Theorem 16 (Euler's Theorem). *Let $\triangle ABC$ have circumcenter O, incenter I, circumscribed circle radius R and inscribed circle radius r. Then $\mathrm{d}(O,I)^2 = R^2 - 2rR$.*

Proof. Interpretation in terms of the power of a point (with respect to the circle ω circumscribing $\triangle ABC$): we have to prove this,

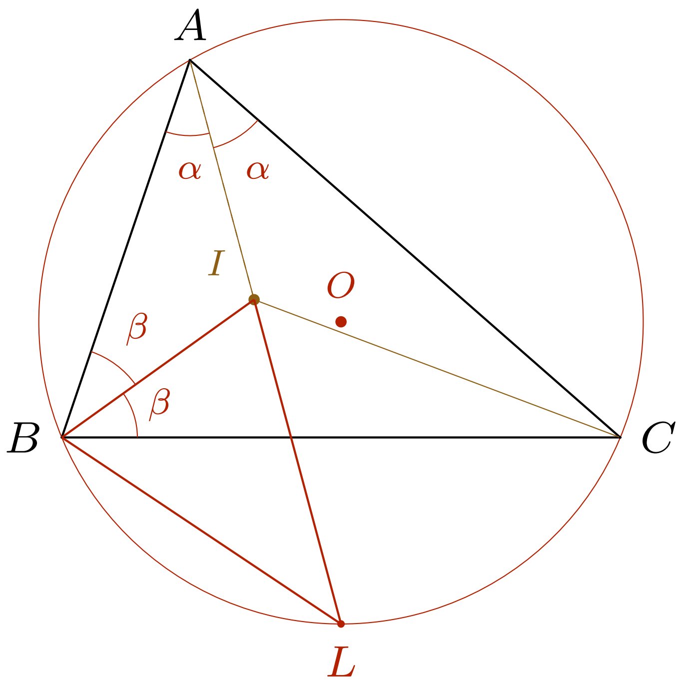
$$\mathrm{pow}(I,\omega) = d(O,I)^2 - R^2 = -2rR.$$

Let L be the *other* point of intersection of AI with ω:



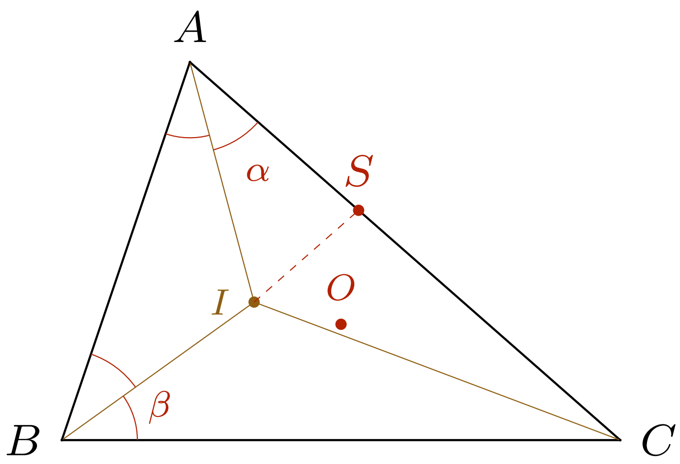
then using the Proposition, $\mathrm{pow}_\omega(I) = -AI \cdot IL$ and we want to prove $AI \cdot IL = 2rR$.

Let $\alpha = \widehat{LAC} = \widehat{LAB}$ and $\beta = \widehat{CBI} = \widehat{IBA}$.

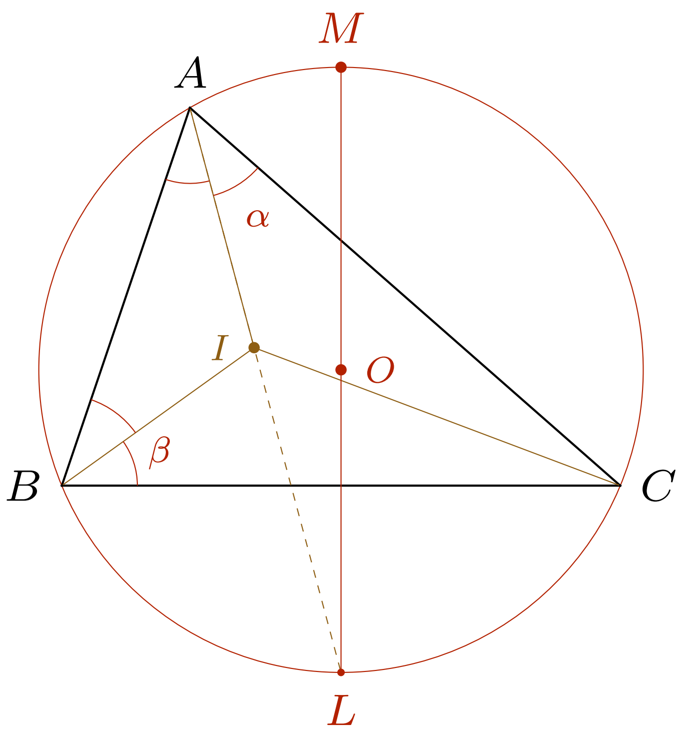


then angles $\widehat{BCA} = \pi - 2\alpha - 2\beta$ and $\widehat{BLA}$ comprise the same arc segment AB, thus must be equal: $\widehat{BLA} = \pi - 2\alpha - 2\beta$ which coupled with $\widehat{ABC} = 2\beta$ yields $\widehat{LBC} = \alpha$, thus $\widehat{LBI} = \alpha + \beta$. $\widehat{BIL} = \alpha + \beta$ as well, thus $\triangle IBL$ is an isosceles triangle and thus $BL = IL$.

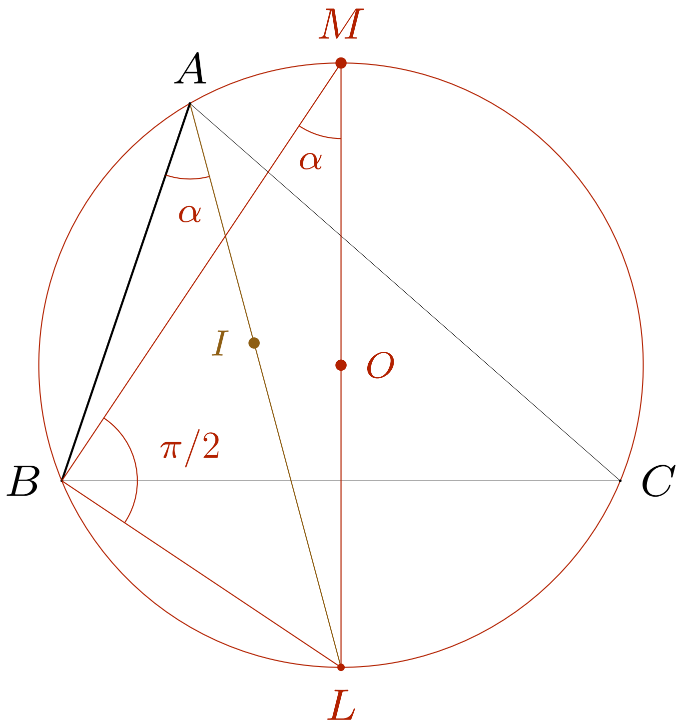
Thus $AL \cdot IL = AI \cdot BL$ and if S is the orthogonal projection of I onto AC,



then $r = IS = IA \sin\alpha$ hence $IA = \frac{r}{\sin\alpha}$. Line LO intersects ω in M,



And given the fact that LM is a diameter, $\widehat{LBM} = \pi/2$ and angles $\widehat{BAL}$ and $\widehat{BML}$ comprise the same arclength BL hence $\widehat{BML} = \alpha$:



$LM = 2R$, thus $BL = 2R\sin\alpha$ which coupled with $IA = \frac{r}{\sin\alpha}$ yields

$$-IA \cdot BL = -\frac{r \cdot 2R}{\sin\alpha} \cdot \sin\alpha = -2Rr. \qquad \square$$

The radical axis

- Let ω, ω' be two non-concentric circles. The **radical axis** of ω, ω' is the geometric locus of all points P of the plane such that $\mathrm{pow}_\omega(P) = \mathrm{pow}_{\omega'}(P)$.

Proposition 17. *The radical axis of ω and ω' is a line orthogonal to the line determined by the centers of ω and ω'.*

PROOF: We will use analytical geometry, i.e. geometry with coordinates. In a Cartesian coordinate system of the plane,

- ω has center (a, b) and radius r, thus is the set of all points $P = (x, y)$ such that $\boxed{\mathrm{pow}_\omega(P) = (x-a)^2 + (y-b)^2 - r^2 = 0}$ i.e. $\boxed{x^2 + y^2 - 2ax - 2by + c = 0}$ where $c = a^2 + b^2 - r^2$,
- similarly, ω' has implicit equation $\boxed{\mathrm{pow}_{\omega'}(P) = x^2 + y^2 - 2a'x - 2b'y + c' = 0}$

- The radical axis is defined as the set of points P such that $\mathrm{pow}_\omega(P) = \mathrm{pow}_{\omega'}(P)$, i.e.

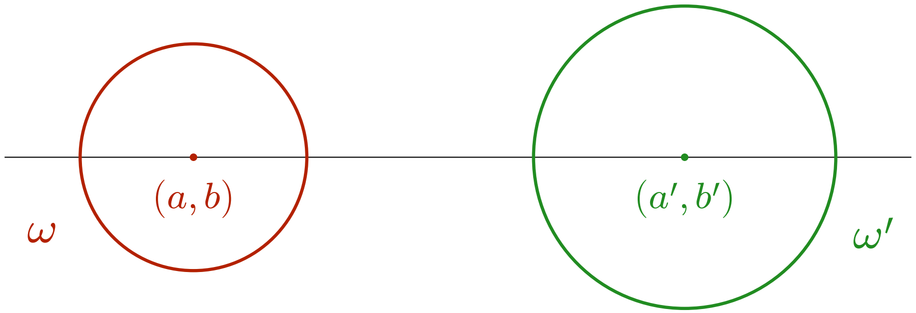
$$x^2 + y^2 - 2ax - 2by + c = x^2 + y^2 - 2a'x - 2b'y + c'.$$

but if we rearrange this we obtain

$$2\left(a - a'\right)x + 2\left(b - b'\right)y + \left(c' - c\right) = 0, \tag{1}$$

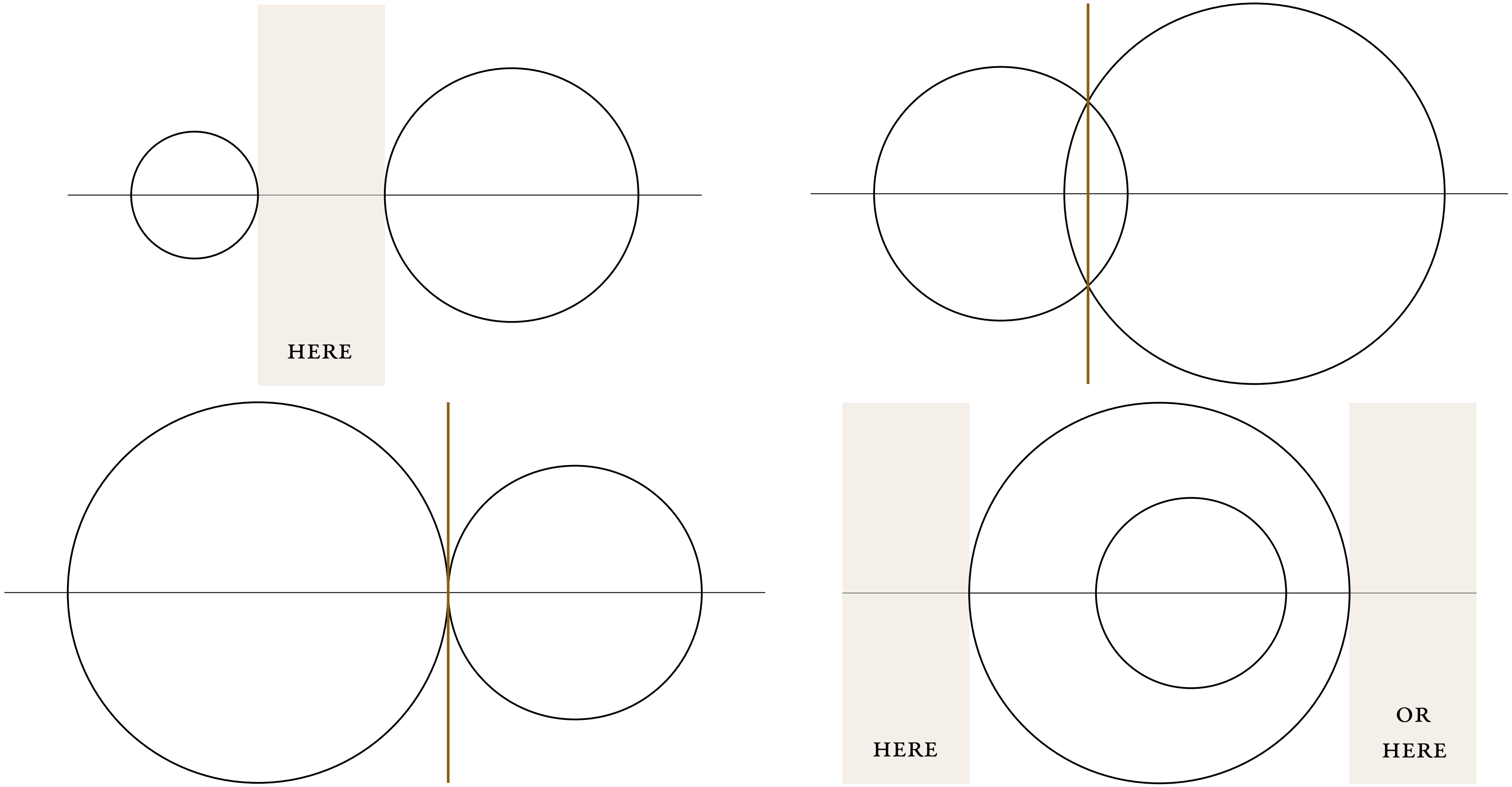
which is the equation of a line.

- To know the inclination of that line with respect to the line joining the centers of ω and ω'



we write $X = (a, b)$ and $Y = (a', b')$, then (1) is the equation of the points $P = (x, y)$ such that the dot product $\overrightarrow{OP} \bullet \overrightarrow{XY}$ is constant (and equal to $\frac{c-c'}{2}$). This implies (EXERCISE: show why) that the set of points in (1) is a line *perpendicular* to the line XY. $\square$
- EXERCISE: prove it without using analytical geometry.

- These are the possible positions of the radical axis depending on the relative situation of ω, ω':



- EXERCISE:
  - in the first case, decide whether the axis will be closer to the circumference with the larger radius, or to the one with the smaller radius.
  - In the final case, decide in which cases the axis will be in the left or right region.

Radical center

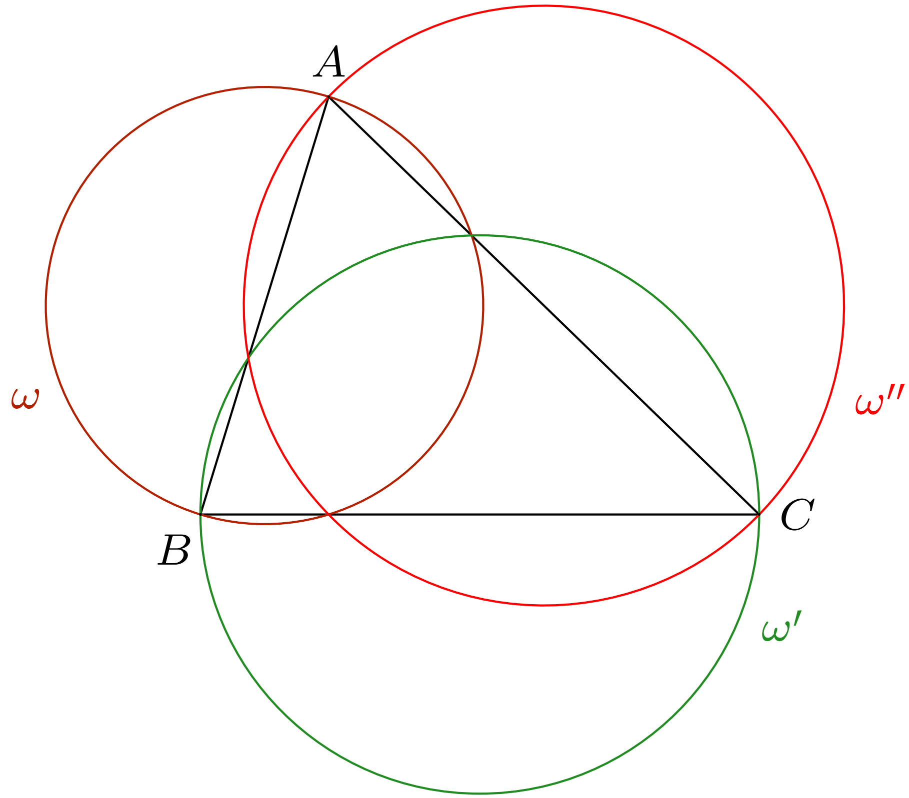
- Let $\omega, \omega', \omega''$ be three circumferences such that no two of them are concentric.
  The **radical center** of $\omega, \omega', \omega''$ is the geometric locus of points P such that

$$\operatorname{pow}_{\omega}(P) = \operatorname{pow}_{\omega'}(P) = \operatorname{pow}_{\omega''}(P).$$

Proposition 18. *The radical center of $\omega, \omega', \omega''$ is a single point and is the intersection of the corresponding radical axes determined by these three circumferences.*

PROOF: EXERCISE.

Proposition 19. *Let ABC be a triangle and $\omega, \omega', \omega''$ having diameters AB, BC, CA:*

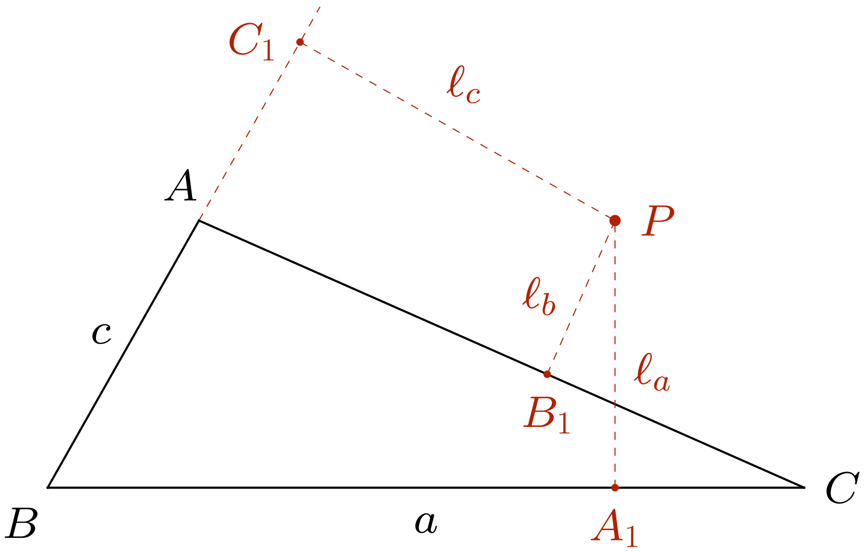


Then the radical center of $\omega, \omega', \omega''$ is the orthocenter of $\triangle ABC$.

PROOF: EXERCISE.

The Simson line

- Let ABC be a triangle and P an exterior point. Take lines ℓ_a, ℓ_b, ℓ_c containing P and orthogonal to sides a, b, c respectively:

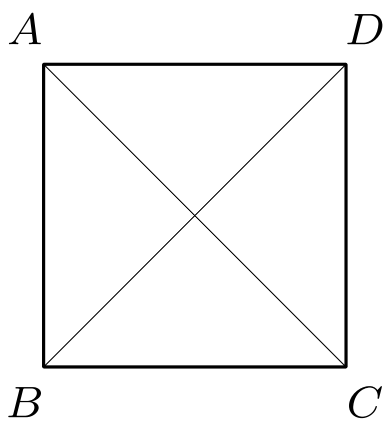


Theorem 20 (Simson)**.** *Points A_1, B_1, C_1 are in the same line if and only if P lies in the circumcircle (i.e. the circle containing A, B, C).*

PROOF: EXERCISE.

Ptolemy's Theorem

- The Pythagoras Theorem (300 BC) is a statement about four points



and these points lie on a circumference:

$$AB \cdot DC + BC \cdot AD = AC \cdot BD, \qquad \text{i.e.} \quad b^2 + a^2 = c^2.$$

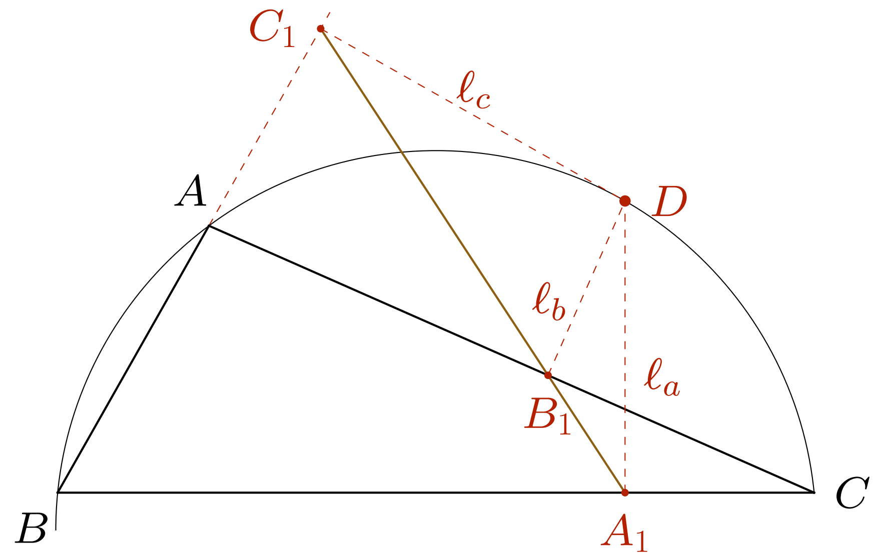
Let us address a slightly more general form of this formula:

Theorem 21 (Ptolemy's Theorem). *Let $\triangle ABC$ and D be a fourth point of the plane Then*

$$AB \cdot DC + BC \cdot AD \geq AC \cdot BD,$$

and equality holds if, and only if, all four points lie on a circumference.

PROOF: Consider the triangle ABC and the circle ω of radius R circumscribed to it. Point D is in ω if and only if the base points of lines orthogonal to the sides are aligned (Simson's Theorem), i.e. $A_1B_1 + B_1C_1 = A_1C_1$ below:



applying the Law of Sines $\dfrac{B_1C_1}{\sin \widehat{B_1AC_1}} = 2R'$ where R' is the radius of the circle circumscribing $\triangle AB_1C_1$. D belongs to this circle as well (a further application of Simson's Theorem) thus $AD = 2R'$ and $\dfrac{B_1C_1}{\sin \widehat{B_1AC_1}} = AD$, hence $B_1C_1 = AD \sin \widehat{B_1AC_1}$.

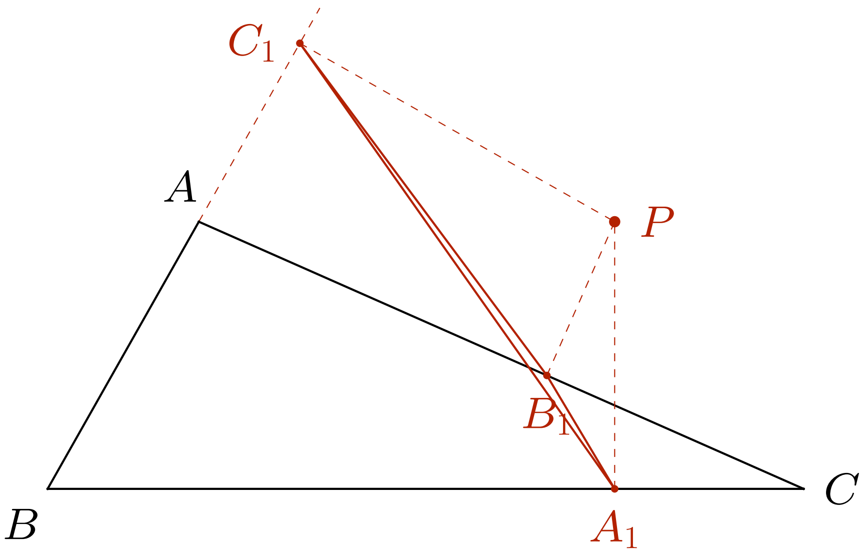
- Applying the Law of Sines to $\triangle ABC$ and the fact $\sin \widehat{B_1AC_1} = \sin\left(\pi - \widehat{B_1AC_1}\right) = \sin \hat{A}$,

$$B_1C_1 = AD\frac{BC}{2R}$$

- We can perform a cyclic permutation of the letters A, B, C (i.e. a cyclic permutation of our choice of vertices of the triangle) and

$$B_1C_1 = \frac{AD \cdot BC}{2R}, \quad A_1C_1 = \frac{BD \cdot CA}{2R}, \quad A_1B_1 = \frac{CD \cdot AB}{2R}$$

hence the inequality $A_1B_1 + B_1C_1 \geq A_1C_1$, which is always true,



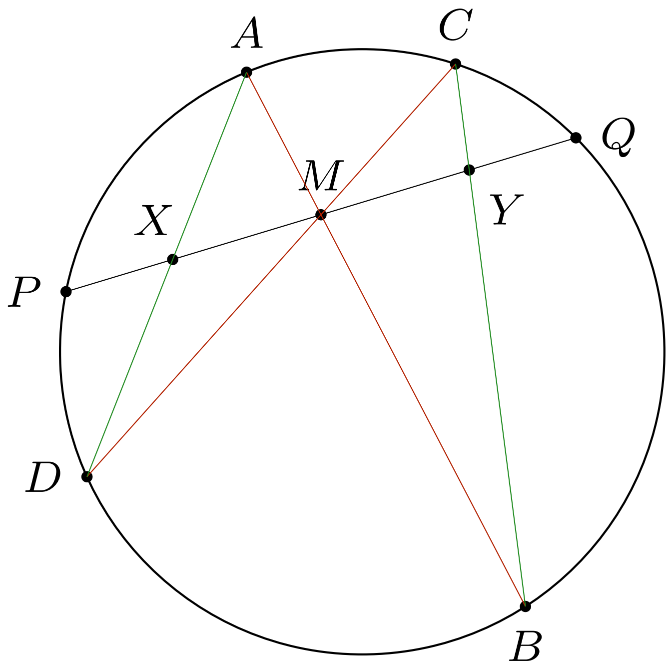
translates into

$$\frac{CD \cdot AB}{2R} + \frac{AD \cdot BC}{2R} = \frac{AB \cdot CD + AD \cdot BC}{2R} \geq \frac{BD \cdot CA}{2R},$$

i.e. $AB \cdot CD + AD \cdot BC \geq BD \cdot CA$, and equality holds if and only if A_1, B_1, C_1 are aligned, i.e. iff D belongs to ω. $\square$

The Butterfly Theorem

Theorem 22 (The Butterfly Theorem)**.** *Let ω be a circle, PQ a secant and M its midpoint:*

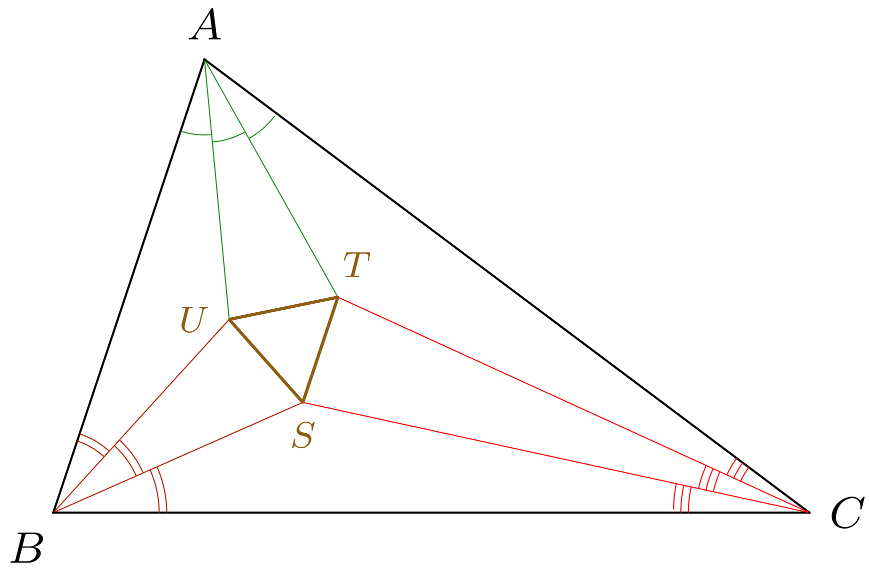


Consider another pair of secants AB and CD, both of them containing M, and let X, Y be the intersections of AD and BC with PQ respectively. Then M is always the midpoint of XY.

PROOF: EXERCISE.

Morley's Theorem

Theorem 23 (Morley, 1904). *Let ABC be a triangle. The intersections of the angle trisectors of the three angles in $\triangle ABC$ define an* ***equilateral*** *triangle.*



More precisely, if $\alpha = 3\tilde{\alpha}$, $\beta = 3\tilde{\beta}$, $\gamma = 3\tilde{\gamma}$, and R is the radius of the incircle (inscribed circle) to ABC, then

$$SU = UT = TS = 8R \sin\tilde{\alpha} \sin\tilde{\beta} \sin\tilde{\gamma}.$$

PROOF: EXERCISE.

Exercises

- What is the smallest possible value for $\text{pow}_{\omega}(P)$ for a circle ω of radius R? What point P has such value?
- Find the set of points whose power is constantwith respect to a given circle of radius R.
- If the power of a point has a positive value t^2, interpret length t geometrically.
- If PT and PU are tangents from P to two concentric circles (T being in the smaller one) and segment PT cuts the larger circle in point Q, prove that $PT^2 - PU^2 = QT^2$.
- Express in terms of r and R the power of the incenter with respect to the circumcircle.
- What is the distance to the horizon if seen from the summit of a 2 km. tall mountain, assuming the Earth is a sphere of diameter 20600 kms?
- Determine the geometric locus of points P such that if line TT' is tangent (in T and T') to two given circles, then $PT = PT'$.
- Prove that, if the distance between the centers of two circles is greater than the sum of their radii, the circles have four common tangent lines. Furthermore, the midpoints of these four segments are collinear.
- Given a and b, determine for which values of c the equation $x^2 + y^2 - 2ax - 2by + c = 0$ corresponds to a circle.

- Describe a method to obtain the radical axis of two non-concentric circumferences, in such a way that the method also works when one of the circles contains the other.
- Two circles are tangent internally on a point T. Let AB be the chord of the larger circle that is tangent to the smaller one at a point P. Prove that line TP bisects angle $\widehat{ATB}$.
- Consider three disjoint circles having radical center O. Prove that the contact points of the six tangents from O to the circles are concyclic.
- Prove that the points where the extended altitudes intersect the circumcircle form a triangle similar to the orthic triangle.

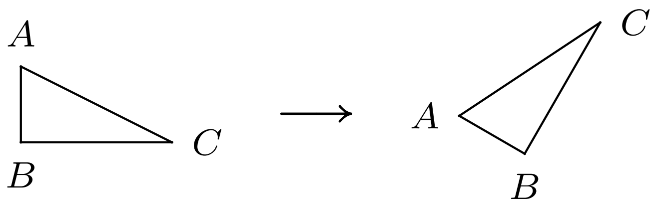
3 Transformations of the plane

3.1. Isometries

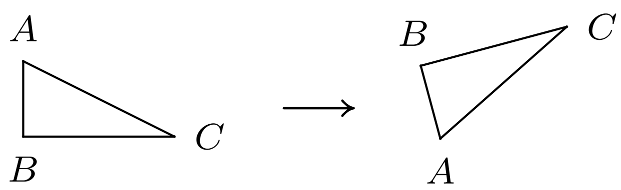
- A bijective affine transformation $f : \mathbb{A}^2_{\mathbb{R}} \to \mathbb{A}^2_{\mathbb{R}}$ is an **isometry** if it preserves distances:

$$d(\boldsymbol{x}, \boldsymbol{y}) = d(f(\boldsymbol{x}), f(\boldsymbol{y})) \text{ for every } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{A}^2_{\mathbb{R}}.$$

- Every isometry of the plane is also a **conformal** map, i.e. it preserves angles (Exercise: why?).
- However, orientation may change, e.g. from clockwise to counterclockwise in triangles:



Direct Isometry



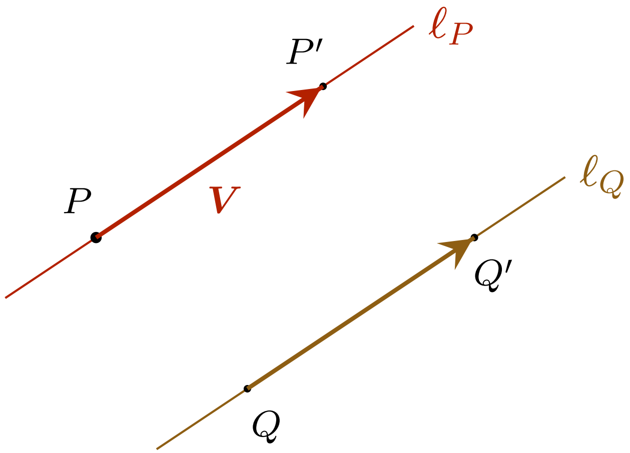
Indirect Isometry

Theorem 24. *Isometries of the plane are classified as follows:*

- *Direct isometries:*
  - ***Rotations** of angle θ, including the identity ($\theta = 2\pi k,\ k \in \mathbb{Z}$)*
  - ***Translations** by a fixed vector $\boldsymbol{v}$ (again including the identity, $\boldsymbol{v} = \boldsymbol{0}$)*
  - ***Central symmetries**,*
- *Indirect isometries:*
  - ***reflection symmetries**,*
  - ***glide reflections***

Translations

- **Translations** are the isometries on the Euclidean plane defined by $\boldsymbol{P} \mapsto f(\boldsymbol{P}) := \boldsymbol{P} + \boldsymbol{V}$, where $\boldsymbol{V}$ is a constant vector.
- If $\boldsymbol{V} \neq \mathbf{0}$ (i.e. $f \neq \mathrm{id}$) we call $\boldsymbol{V}$ the **translation vector**.
- Vector $\boldsymbol{V}$ is uniquely determined if we know one point $\boldsymbol{P}$ and its transformed point $f(\boldsymbol{P})$:



Both lines ℓ_P and ℓ_Q have a common perpendicular, i.e. they are parallel to one another and all such lines are determined by $\boldsymbol{V} := \overrightarrow{PP'} = P' - P$.

Every other image $f(Q)$ is thus determined with a line, its perpendicular, the perpendicular to the perpendicular and a vector of the same magnitude

- Translations do not have double or fixed points (i.e. points P such that $f(P) = P$) but they do possess double lines (namely those parallel to the direction vector $\boldsymbol{V}$).
- Exercise: given a circle ω of radius R, let AB a segment smaller in magnitude than $2R$. Construct a rectangle of size $|AB|$ inscribed in ω.

