Introduction to Topology

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Contents

Part I

General Topology

Chapter 1

Metric Spaces

This first Section will bring forward three of the main properties later generalised into those of topological spaces. Metric spaces are sets endowed with a distance, and on them it is possible to define concepts such as open sets, closed sets, neighborhood, and so on. These concepts will be the model for future definitions within wider topological terminology.

1.1 Definition

The first and foremost paragon of a metric space is the Euclidean space $\mathbb{R}^n$ we are used to dealing with in basic geometry and analysis. Let us recall its definition.

Definition 1.1.1. *Let $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an element of the Cartesian product $\mathbb{R}^n$. We define the **(Euclidean or standard) norm** on $\mathbb{R}^n$ as*

$$\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}, \qquad \boldsymbol{x} \mapsto \|\boldsymbol{x}\|_2 := \sqrt{x_1^2 + \cdots + x_n^2},$$

*and the **(Euclidean or standard) distance** on $\mathbb{R}^n$ as*

$$d_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad (\boldsymbol{x}, \boldsymbol{y}) \mapsto d_2(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\|. \tag{1.1}$$

Remark 1.1.2. We observe that for $n = 1$ this is nothing but the absolute value of the difference on $\mathbb{R}$.

From here onwards it is just a matter of generalising the properties satisfied by d in (1.1) into a wider family of structures:

Definition 1.1.3. *A **metric space** is a set X endowed with a map $d : X \times X \to \mathbb{R}$, called a **distance** or a **metric**, satisfying the following properties:*

a) $d(x, y) \geq 0$ *for every* $x, y \in X$;

b) *for every* $x, y \in X$, $d(x, y) = 0$ *if and only if* $x = y$;

c) *(symmetric property)* $d(x, y) = d(y, x)$ *for every* $x, y \in X$;

d) *(triangle inequality)* $d(x, y) \leq d(x, z) + d(z, y)$ *for every* $x, y, z \in X$.

Examples 1.1.4.

1. First and most relevant, the Euclidean n-space with distance defined as in (1.1), $(\mathbb{R}^n, d_2)$, is a metric space. In particular, so is $\mathbb{R}$ with the norm defined by the absolute value, $d_{|\cdot|}(a, b) := |a - b|$.

2. Define the following on $\mathbb{R}^n$

$$d_\infty\left(\left(x_1,\ldots,x_n\right),\left(y_1,\ldots,y_n\right)\right) := \max\left\{\left|x_i-y_i\right| : i=1,\ldots,n\right\},$$

or $d_\infty\left(\boldsymbol{x},\boldsymbol{y}\right) := \left\|\boldsymbol{x}-\boldsymbol{y}\right\|_\infty$ in terms of the maximum or L^∞ norm.

3. Define the following on $\mathbb{R}^n$:

$$d_1\left(\left(x_1,\ldots,x_n\right),\left(y_1,\ldots,y_n\right)\right) := \sum_{i=1}^n \left|x_i-y_i\right|.$$

4. More in general, on $\mathbb{R}^n$ we define the so-called p-**norm** for any given $p\in\mathbb{R}_{\geq 1}$

$$\left\|\boldsymbol{x}\right\|_p := \left(\sum_{i=1}^n \left|x_i\right|^p\right)^{1/p} \tag{1.2}$$

and the metric given by it,

$$d_p\left(\boldsymbol{x},\boldsymbol{y}\right) := \left\|\boldsymbol{x}-\boldsymbol{y}\right\|_p. \tag{1.3}$$

This is indeed a distance and it extends to d_∞ by setting $p\to\infty$ (Exercise).

- $d_p\left(\boldsymbol{x},\boldsymbol{y}\right) = \left(\text{sum of terms } \geq 0\right)^{1/p} \geq 0$.
- $d_p\left(\boldsymbol{x},\boldsymbol{y}\right) = \left(\sum_{i=1}^n \left|x_i-y_i\right|^p\right)^{1/p} = 0$ if and only if $\left|x_i-y_i\right| = 0$ for every i, hence $\boldsymbol{x}=\boldsymbol{y}$.
- $d_p\left(\boldsymbol{x},\boldsymbol{y}\right) = \left(\sum_{i=1}^n \left|x_i-y_i\right|^p\right)^{1/p} = \left(\sum_{i=1}^n \left|y_i-x_i\right|^p\right)^{1/p} = d_p\left(\boldsymbol{y},\boldsymbol{x}\right)$.
- For every $\boldsymbol{x},\boldsymbol{z}.\boldsymbol{y}\in\mathbb{R}^n$, Exercise,

$$\begin{aligned} d_p\left(\boldsymbol{x},\boldsymbol{y}\right) &= \left(\sum_{i=1}^n \left|x_i-y_i\right|^p\right)^{1/p} = \left(\sum_{i=1}^n \left|x_i-z_i+z_i-y_i\right|^p\right)^{1/p} \\ &\leq \left(\sum_{i=1}^n \left|x_i-z_i\right|^p\right)^{1/p} + \left(\sum_{i=1}^n \left|z_i-y_i\right|^p\right)^{1/p} \end{aligned}$$

In the case $p=\infty$ the above are all easier including the triangle inequality:

$$d_\infty\left(\boldsymbol{x},\boldsymbol{y}\right) = \max_{1\leq i\leq n}\left\{\left|x_i-y_i\right|\right\} \leq \max\left\{\left|x_1-z_1\right|+\left|z_1-y_1\right|,\ldots,\left|x_n-z_n\right|+\left|z_n-y_n\right|\right\} = d_\infty\left(\boldsymbol{x},\boldsymbol{z}\right)+d_\infty\left(\boldsymbol{z},\boldsymbol{y}\right).$$

5. Given any metric space $\left(X,d\right)$ and any subset $Y\subset X$ of the underlying set, the distance on X induces, by restriction, a distance d_Y on Y:

$$d_Y\left(y_1,y_2\right) := d\left(y_1,y_2\right), \qquad \text{for every } y_1,y_2\in Y.$$

6. Hence, for instance, any subset of $\mathbb{R}^n$ is a metric space itself with the induced Euclidean distance.

7. Let $\left(X_1,d_1\right),\left(X_2,d_2\right),\ldots,\left(X_n,d_n\right)$ be n metric spaces. We define the product metric on the Cartesian product $X_1\times\cdots\times X_n$ in the following manner:

$$d_{X_1\times\cdots\times X_n}\left(\left(a_1,\ldots,a_n\right),\left(b_1,\ldots,b_n\right)\right) := \max\left\{d_i\left(a_i,b_i\right) : i=1,\ldots,n\right\}.$$

8. We observe that for $X_1=\cdots=X_n=\mathbb{R}$ and $d_1=\cdots=d_n=d_{|\cdot|}$ we recover the metric structure $\left(\mathbb{R}^n,d_\infty\right)$ in Example **2**.

9. Let

$$\mathbb{S}^2 := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \|(x_1, x_2, x_3)\|_2 = 1\right\},$$

be the sphere of radius 1 in $\mathbb{R}^3$. We define the following distance on $\mathbb{S}^2$. Given two points $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ in $\mathbb{S}^2$, let $C(\boldsymbol{x}, \boldsymbol{y})$ be the only circumference having the same center as the sphere and containing both $\boldsymbol{x}$ and $\boldsymbol{y}$. We define $d(\boldsymbol{x}, \boldsymbol{y})$ as the length of the smaller arc between $\boldsymbol{x}$ and $\boldsymbol{y}$ – or the length of any of both arcs if the points are antipodal, in which case it is easy to check said arclength is equal to π.

10. Let $I := [0, 1]$. We define $X := \{f : I \to \mathbb{R} : f \text{ continuous}\}$. On X we define the distance

$$d(f, g) := \max\{|f(x) - g(x)| : x \in I\}.$$

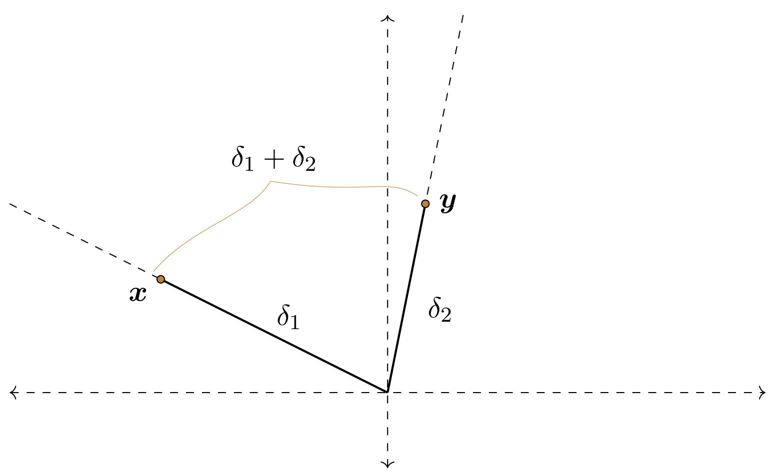
This is a well-defined map in virtue of the **Weierstrass Theorem**, which states that any continuous function defined on a closed and bounded interval of $\mathbb{R}$ has a maximum and a minimum value.

11. Let X be a set. We define d on $X \times X$ by:

$$d(x, y) := \begin{cases} 0 & \text{if} \quad x = y, \\ 1 & \text{if} \quad x \neq y. \end{cases}$$

We call this the **discrete distance** on X and we call (X, d) a **discrete metric space**.

12. The function defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by $d(\boldsymbol{x}, \boldsymbol{y}) = d_p(\boldsymbol{x}, \boldsymbol{0}) + d_p(\boldsymbol{0}, \boldsymbol{y})$



is also a distance, Exercise.

1.2 Open balls

Let (X, d) be a metric space and $x \in X$.

Definition 1.2.1. *For every real number $r > 0$, the open ball of radius r and center x is the subset of X defined as follows:*

$$B(x; r) := \{y \in X : d(x, y) < r\}.$$

Examples 1.2.2.

1. For $X = \mathbb{R}$ and $d = d_{|\cdot|}$, we get $B(0; 2) = \{x \in \mathbb{R} : |x - 0| < 2\} = (-2, 2)$ and $B(10.1; 1) = (9.1, 11.1)$ for instance.

2. For $X = \mathbb{R}^2$ with the Euclidean metric, the open ball of center (a, b) and radius r is nothing but the inside area of circle of center (a, b) and radius r not including the border: for instance,

$$B\left((0,0);1\right) = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}.$$

3. For $X = \mathbb{R}^2$ with the sub-infinity metric, we have $B(0;1)$ equal to the interior of the square having vertices $(\pm 1, \pm 1)$.

4. Let X be a set and d the discrete metric on X. Then

$$B(x;r) = \begin{cases} \{x\} & \text{if} \quad r < 1, \\ X & \text{if} \quad r \geq 1. \end{cases}$$

5. On $\mathbb{S}^2$ with the distance introduced above, let us find $B\left((0,0,1);1/2\right)$. Let us first find $d\left((0,0,1),(x,y,z)\right)$ for $z \geq 0$. This is equal to $\arccos z$ because $\cos\alpha = \sin\left(\pi/2 - \alpha\right) = z$. Hence $(x,y,z) \in B\left((0,0,1);1/2\right)$ is equivalent to $\arccos z < 1/2$, hence $z > \cos 1/2$ (the function cosine being decreasing on $[0, \pi/2]$).

Proposition 1.2.3. *Let (X, d) be a metric space. Open balls satisfy the following:*

a) $B(x;r) \neq \varnothing$ *for every* $x \in X$ *and every* $r > 0$.

b) $X = \bigcup_{x \in X, r > 0} B(x;r)$.

c) For every $x \in X$ *and* $r > 0$, *if* $y \in B(x;r)$ *then there exists* $s > 0$ *such that* $B(y;s) \subset B(x;r)$.

d) For every $x, y \in X$ *and* $r, s > 0$ *the intersection* $B(x;r) \cap B(y;s)$ *is equal to the union of a set of balls.*

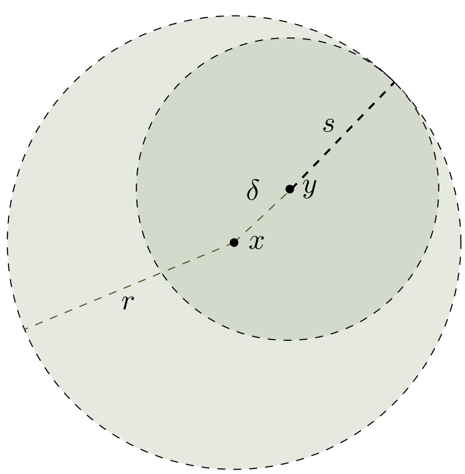
Proof. a) We have $d(x,x) = 0 < r$ for every $r > 0$, hence at least $x \in B(x;r)$.

b) The fact $x \in B(x;r)$ implies $X = \bigcup_{x \in X} \{x\} \subset \bigcup_{x \in X, r > 0} B(x;r)$.

c) Let $y \in B(x;r)$, $\delta := d(x,y)$ and $s = r - \delta$. Then for every $z \in B(y;s)$,

$$d(x,z) \leq d(x,y) + d(y,z) < \delta + s = r.$$

Hence $z \in B(x;r)$.



d) Let $z \in B(x;r) \cap B(y;s)$. Applying the previous item, there must exist $\tilde{r}, \tilde{s} > 0$ such that $B(z;\tilde{r}) \subset B(x,r)$ and $B(z;\tilde{s}) \subset B(s,y)$. Let $\delta_z := \min\{\tilde{r}, \tilde{s}\}$. Then $B(z;\delta_z) \subset B(x;r) \cap B(y;s)$. Fix such a $\delta_z > 0$ for every $z \in B(x;r) \cap B(y;s)$. We obtain

$$\bigcup_{z \in B(x;r) \cap B(y;s)} B(z;\delta_z) = B(x;r) \cap B(y;s).$$

□

Definition 1.2.4. *Let X be a set and d_1, d_2 be two metrics defined on it. We call d_1 and d_2* ***(strongly) equivalent****, and write $d_1 \equiv d_2$, if there exist $\alpha, \beta > 0$ such that*

$$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_2(x,y), \qquad \text{for every } x, y \in X. \tag{1.4}$$

Checking $\equiv$ is an equivalence relation is left as Exercise 17.

Lemma 1.2.5. *Distances d_1, d_2, d_∞ on $\mathbb{R}^n$ in Examples 1.1.4 are all equivalent.*

Proof. Accepting the fact that $\equiv$ is an equivalence relation, it will suffice to prove $d_\infty \equiv d_1$ and $d_\infty \equiv d_2$.

Let us first prove $\alpha = 1$ and $\beta = \sqrt{n}$ fulfill equation (1.4) on d_∞, d_2 for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, i.e.

$$\max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} \leq \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2} \leq \sqrt{n} \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

But this is the same as asking the following to hold for any $r_1, \ldots, r_n \geq 0$:

$$\max\{r_1, \ldots, r_n\} \leq \sqrt{r_1^2 + \cdots + r_n^2} \leq \sqrt{n} \max\{r_1, \ldots, r_n\}. \tag{1.5}$$

The first inequality is immediate: there exists $i = 1, \ldots, n$ such that $r_i = \max\{r_1, \ldots, r_n\}$, hence

$$(\max\{r_1, \ldots, r_n\})^2 = r_i^2 \leq r_1^2 + \cdots + r_n^2,$$

and inequalities are maintained after taking squares or square roots (both functions being increasing), hence

$$\max\{r_1, \ldots, r_n\} \leq \sqrt{r_1^2 + \cdots + r_n^2}.$$

On the other hand, the fact r_i is the maximum of these elements implies

$$r_1^2 + \cdots + r_n^2 \leq r_i^2 + \cdots + r_i^2 = n r_i^2,$$

and taking square roots on both sides yields the second inequality in (1.5).

Let us now prove $\alpha = 1$ and $\beta = n$ fulfill equation (1.4) on d_∞, d_1 for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, i.e.

$$\max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} \leq |x_1 - y_1| + \cdots + |x_n - y_n| \leq n \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

But this is the same as asking the following to hold for any $r_1, \ldots, r_n \geq 0$:

$$\max\{r_1, \ldots, r_n\} \leq r_1 + \cdots + r_n \leq n \max\{r_1, \ldots, r_n\}. \tag{1.6}$$

The first inequality is immediate as well: there exists $i = 1, \ldots, n$ such that $r_i = \max\{r_1, \ldots, r_n\}$, hence

$$(\max\{r_1, \ldots, r_n\}) = r_i \leq r_1 + \cdots + r_n.$$

On the other hand,

$$r_1 + \cdots + r_n \leq r_i + \cdots + r_i = n r_i,$$

which is the second inequality in (1.6). □

In fact:

Lemma 1.2.6. *All metrics d_p defined on $\mathbb{R}^n$ as in (1.2) and (1.3) are equivalent.*

Proof. Exercise 12. □

1.3 Open sets

Definition 1.3.1. *Let (X, d) be a metric space. A subset $U \subset X$ is called* ***open*** *if for every $x \in U$ there exists $r > 0$ such that $B(x; r) \subset U$.*

Examples 1.3.2.

1. Applying one of the items in Proposition 1.2.3 we conclude all open balls in (X, d) are open sets.

2. Let $x \in X$ and $U \setminus \{x\}$. U is open; indeed, for every $y \in U$, let $\delta := d(x, y)$. Then $B(y; \delta/2) \subset U$.

3. $\mathbb{R}^n$ with the different p-distances defined in Examples 1.1.4 have the same open sets, since in every ball for distance d_p we can always inscribe a ball for distance $d_{\tilde{p}}$, for every $p, \tilde{p} > 0$. Hence, different distances may determine the same sets. This is exactly was was proven in Lemmae 1.2.5 and 1.2.6.

4. In all of Examples 1.1.4 save for $X := \{f : I \to \mathbb{R} : f \text{ continuous}\}$ single points are not open sets.

5. Let (X, d) be a metric space. The subset $U = \{y \in X : d(x, y) > \epsilon\}$ is open for any $x \in X$ and any $\epsilon \geq 0$. Indeed, let $y \in U$ and $\delta := d(x, y) > \epsilon$. Let us prove $B(y, \delta - \epsilon) \subset U$. Let $z \in B(y, \delta - \epsilon)$. Then

$$\delta = d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + \delta - \epsilon,$$

hence $d(x, z) > \epsilon$.

6. If (X, d) is a metric space then $\{y \in X : a < d(x, y) < b\}$ is an open set for every $a, b > 0$ and $x \in X$.

Definition 1.3.3. *In the above hypotheses, subset $A \subset X$ is called* ***closed*** *if $X \setminus A$ is an open set.*

The open sets in a metric space satisfy the following three properties. These will be the axioms of definition of a more general type of structure – namely *topologies*.

Proposition 1.3.4. *Let (X, d) be a metric space. Then,*

a) Subsets $\varnothing$ and X are open subsets in X.

b) Let $\{U_i\}_{i \in I}$ be an arbitrary family of open sets in X. Then, the union $\bigcup_{i \in I} U_i$ is an open subset of X.

c) Let $U_1, \ldots, U_n$ be a finite collection of open sets in X. Then intersection $U_1 \cap \cdots \cap U_n$ is an open subset of X.

Proof. a) $\varnothing$ is an open subset because it contains no element, hence poses no obstacle to the thesis in Definition 1.3.1.

For every $x \in X$, $B(x, r) \subset X$ for actually *any* $r > 0$, hence X is open.

b) Let $x \in \bigcup_{i \in I} U_i$. There exists at least one $i \in I$ such that $x \in U_i$, and since U_i is open there must exist $r > 0$ such that $B(x; r) \subset U_i$, hence $B(x; r)$ is contained in the total union.

c) Let $x \in \bigcap_{i=1}^{n} U_i$ and for every $i = 1, \ldots, n$ let $r_i > 0$ be such that $B(x; r_i) \subset U_i$. Then $B(x; r) \subset \bigcap_{i=1}^{n} U_i$ where $r = \min\{r_1, \ldots, r_n\}$.

□

Remark 1.3.5. The finiteness condition in the third item cannot be dropped. For instance, all intervals of the form $(-1/n, 1/n)$, $n \in \mathbb{N}$ are open subsets of $\mathbb{R}$, yet their intersection is equal to $\{0\}$ which is not open.

1.4 Continuous functions

Definition 1.4.1. *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is called* ***continuous in a point*** *$x \in X$, provided for every $\epsilon > 0$ there exists $\delta = \delta_{\epsilon,x} > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Alternatively, if for every $\epsilon > 0$ there exists $\delta = \delta_{\epsilon,x} > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))$.*

Proposition 1.4.2. *f is continuous if, and only if, for every open subset $U \subset Y$, $f^{-1}(U)$ is an open subset of X.*

Proof. Let $U \subset Y$ be open and $x \in f^{-1}(U)$. There exists $\epsilon > 0$ such that $B(f(x), \epsilon) \subset U$ on account of U being open. The continuity of f implies the existence of $\delta = \delta_\epsilon > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon) \subset U$. Hence $B(x, \delta) \subset f^{-1}(U)$, which implies $f^{-1}(U)$ is open.

Conversely, let $x \in X$ and $\varepsilon > 0$. Then $f^{-1}(B(f(x), \varepsilon))$ is an open set, hence there exists $\delta > 0$ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$ □

Examples 1.4.3.

1. Function $d(\cdot, y) : M \to \mathbb{R}$ defined by $x \mapsto d(x, y)$ is continuous as a consequence of $A = \{y\}$ in 10. Similarly $d(x, \cdot)$ defined by $y \mapsto d(x, y)$. Thus, $d : M \times M \to \mathbb{R}$ is continuous.

2. Another way of proving the latter is proving $X_{a,b} = \{(x, y) \in X \times X : a < d(a, b) < b\}$ is continuous. Let $(x, y) \in X_{a,b}$. If $r = d(x, y)$ and $\varepsilon = \min\left\{\frac{b-r}{2}, \frac{r-a}{2}\right\}$ then $B((x, y), \varepsilon) \subset X_{a,b}$, Exercise.

1.5 Solved Exercises

1. Decide which of the following are metrics on $\mathbb{R}$:

a) $d(x, y) := |e^x - e^y|$;

b) $d(x, y) := \frac{|x-y|}{x^2+y^2}$ whenever $x \neq 0$ or $y \neq 0$, and $d(0, 0) := 0$;

c) $d(x, y) := (x - y)^2$.

Whenever the answer is affirmative, find $d(0, 1)$ and $B(0, 1)$.

Solution.

a) Let us check all the items in Definition 1.1.3 are fulfilled:

- $|z| \geq 0$ for every $z \in \mathbb{R}$. $z := e^x - e^y$ is a real number for every $x, y \in \mathbb{R}$, hence $d(x, y) \geq 0$ for every $x, y \in \mathbb{R}$;
- We only need to prove an implication. Assume $x, y \in \mathbb{R}$ are such that $d(x, y) = 0$. Then $|e^x - e^y| = 0$, hence $e^x = e^y$. The exponential function is injective (easily proven, for instance, from the fact its derivative never vanishes), hence $x = y$.

- $|z| = |-z|$ for every $z \in \mathbb{R}$, hence $d(x, y) = |e^x - e^y| = |e^y - e^x| = d(y, x)$ for every $x, y \in \mathbb{R}$.
- Let $x, y, z \in \mathbb{R}$. We have

$$d(x, y) = |e^x - e^y| = |e^x - e^z + e^z - e^y| \le |e^x - e^z| + |e^z - e^y| = d(x, z) + d(z, y),$$

the first inequality on account of the triangular inequality on absolute values, which we know is true.

Hence d is a distance. $d(0, 1) = \left|e^0 - e^1\right| = e - 1$ and a simple exercise proves

$$B(0, 1) = \{x \in \mathbb{R} : d(0, x) < 1\} = \{x \in \mathbb{R} : |1 - e^x| < 1\} = (-\infty, \ln 2).$$

b) Let $d(x, y) := \frac{|x-y|}{x^2+y^2}$.

- For every $x, y \in \mathbb{R}$, $d(x, y)$ is a quotient of two non-negative numbers: an absolute value and a sum of squares; hence $d(x, y) \ge 0$.
- Again, only one of the implications needs work. For every $x, y \in \mathbb{R}$, $d(x, y) = 0$ implies $\frac{|x-y|}{x^2+y^2} = 0$, hence $|x - y| = 0$ and the fact the latter is nothing but the Euclidean metric on $\mathbb{R}$ implies $x = y$.
- For every $x, y \in \mathbb{R}$ we have $x^2 + y^2 = y^2 + x^2$, as well as $|x - y| = |y - x|$, hence $d(x, y) = d(y, x)$.
- Let us assume $x = 0$, $y = 2$, $z = 4$. We have $d(x, y) = |0 - 2| / \left(0^2 + 2^2\right) = 1/2$ and

$$d(x, z) + d(z, y) = \frac{|0 - 4|}{0^2 + 4^2} + \frac{|4 - 2|}{4^2 + 2^2} = \frac{1}{4} + \frac{1}{10} = \frac{7}{20},$$

which is strictly smaller than $d(x, y)$.

Hence d is *not* a metric since it does not fulfil the triangular inequality on at least three of its elements.

c) Let $d(x, y) := (x - y)^2$.

- $d(x, y) = (x - y)^2 \ge 0$ for every $x, y \in \mathbb{R}$, as are all squares of real numbers.
- For every $x, y \in \mathbb{R}$, $d(x, y) = 0$ if and only if $(x - y)^2 = 0$, equivalent to $x - y = 0$ which in turn holds if and only if $x = y$.
- $d(x, y) = (x - y)^2 = (y - x)^2 = d(y, x)$ for every $x, y \in \mathbb{R}$.
- Let $x = 1, y = 3, z = 2$. $d(x, y) = 4$ and $d(x, z) + d(z, y) = 1 + 1 < 4$, hence the triangular inequality is not fulfilled by these elements.

Hence d is not a distance.

2. Consider the following function on $X = (-1, 1)$:

$$d(x, y) = \frac{1}{2} \ln \frac{1 + \left|\frac{y-x}{1-xy}\right|}{1 - \left|\frac{y-x}{1-xy}\right|}$$

a) Prove d defines a metric on X.

b) Compute the following: $d(0, 0.01), d(0.2, 0.3), d(0, -0.2), \lim_{x\to 1} d(0, x)$.

Solution.

a) It is a well-defined function: $\frac{1}{2}\ln\frac{1+\left|\frac{y-x}{1-xy}\right|}{1-\left|\frac{y-x}{1-xy}\right|}>0$ if and only if $1-\left|\frac{y-x}{1-xy}\right|>0$, if and only if $|1-xy|>|y-x|$ which is equivalent to $(1-xy)^2>(y-x)^2$ hence $\left(1-x^2\right)\left(1-y^2\right)>0$ which is true. Let us prove it fulfils the properties

(a) We have $\left|\frac{y-x}{1-xy}\right|\geq-\left|\frac{y-x}{1-xy}\right|$, hence $1+\left|\frac{y-x}{1-xy}\right|\geq 1-\left|\frac{y-x}{1-xy}\right|$ which implies $\frac{1+\left|\frac{y-x}{1-xy}\right|}{1-\left|\frac{y-x}{1-xy}\right|}\geq 1$; taking logarithms on both sides yields the positiveness of d.

(b) $y=x$ implies $d(x,y)=0$ in virtue of the chain of implications shown above. Conversely, the reversed chain of implications implies $\left|\frac{y-x}{1-xy}\right|=-\left|\frac{y-x}{1-xy}\right|$, hence $|y-x|=0$ which implies $y=0$.