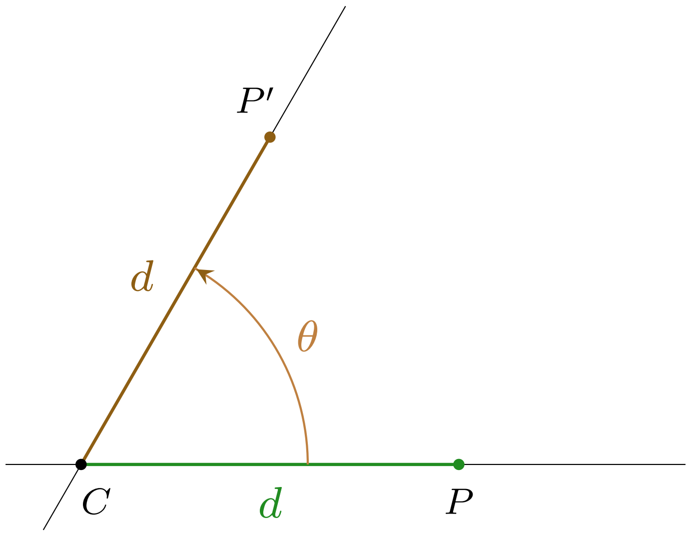
Rotations

- Let $C \in \mathbb{A}^2_{\mathbb{R}}$. A **rotation** of center C and angle θ is a transformation of the plane

$$\begin{aligned} \mathbb{A}^2_{\mathbb{R}} &\xrightarrow{\quad f \quad} \mathbb{A}^2_{\mathbb{R}} \\ P &\longmapsto f(P) := P' \end{aligned}$$

  such that

  - if $P \neq C$, the segment CP forms an angle θ with segment CP' and $d(C,P) = d(C,P')$;
  - if $P = C$, then $P' = P = C$.



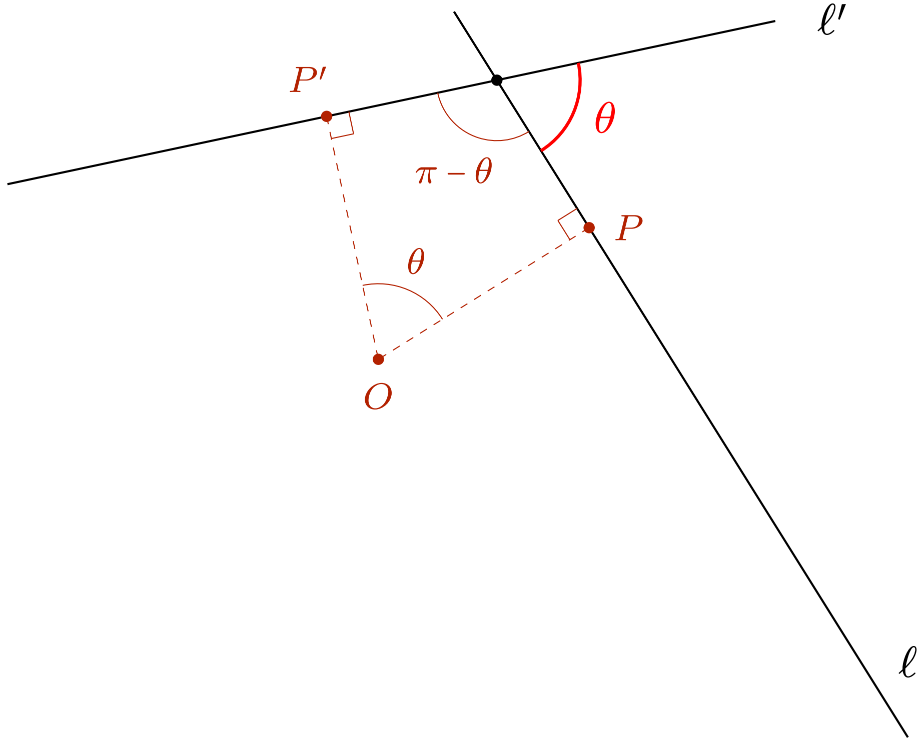
Theorem 25. *The set of all rotations with a given center C forms a group with the operation $\circ$.*

Proof: Exercise.

- A rotation only has one fixed point: its center of rotation.
- A rotation does not have fixed straight lines unless $\theta = 0, \pi$. More precisely,

Lemma 26. *If ℓ is a straight line and ℓ' is its transformed line by the rotation, they form an angle θ.*

PROOF: Let O be the center of rotation, choose a point $P \in \ell$ such that OP is orthogonal to ℓ. Then OP' (where $P' = f(P)$) is orthogonal to ℓ' as well, given that f is bijective and thus the circle of center O and radius $|OP|$ must also be tangent to ℓ' (precisely at P'):

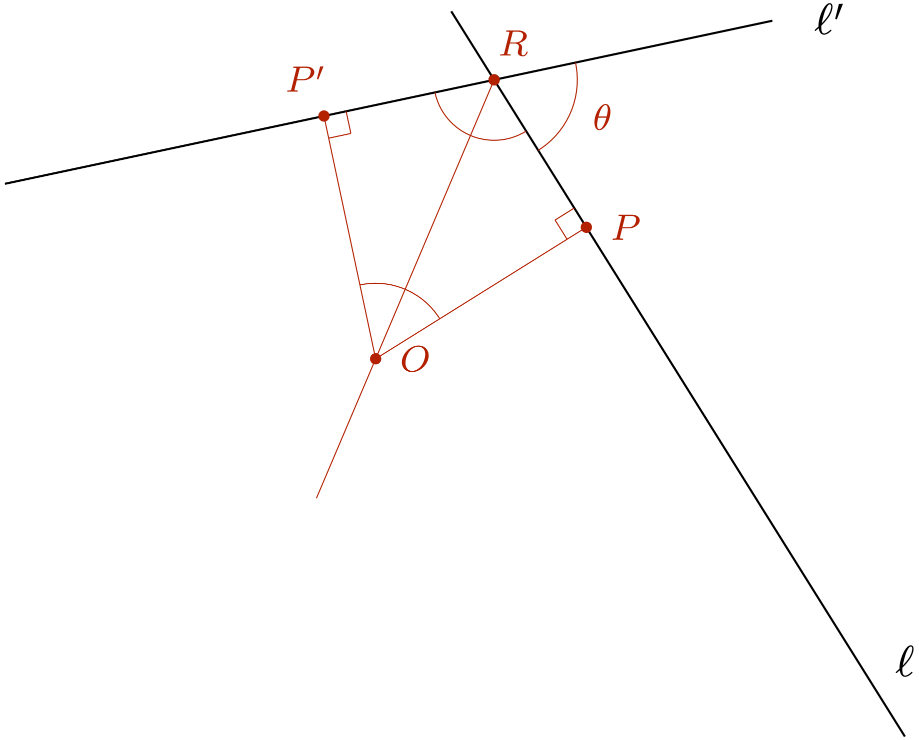


THE TWO RIGHT OPPOSITE ANGLES IMPLY O,P,P' AND $\ell \cap \ell'$ ARE CONCYCLIC.

THUS THE OTHER TWO OPPOSITE ANGLES ARE COMPLEMENTARY, HENCE THE FINAL ANGLE MARKED IN RED.

Lemma 27. *In the above hypotheses, if R is the point of intersection of ℓ and ℓ', then OR is the angle bisector of ℓ and ℓ'.*

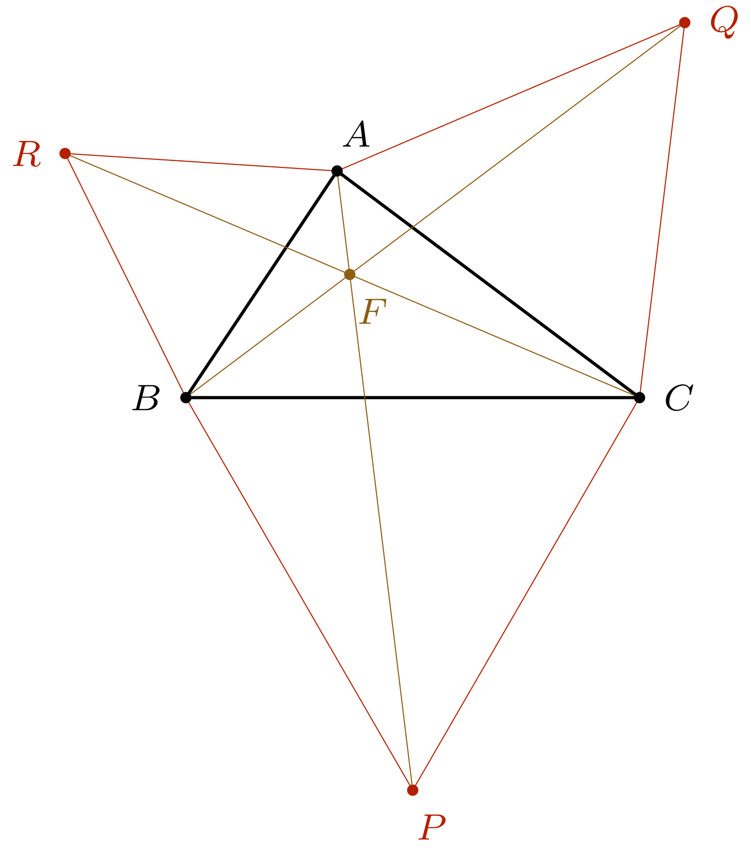
PROOF:



TRIANGLES $OP'R$ AND OPR HAVE TWO SIDES OF EQUAL LENGTH: OP' AND OP, AND THEIR COMMON SIDE OR, AND A COMMON ANGLE $\widehat{OP'R} = \widehat{OPR}$.

THUS BOTH TRIANGLES ARE SIMILAR AND $\widehat{P'RP} = \widehat{ORP}$

Theorem 28. *Let $\triangle ABC$ and place an equilateral triangle on each side:*



(i) Segments RC, AP and BQ intersect and are of equal length. Their point of intersection is called the ***Fermat point*** *of $\triangle ABC$.*

(ii) Furthermore, lines RC, AP and BQ form consecutive angles of $\pi/3$.

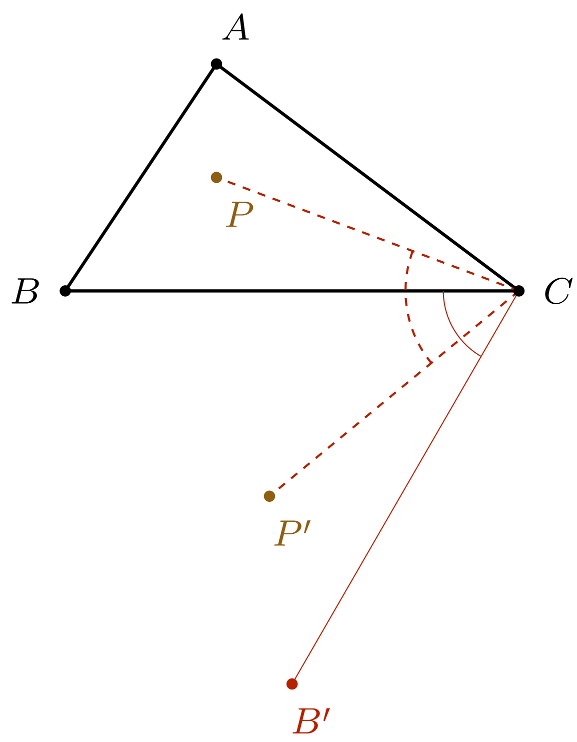
(iii) And each line is an angle bisector of the other two.

PROOF: EXERCISE using properties of rotations.

- An example of an application:

Problem. *Given a triangle ABC, find a point P such that $AP + BP + CP$ is minimal.*

SOLUTION. A rotation of center C and angle $\pi/3$ yields B' and P' below:



Regardless of where this point P that we are looking for lies, what is clear is that $PB = P'B'$ and $P'P = P'C$, thus $AP + BP + CP = AP + PP' + P'B' \geq AB'$ on account of the triangle inequality. The minimal value is obtained if P and P' belong to AB'.

EXERCISE: P is the Fermat point if $\triangle ABC$ satisfies some property with respect to its angles.

Central symmetries

- A **central symmetry** having center O is a transformation of the plane

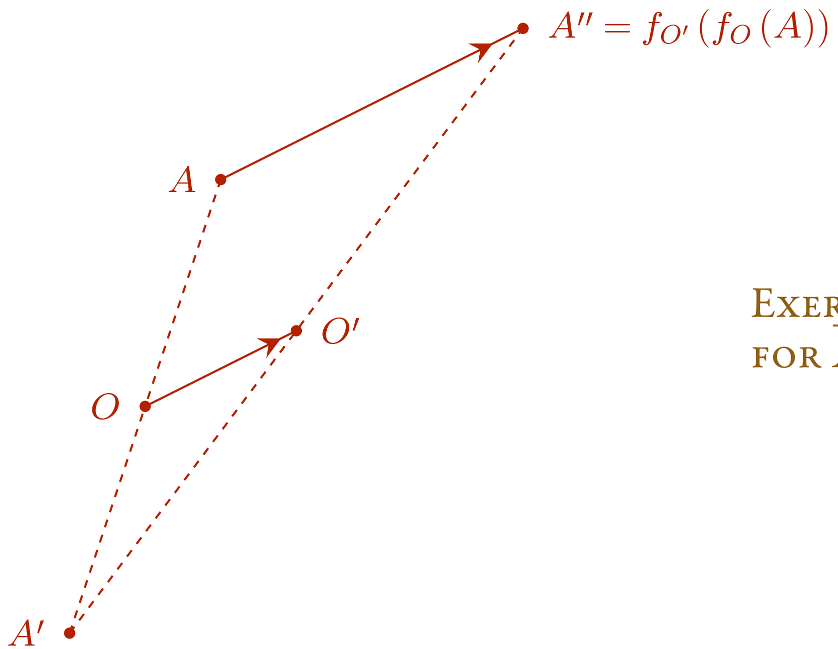
$$\begin{array}{ccc} \mathbb{A}^2_{\mathbb{R}} & \xrightarrow{f_O} & \mathbb{A}^2_{\mathbb{R}} \\ P & \longmapsto & f_O(P) := P' \end{array}$$

such that for every point P, O is the midpoint of segment PP', i.e.

$$d(O,P) = d(O,P') = \frac{1}{2} d(P,P').$$

Proposition 29. *If f_O and $f_{O'}$ are two central symmetries, then their composition $f_{O'} \circ f_O$ is a translation having direction vector $2\overrightarrow{OO'}$.*

PROOF:



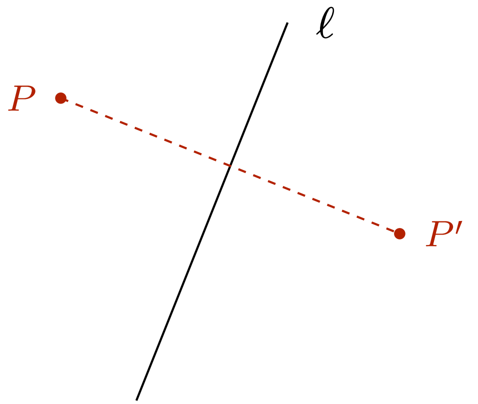
EXERCISE: FILL IN THE BLANKS OF THE ARGUMENT FOR $\overrightarrow{AA''} = 2\overrightarrow{OO'}$

Reflections

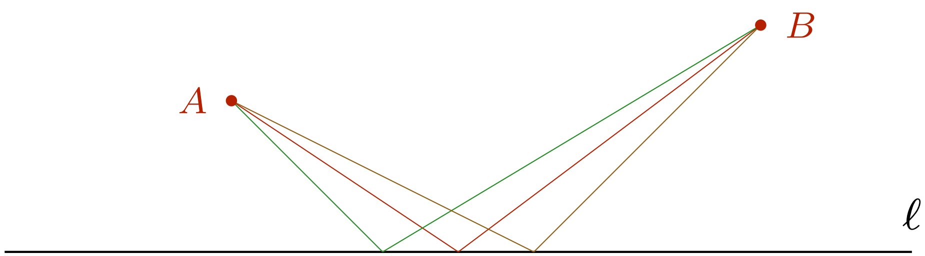
- An **axial symmetry** or **reflection**, with respect to a line ℓ is a transformation of the plane

$$\begin{aligned} \mathbb{A}^2_{\mathbb{R}} &\xrightarrow{\quad f \quad} \mathbb{A}^2_{\mathbb{R}} \\ P &\longmapsto f(P) := P' \end{aligned}$$

  such that ℓ is the perpendicular bisector of segment PP', i.e. the line perpendicular to PP' and intersecting it on its midpoint:

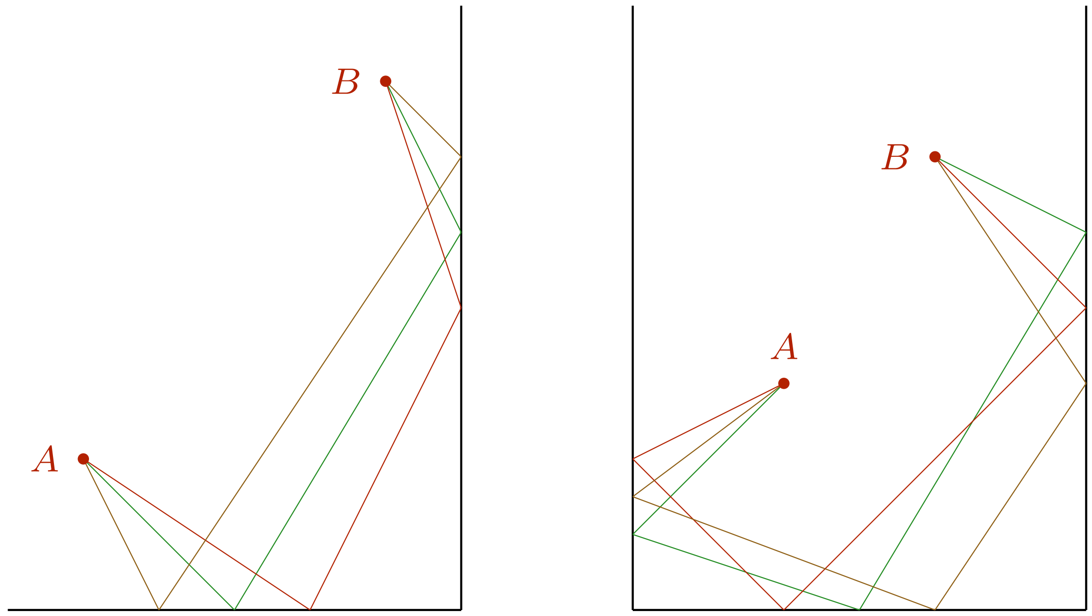


Problem. *Find the minimal-distance trajectory from A to B upon mirror reflection by ℓ:*



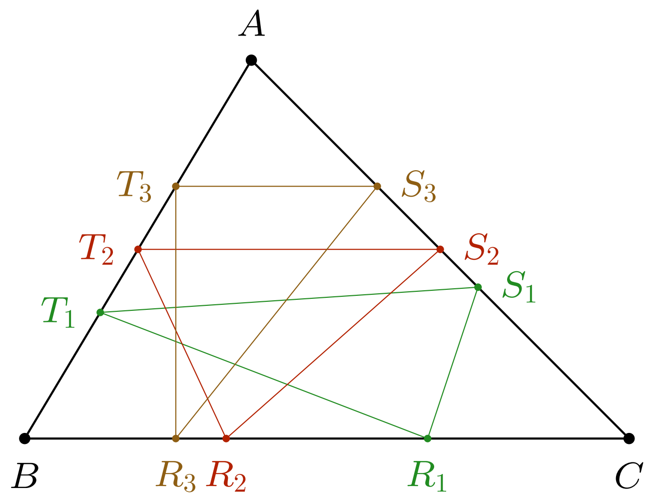
Solution: Exercise.

- Once you solve the above, it is easy to play billiards not only on one side, but on two, three, etc:



Again, EXERCISE.

Problem (Fagnano, circa 1775). *Given a triangle ABC, find an inscribed triangle RST having minimal perimeter.*



Theorem 30. *If ABC is acutangle, the triangle solving Fagnano's problem is the orthic triangle.*

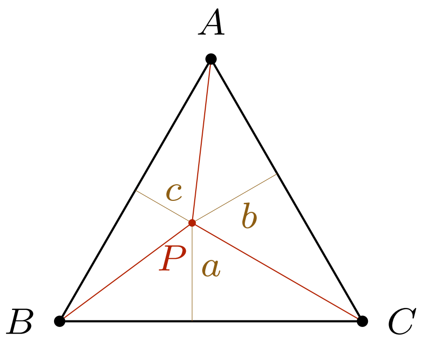
PROOF: EXERCISE.

The Three Jug Problem

- In general, assume we have three containers A, B, C with respectively capacities t_A, t_B, t_C units. There are t units ("liters") distributed among A, B, C hence

$$t = a + b + c, \qquad 0 \leq a \leq t_A, \quad 0 \leq b \leq t_B, \quad 0 \leq c \leq t_C.$$

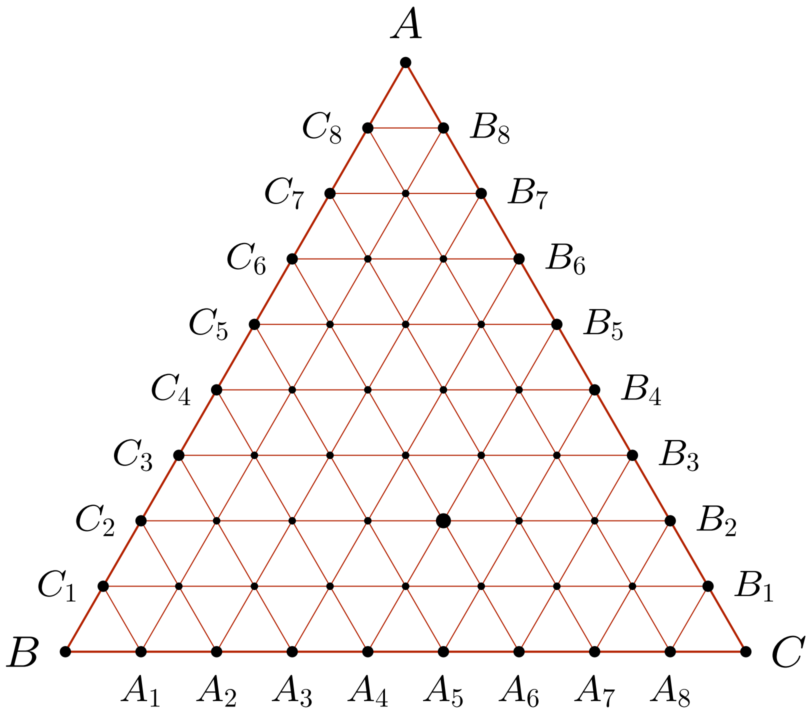
- The problem can be summarized in determining which combinations (a, b, c) are "accessible", knowing that the permissible movements are:
  - emptying one container entirely into another, or
  - entirely filling one container with part of the contents of another.
- Geometrical interpretation: if we consider an equilateral triangle ABC of height t and side ℓ, take a point P such that its distances to the sides are (a, b, c). Given that the area of the total triangle can be expressed in terms of the areas of the three triangles PCA, PAB, PBC,



$$(ABC) = \frac{1}{2} \cdot t \cdot \ell = \frac{1}{2}a\ell + \frac{1}{2}b\ell + \frac{1}{2}c\ell = \frac{1}{2}(a + b + c)\,\ell,$$

hence $a + b + c = t$. (a, b, c) are the **barycentric coordinates** of the triangle.

- Assume we have 9 liters of wine and three jugs holding $6, 5, 8$ liters exactly. How can we sell 7 liters of wine exactly?
  - In the above example, $t = 9$ thus any of the points in the mesh below (including border points A_i, B_j, C_k) could be an admissible integer

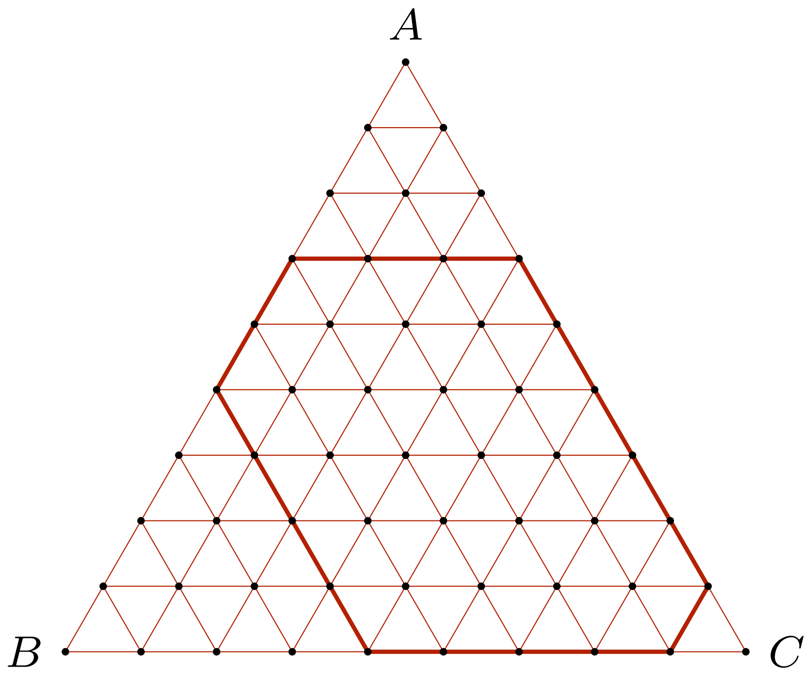


in barycentric coordinates, $A = (9, 0, 0)$, $B = (0, 9, 0)$, $C = (0, 0, 9)$ and the point marked with a larger dot would be $(2, 3, 4)$. Observe that the barycentric coordinates (a, b, c) of the points in this mesh are all the integer nonnegative solutions to $a + b + c = 9$.

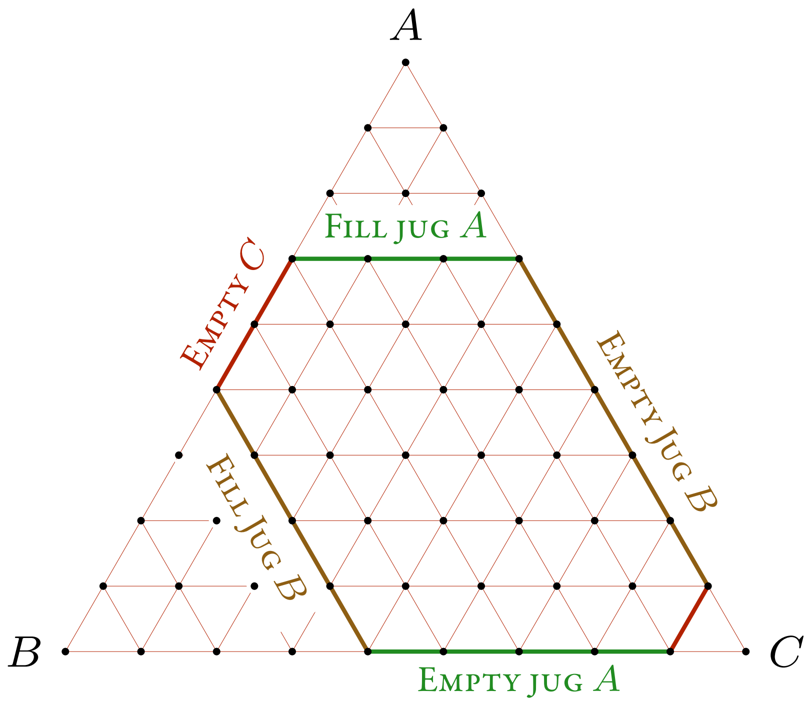
– The condition on the jugs implies

$$0 \leq a = t_A = 6, \qquad 0 \leq b = t_B = 5, \qquad 0 \leq c = t_C = 8,$$

which corresponds to the area delimited below:

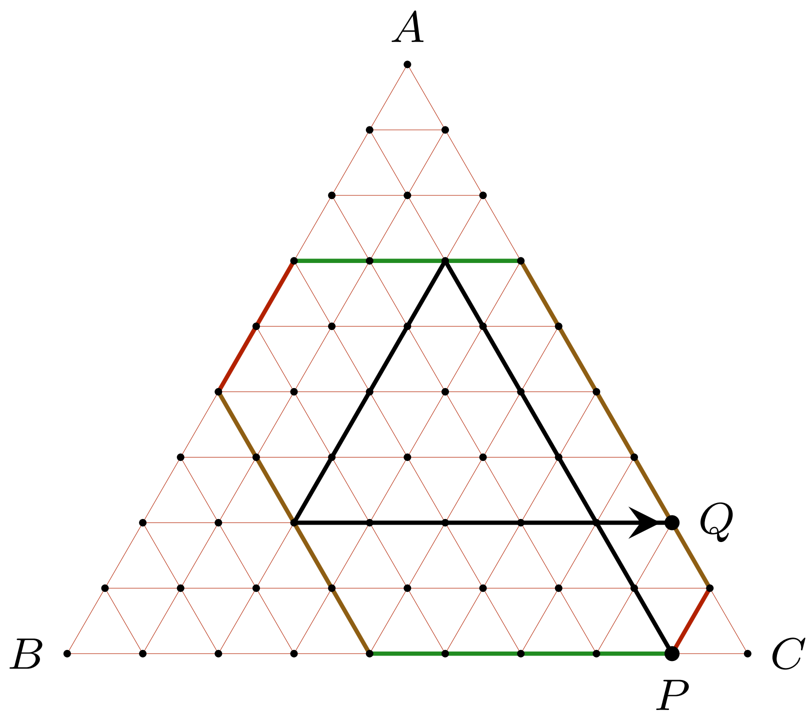


– In terms of what is allowed to be done to the jugs (i.e. filling completely or emptying totally), this would be the representation:



Those actions (i.e. the marked points on the hexagon boundary) are the only ones we are allowed to perform, because they are the only ones that can be performed exactly. Thus, except for all the points *inside* the hexagon are off-limits.

– This is exactly a three-way billiards problem like the one we saw before, only in this case the three "walls" are the sides of an equilateral triangle and our initial point must be a vertex of a hexagon. But our method is still the same: using reflections (Exercise). For instance if we want to obtain seven liters and we start in a situation where jug A is empty and jug C is full, that means jug B has one liter left and our initial point is P below. The method is then:



1. Fill A with contents of C.
2. Fill B with contents of A.
3. Empty B into C.

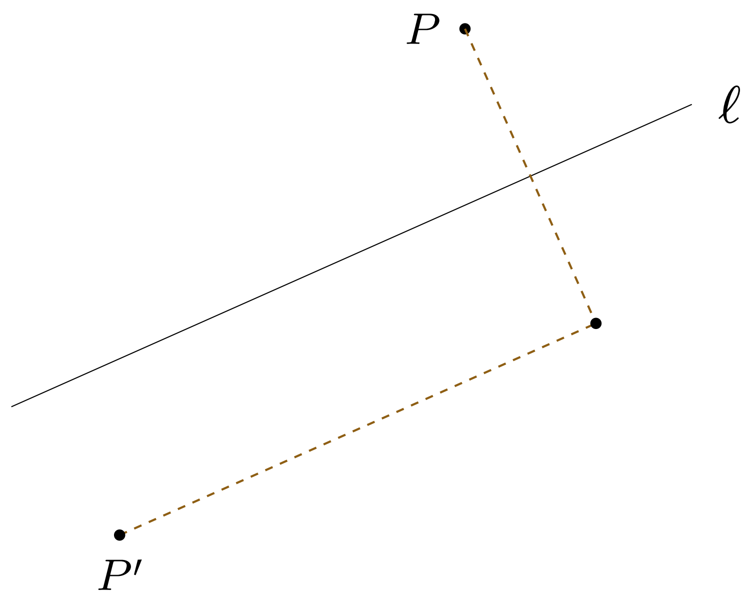
And we know final point Q has barycentric coordinates $(2, 0, 7)$. One of the coordinates, namely that corresponding to C, is 7. Thus we can be sure that the above procedure yields the desired 7 liters in C.

Glide reflections

- A **glide reflection** along a line ℓ is a transformation

$$\begin{aligned} \mathbb{A}^2_{\mathbb{R}} &\xrightarrow{h_{O,k}} \mathbb{A}^2_{\mathbb{R}} \\ P &\longmapsto h_{O,k}(P) := P' \end{aligned}$$

  is a composition of an axial symmetry along ℓ with a translation having a direction vector $\boldsymbol{v} \parallel \ell$.



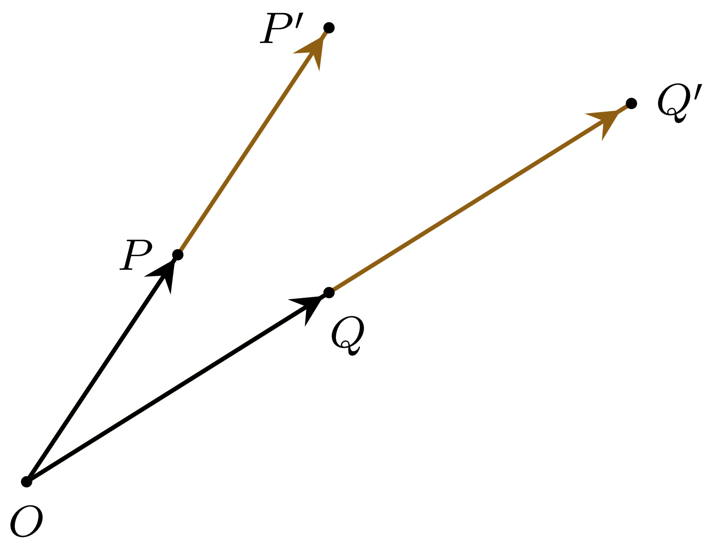
Lemma 31. *Glide symmetries are isometries. They have an invariant line but no invariant points.*

3.2 Homotheties

- A **homothety** or **dilatation** of center O and ratio $k > 0$ is a transformation

$$\begin{aligned} \mathbb{A}^2_{\mathbb{R}} &\xrightarrow{\quad h_{O,k} \quad} \mathbb{A}^2_{\mathbb{R}} \\ P &\longmapsto h_{O,k}(P) := P' \end{aligned}$$

such that $\overrightarrow{OP'} = k\,\overrightarrow{OP}$ for every $P \in \mathbb{A}^2_{\mathbb{R}}$:



HOMOTHETY $h_{O,2}$

Lemma 32. *Homotheties preserve alignment between points and angles.*

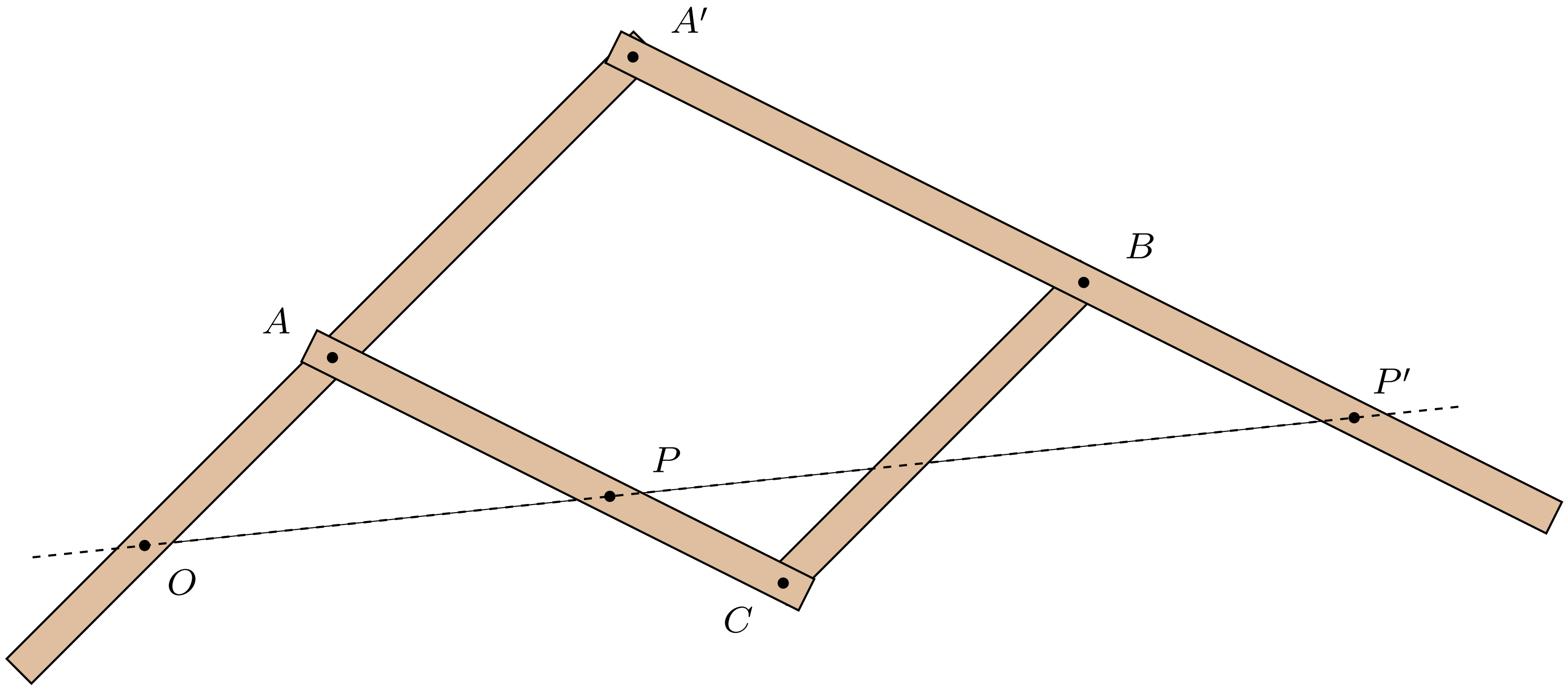
- A transformation of the Euclidean plane that preserves angles is called **conformal**.
  Thus homotheties are not isometries, but they are conformal.
- A **similarity** is a transformation that retains shapes, i.e. a composition of a homothety and an isometry.

Exercises

- Prove everything marked as EXERCISE in this Section.
- In a given triangle ABC, how would you inscribe a segment parallel (and of equal length) to a given segment m?
- What kind of picture do you obtain from an equilateral triangle ABC if we apply translations by vectors that are sums of integer multiples of AB and integer multiples of AC?
- If we build squares over the sides of a parallellogram, their centers are the vertices of a square.
- Build an equilateral triangle such that, given a point in its interior, its distance to the vertices is 2, 3 and 4 units respectively.
- Given a point A exterior to a given circle, build a straight line intersecting the circle in points P and Q in such a way that $AP = PQ$.
- Given the basis and the area of a triangle, prove that the perimeter will be minimal whenever the triangle is isosceles.
- We have a jug containing 12 cubic centimeters full of liquid and two jugs containing 9 and 5 cubic centimeters. How can we divide the liquid in two equal portions?
- Three burglars have stolen a bottle containing 24 ounces of expensive wine. In their flight, they steal three flasks of 13, 11 and 5 ounces respectively. How can they divide it in equal parts?

- If homotheties $h_{O_1,\lambda_1}, h_{O_2,\lambda_2}$ are such that their composition is $h_{O_1,\lambda_1} h_{O_2,\lambda_2} = h_{O,\lambda_1\lambda_2}$, where is O?
- What is the geometric locus described by the midpoint of a segment of variable length if an end point of the segment is fix while the second lies on a circle?
- Given an acutangle triangle ABC, build a square having one side on BC and the other two vertices on CA and AB respectively.
- Prove that the product of three reflections is a reflection if and only if the reflection axes are concurrent or parallel.

- A *pantograph* is an ancient instrument designed to make larger or smaller copies of a given picture. Interior angles of parallellogram $AA'BC$ are variable.



Choose the collinear points O, P, P' are chosen to be collinear. O is fixed, P is draws the original picture, and P' will draw the amplified or smaller copy.

(i) Explain the mechanism.

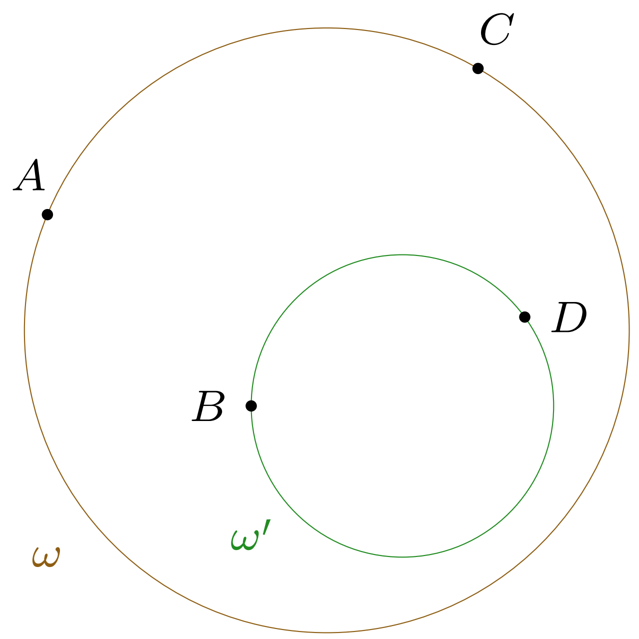
(ii) How does the copy change if we alter the positions of O and P?

(iii) How would you modify the mechanism in order to obtain a dilatation of *negative* ratio k?

4. Inversions

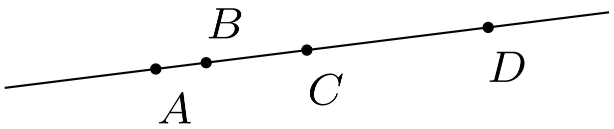
Separation

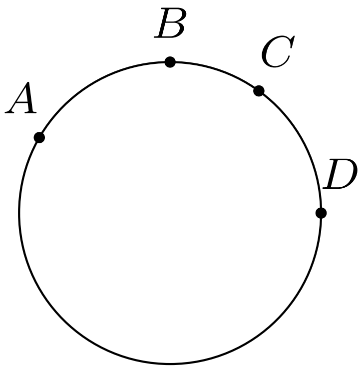
Theorem 33. *If four different points A, B, C, D do not belong to the same line or circle, then there exist two circles ω, ω' of empty intersection such that ω contains A, C and ω' contains B, D.*



PROOF: EXERCISE.

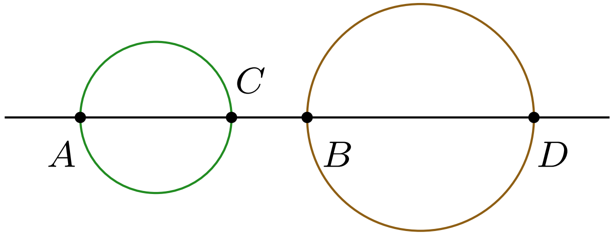
- Two pairs of points A, C and B, D **separate** one another if they all belong to the same circle (or line) and each arc (or segment) joining the points of a pair contains *only one* point of the other:





It is customary to write this as $AC \mathbin{/\!/} BD$.

- If two pairs of collinear or cocyclic points A, C and B, D do not separate one another, one can draw two non-intersecting circumferences containing these respective pairs:
  - if the points are collinear, the circles can be chosen with segments AC and BD as diameters:



  - if the points are concyclic and $AB \mathbin{/\!/} CD$ with arclengths $\widehat{BC} < \widehat{AD}$, we can take the centers of the circles to be the intersection points of BC with angle bisectors of AC and BD, respectively (Exercise: prove this).

- On the other hand, if $AC \not\parallel BD$, any circle ω containing A and C but not B "separates" B and D (i.e. one of them is in, the other is out). Thus ω intersects every circle containing B, D.
- In conclusion,

Proposition 34. *Two pairs of points A, C and B, D separate one another if and only if every circle containing A and C intersects (or coincides with) every circle containing B and D.* □

A different way of characterizing this:

Theorem 35. *Let A, B, C, D be four points in the plane. Then their mutual distances satisfy*

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD,$$

and the above is an equality if and only if $AC \not\parallel BD$.

Proof: Exercise.

- Let A, B, C, D be four points on the plane. The **cross-ratio** is defined as

$$(A, B, C, D) := \frac{AC}{AD} \cdot \frac{BC}{BD}.$$

- So the above theorem translates into

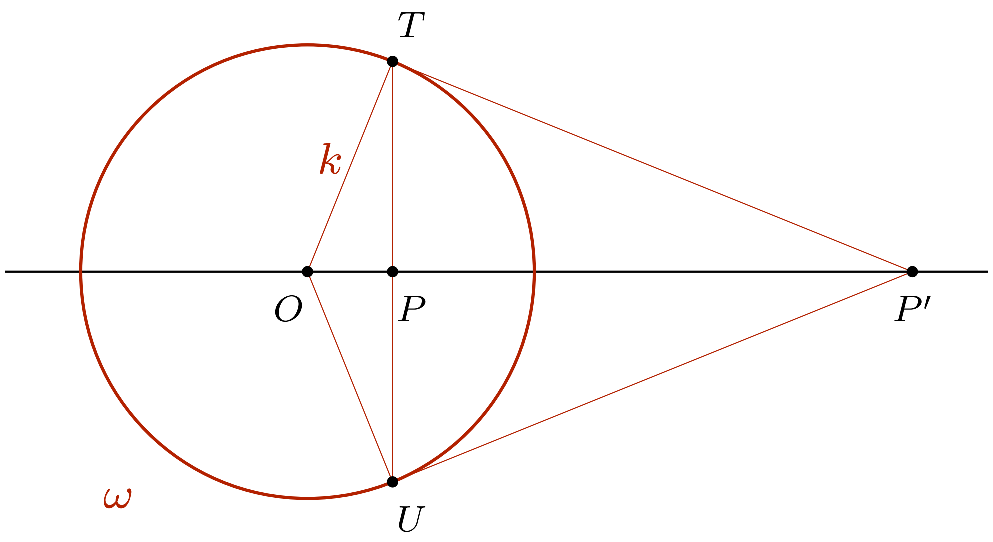
$$\boxed{(A, D, B, C) + (A, B, D, C) = 1 \text{ if, and only if, } AC \not\parallel BD.}$$

Inversions

- Let O be a point of the Euclidean plane and $k > 0$. The **inversion of center O and ratio k** is

$$\begin{aligned} \mathbb{A}^2_{\mathbb{R}} &\xrightarrow{\quad f \quad} \mathbb{A}^2_{\mathbb{R}} \\ P &\longmapsto f(P) := P' \end{aligned}$$

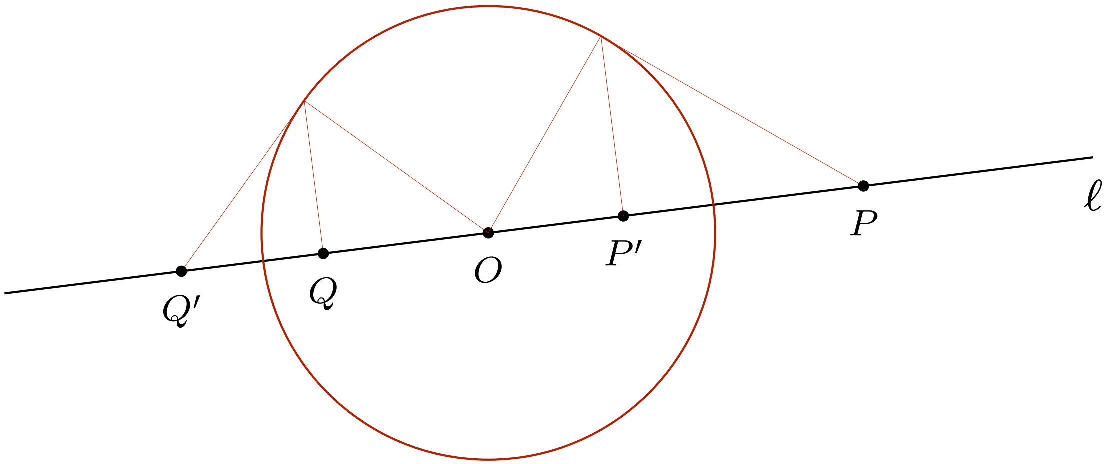
  such that P' belongs to the half-line or ray OP in such a way that $OP \cdot OP' = k^2$.
- This generalizes inversion in one dimension $f(x) = 1/x$ if $O = 0 \in \mathbb{R}$ and $k = 1$.
- A point P is called **self-inverse** or **double** with respect to the inversion if $OP \cdot OP' = OP \cdot OP = k^2$. Thus self-inverse points form a circumference ω called the **circle of inversion**.
- Given the circle of inversion ω, the inverse P' of a point P can be determined geometrically:



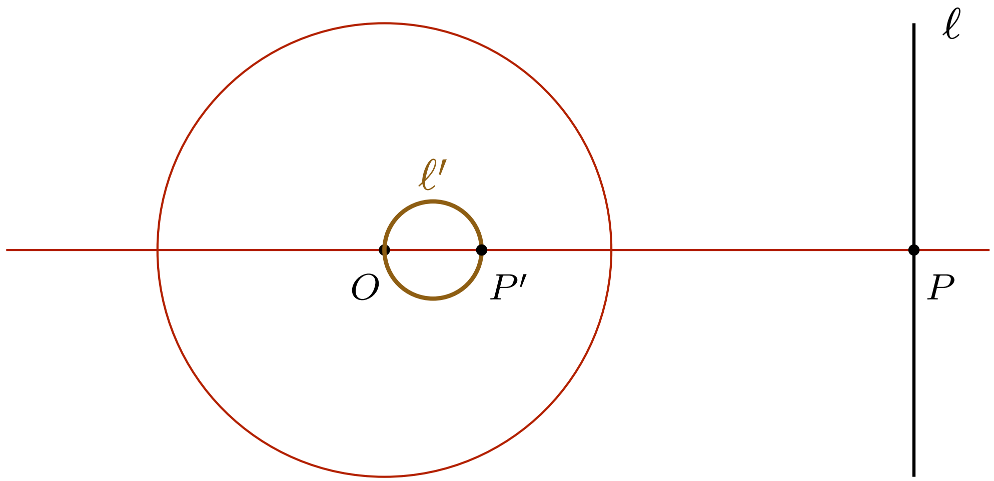
  Exercise: fill in the blanks of the explanation using any of the two tangent segments.
- Inversions are not similarities, thus in particular they are not isometries.

Inverse of a line

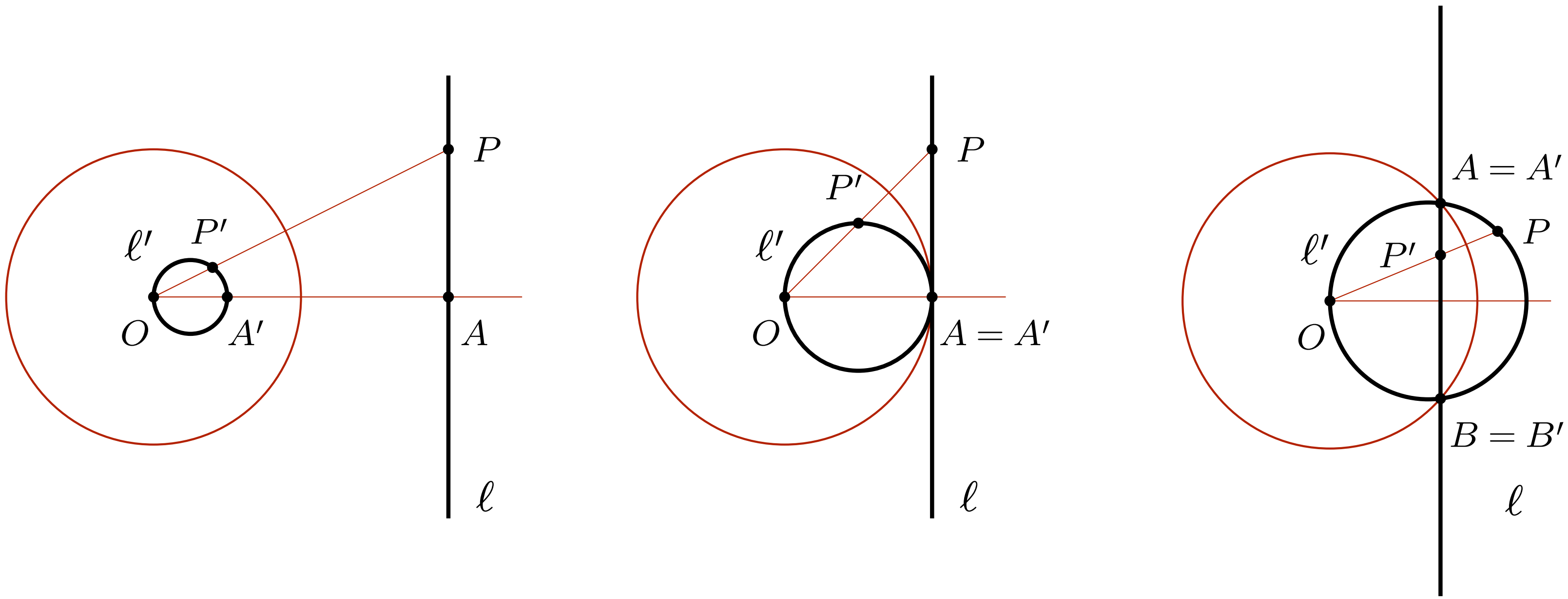
- Let ℓ be a line in $\mathbb{A}^2_{\mathbb{R}}$ and $\ell' = \{P' = f(P) : P \in \ell\}$ be its inverse locus, f being the inversion of center O and ratio k. The question is obvious: $\boxed{\text{what is } \ell'\text{?}}$
  - Case 1: $O \in \ell$. Then $\ell' = \ell$.



  - Case 2: $O \notin \ell$. Then ℓ' is the **circle** having OP' as a diameter, P' being the inverse of the point P of intersection of ℓ with its perpendicular containing O (Exercise: prove this):



- As a consequence, the inverse transform of a circle containing O is a line parallel to the tangent at O to said circle.
- All of the above makes it simple to compute the inverse of any point P:

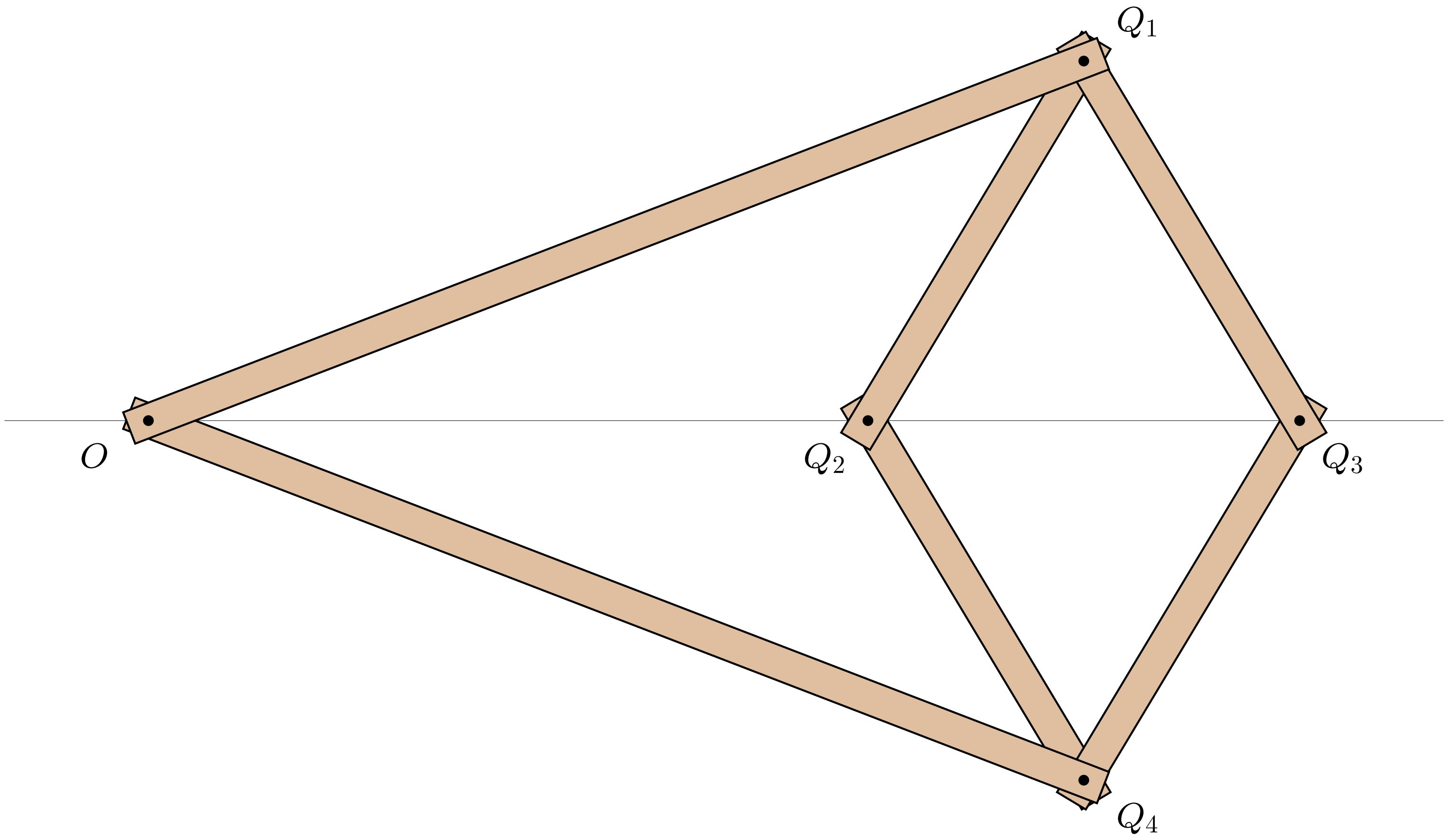


- We can exchange the $'$ indicating inversion between $\star$ and $\star'$ above, because

Lemma 36. *Every inversion f is an **involution**, i.e. $f\left(f\left(P\right)\right) = P$ for every P.*

PROOF: EXERCISE.

- **Peaucellier's inversor** (1864) is a mechanism that can be used to draw lines without a ruler (or circles without a compass):



all that the mechanism needs is the assumption that $d(O, Q_1) = d(O, Q_4) = a$ and

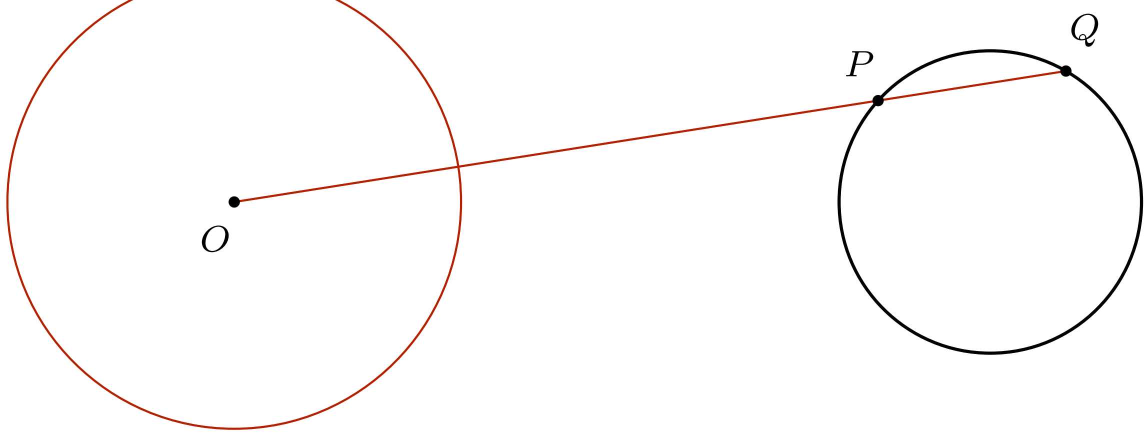
$$d(Q_1, Q_2) = d(Q_4, Q_2) = d(Q_1, Q_3) = d(Q_4, Q_3) = b.$$

EXERCISE:

- find two points above that are mutual inverses; identify the inversion circle ω and ratio k;
- describe how you would draw a line with a compass and the above mechanism.

Inversion of circles

- Let ω be an inversion circle having center O and ratio k.
- Assume χ is another circle not containing O (hence not as easily invertible as the examples above). We want to determine the geometric locus χ'.
- Recall the geometric construction leading to the computation of $p = \operatorname{pow}_\chi(O)$:



$p = OP \cdot OQ$ IF O EXTERIOR TO χ
$p = -OP \cdot OQ$ IF O INTERIOR TO χ

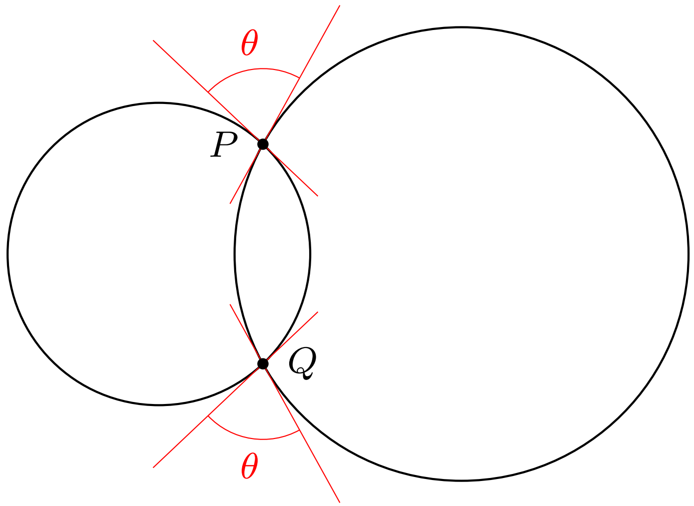
Theorem 37. *Let $p = \operatorname{pow}_\chi(O)$ and χ' the inverse image of χ. Then,*
(i) if O is exterior to χ, χ' is the outcome of applying a homothety of center O and ratio k^2/p to χ.
(ii) If O is interior to χ, χ' is the outcome of applying a homothety of center O and ratio k^2/p, and a central symmetry to χ.

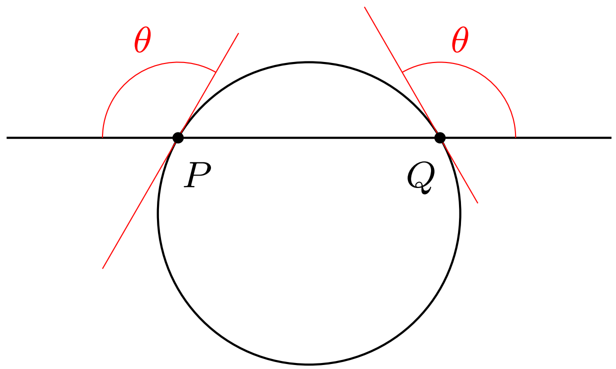
Corollary 38. *The inverse image χ' of a circle is a circle.*

PROOF OF BOTH RESULTS: EXERCISE.

Preservation of angles

- Assume we have two curves α, β intersecting in a point P. The angle formed by α and β is that of their tangents at that point.
- If α and β are circles and intersect in two points, then their angles are the same for both points of intersection. Ditto if α or β is a line:





Theorem 39. *Let ω be a circle of inversion.*

(i) If ℓ_1 and ℓ_2 are two lines intersecting at a point P, then the angle formed by ℓ_1' and ℓ_2' at P' equals the angle formed by ℓ_1 and ℓ_2 at P.

(ii) If circle χ and line ℓ intersect in P, their angle at P is also preserved in χ' and ℓ' at P'.

PROOF: EXERCISE.

Preservation of cross-ratio and separation

- Let ω be a circle of inversion having center O and radius k. Let $A, B, C, D \in \mathbb{A}^2_{\mathbb{R}} \setminus \{O\}$ and A', B', C', D' be their inverses.

Lemma 40. *In the above hypotheses,* $\dfrac{A'B'}{AB} = \dfrac{k^2}{OA \cdot OB}$.

Proof: Triangles $OA'B'$ and OAB have a common angle θ and

$$OA \cdot OA' = k^2 = OB \cdot OB'.$$

Thus $\dfrac{OA}{OB'} = \dfrac{OB}{OA'}$ hence $\triangle OA'B' \sim \triangle OAB$ thus

$$\frac{AB}{A'B'} = \frac{OB}{OA'} = \frac{OB}{OA'} \cdot \frac{OA}{OA} = \frac{OA \cdot OB}{k^2}.$$

Immediate consequence: the preservation of cross-ratio upon inversion:

Theorem 41. *In the above hypotheses,* $(A', B', C', D') = (A, B, C, D)$.

Proof:

$$(A', B', C', D') = \frac{A'C' \cdot B'D'}{A'D' \cdot B'C'} = \frac{AC\frac{OA \cdot OC}{k^2} \cdot BD\frac{OB \cdot OD}{k^2}}{AD\frac{OA \cdot OD}{k^2} \cdot BC\frac{OB \cdot OC}{k^2}} = \frac{AC \cdot BD}{AD \cdot BC} = (A, B, C, D).$$

- This, in particular, implies the following:

Theorem 42. *In the above hypotheses, if $AC \mathbin{/\!/} BD$, then $A'C' \mathbin{/\!/} B'D'$.*

Proof: the fact $AC \mathbin{/\!/} BD$ implies

$$(A,D,B,C)+(A,B,D,C)=1,$$

but the preservation of cross-rations means

$$(A,D,B,C)=(A',D',B',C'), \quad (A,B,D,C)=(A',B',D',C'),$$

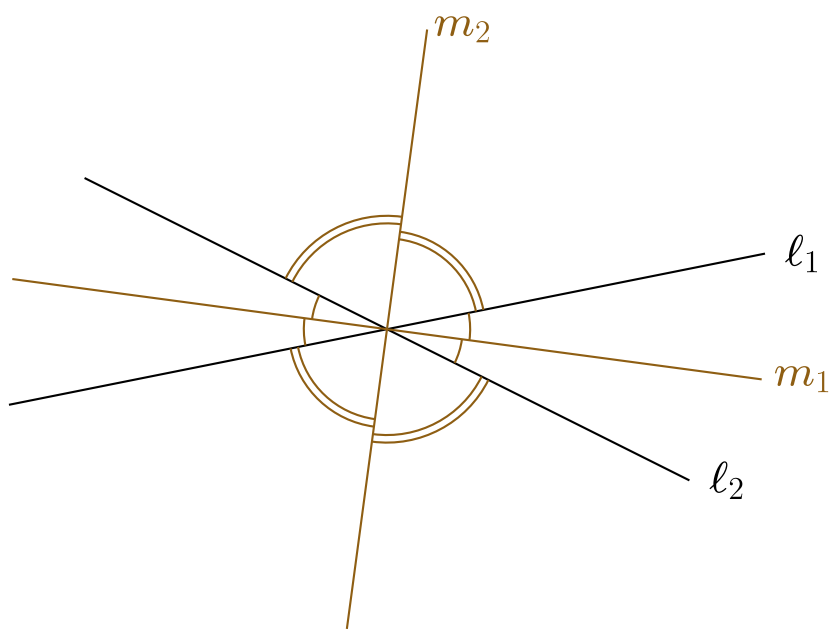
thus

$$(A',D',B',C')+(A',B',D',C')=1$$

and thus A, D separate B, C.

Angle bisectors of circles

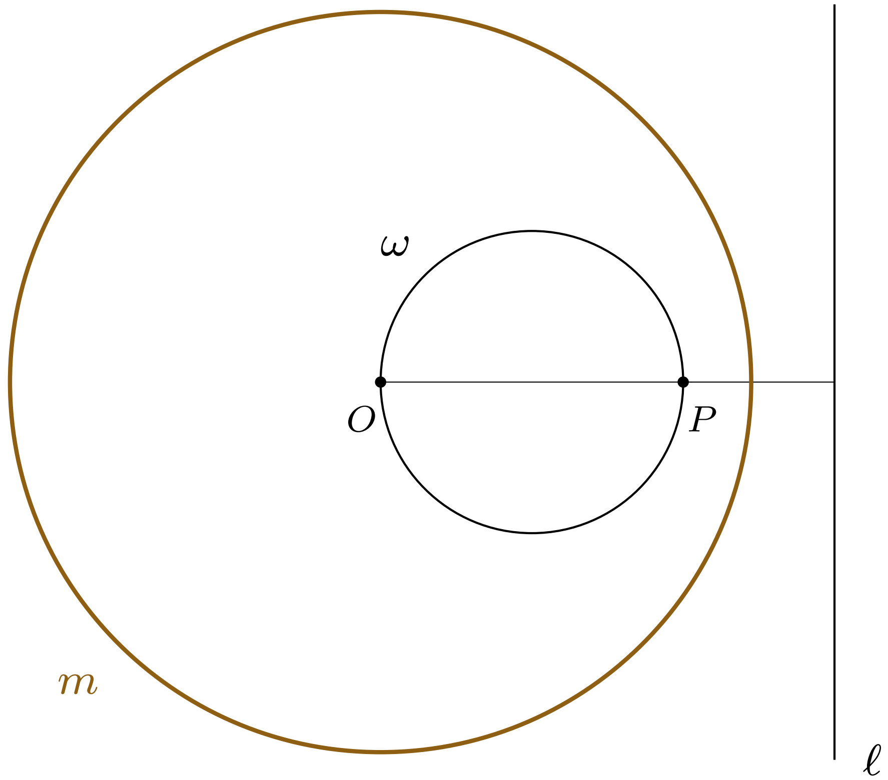
- Let ℓ_1 and ℓ_2 be two lines. If they intersect in a given point, they determine two angle bisectors m_1 and m_2 that are perpendicular to one another:



in other words, reflection by m_i $(i = 1, 2)$ swaps ℓ_1 and ℓ_2.

- The question becomes, how can we define an analogue for two circles or a line and a circle?
  - if $m, n \in \{\text{all circumferences and straight lines}\} = S$, then an angle bisector is a $p \in S$ such that the "symmetry" with respect to p swaps m and n.
  - And if m, n happen to be of the same type and parallel, then p is of the same type and "equidistant between" them.

Theorem 43. *If ω is a circle and ℓ is a line and they have empty intersection, then they have a unique angle bisector m which is the circumference having center O (point in ω farthest from ℓ) and radius k such that $k^2 = OP \cdot OP'$:*

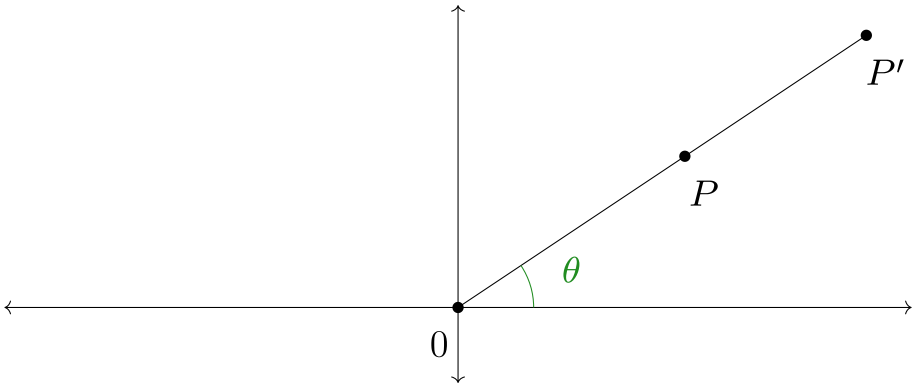


Proof: Exercise.

- Exercise: describe the angle bisector of m and n in the following cases:
  - m and n are two parallel lines.
  - m and n are two circles intersecting in two points.
  - m and n are two circles tangent in one point.
  - m and n are two circles that do not intersect.