(c) We have

$$\frac{1}{2}\ln\frac{1+\left|\frac{y-x}{1-xy}\right|}{1-\left|\frac{y-x}{1-xy}\right|}=\frac{1}{2}\ln\frac{1+\left|\frac{x-y}{1-yx}\right|}{1-\left|\frac{x-y}{1-yx}\right|}.$$

(d) Left as an Exercise.

b)
- $d(0,0.01)=\frac{1}{2}\ln\frac{1+\left|\frac{0.01}{1}\right|}{1-\left|\frac{0.01}{1}\right|}=\frac{1}{2}\ln\frac{1.01}{0.99}=0.01000033326.$
- $d(0.2,0.3)=\frac{1}{2}\ln\frac{1+\left|\frac{0.1}{0.94}\right|}{1-\left|\frac{0.1}{0.94}\right|}=0.1067870501.$
- $d(0,-0.2)=0.2027325540.$
- $\lim_{x\to 1}d(0,x)=\lim_{x\to 1}\frac{1}{2}\ln\frac{1+x}{1-x}=+\infty$

3. (i) Is $d(\boldsymbol{x},\boldsymbol{y}):=\min\{|x_i-y_i|:i=1,\ldots,n\}$ a distance on $\mathbb{R}^n$?

(ii) Ditto for $d(\boldsymbol{x},\boldsymbol{y}):=\max\{|x_i-y_i|:i=1,\ldots,n\}$

Solution.

(i) It satisfies the first item in Definition 1.1.3. However, it does not satisfy the second. Let $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$ such that they both are equal in one of their entries, e.g. the first one: $x_1=y_1$, and different in some or all of the remaining entries. Then $\min\{|x_i-y_i|:i=1,\ldots,n\}=0$ yet $\boldsymbol{x}\neq\boldsymbol{y}$.

(ii) $d(\boldsymbol{x},\boldsymbol{y})$ is the maximum of a set of absolute values, hence is ≥ 0. The second property is immediate on account of maximality. The third property is a consequence of the symmetry of $|\cdot|$. Finally let j,k such that $|x_j-z_j|=\max_i|x_i-z_i|$ and $|z_k-y_k|=\max_i|z_k-y_k|$. For every i,

$$|x_i-y_i|\leq|x_i-z_i|+|z_i-y_i|\leq|x_j-z_j|+|z_k-y_k|,$$

which means $\max_i|x_i-y_i|\leq|x_j-z_j|+|z_k-y_k|=\max_i|x_i-z_i|+\max_i|z_i-y_i|$.

4. Let X be a set. For every $x,y\in X$ define

$$d(x,y):=\begin{cases}0 & \text{if} \quad x=y,\\ 1 & \text{if} \quad x\neq y.\end{cases}$$

Prove d is a distance on $\mathbb{R}^n$. It is called the **discrete metric** on X.

Solution.

a) $d(x,y) \in \{0,1\}$ for every x, y, in particular $d(x,y) \geq 0$.

b) $d(x,y) = 0$ can only happen in one case by definition: $x = y$.

c) $=$ is a relation of equivalence, hence symmetric and the result follows.

d) Let $x, y, z \in X$. Assume $x = y$. Then $d(x,y) = 0$ and $d(x,z) + d(z,y)$ is a sum of numbers ≥ 0, hence $\geq d(x,y)$. Assume $x \neq y$; then $d(x,y) = 1$ and z must be different from at least one of them, say x: $d(x,z) = 1$. Which means $d(x,z) + d(z,y) \geq 1 = d(x,y)$.

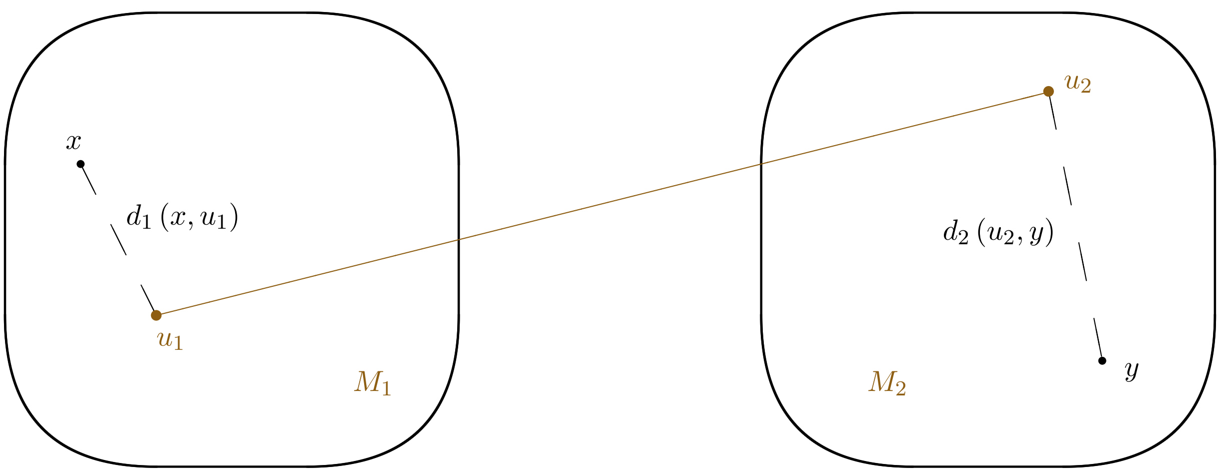
5. Let $(M_1, d_1), (M_2, d_2)$ be two metric spaces. Can we endow the disjoint union of X and Y, $X \amalg Y$, with a metric space structure?

Solution. Fix $u_1 \in M_1$ and $u_2 \in M_2$ and define

$$(X \amalg Y) \times (X \amalg Y) \xrightarrow{\quad d \quad} X \amalg Y$$

$$(x,y) \longmapsto d(x,y) := \begin{cases} d_1(x,y), & x, y \in M_1, \\ d_2(x,y), & x, y \in M_2, \\ d_1(x,u_1) + d_2(u_2,y) + 1, & x \in M_1,\ y \in M_2, \\ d_1(y,u_1) + d_2(u_2,x) + 1, & y \in M_1,\ x \in M_2. \end{cases}$$

We can replace 1 by any $\alpha > 0$; this value will be, by construction, $d(u_1, u_2)$ in $X \amalg Y$.



Let us prove d defines a metric on $X \amalg Y$.

a) $d(x,y) \geq 0$ for all four possibilities, being defined either as an element ≥ 0 or a sum thereof (on account of d_1 and d_2 being metrics).

b) $d(x,y) = 0$ if and only if x, y are in the first two possibilities for the above definition, and in both cases this can only happen if $x = y$ (again because d_1 and d_2 are metrics).

c) Symmetry is immediate from the definition.

d) Let $x, y, z \in X \amalg Y$. Assume without loss of generality $x \in M_1$. Then

- if $y, z \in M_1$, we have
$$d(x,y) = d_1(x,y) \leq d_1(x,z) + d_1(z,y),$$
in virtue of d_1 being a metric.
- If $y, z \in M_2$, then $d_2(u_2,y) \leq d_2(u_2,z) + d_2(z,y)$ on account of d_2 being a metric, implying
$$d(x,y) = d_1(x,u_1)+1+d_2(u_2,y) \leq d_1(x,u_1)+1+d_2(u_2,z)+d_2(z,y) = d(x,z)+d(z,y).$$
- If $y \in M_1$ and $z \in M_2$, $d_1(x,y) \leq d_1(x,u_1) + d_1(u_1,y)$ on account of d_1 being a metric, implying
$$d(x,y) = d_1(x,y) \leq d_1(x,u_1)+d_1(u_1,y) < d_1(x,u_1)+1+d_2(u_2,z)+d_1(u_1,y)+1+d_2(u_2,z).$$
- If $z \in M_1$ and $y \in M_2$, then $d_1(x,u_1) \leq d_1(x,z) + d_1(z,u_1)$ hence
$$d(x,y) = d_1(x,u_1)+1+d_2(u_2,y) \leq d_1(x,z)+d_1(z,u_1)+1+d_2(u_2,y) = d(x,z)+d(z,y).$$

6. (*May 2014*) A metric set (X,d) is called *ultrametric* if it satisfies the following property, stronger than the triangle inequality:
$$d(x,y) \leq \max\{d(x,z), d(z,y)\}, \qquad \text{for every } x,y,z \in X. \tag{1.7}$$
Prove every point in such a metric set is the centre of any ball containing it.

Solution. Let $B_d(x,r)$ be an open ball for this metric, and $y \in B_d(x,r)$. We are going to prove $B_d(x,r) = B_d(y,r)$.

Let $z \in B_d(y,r)$. We have $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ by hypothesis. Since both quantities inside the brackets are smaller than r, so is $d(x,z)$ and we conclude $z \in B_d(x,r)$.

The other inclusion is proven in the same way. Given $t \in B_d(x,r)$, we have $d(t,y) \leq \max\{d(t,x), d(x,y)\}$, again the maximum of two quantities that are smaller than r. Hence $d(t,y) < r$ and we obtain $t \in B_d(y,r)$.

Hence $B_d(x,r) = B_d(y,r)$.

7. Let p be a prime number. Define the p-adic distance on $\mathbb{Z}$ as follows:
$$\begin{aligned} \mathbb{Z}\times\mathbb{Z} &\xrightarrow{\quad d \quad} \mathbb{R} \\ (x,y) &\longmapsto d_p(x,y) := \begin{cases} \frac{1}{p^n}, & \text{if } x \neq y, n \in \mathbb{N}\cup\{0\}, p^n \mid x-y, p^{n+1} \nmid x-y, \\ 0, & x = y. \end{cases} \end{aligned}$$

(i) Prove d_p is a distance and $(\mathbb{Z}, d_p)$ is an *ultrametric* space as per 6, i.e. it fulfils equation (1.7): $d_p(x,y) \leq \max\{d_p(x,z), d_p(z,y)\}$ for every $x,y,z \in \mathbb{Z}$.

(ii) Compute:

(a) $d_5(1,26)$.

(b) $d_5(1,476)$.

(c) $B_{d_3}(0,1)$.

(d) $d_3(1,26)$.

(e) $d_3(1,28)$.

(f) $B_{d_5}(4, 0.01)$.

SOLUTION.

(i) Let us prove the first three properties of a metric:

1. $d_p(x, y)$ is either 0 or the reciprocal of a prime positive integer; either way it is always ≥ 0.
2. $\frac{1}{p^n} \neq 0$ for every $n \in \mathbb{N}$, hence the only possibility for $d_p(x, y)$ to be 0 is $x = y$ as per the definition.
3. For every $x, y \in \mathbb{Z}$, assume $x \neq y$ (otherwise the proof is trivial). Then $p^n \mid x - y$ (resp. $p^{n+1} \nmid x - y$) if and only if $p^n \mid y - x$ (resp. $p^{n+1} \nmid y - x$), hence $n \in \mathbb{Z}$ is the same for both ordered pairs (x, y) and (y, x), which means $d_p(x, y) = d_p(y, x)$.

Let us now prove the fourth property of a metric by actually proving the stronger property (1.7). Let $x, y, z \in \mathbb{Z}$.

- Assume $x = y$. Then $d_p(x, y) = 0 \leq \max\{d_p(x, z), d_p(z, y)\} = d_p(x, z)$ for every $z \in \mathbb{Z}$ on account of property 1. above.
- Assume $x \neq y$. That means $d_p(x, y) \neq 0$, hence there exists a maximal $n \in \mathbb{N}$ such that $p^n \mid x - y$ and we define $d_p(x, y) = \frac{1}{p^n}$.
  - If $x = z$ or $z = y$, then $\max\{d_p(x, z), d_p(z, y)\} = \frac{1}{p^n}$ as well and we have an equality (i.e. a special case of (1.7)).
  - If $x \neq z$ and $z \neq y$, let $m, \tilde{m}$ such that $p^m \mid x - z$ and $p^{\tilde{m}} \mid z - y$ and let $m, \tilde{m}$ be maximal with these properties. Then $d_p(x, z) = \frac{1}{p^m}$, $d_p(z, y) = \frac{1}{p^{\tilde{m}}}$. Assume $d_p(x, y) > \max\{d_p(x, z), d_p(z, y)\}$ and let us arrive to a contradiction. It would imply $\frac{1}{p^n} > \frac{1}{p^m}$, hence $n < m$, and $\frac{1}{p^n} > \frac{1}{p^{\tilde{m}}}$ which means $n < \min\{m, \tilde{m}\} =: \mu$. Thus $p^m \mid x - z$ implies $p^\mu \mid x - z$ and $p^{\tilde{m}} \mid z - y$ implies $p^\mu \mid z - y$, thus $p^\mu \mid x - z + z - y = x - y$ which would imply $r \leq n$, contradicting maximality of n.

Thus $d_p(x, y) \leq \max\{d_p(x, z), d_p(z, y)\} \leq d_p(x, z) + d_p(z, y)$, rendering $(\mathbb{Z}, d_p)$ (ultra)metric.

(ii) (a) $d_5(1, 26) = \frac{1}{5^n}$, where n is the largest natural number such that $5^n \mid 26 - 1 = 25$. Clearly $n = 2$ and $d_5(1, 26) = \frac{1}{25}$.

(b) $d_5(1, 476) = \frac{1}{5^n}$ and let us find n. We have $476 - 1 = 475 = 5^2 \cdot 19$, thus once again $d_5(1, 476) = \frac{1}{5^2}$.

(c) We have

$$B_{d_3}(0, 1) = \{x \in \mathbb{Z} : d_3(x, 0) < 1\} = \left\{x \in \mathbb{Z} : d_3(x, 0) \leq \frac{1}{3}\right\} = \{x \in \mathbb{Z} : 3 \mid x - 0\} = \{x \in \mathbb{Z} : 3 \mid x\} = \dot{3}.$$

(d) $d_3(1, 26) = \frac{1}{3^n}$ where n is maximal such that $3^n \mid 26 - 1 = 25$. Hence $n = 0$ and $d_3(1, 26) = \frac{1}{3^0} = 1$.

(e) $d_3(1, 28) = \frac{1}{3^3} = \frac{1}{27}$ since $3^3 \mid 28 - 1$.

(f) We have

$$B_{d_5}(4, 0.01) = \{x \in \mathbb{Z} : d_5(x, 4) < 0.01\} = \left\{x \in \mathbb{Z} : d_5(x, 4) < \frac{1}{5^3}\right\} = \left\{x \in \mathbb{Z} : 5^3 \mid x - 4\right\} = 4 + 125\mathbb{Z}.$$

8. Let (M, d) be a metric space and $A \subset M$. Define the **diameter** of A by $\operatorname{diam}(A) := \sup_{x,y\in A} d(x, y)$.

(i) What are the subsets of M having diameter zero?

(ii) Prove that $A \subset B$ implies $\operatorname{diam}(A) \leq \operatorname{diam}(B)$.

(iii) Find two subsets A, B of a metric space such that $A \subsetneq B$ and $\operatorname{diam}(A) = \operatorname{diam}(B)$.

(iv) Using the p-adic distance defined in 7 find $\operatorname{diam}(B_{d_5}(0, 0.5))$.

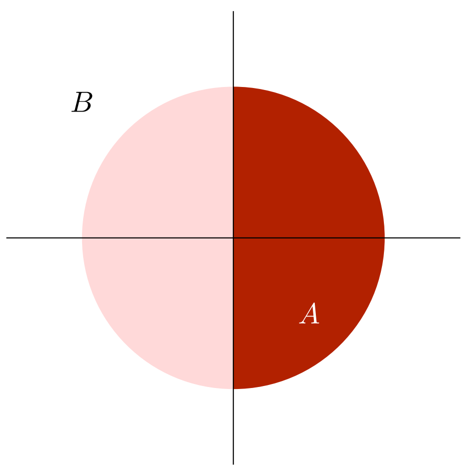
Solution.

(i) $\operatorname{diam}(A) = 0$ if and only if $\sup_{x,y\in A} d(x, y) = 0$, if and only if $d(x, y) = 0$ for every $x, y \in A$. That means $x = y$ for every $x, y \in A$, hence sets with diameter zero are simpletons $A = \{x\}$ (sets of cardinality 1).

(ii) $\operatorname{diam}(A) = \sup_{x,y\in A} d(x, y) \leq \sup_{x,y\in B} d(x, y)$.

(iii) Three examples. For $(\mathbb{R}, d_2)$, $d_2(x, y) = |x - y|$ being the standard metric on $\mathbb{R}$, let $B = [0, 1] \subset \mathbb{R}$. We have $\operatorname{diam}(B) = 1$ and any of the subsets $A = (0, 1), (0, 1], [0, 1)$ has diameter 1 as well.

Let $M = \mathbb{R}^2$ with the Euclidean distance d_2. Let $B = \overline{B_{d_2}(\mathbf{0}, 1)} := \left\{(x, y) : x^2 + y^2 \leq 1\right\}$ and $A := B \cup \{x \geq 0\} \subsetneq B$.



In both subsets, the diameter is equal to 2. Indeed,

$$d(\boldsymbol{x}, \boldsymbol{y}) \leq d(\boldsymbol{x}, \mathbf{0}) + d(\mathbf{0}, \boldsymbol{y}) \leq 1 + 1 = 2,$$

and they both contain points $(0, 1), (0, -1)$ lying at distance exactly 2.

Finally, for any (M, d) let $A = \varnothing$ and $B = \{x\}$. We have $\operatorname{diam}(\varnothing) = 0 = \operatorname{diam}(\{x\})$.

(iv) We have

$$B_{d_5}(0, 0.5) = \{x \in \mathbb{Z} : d_5(0, x) < 0.5\} = \left\{x \in \mathbb{Z} : d_5(0, x) \leq \frac{1}{5}\right\} = \{x \in \mathbb{Z} : 5 \mid x - 0\} = \dot{5}.$$

Hence $x, y \in B_{d_5}(0, 0.5)$ if and only if $x = 5n$ and $y = 5m$ for some $n, m \in \mathbb{Z}$. Thus $d_5(5n, 5m) = \frac{1}{5^l}$ where $5^l \mid 5(n - m)$ and $5^{l+1} \nmid 5(n - m)$ which means $5^{l-1} \mid n - m$ and l is maximal with this property. For the previous division to be possible, $l \geq 1$ must hold, hence $\operatorname{diam}(B_{d_5}(0, 0.5)) \leq \frac{1}{5}$. On the other hand, $d_5(5, 0) = \frac{1}{5}$ which implies $\operatorname{diam}(B_{d_5}(0, 0.5)) \geq \frac{1}{5}$. Both inequalities yield $\operatorname{diam}(B_{d_5}(0, 0.5)) = \frac{1}{5}$.

9. Consider on $\mathbb{R}^2$ the following two distances:

$$d\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right) := \max\left\{\left|x_1-y_1\right|,\left|x_2-y_2\right|\right\}, \qquad D\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right) := \sqrt{\left(x_1-y_1\right)^2+\left(x_2-y_2\right)^2}.$$

Prove that the functions

$$f,g:\mathbb{R}^2\to\mathbb{R}, \qquad f\left(x,y\right):=x+y, \quad g\left(x,y\right):=xy,$$

are continuous whenever we consider $\mathbb{R}^2$ endowed with either d or D. Hint: if you feel you only need to prove this for one of the two metrics, make sure you justify your decision with a general statement as well as its proof.

Solution. We will prove it for d. Let $\boldsymbol{x}=\left(x_1,x_2\right)\in\mathbb{R}^2$ and $\varepsilon>0$. We need to prove there exist $\delta_f,\delta_g>0$ such that $f\left(B_d\left(\boldsymbol{x},\delta_f\right)\right)\subset B_{d_2}\left(f\left(\boldsymbol{x}\right),\varepsilon\right)$ and $g\left(B_d\left(\boldsymbol{x},\delta_g\right)\right)\subset B_{d_2}\left(g\left(\boldsymbol{x}\right),\varepsilon\right)$ where $d_2\left(\star,\star\right)=\left|\star-\star\right|$.

a) Define $\delta_f:=\frac{\varepsilon}{2}$. For every $\boldsymbol{y}=\left(y_1,y_2\right)\in B_d\left(\boldsymbol{x},\delta_f\right)$ then by definition of d both $\left|x_1-y_1\right|<\delta_f$ and $\left|x_2,y_2\right|<\delta_f$. Let us prove $f\left(y_1,y_2\right)\in B_{d_2}\left(f\left(x\right),\varepsilon\right)$. Indeed, $d_2\left(f\left(\boldsymbol{x}\right),f\left(\boldsymbol{y}\right)\right)$ is equal to

$$\left|f\left(\boldsymbol{x}\right)-f\left(\boldsymbol{y}\right)\right|=\left|\left(x_1+x_2\right)-\left(y_1+y_2\right)\right|=\left|\left(x_1-y_1\right)+\left(x_2+y_2\right)\right|\le\left|x_1-y_1\right|+\left|x_2-y_2\right|<2\delta_f=\varepsilon.$$

Hence f is continuous.

b) Let us find δ_g. Again, such a number must fulfil the following: if $\left|x_1-y_1\right|<\delta_g$ and $\left|x_2,y_2\right|<\delta_g$, then $\left|g\left(\boldsymbol{x}\right)-g\left(\boldsymbol{y}\right)\right|<\varepsilon$. But $\left|g\left(\boldsymbol{x}\right)-g\left(\boldsymbol{y}\right)\right|$ is equal to

$$\left|x_1x_2-y_1y_2\right|=\left|x_1x_2-x_1y_2+x_1y_2-y_1y_2\right|=\left|x_1\left(x_2-y_2\right)+\left(x_1-y_1\right)y_2\right|\le\left|x_1\left(x_2-y_2\right)\right|+\left|\left(x_1-y_1\right)y_2\right|$$

which ought to be smaller than $\left|x_1\right|\delta_g+\left|y_2\right|\delta_g$. Now

$$\left|y_2\right|=\left|y_2-x_2+x_2\right|\le\left|y_2-x_2\right|+\left|x_2\right|<\delta_g+\left|x_2\right|,$$

hence

$$\left|x_1x_2-y_1y_2\right|<\left|x_1\right|\delta_g+\left|y_2\right|\delta_g<\left|x_1\right|\delta_g+\left(\delta_g+\left|x_2\right|\right)\delta_g.$$

Exercise: finish this by majoring the above by ε.

10. Let M be a metric space and $A\subseteq M$. Define the distance from a point $x\in M$ to A as

$$d\left(x_0,A\right):=\inf_{y\in A}d\left(x,y\right).$$

(i) Prove that the map

$$f:M\to\mathbb{R}, \qquad x\mapsto d\left(x,A\right),$$

is continuous.

(ii) If $x\in M$, prove $d\left(x,A\right)=0$ if and only if every open set containing x contains at least an element of A.

Solution.

(i) f is continuous on M if it is continuous on every $x\in M$, hence if for every $x\in M$ and every $\varepsilon>0$ there exists a $\delta>0$ such that $d\left(x,y\right)<\delta$ implies $\left|f\left(x\right)-f\left(y\right)\right|<\varepsilon$. This is the same as saying that for every $x\in M$ and every $\varepsilon>0$ there exists a $\delta>0$ such that $d\left(x,y\right)<\delta$ implies $\left|d\left(x,A\right)-d\left(y,A\right)\right|<\varepsilon$. We wil prove $\left|d\left(x,A\right)-d\left(y,A\right)\right|\le d\left(x,y\right)$ for every $x,y\in A$ and this will allow $\delta=\varepsilon$.

First of all the fact d is a distance implies

$$d(x,A)=\inf_{z\in A} d(x,z)\le \inf_{z\in A}\{d(x,y)+d(y,z)\}=d(x,y)+\inf_{z\in A} d(y,z)=d(x,y)+d(y,A),$$

hence $d(x,A)-d(y,A)\le d(x,y)$.

On the other hand

$$d(y,A)=\inf_{z\in A} d(y,z)\le \inf_{z\in A}\{d(y,x)+d(x,z)\}=d(x,y)+\inf_{z\in A} d(x,z)=d(x,y)+d(x,A),$$

from which $d(x,A)-d(y,A)\ge -d(x,y)$.

(ii) We want to prove $d(x,A)=0$ if and only if for every $\varepsilon>0$, $B(x,\varepsilon)\cap A\ne\varnothing$.

$\Rightarrow$) Assume $d(x,A)=0$. Assume there exists $\varepsilon>0$ such that $B(x,\varepsilon)\cap A=\varnothing$ Then $d(x,y)\ge\varepsilon$ for every $y\in A$ and $\inf_{y\in A} d(x,y)\ge\varepsilon$. Thus $d(x,A)\ge\varepsilon>0$, absurd.

$\Leftarrow$) Assume $B(x,\varepsilon)\cap A\ne\varnothing$ for every $\varepsilon>0$. Assume $d(x,A)=r>0$. Then $\inf_{y\in A} d(x,y)>0$, which means $d(x,y)\ge\inf_{y\in A} d(x,y)=r>0$ for every $y\in A$, which means there is no $y\in A$ such that $d(x,y)<r$ which would contradict $B(x,\varepsilon)\cap A\ne\varnothing$.

11. (*July 2014*) Let (X,d) be a metric space, X being a set of size at least 3. For any two subsets $S_1,S_2\subset X$, define

$$D(S_1,S_2):=\inf_{x\in S_1,y\in S_2} d(x,y).$$

Is the function $D:\mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}$ thus defined a metric on $\mathcal{P}(X)$? Justify your answer.

Solution. Obviously not. For it to be a metric, it should satisfy the following properties:

a) $D(S_1,S_2)\ge 0$ for every $S_1,S_2\in\mathcal{P}(X)$;

b) $D(S_1,S_2)=0$ if and only if $S_1=S_2$;

c) $D(S_1,S_2)=D(S_2,S_1)$ for every $S_1,S_2\in\mathcal{P}(X)$;

d) $D(S_1,S_2)\le D(S_1,S_3)+D(S_3,S_1)$ for every $S_1,S_2,S_3\in\mathcal{P}(X)$.

It satisfies the first and third properties, but it does not satisfy the second and fourth. It is easiest to check for the second: consider two different sets having non-empty intersection, e.g. $S_1=\{x,y\}$ and $S_2=\{y,z\}$ with x,y,z pairwise different (which is possible because $\#X\ge 3$). Then $\inf_{s_1\in S_1,s_2\in S_2} d(s_1,s_2)=d(y,y)=0$, yet $S_1\ne S_2$.

Another example would be $S_1=(a,b)$, $S_2=(c,d)$ real intervals with $a<c<b<d$. Their intersection (c,b) is different from $\varnothing$, hence $D(S_1,S_2)=0$, yet $S_1\ne S_2$. □

12. Prove Lemma 1.2.6 stating the equivalence of all Hölder p-metrics on $\mathbb{R}^n$. Hint: prove them all equivalent to the ∞-metric.

Solution. Define $||\boldsymbol{x}||^p=\left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, $||\boldsymbol{x}||^\infty:=\max_{1\le i\le n}|x_i|$ and $d_p(\boldsymbol{x},\boldsymbol{y}):=||\boldsymbol{x}-\boldsymbol{y}||_p$ for every $p\in\mathbb{R}_+\cup\{\infty\}$.

Let $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$ and $p\in\mathbb{R}_+$. First of all, let $j\in\{1,\dots,n\}$ such that $\max_{1\le i\le n}|x_i-y_i|=|x_j-y_j|$.

$$\begin{aligned}
||\boldsymbol{x}-\boldsymbol{y}||_\infty &= \left[\left(\max_{1\le i\le n}|x_i-y_i|\right)^p\right]^{1/p}\le\left[\left(\max_{1\le i\le n}|x_i-y_i|\right)^p+\sum_{i\ne j}|x_i-y_i|^p\right]^{1/p}\\
&= \left[\sum_{i=1}^n |x_i-y_i|^p\right]^{1/p}=||\boldsymbol{x}-\boldsymbol{y}||_p
\end{aligned}$$

Hence we can choose $\beta = 1$ in (1.4).