Inversions vs homographies

- Let us consider "the" complex structure available to $\mathbb{A}^2_{\mathbb{R}}$ by identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the obvious manner $a + b\mathrm{i} \leftrightarrow (a, b)$.
- Assume we start by studying the inversion with respect to $0 \in \mathbb{C}$ and a given radius:



identifying P and P' with two complex numbers z, z' by way of the aforementioned manner and imposing $OP \cdot OP' = k^2$, we have $|z|\,|z'| = k^2$ and thus

$$z = \rho e^{\mathrm{i}\theta} \quad \Rightarrow \quad z' = \frac{k^2}{\rho} e^{\mathrm{i}\theta} = \frac{k^2}{\rho e^{-\mathrm{i}\theta}},$$

in other words, we have proved the following:

Lemma 44. *The inverse with respect to center $0 \in \mathbb{C}$ and radius k can be expressed as* $\boxed{z' = \dfrac{k^2}{\overline{z}}}$.

- A similar argument yields the following whenever we change the center of inversion:

Lemma 45. *The inverse with respect to center $u \in \mathbb{C}$ and radius k can be expressed as*

$$\boxed{z' = \frac{u\overline{z} + \left(k^2 - u \cdot \overline{u}\right)}{\overline{z} - \overline{u}}.}$$

Proof: Exercise.

Corollary 46. *All inversions are transformations of the (extended) complex plane $\mathbb{C}^\star = \mathbb{C} \cup \{\infty\}$*

$$z' = \frac{a\overline{z} + b}{c\overline{z} + d}, \qquad \textit{with } ad - bc \neq 0.$$

- In general, a **homography** and an **anti-homography** on $\mathbb{C}^\star$ are defined, respectively, as

$$z' = \frac{az + b}{cz + d}, \qquad z' = \frac{a\overline{z} + b}{c\overline{z} + d}.$$

Proposition 47.

(i) The composition of an even number of inversions is a homography.

(ii) The composition of an odd number of inversions is an anti-homography.

Proof: Exercise.

Proposition 48. *Homographies and anti-homographies preserve angles, i.e. are* conformal.

PROOF: EXERCISE.

- Observe that an anti-homography can be expressed as the composition of axial symmetry $z \mapsto \overline{z}$ and a homography.
- In general, there is a theorem in Analysis that will not be proved here, that states that every $\mathcal{C}^\infty$ function preserving angles is either a homography or an anti-homography. In other words: if $\mathrm{Conf}(2, \mathbb{R})$ is the group formed by all conformal transformations of the extended plane, then

$$\mathrm{Conf}(2, \mathbb{R}) = \underbrace{\mathrm{Conf}^+(2, \mathbb{R})}_{\text{homographies}} \cup \underbrace{\mathrm{Conf}^-(2, \mathbb{R})}_{\text{anti-homographies}},$$

and we can represent the transformations we have been seeing so far as elements of $\mathrm{Conf}(2, \mathbb{R})$:
  - **Similarities**,
    - **Isometries**,
      - Translations: $f(z) = z + b$.
      - Rotations: e.g. $f(z) = az, \quad |a| = 1$.
      - Reflections: e.g. $f(z) = \overline{z}$.
    - **Homotheties**, e.g. $f(z) = az, \quad a \in \mathbb{R}_+$.
  - **Inversions**, e.g. $f(z) = \dfrac{a}{\overline{z}}, \quad a \in \mathbb{R}_+$.

Exercises

- Prove everything marked as an EXERCISE in the section.
- Build the inverse image of a square circumscribing the circle of inversion.
- Find the coordinates of the inverse of a point (x, y) with respect to the circle $x^2 + y^2 = k^2$.
- What are the positions of the center O for which the sides of a triangle invert into three congruent (i.e, equal-radius) circles?
- Let A be external to ω, A' its inverse and P a variable point on ω. Prove that the ratio PA/PA' is constant. Conversely, if B and C divide a segment AA' internally and externally by a given ratio (different from 1), prove that the circle having diameter BC is the geometric locus of points whose distances to A and B are in the given ratio (this circle is known as the *circle of Apollonius*).
- Prove that two points P and Q interior to a circle ω, lie on only two circles tangent to ω.
- Prove that given a circle ω having center O, the inverse of another circle α containing O is the radical axis of α and ω.
- Given a circle ω and an external point A, construct the circle having center A, orthogonal to ω.
- Given a circle ω and two external points P and Q, not mutually inverse, construct the circle containing both points and orthogonal to ω.

II Axiomatic Geometry

1. Absolute and neutral geometries

- There are roughly four families of axioms in Geometry, namely those due to:
  - Euclid of Alexandria ($\sim$ 300 BC),
  - David Hilbert (1862–1943),
  - George David Birkhoff (1884–1944),
  - Felix Klein (1849–1925).

  The first two did not use the existence of $\mathbb{R}$ and its own axiomatics as an assumption, whereas the latter two did. We will focus mainly on the third of these families.
- A **geometry (of the plane)** consists of two sets $\mathcal{P}$ and $\mathcal{L}$.
  - Elements P of $\mathcal{P}$ are called **points**. The entire set $\mathcal{P}$ is called the **plane**.
  - Elements ℓ of $\mathcal{L}$ are called **lines**.
  - There exist three types of relations that need to be explored:
    * **incidence**, i.e. "belonging";
    * **order**, i.e. "being in between",
    * **congruence**, i.e. "being equivalent".
  - and three sets of axioms must be satisfied by these relations:
    axioms of incidence, **axioms of order** and **axioms of congruence**.

Axioms of incidence

- There exists a relation between points $P \in \mathcal{P}$ and lines $\ell \in \mathcal{L}$, named **incidence relation** $A \subseteq \mathcal{P} \times \mathcal{L}$, i.e. comprised of elements of the form (P, ℓ). We write $\boxed{P \in \ell}$ and say e.g.
  - P **belongs to** or **is a point of** or **lies upon** ℓ, or
  - ℓ **goes through** or **contains** P.
- This relation must satisfy the following axioms:

(IA.1) For every two points $P, Q \in \mathcal{P}$, $P \neq Q$, there is a unique line $\ell \in \mathcal{L}$ containing P and Q.

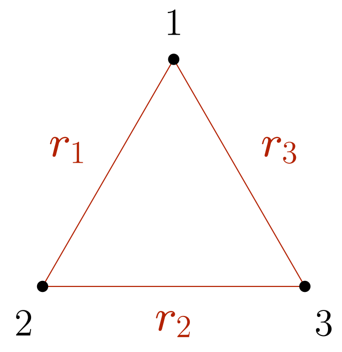
(IA.2) There exist at least 3 non-aligned points, i.e. P, Q, R for which there exists no $\ell \in \mathcal{L}$ such that $(P, \ell), (Q, \ell), (R, \ell) \in A$.

(IA.3) Every line has at least two different points, i.e. for every $\ell \in \mathcal{L}$ there are at least two points $P, Q \in \mathcal{P}$ such that $(P, \ell), (Q, \ell) \in A$.

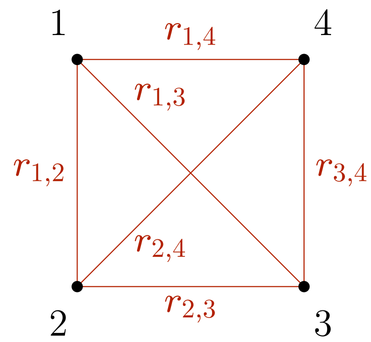
- Any pair $\mathcal{P}, \mathcal{L}$ with an incidence relation A satisfying axioms (IA.1), (IA.2), (IA.3) is called an **incidence geometry**.
- Two lines r, s in an incidence geometry $(\mathcal{P}, \mathcal{L}, A)$ **intersect** if there exists a point $P \in \mathcal{P}$ such that $(P, r), (P, s) \in A$.

Examples:

- The **minimal** incidence geometry is $\mathcal{P} = \{1,2,3\}$, $\mathcal{L} = \{r_1, r_2, r_3\}$ given by $A = \{(1,r_1), (2,r_1), (2,r_2), (3,r_2), (3,r_3), (1,r_3)\}$:



- If we increase the number of points by one, this is the smallest geometry of incidence we can build from $\mathcal{P}$: $A = \left\{(i, r_{i,j}) : 1 \leq i < j \leq 4\right\}$,

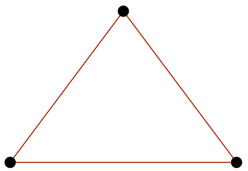


Theorem 49. *Two different lines in an incidence geometry either have no point in common, or intersect in a single point.*

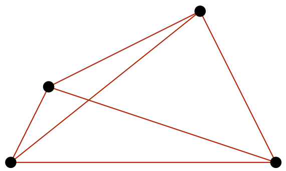
PROOF: if r and s have two different points in common, then $(\text{IA}.1)$ would imply $r = s$, contradiction. □

- If two lines $r, s \in \mathcal{L}$ do not intersect in a given incidence geometry, we say they are **parallel**.
- A geometry is called:
  - **Euclidean** if for every $r \in \mathcal{L}$ and $P \in \mathcal{P}$, if $P \notin r$ then there exists a *unique* line s through P parallel to r;
  - **elliptic** if for every $r \in \mathcal{L}$ and $P \in \mathcal{P}$, if $P \notin r$ then there exists *no* line s through P parallel to r;
  - **hyperbolic** if for every line r and point P, if $P \notin r$ then there exists *more than one* line s through P parallel to r.

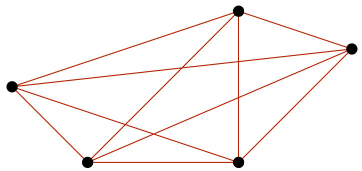
for instance the first two examples in the previous page were elliptic and Euclidean, respectively:



ELLIPTIC



EUCLIDEAN



HYPERBOLIC

Axioms of order

- Remember that the straight line $\mathbb{R}$ is (totally) ordered by a well-known relation $\leq$. In such context, "betweeness" can be characterized as:
  - $x \in \mathbb{R}$ is between y and z if, and only if, $y \leq x \in z$ or $z \leq x \leq y$.
- In the wider context of a geometry of incidence $\mathcal{P}, \mathcal{L}$, define for every $r \in \mathcal{L}$ the following subset of $\mathcal{P}$:

$$\mathcal{P}_r := \{P \in \mathcal{P} : P \in r\}.$$

  and the ternary relation $B_r \subseteq \mathcal{P}_r \times \mathcal{P}_r \times \mathcal{P}_r$ such that if $(x, y, z) \in B_r$, then $x, y, z \in r$. Then we say **x is (or lies) between y and z**.
- Before we specify the axioms of this relation, we need to define the following concepts:
  - A **parametrization** of a line $r \in \mathcal{L}$ is a bijective function $\varphi : \mathcal{P}_r \to \mathbb{R}$ preserving "betweeness", i.e. $P \in r$ lies between $Q_1, Q_2 \in r$ if and only if $\varphi(P)$ lies between $\varphi(Q_1)$ and $\varphi(Q_2)$.
  - Given $P, Q \in \mathcal{P}$, the **segment** defined by P and Q is

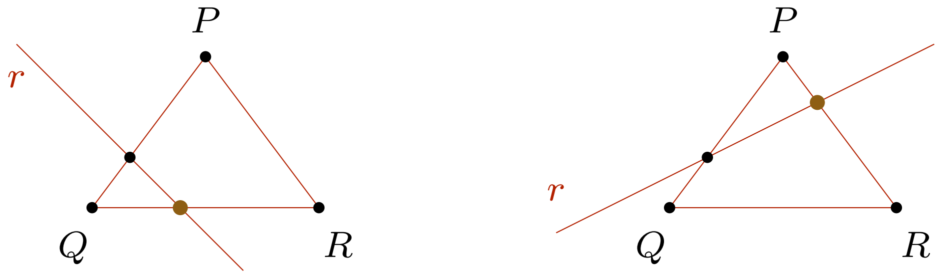
$$\overline{PQ} := \{R \in \mathcal{P}_r : R \text{ lies between } P \text{ and } Q\}.$$

    Alternatively, if $\varphi : \mathcal{P}_r \to \mathbb{R}$ is a parametrization, then the segment in the geometry is the preimage of the segment in $\mathbb{R}$: $\overline{PQ} = \varphi^{-1}\left(\overline{\varphi(P)\varphi(Q)}\right)$ where $\overline{\varphi(P)\varphi(Q)} \subset \mathbb{R}$ is either equal to $[\varphi(P), \varphi(Q)]$ or $[\varphi(Q), \varphi(P)]$.

- We are now in a position to list the two axioms that need to be satisfied by the relation:

(OA.1) For every $r \in \mathcal{L}$, there exists a parametrization $\varphi : \mathcal{P}_r \to \mathbb{R}$ of r.

(OA.2) (*Pasch's axiom*) Let $P, Q, R \in \mathcal{P}$ not aligned, and r a line not containing any of the three points. If r intersects the segment $\overline{PQ}$, then r intersects $\overline{PR}$ or $\overline{QR}$, but only one of them.



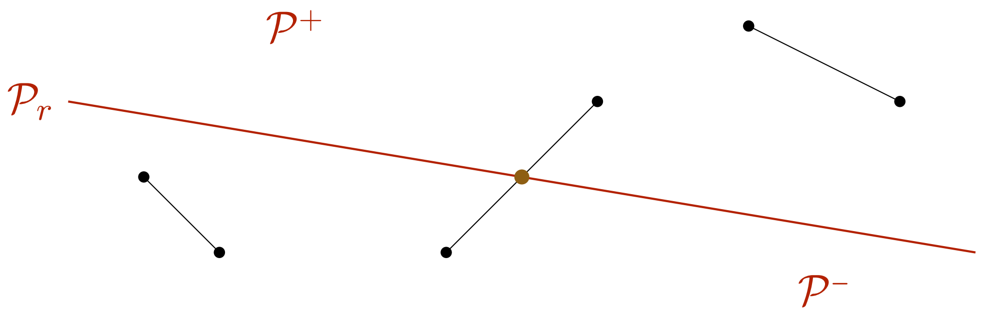
Theorem 50. *If r is a line, then the plane $\mathcal{P}$ has the following half-plane partition:*

$$\mathcal{P} = \mathcal{P}^+ \cup \mathcal{P}_r \cup \mathcal{P}^-.$$

where $\mathcal{P}_r := \{P \in \mathcal{P} : P \in r\}$ has already been defined and

(i) if $A, B \in \mathcal{P}^+$ or $A, B \in \mathcal{P}^-$, then $\overline{AB}$ does not intersect $\mathcal{P}_r$;

(ii) if $A \in \mathcal{P}^+$ and $B \in \mathcal{P}^-$, then $\overline{AB}$ intersects $\mathcal{P}_r$.



PROOF: incidence axiom (IA.2) implies the existence of at least three non-aligned points, thus there exists a $P \notin \mathcal{P}_r$. Define:

$$\begin{aligned}
\mathcal{P}^+ &:= \left\{A \in \mathcal{P} \setminus \mathcal{P}_r : A \neq P \text{ and } \overline{AP} \cap \mathcal{P}_r = \varnothing\right\} \cup \{P\}, \\
\mathcal{P}^- &:= \left\{A \in \mathcal{P} \setminus \mathcal{P}_r : A \neq P \text{ and } \overline{AP} \cap \mathcal{P}_r \neq \varnothing\right\}.
\end{aligned}$$

Then $\boxed{\mathcal{P} = \mathcal{P}^+ \,\dot\cup\, \mathcal{P}_r \,\dot\cup\, \mathcal{P}^-}$ because:

- $\mathcal{P}^+ \cap \mathcal{P}^- = \varnothing$ because $\mathcal{P}^\pm$ are defined by mutually contradicting conditions.
- $\mathcal{P}_r$ has empty intersection with either of them because they are defined as subsets of $\mathcal{P} \setminus \mathcal{P}_r$.
- And every $A \in \mathcal{P}$ belongs to one of the subsets.

Let $A, B \in \mathcal{P}^+$, both different from P. We must prove that $\boxed{\overline{AB} \cap \mathcal{P}_r = \varnothing.}$ Indeed, our hypothesis concerning A, B implies $\overline{PA} \cap \mathcal{P}_r = \overline{PB} \cap \mathcal{P}_r = \varnothing$. If $\mathcal{P}_r \cap \overline{AB} \neq \varnothing$, then Pasch's axiom (OA.2) would imply $\mathcal{P}_r \cap \overline{PA} \neq \varnothing$ or $\mathcal{P}_r \cap \overline{PB} \neq \varnothing$, contradiction.

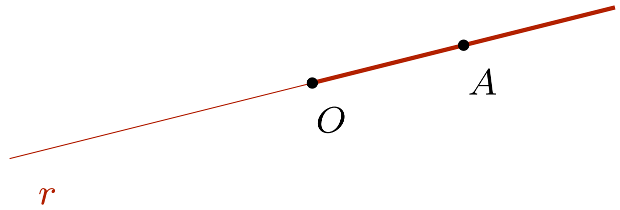
If $A, B \in \mathcal{P}^-$, then $\overline{AB} \cap \mathcal{P}_r = \varnothing$. Indeed, again due to Pasch's axiom (OA.2), the fact that $\mathcal{P}_r$ intersects both segments and contains none of the three points implies it cannot intersect the third segment AB.

Finally, assume $A \in \mathcal{P}^+, B \in \mathcal{P}^-$. Then $\boxed{\overline{AB} \cap \mathcal{P}_r \neq \varnothing.}$ Indeed, $\overline{PA} \cap \mathcal{P}_r = \varnothing$ and $\overline{PB} \cap \mathcal{P}_r \neq \varnothing$, hence once again (OA.2) implies $\overline{AB} \cap \mathcal{P}_r \neq \varnothing$.

Angles

- Let r be a line, and $O, A \in r$ two points. The **ray** or **half-line** having origin O and determined by A is the set of points

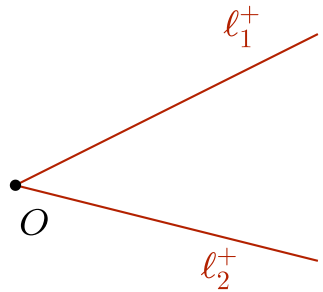
$$\{P \in \mathcal{P}_r : O \text{ is not between } A \text{ and } P\} \cup \{O\}$$



- The following is immediate:

Lemma 51. *If $\varphi : \mathcal{P}_r \to \mathbb{R}$ is a parametrization of $\mathcal{P}_r$ such that $\varphi(O) = 0$, $\varphi(A) = a$ and $0 < a$, then $O \neq A$ and the ray having origin O and determined by A is $\varphi^{-1}([0, \infty))$.*

- We will denote rays with a $+$ superscript, e.g. r^+, s^+, ℓ^+, etc.
- Let O be a point. An **angle** of vertex O is a pair of rays ℓ_1^+, ℓ_2^+ having origin O and such that they are not contained in the same line. It is customary to write it as $\widehat{\ell_1^+ \ell_2^+}$.



- Note that this definition prevents the existence of angle 0, flat angles ($2\pi k$, $k \in \mathbb{Z}$) or angles $\geq \pi$.

Lemma 52. *Let r be a line, O a point of r and ℓ^+ a ray having origin O. Then either*

(i) $\ell^+ \setminus O \subseteq \mathcal{P}_r^+$,

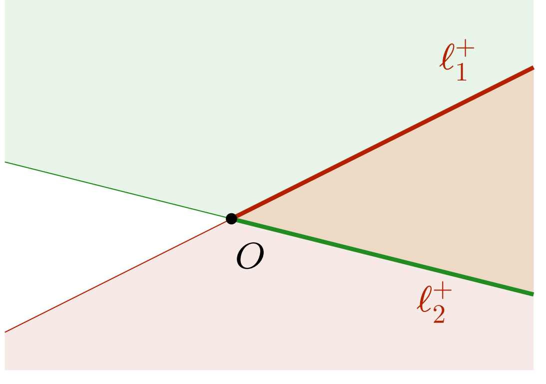
(ii) *or* $\ell^+ \subseteq \mathcal{P}_r$,

(iii) *or* $\ell^+ \setminus O \subseteq \mathcal{P}_r^-$.

PROOF: let $A \in \ell^+ \setminus \{O\}$. Then either $A \in \mathcal{P}_r^+$ or $A \in \mathcal{P}_r$ or $A \in \mathcal{P}_r^-$ because all three sets form a partition of $\mathcal{P}$.

- If $A \in \mathcal{P}_r^+$ and $B \in \ell^+ \setminus \{O\}$ then $\overline{AB} \cap \mathcal{P}_r = \varnothing$, thus $B \in \mathcal{P}_r^+$.
- If $A \in \mathcal{P}_r$ then $O \in r$ and $A \in r$, which means $\ell = r$.
- If $A \in \mathcal{P}_r^-$ then again for $B \in \ell^+ \setminus O$, $\overline{AB} \cap \mathcal{P}_r = \varnothing$, thus $B \in \mathcal{P}_r^-$. $\square$

- Assume $\widehat{\ell_1^+ \ell_2^+}$ is an angle having vertex O, and let $\mathcal{P}_1^+$ be the half-plane such that $\ell_2^+ \subseteq \mathcal{P}_1^+$. Let $\mathcal{P}_2^+$ the half-plane such that $\ell_1^+ \subseteq \mathcal{P}_2^+$. The **angular region** of angle $\widehat{\ell_1^+ \ell_2^+}$ is defined as the set $\mathcal{P}_1^+ \cap \mathcal{P}_2^+ \cup \ell_1^+ \cup \ell_2^+$:



Axioms of congruence

- The set of segments in the plane will be denoted as

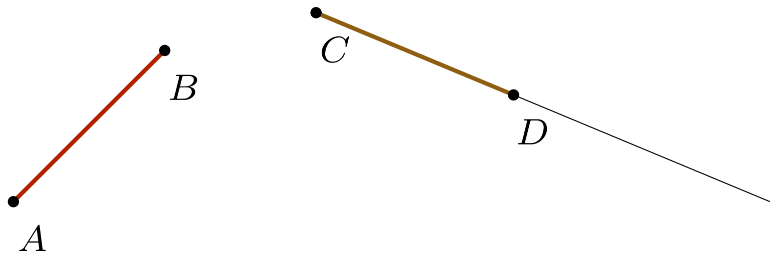
$$\mathcal{S} = (\mathcal{P} \times \mathcal{P} \setminus \Delta) / \sim,$$

where $\Delta = \{(P, Q) : P = Q\}$ and $\sim$ is the equivalence relation whereby $(P, Q) \sim (Q, P)$.

- On this set $\mathcal{S}$, we intend to define a binary relation: $\equiv \subseteq \mathcal{S} \times \mathcal{S}$. We will call this a **congruence relation**, hence if $\overline{AB}, \overline{CD} \in \mathcal{S}$, and $\overline{AB} \equiv \overline{CD}$, we will say "$\overline{AB}$ is congruent to $\overline{CD}$".
- This congruence relation must satisfy the following axioms:

(CA.1) Relation $\equiv$ must be a relation of equivalence.

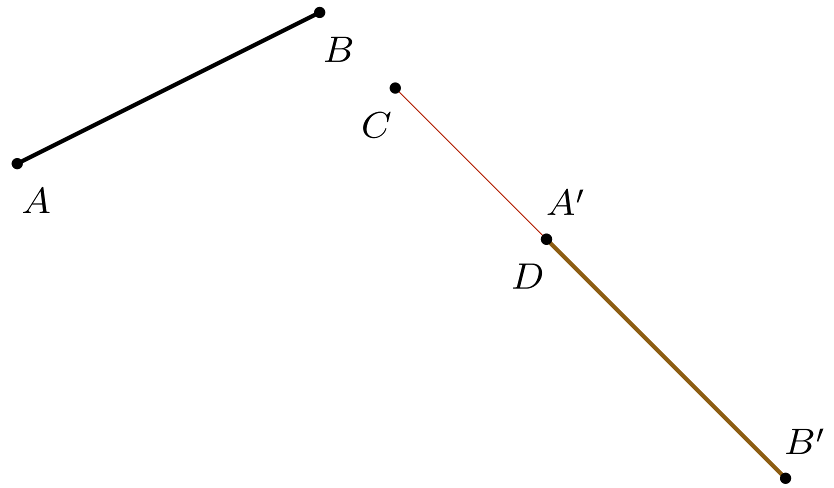
(CA.2) If $\overline{AB}$ is a segment and ℓ^+ is a ray having origin C, there exists a unique point $D \in \ell^+$ such that $\overline{AB} \equiv \overline{CD}$.



(CA.3) If A, B, C (resp. A', B', C') are three aligned points, and B (resp. B') lies between A and C (resp. A' and C') then $\overline{AB} \equiv \overline{A'B'}$ and $\overline{BC} \equiv \overline{B'C'}$ imply $\overline{AC} \equiv \overline{A'C'}$.

Corollary 53. *If the congruence axioms we have seen so far hold, then segment addition is well-defined on* $\mathcal{S}/\equiv$.

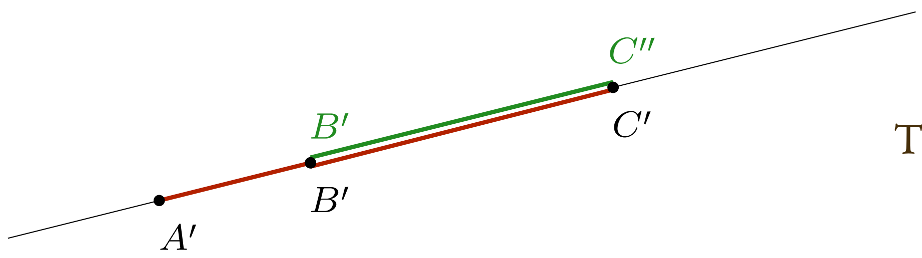
PROOF: let $\overline{AB} \in \mathcal{S}$, then we can add a segment congruent to it to $\overline{CD}$:



because there $(\mathrm{CA}.2)$ exists a unique B' such that $\overline{DB'} \equiv \overline{AB}$ and D lies between C and B'. Thus we can add classes naturally by defining $\overline{AB} + \overline{CD} := \overline{CB'}$.

Lemma 54. *If A, B, C (resp. A', B', C') are three aligned points, and B (resp. B') lies between A and C (resp. A' and C') then $\overline{AB} \equiv \overline{A'B'}$ and $\overline{AC} \equiv \overline{A'C'}$ imply $\overline{BC} \equiv \overline{B'C'}$.*

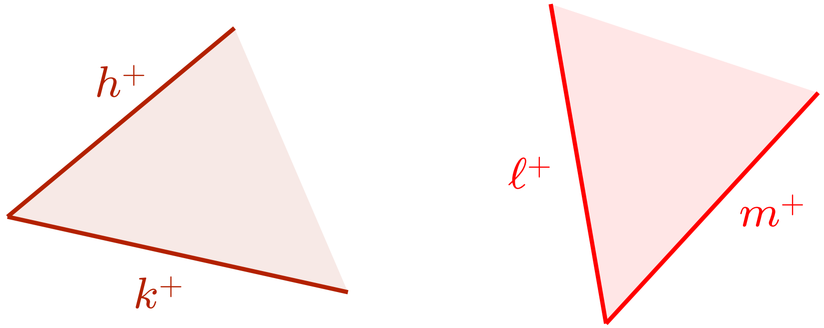
Proof: on the ray having origin B' and determined by C', transport the segment BC by way of congruent addition:



There exists a unique C'' such that $\overline{BC} \equiv \overline{B'C''}$

and the fact $\overline{AB} \equiv \overline{A'B'}$ and $\overline{BC} \equiv \overline{B'C''}$, along with axiom (CA.3), imply $\overline{AC} \equiv \overline{A'C''}$. Given that $\overline{AC} \equiv \overline{A'C'}$ and C', C'' are on the same ray of origin A defined by B', axiom (CA.2) implies $C' = C''$.

- We also need a congruence relation for angles,

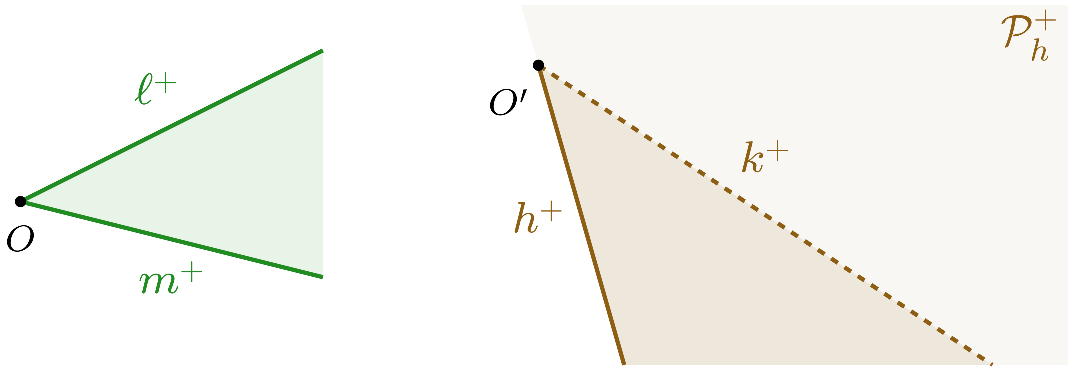


$$\widehat{h^+k^+} \equiv \widehat{\ell^+m^+}$$

and we require the following axioms to hold:

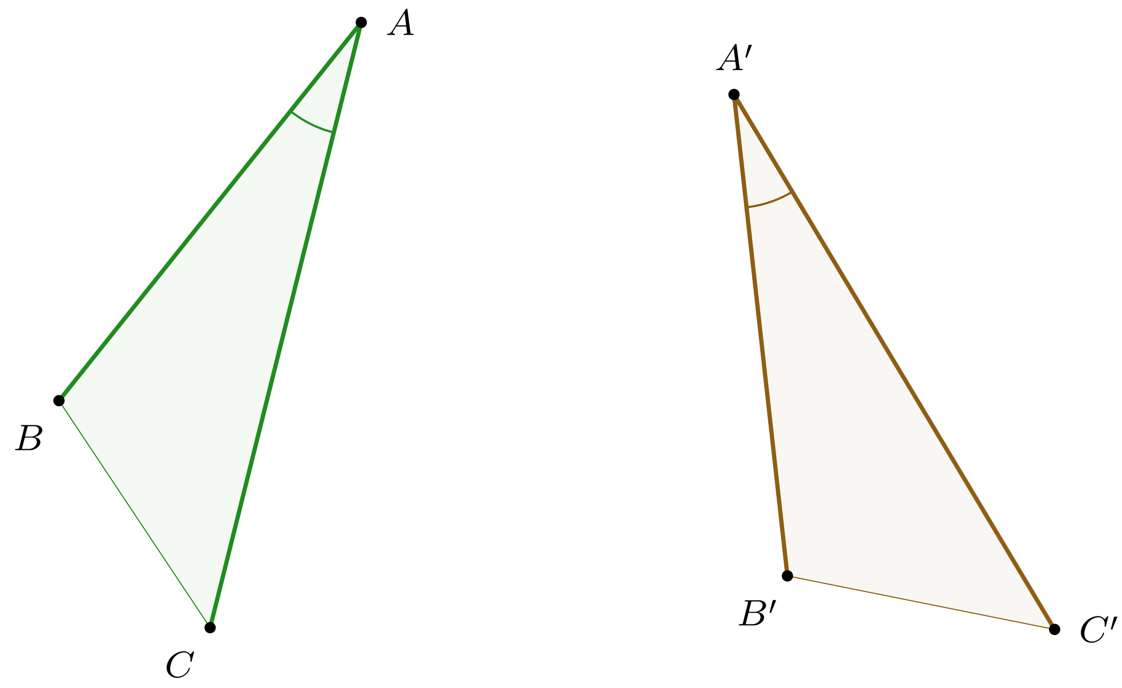
(CA.4) Congruence of angles is a relation of equivalence.

(CA.5) Angles can be transported: let $\widehat{\ell^+m^+}$ be an angle having vertex O and h^+ a ray of origin O'. Let $\mathcal{P}_h^+$ be any of the two half-planes defined by h. Then there exists a unique ray k^+ such that $\widehat{h^+k^+} \equiv \widehat{\ell^+m^+}$ and $k^+ \smallsetminus O' \subseteq \mathcal{P}_h^+$.



(CA.6) (*Pre-SAS*, i.e. Pre-"Side-Angle-Side") Let A, B, C (resp. A', B', C') be three non-aligned points. Let $\alpha = \widehat{A}, \beta = \widehat{B}, \gamma = \widehat{C}$ (resp. $\alpha' = \widehat{A'}, \beta' = \widehat{B'}, \gamma' = \widehat{C'}$) be the angle formed by the corresponding rays having vertices A, B, C (resp. A', B', C'). Then

$$\left.\begin{array}{l} \overline{AB} \equiv \overline{A'B'} \\ \overline{AC} \equiv \overline{A'C'} \\ \alpha \equiv \alpha' \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \beta \equiv \beta' \\ \gamma \equiv \gamma' \end{array}\right.$$



Absolute geometries

- A **(planar) neutral or absolute geometry** $\mathcal{G}$ is formed by $(\mathcal{P}, \mathcal{L}, \in, \star, \equiv)$ where
  - $\mathcal{P}, \mathcal{L}$ are sets, whose elements are referred to **points** and **lines** respectively;
  - $\in$ is a relation satisfying the aforementioned **incidence axioms** $(\mathrm{IA}.1), (\mathrm{IA}.2), (\mathrm{IA}.3)$.
  - $\star$ is a relation satisfying the aforementioned **order axioms** $(\mathrm{OA}.1), (\mathrm{OA}.2)$.
  - $\equiv$ is a relation satisfying the aforementioned **congruence axioms** $(\mathrm{CA}.1), (\mathrm{CA}.2), (\mathrm{CA}.3), (\mathrm{CA}.4), (\mathrm{CA}.5), (\mathrm{CA}.6)$.
- An geometry, absolute or not, is called
  - **Euclidean** if given a point $P \in \mathcal{P}$ and a line $r \in \mathcal{L}$ such that $P \notin r$, there exists a unique line ℓ parallel to r (i.e. non-intersecting) such that $P \in \ell$
  - **hyperbolic** if given a point $P \in \mathcal{P}$ and a line $r \in \mathcal{L}$ such that $P \notin r$, there exist two lines ℓ_1, ℓ_2 parallel to r such that $P \in \ell_1, \ell_2$
  - **elliptic** if given a point $P \in \mathcal{P}$ and a line $r \in \mathcal{L}$ such that $P \notin r$, there exists no line parallel to r and containing P.
- We will study the existence of each of these geometries if we impose the axiomatics defining neutrality.

- Let us study some consequences of the axioms. First of all, let us characterize triangle similarity:

Theorem 55 (SAS, i.e. Side-Angle-Side Criterion)**.** *Let (A, B, C) and (A', B', C') be triples of non-aligned points. Then*

$$\left.\begin{array}{l} \overline{AB} \equiv \overline{A'B'} \\ \overline{AC} \equiv \overline{A'C'} \\ \widehat{A} \equiv \widehat{A'} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \widehat{B} \equiv \widehat{B'} \\ \widehat{C} \equiv \widehat{C'} \\ \overline{BC} \equiv \overline{B'C'} \end{array}\right.$$

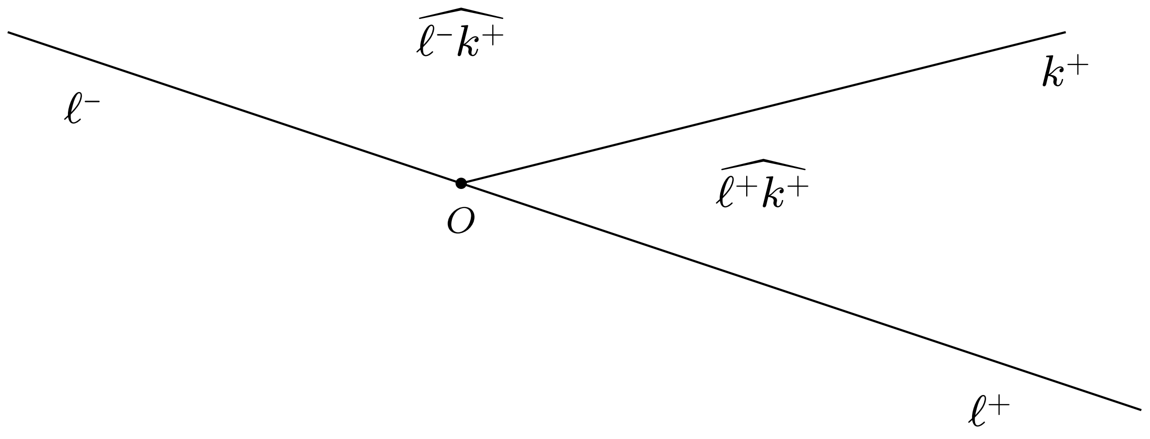
PROOF: EXERCISE using the Pre-SAS axiom.

- A triangle (A, B, C) is called **isosceles** at a given vertex if it has two congruent sides containing the vertex.

Theorem 56. *A triangle (A, B, C) is isosceles at $P \in \{A, B, C\}$ if, and only if, it has two congruent angles, namely those not having O as their vertex.*

PROOF: EXERCISE.

- Let $\mathcal{G}$ be an absolute geometry. Two angles are **adjacent** if they are of the form $\widehat{\ell^+k^+}$ and $\widehat{\ell^-k^+}$, i.e. if they share their vertex and a side and the remaining sides are collinear:



We write $\widehat{\ell^+k^+} \angle \widehat{\ell^-k^+}$. The following is immediate:

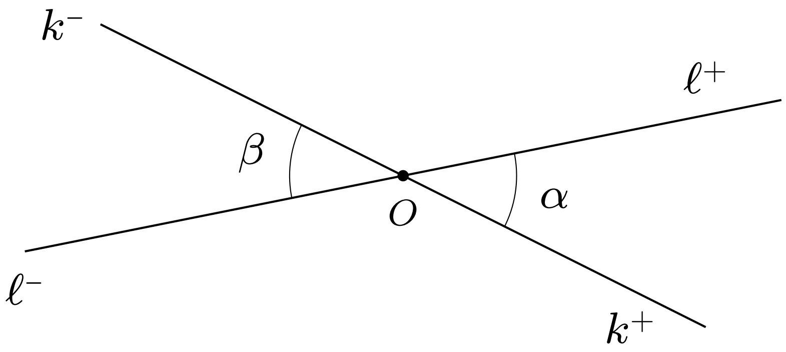
Lemma 57. *Every angle α has an adjacent angle $\beta \angle \alpha$.*

- An angle is called **right** if it is congruent to an adjacent angle.

Theorem 58. *If $\alpha \mathrel{-\!\!\!\angle} \beta$, $\alpha' \mathrel{-\!\!\!\angle} \beta'$ and $\alpha \equiv \alpha'$, then $\beta \equiv \beta'$.*

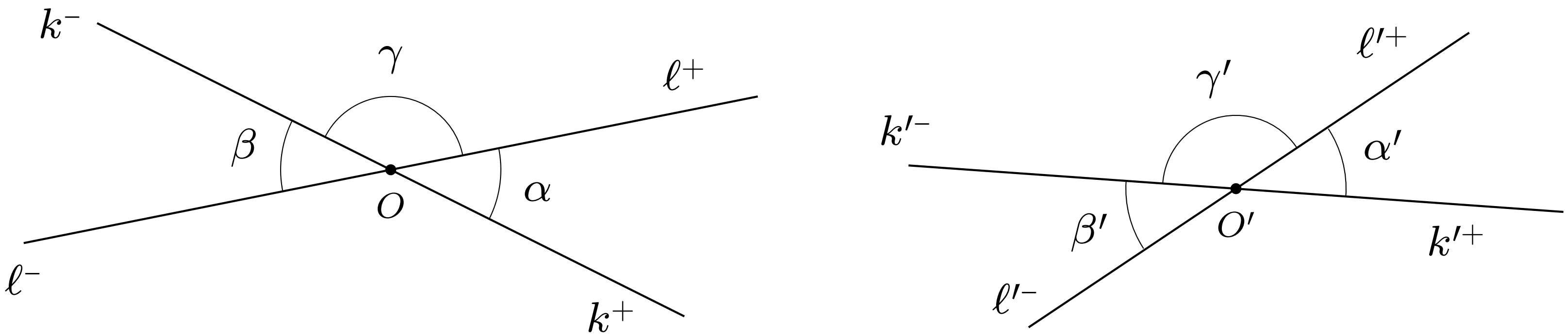
PROOF: EXERCISE.

- Let ℓ, k be two lines intersecting at O. We say $\alpha = \widehat{\ell^+ k^+}$ and $\beta = \widehat{\ell^- k^-}$ are **opposite angles** and write $\alpha \mathrel{>\!\!\!<} \beta$.



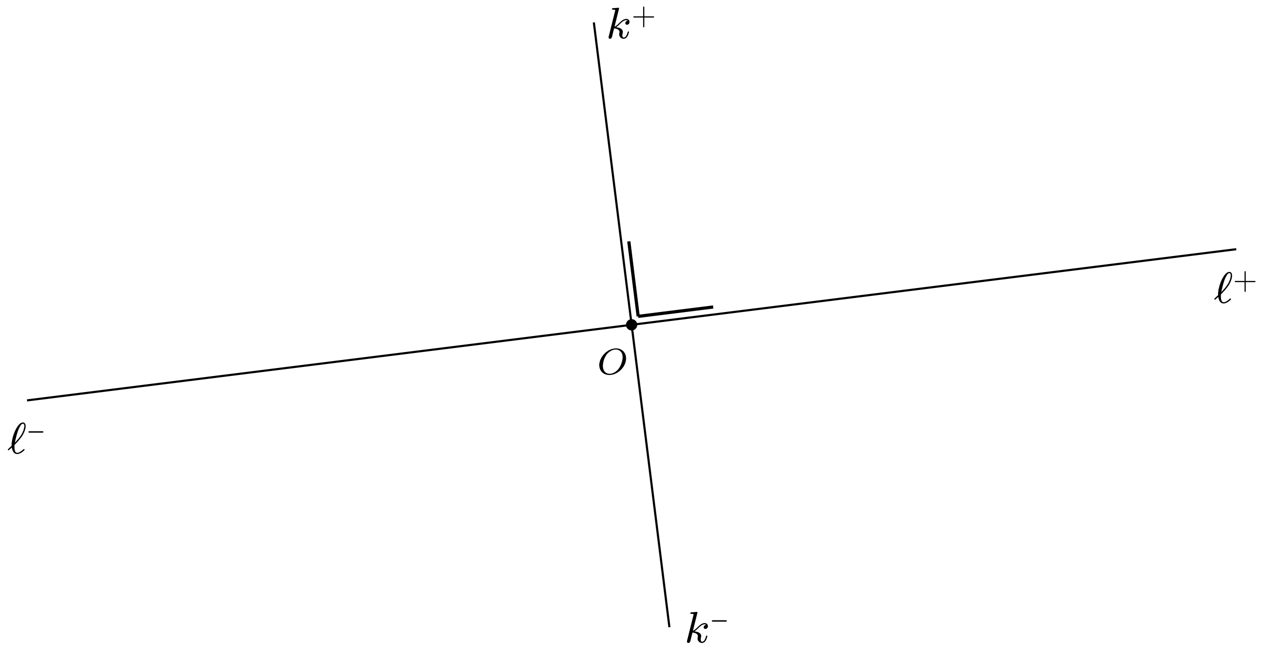
Theorem 59. *If $\alpha \mathrel{>\!\!\!<} \beta$, $\alpha' \mathrel{>\!\!\!<} \beta'$ and $\alpha \equiv \alpha'$, then $\beta \equiv \beta'$.*

PROOF: if $\gamma = \widehat{k^- \ell^+}$ and $\gamma = \widehat{k'^- \ell'^+}$, then $\alpha \mathrel{-\!\!\!\angle} \gamma$, $\alpha' \mathrel{-\!\!\!\angle} \gamma'$, and $\alpha \equiv \alpha'$ and Theorem 58 imply $\gamma \equiv \gamma'$; we also have $\gamma \mathrel{-\!\!\!\angle} \beta$, and $\gamma \equiv \gamma'$ and again Theorem 58 imply $\beta \equiv \beta'$. $\square$



Corollary 60. *If ℓ and k are two lines and one of the angles they form is right, then all of them are.*

PROOF:



If $\widehat{\ell^+k^+}$ is right, then $\widehat{\ell^+k^+} \equiv \widehat{\ell^-k^+}$ and thus $\widehat{\ell^-k^+}$ is right as well. The same argument applied successively to $\widehat{\ell^-k^+} \equiv \widehat{\ell^-k^-}$ and $\widehat{\ell^-k^-} \equiv \widehat{\ell^+k^-}$ implies all of them are right.

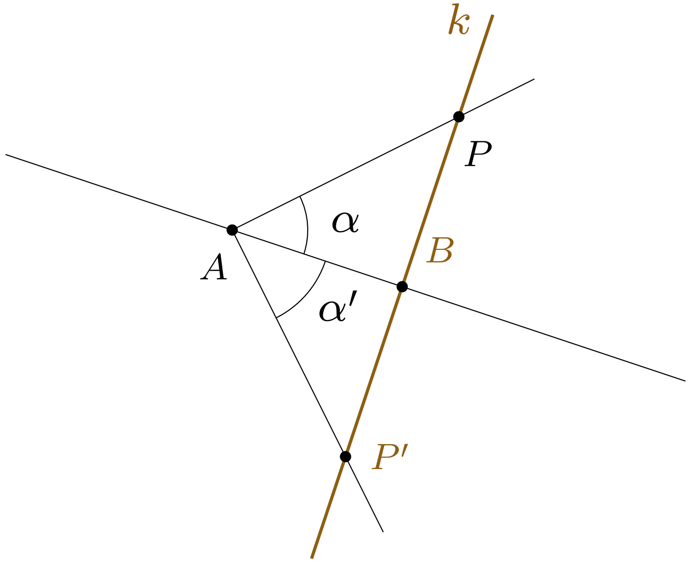
- Two lines ℓ, k are said to be **perpendicular** or **orthogonal** if their four angles are right angles. We write this as $\ell \perp k$.

Theorem 61. *Assume $\ell \neq m$ are two perpendicular lines that are perpendicular to a third line k. Then ℓ and m do not intersect in this geometry, i.e. they are parallel. In this case we call them* ***ultraparallel*** *or* ***hyperparallel****.*

PROOF: EXERCISE.

Theorem 62. *Let ℓ be a line and P be a point exterior to this line, $P \notin \ell$. Then there exists a unique line k such that $k \perp \ell$ and $P \in k$.*

PROOF: uniqueness results from the previous Theorem. Let us prove existence. Let A be a point of the line ℓ and let h^+ be the ray having origin A and determined by P. We can transport angle $\alpha = \widehat{\ell^+ h^+}$ to the half-plane of ℓ not containing P, i.e. $\alpha \equiv \alpha' = \widehat{\ell^+ m^+}$ below:



on m^+ transport segment $\overline{AP}$, i.e. let $P' \in m^+$ such that $\overline{AP} \equiv \overline{AP'}$. The fact P, P' belong to separate half-planes implies $P \neq P'$ thus there exists a unique line k containing P, P'.

Let us prove that $k \perp \ell$. Let $B = k \cap \ell$. Consider triangles APB and $AP'B$. The facts $\overline{AP} \equiv \overline{AP'}$ and $\alpha \equiv \alpha'$ imply, along with the SAS criterion, that $\widehat{ABP} \equiv \widehat{ABP'}$. Given that $\widehat{ABP} \mathrel{\underline{\angle}} \widehat{ABP'}$, this implies both angles are right. □

Theorem 63. *Let ℓ be a line and $P \in \ell$. Then there exists a unique line m perpendicular to ℓ such that $P \in m$.*

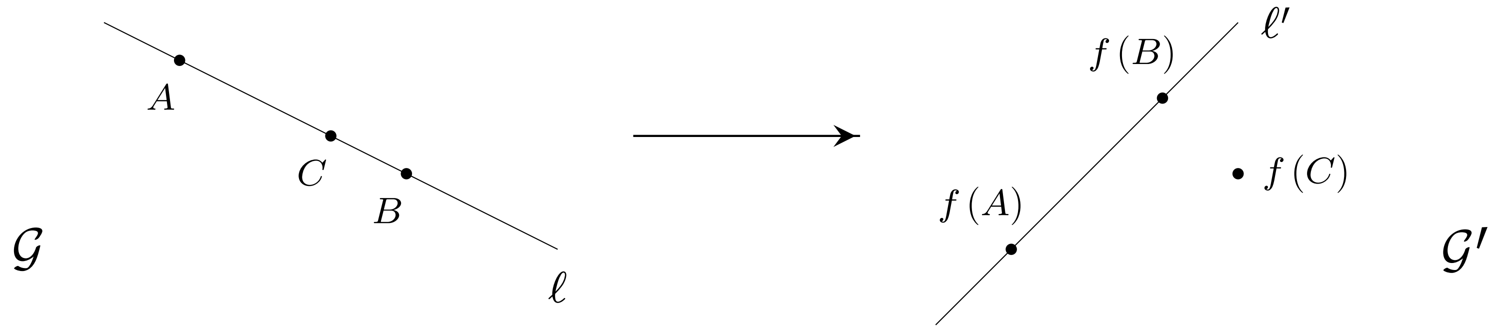
Theorem 64. *If ℓ is a line and P is a point exterior to ℓ, there always exists at least one line m parallel to ℓ, containing P.*

Proof of both results: Exercise.

Corollary 65. *An absolute geometry $\mathcal{G}$ is either Euclidean or hyperbolic, but never elliptic.*

Isomorphic geometries

- Let $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \in, \star, \equiv)$, $\mathcal{G}' = (\mathcal{P}', \mathcal{L}', \in', \star', \equiv')$ be two absolute geometries.
- Let $f : \mathcal{P} \to \mathcal{P}'$ be a bijective function. Given a line $\ell \in \mathcal{P}$ determined by two points $A \neq B$ in $\mathcal{P}$, points $f(A)$ and $f(B)$ determine a line $\ell' \in \mathcal{P}'$. However, that in itself is not enough, in general, to induce a map $\mathcal{L} \to \mathcal{L}'$:



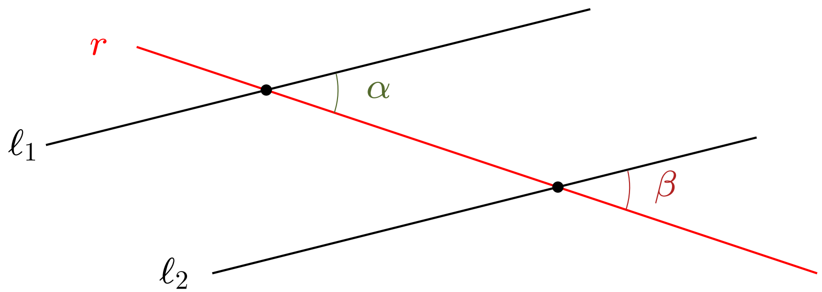
- Assume, however, that bijection f *does* induce a map $\mathcal{L} \to \mathcal{L}'$. In other words, for every $A, B, C \in \mathcal{P}$, $f(A), f(B), f(C)$ are aligned. In such case, we would still need to check the preservation of the rest of relations, e.g. congruence $\equiv$ or "betweeness" $\star$.
- An **isomorphism** between geometries $\mathcal{G}$ and $\mathcal{G}'$ is a bijective map $f : \mathcal{P} \to \mathcal{P}'$ such that f preserves alignment, being between points (i.e. incidence) and congruence.
- Isomorphisms of a geometry $\mathcal{G}$ to itself, $\mathcal{P} \to \mathcal{P}$ are called **automorphisms** of $\mathcal{G}$.

Proposition 66. *Automorphisms of a geometry $\mathcal{G}$ form a group with $\circ$, denoted* $\mathrm{Aut}(\mathcal{G})$.

Exercises

Using the axioms of incidence, order and congruence, prove the following results:

- (SSS criterion) If on triangles ABC and $A'B'C'$ we have $\overline{AB} \equiv \overline{A'B'}$, $\overline{AC} \equiv \overline{A'C'}$ and $\overline{BC} \equiv \overline{B'C'}$, then $\triangle ABC \equiv \triangle A'B'C'$.
- (ASA criterion) If on triangles ABC and $A'B'C'$ we have $\overline{AB} \equiv \overline{A'B'}$, $\widehat{A} \equiv \widehat{A'}$ and $\widehat{B} \equiv \widehat{B'}$, then both triangles are congrent, i.e. the remaining angles and sides are congruent.
- If two angles are congruent, the adjacent angles are congruent as well.
- Opposite angles are congruent.
- All right angles are congruent to one another.
- In an isosceles triangle, the median to the base is an altitude and is also the angle bisector of the opposite vertex.
- An exterior angle of a triangle is larger than each of the interior non-adjacent angles.
- Prove the converse of the *Corresponding Angles Theorem*: in a situation such as this,



If $\alpha \equiv \beta$, then ℓ_1 and ℓ_2 are parallel.

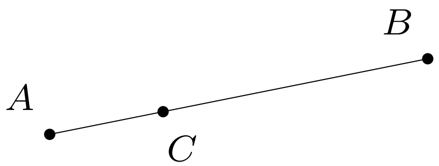
2 Euclidean Geometry

Parallel axiom

- As seen in earlier pages, a geometry $\mathcal{G}$ is called **Euclidean** if it satisfies the following:
  - (*Parallel Axiom*) for every $r \in \mathcal{L}$ and $P \in \mathcal{P}$ such that $P \notin r$, there exists a unique line parallel to (i.e. non-intersecting with) r containing P.
- Let us see that there *exists* a Euclidean geometry. Take

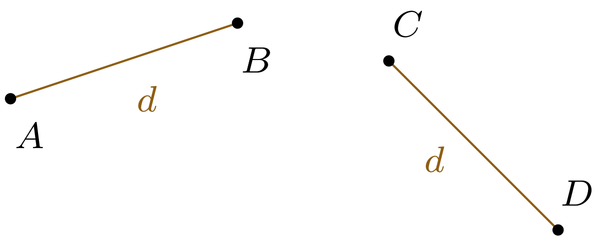
$$\begin{aligned} \mathcal{P} &:= \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}, \\ \mathcal{L} &:= \{\text{lines of the plane}\} = \left\{(x, y) \in \mathbb{R}^2 : ax + by + c = 0,\ a \neq 0 \text{ or } b \neq 0\right\}/\mathbb{R}^* \end{aligned}$$

  and introduce the following relations: $\in$, "betweeness" and $\equiv$ by
  - $P \in r$ if $P = (x_0, y_0)$, $r = \{ax + by + c = 0,\ a \neq 0 \text{ or } b \neq 0\}$ and $ax_0 + by_0 + c = 0$.
  - C between A and B if $C = \lambda A + (1 - \lambda) B$ for some $\lambda \in (0, 1)$:



$$C = (\lambda(x_0 - X_0) + X_0, \lambda(y_0 - Y_0) + Y_0)$$
$$\text{IF } A = (x_0, y_0) \text{ AND } B = (X_0, Y_0)$$

  - $\overline{AB} \equiv \overline{CD}$ if $d(A, B) = d(C, D)$ where d is the usual straight-line distance:



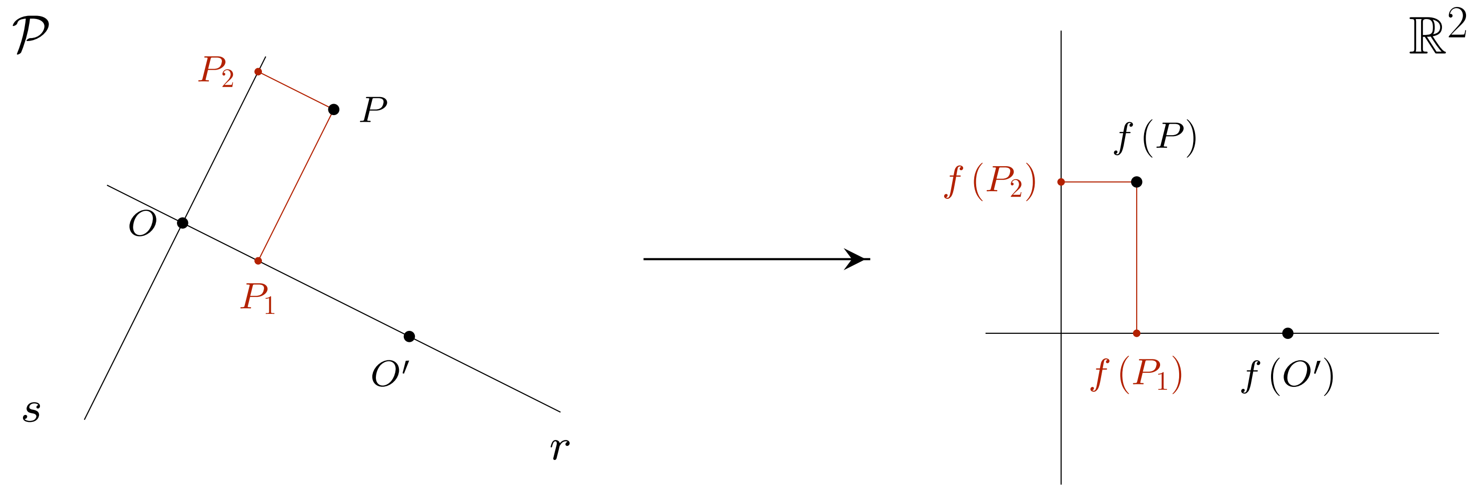
$$d = \sqrt{(A_1 - B_1)^2 + (A_2 - B_2)^2} = \sqrt{(C_1 - D_1)^2 + (C_2 - D_2)^2}$$

EXERCISE: prove that $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \dots)$ defined above is indeed a Euclidean geometry.

EXERCISE: determine $\mathrm{Aut}(\mathcal{G})$ and prove that it is the group of motions of the plane, i.e. what we defined to be **isometries**.

Theorem 67 (Uniqueness up to isomorphisms of the Euclidean geometry)**.** *Let $\mathcal{G}$ be a Euclidean geometry. Then it is isomorphic to the Euclidean geometry on $\mathbb{R}^2$ defined in the previous page.*

IDEA OF THE PROOF: all we need to do is find a bijective map $f : \mathcal{P} \to \mathbb{R}^2$ that preserves alignment, order and congruence. For instance, let O and O' be two different points of $\mathcal{P}$, r the line determined by O and O' and s the line orthogonal to r containing O. Then any point $P \in \mathcal{P}$ is contained in (uniquely determined) lines parallel to r and s containing P, and a bijection $\mathcal{P} \to \mathbb{R}^2$ mapping $O \mapsto (0,0)$, $O' \mapsto (1,0)$ and parallel incidence points of any other P to the corresponding points in $\mathbb{R}^2$ will do the job (EXERCISE: fill in the blanks of the proof):



Exercises

Using the axioms of Euclidean geometry, prove the following:

- The sum of the interior angles of a triangle equals two right angles.
- Non-intersecting lines are equidistant.
- (*Thales' Theorem*) Given a triangle $\triangle ABC$, a line not intersecting BC will divide the other two sides in proportional segments.
- Two triangles with two equal angles are similar (two triangles are called *similar* if their three angles are equal and their corresponding sides are proportional).
- (*Pythagoras' Theorem*) In a right triangle (i.e. a triangle containing one right angle) the hypotenuse squared equals the sum of the squares of the catheti.

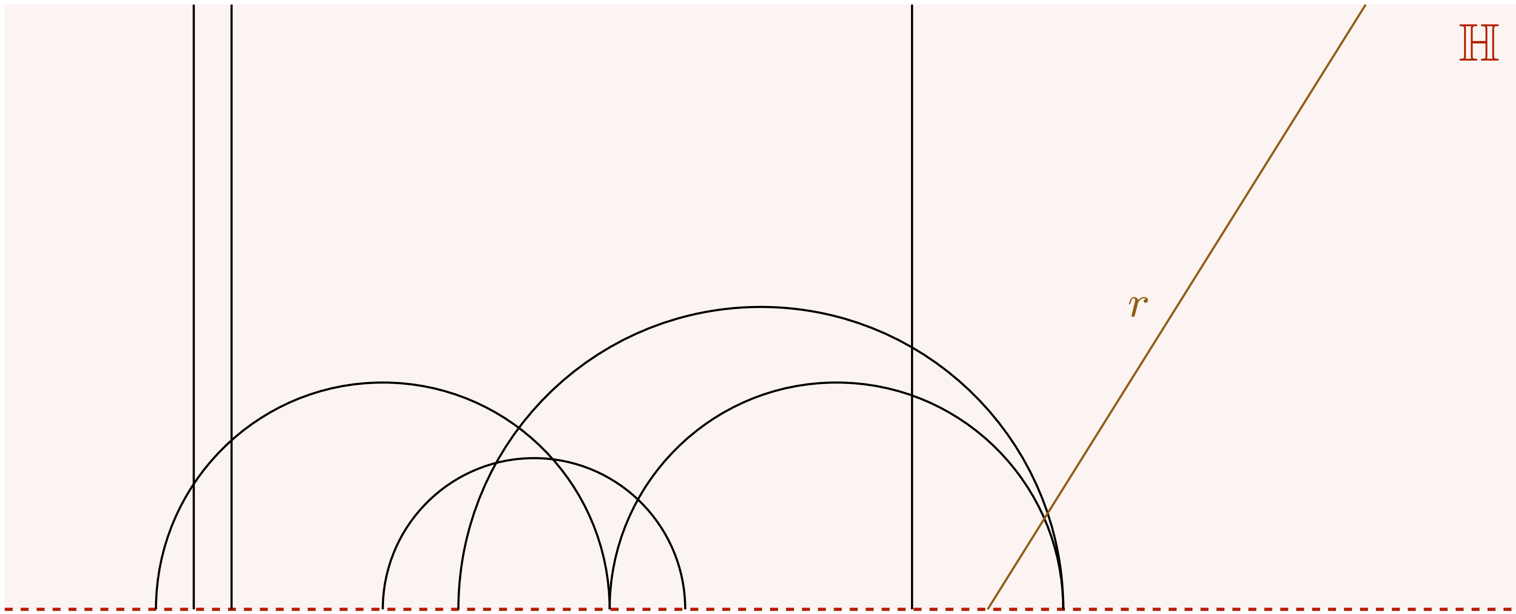
3 Hyperbolic Geometry

Lobachevsky's axiom

- A geometry $\mathcal{G}$ is called **hyperbolic** if it satisfies the following:
  - (*Lobachevsky's axiom*) for every point $P \in \mathcal{P}$ and line $r \in \mathcal{L}$ not containing P, there exist at least two lines parallel to r containing P.
- Let us prove there exists one such geometry. Define $\mathcal{P}$ to be **Poincaré's upper half-plane**,

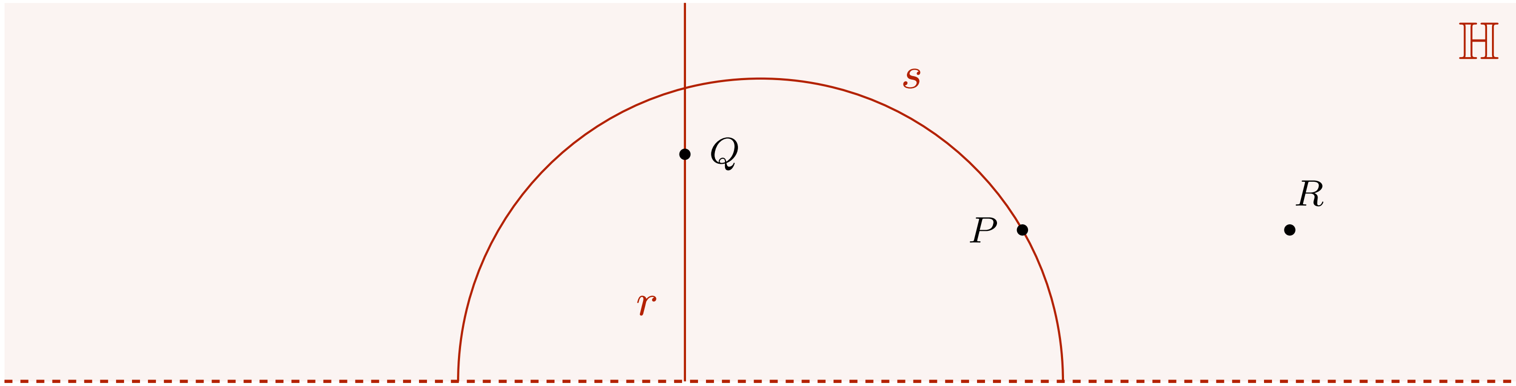
$$\mathcal{P} = \left\{(x, y) \in \mathbb{R}^2 : y > 0\right\} =: \mathbb{H}$$

and let $\mathcal{L} = \{\text{upper half-circles with their center on the } x\text{-axis}\} \cup \{\text{vertical rays}\}$, e.g.



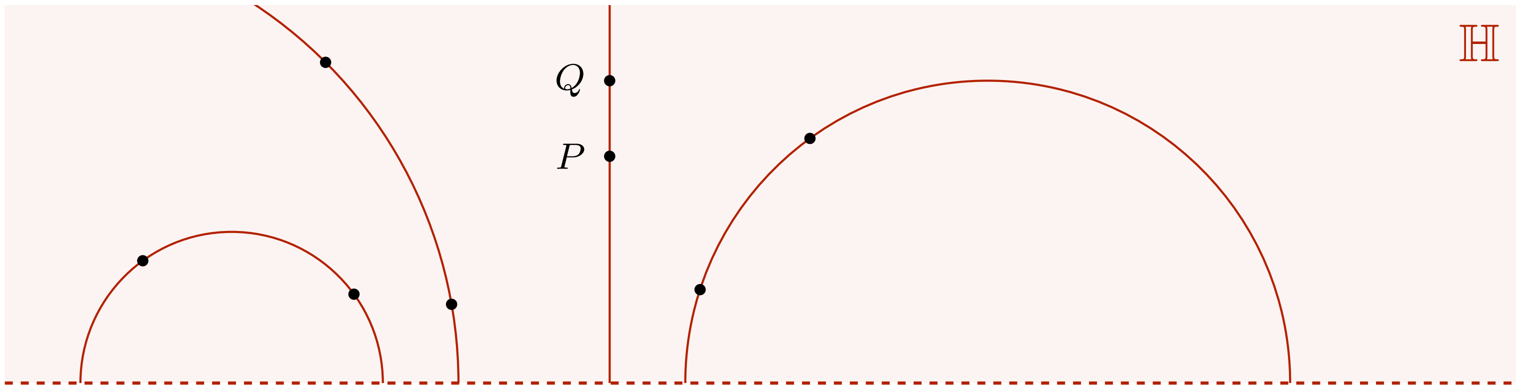
ALL OF THE SUBSETS HIGHLIGHTED ABOVE ARE LINES IN THIS GEOMETRY EXCEPT FOR r

- We say that a $P \in \mathcal{P}$ belongs to a line r if it belongs to it in the usual Euclidean sense, e.g.