On the other hand, with the same definition for index j we have

$$\left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} |x_j - y_j|^p\right]^{1/p} = (n\,|x_j - y_j|^p)^{1/p} = n^{1/p}\,||\boldsymbol{x} - \boldsymbol{y}||_\infty$$

and thus $\alpha = n^{-1/p}$ in (1.4):

$$\frac{1}{n^{1/p}}\,||\boldsymbol{x} - \boldsymbol{y}||_\infty \leq ||\boldsymbol{x} - \boldsymbol{y}||_p \leq ||\boldsymbol{x} - \boldsymbol{y}||_\infty\,, \qquad \text{for every } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

13. (*July 2014*) Let (M, d) be a metric space and define

$$\overline{D} : \mathcal{P}(M) \times \mathcal{P}(M) \to \mathbb{R}, \qquad \overline{D}(A, B) := \sup_{y \in B} d(y, A)\,.$$

Is $\overline{D}$ a metric on $\mathcal{P}(M)$?

Solution. Let us check the properties.

- $\overline{D}(A, B) = \sup_{y \in B} d(x, y) = \sup_{y \in B} \inf_{x \in A} d(y, x) \geq 0$ on account of $d(x, y) \geq 0$ for every $x, y \in M$ because d is a metric.
- Symmetry. Exercise: look for a counterexample if there is one.
- Let $A, B \subset M$ such that $B \subsetneq A$, e.g. $(0, 1) \subset [0, 1]$ in $\mathbb{R}$. Then for every $y \in B$, $d(y, A) = \inf_{x \in A} d(y, x) = d(y, y)$ since $y \in A \subset B$. Thus $\overline{D}(A, B) = \sup_{y \in B} d(y, A) = \sup_{y \in B} 0 = 0$, and yet $A \neq B$.
- Triangle inequality. Exercise: look for a counterexample if there is one.

It does not fulfil the third property hence it is not a metric.

14. (*May 2014*) Let M be a metric space and $A, B \subset M$ be two closed, disjoint subsets. Define function

$$f : M \to \mathbb{R}, \qquad x \mapsto \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Prove f is continuous and its range lies between 0 and 1. Find $f^{-1}(0)$ and $f^{-1}(1)$. Hint: feel free to use Exercises 19 and 10

Solution. In virtue of 19 and 10 all we have to prove is $d(x, A) + d(x, B) \neq 0$ for every $x \in M$. We know $A \cap B = \varnothing$ and $d(x, A) + d(x, B) = \inf_{y \in A} d(x, y) + \inf_{y \in B} d(x, y)$. We know for every $x \in M$ that x cannot belong to both A and B. Hence assume, without loss of generality, $x \notin A$. Let us prove $d(x, A) > 0$. Assuming otherwise would imply (Exercise 10 (ii)) that for every $\varepsilon > 0$, $B(x, \varepsilon) \cap A \neq \varnothing$. But the fact $M \setminus A$ is open implies the existence of $\delta > 0$ such that $B(x, \delta) \subset X \setminus A$, which means $B(x, \delta) \cap A = \varnothing$, absurd.

Hence $d(x, A) > 0$ which means $d(x, A) + d(x, B) > 0$.

Let us prove $0 \leq f(x) \leq 1$ for every x. The fact $d(x, A), d(x, B) \geq 0$ yields $d(x, A) \leq d(x, A) + d(x, B)$, hence $\frac{d(x,A)}{d(x,A)+d(x,B)} \leq 1$, and $\frac{d(x,A)}{d(x,A)+d(x,B)} \geq \frac{0}{d(x,A)+d(x,B)} = 0$.

$f^{-1}(0)$ is equal to

$$\left\{x \in M : \frac{d(x, A)}{d(x, A) + d(x, B)} = 0\right\} = \{x \in M : d(x, A) = 0\} = A,$$

on account of the reasoning described at the beginning of this solution ($d(x, S) = 0$ if and only if $x \in S$ for any closed S).

$f^{-1}(1)$ is equal to

$$\left\{x \in M : \frac{d(x, A)}{d(x, A) + d(x, B)} = 1\right\} = \{x \in M : d(x, B) = 0\} = B,$$

again for the same reason.

15. Prove a function $d : X \times X \to \mathbb{R}$ is a metric if, and only if, the two following properties hold:

a) $d(x, y) = 0$ iff $x = y$;

b) $d(x, z) \leq d(x, y) + d(z, y)$ for every $x, y, z \in X$.

SOLUTION. A function $d : X \times X \to \mathbb{R}$ is a metric if and only if

(i) $d(x, y) \geq 0$ for every $x, y \in X$;

(ii) $d(x, y) = 0$ if and only if $x = y$;

(iii) $d(x, y) = d(y, x)$ for every $x, y \in X$;

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \in X$.

$\Rightarrow$) Let us first prove a,b imply (i),(ii),(iii),(iv). Assuming a and b we trivially have (ii) which is nothing but a. Setting $x = y = z$ in b we have $d(x, x) \leq d(x, x) + d(x, x)$, and subtracting $d(x, x)$ from both sides, $d(x, x) \geq 0$ which is (i). Given $x, y, z \in X$, the chains of inequalities $d(x, y) \leq d(x, z) + d(y, z)$ and $d(y, x) \leq d(y, z) + d(x, z)$ can be subtracted in both directions: $d(x, y) - d(y, x) \leq 0$ and $d(y, x) - d(x, y) \leq 0$, which entail the identity $d(x, y) = d(y, x)$ in (iii). Finally (iv) is the consequence of applying the symmetry (iii) we just proved to b.

$\Leftarrow$) This is the easy implication. (ii) implies (equates to) a, (iii) and (iv) imply b.

16. Let M be a metric space, $x_0 \in M$ and $S = \{a_n\}_{n \geq 1}$ a sequence of pairwise different points of M such that

$$d(a_k, x_0) = \frac{1}{2} d(a_{k-1}, x_0), \qquad \text{for every } k \geq 2.$$

(i) Prove that every ball of centre x_0 contains a point in S different from x_0.

(ii) Prove x_0 is the only point with this property.

SOLUTION.

(i) We need to prove, for every $\varepsilon > 0$, $B(x_0, \varepsilon) \cap S \neq \varnothing$. This is the same, in the terms described in Exercise 10, as $d(x_0, S) = 0$. And indeed $d(x_0, S) = \inf_{n \in \mathbb{N}} d(x_0, a_n) = \inf_{n \in \mathbb{N}} \frac{1}{2^n} r$ due to $d(x_0, a_n) = r, d(x_0, a_1) = \frac{1}{2} d(x_0, a_0) = \frac{1}{2} r, \ldots$, hence $d(x_0, S) = \inf_{n \in \mathbb{N}} \frac{1}{2^n} r = 0$.

Remark 1.5.1. Let us remind that $x_0 \notin S$. If $x_0 = a_m$ for some m, then $d(x_0, a_m) = 0$ and for every $n > m$ $d(x_0, a_m) = 0$ which means $a_m = x_0 = a_n$ for every $n > m$, contradicting the assumption that $\{a_n\}$ are pairwise different.

Thus the Remark, along with the preceding argument, proves the statement of the Exercise.

(ii) Let $x \in M$ such that $x \neq x_0$. Let us try to find $\varepsilon > 0$ such that $B(x, \varepsilon) \cap S = \varnothing$. Assume $a_{n_0} \in B(x_0, r)$ for some n_0. Let $s := d(x_0, a_{n_0})$. The fact $r := d(x, x_0) > 0$ and $d(a_n, x_0) < d(a_{n_0}, x_0) < r$ for every $n > n_0$ implies $a_n \in B(x_0, r)$. Let $\varepsilon < \min\{r - s, d(x, a_0), \ldots, d(x, a_{n_0 - 1})\}$. Let us prove $d(x, a_n) \geq \varepsilon$ for every $n \geq 1$. Indeed, for every $0 \leq k \leq n_0 - 1$

$$d(x, a_k) \geq \min\{r - s, d(x, a_0), \ldots, d(x, a_{n_0-1})\} = \varepsilon$$

and for every $k > n_0$ the fact

$$d(x, x_0) \leq d(x, a_n) + d(a_n, a_0) \leq d(x, a_n) + d(a_{n_0}, x)$$

implies

$$d(x, a_n) \geq d(x, x_0) - d(a_{n_0}, x) = r - s \geq \min\{r - s, \cdots\} = \varepsilon.$$

1.6 Exercises

17. Prove $\equiv$ defined as in Definition 1.2.4 is indeed an equivalence relation.

18. Is

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad (\boldsymbol{x}, \boldsymbol{y}) \mapsto d(\boldsymbol{x}, \boldsymbol{y}) := \min_{1 \leq i \leq n} |x_i - y_i|$$

a distance?

19. Let (M, d) be a metric space and $f, g : M \to \mathbb{R}$ continuous functions. Prove the following are continuous:

$$\begin{aligned}
M &\xrightarrow{\quad f \pm g,\, f \cdot g,\, f/g \quad} \mathbb{R} \\
x &\longmapsto (f + g)(x) := f(x) + g(x), \\
x &\longmapsto (f - g)(x) := f(x) - g(x), \\
x &\longmapsto (f \cdot g)(x) := f(x) \cdot g(x), \\
x &\longmapsto (f/g)(x) := f(x)/g(x) \ \text{ if } g(x) \neq 0 \text{ for every } x \in M.
\end{aligned}$$

20. Let (X, d) be a metric space. An **open cover** of a subset $A \subset X$ is a collection of open subsets $\{U_i\}_{i \in I}$ overall containing the subset: $A \subset \bigcup_{i \in I} U_i$. We call A **compact** if for *any* open cover $\{U_i\}_{i \in I}$ of A there exist finitely many U_i, $i = i_1, \ldots, i_n$, already covering the subset: $A = U_{i_1} \cup \cdots \cup U_{i_n}$.

(i) Prove that for any compact set A and any $\varepsilon > 0$, there exist finitely many $a_1, \ldots, a_k \in A$ such that $A \subset B(a_1, \varepsilon) \cup \cdots \cup B(a_k, \varepsilon)$.

(ii) Let A be a countable infinite subset of X. Can A be compact? Explain.

Chapter 2

Topological spaces

Our intention is to generalise the concept of a metric space introduced in the previous Chapter by means of a definition allowing us to handle the concept of continuity of functions within a wider landscape. This was achieved in the early twentieth century by Poincaré, who laid the groundwork for a newborn theory originally called *Analysis Situs* in reference to his original 1895 paper.

Roughly divided into two parts, Topology first addresses point-set concepts requiring large amounts of background in Analysis. This is followed by a second part addressing deformations of surfaces and their classification, especially concerning connected, compact surfaces (e.g. Homotopy Theory).

The operative results this wider setting will be couched on will be two Propositions involving open sets in metric spaces, shown in Sections 1.3 and 1.4.

2.1 Definition

Let us first retain the one significant set of properties of open sets in Section 1.3 that made no mention of the concept of metric in its statement, namely Proposition 1.3.4.

Definition 2.1.1. *Let X be a set and $\tau \subset \mathcal{P}(X)$ a collection of subsets of X. We say τ is a **topology** on X and (X, τ) is a **topological space** if the following properties are met:*

T_1*: $\varnothing, X \in \tau$;*

T_2*: for every collection $\{U_i\}_{i \in I}$ of elements in τ, $\bigcup_{i \in I} U_i$ is an element of τ;*

T_3*: for every* finite *collection $\{U_1, \ldots, U_n\}$ of elements in τ, $\bigcap_{i=1}^{n} U_i$ is an element of τ.*

The elements of τ are called open sets *of (X, τ) (or of X if there is no ambiguity concerning the topology).*

Examples 2.1.2.

1. A first natural example, given the very specific choice of three axioms any topology must follow, is the set of all open subsets of a metric space (X, d) in virtue of Proposition 1.3.4. A topological space whose open sets are those of a metric is usually called **associated** to the metric and belongs to a wider class of topological spaces called **metrizable** which will be dealt with later.

2. Let X be a set. $\{X, \varnothing\}$ is a topology. It is called the **coarse** or **trivial** topology on X.

3. Given a set X, $\mathcal{P}(X)$ is also a topology. It is called the **discrete topology** on X.

4. Given a set X, the **finite complement topology** on X is defined by

$$\tau_X := \{S \subset X : X \setminus S \text{ is finite}\} \cup \{\varnothing\}.$$

It is indeed a topology. $\varnothing \in \tau_X$ by definition, $\varnothing$ is finite (its size being 0) hence $X = X \setminus \varnothing \in \tau_X$. If $\{U_i\}_{i \in I}$ is a collection of elements of τ_X then $X \setminus \bigcup_{i\in I} U_i = \bigcap_{i \in I} (X \setminus U_i)$, which is finite because every $X \setminus U_i$ is finite. And finally, given $U_1, \ldots, U_n$ non-empty elements of τ_X their intersection is such that $X \setminus (U_1, \ldots, U_n) = (X \setminus U_1) \cup \cdots \cup (X \setminus U_n)$, a finite union of finite sets, hence finite.

5. Let us find all possible topologies on a set of 2 elements $X = \{a, b\}$:

$$\tau_1 = \{\varnothing, \{a, b\}\}, \quad \tau_2 = \{\varnothing, \{a, b\}, \{a\}\}, \quad \tau_3 = \{\varnothing, \{a, b\}, \{b\}\}, \quad \tau_4 = \{\varnothing, \{a, b\}, \{a\}, \{b\}\}.$$

6. A set of 3 elements has 27 different topologies. Exercise: find them all. However, not every set of subsets is a topology on $X = \{a, b, c\}$. For instance $\{X, \varnothing, \{a\}, \{b\}\}$ or $\{X, \varnothing, \{a, b\}, \{b, c\}\}$.

7. Let $X = \mathbb{N}$ and $\tau = \{X, \varnothing, U_n = \{n, n+1, \ldots\} : n > 1\}$. Then τ is a topology on $\mathbb{N}$. Exercise.

8. Given X the collection $\tau = \{A \subset X : A = X \text{ or } X \setminus A \text{ countable}\}$ is a topology. Exercise.

Remarks 2.1.3.

1. The discrete topology in Example **3** is exactly the topology associated to the discrete metric in Example 1.1.4 (**9**). Indeed, the fact $\{x\} = B\left(x, \frac{1}{2}\right)$ for every $x \in X$, as seen in Example 1.2.2 (**4**), implies points are open sets for the topology associated to this metric. Property T_2 in the Definition thus implies all arbitrary unions of points of X (i.e. all subsets of X) are open sets for this topology as well.

2. As seen in Example **1** and recounted in the above Remark, metric spaces are automatically topological spaces. However, not all topological spaces need be associated to a metric. For instance, if X has more than one element the coarse topology defined in Example **2** is not associated to any metric. See Exercise 52

2.2 Comparison of topologies

Let X be a set and $\tau_1, \tau_2 \subset \mathcal{P}(X)$ two topologies on X.

Definition 2.2.1. *We say τ_2 is **finer** (or **stronger**) than τ_1, or alternatively τ_1 is **coarser** or **weaker** than τ_2, if $\tau_1 \subseteq \tau_2$.*

The following is so trivial that it merits no space for a proof:

Lemma 2.2.2. *The coarse topology in Example 2.1.2 (**2**) is the coarsest possible topology on X, and the discrete topology in Example 2.1.2 (**3**) is the finest among all topologies on X.*

Remark 2.2.3. Two topologies are not necessarily comparable. Indeed, if $X = \{a, b, c, \ldots\}$ and $\tau_1 = \{X, \varnothing, X \setminus \{c\}, X \setminus \{a\}, X \setminus \{a, c\}\}$, $\tau_2 = \{X, \varnothing\}$, $\tau_3 = \mathcal{P}(X)$ and $\tau_4 = \{X, \varnothing, \{a\}, X \setminus \{a\}\}$, then $\tau_{2,3}$ can be compared to the other two and to one another (one is finer than the rest, the other is coarser than the rest) but τ_1 and τ_4 cannot be compared: neither of them is finer or coarser than the other.

2.3 Topology induced on subsets

Definition 2.3.1. *Let (X,τ) be a topological space and $Y \subseteq X$ a subset of X. The* ***subspace topology*** *or* ***induced topology*** *on Y is defined by*

$$\tau_Y := \{U \cap Y : U \in \tau\} \subseteq \mathcal{P}(Y).$$

Let us prove it is indeed a topology on Y:

T_1: The fact $\varnothing, X \in \tau$ implies $\varnothing \cap Y = \varnothing$ and $X \cap Y = Y$ belong to τ_Y.

T_2: For every collection $\{Y \cap U_i\}_{i\in I}$ of elements in τ_Y, $\bigcup_{i\in I}(Y \cap U_i) = \left(\bigcup_{i\in I} U_i\right) \cap Y$ is an element of τ_Y.

T_3: For every *finite* collection $\{Y \cap U_1, \dots, Y \cap U_n\}$ of elements in τ_Y,

$$\bigcap_{i=1}^{n}(Y \cap U_i) = (U_1 \cap \cdots \cap U_n) \cap Y$$

which is an element of τ_Y.

Examples 2.3.2.

1. Let X be any set and τ the discrete (resp. coarse) topology on X. Then for every subset $Y \subset X$, τ_Y is the discrete (resp. coarse) topology on Y.

2. The converse to the above definition need not be true, i.e. open subsets of τ_Y need not be open sets of τ. Let $Y := [0,1] \subset \mathbb{R}$. With the induced Euclidean metric topology $\left[0, \frac{1}{3}\right) = \left(-1, \frac{1}{3}\right) \cap Y$ is an open set of Y, yet it is not an open set of $\mathbb{R}$.

2.4 Closed sets, interiors, closures and boundaries

The notion of relative position between points so easy to characterise in metric spaces needs further introduction of new concepts in the wider setting of topological spaces (X,τ).

Definition 2.4.1. *Let C be a subset of X. We say C is* ***closed*** *in (X,τ) if its complementary is an open set: $X \setminus C \in \tau$.*

Taking complementaries of the properties defining open sets in Definition 2.1.1 and using De Morgan's laws $X \setminus \bigcap_i A_i = \bigcup_i X \setminus A_i$ and $X \setminus \bigcup_i A_i = \bigcap_i X \setminus A_i$, we have the following properties for closed sets:

F_1: $\varnothing, X$ are closed;

F_2: for every collection $\{C_i\}_{i\in I}$ of closed subsets of X, $\bigcap_{i\in I} C_i$ is a closed subset of X;

F_3: for every *finite* collection $\{C_1, \dots, C_n\}$ of closed subsets of X, $\bigcup_{i=1}^{n} C_i$ is a closed subset of X.

Given a collection $\mathcal{C} \subseteq \mathcal{P}(X)$ of subsets of X fulfilling the above three properties, there is a unique topology on X, namely $\tau := \{X \setminus C : C \in \mathcal{C}\}$, for which $\mathcal{C}$ are all the closed subsets. In other words, the set of closed sets of (X,τ) is also useful to determine the topology τ on X. See the Example below.

Examples 2.4.2.

1. In Example 1.3.2 (**2.**) we established that if (X, τ) is the topological space associated to a metric space, set $X \setminus \{x\}$ is open for every $x \in X$. Hence in this case all points are closed sets.

2. Let $Y = [0,1] \cup (2,3) \subset \mathbb{R}$ with the induced Euclidean topology. $[0,1]$ is open in Y since it is the intersection of Y with open set $(-1/2, 3/2)$. Similarly $(2,3)$ is open in Y and even in $\mathbb{R}$. The fact $(0,1)$ and $[2,3]$ are complementaries of each other implies they are both open and closed.

3. In the discrete topology, every set is both open and closed.

4. In $(\mathbb{R}, |\cdot|)$, every interval of the form $[a, +\infty)$ is closed since its complementary $(-\infty, a)$ is open. Ditto for intervals of the form $(-\infty, a]$. Hence every interval of the form $[a, b]$ is closed due to the fact it is the intersection of intervals $[a, +\infty)$, $(-\infty, b]$.

5. $\{(x, y) : x \geq 0, y \geq 0\}$ is closed since its complementary is the union of two open sets $(-\infty, 0) \times \mathbb{R}$ and $\mathbb{R} \times (-\infty, 0)$. The same reasoning entails that every subset of $\mathbb{R}^n$ defined by a finite amount of non-strict (resp. strict) inequalities is closed (resp. open). Exercise

We have to exercise caution in the use of the word "closed" in the context of subspace topology. If $Y \subset X$ is a subspace of X the reader must remember that $A \subset Y$ will be called *closed* only as long as it is closed in the induced topology τ_Y.

Lemma 2.4.3. *Let $Y \subset X$ be a subspace with the induced topology. Then $A \subset Y$ is closed in Y if, and only if, $A = C \cap Y$ where C is closed in X.*

Proof. Assume $A = C \cap Y$ where C is closed in X. Then $X \setminus C$ is open in X, hence $(X \setminus C) \cap Y$ is open in τ_Y. But $(X \setminus C) \cap Y = Y \setminus A$, hence $Y \setminus A$ is an element of τ_Y which means A is closed in Y.

Assume A is closed in Y. Then $Y \setminus A$ is open in Y, hence $Y = U \cap Y$ for some $U \in \tau_X$. Thus $X \setminus U$ is closed in X, and the fact $A = Y \cap (X \setminus U)$ implies we can choose $C = X \setminus A$. □

Recall $(0,1)$ was closed in Y in Example 2.4.2 (**2.**), yet not in $\mathbb{R}$. We do have a sufficient condition, though:

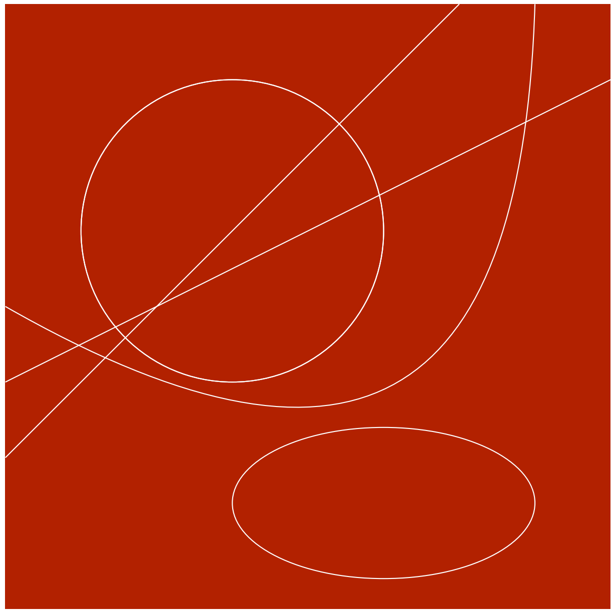
Proposition 2.4.4. *Let $Y \subset X$ be a subspace. If $A \subset Y$ is closed in Y and Y is closed in X, then A is closed in X.*

Proof. If A is closed in Y, then $A = Y \cap V$ for some closed set V of X due to the Lemma. Hence, Y and V are both closed sets of X, thus so is their intersection. □

Example 2.4.5. An example of a topology best defined by its closed sets. Let $X = \mathbb{R}^n$ and $R = \mathbb{R}[X_1, \ldots, X_n]$ the set of all polynomials in n indeterminates. Let $S \subseteq R$ be a set of such polynomials and define

$$V(S) := \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : f(x_1, \ldots, x_n) = 0 \text{ for every } f \in S\} \subset \mathbb{R}^n,$$

the set of zeroes of all polynomials in S. Such sets $V(S)$ are called **algebraic varieties** and the set of all of them, $\mathcal{C} := \{V(S) : S \subset \mathbb{R}[X_1, \ldots, X_n]\}$ is the set of closed sets for a topology, called the **Zariski topology** on $\mathbb{R}^n$. Open sets are thus all complementaries of such subsets, for instance an open set for the Zariski topology on $\mathbb{R}^2$ would be the shaded area a portion of which is shown below:



In general, for every field K the same topology can be defined on K^n for sets of zeros $V(S) \subset K^n$ of polynomials $f(x) \in S \subset K[x_1, \ldots, x_n]$ although its closed and open sets will not be as easy to represent as the ones above.

Definition 2.4.6. *Let (X, τ) be a topological space and $A \subseteq X$ a subset.*

*a) The **closure** of X, written $\overline{A}$, is the intersection of all closed subsets of X containing A.*

*b) The **interior** of X, written $\mathring{A}$, is the union of all open sets contained in A.*

*c) The **boundary** of A is $\partial A := \overline{A} \setminus \mathring{A}$.*

*In virtue of properties T_2 for open sets and F_2 for closed sets, $\overline{A}$ is a closed set containing A and $\mathring{A}$ is an open set contained in A. An element $x \in X$ is called a **closure point** or **adherent point** of A if $x \in \overline{A}$, and an **interior point** of A if $x \in \mathring{A}$.*

In general $\mathring{A} \subset A \subset \overline{A}$. The following is trivial:

Lemma 2.4.7. *The closure of a subset A is the smallest closed subset containing A. The interior of A is the largest open subset contained in A.* □

Lemma 2.4.8. *Let $A \subset Y \subset X$ where X is a topological space and Y is endowed with the subspace topology. Then the closure of A in Y equals $\overline{A} \cap Y$, where $\overline{A}$ is the closure of A in X.*

Proof. Let B be the closure of A in Y. $\overline{A}$ is closed in X, thus $\overline{A} \cap Y$ is closed in Y. We have $A \subset \overline{A} \cap Y$ and B is the intersection of all closed sets of Y containing A, hence $B \subset \overline{A} \cap Y$.

On the other hand B is a closed set of Y, hence $B = C \cap Y$ for some C closed set of X. But C, a closed set containing A, must therefore contain $\overline{A}$. Which means $\overline{A} \cap Y \subset C \cap Y = B$. □

Lemma 2.4.9. *In the same hypotheses as the previous Lemma, the interior $\mathring{A}$ of A in X equals $\left(\mathring{A}\right)_Y \cap \mathring{Y}$, where $\left(\mathring{A}\right)_Y$ is the interior of A in Y.*

Proof. Exercise. □

Examples 2.4.10.

1. $A = \mathring{A}$ if and only if A is open, and $A = \overline{A}$ if and only if A is closed.

2. d_2 being the usual Euclidean metric on $\mathbb{R}^2$, the closure of $B\left((a,b),r\right)$ is
$$\overline{B\left((a,b),r\right)} = \left\{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 \leq r^2\right\},$$
and $\partial B\left((a,b),r\right) = \left\{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 = r^2\right\}$. In particular, $\partial B\left((0,0),1\right) = \mathbb{S}^1$, the unit circle centered at the origin.

3. With the discrete topology, every subset is both open and closed, hence for every $A \subseteq X$, $A = \mathring{A} = \overline{A}$ and $\partial A = \varnothing$.

We have the following, usually simpler way of characterising closure and interior points.

Proposition 2.4.11. *Let (X, τ) be a topological space. Let $x \in X$ and $A \subseteq X$.*

a) x is a closure point of A if, and only if, for every open set U such that $x \in U$, $U \cap A \neq \varnothing$.

b) x is an interior point of A if, and only if, there exists an open set U such that $x \in U \subset A$.

Proof. a) Assume $x \in \overline{A}$ and let $U \in \tau$ containing x. Assume $U \cap A = \varnothing$. Then $X \setminus U =: C$ would be a closed set containing A, which in virtue of Definition 2.4.6 or Lemma 2.4.7 implies $\overline{A} \subset C$. Which means $\overline{A} \cap U = \varnothing$, contradicting $x \in \overline{A} \cap U$.

Conversely let $x \in X$ such that $U \cap A \neq \varnothing$ for every $U \in \tau$ containing x, yet $x \notin \overline{A}$. This means $X \setminus \overline{A}$ is an open set containing x, which implies $U \cap A \neq \varnothing$, contradicting $U \cap \overline{A} = \varnothing$ and $A \subset \overline{A}$.

b) $\mathring{A}$ is an open set and $x \in \mathring{A} \subset A$ for every interior point of A.