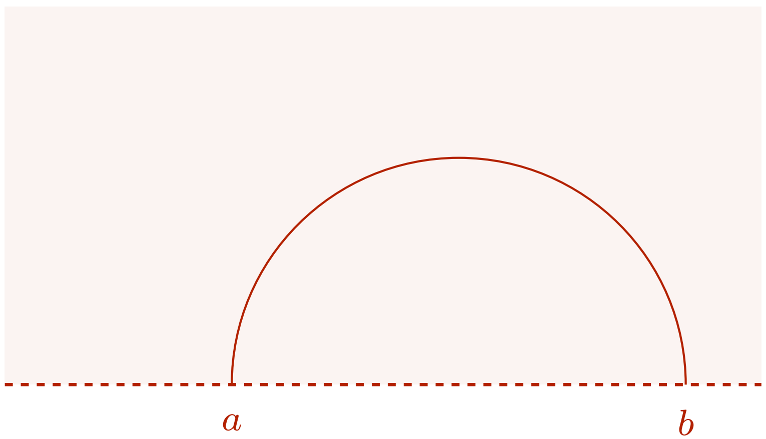
$P \in s$, $Q \in r$ AND R DOES NOT BELONG TO EITHER LINE

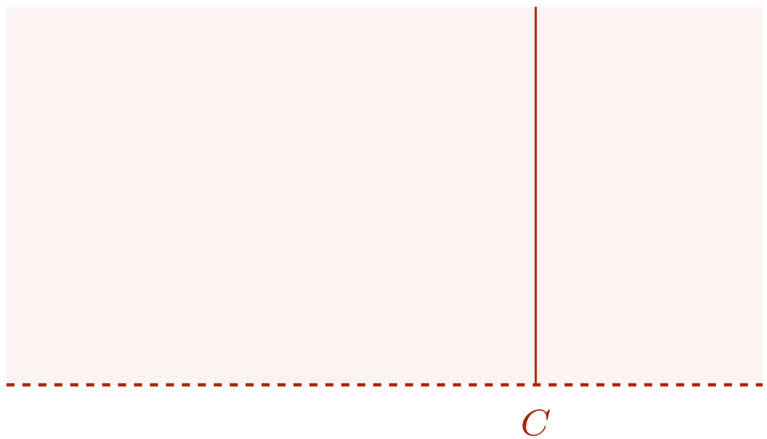
- Checking the axioms of incidence is now simple:
  - for (IA.1) we only need to distinguish between pairs of points having the same x coordinate (e.g. P and Q below), and all other cases (e.g. other pairs shown below):



ANY TWO POINTS IN $\mathbb{H}$ ARE CONTAINED IN A UNIQUE VERTICAL LINE OR A UNIQUE HALF-CIRCLE

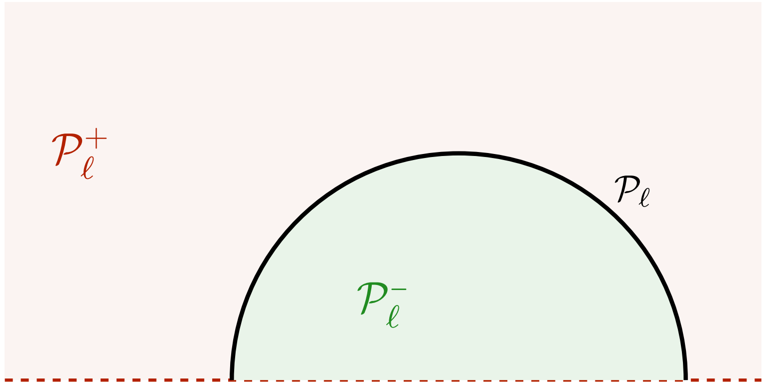
  - (IA.2) is a consequence of the fact that there are at least three points (infinitely many, in fact) that are non-aligned. Choosing three points such that two of them are on the same vertical line and the third is not proves this (Exercise).
  - (IA.3) is trivially true because every line has at least two points (infinitely many, in fact).
- Axioms of order:
  - (OA.1) let $r \in \mathcal{L}$ be a line. Then it is either a vertical ray $x = C$, in which case a parametrization is f_C below, or it is a half-circle wherein $x \in (a, b)$, in which case it would be $g_{a,b}$:



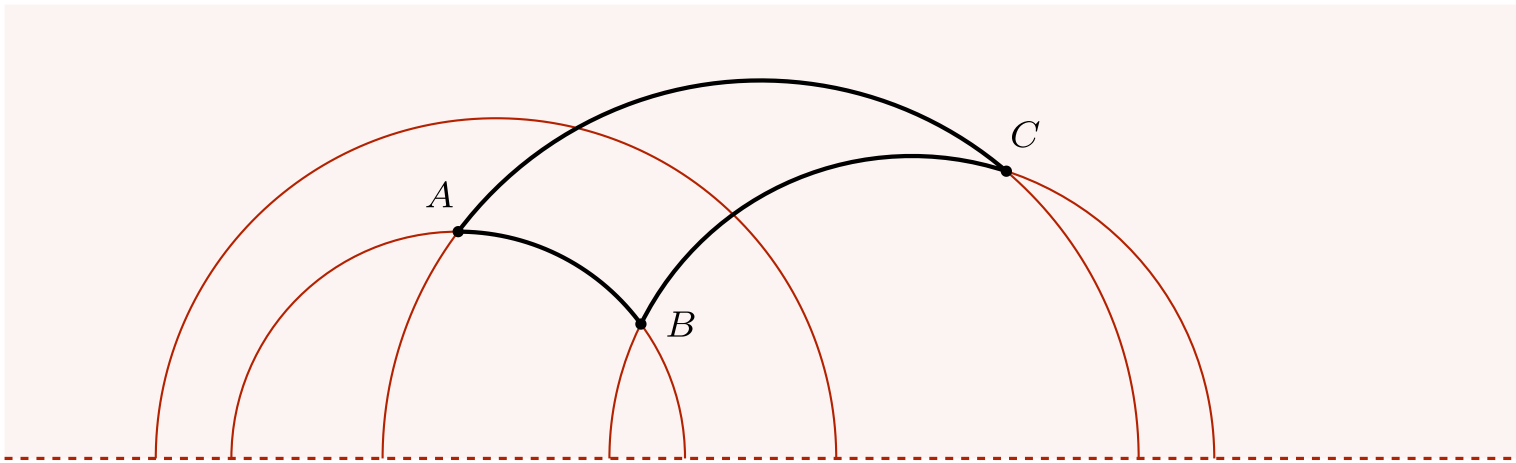


$$\begin{array}{rcl} \mathcal{P} & \xrightarrow{\quad g_{a,b} \quad} & \mathbb{R} \\ (x,y) & \longmapsto & g_{a,b}(x,y) := \frac{1}{a-x} + \frac{1}{b-x} \end{array} \qquad \begin{array}{rcl} \mathcal{P} & \xrightarrow{\quad f_C \quad} & \mathbb{R} \\ (x,y) & \longmapsto & f_C(x,y) := y - \frac{1}{y} \end{array}$$

– Pasch's axiom (OA.2): every line ℓ divides $\mathcal{P}$ into three subsets (the line itself and the two half-planes we have been seeing before):

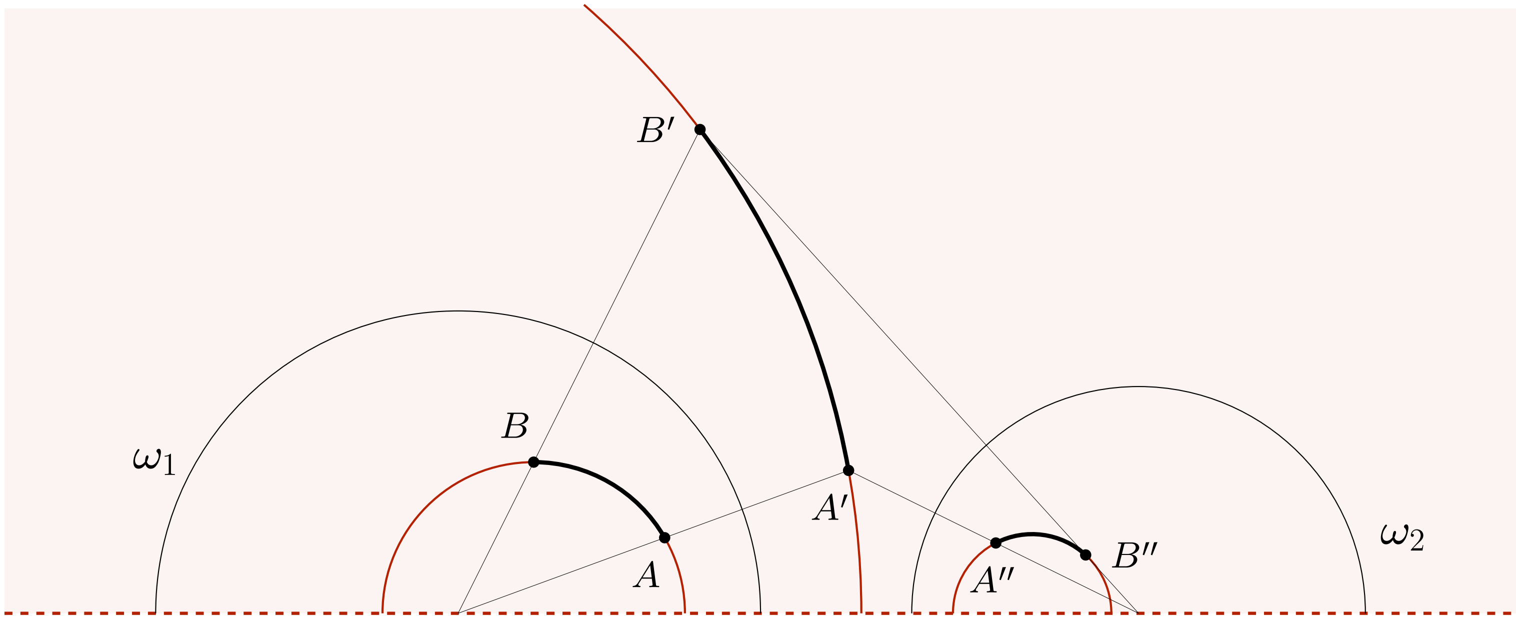


each of these subsets is **path-connected**, i.e. any two points in it can be joined by a continuous curve. Pasch's axiom is a consequence of this, e.g.



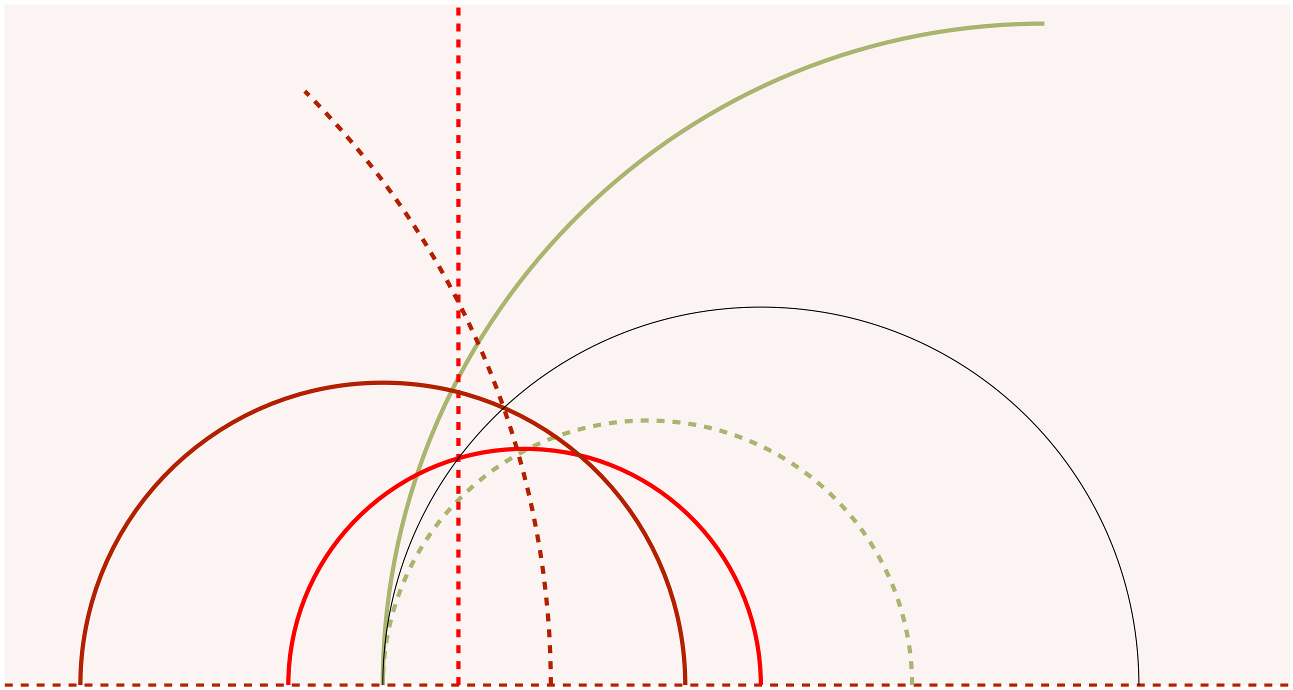
Exercise: finish the formal proof of this.

- In order to state congruence axioms, we first need to define a congruence of hyperbolic segments. Two segments $\overline{AB}$ and $\overline{CD}$ are said to be **congruent**, $\overline{AB} \equiv \overline{CD}$, if there exists a sequence of inversions with respect to circles centered at the x-axis, transforming $A \mapsto C$ and $B \mapsto D$ or $A \mapsto D$ and $B \mapsto C$. For example, $\overline{AB} \equiv \overline{A'B'} \equiv \overline{A''B''}$ below with inversion circles ω_1, ω_2:



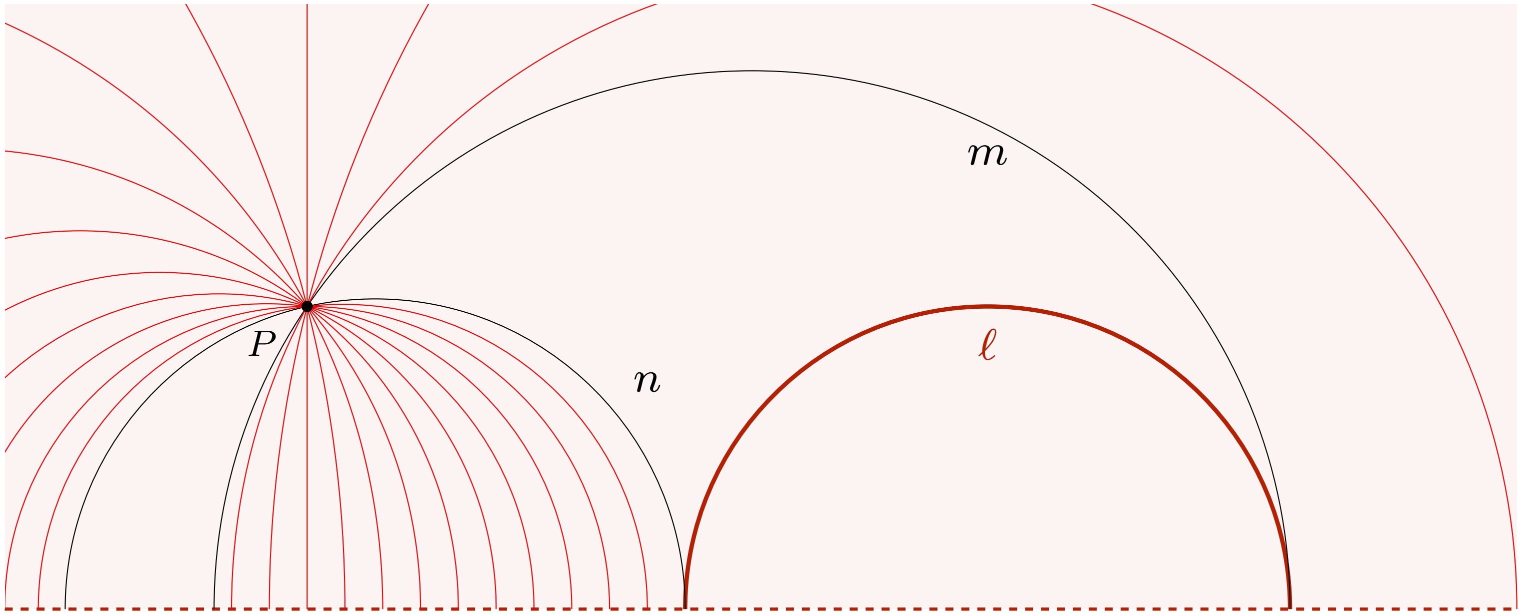
$$(A, B) \xrightarrow{\omega_1} (A', B') \xrightarrow{\omega_2} (A'', B'')$$

- You can now check it satisfies the congruence axioms:
  - (CA.1) immediate. $\equiv$ is a relation of equivalence (Exercise).
  - (CA.2) if $\overline{AB}$ is a segment and ℓ^+ a ray having origin C, there exists a unique $D \in \ell^+$ such that $\overline{AB} \equiv \overline{CD}$. Exercise: prove this by "transporting" the proof you would use in Euclidean geometry.
  - (CA.3) if a succession of inversions transforms $\overline{AB}$ into $\overline{A'B'}$, this same succession of inversions will transform $\overline{BC}$ into $\overline{B'C'}$. Again, Exercise.
  - the rest of axioms follows (Exercise) from the correct definition of what congruent angles mean in $\mathbb{H}$ but that is obvious from the above definition via sequences of inversions, and the fact that inversions preserve angles:



Lines of the same color are mutual inverses (one dashed, one solid) with respect to the black line. Observe the angle preservation between any lines of different colors

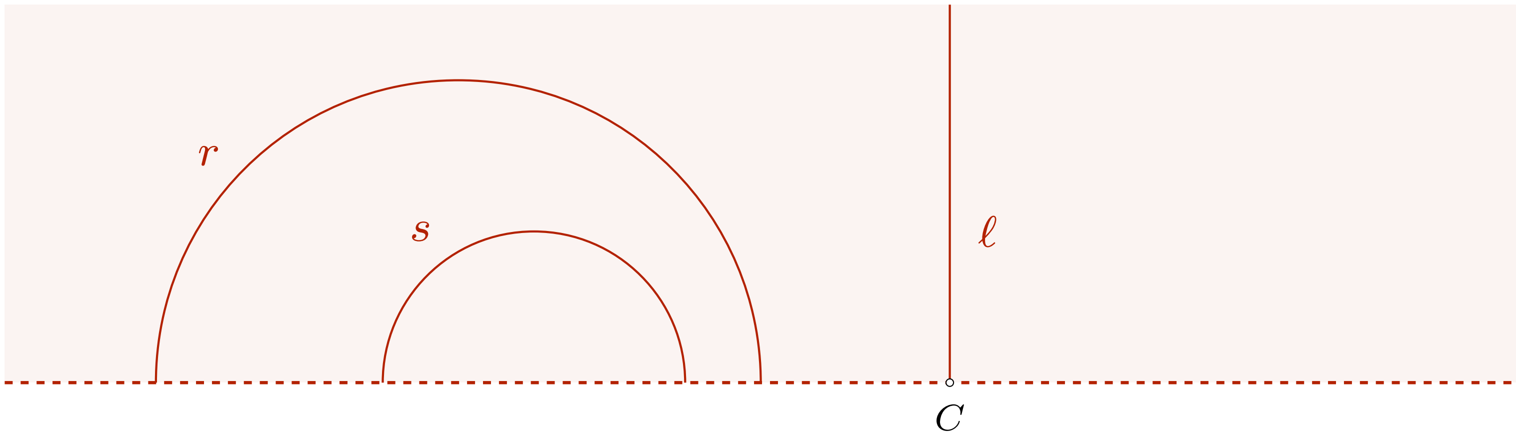
- Let us check that the geometry thus constructed is indeed hyperbolic. Let ℓ be a line and $P \notin \ell$ a point exterior to it. Then there are infinitely many lines containing P, not intersecting ℓ (EXERCISE: describe a general procedure that does not require pictures)



lines m and n above also count as non-intersecting lines (they "intersect" at the line of infinity).

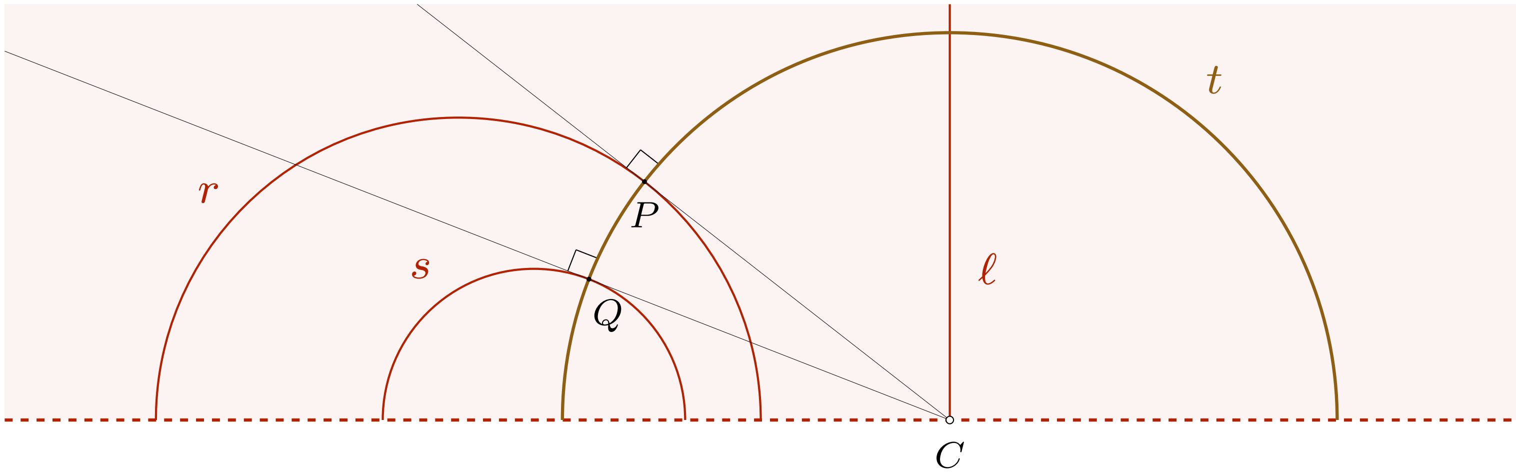
- In other words, m, n are limit cases and thus notable parallel lines in that they do not intersect ℓ in $\mathbb{H}$, but would intersect it in the Euclidean plane $\mathbb{R}^2$.
- In the next page we will see that the infinitely many remaining lines are not only parallel to ℓ, but also **hyperparallel** to it (remember Theorem 61), i.e. they share a common perpendicular line with ℓ.

- In Euclidean geometry, there is no difference between parallel and hyperparallel (EXERCISE.
- However, hyperbolic geometry, we have
  - hyperparallel always implies parallel,
  - but parallel does not in general imply hyperparallel.
- In order to check that all lines in $\mathbb{H}$ that do not intersect r in $\mathbb{R}^2$ (i.e. lines that do not meet r at infinity, such as m and n in the previous page) all you have to do is take any two lines r and s having empty Euclidean intersection as Euclidean half-circles. Let us first start with the case in which they are not concentric as half-circles. In that case they have a well-defined radical axis ℓ, orthogonal to the real line $\mathbb{R}$ (line of infinity for $\mathbb{H}$) at a certain point C:

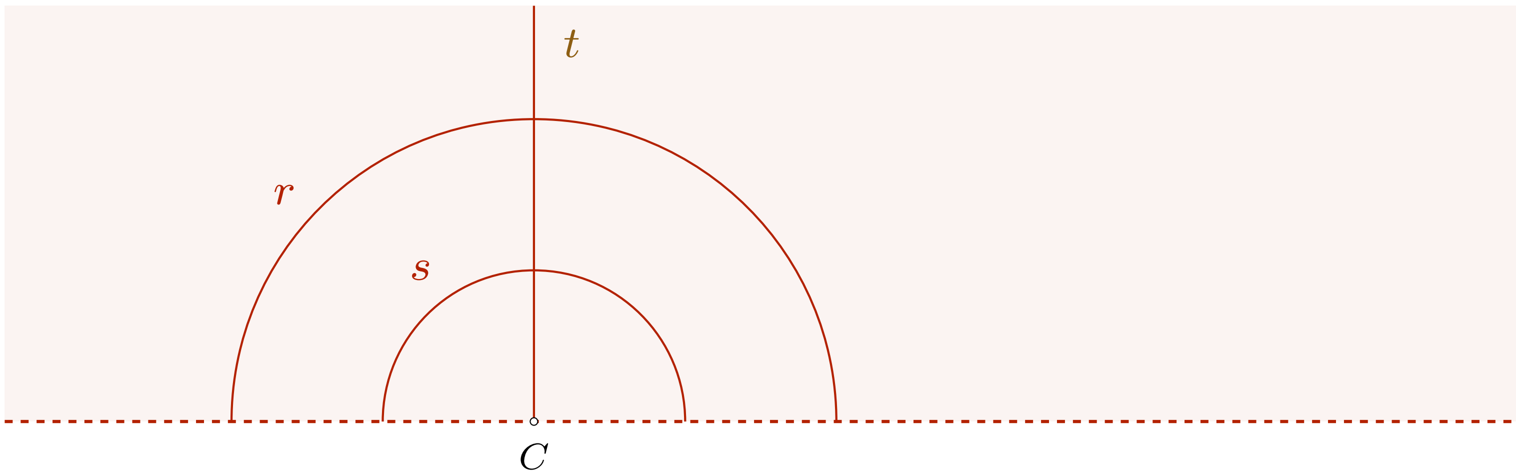


the radical axis ℓ can be characterized (EXERCISE) as follows: a circle is orthogonal to r and s if and only its center lies on ℓ. Thus all we need to find is the ray from C tangent to one of the lines.

- Let P be the tangency point. If we repeat the procedure with the other circle and obtain another tangency point Q, we know $\overline{CP} \equiv \overline{CQ}$ (the radical axis being the set of points whose power with respect to r equals that with respect to s) and the fact that both of them are tangency points implies that the circle t of center C and radius $|OP|$ is orthogonal to both r and s:



- In the case in which r, s are concentric as Euclidean half-circles, the common perpendicular is the vertical Euclidean ray determined by the common center of r, s:



- Thus if you fill in the blanks of the above, you prove the following:

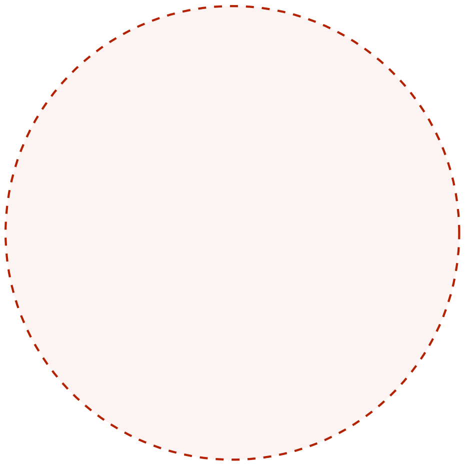
Proposition 68. *If two hyperbolic lines do not intersect anywhere, not even asymptotically at infinity, then they are ultraparallel and the common perpendicular is unique (unlike in the Euclidean case).*

- The following is left as an EXERCISE:

Theorem 69 (Uniqueness up to isomorphisms of the hyperbolic geometry)**.** *Let $\mathcal{G}$ be a hyperbolic geometry. Then it is isomorphic to the hyperbolic geometry on $\mathbb{H}^2$ defined in the previous pages.*

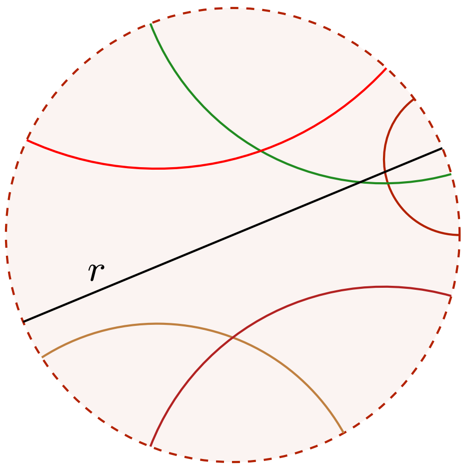
- There are other models of hyperbolic geometries (i.e. geometries isomorphic to $\mathbb{H}$). We will briefly see three of these:
  - the POINCARÉ DISK MODEL;
  - the KLEIN DISK MODEL;
  - the HEMISPHERE MODEL.

- The **Poincaré disk model** is a geometry wherein the set of points is the open disk of $\mathbb{R}^2$ or $\mathbb{C}$ having center $(0,0)$ or $0+0\mathrm{i}$:



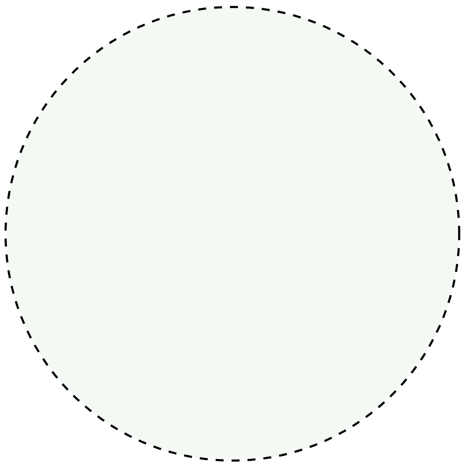
$$\mathcal{P} = \{z \in \mathbb{C} : |z| < 1\}$$

and the subsets playing the role of lines are all circular arcs contained in $\mathcal{P}$ and orthogonal to its boundary $\{z \in \mathbb{C} : |z| = 1\}$:



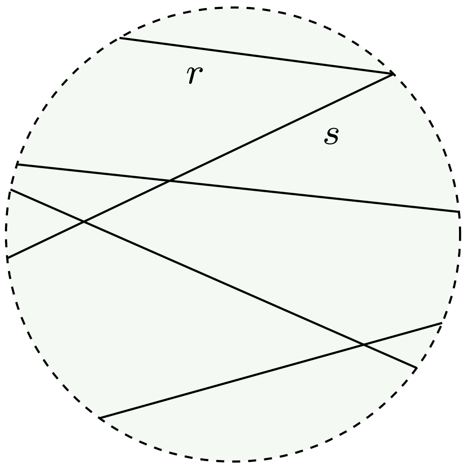
LINE r IS A HYPERBOLIC LINE (ITS CENTER LIES AT INFINITY)

- In the **Klein disk model**, the set of points is still the same as before:



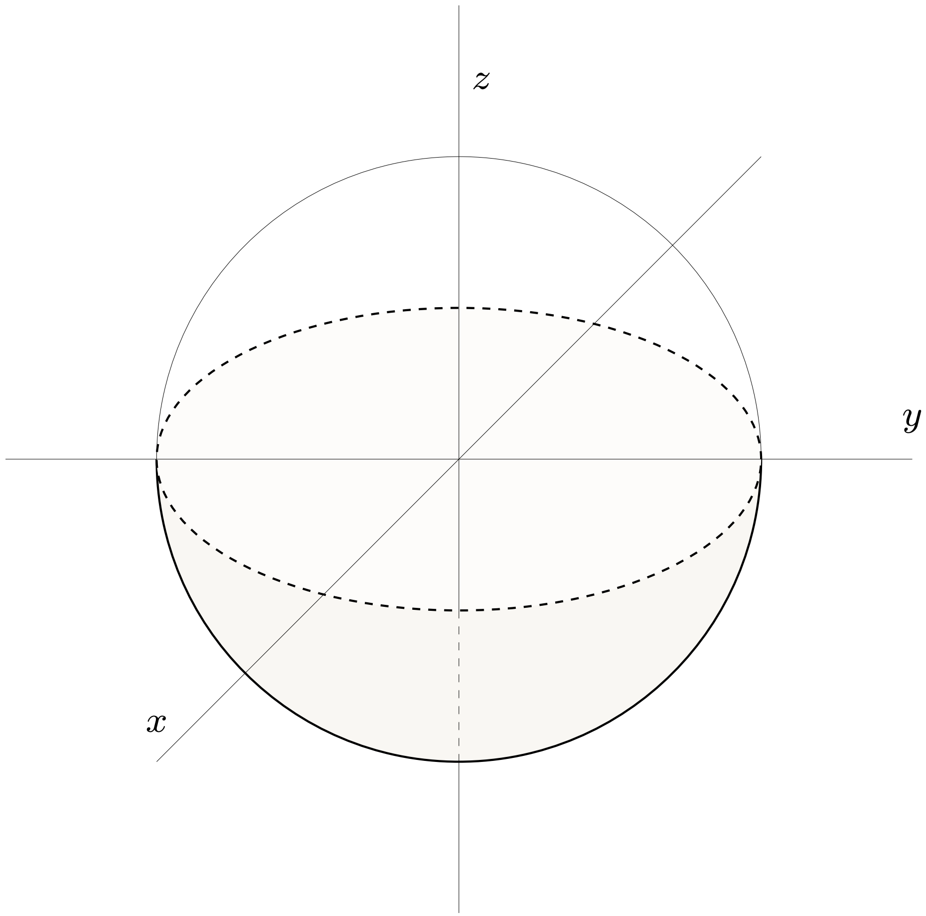
$$\mathcal{P} = \{z \in \mathbb{C} : |z| < 1\}$$

but the set of lines is represented by Euclidean straight line segments totally contained in $\mathcal{P}$:



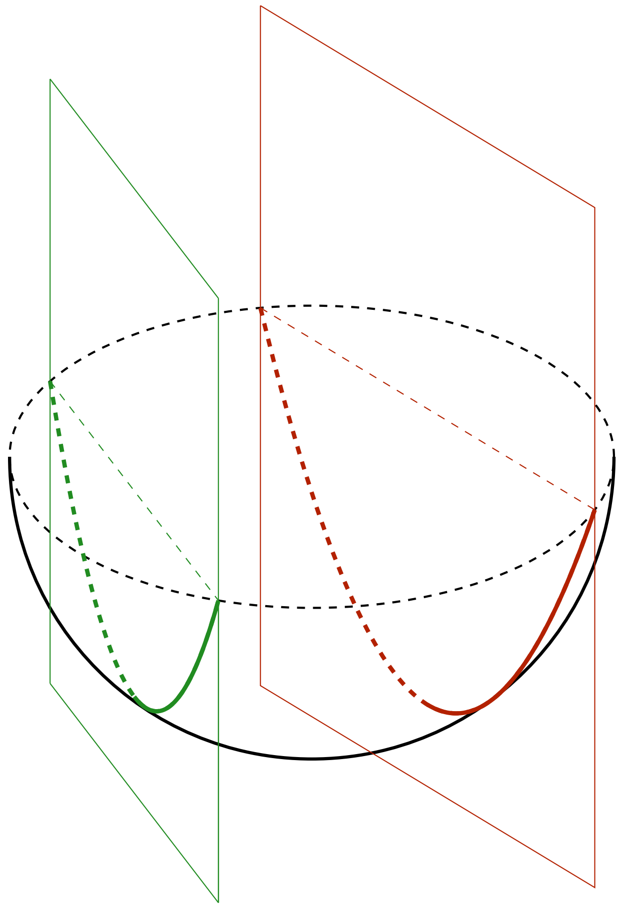
r IS PARALLEL TO ALL LINES SHOWN IN THIS FIGURE, INCLUDING s

- The **hemisphere model** is represented in three dimensions but is still easily seen (with some further details) to be isomorphic to a hyperbolic planar geometry whose set of points is the bottom half (excluding the boundary $z = 0$) of the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$:

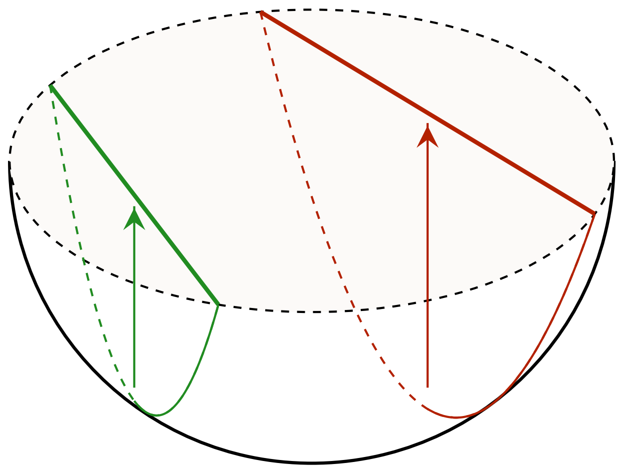


$$\mathcal{P} = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z < 0\right\}$$

The subsets playing the role of lines are the circular arcs resulting from intersecting the sphere with planes parallel to the z-axis:



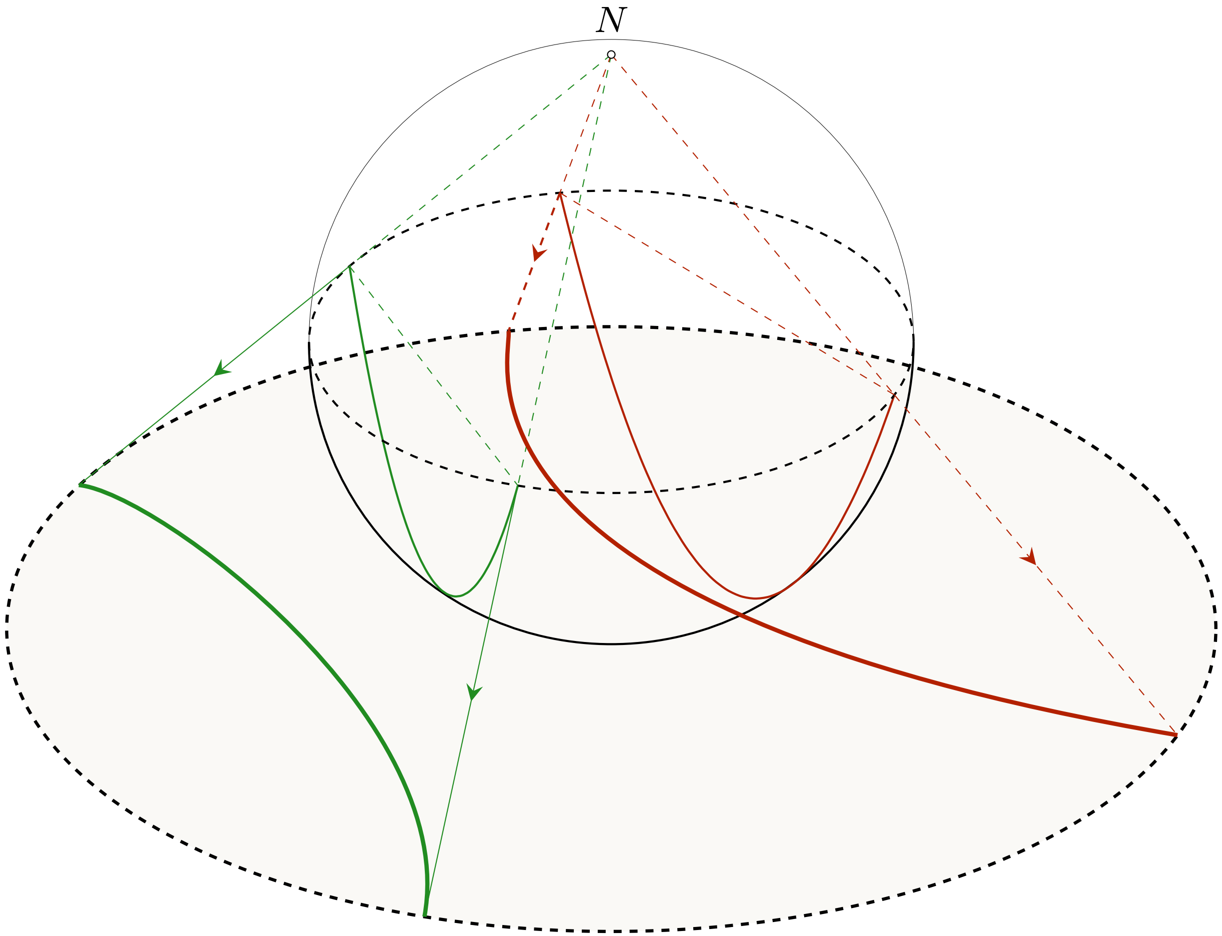
- Let us outline the isomorphisms between these models and the half-plane model we have been studying more closely. Firstly, if we project the hemisphere vertically on the (x, y)-plane, we obtain the Klein disk model:



vertical projections are nothing but intersections of the plane $z = 0$ with planes parallel to the z axis containing each of these hyperplane lines, thus the function is well-defined.

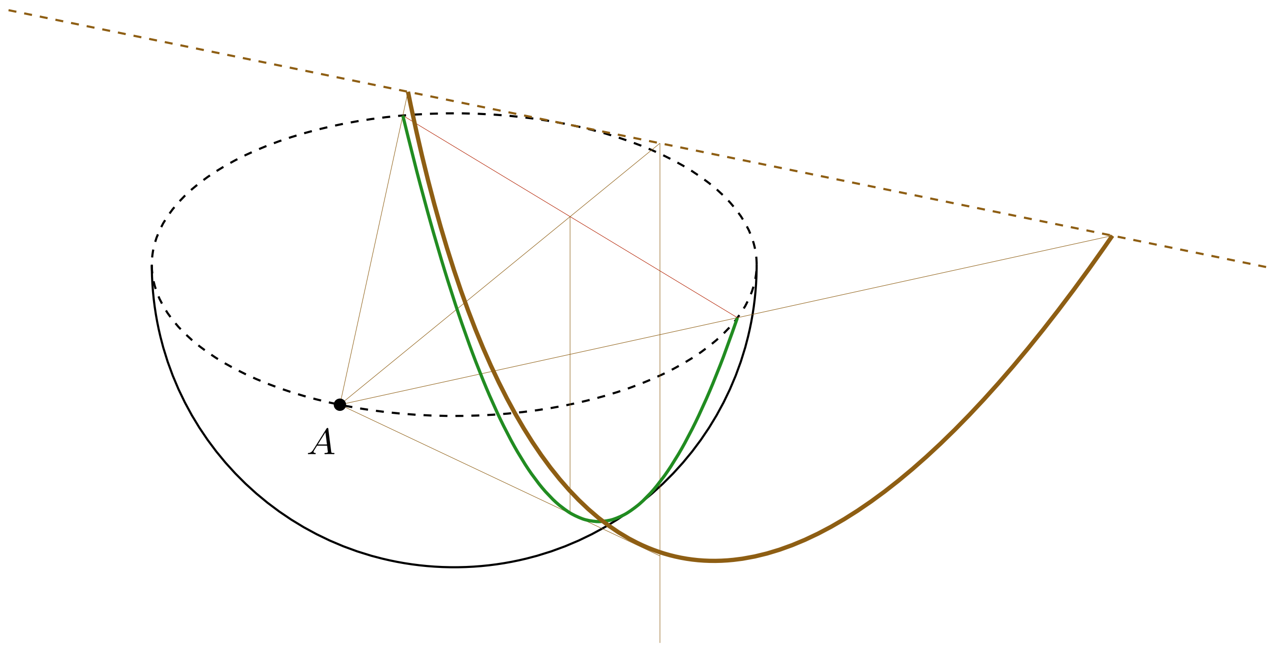
- EXERCISE: prove it is an isomorphism of geometries.

- If we project the hemisphere stereographically on the plane $y = -1$ from the north pole $N = (0, 0, 1)$ of the unit sphere, we obtain the Poincaré disk model (after adjusting homothety):



- EXERCISE: prove this is also an isomorphism of geometries.

- If we project the hemisphere stereographically from any fixed point in $z = 0$ to a plain tangent to the sphere and diametrically opposite to such point, we obtain the Poincaré half-plane model reverted:



- Exercise: prove that this is also an isomorphism of geometries.

Circles, horocycles and hypercycles

- We know the definition of a circle in a Euclidean geometry, but we need a more general definition in an absolute geometry $\mathcal{G}$ so that we can define it in $\mathbb{H}$ as well. If $P \in \mathcal{P}$ and $\overline{AB}$ is a segment of the geometry $\mathcal{G}$, then the **circle of center P and radii congruent to $\overline{AB}$** is the set of points of

$$\left\{Q \in \mathcal{P} : \overline{PQ} \equiv \overline{AB}\right\}.$$

  Example: if $\mathcal{G}$ is $\mathbb{R}^2$ with the usual Euclidean geometry then $\overline{AB} \equiv \overline{CD}$ iff $d(A,B) = d(C,D)$ and the circle of center P and radius $r = d(A,B)$ is the usual set of points we already know: $\left\{Q \in \mathbb{R}^2 : d(P,Q) = r\right\}$.
- The question arises: what are the circles in, say, the hyperbolic geometry of the half-plane $\mathbb{H}$?
- We first need a concept related to inversion:

Proposition 70. *Given two circles ω_1, ω_2 in the Euclidean geometry of $\mathbb{R}^2$. The following are equivalent for a given point C:*

(i) every circle or line orthogonal to both ω_1 and ω_2 contains C;

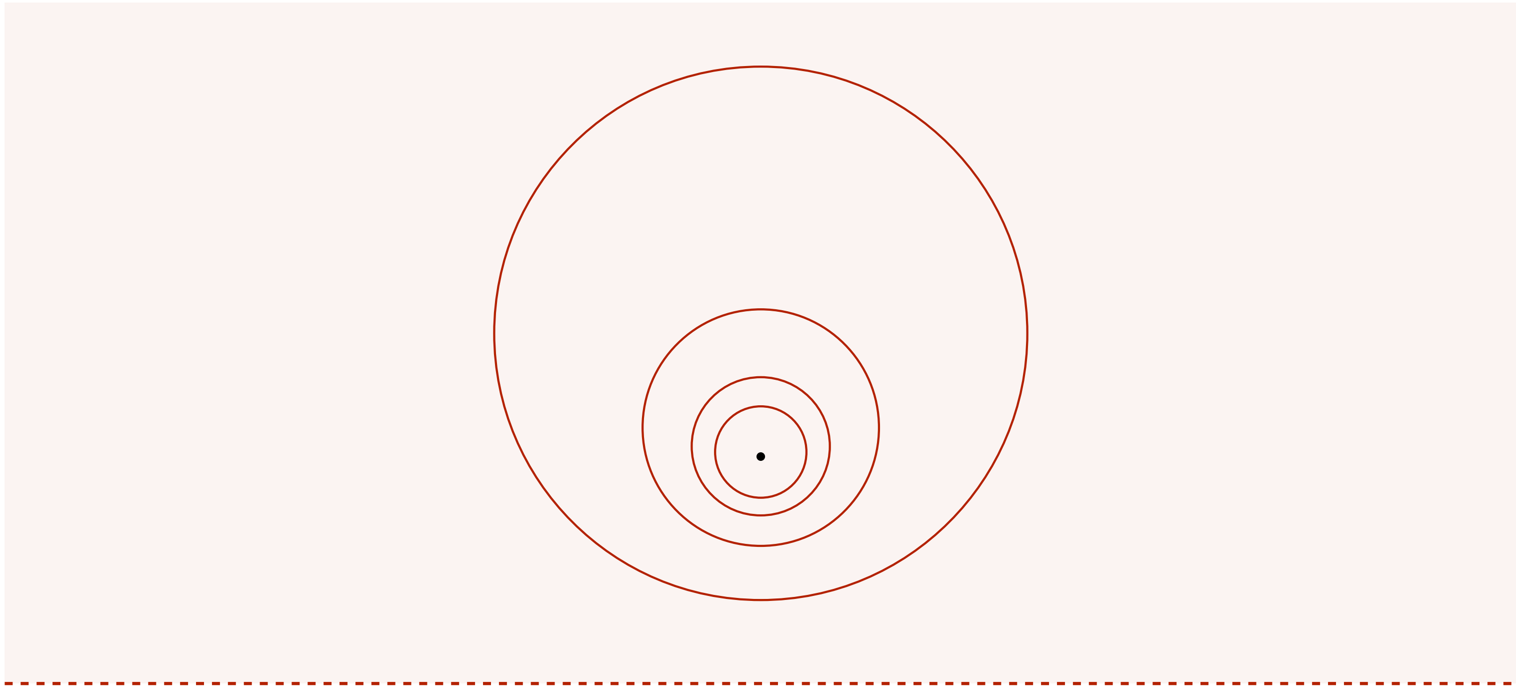
(ii) an inversion centered at C transforms ω_1 and ω_2 into concentric circles.

*We call C a **limiting point** of ω_1 and ω_2.*

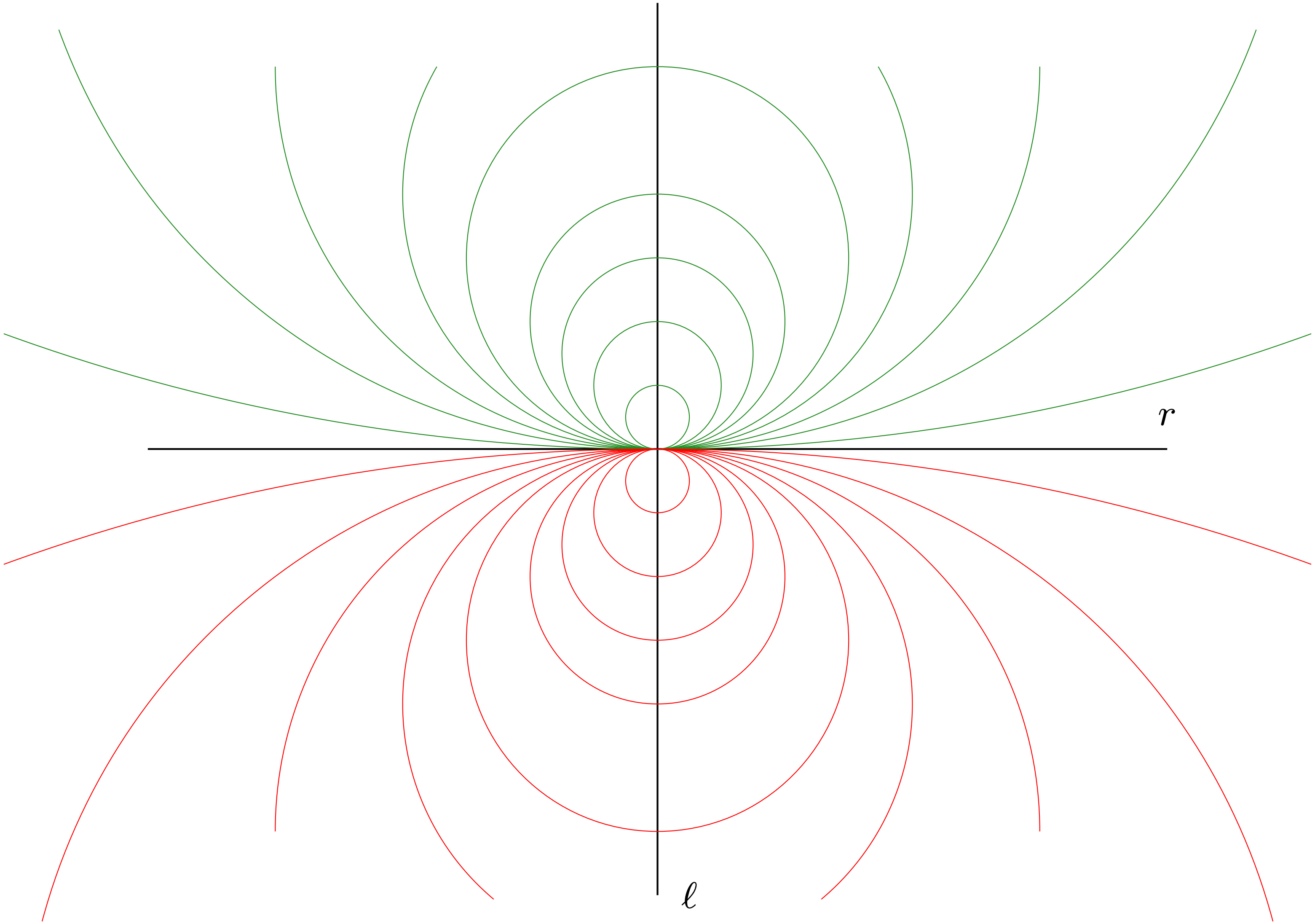
Theorem 71. *Let P be a point of the Poincaré half-plane $\mathbb{H}$. Every hyperbolic circle of center P is a Euclidean circle whose limiting point with the x-axis is point P.*

PROOF: EXERCISE.

- In other words, these would be concentric hyperbolic circles in $\mathbb{H}$ having the highlighted point as their center:

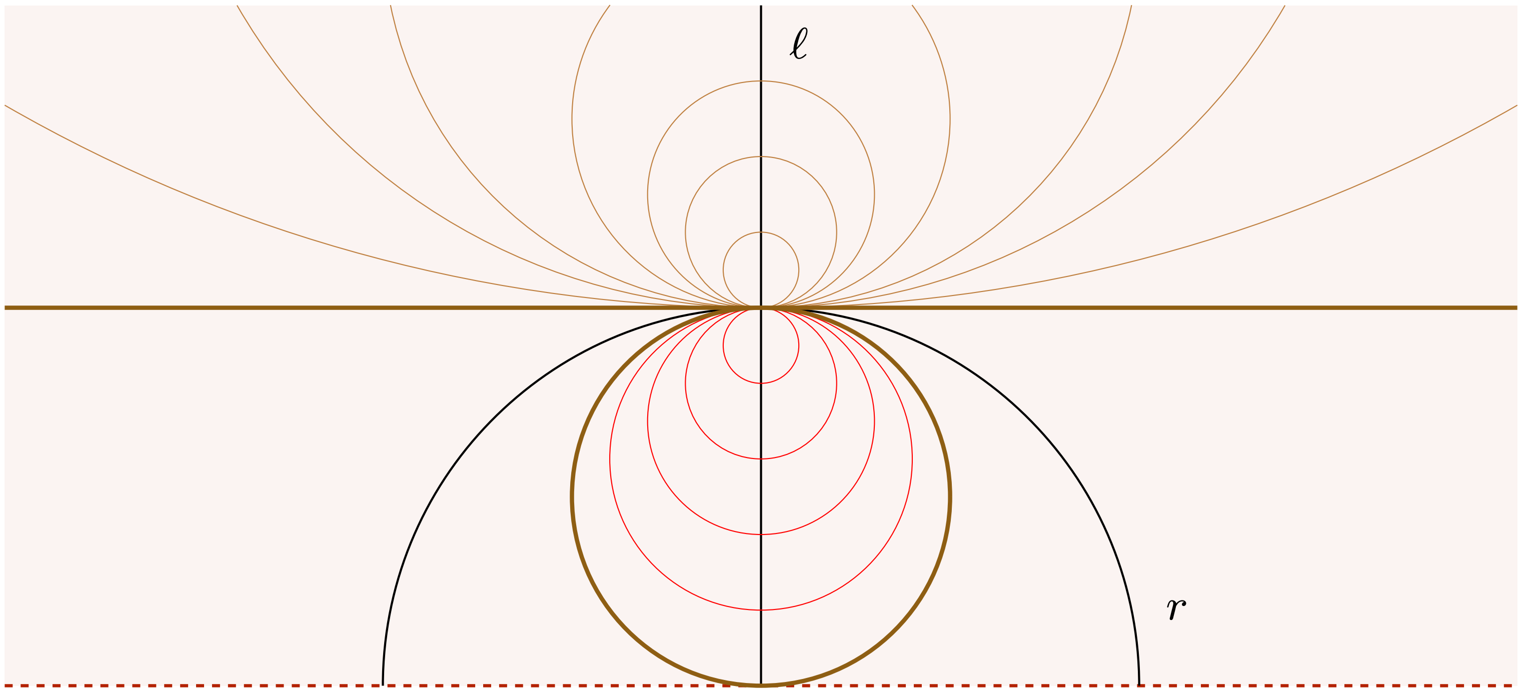


- Assume we have a Euclidean line r, an orthogonal to it ℓ and the pencil of circles tangent to r having center in ℓ:



it becomes clear that r appears geometrically as the limit case from both sides whenever the radius of the circles tends to ∞. We call r a (Euclidean) **horocycle**.

- Similarly, in hyperbolic geometry if two lines r and ℓ are orthogonal (in this case one or both can be half-circles as Euclidean subsets), the **horocycle** defined by r and ℓ can be a Euclidean circumference tangent to the x-axis, or a horizontal line, e.g. these highlighted below:

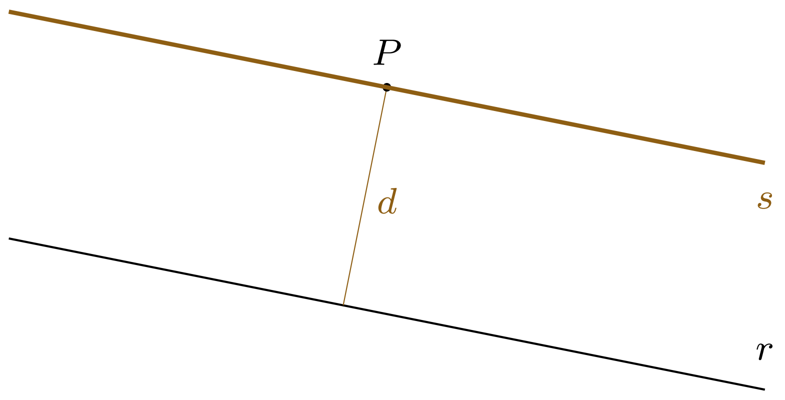


EXERCISE: draw examples in which both hyperbolic lines manifest as Euclidean half-circles.

- In Euclidean geometry, given line r and point $P \notin r$, there exists a well-defined set

$$\{Q \in \mathcal{P} : d(Q, r) = d(P, r)\} \tag{2}$$

which in this context is the (unique) line parallel to r containing P:



If we characterize s by distance condition (2) we call it a **hypercycle** in this geometry.

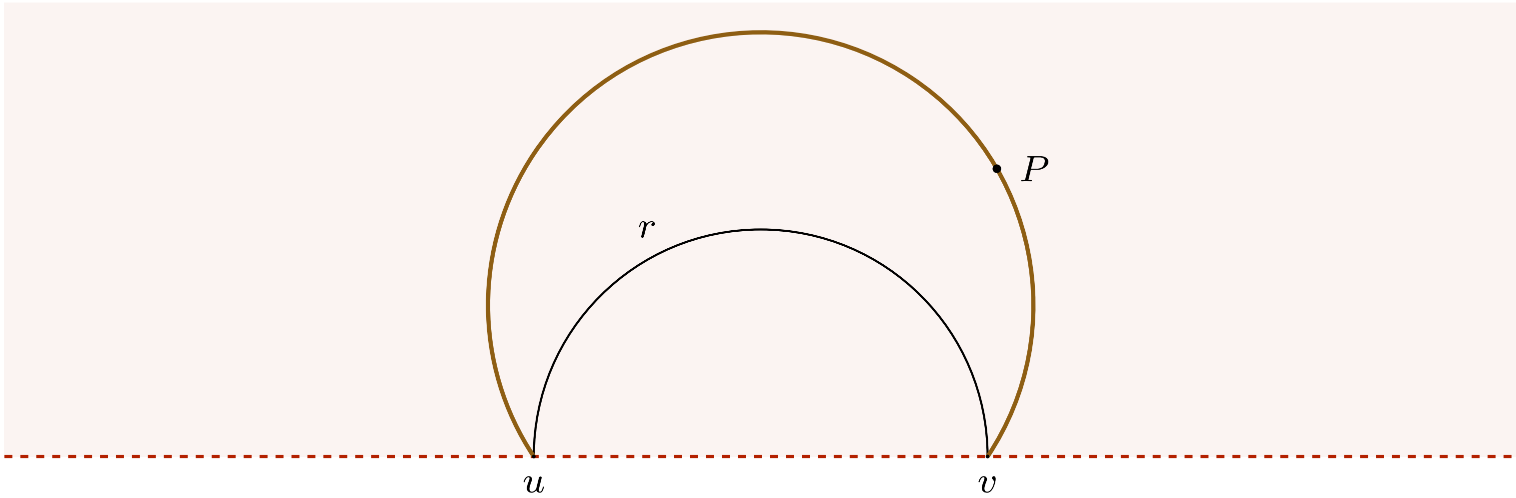
- In a hyperbolic geometry $\mathcal{G}$, we can impose a similar condition on this geometric locus to be a **hypercycle** of $\mathcal{G}$:

$$s = \left\{Q \in \mathcal{P} : \overline{PP'} \equiv \overline{QQ'}\right\},$$

where P' (resp. Q') is the intersection of the unique perpendicular to r through P (resp. P') with s.

- In the hyperbolic geometry of $\mathbb{H}$ this translates in the following way:

Theorem 72. *Let u, v the points of intersection of line r with the x-axis. Then the hypercycle defined by r and P is the Euclidean circular arc containing P, u and v.*



PROOF: EXERCISE. With the understanding that if r is modeled as a vertical Euclidean ray, then one of u, v equals ∞ and this circular arc becomes something else (think about what it could be).

- It is important to remark: horocycles and hypercycles in $\mathbb{H}$ are *not* hyperbolic lines.

Metric properties in an absolute geometry

- Let $\mathcal{G}$ be an absolute geometry. Recall the set of segments as defined by

$$\mathcal{S} = (\mathcal{P} \times \mathcal{P} \smallsetminus \Delta) \,/ \sim,$$

where $\Delta = \{(P,Q) : P = Q\}$ and $\sim$ is the equivalence relation whereby $(P,Q) \sim (Q,P)$. A **length function** on $\mathcal{G}$ is any map

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{L} & \mathbb{R}_{>0} \\ \overline{AB} & \longmapsto & L(\overline{AB}) \end{array}$$

satisfying the following:

- if $\overline{AB} \equiv \overline{A'B'}$, then $L(\overline{AB}) = L(\overline{A'B'})$;
- if A, B, C are aligned and B is between A and C, then

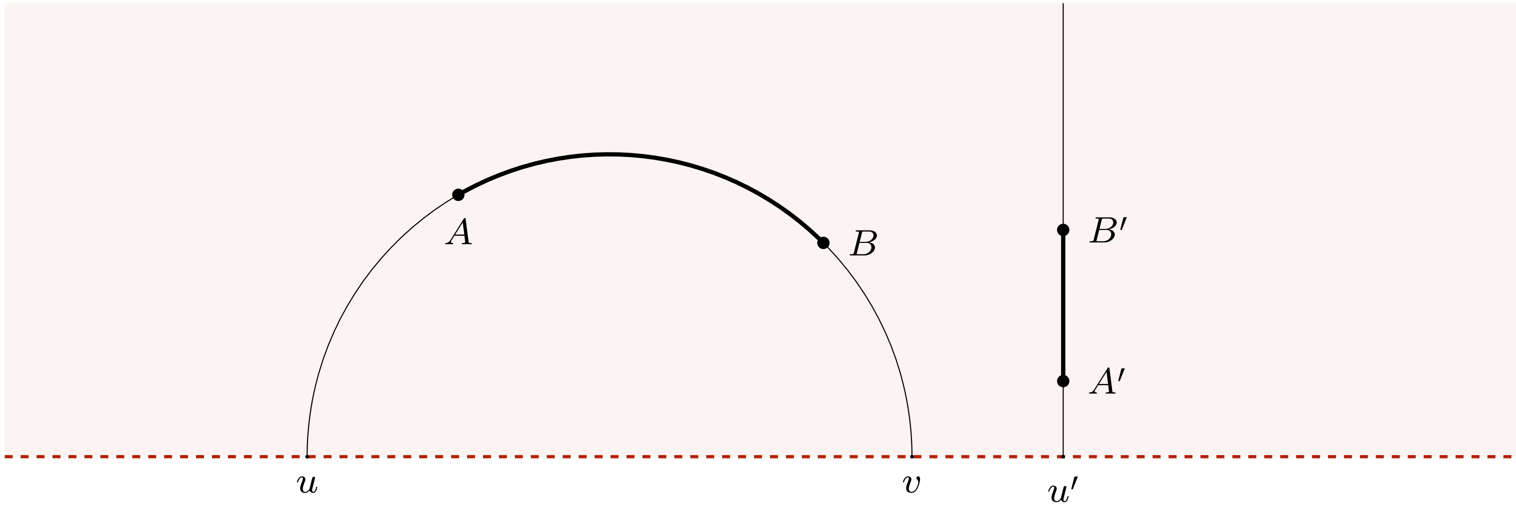
$$L(\overline{AC}) = L(\overline{AB}) + L(\overline{BC}).$$

Theorem 73. *Let $\mathcal{G}$ be an absolute geometry, O, O' two points of $\mathcal{P}$ and $k > 0$ a fixed real constant. Then there exists a unique length function L on $\mathcal{G}$ such that $L(\overline{OO'}) = k$.*

Corollary 74. *There are infinitely many length functions defined on $\mathcal{G}$.*

Proof: Exercise.

- In Euclidean geometry, there is nothing new to explain: a valid example of a length function is the usual Euclidean distance between two points in the plane.
- In hyperbolic geometry, let $A, B \in \mathbb{H}$, $k \in (0, \infty)$. Let u, v be the intersections of the hyperbolic line containing A, B with the x-axis ($u = \infty$ or $v = \infty$ means a vertical Euclidean ray).



Then L defined by $\boxed{L(\overline{AB}) := k\,|\ln(A, B, u, v)|}$ is a length function on $\mathbb{H}$. Again, with the understanding that for any four points $A, B, C, D \in \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, their cross-ratio is

$$(A, B, C, D) = \frac{AC}{AD}\frac{BD}{BC} \text{ and if } D = \infty, \text{ then } (A, B, C, D) = (A, B, C) = \frac{AC}{BC}.$$

Exercise: prove the above.

Theorem 75. *Let $s \subset \mathbb{H}$ be a hyperbolic circle of radius r, where the length function $L(\overline{AB}) := k\,|\ln(A, B, u, v)|$ described above is considered. Then the circumference of s equals*

$$L_r = k \cdot 2\pi \sinh \frac{r}{k}.$$

PROOF: EXERCISE.

- Observe for instance that if $k \to \infty$,

$$L_r = k \cdot 2\pi \sinh \frac{r}{k} = k \cdot 2\pi \sinh \left(\frac{r}{k} + \frac{1}{3!} \left(\frac{r}{k} \right)^3 + \dots \right) \to 2\pi r.$$

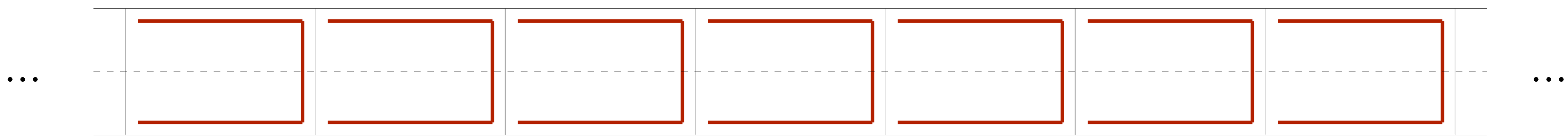
EXERCISE: interpret this.

Exercises

- Prove everything marked as an EXERCISE in this section.
- Let $\mathcal{G}$ be a hyperbolic geometry. Let A be a point exterior to a line t. Using the axioms of Absolute Geometry and Lobachevsky's Axiom, prove that there exists an angle α with origin in A such that the lines s_α and s'_α adjacent to α do not intersect t and the angle bisector of α is perpendicular to t.
- Let $(x-a)^2 + (y-b)^2 = r^2$ $(b > r > 0)$ be a circle totally contained in $\mathbb{H}$. Find the center of this circle as a non-Euclidean center.
- Conversely, if you are given a hyperbolic circle with center P and radius ρ in $\mathbb{H}$, find its Euclidean center and radius.
- Prove that $\cosh c = \cosh a \cosh b$ (*Hyperbolic Pythagoras*).

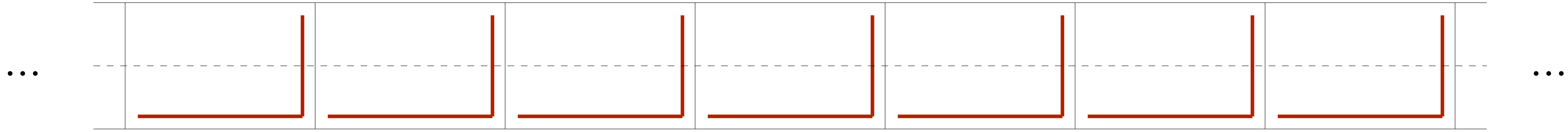
III Symmetries

- Let us first delineate the problem: let $\mathcal{G}$ be an absolute geometry and $C \subset \mathcal{P}$ a configuration of points and lines in $\mathcal{G}$. Fix a length function on $\mathcal{G}$ (e.g. Euclidean distance if the geometry is Euclidean). Recall the set of automorphisms of $\mathcal{G}$ (Proposition 66), i.e. isomorphisms of a geometry to itself. Let $G \subset \mathrm{Aut}(\mathcal{G})$ be the subgroup consisting of **isometries**, i.e. length-preserving automorphisms.
- $\mathrm{Sym}(C)$ be the **group of symmetries of configuration C**, is the subgroup of G consisting of all those transformations leaving C fixed.
- Our aim is to study these symmetries of C whenever $\mathcal{G}$ has a tractable, familiar structure that makes the study relatively simple.
- First example: consider the configuration defined by the dark red segments below:

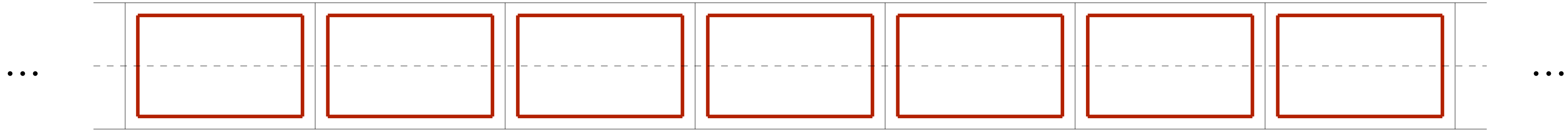


then the symmetries of this configuration would consist of the identity, an reflection α with respect to the dashed line, and τ^n, $n \in \mathbb{Z}$ where τ is the horizontal translation whose direction vector has the same length as that of one of the individual cells.

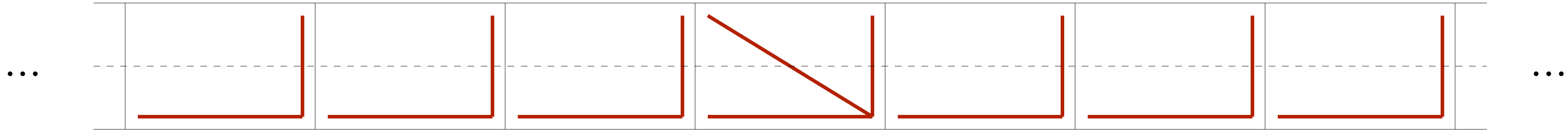
- It is easy to see that the symmetry group will change if we modify the configuration a bit, e.g. this one which does not admit reflections as symmetries (i.e. $\mathrm{Sym}\,(C) = \{\tau^k : k \in \mathbb{Z}\}$):



or like this one, in which certain rotations (think which) of angle $k\pi$, $k \in \mathbb{Z}$ are also allowed,

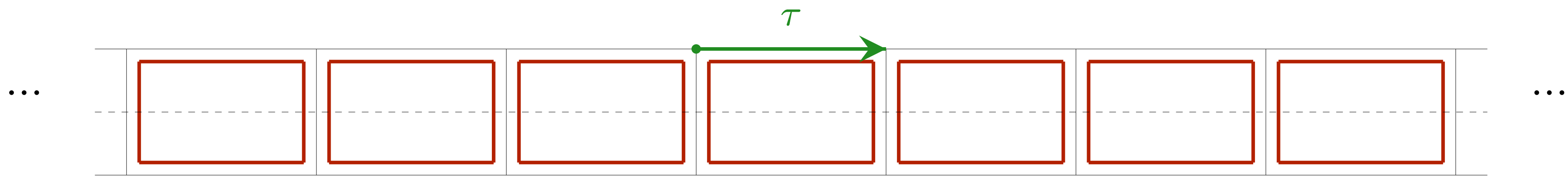


and that in some cases, e.g. the one below in which one of the cells is different from all the others,

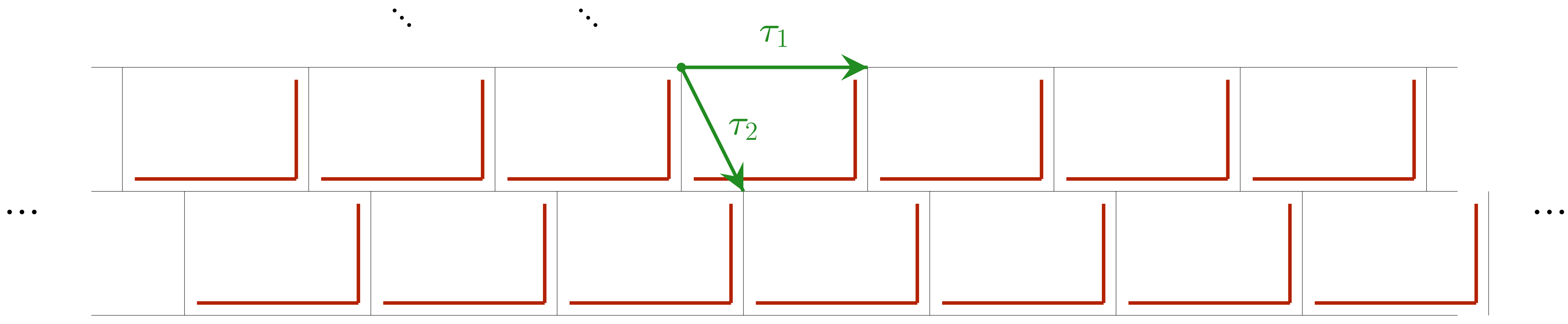


the symmetry group will be trivial: $\mathrm{Sym}\,(C) = \{\mathrm{id}\}$.

- Our classification will study three types of configurations:
  - those in which $\operatorname{Sym}(C)$ is finite, e.g. the last pattern in the previous page;
  - those with an infinite symmetry group S, including a subgroup $H = S \cap \{\text{translations}\}$ such that $H \cong \mathbb{Z}$, i.e. patterns repeating in *one* direction, e.g. most of the patterns seen in the previous two pages. We call such configurations **friezes** or **frieze patterns**:

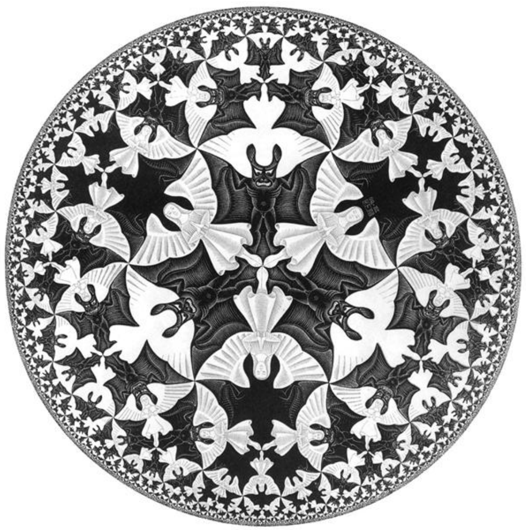


  - those with an infinite symmetry group S, such that $H = S \cap \{\text{translations}\} \cong \mathbb{Z}^2$, i.e. patterns repeating in *two* directions. We call them **mosaics** or **wallpapers**.