Conversely, if there is an open set U such that $x \in U \subset A$, U must be contained in the union of all open sets contained in A, hence $x \in U \subset \mathring{A}$.

□

Proposition 2.4.12. *Let (X, τ) be a topological space and $A, B \subset X$ two subsets. Then,*

(i) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(ii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

(iii) $\overset{\circ}{\overbrace{A \cap B}} = \mathring{A} \cap \mathring{B}$.

(iv) $\overline{\overline{A}} = \overline{A}$.

(v) $X \setminus \mathring{A} = \overline{X \setminus A}$.

(vi) $\partial A = \overline{A} \cap \overline{X \setminus A}$.

(vii) The interior, the exterior $X \setminus A$ and the boundary of A are disjoint and their union is equal to X.

(viii) If A is open in X and B is any subset of X, then $A \cap \overline{B} \subset \overline{A \cap B}$.

(ix) $\mathring{A} \cup \mathring{B} \subset \overset{\circ}{\overbrace{A \cup B}}$, *the opposite inclusion not true in general.*

Proof. (i) $\overline{A} \cup \overline{B}$ is a closed set containing both A and B, hence $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Let $x \in \overline{A} \cup \overline{B}$ and U an open set containing x. Assume (without loss of generality, proof for $\overline{B}$ being similar) $x \in \overline{A}$. Then $U \cap A \neq \varnothing$ which means $U \cap (A \cup B) \neq \varnothing$.

(ii) $\overline{A} \cap \overline{B}$ is a closed set containing $A \cap B$, hence $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

(iii) $\overbrace{A \cap B}^{\circ}$ is an open set contained in A and B, hence it must be contained in $\mathring{A}$ and $\mathring{B}$ and thus in their intersection. For the other inclusion, any element $x \in \mathring{A} \cap \mathring{B}$ is contained in open sets $U \subset A$ and $V \subset B$. Thus $U \cap V$ is an open set containing x and contained in $A \cap B$.

(iv) The fact $\overline{A}$ is closed implies it is equal to its closure.

(v) The fact $\mathring{A} \subset A$ implies $X \setminus A \subset X \setminus \mathring{A}$ and the latter is a closed set, hence $\overline{X \setminus A} \subset X \setminus \mathring{A}$.
In order to prove the opposite inclusion, let C be a closed set such that $X \setminus A \subset C$, then open set $X \setminus C$ is contained in A, hence (being open) in $\mathring{A}$. Taking complementaries of $X \setminus C \subset \mathring{A}$ we obtain $X \setminus \mathring{A} \subset C$. This is true for every closed set C containing $X \setminus A$, hence for $C := X \setminus A$.

(vi) $\partial A = \overline{A} \setminus \mathring{A}$ which is equal to $\overline{A} \cap \left(X \setminus \mathring{A}\right)$, which in virtue of the previous item is equal to $\overline{A} \cap \overline{X \setminus A}$.

(vii) Immediate.

(viii) Let $x \in A \cap \overline{B}$. Then for any open set U containing x, $U \cap A$ is an open set containing x as well, hence $U \cap A \cap B \neq \varnothing$. Thus $x \in \overline{A \cap B}$.

(ix) Exercise.

□

Definition 2.4.13. *Given $x \in X$ and $A \subset X$, we say x is a* ***limit point*** *or an* ***accumulation point*** *if every $U \in \tau$ containing x intersects A in at least one point* different from x. *In other words x is a limit point if $x \in \overline{A \setminus \{x\}}$.*

Examples 2.4.14.

1. Every point in a set X is a limit point thereof.

2. Every limit point is a closure point of X.

Proposition 2.4.15. *Let $A \subset X$ and $L(A)$ be the set of limit points of A. Then $\overline{A} = A \cup L(A)$. Hence A is closed if and only if it contains all its limit points.*

Proof. Let $x \in L(A)$. Then every open set U containing x intersects A in a point $y \neq x$. If we drop the condition $y \neq x$, however true, we obtain the alternative definition for x being a closure point.

Let $x \in \overline{A}$. If $x \in A$ then we are finished. Assume $x \notin A$; then every open set U containing x intersects A in at least one point and this point cannot be x, hence x is a limit point of A. □

Definition 2.4.16. *Let X be a topological space, $A \subset X$ and $x \in S$. We say x is* ***isolated*** *if there exists an open set U containing x such that $S \cap U = \{x\}$, in other words if x is not a limit point of A.*

Definition 2.4.17. *We say A is* ***dense*** *in X if $A \subset X$ and $\overline{A} = X$.*

Examples 2.4.18.

1. $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

2. In any discrete space $(X, \mathcal{P}(X))$ the only dense subspace of X is X. In the coarse space $(X, \{\varnothing, X\})$ every non-empty subset of X is dense in X.

2.5 Bases

Let us generalise the role played in metric spaces by open balls.

Definition 2.5.1. *Let (X, τ) be a topological space. A subset $\beta \subset \tau$ is called a* ***base*** *or a* ***basis*** *for the topology if every open set $U \in \tau$ is equal to the union of a collection of elements in β: there exist $\{V_i\}_{i \in I} \subset \beta$ such that $U = \bigcup_{i \in I} V_i$.*

Examples 2.5.2.

1. If the topology τ is associated to a metric d, then

$$\beta = \{\text{all open balls of all centres and radii for metric } d\}$$

is a basis of τ. In particular, interiors of both squares and circles form bases of $\mathbb{R}^2$ with the usual Euclidean topology. For the same reason (a, b), $a < b$ form a basis for $\mathbb{R}$.

2. The same as above applies to $\mathbb{Q}^2$ with the induced topology: $\left\{B\left(\boldsymbol{p}_i, r_i\right) : \boldsymbol{p}_i \in \mathbb{Q}^2, r_i \in \mathbb{Q}^+\right\}$ is a countable basis of $\mathbb{Q}^2$. This happens to be a countable basis for $\mathbb{R}^2$ as well.

3. Any topology τ is a basis for itself.

4. $\{x\}$ for any $x \in X$ are a basis for the discrete topology. So is the collection of all finite subsets of X.

Proposition 2.5.3. *Let (X, τ) be a topological space and $\beta \subset \tau$. β is a basis of τ if, and only if, for every $x \in X$ and every $U \in \tau$ containing x, there exists $V \in \beta$ such that $x \in V \subset U$.*

Proof. Assume β is a basis and U an open set. We may then write $U = \bigcup_{i \in I} V_i$ where I is an index set and $V_i \in \beta$. Given $x \in U$, there will exist at least one index i such that $x \in V_i$, hence $x \in V_i \subset U$.

Conversely, let U be an open set. For every $x \in U$ we may denote an open set containing x and contained in U (which we know exists) by V_x. The equality $U = \bigcup_{x \in X} V_x$ follows. $\square$

Rather than having τ as a starting point and assuming $\beta \subset \tau$ in our conditions, let us now start from any subset β of $\mathcal{P}(X)$ and decide when β will be a basis for *some* topology τ on X. If such τ exists, it will be unique.

Compare B_1 and B_2 below to conditions (b) and (d) on open balls in Proposition 1.2.3:

Proposition 2.5.4. *Let X be a set and $\beta \subset \mathcal{P}$ a collection of subsets of X. Then*

$$\tau = \left\{\bigcup_{i \in I} V_i : V_i \in \beta, \ \textit{and } I \textit{ is any index set}\right\}, \tag{2.1}$$

is a topology over X, and thus β a basis for τ, if and only if the following hold:

B_1*: $\varnothing \in \beta$ and $X = \bigcup_{V \in \beta} V$.*

B_2*: For every $V, W \in \beta$, $V \cap W$ is a union of elements of β.*

Thus τ is unique satisfying the above two. We say τ is ***generated by basis*** *β.*

Proof. Assume τ defined as in (2.1) is a topology. Condition T_1 in Definition 2.1.1 implies $\varnothing, X \in \tau$, hence $\varnothing$ must be one of the elements of β and X must be equal to $X = \bigcup_{V \in \beta} V$. Furthermore, given $V, W \in \beta$, both are open sets of τ, hence so is their intersection. Which means $V \cap W$ must be a union of elements of β by (2.1).

Conversely, T_1 in Definition 2.1.1 is verified in virtue of B_1. In order to prove T_2 assume $U_i \in \tau$ for every i in some index set I. We may write $U_i = \bigcup_{k \in K_i} V_k$ where K_i is an index set and $V_k \in \beta$. Hence $\bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup_{k \in K_i} V_k \in \tau$. Let us prove T_3: given $U_1, \ldots, U_n \in \tau$, we may express each of them $U_i = \bigcup_{k \in K_i} V_k$ for every $i = 1, \ldots, n$. Then

$$U_1 \cap \cdots \cap U_n = \left(\bigcup_{k \in K_1} V_k \right) \cap \cdots \cap \left(\bigcup_{k \in K_n} V_k \right) = \bigcup_{k_1 \in K_1, \ldots, k_n \in K_n} V_{k_1} \cap \cdots \cap V_{k_n},$$

and, applying B_2 $n-1$ times, $V_{k_1} \cap \cdots \cap V_{k_n}$ is a union of elements of β, hence so is the total union $U_1 \cap \cdots \cap U_n$. □

Yet another definition:

Proposition 2.5.5. *Let (X, τ) a topological space and $\beta \subset \mathcal{P}(X)$. Then β is a basis for a topology τ (which will be given by (2.1)) if and only the following hold:*

(i) for every $x \in X$ there is at least one $V \in \beta$ such that $x \in V$.

(ii) For any two $V, W \in \beta$ and any $x \in V \cap W$, there exists a $B \in \beta$ such that $x \in B \subset V \cap W$.

Proof. Assume β is a basis. Then for every $x \in X$ there exists an open set $U \in \tau$ such that $x \in U$ and Proposition 2.5.3 yields $V \in \beta$ such that $x \in V \subset U$. To prove (ii) let $x \in V \cap W$. Proposition 2.5.4 implies $V \cap W$ is a union of elements of β, hence one of these elements, say B, will contain x.

Assume (i), (ii) hold. Then for every $U \in \tau$, U is the union of a family of $\{B_i\}_{i \in I}$ in β, hence for every $x \in U$, there exists one of these subsets, say B_{i_0} such that $x \in B_{i_0} \subset U$.

□

Bases are useful to simplify comparison of topologies on occasion:

Proposition 2.5.6. *Let X be a set and τ_1, τ_2 be two topologies over X. Let β_1, β_2 be two bases of τ_1, τ_2 respectively. Then $\tau_1 \subset \tau_2$ if and only if for every $U \in \beta_1$ and every $x \in U$ there exists a basis element $V \in \beta_2$ such that $x \in V \subset U$.*

Proof. Assume $\tau_1 \subset \tau_2$. Let $x \in X$ and $U \in \beta_1$ containing x. The fact $U \in \beta_1 \subset \tau_1 \subset \tau_2$ and β_2 is a basis of τ_2 implies $U = \bigcup_{i \in I} V_i$, where $V_i \in \beta_2$ for every $i \in I$. One of these basis elements V_i will contain x, hence $x \in V_i \subset U$.

Conversely, let $U \in \tau_1$. For every $x \in U$ there exists an open set $V_x \in \beta_1$ such that $x \in V_x \subset U$. Due to hypothesis, there is an open set $W_x \in \beta_2$ such that $x \in W_x \subset V_x$. Thus $U = \bigcup_{x \in U} W_x \in \tau_2$. □

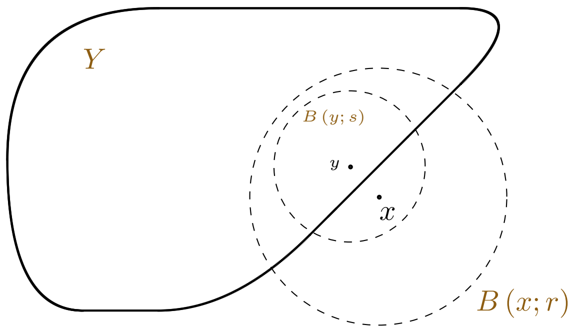
Corollary 2.5.7. *Let (X, d) be a metric space and $Y \subseteq X$ a subset. Then the topology τ_1 associated to the induced distance on Y is equal to the topology τ_2 obtained by inducing the topology associated to (X, d) to the subset Y.*

Proof. A basis for τ_1 may be obtained from balls for the induced distance $d_Y = d|_{Y \times Y}$,

$$\beta_1 = \{B_{d_Y}(y; r) = B_d(y; r) \cap Y : y \in Y, r \geq 0\}.$$

A basis for τ_2 is $\beta_2 := \{B_d(x; r) \cap Y : x \in X, r \geq 0\}$. We obviously have $\beta_1 \subset \beta_2$. Hence $\tau_1 \subset \tau_2$. In order to prove $\tau_2 \subset \tau_1$, let $y \in Y$ and $U = B_d(x; r) \cap Y \in \beta_2$ such that $y \in U$.

As seen in Proposition 1.2.3 (c), there exists $s > 0$ such that $B_d(y;s) \subset B_d(x,r)$. Hence $y \in B_d(y;s) \cap Y \subset U$.



□

Sometimes a weaker property than those for bases is necessary:

Definition 2.5.8. *Let (X,τ) be a topological space. A* ***sub-base*** *or* ***sub-basis*** *(hyphen optional) is a subset $\mu \subset \mathcal{P}(X)$ such that the set of all possible finite intersections of elements in it, along with the empty set,*

$$\beta = \{U_1 \cap \cdots \cap U_n : n \in \mathbb{N}, U_1, \ldots, U_n \in \mu\} \cup \{\varnothing\},$$

is a basis for τ.

Trivially every basis is a sub-basis but the opposite implication does not hold in general, as can be seen in the Solved Exercises. The following is left as an Exercise:

Lemma 2.5.9. *$\mu \subset \mathcal{P}(X)$ is a subbasis for a topology τ on X if, and only if, τ is the smallest topology on X containing μ.*

Proof. Exercise 2.9. □

Examples 2.5.10.

1. Any basis is also a subbasis.

2. $\{(a,\infty), (-\infty,b)\}_{a,b\in\mathbb{R}}$ is a sub-basis for $(\mathbb{R}, |\cdot|)$.

3. The set $\{\{x,y\} : x,y \in X\}$ is a sub-basis for the discrete topology.

4. $\{X \setminus \{x\} : x \in X\}$ is a subbase of the finite complementary topology.

Proposition 2.5.11. *Let X be a set and $\mu \subset \mathcal{P}(X)$ such that $X = \bigcup_{V\in\mu} V$. There exists a unique topology τ on X such that μ is its subbasis.*

Proof. Define $\beta = \{\text{finite intersections of elements in } \mu\} \cup \{\varnothing\} \subset \mu$. We need to check β satisfies B_1 and B_2 in Proposition 2.5.4. First of all, $\varnothing \in \beta$ by definition and $X = \bigcup_{V\in\mu} V = \bigcup_{V\in\beta} V$. Let $V = V_1 \cap \cdots \cap V_n$ and $W = W_1 \cap \cdots \cap W_m$ be two elements of β. Then $V \cap W = V_1 \cap \cdots \cap V_n \cap W_1 \cap \cdots \cap W_m$ is a finite intersection of elements of μ, hence an element of β. □

2.6 Neighbourhoods and neighbourhood systems

Definition 2.6.1. *Let (X, τ) be a topological space and $x \in X$. A* ***neighbourhood*** *of x is a subset $N \subset X$ such that $x \in \overset{\circ}{N}$, i.e. such that there exists $U \in \tau$ for which $x \in U \subset N$. N is called an* ***open neighbourhood*** *if the obvious holds, $N = \overset{\circ}{N}$.*

Examples 2.6.2.

1. In a metric space open balls are open neighbourhoods of each of the points contained in them.

2. In general, in a topological space every open set is an open neighbourhood of all the points contained in it.

3. Conversely, any set which is a neighbourhood of all its points is open.

4. Every neighbourhood of a point contains an open neighbourhood thereof.

5. In $X = \mathbb{R}$ with the Euclidean topology the subset $\left[-\frac{1}{n}, \frac{1}{n}\right]$, for any $n \in \mathbb{N}$, is a neighbourhood of 0 but it is not one for, say, $\frac{1}{n}$. In general $[p - \varepsilon, p + \varepsilon]$ for $\varepsilon > 0$ is a neighbourhood of p. So is $[p - \varepsilon, p + \varepsilon] \cup S$ for any other subset of $\mathbb{R}$.

5. In the discrete topology every $\{x\}$ is an open neighbourhood of $x \in X$.

Definition 2.6.3. *Let $x \in X$.*

a) The set of all neighbourhoods of x in X is called the ***neighbourhood system*** *of x, $\mathcal{N}_x$.*

b) A ***neighbourhood basis***, ***fundamental system of neighbourhoods*** *or* ***local basis*** *is a set $\{N_i\}_{i \in I}$ of neighbourhoods of x such that for any neighbourhood N of x, there exists at least one $N_i \subset N$.*

Example 2.6.4. In a metric space the open balls $\left\{B\left(x, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ is an open neighbourhood basis of x. This is a consequence of the Archimedean property of natural numbers, as the reader may check.

Proposition 2.6.5. *Let X be a set. For every $x \in X$, let $\mathcal{N}_x$ be a set of subsets of X such that*

(i) $\mathcal{N}_x \neq \varnothing$.

(ii) for every $U \in \mathcal{N}_x$ and every $M \subset X$ such that $U \subset M$, then $M \in \mathcal{N}_x$.

(iii) for every two $U_1, U_2 \in \mathcal{N}_x$, there exists $V \in \mathcal{N}_x$ such that $V \subset U_1 \cap U_2$.

(iv) For every $U \in \mathcal{N}_x$, there exists $V \in \mathcal{N}_x$ such that for every $y \in V$, $U \in \mathcal{N}_y$.

Then there exists a unique topology τ on X for which $\mathcal{N}_x$ is the neighbourhood system of x for every $x \in X$.

Corollary 2.6.6. *For every $x \in X$, let $\mathcal{N}_x$ be defined as above and $\mathcal{B}_x \subset \mathcal{N}_x$ such that for every $V \in \mathcal{N}_x$ there exists $W \in \mathcal{B}_x$ with $x \in W \subset V$. There exists a unique topology τ on X for which $\mathcal{B}_x$ is a local basis for every $x \in X$.*

Among the properties satisfied by $\mathcal{B}_x$ we obviously have three:

(i) $\mathcal{B}_x \neq \varnothing$ for every $x \in X$.

(ii) For every $V_1, V_2 \in \mathcal{B}_x$, there exists $V_3 \in \mathcal{B}_x$ such that $V_3 \subset V_1 \cap V_2$.

(iii) For every $V \in \mathcal{B}_x$ and every $y \in V$, there exists $W \in \mathcal{B}_y$ such that $W \subset V$.

Corollary 2.6.7. *Given a correspondence $x \mapsto \mathcal{B}_x \subset \mathcal{P}(X)$ satisfying (i)-(iii), there exists a unique topology on X such that $\mathcal{B}_x$ is a local basis of x for every $x \in X$.*

Note the similarities with the properties satisfied by topological bases.

2.7 Countability axioms

Neighbourhood bases allow us to reduce our study topological properties on more local or "smaller" collections of neighbourhoods of a point. It only takes a look at the neighbourhoods of zero in $\mathbb{R}$ to realise there exist, in general, no finite neighbourhood bases. However, we may ensure the existence of *countably infinite* neighbourhood bases for any point of a metric space, as seen in Example 2.6.4 above.

Definition 2.7.1. *A topological space satisfies the **first countability axiom** or is **first countable** if every point $x \in X$ possesses at least one countable neighbourhood basis.*

Example 2.7.2. Topological spaces whose topology comes from a metric satisfy this axiom as seen in Example 2.6.4.

From a non-local point of view, we may still call for the existence of countable bases on occasion:

Definition 2.7.3. *A topological space (X, τ) satisfies the **second countability axiom** or is **second countable** if it has at least one countable basis.*

Examples 2.7.4.

1. $\mathbb{R}^n$ with the usual Euclidean topology satisfies this axiom. Indeed,

$$\beta = \left\{ B\left(\boldsymbol{q}, \frac{1}{m}\right) : \boldsymbol{q} = (q_1, \ldots, q_n) \in \mathbb{Q}^n, m \in \mathbb{N} \right\},$$

   is a basis for $\mathbb{R}^n$ (again in virtue of the Archimedean property, same as Example 2.6.4) and it is countable on account of $\mathbb{Q}^n$ and $\mathbb{N}$ being so.

2. Every non-countable set X endowed with the discrete topology $\tau = \mathcal{P}(X)$ does not satisfy the second countability axiom. Indeed, every point is open in X and every open set is a union of elements of a basis, hence every basis must contain individual points $\{x\}$, $x \in X$, which renders it automatically non-countable.

   However, the discrete topology $\tau = \mathcal{P}(X)$ is the topology associated to the discrete metric (see Example 1.1.4), which implies (X, τ) does satisfy the first countability axiom.

Proposition 2.7.5. *If a topological space (X, τ) is second countable, then it is also first countable.*

Proof. $\beta = \{V_i : i \in \mathbb{N}\} \subset \tau$ be a countable basis of (X, τ). Let us prove that for every $x \in X$,

$$N_x := \{V_i : x \in V_i\} \subset \beta$$

is a (logically countable) neighbourhood basis of x. Let $U \in \tau$ such that $x \in U$. Then $U = \bigcup_{i \in I} V_i$ for some index set $I \subset \mathbb{N}$. There exists a $i \in I$ such that $x \in V_i$, hence $x \in V_i \subset U$ and $V_i \in N_x$. Hence N_x is a countable local basis of x and X satisfies the first countability axiom. $\square$

Definition 2.7.6. *A topological space X is **separable** if it contains a dense countable subset $Q \subset X$, i.e. $\overline{Q} = X$.*

Lemma 2.7.7. *Every second-countable space is separable.*

Proof. Indeed, given a countable basis $\{U_n\}_{n\in\mathbb{N}}$ and choosing an element $x_n \in U_n$ for every $U_n \neq \varnothing$, set $\{x_n\}_{n\in\mathbb{N}}$ is dense in X. $\square$

Examples 2.7.8.

1. $\left(\mathbb{R}, \tau_{\mathbb{R}}^{\text{Eucl}}\right)$ is separable and so is any finite product $\mathbb{R}^n$ thereof.

2. $(\mathbb{R}, \tau)$ where τ is the finite complement topology is separable, yet not second countable. Indeed, if it were there would be a countable basis β. Then for every $x \in \mathbb{R}$, we would have $\{x\} = \bigcap_{V\in\beta, x\in V} V$, and $\mathbb{R} \setminus \bigcap V = \mathbb{R} \setminus \{x\} = \bigcup (\mathbb{R} \setminus V)$; one of these sets is countable (being a union of countable sets), the other is not.

2.8 Solved Exercises

21. In $\mathbb{R}^2$ with the Euclidean topology, decide whether the following subsets are open or closed:

a) $S_1 := \left\{(x, y) : x^2 + y^2 < 1\right\}$.

b) $S_2 := \left\{\left(\frac{1}{n}, 0\right) : n \in \mathbb{N}\right\}$.

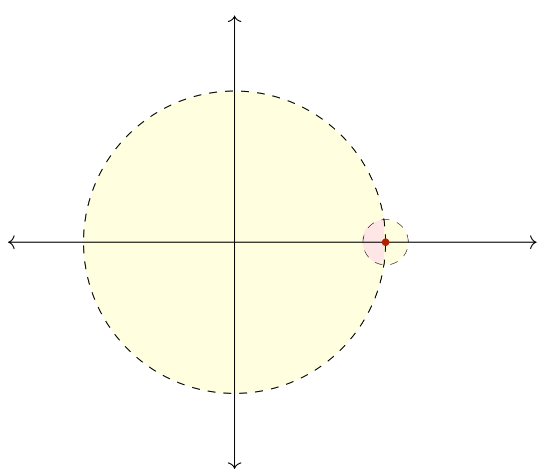
c) $S_3 := \{(x, y) : x \leq 6, y \geq 0\}$.

d) $S_4 := \mathbb{R}^2$.

Solution.

a) $S_1 = \left\{(x, y) : x^2 + y^2 < 1\right\}$ is trivially open because it is an open ball of $\mathbb{R}^2$: indeed, for every point $(x, y) \in S_1$ Proposition 1.2.3 (c) holds and there exists $s > 0$ such that $B\left((x, y), s\right) \subset S_1$.

It is not closed because $(1, 0) \in \overline{S_1}$, yet $(1, 0) \notin S_1$. To check $(1, 0) \in \overline{S_1}$ all we need to do is consider any ball centred at $(0, 1)$ of any radius $\varepsilon > 0$. Regardless of ε, $B\left((1, 0), \varepsilon\right) \cap S_1 \neq \varnothing$:

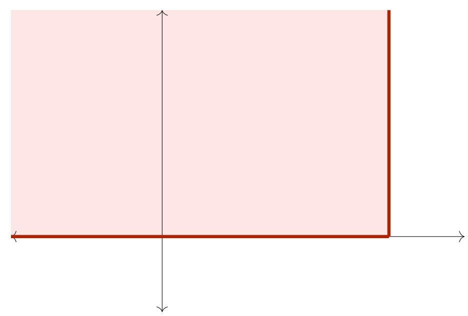


for instance, $\left(1 - \frac{\varepsilon}{2}, 0\right)$ belongs to said intersection.

b) $S_2 = \left\{ \left(\frac{1}{n}, 0\right) : n \in \mathbb{N} \right\}$ is not open. In fact, its interior is equal to $\varnothing$ since, for every $\left(\frac{1}{n}, 0\right)$ and every $\varepsilon > 0$, $B\left(\left(\frac{1}{n}, 0\right), \varepsilon\right)$ contains points not in S_2 (for instance, points with non-zero y coordinate).

S_2 is not closed, either, since $(0,0) \in \overline{S_2}$ yet $(0,0) \notin S_2$. In order to prove the first assertion let $\varepsilon > 0$; regardless of ε, $B\left((0,0), \varepsilon\right) \cap S_2 \neq \varnothing$ since there is some n_0 for which $\frac{1}{n_0} < \varepsilon$, hence $\left(\frac{1}{n_0}, 0\right) \in B\left((0,0), \varepsilon\right)$.

c) Let $S_3 = \{(x,y) : x \leq 6, y \geq 0\}$.



S_3 is not open: for every point $(6, y) \in \{(x,y) \in S_3 : x = 6\}$, and any $\varepsilon > 0$, $B\left((6,y), \varepsilon\right)$ contains points not in S_3, for instance $(6 + \varepsilon/2, y)$.

S_3 is open, however:

$$\mathbb{R}^2 \setminus S_3 = \left\{ (x,y) \in \mathbb{R}^2 : x > 6 \right\} \cup \left\{ (x,y) \in \mathbb{R}^2 : y < 0 \right\},$$

i.e. a union of two open sets of $\mathbb{R}^2$, hence an open set itself. You may check yourselves each of the two above subsets is open by guessing the adequate radius of a ball centred at any point therein in terms of the infimum distance from the point to the boundary. Exercise.

d) $\mathbb{R}^2$ is open and closed because it is the set our topology is defined on.

22. Let $X = \{a, b, c, d\}$ and define the two following collections of subsets of X:

$$\tau_1 := \{\varnothing, X, \{d\}, \{c,d\}, \{a,b,d\}\}, \qquad \tau_2 := \{\varnothing, X, \{c\}, \{d\}, \{a,b\}\}.$$

a) Prove (X, τ_1) is a topological space and find its closed sets.

b) Prove (X, τ_2) is not a topological space. Find the smallest subset $\tau_3 \subset \mathcal{P}(X)$ such that $\tau_2 \subset \tau_1$ and (X, τ_3) is a topological space.

Solution.

a) Let us prove it fulfils the properties of a topology:

T$_1$: $\varnothing, X \in \tau$ by definition.

T$_2$: let us study arbitrary unions (which happen to be finite in this case because X is):

$$\begin{aligned}
&\varnothing \cup U = U \in \tau_1, \quad \text{for every } U \in \tau_1, \\
&X \cup U = X \in \tau_1, \quad \text{for every } U \in \tau_1, \\
&\{d\} \cup \{c,d\} = \{c,d\} \in \tau_1, \\
&\{d\} \cup \{a,b,d\} = \{a,b,d\} \in \tau_1, \\
&\{c,d\} \cup \{a,b,d\} = \{a,b,c,d\} = X \in \tau_1.
\end{aligned}$$

T$_3$: Let us check finite intersections of elements of τ_1 belong to τ_1 as well:

$$\begin{aligned}
&\varnothing \cap U = \varnothing \in \tau_1, \quad \text{for every } U \in \tau_1,\\
&X \cap U = U \in \tau_1, \quad \text{for every } U \in \tau_1,\\
&\{d\} \cap \{c, d\} = \{d\} \in \tau_1,\\
&\{d\} \cap \{a, b, d\} = \{d\} \in \tau_1,\\
&\{c, d\} \cap \{a, b, d\} = \{d\} \in \tau_1.
\end{aligned}$$

Hence τ_1 is a topology on X.

The closed sets of (X, τ_1) are the complementaries of its open sets:

$$\{X \setminus U : U \in \tau_1\} = \{\varnothing, X, \{a, b, c\}, \{a, b\}, \{c\}\}.$$

b) (X, τ_2) fulfils property T$_1$ but it does not fulfil property T$_2$:

$$\{c\} \cup \{d\} = \{c, d\} \notin \tau_2.$$

Hence τ_2 is not a topology on X. In order to find the smallest completion of τ_2 into a topology on X, all we need to do is impose the belonging of unions and intersections of already-existing elements as well as of any new elements obtained therefrom:

$$\begin{aligned}
&\{c\} \cup \{d\} = \{c, d\},\\
&\{c\} \cup \{a, b\} = \{a, b, c\},\\
&\{d\} \cup \{a, b\} = \{a, b, d\},
\end{aligned}$$

hence $\{c, d\}, \{a, b, c\}, \{a, b, d\}$ must belong to τ_3. The union of any of these three elements is equal to X. Furthermore the only non-empty intersections are

$$\begin{aligned}
&\{c\} \cup \{c, d\} = \{c\} \cup \{a, b, c\} = \{c\},\\
&\{d\} \cup \{c, d\} = \{d\} \cup \{a, b, d\} = \{d\},\\
&\{a, b\} \cup \{a, b, c\} = \{a, b\} \cup \{a, b, d\} = \{a, b\},
\end{aligned}$$

which means we do not need to add any further elements. We have

$$\tau_3 = \{\varnothing, X, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}.$$

23. Let $X = \mathbb{N}$ and $\tau = \{\varnothing, X\} \cup \{U_n := \{n, n+1, \ldots\} : n > 1\}$. Prove τ is a topology on X.

Solution.

T$_1$: $\varnothing, X \in \tau$ by definition.

T$_2$: let us study arbitrary unions. Since τ is a countable set, said unions can only be either countable or finite. Let $\{U_i : i \in I\}$ be a set of elements of τ, I being a countable or finite index set. For every two indices $i_1, i_2 \in I$ such that $i_1 \leq i_2$, it is clear that $U_{i_1} \supset U_{i_2}$ since

$$U_{i_1} = \{i_1, i_1 + 1, \ldots\} = \{i_1, i_1 + 1, \ldots, i_2, i_2 + 1, \ldots\} \supset \{i_2, i_2 + 1, \ldots\}.$$

Hence $\bigcup_{i \in I} U_i = U_{\min I}$, and $\min I$ does exist since I is countable or finite and bounded below. Hence there exists an index k (namely $\min I$) such that $\bigcup_{i \in I} U_i = U_k$, implying the union belongs to τ.

T$_3$: Let us check finite intersections of elements of τ belong to τ as well: we already know $U_2 \supset U_3 \supset \dots$, hence for any given finite index set I we have $\bigcap_{i\in I} U_i = U_{\max I}$, and $\max I$ exists and is a natural number on account of the finiteness of I; thus $\bigcap_{i\in I} U_i = U_{\max I} \in \tau$.

24. On $\mathcal{P}\left(\mathbb{R}^2\right)$ we define the following sets:

a) $\tau_1 := \left\{\varnothing, \mathbb{R}^2\right\} \cup \left\{\text{all straight lines in } \mathbb{R}^2\right\}$.

b) $\tau_2 := \left\{\varnothing, \mathbb{R}^2\right\} \cup \left\{\text{interiors of balls in } \mathbb{R}^2\right\}$.

c) $\tau_3 := \left\{\bigcup_{j=1}^{n} C\left(a_j, b_j\right), \text{ where } n \in \mathbb{N} \text{ and } a_1, b_1, \dots, a_n, b_n \in \mathbb{R}\right\}$, where

$$C\left(a_j, b_j\right) := \left\{(x, y) \in \mathbb{R}^2 : a_j < y < b_j\right\}.$$

d) τ_4 defined as τ_3, albeit allowing for *arbitrary* unions of subsets $C(a, b)$, $a, b \in \mathbb{R}$.

e) $\tau_5 := \left\{\varnothing, \mathbb{R}^2\right\} \cup \{C_r : r \in \mathbb{R}_+\}$, where $C_r := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r\right\}$.

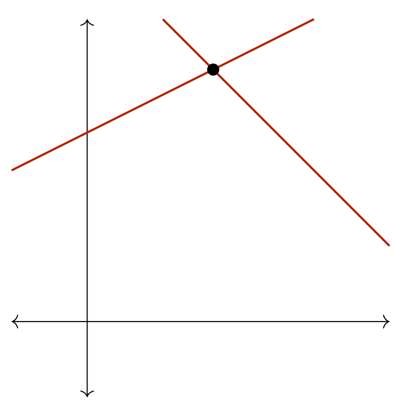
f) $\tau_6 := \left\{\varnothing, \mathbb{R}^2\right\} \cup \{C_r : r \in \mathbb{Q}_+\}$, C_r defined as in the previous item.

Answer the following:

- Which of them defines a topology? Whenever the answer is affirmative, find a neighbourhood basis for $(0, 0)$.
- Which of them are bases for a topology on $\mathbb{R}^2$? Which of them are subbases?

SOLUTION.

a) Let $\tau_1 := \left\{\varnothing, \mathbb{R}^2\right\} \cup \left\{\text{all straight lines in } \mathbb{R}^2\right\}$. τ_1 does not define a topology, since the intersection of any finite set of lines (e.g. two of them) is not a line in general:



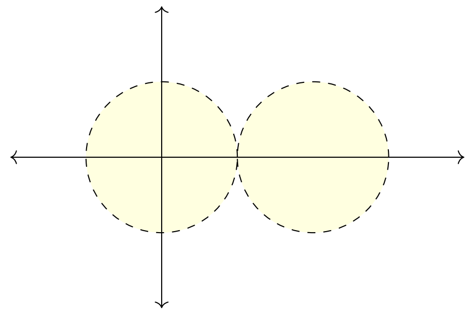
It is not a basis for a topology either, since the intersection of two lines (e.g. the lines shown in the picture) is in general not equal to a union of lines, hence it does not fulfil the properties in Proposition 2.5.4.

It is a subbase, however, since the set of all finite intersections of elements in τ_1 is precisely the set of all lines *and* points of $\mathbb{R}^2$, and this is a basis for the topology

$$\Lambda \cup \Pi := \left\{\text{arbitrary unions of lines of } \mathbb{R}^2\right\} \cup \left\{\text{arbitrary unions of points of } \mathbb{R}^2\right\},$$

i.e. the discrete topology $\mathcal{P}\left(\mathbb{R}^2\right)$ on $\mathbb{R}^2$ since lines themselves are already unions of points (hence $\Lambda \subset \Pi = \mathcal{P}\left(\mathbb{R}^2\right)$).

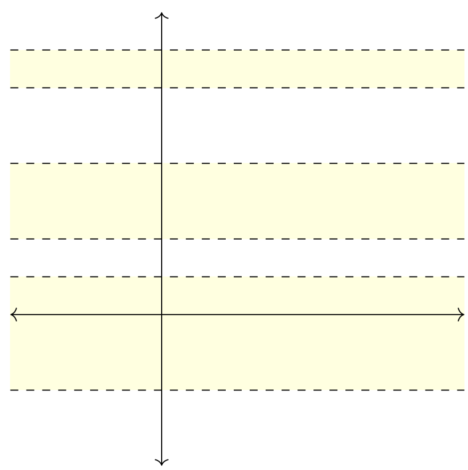
b) Let $\tau_2 := \{\varnothing, \mathbb{R}^2\} \cup \{\text{interiors of balls in } \mathbb{R}^2\}$. $\varnothing$ and $\mathbb{R}^2$ do belong to τ_2, but the union of balls is in general not a ball, e.g. $B((0,0),1) \cup B((2,0),1)$:



Hence it is not a topology. It is, however, a basis for a topology (namely the Euclidean topology on $\mathbb{R}^2$) since it fulfils both properties of Proposition 2.5.4, which in this case is also Proposition 1.2.3.

And since every basis is also a subbase, it is also a subbase for the same Euclidean topology.

c) Let $\tau_3 := \left\{\bigcup_{j=1}^{n} C(a_j, b_j),\ \text{where } n \in \mathbb{N} \text{ and } a_1, b_1, \dots, a_n, b_n \in \mathbb{R}\right\}$. $\varnothing$ does belong to τ_3, but $\mathbb{R}^2$ does not: we cannot cover $\mathbb{R}^2$ with a finite set of horizontal bands of finite width.



Hence τ_3 is not a topology.

It is a basis, though: it contains both $\mathbb{R}^2$ and the empty set, and the intersection of any two horizontal bands, whenever non-empty, can be expressed as the union of bands because it is a band itself.

Being a basis, it is also a subbase for the same topology.

d) Assume τ_4 is defined as τ_3, albeit allowing for *arbitrary* unions of bands. Arbitrary unions of arbitrary unions are obviously arbitrary unions themselves, hence T_2 is satisfied. Given any two elements of τ_4,

$$C_1 = \bigcup_{i\in I} C(a_i, b_i), \qquad C_2 = \bigcup_{j\in J} C(A_j, B_j),$$

we have

$$C_1 \cap C_2 = \left(\bigcup_{i\in I} C(a_i, b_i)\right) \cap \left(\bigcup_{j\in J} C(A_j, B_j)\right) = \bigcup_{i\in I, j\in J} \left(C(a_i, b_i) \cap C(A_j, B_j)\right);$$

each subset $C(a_i, b_i) \cap C(A_j, B_j)$ is either empty or a band itself, hence the union written above is either empty or a union of bands. This means T_3 is also fulfilled.

Hence τ_4 is a topology on $\mathbb{R}^2$. Every topology is also a base and a subbase for itself, hence τ_4 is both.

Let us find a neighbourhood base for $(0,0)$. Define $\beta := \{V_n : n \in \mathbb{N}\}$, where

$$V_n := \left\{(x, y) \in \mathbb{R}^2 : -\frac{1}{n} < y < \frac{1}{n}\right\}.$$

Each V_n is trivially an element of τ_4 and contains the origin, hence it is an open neighbourhood of $\{0,0\}$. Let W be any neighbourhood of $(0,0)$; this means $(0,0) \in \mathring{W}$, hence there exists an open set $U \in \tau_4$ such that $(0,0) \in U \subset W$. If $U = \mathbb{R}^2$, then $V_n \subset U$ for every $n \geq 1$ and we are finished. If $U \subsetneq \mathbb{R}^2$ this means $U = \bigcup_{i \in I} C(a_i, b_i)$ for a given index set I and elements $a_i, b_i \in \mathbb{R}$. The fact $(0,0) \in \bigcup_{i \in I} C(a_i, b_i)$ means $(0,0) \in C(a_i, b_i)$ for some $i \in I$, i.e. $a_i < 0 < b_i$. The Archimedean property of natural numbers implies the existence of $n \in \mathbb{N}$ such that $\frac{1}{n} < a_i + b_i$; it is left as an Exercise to conclude $V_n \subset C(a_i, b_i)$ for this value of n.

Hence for every neighbourhood W of $(0,0)$ there is an element $V_n \in \beta$ such that $(0,0) \in V_n \subset W$. β is therefore a local base for τ_4.

e) Let $\tau_5 := \{\varnothing, \mathbb{R}^2\} \cup \{C_r : r \in \mathbb{R}_+\}$, where C_r is the open ball of centre $(0,0)$ and radius r. T_1 is fulfilled by definition. Given any set of elements $\{C_{r_i} : i \in I\}$ of τ_5 (we discard $\varnothing$ and $\mathbb{R}^2$ which either add nothing to the union or render it equal to all of $\mathbb{R}^2$) we have

$$\bigcup_{i \in I} C_{r_i} = C_r, \qquad \text{where } r = \sup_{i \in I} r_i.$$

r is well-defined since $\{r_i : i \in I\}$ is bounded above – otherwise $\bigcup_{i \in I} C_{r_i} = \mathbb{R}^2$. Hence T_2 is satisfied. Finally, given any two elements $C_r, C_s \in \tau_5$ we have $C_r \cap C_s = C_{\min\{r,s\}} \in \tau_5$, hence the intersection of two elements in τ_5 belongs to τ_5 – hence the same holds for any finite intersection.

τ_5 is thus a topology. A neighbourhood base for the origin is, for instance, $\beta = \left\{C_{\frac{1}{n}} : n \in \mathbb{N}\right\}$ as was the case in Example 2.6.4.

f) $\tau_6 := \{\varnothing, \mathbb{R}^2\} \cup \{C_r : r \in \mathbb{Q}_+\}$, C_r defined as in the previous item, is not a topology. Exercise: choose a set $\{r_i :\in I\}$ whose supremum is not a rational number; $\bigcup_{i \in I} C_{r_i}$ will be equal to $C_{\sup\{r_i : i \in I\}} \notin \tau_6$. It is not a base, however – for the topology τ_5 defined above.

25. Let (X, d) be a metric space and define the following function:

$$\overline{d} : X \times X \to \mathbb{R}, \qquad (x, y) \mapsto \overline{d}(x, y) := \min\{d(x, y), 1\}.$$

Prove $\overline{d}$ is a distance and it defines the same topology on X as d.

Solution. Let us first prove it is a distance.

- For every $x, y \in X$, $\overline{d}(x, y) = \min\{d(x, y), 1\}$, the minimum of two real numbers both of which are ≥ 0. Hence so is the minimum.
- $\overline{d}(x, y) = 0$ if, and only if, $\min\{d(x, y), 1\} = 0$, which is equivalent to saying $d(x, y) = 0$ and this holds (on account of d being a distance) if and only if $x = y$.
- symmetric property on $\overline{d}$: $\overline{d}(x, y) = \min\{d(x, y), 1\} = \min\{d(y, x), 1\} = \overline{d}(y, x)$, the second equality being true because of the symmetric property on d.

- Let $x, y, z \in X$. If $\overline{d}(x,z) \geq 1$ or $\overline{d}(z,y) \geq 1$, we have
$$\overline{d}(x,y) = \min\{d(x,y), 1\} \leq 1 \leq \overline{d}(x,z) + \overline{d}(z,y).$$
If both $\overline{d}(x,z) < 1$ and $\overline{d}(y,z) < 1$, then using the triangle inequality on d
$$\overline{d}(x,y) = \min\{d(x,y), 1\} \leq d(x,y) \leq d(x,z) + d(z,y) = \overline{d}(x,z) + \overline{d}(z,y).$$

Let τ_d be the topology defined by d and $\tau_{\overline{d}}$ the topology defined by $\overline{d}$. Let us prove $\tau_d = \tau_{\overline{d}}$

$\supseteq$: Let us prove $\tau_{\overline{d}} \subset \tau_d$, i.e. every subset of X which is open according to $\overline{d}$ is also open for d. Assuming $S \subset X$ open for $\overline{d}$ means for every point $x \in S$ there exists $\varepsilon > 0$ such that $B_{\overline{d}}(x,\varepsilon) := \left\{y \in X : \overline{d}(x,y) < \varepsilon\right\} \subset S$. But in this case $\overline{d}(x,y) \leq d(x,y)$ for every $x, y \in X$, hence
$$\{y \in X : d(x,y) < \varepsilon\} \subset \left\{y \in X : \overline{d}(x,y) < \varepsilon\right\}.$$
This means every ball for $\overline{d}$ contains a ball for d having the same centre, $B_d(x,\varepsilon) \subset B_{\overline{d}}(x,\varepsilon)$, hence the subset S open for $\overline{d}$ is also open with respect to d. For this inclusion we did not even have to change radius ε.

$\subseteq$: Let us prove $\tau_d \subset \tau_{\overline{d}}$. Let $S \in \tau_d$. This means for every $x \in S$ there exists a ball $B_d(x,\varepsilon)$ totally contained in S. If we define $\delta := \min\{\varepsilon, 1\}$, we have
$$B_{\overline{d}}(x,\delta) = \left\{\overline{d}(x,y) < \delta\right\} = \begin{cases} \left\{y \in X : \overline{d}(x,y) < 1\right\} = \{y \in X : d(x,y) < 1\} & \text{if} \quad \varepsilon \geq 1, \\ \left\{y \in X : \overline{d}(x,y) < \varepsilon\right\} = \{y \in X : d(x,y) < \varepsilon\} & \text{if} \quad \varepsilon < 1, \end{cases}$$
hence if $\varepsilon \geq 1$ then $B_{\overline{d}}(x,\delta) \subset B_d(x,1) \subset B_d(x,\varepsilon)$, and if $\varepsilon < 1$ then $B_{\overline{d}}(x,\delta) = B_d(x,\varepsilon)$. In either case, $B_{\overline{d}}(x,\delta) \subset B_d(x,\varepsilon)$ for an adequate radius δ, meaning every ball for d contains an adequately resized ball for $\overline{d}$ having the same centre. Hence, set S open for d will also be open for $\overline{d}$. Exercise: draw a picture and it should become clearer to you.

26. Consider $\mathbb{R}$ with the Euclidean topology. Prove the set
$$\mu := \{(-\infty, a), (b, \infty) : a, b \in \mathbb{R}\} \cup \{\varnothing\}$$
is a subbase, but not a base for this topology.

Solution. For it to be a subbase, the set of all finite intersections of elements of μ,
$$\beta := \{U_1 \cap \cdots \cap U_n : U_1, \ldots, U_n \in \mu, n \in \mathbb{N}\},$$
must be a basis of the Euclidean topology, i.e. fulfil Proposition 2.5.3 or 2.5.4 (whichever comes in handier).

For every two $U, V \in \mu$, assume $U = (-\infty, a)$ and $V = (b, \infty)$ (all other combinations yield elements in μ). Then
$$U \cap V = (-\infty, a) \cap (b, \infty) = \begin{cases} \varnothing & \text{if} \quad a \leq b, \\ (b, a) & \text{if} \quad a > b, \end{cases}$$
hence all possible finite intersections of elements of μ make up the set
$$\beta = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\} \cup \{(a, b) : a, b \in \mathbb{R}, a < b\} \cup \{\varnothing\}. \tag{2.2}$$

Let us prove β is, indeed, a basis for the Euclidean topology τ on $\mathbb{R}$. Let us use for instance Proposition 2.5.3. A subset $U \subset \mathbb{R}$ is an open set of this topology if, and only if, for every $x \in U$

there exists $\delta > 0$ such that $B(x, \delta) \subset U$. This open ball is equal to an open interval centered at x, $B(x, \delta) = (x - \delta, x + \delta)$, which definitely belongs to β as seen in (2.2).

We have just proven that for every open set $U \in \tau$ and every element $x \in U$, there exists $V \in \beta$ such that $x \in V \subset U$. This, along with Proposition 2.5.3, implies β is a basis, hence μ is a subbase.

Let us prove μ is not a basis, however. This is immediate since the intersection of $U = (-\infty, 1)$ and $V = (-2, \infty)$, both of which belong to μ, is $(-2, 1)$ which cannot be expressed as a union of elements of μ because it is an interval of finite length; hence it does not fulfil the second property in Proposition 2.5.4.

27. Consider the following subsets of $\mathcal{P}(\mathbb{R})$:

- $\beta_1 := \{(a, b) : a < b\} \cup \{\varnothing\}$;
- $\beta_2 := \{[a, b) : a < b\} \cup \{\varnothing\}$;
- $\beta_3 := \{(a, b] : a < b\} \cup \{\varnothing\}$;
- $\beta_4 := \{(a, \infty) : a \in \mathbb{R}\} \cup \{\varnothing\}$;
- $\beta_5 := \{(-\infty, b) : b \in \mathbb{R}\} \cup \{\varnothing\}$.

a) Prove every one of them is the basis of some topology on $\mathbb{R}$.

b) Compare the topologies among them whenever possible.

c) Prove $\beta_4 \cup \beta_5$ is a subbase generating the same topology as β_1.

Solution.

a) Proposition 2.5.4 is easier to use than Proposition 2.5.3 in this Exercise.

- β_1 contains $\varnothing$, and we can express $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$, hence β_1 fulfils property B_1 in Proposition 2.5.4. For any $a, b, c, d \in \mathbb{R}$, $(a, b) \cap (c, d)$ is equal to either $\varnothing$ or an open interval, depending on the relative position of a, b, c, d (Exercise: write this in detail) which means the intersection of any two elements in β_1 is an element of β_1 (hence a union of elements, namely of one, in β_1). Property B_2 in Proposition 2.5.4 is thus fulfilled.
- β_2 contains $\varnothing$ and $\mathbb{R} = \bigcup_{a \in \mathbb{R}} [a, a + 1)$. On the other hand, the intersection of $[a, b)$ and $[c, d)$ is equal to $\varnothing$ if $a \le b \le c \le d$, and $[\max\{a, c\}, \min\{b, d\})$ otherwise. Either way, it is definitely a union of elements of β_2.
- β_3 contains $\varnothing$ and $\mathbb{R} = \bigcup_{a \in \mathbb{R}} (a, a + 1]$. Exercise: verify that the intersection of two intervals $(a, b]$ and $(c, d]$ is an element of β_3, hence a union of elements thereof, and B_2 is thus satisfied.
- β_4 contains $\varnothing$ and $\mathbb{R} = \bigcup_{x \in \mathbb{R}} (x, \infty)$, hence B_1 is satisfied. The intersection of any two elements in β_4 is either $\varnothing$ if one of them is $\varnothing$, or an interval of the form (c, ∞) otherwise. This implies property B_2.
- β_5 works along the same lines as β_4 and is left as an Exercise.

b) Let us study possible comparisons. For every $i = 1, \ldots, 6$, we denote by τ_i the topology generated by basis β_i. We will use Proposition 2.5.6 which allows us to restrict comparison to basis elements.

- τ_1 is strictly coarser than τ_2 (or τ_2 is strictly finer than τ_1), i.e. $\tau_1 \subsetneq \tau_2$. Indeed, we can express any element $(a,b) \in \beta_1$ as a union of elements of β_2:

$$(a,b) = \bigcup_{n\in\mathbb{N}} \left[a + \frac{1}{n}, b\right),$$

  hence for every $x \in (a,b)$ there is a $\left[a+\frac{1}{n}, b\right) \in \beta_2$ such that $x \in \left[a+\frac{1}{n}, b\right) \subset (a,b)$. The inclusion of τ_1 in τ_2 is strict since given $[a,b)$, $a,b \in \mathbb{R}$, $a < b$, there is no open set $(c,d) \in \beta_1$ such that $a \in (c,d) \subset [a,b)$.

- Using the same procedure, we can prove $\tau_1 \subsetneq \tau_3$. Indeed,

$$(a,b) = \bigcup_{n\in\mathbb{N}} \left(a, b - \frac{1}{n}\right],$$

  and the reasoning follows as above.

- $\tau_4 \subsetneq \tau_1$ since every infinite interval (a,∞) can be expressed, for instance, in the form $\bigcup_{n\in\mathbb{N}} (a,n)$, hence for every point $x \in (a,\infty)$ there is at least one interval (a,n) containing it and contained in (a,∞).

  The inclusion $\tau_4 \subset \tau_1$ is strict, i.e. the other inclusion does not hold, because (a,b), a finite-length interval, cannot contain non-empty elements of β_4.

- $\tau_5 \subsetneq \tau_1$ for essentially the same reasons as the previous comparison and is left as an Exercise.

- τ_2 and τ_3 are not comparable. Indeed, $a \in [a,b)$ and yet there is no open set $(c,d] \in \tau_3$ such that $a \in (c,d] \subset [a,b)$. Hence using Proposition 2.5.6 τ_3 is not finer than τ_2. A similar argument, left as an Exercise, proves τ_2 is not finer than τ_3 either.

- Proving τ_4 and τ_5 cannot be compared is even easier since there is no way of fitting basis open sets of one topology into those of the other, hence 2.5.6 cannot be applied on any point of either open set.

c) We have $\beta_4 \cup \beta_5 = \{(-\infty, b) : b \in \mathbb{R}\} \cup \{(a,\infty) : b \in \mathbb{R}\} \cup \{\varnothing\}$ which is exactly the subbase in Exercise 26 generating the Euclidean topology. The topology τ_1 having β_1 as a basis is precisely the Euclidean topology.

28. Let $C_{m,n} := \{m + kn : m \in \mathbb{Z}\}$ for every $n, m \in \mathbb{Z}$ and $\mathcal{B} = \{C_{m,n} : n > 0, m \in \mathbb{Z}\}$. Let $\mathfrak{P}$ be the set of all prime numbers in $\mathbb{Z}$ and, for every $p \in \mathfrak{P}$, $M_p := C_{0,p}$.

a) Prove that for every $z \in C_{n,m} \cap C_{N,M}$ there exists $t \in \mathbb{N}$ such that $C_{z,t} \subseteq C_{n,m} \cap C_{N,M}$.

b) Prove $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

c) Prove that for every $n > 0$ there exists $p \in \mathfrak{P}$ such that $p \nmid n$.

d) Prove $\bigcap_{p\in\mathfrak{P}} M_p = \{0\}$ without assuming $\mathfrak{P}$ is infinite.

e) Prove the set of prime numbers is infinite.

Solution.

a) Let $z \in C_{n,m} \cap C_{N,M}$. This means there exist $k, K \in \mathbb{Z}$ such that $z = m + kn$ and $z = M + KN$. On the other hand, calling for $x \in C_{z,t}$ implies the existence of $r \in \mathbb{N}$ such that $x = z + tr$, which coupled with the above two equations implies $x = m + kn + tr = M + KN + tr$; hence, choosing $t := nN$, we have

$$x = m + n\,(k + Nr) = M + N\,(k + nr) \in C_{n,m} \cap C_{N,M}.$$

b) Let us prove $\mathcal{B}$ satisfies the properties of Proposition 2.5.4. $\mathbb{Z}$ can be obtained as a union of elements of $\mathcal{B}$, e.g. a single one of them: $C_{0,1} = \{0 + 1k : k \in \mathbb{Z}\} = \mathbb{Z}$. In the previous item we just proved that for every two $C, C' \in \mathcal{B}$ and for every $z \in C \cap C'$ there exists $C_z \in \mathcal{B}$ such that $z \in C_z \subset C \cap C'$. This is enough to prove B_2; indeed,

$$C \cap C' = \bigcup_{z \in C \cap C'} \{z\} \subseteq \bigcup_{z \in C \cap C'} C_z \subseteq C \cap C',$$

hence $\bigcup_{z \in C \cap C'} C_z = C \cap C'$.

c) For every $n > 2$ there exists a prime number p such that $p \mid n - 1$. This prime number p does not divide n; indeed, otherwise we would have $n - 1 = kp$ and $n = mp$ for some $k, m \in \mathbb{Z}$, hence $(m - k)\, p = 1$, which is absurd since $|p| > 1$.

d) We have $\{0\} \subseteq \bigcap_{p \in \mathfrak{P}} M_p$ because $0 \in M_p$ for every $p \in \mathfrak{P}$. Conversely, $1 \notin M_2$ implies $1 \notin \bigcap_{p \in \mathfrak{P}} M_p$, $2 \notin M_3$ implies $2 \notin \bigcap_{p \in \mathfrak{P}} M_p$ and in general, see previous item, for every $n \in \mathbb{N}$ there exists $p \in \mathfrak{P}$ such that $n \notin M_p$ which implies $p \notin \bigcap_{p \in \mathfrak{P}} M_p$.

e) Every basis element $C_{m,n} \in \mathcal{B}$ is infinite, hence $\{0\}$ cannot be an open set in the topology generated by $\mathcal{B}$. We thus have $\bigcap_{p \in \mathfrak{P}} M_p = \{0\}$, an intersection of open sets equal to a non-open set. The fact finite intersections of open sets are open implies $\bigcap_{p \in \mathfrak{P}} M_p$ must be the intersection of an *infinite* amount of open sets, indexed by the elements of $\mathfrak{P}$. Hence $\mathfrak{P}$ is infinite.

29. Let X be a set and $f : \mathcal{P}(X) \to \mathcal{P}(X)$ such that

a) $f(\varnothing) = \varnothing$;

b) $A \subseteq f(A)$ for every $A \subseteq X$;

c) $f(A \cup B) = f(A) \cup f(B)$ for every $A, B \subseteq X$;

d) $f \circ f = f$.

Prove there exists a unique topology on X satisfying $\overline{A} = f(A)$ for every $A \in \mathcal{P}(X)$. We call f the *closure operator* and the four properties the **Kuratowski closure axioms**.

Decide which topologies are obtained in the following cases:

- $f_1 = \mathrm{Id}$;
- $f_2(Y) = X$ if $Y \neq \varnothing$.

Solution. In a topological space, every closed set is of the form $\overline{A}$ for some subset $A \subseteq X$. With this in mind, let $\mathcal{C}_f = \{f(A) : A \subseteq X\}$. Let us prove it is the set of closed sets of a topology.

F_1: $\varnothing = f(\varnothing) \in \mathcal{C}_f$ and $X \subseteq f(X) \subseteq X$, hence $X = f(X)$.

F_2: For every collection $\{C_i\}_{i \in I}$ of elements in $\mathcal{C}_f$, $\bigcap_{i \in I} C_i \subseteq C_i$, hence $f\left(\bigcap_{i \in I} C_i\right) \subseteq f(C_i)$ for every i, which means $f\left(\bigcap_{i \in I} C_i\right) \subseteq \bigcap_{i \in I} f(C_i) = \bigcap_{i \in I} C_i$. The other inclusion $\bigcap_{i \in I} C_i \subseteq f\left(\bigcap_{i \in I} C_i\right)$ is a consequence of the second property of f. Hence $\bigcap_{i \in I} C_i = f\left(\bigcap_{i \in I} C_i\right)$ which implies $\bigcap_{i \in I} C_i \in \mathcal{C}_f$.

F_3: for every $C_1, C_2 \in \mathcal{C}_f$, $f(C_1 \cup C_2) = f(C_1) \cup f(C_2) = C_1 \cup C_2$ in virtue of the third and fourth properties of f, hence $C_1 \cup C_2 = f(C_1 \cup C_2) \in \mathcal{C}_f$.

We still need to check $f(A) = \overline{A}$ for every $A \subset X$ in this topology. $f(A)$ is closed by definition. $A \subset f(A)$ by the second property of f. And for any closed set C such that $A \subseteq C \subseteq f(A)$, this implies $f(A) \subseteq f(C) \subseteq f(f(A)) = f(A)$, hence $C = f(C) = f(A)$. Hence $f(A)$ is the smallest closed set containing A, hence $\overline{A}$.

For $f_1 = \mathrm{Id}_X$, we have

$$\mathcal{C}_f = \{f(A) : A \subseteq X\} = \{A : A \subseteq X\} = \mathcal{P}(X),$$

the discrete topology on X.

For f_2 defined as X on every non-empty subset Y, and $f_2(\varnothing) = \varnothing$,

$$\mathcal{C}_f = \{f(A) : A \subseteq X\} = \{X, \varnothing\},$$

the coarse topology on X.

30. Given an uncountable set X with the discrete topology, is it second countable?

Solution. In the discrete topology, simpletons $\{x\}$, $x \in X$, are open sets. Given a basis β of X, for every such open set $\{x\}$ there must be a basis element $V \in \beta$ such that $x \in V \subseteq \{x\}$, hence $V = \{x\}$. Thus we have an uncountable set among the basis elements: $\{\{x\} : x \in X\} \subseteq \beta$, which implies β is uncountable.

There is no countable basis for X, hence X does not satisfy the second axiom of countability.

31. Let X be a topological space. Decide which of the following properties are true for arbitrary subsets A, B, A_i of X.

a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b) $\overline{\bigcup_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$

c) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

d) $\overline{\bigcap_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$

e) $\overline{X \setminus A} = \overline{X} \setminus \overline{A}$.

Solution.

a) True.

$\subseteq$: $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, hence $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$. Finite unions of closed sets are closed, hence $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$

$\supseteq$: For every $x \in \overline{A} \cup \overline{B}$, the fact $x \in \overline{A}$ (resp. $x \in \overline{B}$) implies for every open set U containing x (resp. for every open set V containing x) $U_x \cap A \neq \varnothing$ (resp. $U_x \cap B \neq \varnothing$. Hence for every open set W containing x, $W \cap (A \cup B) \neq \varnothing$, which is the same as saying $x \in \overline{A \cup B}$.

b) False.

$\subseteq$: This is the false inclusion in general. For instance, $A_n = \left[\frac{1}{n}, 1\right]$ are closed subsets of $\mathbb{R}$ with the subspace Euclidean topology, yet their union is $(0, 1]$ whereas the closure of the union is $[0, 1]$.

$\supseteq$: This is true. $A_i \subset \bigcup_{i\in I} A_i$ for every $i \in I$, hence $\overline{A_i} \subset \overline{\bigcup_{i\in I} A_i}$ and thus $\bigcup_{i\in I} \overline{A_i} \subset \overline{\bigcup_{i\in I} A_i}$

c) False. Consider $\mathbb{R}$ with the Euclidean topology. Let $A = (0,3)$ and $B = (3,5)$. $\overline{A} = [0,3]$ and $\overline{B} = [3,5]$, and $A \cap B = (0,3) \cap (3,5) = \varnothing$, implying $\overline{A \cap B} = \varnothing$, whereas $\overline{A} \cap \overline{B} = \{3\}$.

One of the inclusions, though, does hold: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ (EXERCISE).

d) False because it does not even hold for a finite amount of subsets, as seen above. Inclusion $\overline{\bigcap_{i \in I} A_i} \subset \bigcap_{i \in I} \overline{A_i}$ does hold, though (EXERCISE).

e) This is false. Let us find a counterexample. Let $A = [0,1]$ and $B = (0,1]$. Then $\overline{A \setminus B} = \{0\}$ whereas $\overline{A} \setminus \overline{B} = [0,1] \setminus [0,1] = \varnothing$.

Let us prove one of the inclusions holds: $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$. Let $x \in \overline{A} \setminus \overline{B}$. This means for every open set U containing x, $U \cap A \neq \varnothing$, whereas there exists an open set V such that $x \in V$ and $V \cap B = \varnothing$. Let W be an open set such that $x \in W$ and $U' = W \cap V$, which is an open set containing x as well. The fact $x \in \overline{A}$ implies $U' \cap A \neq \varnothing$. This means there exists $y \in U' \cap A$.

If $y \in B$ then $y \in U' \subset V$ would mean $y \in B \cap V \neq \varnothing$, absurd. Hence $y \notin B$, implying $y \in A \setminus B$. Plus $y \in U'$ which implies $y \in W$. Thus $W \cap (A \setminus B) \neq \varnothing$.

Thus every open set W containing an element $x \in \overline{A} \setminus \overline{B}$ must intersect $A \setminus B$.

32. Given the following sets X and subsets $S \subset \mathcal{P}(X)$, decide whether S can be a base or a subbase for some topology on X.

a) $X = \{1,2,3,4,5\}$ and $S = \{\{1,2,3\}, \{1,4,5\}, \{3,4,5\}\} \cup \{\varnothing\}$.

b) $X = \mathbb{R}^n$ and $S = \left\{C_{\boldsymbol{p},m} : \boldsymbol{p} = (p_1, \dots, p_n) \in \mathbb{R}^n, m \in \mathbb{N}\right\} \cup \{\varnothing\}$ where

$$C_{\boldsymbol{p},m} = \left\{\boldsymbol{x} \in \mathbb{R}^n : x_i = p_i \text{ for } i = 2, \dots, n \text{ and } |x_1 - p_1| < \frac{1}{m}\right\}.$$

c) $X = \mathbb{R}^n$ and $S = \{S(\boldsymbol{q}) : \boldsymbol{q} \in \mathbb{Q}^n\} \cup \{\varnothing\}$ where $S(\boldsymbol{q}) = \{\boldsymbol{x} \in \mathbb{R}^n : x_i < q_i \text{ for } i \leq n\}$.

In those where the answer is affirmative, decide whether it satisfies any countability axiom. Draw sample open subsets for $n = 2$ in the latter two examples.

SOLUTION.

a) Let us use Proposition 2.5.4. The first property is fulfilled: $\varnothing \in S$ and $X = \{1,2,3\} \cup \{1,4,5\}$, hence X can be expressed as a union of elements of S. However the second property is not fulfilled. $\{1,2,3\} \cap \{1,4,5\}$ which cannot be expressed as a union of elements of S. Hence S is not a basis.

In order to check whether it is a subbase, we need to study the set of finite intersections of elements of S (in this case they have to be finite since S is) and check whether the resulting set is a basis. We have

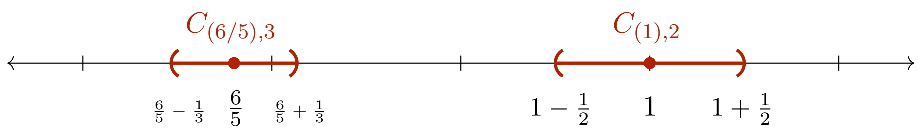
$$\{1,2,3\} \cap \{1,4,5\} = \{1\}, \quad \{1,2,3\} \cap \{3,4,5\} = \{3\}, \quad \{1,4,5\} \cap \{3,4,5\} = \{4,5\},$$

hence the set of intersections is

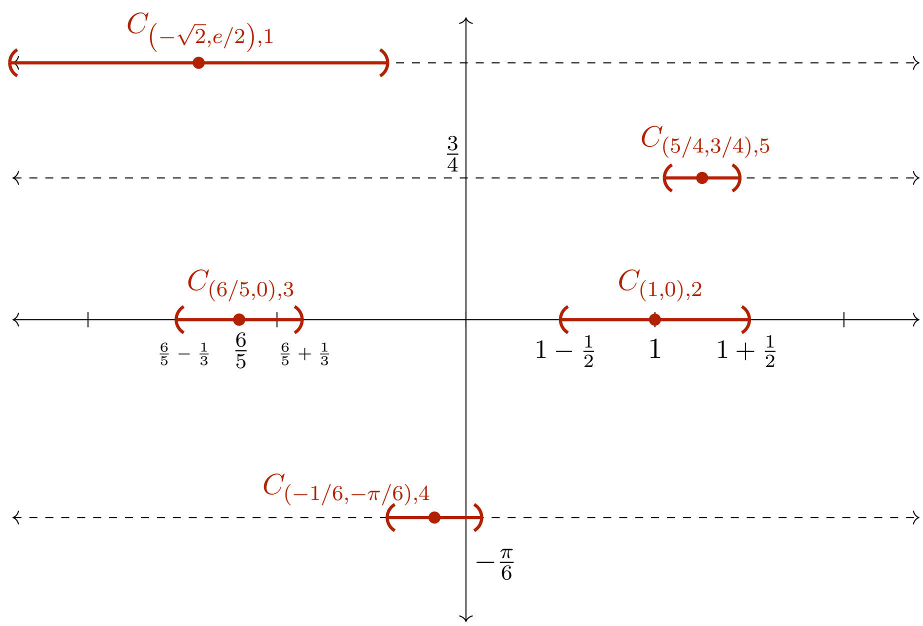
$$\beta = \{\{1\}, \{3\}, \{4,5\}, \{1,2,3\}, \{1,4,5\}, \{3,4,5\}\} \cup \{\varnothing\}.$$

It is closed under intersection, and X can be expressed as a union of elements of it (this was already true for S). Hence β is a basis, meaning S is a subbase.

b) In $\mathbb{R}$ sample subsets would be for instance



In $\mathbb{R}^2$ the subsets are simply the same as those in $\mathbb{R}$ at different y-levels:



Let us check whether S is a basis for $\mathbb{R}^n$. $\varnothing \in S$ and $\mathbb{R}^n$ can be expressed as a union of elements in S, e.g. $\mathbb{R}^n = \bigcup_{\boldsymbol{p}\in\mathbb{R}^n} C_{\boldsymbol{p},1}$. The intersection $C_{\boldsymbol{p},m} \cap C_{\boldsymbol{q},n}$ of any two elements of S is a union of elements in S as well:

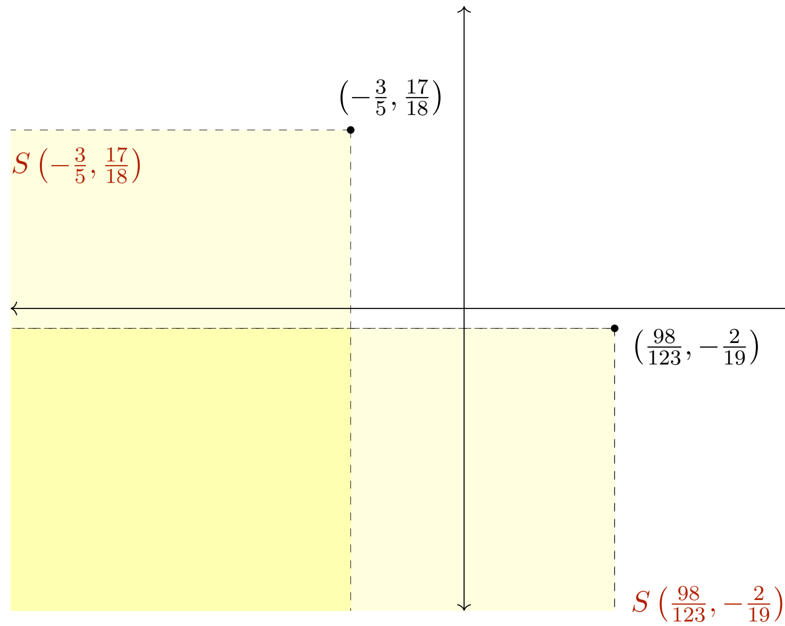
$$\begin{cases} \varnothing & \text{if} \quad p_i \neq q_i \text{ for some } i = 2,\dots,n, \\ \varnothing & \text{if} \quad p_i = q_i \text{ for } i \geq 2 \text{ and } \left(p_1 - \frac{1}{m}, p_1 + \frac{1}{m}\right) \cap \left(q_1 - \frac{1}{n}, q_1 + \frac{1}{n}\right) = \varnothing, \\ \left(p_1 - \frac{1}{m}, p_1 + \frac{1}{m}\right) \cap \left(q_1 - \frac{1}{n}, q_1 + \frac{1}{n}\right) & \text{if} \quad p_i = q_i \text{ for } i \geq 2 \text{ and } \left(p_1 - \frac{1}{m}, p_1 + \frac{1}{m}\right) \cap \left(q_1 - \frac{1}{n}, q_1 + \frac{1}{n}\right) \neq \varnothing. \end{cases}$$

In the latter case, $\left(p_1 - \frac{1}{m}, p_1 + \frac{1}{m}\right) \cap \left(q_1 - \frac{1}{n}, q_1 + \frac{1}{n}\right)$ can obviously be expressed as a union $\bigcup C_{\boldsymbol{r},k}$ (Exercise). Hence S is a basis, and thus a subbase for a topology τ_S on $\mathbb{R}^n$.

For every $\boldsymbol{p} \in \mathbb{R}^n$, $\mathcal{N}(\boldsymbol{p}) := \left\{C_{\boldsymbol{p},m} : m \in \mathbb{N}\right\}$ is a neighbourhood basis for $\boldsymbol{p}$. Indeed, for every open set $U \in \tau_S$ there exists $C_{\boldsymbol{p},m}$ for m small enough such that $\boldsymbol{p} \in C_{\boldsymbol{p},m} \subset U$. $\mathcal{N}(\boldsymbol{p})$ is countable, hence $(\mathbb{R}^n, \tau_S)$ is first countable.

$(\mathbb{R}^n, \tau_S)$ does not satisfy the second axiom of countability, however. The existence of a countable basis β would imply, for every point $\boldsymbol{p} \in \mathbb{R}^n$ and every open set $U(\boldsymbol{p})$ containing $\boldsymbol{p}$, the existence of a basis element $V(\boldsymbol{p})$ containing $\boldsymbol{p}$ and contained in $U(\boldsymbol{p})$. Let $U(\boldsymbol{p}) = \mathbb{R}^{n-1} \times \{p_1\}$ (e.g. the dashed lines in the latter figure); there will be as many different open sets $U(\boldsymbol{p})$ as elements $p_1 \in \mathbb{R}$, i.e. uncountably many. Hence same applies to open sets $V(\boldsymbol{p})$ in β.

c) For $n = 1$ all we have is intervals open-ended to the left. For $n = 2$,



S definitely contains $\varnothing$ and X can be expressed as a union of elements of S: $\mathbb{R}^n = \bigcup_{\boldsymbol{q}\in\mathbb{Q}^n} S(\boldsymbol{q})$. The intersection of any two such subsets is another subset belonging to the same collection S:

$$S_{q_1,\dots,q_n} \cap S_{p_1,\dots,p_n} = S_{Q_1,\dots,Q_n}, \qquad \text{where } Q_i = \min\{q_i, p_i\}, \quad i = 1, \dots, n.$$

Therefore S satisfies properties B_1 and B_2 in Proposition 2.5.4 and is thus a basis (and a subbasis) for some topology τ_S on $\mathbb{R}^n$.

Let us prove (X, τ_S) satisfies both countability axioms by proving it satisfies the second. S is a basis, and it is countable since it is indexed by countable set $\mathbb{Q}^n$. Thus X is second countable, which implies it is also first countable.

33. (*May 2013* and *Coursework January 2014*) Let (M, d) be a metric space. Define the following function:

$$\overline{d}(x, y) := \frac{d(x, y)}{d(x, y) + 1}, \qquad \text{for every } x, y \in M.$$

(i) (10 MARKS) Prove $\overline{d}$ is a distance.

(ii) (10 MARKS) Does it define the same topology as d?

SOLUTION.

(i) Let us check the properties of a metric on this new function:

- For any $x, y \in M$, $\overline{d}(x, y)$ is a fraction whose numerator is $d(x, y) \geq 0$ (because d is a distance) and whose denominator is $1 + d(x, y) > 0$. Hence it is always a real number (division by zero is avoided) and it is always ≥ 0.
- $\overline{d}(x, y) = 0$ if and only if $\frac{d(x,y)}{d(x,y)+1} = 0$; a real quotient of real numbers is zero if and only if the numerator is zero, hence if and only if $d(x, y) = 0$ on account of d, once again, being a metric.

- For any $x, y \in M$, using symmetry of metric d ,

$$\overline{d}(x,y) = \frac{d(x,y)}{d(x,y)+1} = \frac{d(y,x)}{d(y,x)+1} = \overline{d}(y,x).$$

- For any $x, y, z \in M$, we know $d(x,y) \leq d(x,z) + d(z,y)$ because d is a distance. On the other hand, map $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by $f(t) := \frac{t}{t+1}$ is strictly increasing:

$$f'(t) = \frac{1}{(t+1)^2} > 0, \qquad \text{for every } t \geq 0.$$

It is clear that $\overline{d}(x,y) = f(d(x,y))$, and all of the above implies $f(d(x,y)) \leq f(d(x,z) + d(z,y))$ which translates into

$$\frac{d(x,y)}{d(x,y)+1} \leq \frac{d(x,z)+d(z,y)}{d(x,z)+d(z,y)+1} = \frac{d(x,z)}{d(x,z)+d(z,y)+1} + \frac{d(z,y)}{d(x,z)+d(z,y)+1};$$

the fact $d(z,y) \geq 0$ and $d(x,z) \geq 0$ implies

$$\frac{d(x,z)}{d(x,z)+d(z,y)+1} + \frac{d(z,y)}{d(x,z)+d(z,y)+1} \leq \frac{d(x,z)}{d(x,z)+1} + \frac{d(z,y)}{d(z,y)+1};$$

which entails $\overline{d}(x,y) \leq \overline{d}(x,z) + \overline{d}(z,y)$.

(ii) M is a metric space with both d and $\overline{d}$. This means we have two topologies τ, $\overline{\tau}$ given by these metrics, having respective bases

$$\beta := \{\varnothing\} \cup \{B_d(x,\varepsilon) : x \in M, \varepsilon > 0\}, \qquad \overline{\beta} := \{\varnothing\} \cup \left\{B_{\overline{d}}(x,\varepsilon) : x \in M, \varepsilon > 0\right\};$$

our intention is to use Proposition 2.5.6 in order to prove $\tau = \overline{\tau}$. In other words: for any open set U of one of the bases and any point x therein, there is an open set of the other basis containing x and contained in U.

The fact that every point in a ball is the centre of a ball contained in the previous ball (Proposition 1.2.3 (c)) allows us to simplify this proof a bit further. Indeed, since for every metric D on M, any $B_D(x,r)$ and any $y \in B_D(x,r)$ there exists a ball $B_D(y,s)$ *centered at* y such that $x \in B_D(y,s) \subset B_D(x,r)$, all we need to check is the following (ε playing the role of s and z the role of y in the above argument):

(a) for every $B_{\overline{d}}(z,\varepsilon)$, there exists $\delta > 0$ such that $B_d(z,\delta) \subset B_{\overline{d}}(z,\varepsilon)$;

(b) for every $B_d(z,\varepsilon)$, there exists $\delta > 0$ such that $B_{\overline{d}}(z,\delta) \subset B_d(z,\varepsilon)$.

In Exercise 34, for example, we have chosen not to use this shortcut; the reader may see this only entails a bit more notation (i.e. a smaller ball of radius ε inside the original ball (2.3)).

Let us first check (a). For any $x, y \in M$, $\frac{d(x,y)}{d(x,y)+1} \leq d(x,y)$ which means that if $d(x,y) < \varepsilon$ for some $\varepsilon > 0$, then $\overline{d}(x,y) < \varepsilon$ as well. This implies we can take $\delta := \varepsilon$ since $B_d(x,\varepsilon) \subset B_{\overline{d}}(x,\varepsilon)$. Thus, $\overline{\tau} \subset \tau$.

Let us finally check $\tau \subset \overline{\tau}$. Let $x \in M$ and $\varepsilon > 0$. We need $\delta = \delta_{x,\varepsilon} > 0$ such that $\overline{d}(x,y) < \delta$ implies $d(x,y) < \varepsilon$. We can rewrite our hypothesis for δ as $\frac{d(x,y)}{d(x,y)+1} < \delta$ which is equivalent to $d(x,y) < \frac{\delta}{1-\delta}$. Asking the right-hand side to be equal to our original radius, $\frac{\delta}{1-\delta} = \varepsilon$ we may undo the operations and obtain $\delta := \frac{\varepsilon}{1+\varepsilon}$. This radius is such, by construction, that $B_{\overline{d}}(x,\delta) \subset B_d(x,\varepsilon)$. Hence $\overline{\tau}$ is finer than τ.

Each topology is finer than the other. This means they are one and the same: $\overline{\tau} = \tau$.

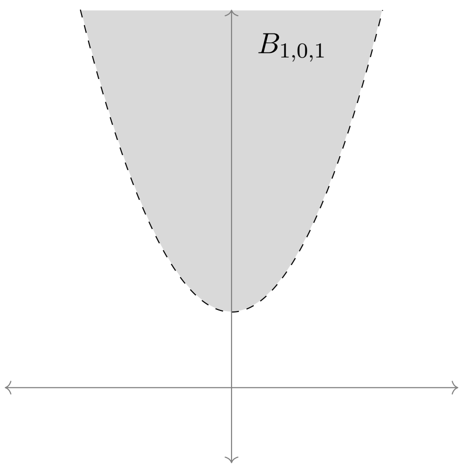
34. (*Coursework January 2014*) Let $X = \mathbb{R}^2$ and

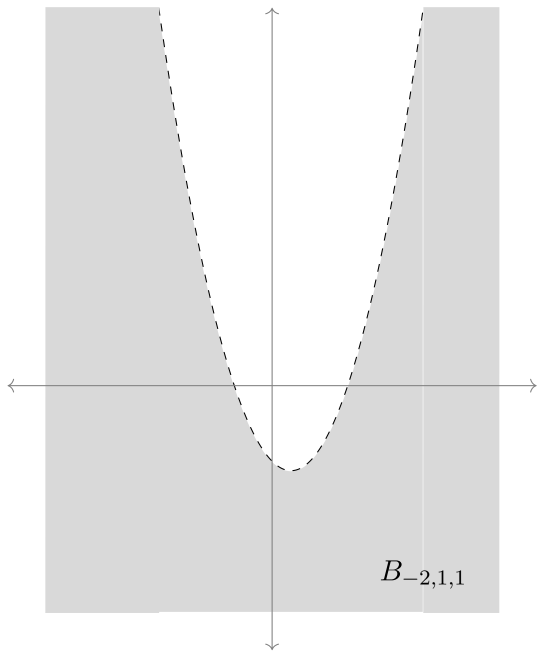
$$\mu := \{B_{a,b,c} : a, b, c \in \mathbb{R}, a \neq 0\} \cup \{\varnothing\}, \text{ where } B_{a,b,c} := \left\{(x, y) \in X : \operatorname{sign}(a) \cdot y > ax^2 + bx + c\right\}.$$

Is μ a topology, a basis for some topology or a subbasis for some topology? If any of these questions has an affirmative answer, is this topology equivalent to the Euclidean topology?

Solution. Sets defined as $B_{a,b,c}$ are regions strictly above ($a > 0$) or strictly below ($a < 0$) upward-facing parabola having equation $y = \frac{|a|}{a}\left(ax^2 + bx + c\right)$.

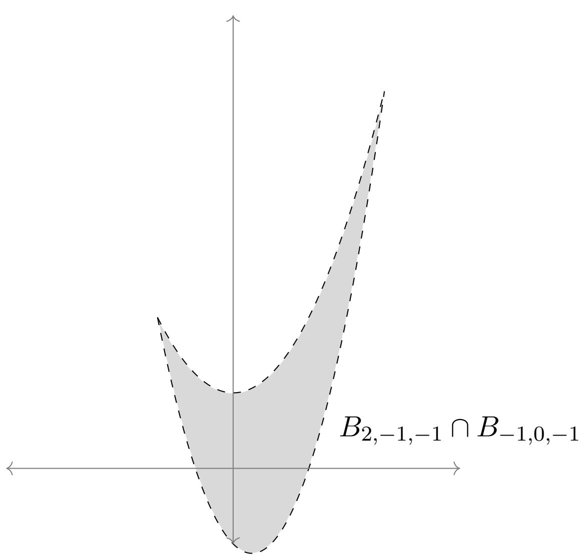
Some examples of elements of this subset of $\mathcal{P}(X)$ are



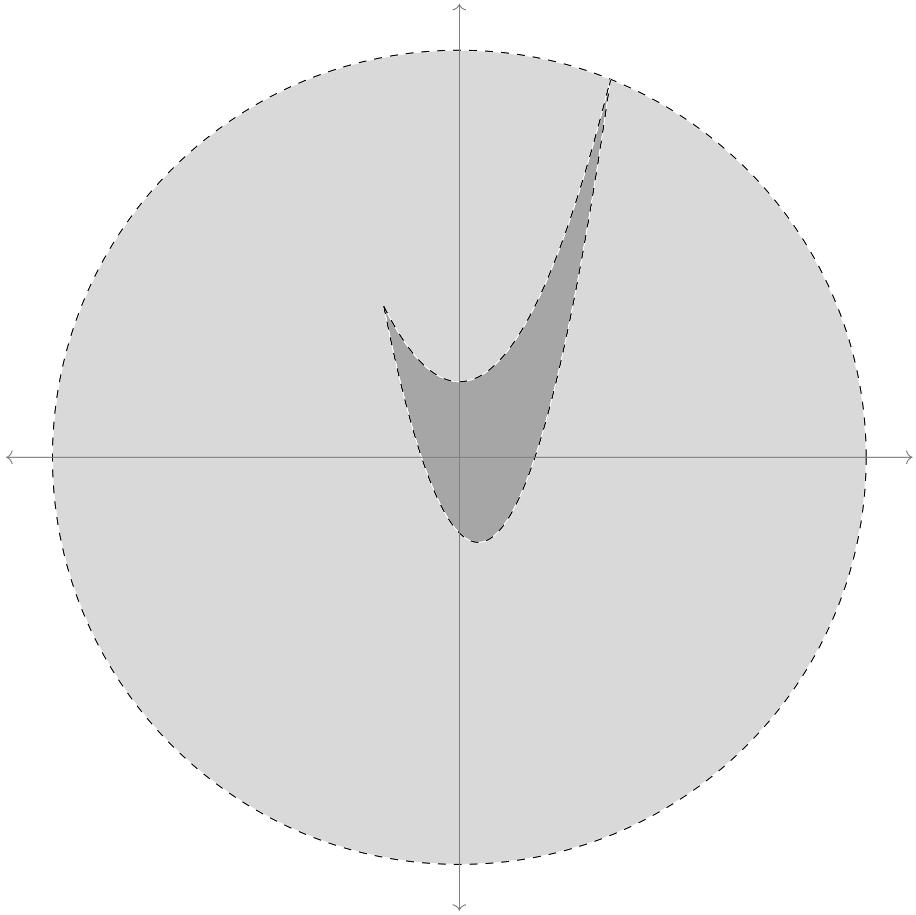


a) μ cannot be a topology since it does not contain $X = \mathbb{R}^2$ which cannot be expressed as a single region $B_{a,b,c}$ because for any $x \in \mathbb{R}$ there will always be an element not in $B_{a,b,c}$, e.g. $\left(x, \frac{|a|}{a}\left(ax^2 + bx + c\right) - \frac{|a|}{a}\varepsilon\right)$ for any $\varepsilon > 0$.

An alternative way of proving this is using the fact that finite intersections (or arbitrary unions) of elements in μ are generally not elements of μ. For instance, shifting signs in the parameters in the above figures, the intersection of $B_{2,-1,-1}$ and $B_{-1,0,-1}$,



is not a region of the form $B_{a,b,c}$ for any a, b, c. Indeed, $B_{2,-1,-1} \cap B_{-1,0,-1}$ is *bounded* in the Euclidean topology (i.e. it fits in a single open ball, e.g. $B_{d_2}\left((0,0), \sqrt{29}\right)$:



whereas one of the defining features of regions of the form $B_{a,b,c}$ is their unboundedness; otherwise there would be a ball $B\left((0,0), r\right)$ of radius r large enough containing one $B_{a,b,c}$, and the reader may check, defining $f(x) := ax^2 + bx + c$, that the point $\left(-\frac{b}{2a}, \frac{|a|}{a} f\left(-\frac{b}{2a}\right) + 2\frac{|a|}{a} r + \varepsilon\right)$, $\varepsilon > 0$, belongs to $B_{a,b,c}$ but does not belong to $B\left((0,0), r\right)$.

b) It is not a basis, either. It does fulfil the first property B_1 in Proposition 2.5.4: $\varnothing \in \mu$ by definition and, e.g. $X = \bigcup_{n\in\mathbb{Z}} B_{-1,0,-n}$ or an even smaller covering: $X = B_{-1,0,1} \cup B_{1,0,0}$. It does not fulfil, however, property B_2; indeed, as seen above all regions $B_{a,b,c}$ are unbounded, hence bounded set $B_{2,-1,-1} \cap B_{-1,0,-1}$ cannot contain, let alone be equal to a union of such regions.

c) Let us prove μ is a *subbasis* for the topology. Define

$$\beta := \left\{U_1 \cap \cdots \cap U_n : U_i \in \mu, n \in \mathbb{N}\right\}.$$

Let us check properties $\mathrm{B}_{1,2}$.

B_1: $\varnothing \in \mu \subset \beta$ and any of the two coverings $X = \bigcup_{n\in\mathbb{Z}} B_{-1,0,-n}$ or $X = B_{-1,0,1} \cup B_{1,0,0}$ described above is also a covering of X by elements of β since all elements in μ are also elements of β.

B_2: Given $V, W \in \beta$, we can express them as

$$V = V_1 \cap \cdots \cap V_n, \qquad W = W_1 \cap \cdots \cap W_n,$$

hence $V \cap W = V_1 \cap \cdots \cap V_n \cap W_1 \cap \cdots \cap W_n$ which is also a finite intersection of elements of μ, hence an element of β. Any element of β is a union of elements of β (namely, a union consisting only of itself).

Thus μ is a sub-basis for topology

$$\tau := \left\{ \bigcup_{i \in I} V_i : V_i \text{ is a finite intersection of elements of } \mu \text{ and } I \text{ is an arbitrary index set} \right\}$$

having basis β.

Let

$$\tau_2 = \left\{ U \subset \mathbb{R}^2 : \text{ for every } (x, y) \in U, \text{ there exists } r > 0 \text{ such that } B_{d_2}\left((x, y), r\right) \subset U \right\}$$

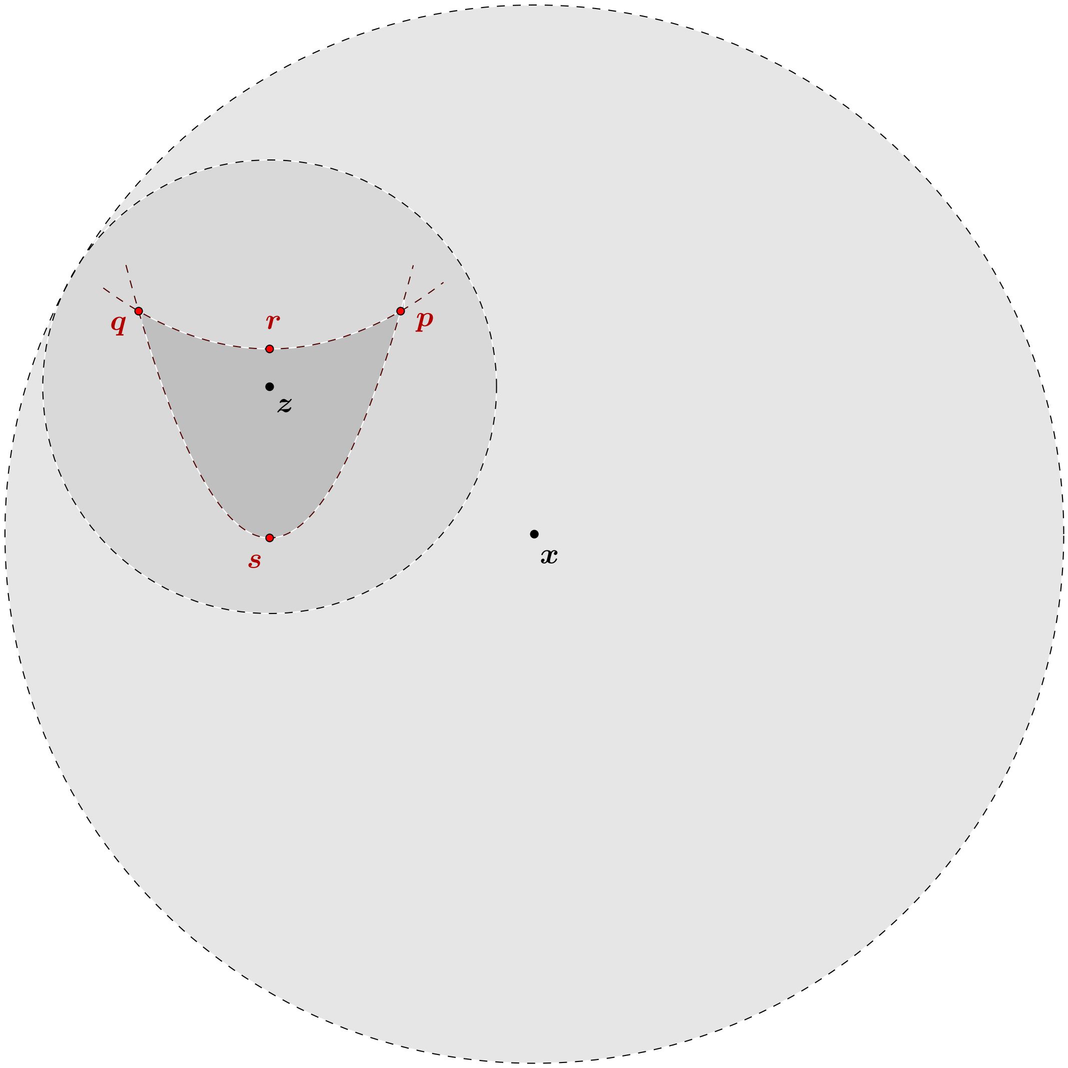
be the Euclidean topology on X. Let us prove this is equivalent to our previous topology: $\tau = \tau_2$. Indeed,

$\tau \subset \tau_2$: Elements in β are finite intersections of subsets which are either $\varnothing$ (which trivially belongs to τ_2) or regions defined by strict inequalities. Let us elaborate on why the latter, non-empty type are elements of τ_2 as well. If $V = B_{a_1,b_1,c_1} \cap \cdots \cap B_{a_n,b_n,c_n}$ is any basis element of β and $\boldsymbol{x} = (x_1, x_2) \in V$, for every $i = 1, \ldots, n$, let $\varepsilon_i = \min_{y \in \partial B_{a_i,b_i,c_i}} d_2(\boldsymbol{x}, \boldsymbol{y})$ be the distance between $\boldsymbol{x}$ and the parabola $y = \frac{|a_i|}{a_i}\left(a_i x^2 + b_i x + c_i\right)$. Define $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then

$$B_{d_2}(\boldsymbol{x}, \varepsilon) \subset B_{a_1,b_1,c_1} \cap \cdots \cap B_{a_n,b_n,c_n} = V,$$

hence Proposition 2.5.6 and the fact open Euclidean balls are a basis β_2 of τ_2 imply τ_2 is finer than τ.

$\tau_2 \subset \tau$: Let $B_{d_2}((x_1, x_2), r)$ be any basis element of β_2. Let $\boldsymbol{z} = (z_1, z_2) \in B_{d_2}((x_1, x_2), r)$. Let us prove there exists an element V of basis β such that $\boldsymbol{z} \in V \subset B_{d_2}((x_1, x_2), r)$. We will use four significant points $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$ in order to find two parabolas enclosing $\boldsymbol{z}$. Let $B_{d_2}((z_1, z_2), \varepsilon)$ $(\varepsilon = r - d_2(\boldsymbol{x}, \boldsymbol{z}))$ be the open ball provided by Proposition 1.2.3 (c), centred at $\boldsymbol{z}$ and contained in $B_{d_2}((x_1, x_2), r)$. $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ will be the vertices of an inverted equilateral triangle having $\boldsymbol{z}$ as its centroid. We could choose the inverted triangle to be circumscribed in $B_{d_2}((z_1, z_2), \varepsilon)$ but the fact it is totally contained in $B_{d_2}((x_1, x_2), r)$ would not be evident from the figure below, hence let us choose e.g. $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ at distance $\frac{2}{3}\varepsilon$ from $\boldsymbol{z}$, and $\boldsymbol{s}$ can be chosen to be a point lying above $\boldsymbol{z}$, at distance $\varepsilon/6$ and in the same vertical line. The parabolas we are looking for are sketched below:



(2.3)

The coordinates of these points are

$$
\begin{aligned}
\boldsymbol{p} &= \boldsymbol{z} + \frac{2\varepsilon}{3}\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(\frac{\varepsilon}{\sqrt{3}} + z_1, \frac{\varepsilon}{3} + z_2\right), \\
\boldsymbol{q} &= \boldsymbol{z} + \frac{2\varepsilon}{3}\left(\cos\frac{5\pi}{6}, \sin\frac{5\pi}{6}\right) = \left(-\frac{\varepsilon}{\sqrt{3}} + z_1, \frac{\varepsilon}{3} + z_2\right), \\
\boldsymbol{s} &= \boldsymbol{z} + \frac{2\varepsilon}{3}\left(\cos\frac{9\pi}{6}, \sin\frac{9\pi}{6}\right) = \left(z_1, -\frac{2\varepsilon}{3} + z_2\right), \\
\boldsymbol{r} &= \boldsymbol{z} + \frac{\varepsilon}{6}\left(\cos\frac{\pi}{2}, \sin\frac{\pi}{2}\right) = \left(z_1, \frac{\varepsilon}{6} + z_2\right).
\end{aligned}
$$

Parabolas can be found either using basic calculus or more sophisticated techniques, e.g. interpolating polynomials. The curve containing $\boldsymbol{p}, \boldsymbol{r}, \boldsymbol{q}$ is

$$y = \frac{x^2}{2\varepsilon} - \frac{xz_1}{\varepsilon} + \frac{\varepsilon^2 + 3z_1^2 + 6\varepsilon z_2}{6\varepsilon},$$

and the parabola containing $\boldsymbol{p}, \boldsymbol{s}, \boldsymbol{q}$ is

$$y = \frac{3x^2}{\varepsilon} - \frac{6xz_1}{\varepsilon} + \frac{-2\varepsilon^2 + 9z_1^2 + 3\varepsilon z_2}{3\varepsilon}.$$

Region

$$V = B_{\frac{3}{\varepsilon}, -\frac{6z_1}{\varepsilon}, \frac{-2\varepsilon^2+9z_1^2+3\varepsilon z_2}{3\varepsilon}} \cap B_{-\frac{1}{2\varepsilon}, \frac{z_1}{\varepsilon}, -\frac{\varepsilon^2+3z_1^2+6\varepsilon z_2}{6\varepsilon}},$$

by construction, contains $\boldsymbol{z}$ and is totally contained in the original Euclidean ball $B_{d_2}(\boldsymbol{x}, r)$. Proposition 2.5.6 implies $\tau_2 \subset \tau$.

35. (*July 2014*) For any given set X, define the *finite complement topology* on X by

$$\tau_X := \{S \subset X : X \setminus S \text{ is finite}\} \cup \{\varnothing\}.$$

Find the interior and the closure of the subset $\{10n : n \in \mathbb{N}\}$ in $\tau_{\mathbb{N}}$.

Solution. The interior $\operatorname{Int}\{10n : n \in \mathbb{N}\}$ is the largest open set contained in $\{10n : n \in \mathbb{N}\}$; the fact it is open in $\tau_{\mathbb{N}}$ implies it can only be equal to either $\varnothing$ (option 1) or the complementary of a finite set (option 2). Let us discard option 2. The fact $\operatorname{Int}\{10n : n \in \mathbb{N}\} \subset \{10n : n \in \mathbb{N}\}$ implies $\mathbb{N} \setminus \{10n : n \in \mathbb{N}\} \subset \mathbb{N} \setminus \operatorname{Int}\{10n : n \in \mathbb{N}\}$, hence the fact $\mathbb{N} \setminus \{10n : n \in \mathbb{N}\}$ is not finite (easily proven by *reductio ad absurdum*) implies that the set containing it, $\mathbb{N} \setminus \operatorname{Int}\{10n : n \in \mathbb{N}\}$, is infinite as well. Which only leaves us with option 1:

$$\operatorname{Int}\{10n : n \in \mathbb{N}\} = \varnothing.$$

The closure $\overline{\{10n : n \in \mathbb{N}\}}$ is the smallest closed set containing $\{10n : n \in \mathbb{N}\}$. The fact it is closed, in virtue of the definition of $\tau_{\mathbb{N}}$, only leaves two options: either $\overline{\{10n : n \in \mathbb{N}\}} = \mathbb{N}$ (option 1) or $\overline{\{10n : n \in \mathbb{N}\}}$ is finite (option 2). Again, $\{10n : n \in \mathbb{N}\}$ is an infinite set and the fact $\{10n : n \in \mathbb{N}\} \subset \overline{\{10n : n \in \mathbb{N}\}}$ implies $\overline{\{10n : n \in \mathbb{N}\}}$ is an infinite set as well, hence discarding option 2:

$$\overline{\{10n : n \in \mathbb{N}\}} = \mathbb{N}.$$

36. (*May 2013*) Let X be a set. Recall the finite complement topology on X described in 35:

$$\tau := \{S \subset X : X \setminus S \text{ is finite}\} \cup \{\varnothing\}.$$

(i) Prove all neighbourhoods are open in this topology.

(ii) Prove $\{X \setminus \{x\} : x \in X\}$ is a subbase of this topology.

(iii) Prove $\mathbb{R}$ with this topology does not satisfy the second axiom of countability.

Solution.

(i) Let $x \in X$ and $N \subset X$ be a neighbourhood of x, i.e. $x \in \mathring{N}$. This implies: there exists $U \in \tau$ such that $x \in U \subset N$, i.e. $X \setminus N \subset X \setminus U$ which by definition is finite, i.e. $X \setminus N$ is finite; again by definition this entails $N \in \tau$.

(ii) Let $\mu = \{X \setminus \{x\} : x \in X\}$. For this to be a subbase of τ, we need the set of finite intersections

$$\beta = \{U_1 \cap \cdots \cap U_n : U_1, \dots, U_n \in \mu, n \in \mathbb{N}\} \cup \{\varnothing\}$$

to be a basis of τ. For every finite set of elements of μ,

$$U_1 = X \setminus \{x_1\}, \quad U_2 = X \setminus \{x_2\}, \qquad \cdots \qquad U_n = X \setminus \{x_n\},$$

we have $U_1 \cap \cdots \cap U_n$ equal to $X \setminus \{x_1, \dots, x_n\}$ ($= \varnothing$ if $X = \{x_1, \dots, x_n\}$). Thus

$$\beta = \{X \setminus \{x_1, \dots, x_n\} : x_1, \dots, x_n \in X, n \in \mathbb{N}\} \cup \{\varnothing\}.$$

Let $U \in \tau$. We have $X \setminus U$ is finite, say $X \setminus U = \{z_1, \dots, z_k\}$ for some $k \in \mathbb{N}$ and $z_1, \dots, z_k \in X$. Hence $U = X \setminus \{z_1, \dots, z_k\} \in \beta$, hence a union of elements of β. We have in fact proven $\tau = \beta$.

μ is not a basis though, since it does not contain the empty set.

(iii) If we prove $(\mathbb{R},\tau)$ does not satisfy the *first* countability axiom, this will imply it does not satisfy the second. Assume $(\mathbb{R},\tau)$ is first countable. This would imply that for every $x\in\mathbb{R}$, there exists a countable neighbourhood basis

$$\mathcal{N}_x = \{N_n : n\in\mathbb{N}\},$$

where $x\in \overset{\circ}{N}_n$ for every $n\in\mathbb{N}$. Since all neighbourhoods are open (in virtue of (i)) this implies the existence of an *open* neighbourhood basis $\{U_n : n\in\mathbb{N}\}$ such that $x\in U_n\in\tau$ for every $n\in\mathbb{N}$. Thus

$$\bigcap_{n\geq 1} U_n = \{x\} \tag{2.4}$$

indeed, the leftward inclusion is immediate and the rightward inclusion comes as follows: if $y\neq x$ then $x\in\mathbb{R}\setminus\{y\}\in\tau$ hence there exists an open neighbourhood U_n such that $x\in U_n\subset\mathbb{R}\setminus\{y\}$, hence $y\notin U_n$ which means $y\notin\bigcap_n U_n$.

Thus $\bigcap_{n\geq 1} U_n=\{x\}$ which means

$$\bigcup_{n\geq 1}\mathbb{R}\setminus U_n = \mathbb{R}\setminus\bigcap_{n\in\mathbb{N}} U_n = \mathbb{R}\setminus\{x\};$$

the leftmost set is a countable union of finite sets, thus countable, whereas the rightmost set is uncountable. Absurd.

Alternatively we can prove it is not second countable directly, without using first countability. For any countable basis β of τ, and any $x\in\mathbb{R}$, $\bigcap_{V\in\beta, x\in V} V=\{x\}$ for the same reasons leading to (2.4). Hence taking complementaries once again we have

$$\bigcup_{V\in\beta,x\in V}\mathbb{R}\setminus V = \mathbb{R}\setminus\bigcap_{V\in\beta,x\in V} V = \mathbb{R}\setminus\{x\},$$

again rendering a countable set equal to an uncountable which is absurd.

37. (*July 2013*) Prove that equivalent distances define the same topology on a given set.

Solution. Assume there exist $\alpha,\beta>0$ such that

$$\alpha d(x,y)\leq D(x,y)\leq \beta d(x,y), \qquad x,y\in X. \tag{2.5}$$

Let $x\in X$ and $\varepsilon>0$. Let us first find $\delta>0$ such that $B_D(x,\delta)\subset B_d(x,\varepsilon)$. Define $\delta:=\alpha\varepsilon$. Then

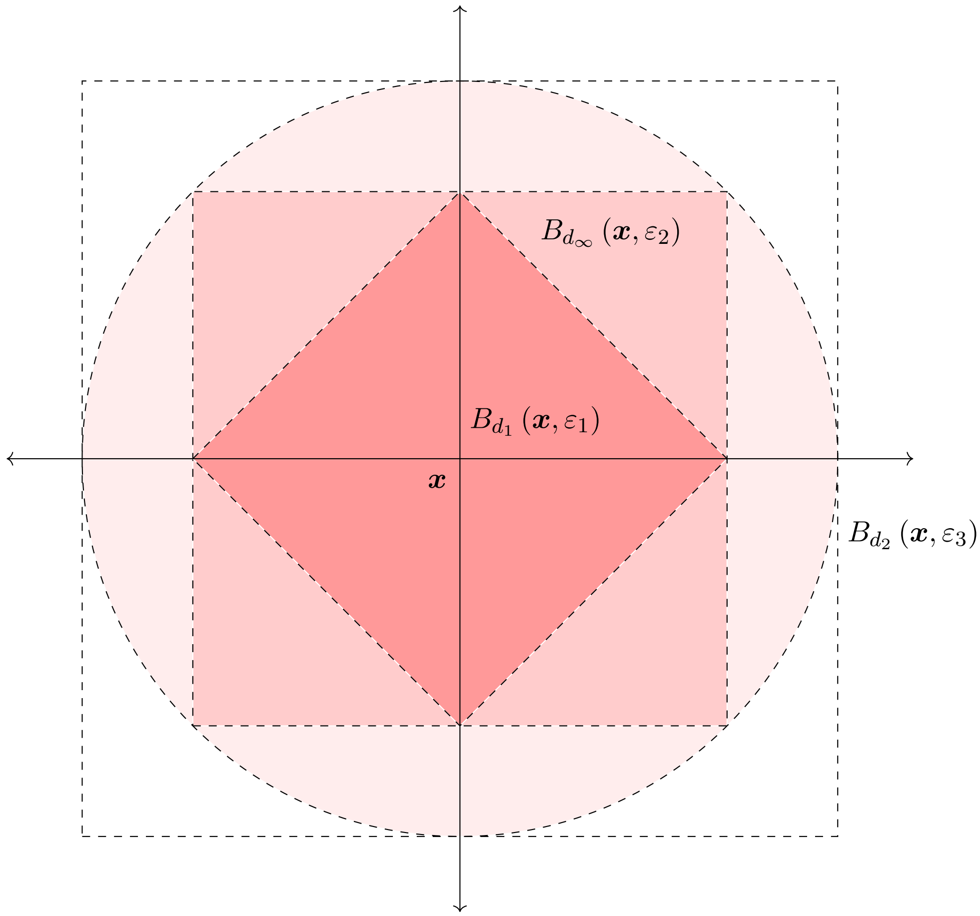
$$B_D(x,\delta)=\{y\in X : D(x,y)<\delta\}=\{y\in X : \alpha d(x,y)\leq D(x,y)<\alpha\varepsilon\}\subset\{y\in X : d(x,y)<\varepsilon\}=B_d(x,\varepsilon).$$

Let us find $\delta>0$ such that $B_d(x,\delta)\subset B_D(x,\varepsilon)$. Define $\delta:=\frac{\varepsilon}{\beta}$. Then

$$B_d(x,\delta)=\{y\in X : d(x,y)<\delta\}=\left\{y\in X : d(x,y)<\frac{\varepsilon}{\beta}\right\}\overset{D(x,y)\leq\beta d(x,y)}{\subset}\left\{y\in X : D(x,y)<\beta\frac{\varepsilon}{\beta}\right\}=B_D(x,\varepsilon).$$

38. Prove all p-Hölder metrics on $\mathbb{R}^n$ define the same topology, i.e. the Euclidean topology.

Solution. This is nothing but the application of Exercise 12 to Exercise 37 above. Geometrically this is obvious, for instance for $p=1,2,\infty$, from the fact that every one of the following shapes can be embedded in the other two: circle, square and rhombus (Exercise: find $\varepsilon_1, \varepsilon_2, \varepsilon_3$ in terms of a single ε below):



39. (July 2015) Let (X, d) be a metric space and Y be a subspace of X such that the induced metric is the discrete metric. Prove Y is closed in X.

Is this true for any topological space (induced topology is discrete implies the subspace is closed)?

Solution. The induced metric is discrete iff

$$d|_Y(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases} \qquad x, y \in Y.$$

And $\tau_Y = \{U \cap Y : U \in \tau_X\} = \tau_{d|_Y}$. We would like to prove $\overline{Y} = Y$ in τ_X. We know $Y \subset \overline{Y}$. Assume $x \in \overline{Y}$. This means $B\left(x, \frac{1}{n}\right) \cap Y \neq \varnothing$ for every $n \in \mathbb{N}$. Let us prove, for every $n \geq 2$, $B\left(x, \frac{1}{n}\right) \cap Y$ has cardinality one. Indeed, if $y_1, y_2 \in B\left(x, \frac{1}{n}\right) \cap Y$ then

$$d(y_1, y_2) \leq d(y_1, x) + d(x, y_2) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n} \leq 1,$$

hence $y_1 = y_2$ due to the fact that $d(y_1, y_2) \in \{0, 1\}$. Thus $B\left(x, \frac{1}{n}\right) \cap Y = \{y\}$ for every $n \geq 2$ and $y \in \bigcap_{n \geq 2} B\left(x, \frac{1}{n}\right) \cap Y \subset \bigcap_{n \geq 2} B\left(x, \frac{1}{n}\right) = \{x\}$ which means $y = x$. Thus $x \in Y$.

This is not true in general. Assume $X = \{1, 2\}$ with the coarse topology, i.e. $\tau_X = \{\varnothing, X\}$. For $Y = \{1\}$, $\tau_Y = \{\varnothing, \{1\}\}$ which turns out to be the discrete topology on Y. But Y is not closed in X since the only closed (and open) sets in τ_X are $\varnothing$ and X.

40. (July 2015) Is any of the following a basis for some topology on $\mathbb{R}$?

(i) $\{[a,b] : a < b, a, b \in \mathbb{Q}\} \cup \{\varnothing\}$.

(ii) $\{[a,b] : a < b, a \notin \mathbb{Q}, b \in \mathbb{Q}\} \cup \{\varnothing\}$.

Solution.

a) $\mathbb{R}$ is obviously equal to $\bigcup_{U\in\beta_1} U$ and the empty set is contained in β_1, hence it fulfils the first property of a basis. The intersection of two intervals with rational extremes could possibly be a single point, e.g. $[a,b] \cap [b,c] = \{b\}$ which cannot be obtained as a union of intervals in β_1. This is not a basis.

b) The first property is also satisfied. But given two intervals $[a,b], [c,d] \in \beta_2$, their intersection is either another interval with the same constraints (left extreme irrational, right extreme rational) or the empty set because the situation described in the first item is impossible. This is a basis.

41. (Coursework 2015) Let $X := \mathcal{C}([a,b])$ be the set of all continuous functions $[a,b] \to \mathbb{R}$ on a closed, bounded interval with extremes $a < b$. Consider metrics $d, D : \mathcal{C}([a,b]) \times \mathcal{C}([a,b]) \to \mathbb{R}$ defined by:

$$d(f,g) := \max_{x\in[a,b]} |f(x) - g(x)|, \qquad D(f,g) := \int_a^b |f(x) - g(x)|\,dx, \qquad f,g \in \mathcal{C}([a,b]).$$

(i) (10 Marks) Sketch or describe open balls for these metrics.

(ii) (15 Marks) Study the comparability of the metric topologies $\tau_X^d, \tau_X^D \subset \mathcal{P}(X)$.

Solution: handwritten at the end of this chapter

42. Let $X = \left\{(x,y) \in \mathbb{R}^2 : y \geq 0\right\}$. Define $\beta = \{\varnothing\} \cup \{V((x,y),r) : (x,y) \in X, r > 0\}$, where

$$V((x,y),r) = \begin{cases} B_{d_2}((x,y),r), & y > r, \\ B_{d_2}((x,y),y) \cup \{\text{tangency point}\}, & \text{otherwise} \end{cases}$$

(i) Prove β is a basis for a topology on X.

(ii) Can the resulting topology be compared to the subspace Euclidean topology on X? Answer in detail.

Solution.

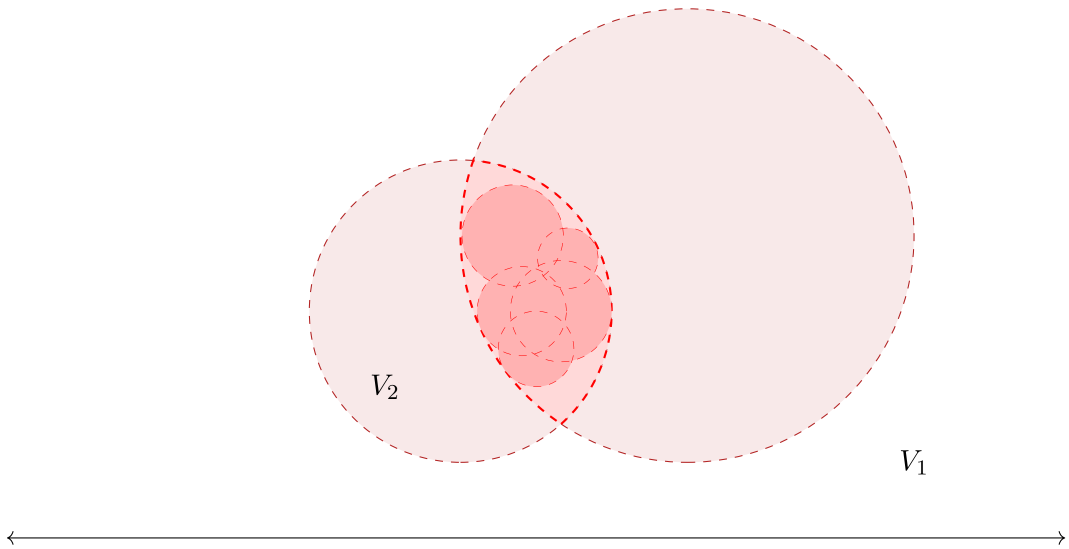
(i) For it to be a basis of a topology it must fulfil

(a) $X = \bigcup_{V\in\beta} V$ and $\varnothing \in \beta$.

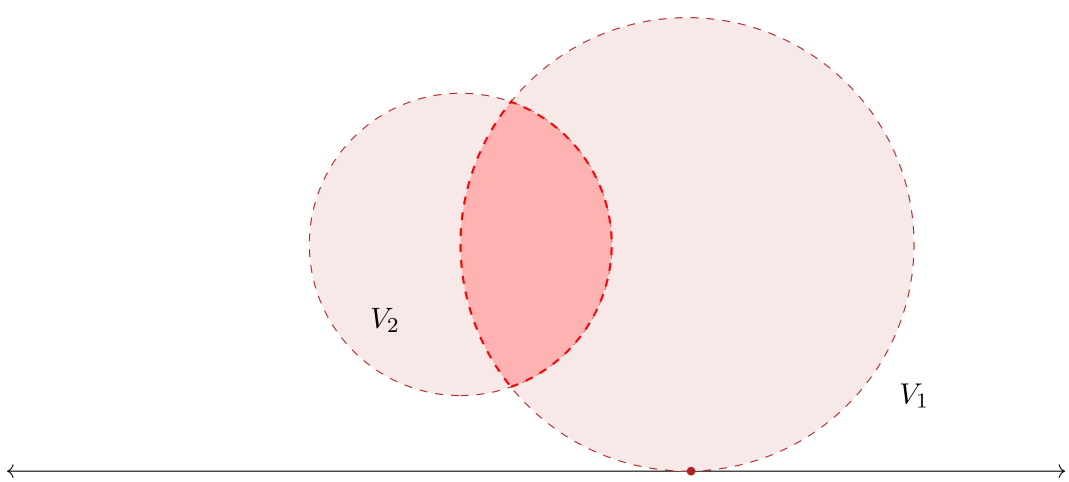
(b) For every $V, W \in \beta$, $V \cap W$ is a union of elements in β.

The first item is a consequence, e.g. of $X = \left(\bigcup_{x\in\mathbb{R}} V((x,1),1)\right) \cup \bigcup_{x\in\mathbb{R}, y\geq 2, r\geq 1} V((x,y),r)$. $\varnothing \in \beta$ by definition.

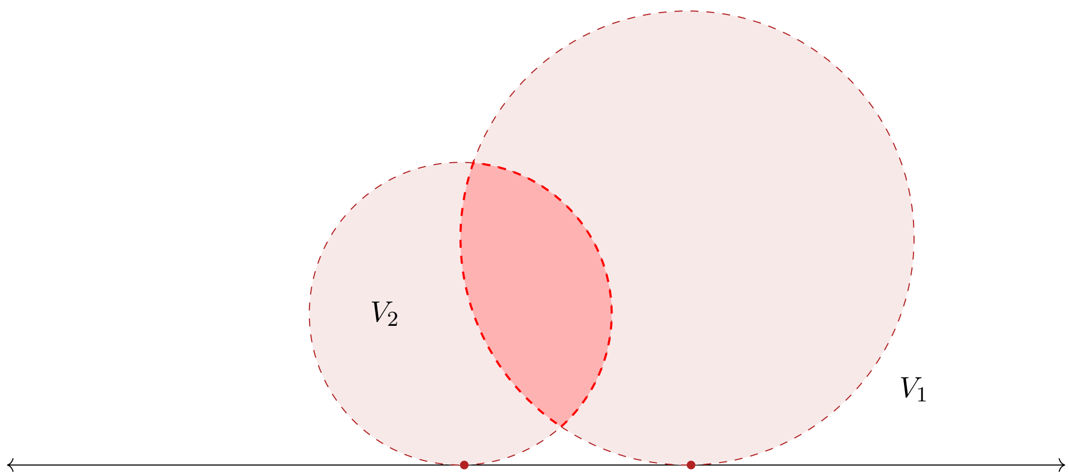
The second item is a consequence of the fact that Euclidean balls themselves are a basis for a topology. Let $V_1, V_2 \in \beta$. If V_1, V_2 are two full Euclidean balls with no tangency point, then their intersection is a union of Euclidean balls not touching the x axis, hence belonging to β:



If, say, V_1 is the union $B\left(\left(x,y\right),y\right)\cup\left\{\left(x,0\right)\right\}$ of a Euclidean ball and its tangency point, and V_2 is a full Euclidean ball with no tangency point, then V_2 cannot contain the tangency point of V_1 because they are not equal, hence their intersection is still an intersection of Euclidean balls in $\mathbb{R}^2$, which can be expressible as a union of Euclidean balls not tangent to the x axis, hence belonging to β:



And finally if both V_1 and V_2 are Euclidean balls tangent to the x axis, along with their tangency points, they are either equal (hence their intersection belongs to β or they are different and thus do not share their respective tangency points, and their intersection is still expressible as a union of Euclidean usual balls not touching the x axis

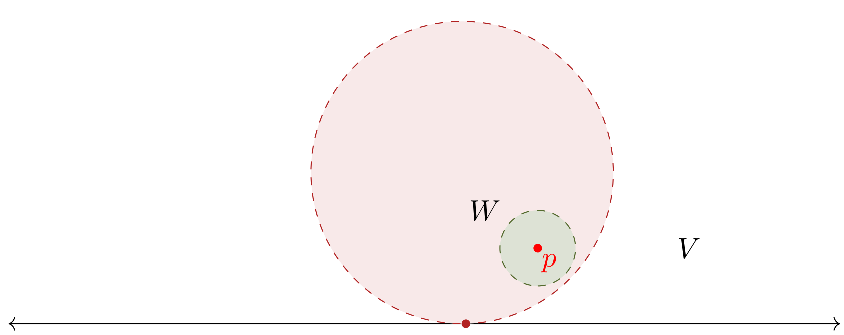


And thus it is a union of sets in β.

(ii) Two topologies τ_1, τ_2 are comparable iff given bases β_1, β_2, for every $V \in \beta_1$ and $x \in V$, there exists $W \in \beta_2$ such that $x \in W \subset V$, and viceversa. A basis for $\tau_1 = \tau_X^{\mathrm{Eucl}}$ is $\beta_1 = \{B((x,y), r) \cap X : (x,y) \in X, r > 0\}$. Let τ_2 be the topology with basis β studied in (i) and assume $V \in \beta_2$.

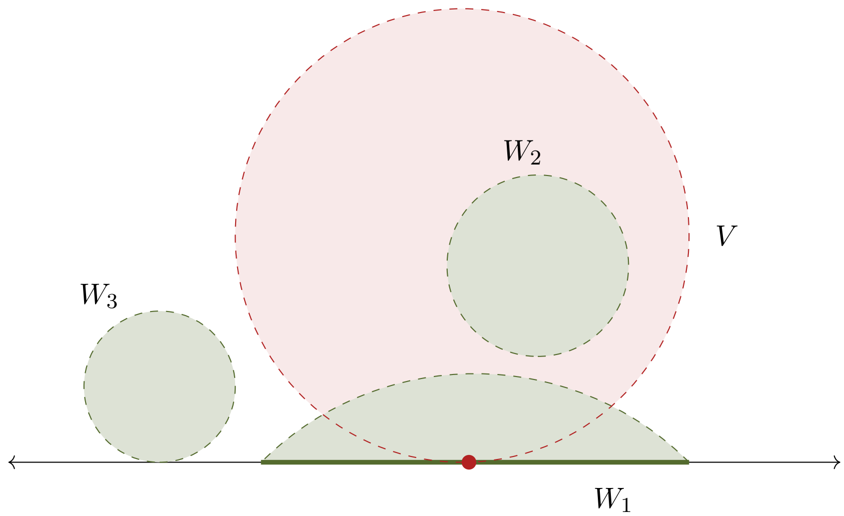
(a) If $V = B((x,y), r)$ and $y > r$, then V is a Euclidean ball completely contained in X and having no intersecting boundary with it and thus it is also a basis open set in β_2.

(b) If $V = B((x,y), y) \cup \{(x,0)\}$, i.e. a ball union its tangency point, then every point $\boldsymbol{p} = (\overline{x}, \overline{y})$ in V not equal to the tangency point can be the centre of a full Euclidean ball $W \in \beta_1$ small enough not to touch the x axis with its boundary,



but the tangency point itself $(x,0)$ cannot be contained in any full or intersected Euclidean ball $W \in \beta_1$. Indeed, any intersected Euclidean ball $W_1 = B((\overline{x}, \overline{y}), r) \cap X$

containing $(x, 0)$ would necessarily contain points in the x axis outside of V,



and any non-intersected Euclidean ball $W = B((\overline{x}, \overline{y}), r)$ such as $W_{2,3}$ above (regardless of whether its boundary intersects the x axis) could never contain this point on the x axis.

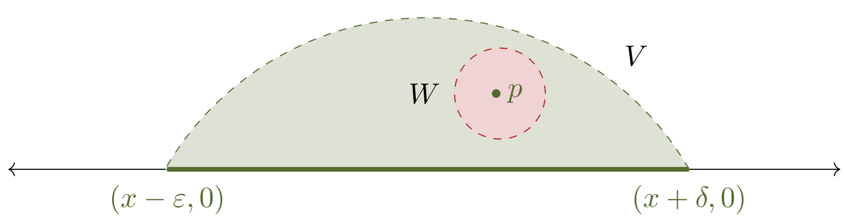
Hence given $V \in \beta_2$ and $\boldsymbol{p} \in V$, there may not exist a $W \in \beta_1$ such that $\boldsymbol{p} \in W \subset V$. Which means $\tau_2 \nsubseteq \tau_1$.

Let us prove $\tau_1 \subseteq \tau_2$. Let $V \in \beta_1$.

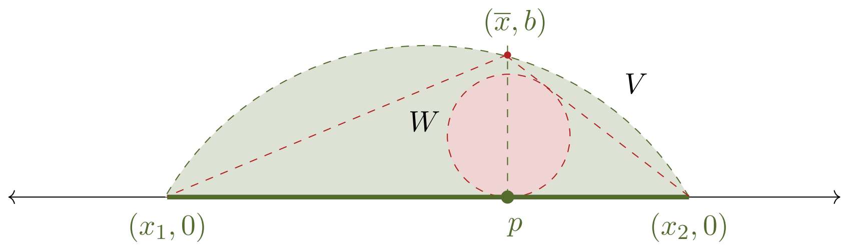
(a) If $V = B((x, y), r)$ is a full Euclidean ball $(y > r)$ it is also a basis open set for β_2.

(b) If $V = B((x, y), y)$ is a full Euclidean ball tangent to the x axis, then it might not be an open set for β_2 itself, but since it is an open set for the Euclidean topology, it can be obtained as an infinite union of Euclidean balls, none of them tangent to the x axis: indeed, for every $\boldsymbol{p} \in V$, the fact $\boldsymbol{p} \notin \{y = 0\}$ implies $\boldsymbol{p} = (\overline{x}, \overline{y})$ with $\overline{y} > 0$, and the ball of centre $\boldsymbol{p}$ and radius $\overline{y}/2$ is totally contained in V, and not tangent to the x axis.

(c) Finally, assume $V = B((x, y), r) \cap X$. If $\boldsymbol{p} = (\overline{x}, \overline{y})$ does not belong to the x axis, it is obviously the centre of a Euclidean ball $W \in \beta_2$ totally contained in V, not tangent to the x axis:



And if $\boldsymbol{p}$ belongs to the x axis, it must belong to an interval $(x_1, x_2) = (x - \varepsilon, x + \delta)$, $\boldsymbol{p} = (\overline{x}, 0)$ with $\overline{x} > x_1$. Hence the vertical line containing $\boldsymbol{p}$ is $\{x = \overline{x}\}$ which intersects ∂V at a point $(\overline{x}, b)$ above the x axis. The open line segments $\overline{(x_1, 0)(\overline{x}, b)}$ and $\overline{(x_2, 0)(\overline{x}, b)}$ are thus totally contained in V, and of two circles C_1, C_2 tangent to the x axis (at $\boldsymbol{p}$) and to $\overline{(x_1, 0)(a, b)}$ and $\overline{(x_2, 0)(a, b)}$ respectively, at least one is totally contained in V:



which means that the union of this circle, the region bounded by it and $\boldsymbol{p}$ is a set W belonging to β_2, totally contained in V.

Hence for every $V \in \beta_1$ and every $\boldsymbol{p} \in V$ there exists $W \in \beta_2$ such that $\boldsymbol{p} \in W \subset V$. Which means $\tau_1 \subseteq \tau_2$.

The topologies can be compared: τ_2 is strictly finer than τ_1.

43. Let (X, d) be a metric space and $A, B \subset X$. Prove $\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B) + d(A, B)$ and $\operatorname{diam}\left(\overline{A}\right) = \operatorname{diam}(A)$

44. Let (X, d) be a metric space and $A, B \subset X$. Prove:

(i) $\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B) + d(A, B)$.

(ii) $\operatorname{diam}\left(\overline{A}\right) = \operatorname{diam}(A)$.

Reminder: $\operatorname{diam}(A) := \sup_{x,y \in A} d(x, y)$ and $d(A, B) = \inf_{x \in A, y \in B} d(x, y)$.
Hint: Minima and suprema preserve inequalities (albeit not their strictness).

Solution.

(i) For every $x, y \in A \cup B$, $x \in A$ or $x \in B$ and $y \in A$ or $y \in B$.

- If $x, y \in A$, then $d(x, y) \leq \operatorname{diam}(A) \leq \operatorname{diam}(A) + \operatorname{diam}(B) + d(A, B)$.
- If $x, y \in B$, then $d(x, y) \leq \operatorname{diam}(B) \leq \operatorname{diam}(A) + \operatorname{diam}(B) + d(A, B)$.
- If $x \in A \setminus B, y \in B \setminus A$, then for every $z_1 \in A$, $z_2 \in B$, applying the triangle inequality twice

$$d(x, y) \leq d(x, z_1) + d(z_1, z_2) + d(z_2, y) \leq \operatorname{diam}(A) + d(z_1, z_2) + \operatorname{diam}(B)$$

and given the fact that infima maintain inequalities,

$$\inf_{z_1 \in A, z_2 \in B} d(x, y) \leq \operatorname{diam}(A) + \inf_{z_1 \in A, z_2 \in B} d(z_1, z_2) + \operatorname{diam}(B) = \operatorname{diam}(A) + d(A, B) + \operatorname{diam}(B).$$

- If $x \in B \setminus A, y \in A \setminus B$ then a similar argument proves $d(x, y) \leq \operatorname{diam}(B) + d(A, B) + \operatorname{diam}(A)$.

Hence $d(x, y) \leq \operatorname{diam}(A) + d(A, B) + \operatorname{diam}(B)$ for every $x, y \in A \cup B$. Taking suprema in this inequality,

$$\operatorname{diam}(A \cup B) = \sup_{x,y \in A \cup B} d(x, y) \leq \operatorname{diam}(A) + d(A, B) + \operatorname{diam}(B).$$

(ii) We know $A \subset \overline{A}$, hence $\operatorname{diam}(A) = \sup_{x,y\in A} d(x,y) \leq \sup_{x,y\in \overline{A}} d(x,y) = \operatorname{diam}\left(\overline{A}\right)$. Let us prove the other inequality: $\operatorname{diam}\left(\overline{A}\right) \leq \operatorname{diam}(A)$. For every $\varepsilon > 0$ and any $x, y \in \overline{A}$, we know there exist $z_1, z_2 \in A$ such that $d(x, z_1) < \varepsilon/2$, $d(x, z_2) < \varepsilon/2$, hence applying the triangle inequality twice $d(x,y) \leq d(x,z_1) + d(z_1,z_2) + d(z_2,y) < \varepsilon + \operatorname{diam}(A)$. Taking suprema, $\operatorname{diam}\left(\overline{A}\right) \leq \varepsilon + \operatorname{diam}(A)$. This is true for any $\varepsilon > 0$, hence taking infima with respect to ε we have $\operatorname{diam}(\overline{A}) \leq \operatorname{diam}(A)$.

45. Is $\tau_1 = \{A \in \mathcal{P}(\mathbb{R}) : \mathbb{R}\setminus A \text{ finite }\} \cup \{\varnothing\}$ a topology on $\mathbb{R}$? Is $\tau_2 = \{A \in \mathcal{P}(\mathbb{R}) : \mathbb{R}\setminus A \text{ infinite }\} \cup \{\varnothing\}$ a topology on $\mathbb{R}$?

Solution. τ_1 is a topology:

T_1 $\varnothing \in \tau_1$ and $\mathbb{R}\setminus\mathbb{R} = \{\varnothing\}$ which is a finite set, hence $\mathbb{R} \in \tau_1$.

T_2 Let $A_i \in \tau_1$ for every i at least one of which is not empty. Then $\mathbb{R}\setminus\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I}(\mathbb{R}\setminus A_i)$, the terms in the brackets being finite, hence so being their intersection.

T_3 For any $A_1, \ldots, A_n \in \tau_1$, $\mathbb{R}\setminus(A_1\cap\cdots\cap A_n) = \bigcup_{i=1}^{n}(\mathbb{R}\setminus A_i)$ which is a finite union of finite sets, hence finite.

τ_2 is not a topology. For instance let us prove the union of any collection of its elements is generally not an element thereof. Given for instance any collection $\{A_i\}_{i\in I}$ such that $\bigcup_{i\in I} A_i = \mathbb{R}\setminus\{0\}$ its complementary is $\{0\}$ which is finite.

46. Let $A \subset X$.

(i) Prove $\mathring{A}$ and ∂A are disjoint and $\overline{A} = \mathring{A} \cup \partial A$.

(ii) Prove $\partial A = \varnothing$ iff A is open and closed.

(iii) Prove A is open iff $\partial A = \overline{A}\setminus A$.

(iv) If A is open, is it true that $A = \mathring{\overline{A}}$?

Solution.

(i) By definition, one of the sets is defined as a complement of the other, hence they have to be disjoint: $\partial A \cap \mathring{A} = \left(\overline{A} \cap \overline{(X\setminus A)}\right) \cap \mathring{A} = \mathring{A} \cap \overline{X\setminus A} = \mathring{A} \cap X \setminus \mathring{A} = \varnothing$. For the same reason their union must be $\overline{A}$: $\mathring{A} \subset A \subset \overline{A}$ and $\partial A = \overline{A}\setminus\mathring{A}$ both imply $\partial A \cup \mathring{A} \subset \overline{A}$, and for the other inclusion let any $x \in \overline{A}$ which means every open neighbourhood U_x of x intersects A, hence either $U_x \cap X\setminus A \neq \varnothing$ (which implies $x \in \partial A$) or for every neighbourhood V_x of x $V_x \cap (X\setminus A) = \varnothing$ (which implies $x \in \mathring{A}$).

(ii) $\partial A = \varnothing$ iff $\overline{A} = \mathring{A}$ and remember: $\overline{A} \supset A \subset \mathring{A}$.

(iii) A is open iff $A = \mathring{A}$ iff $\partial A = \overline{A}\setminus\mathring{A} = \overline{A}\setminus A$.

(iv) It is false. $A \subset \overline{A}$ implies, due to the openness of A, $A = \mathring{A} \subset \mathring{\overline{A}}$. But the other inclusion is false. For instance in $(\mathbb{R}, \tau_{\mathrm{Eucl}})$ we have $\mathbb{R}\setminus\{0\}$ open thus $\mathbb{R}\setminus\{0\} = \mathbb{R}\setminus\mathring{\{0\}}$ but $\overline{\mathbb{R}\setminus\{0\}} = \mathbb{R}$ whose interior is $\mathbb{R}$. Other examples: $A = (0, 1/2)\cup(1/2, 1)$ and $\overline{A} = [0,1]$ whose interior $(0,1)$ is not equal to $\mathring{A} = A$.

47. Consider the following topology for $\mathbb{R}$

$$\tau = \{A \in \mathbb{R} : \mathbb{R}\setminus A \text{ finite or countable}\} \cup \{\varnothing\}.$$

If $a \leq b$, find the closure of $[a,b]$, (a,b).

Solution. Assume $a < b$. The closed sets of this topology are those subsets either finite or countable or $= \mathbb{R}$. $\overline{[a,b]}$ must be such a set, and contain all of the uncountable $[a,b]$. Hence $\overline{[a,b]} = \mathbb{R}$. For the same reason $\overline{(a,b)} = \mathbb{R}$.

$\mathring{(a,b)}$ is open and contained in (a,b). Thus $\mathbb{R} \setminus (a,b) \subset \mathbb{R} \setminus \mathring{(a,b)}$, the former uncountable, the latter either countable or finite or $\mathbb{R}$. Hence $\mathbb{R} \setminus \mathring{(a,b)} = \mathbb{R}$ and $\mathring{(a,b)} = \varnothing$. For the same reason $\mathring{[a,b]} = \varnothing$.

Finally $[a,a] = \{a\}$ which is finite or countable, hence $\overline{[a,a]} = \{a\}$. Its interior is empty because it is not open. And $(a,a) = \varnothing$ hence both its interior and closure are empty.

48. Let (X, τ) be a topological space. Prove that if $A \subseteq X$, the following are equivalent:

(i) $\overline{A} = X$.

(ii) for every $U \in \tau$, if $U \neq \varnothing$, $A \cap U \neq \varnothing$.

We say A is **dense** in X.

Solution. (i) implies (ii): for any $U \neq \varnothing$, U is an open neighbourhood of any of its points $x \in U$ which are also points in $X = \overline{A}$ hence $x \in U \cap A \neq \varnothing$.

(ii) implies (i): for every $x \in X$, let U be an open neighbourhood of x. This means $U \neq \varnothing$, hence $U \cap A \neq \varnothing$, thus $x \in \overline{A}$.

49. (Coursework 2016) Define the following metric on $\mathbb{N}$:

$$\begin{array}{rcl} \mathbb{N} \times \mathbb{N} & \xrightarrow{\quad d \quad} & \mathbb{R} \\ (n,m) & \longmapsto & d(n,m) := \begin{cases} 0, & n = m, \\ 3^{-k_{n,m}}, & n \neq m, \end{cases} \end{array}$$

where $k_{n,m}$ is the largest non-negative integer such that $3^{k_{n,m}} \mid n - m$.

(i) (10 Marks) Find a couple of open sets from a sample basis for the topology $\tau_{\mathbb{N}}^{d}$. Deduce a general form for all the elements of such a basis.

(ii) (15 Marks) Find closure, interior and boundary, in $\tau_{\mathbb{N}}^{d}$, for the following subsets of $\mathbb{N}$:

$$S = \{2n : n \in \mathbb{N}\}, \qquad T = \{1, 2, 3, 4\}.$$

(iii) (Bonus 10 Marks) Prove that this is not the discrete topology: $\left(\mathbb{N}, \tau_{\mathbb{N}}^{d}\right) \ncong (\mathbb{N}, \mathcal{P}(\mathbb{N}))$. In other words: not having the cardinality of the continuum (i.e. not "as infinitely many elements" as $\mathbb{R}$), does not preclude from having a wide array of possible topologies.

Solution: Handwritten at the end of this section.

2.9 Exercises

50. In $\mathbb{R}^2$ with the