$\langle \tau_1, \tau_2 \rangle \cong \mathbb{Z}^2$

- We will study these symmetries in the Euclidean case because we have a classification Theorem 24 for isometries of the plane, i.e. transformations of $\mathrm{Aut}(\mathbb{R}^2)$ preserving distances:
  - Direct (or even) isometries:
    - **Rotations**, including the identity
    - **Translations** by a fixed vector
    - **Central symmetries**,
  - Indirect (or odd) isometries:
    - reflections symmetries,
    - glide reflections
- In other geometries, e.g. hyperbolic geometries, the problem becomes more complicated:



M. C. ESCHER, *CIRCLE LIMIT IV* (1960)
SOURCE: WIKIART

1. Leonardo Da Vinci's Theorem

- Remember that a finite group G is called **cyclic** of order n if it is isomorphic to $(\mathbb{Z}_n, +)$, i.e. it is generated by one element, $G = \langle \sigma : \sigma^n = e \rangle$ wherein $\sigma^m \neq e$ for $m < n$. We usually write (any group isomorphic to) one such group as C_n.
- Also a group is called **dihedral** of order $2n$ and written D_n if

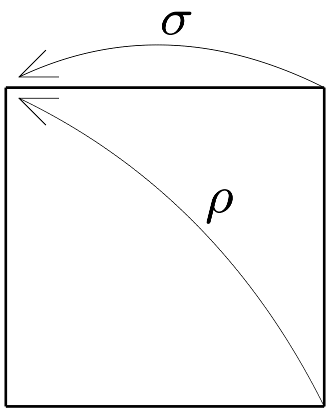
$$G \cong \left\langle \rho, \sigma : \sigma^n = \rho^2 = (\rho\sigma)^2 = e \right\rangle, \qquad \text{where } \sigma^m \neq e \text{ for every } m < n.$$

Theorem 76 (Leonardo Da Vinci). *Let $G \subset \mathrm{Aut}(\mathbb{R}^2)$ be a subgroup of isometries. If G is finite, then it is isomorphic to either C_n or D_n for some $n \in \mathbb{N}$.*

PROOF: EXERCISE.

Remark: every group C_n or D_n is the group of symmetries of a certain configuration. Indeed,

- D_n is the group of $2n$ symmetries of a regular polygon of n sides,
- and C_n is the subgroup of D_n corresponding of the n rotations by angles of the form $\frac{2\pi k}{n}$.



ρ: REFLECTION

σ: ROTATION OF ANGLE $\frac{2\pi}{4} = \frac{\pi}{2}$

$$D_4 = \langle \sigma, \rho \rangle = \left\{ \mathrm{id}, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3 \right\}$$
$$C_4 = \langle \sigma \rangle = \left\{ \mathrm{id}, \sigma, \sigma^2, \sigma^3 \right\} \subseteq D_4$$

2. Frieze classification

- A configuration C in an absolute geometry $\mathcal{G}$ is a **frieze** if it has an invariant line, and the group of translations that are symmetries of C is infinite cyclic, i.e. $\{\text{translations of } \mathbb{R}^2\} \cap \operatorname{Sym}(C) \cong \mathbb{Z}$. Given that the group of translations of two-dimensional vectors is nothing but the additive group $(\mathbb{R}^2, +)$, this is the same as saying $\mathbb{R}^2 \cap \operatorname{Sym}(C) \cong \mathbb{Z}$.
- Let C be a frieze and translation τ a generator of the infinite cyclic subgroup $\mathbb{R}^2 \cap \operatorname{Sym}(C)$. The **International Union of Crystallography** (IUCr) has its own notation to classify friezes according to their symmetries:

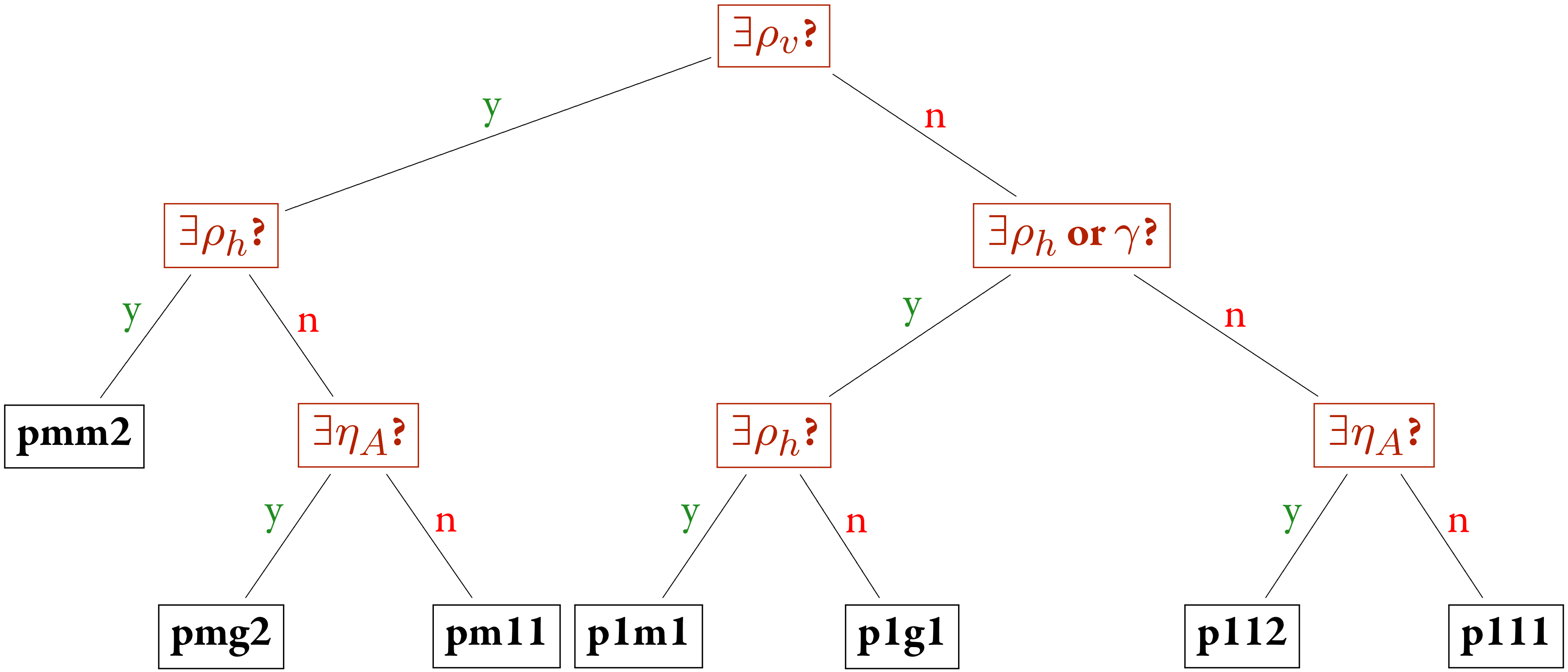
<table>
<tr><td rowspan="6">p</td><td rowspan="3">m</td><td rowspan="2">m</td><td rowspan="3">2</td></tr>
<tr></tr>
<tr><td rowspan="2">g/a</td></tr>
<tr><td rowspan="3">1</td><td rowspan="3">1</td></tr>
<tr><td rowspan="2">1</td></tr>
<tr></tr>
</table>

  (i) First column:
  - **m** if it admits a vertical symmetry ρ_v;
  - **1** if it does not.

  (ii) Second column:
  - **m** if it admits a horizontal symmetry ρ_h;
  - **g** (sometimes **a**) if it admits a glide reflection γ;
  - **1** if neither of the above.

  (iii) Third column:
  - **2** if it admits a central symmetry η_A for some point A;
  - **1** if it does not.

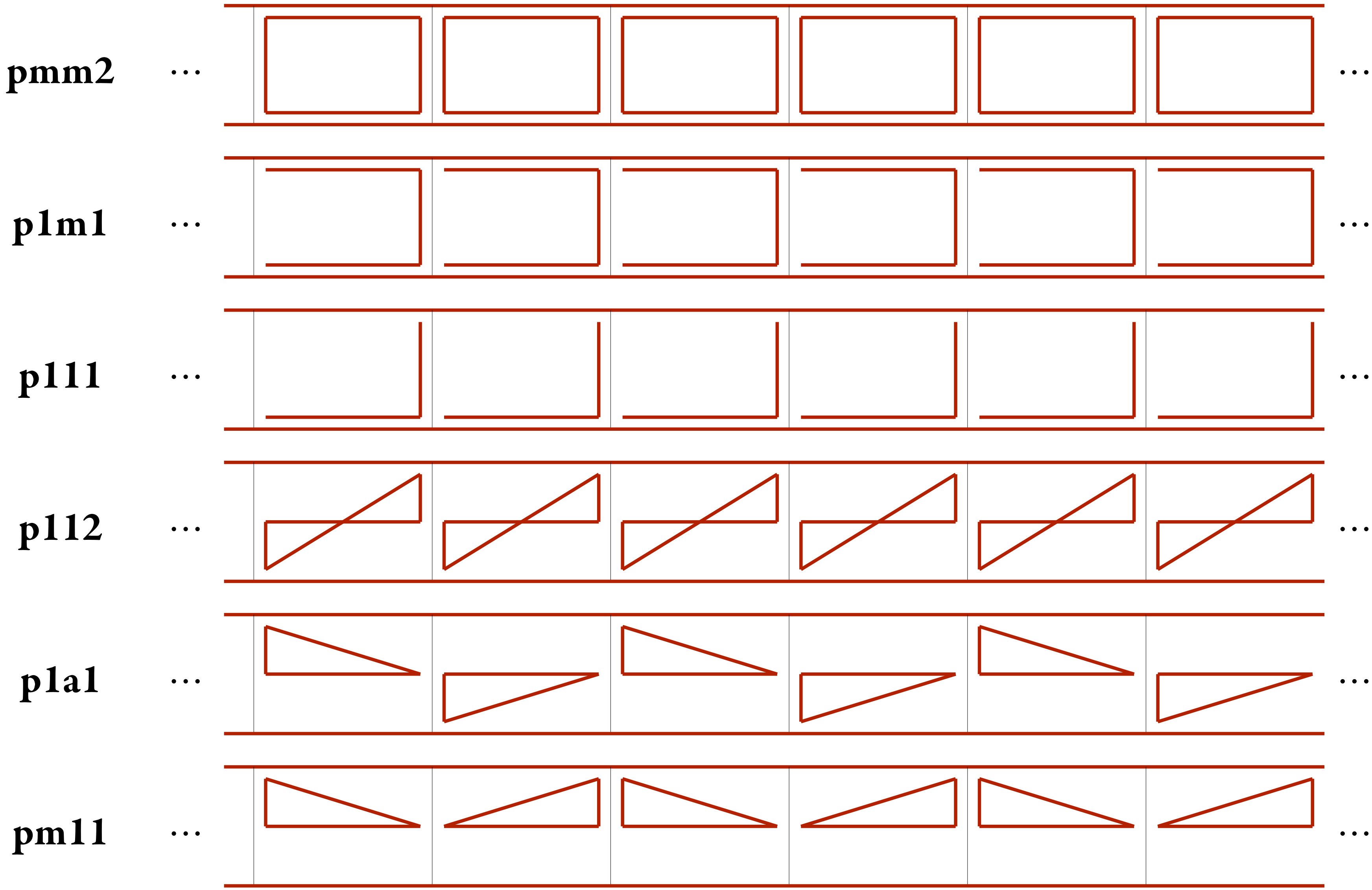
Classification algorithm



Theorem 77. *These (along with translation τ) are all the possible frieze symmetries, i.e. the algorithm works.*

Proof: Exercise.

Examples:



3. Wallpaper classification

- Let G be a subgroup of $\mathrm{Aut}(\mathbb{R}^2)$ such that $G \cap \mathbb{R}^2$ is a free abelian group of rank 2, i.e.
$$\langle \tau_1, \tau_2 \rangle, \qquad \tau_1, \tau_2 \in \mathbb{R}^2 \text{ linearly independent vectors.}$$
A **wallpaper** is a configuration having translation lattice $\{n_1\tau_1 + n_2\tau_2 : n_1, n_2 \in \mathbb{Z}\} \subset \langle \tau_1, \tau_2 \rangle$ among its symmetries. Following the IUCr notation, we need the following table now:

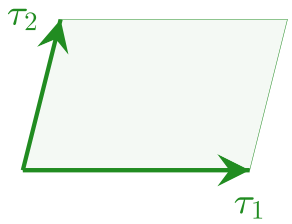
p c	1 2 3 4 6	m a/g 1	m a/g 1

summarizing in a total of 17 symmetry groups which will be explained in the next few pages.
- The choice of **p** or **c** will be given by the shape of the fundamental domain, i.e. the parallellogram determined by translations τ_1, τ_2 leaving the wallpaper invariant, see next page.
- The choice of the second column will be given by the following

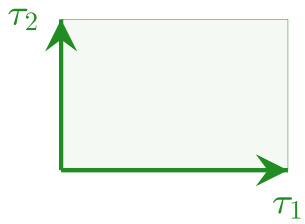
Lemma 78. *If a wallpaper is also invariant by a rotation of angle $\frac{2\pi}{k}$, then $k \in \{1, 2, 3, 4, 6\}$.*

- The choice of the third column will be given by whether the wallpaper is invariant by one reflection (**m**), glide reflection (**a** or **g**) or neither (**1**)
- The choice of the fourth column will be given by whether the wallpaper is invariant by two linearly independent reflections (**m**), two linearly independent glide reflections (**a** or **g**) or neither (**1**).

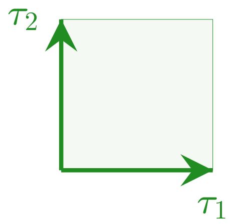
- The group will have a **p** assigned to it if the primitive or fundamental parallellogram determined by τ_1, τ_2 is one of the following:



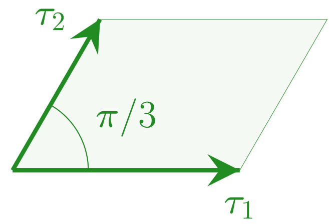
General



Rectangular

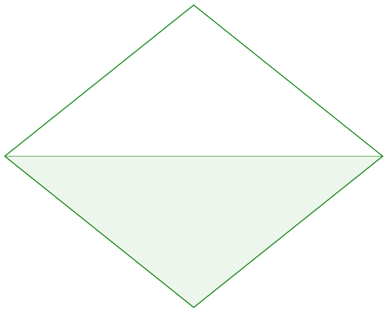


Square

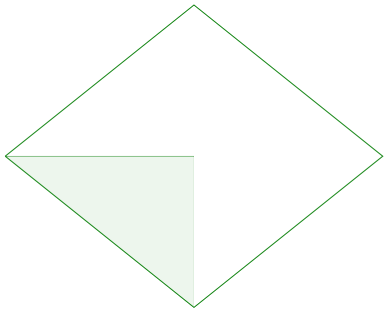


Hexagonal

- And it will have a **c** attached to it if the fundamental parallellogram is rhombic and *centered*, i.e. its relation to a *primitive cell* (meaning a subconfiguration that is sufficient to reconstruct the entire wallpaper with translations, reflections and rotations) is as follows:



Reflection of primitive cell yields entire fundamental parallellogram



Reflection and angle π rotation of primitive cell yields entire fundamental parallellogram

- So the algorithm works as follows. Is the mosaic invariant by a rotation angle $\theta = \frac{2\pi}{k}$?

  $\boxed{k=1}$ and we assume this is the largest value of k satisfying the above.

  * **Invariant by one reflection ρ_1?**
    - YES: in which case **Is the cell centered?**
      - YES: **c1m1**
      - NO: **p1m1**
    - NO: in which case **Is the wallpaper invariant by a glide reflection?**
      - YES: **p1a1**
      - NO: **p111**

  $\boxed{k=2}$ and assume it is the largest value of k for which a rotation of angle $\frac{2\pi}{k}$ is a symmetry.

  * **Invariant by one reflection ρ_1?**
    - YES: in which case **Invariant by a ρ_2 linearly independent with ρ_1?**
      - YES: in which case **Is the cell centered?**
        - YES: **c2mm**
        - NO: **p2mm**
      - NO: **p2ma**
    - NO: in which case **Is the wallpaper invariant by a glide reflection?**
      - YES: **p2aa**
      - NO: **p211**

$\boxed{k=3}$ and assume it is the largest value of k for which a rotation of angle $\frac{2\pi}{k}$ is a symmetry.

* **Invariant by one reflection ρ_1?**
  - YES: in which case **are all rotation centers on the reflection axes?**
    - YES: **p3m1**
    - NO: **p31m**
  - NO: **p311**

$\boxed{k=4}$ and assume it is the largest value of k for which a rotation of angle $\frac{2\pi}{k}$ is a symmetry.

* **Invariant by one reflection ρ_1?**
  - YES: in which case **invariant by reflections in four directions?**
    - YES: **p4mm**
    - NO: **p4am**
  - NO: **p411**

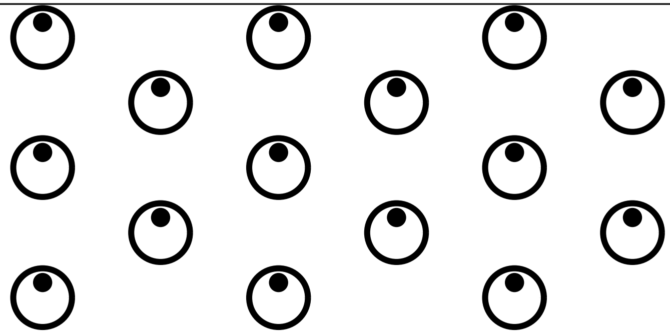
$\boxed{k=6}$ and assume it is the largest value of k for which a rotation of angle $\frac{2\pi}{k}$ is a symmetry.

* **Invariant by one reflection ρ_1?**
  - YES: **p6mm**
  - NO: **p611**

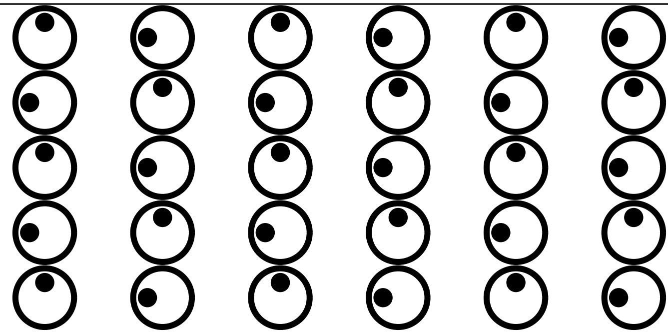
Theorem 79. *These (along with translations τ_1, τ_2) are all the possible wallpaper symmetries, i.e. the algorithm works.*

PROOF: EXERCISE.

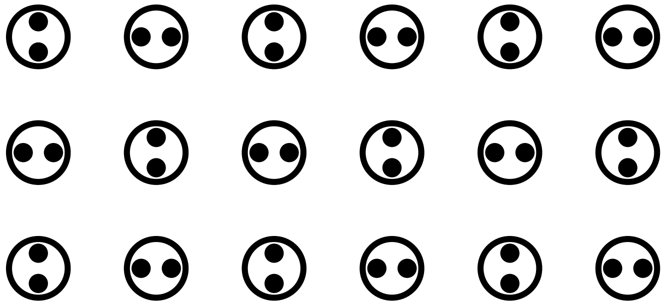
Examples:



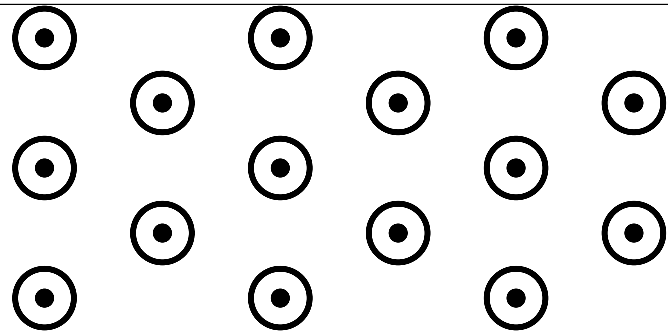
c1m1



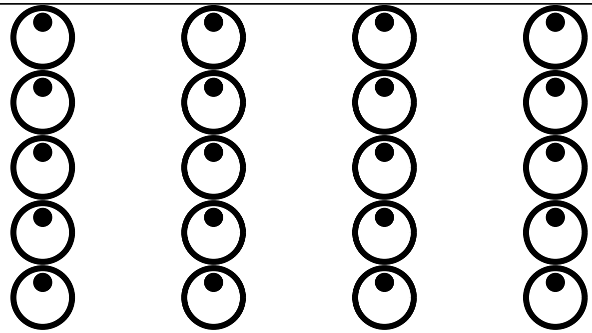
p111



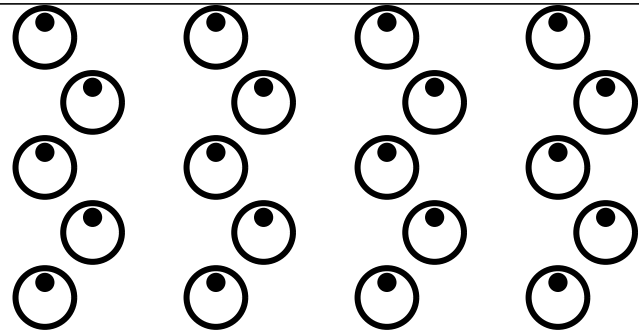
p4mm



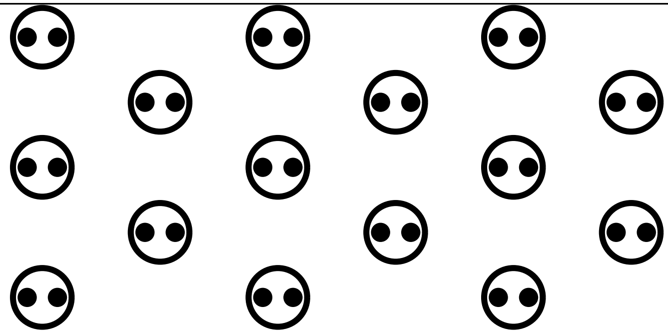
p6mm



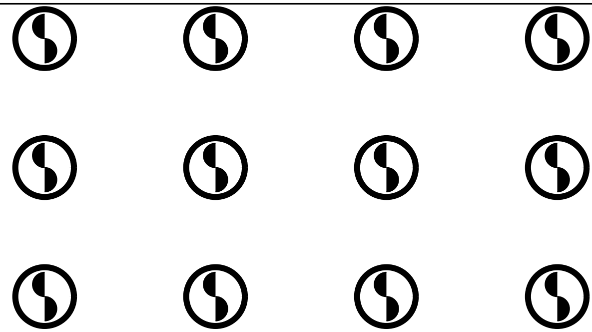
p1m1



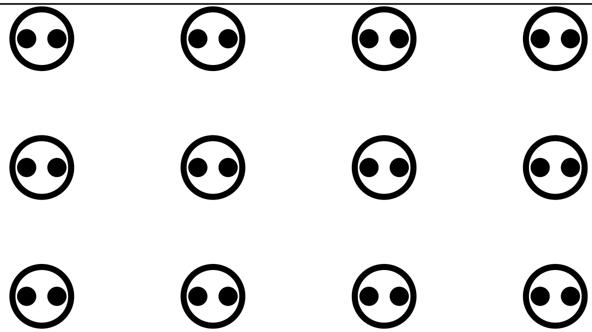
p111



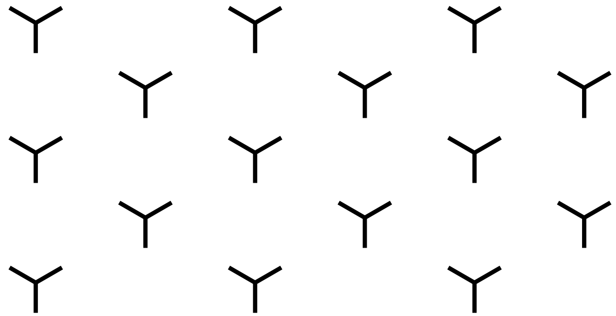
c2mm



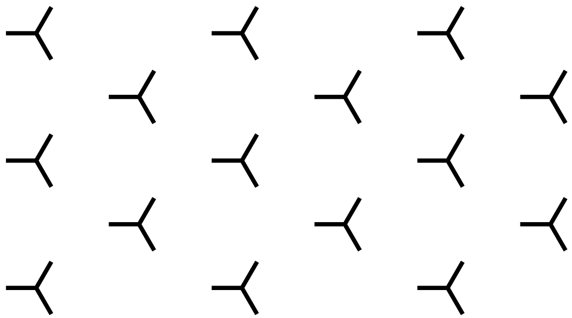
p211



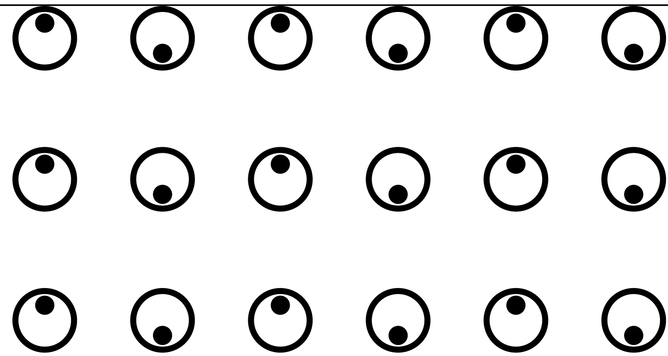
p2mm



p31m



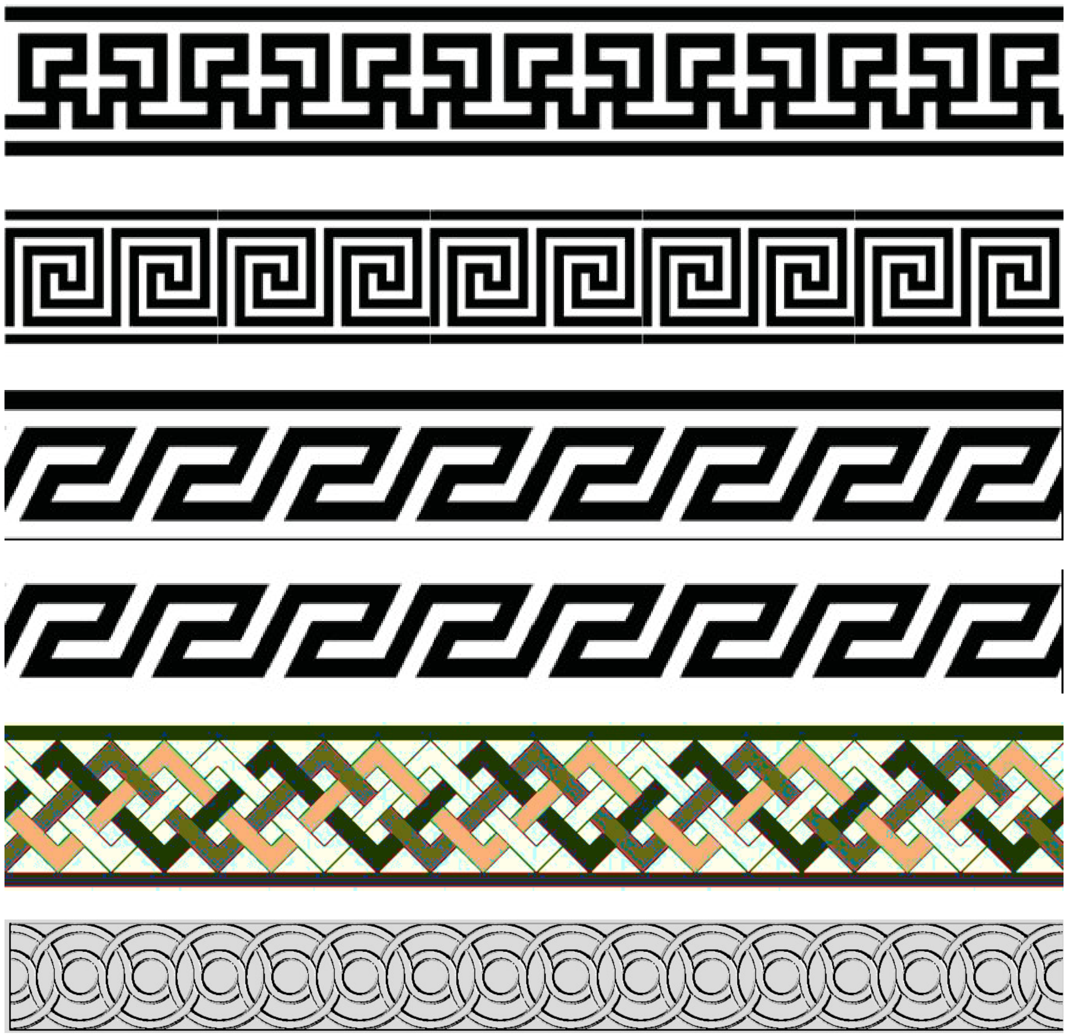
p3m1

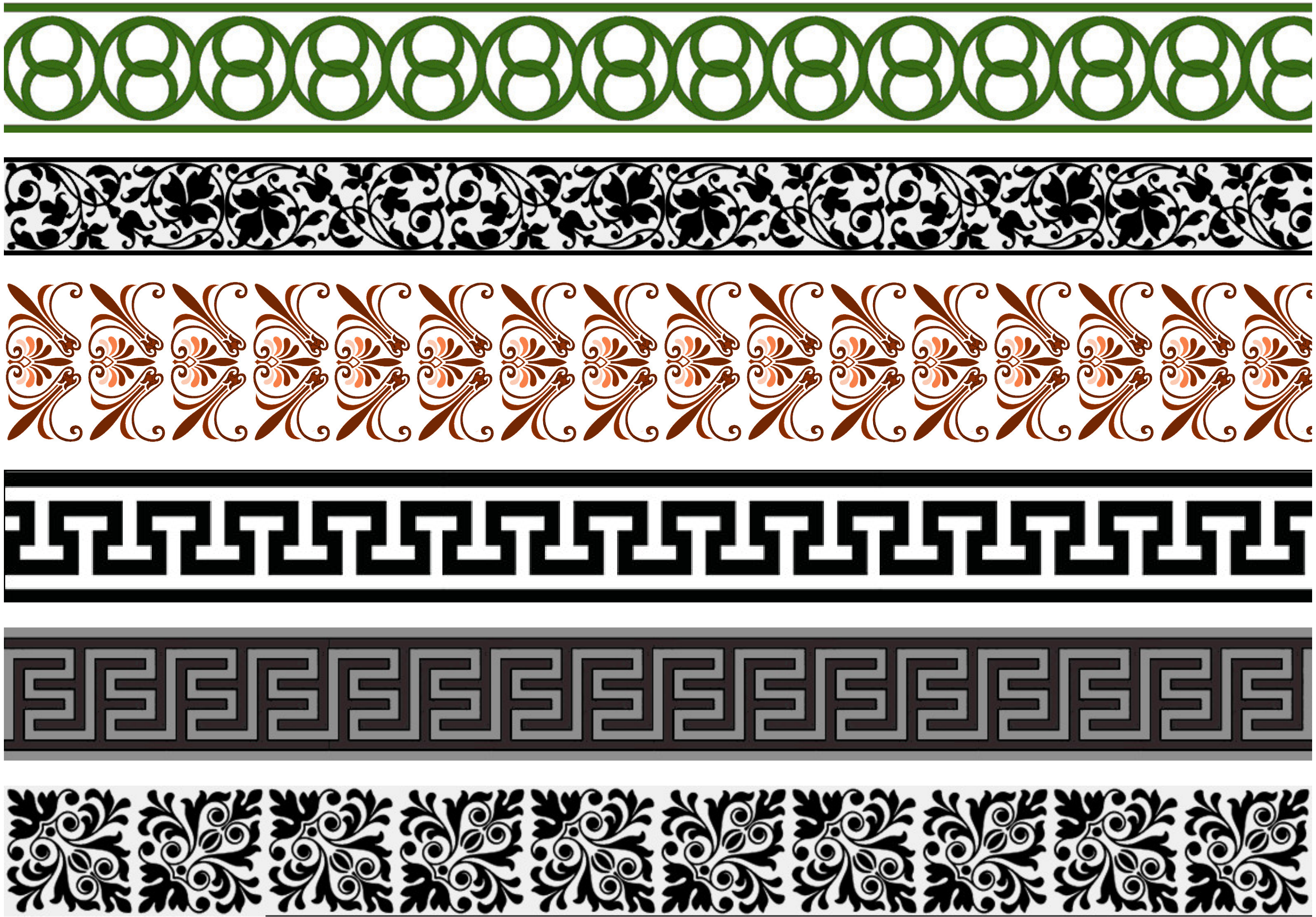


p2ma

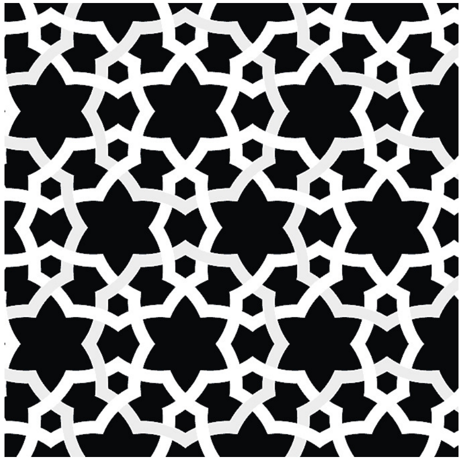
Exercises

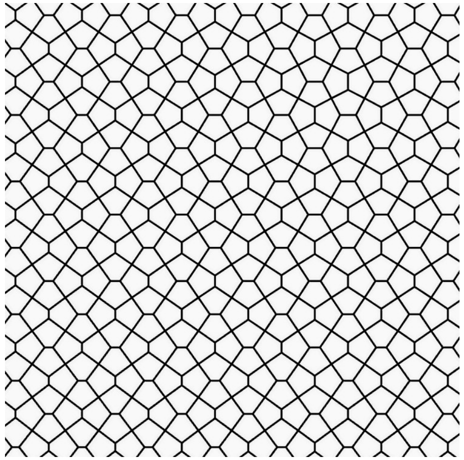
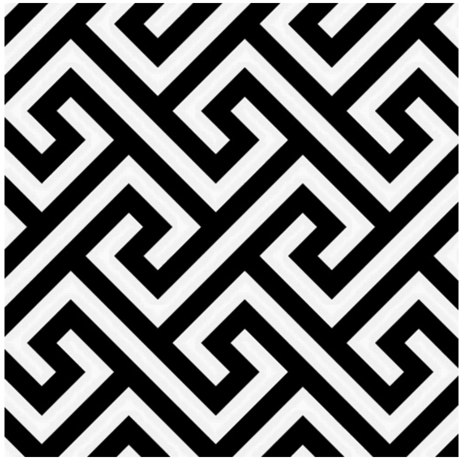
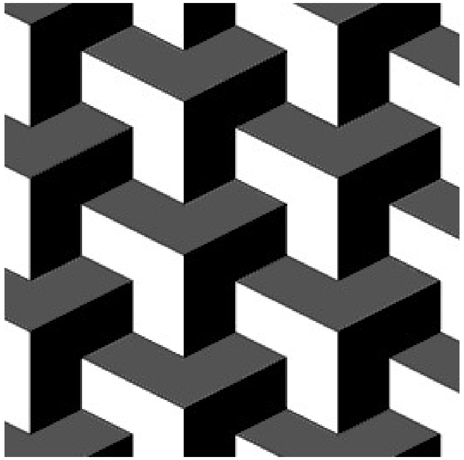
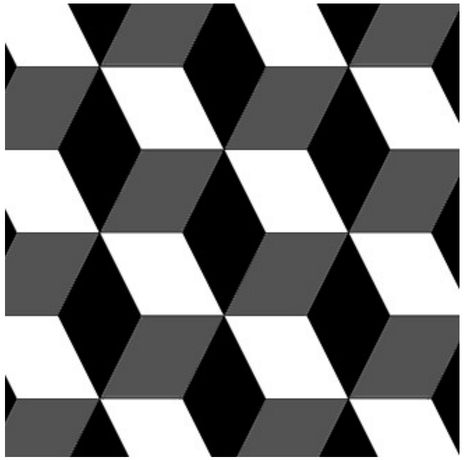
- Classify the following linear designs by frieze symmetry:
  - The graphs of the sine, cosine and tangent functions.
  - The following configurations:





- Classify the following two-dimensional designs by wallpaper symmetry:



- Classify the following two-dimensional designs by wallpaper symmetry: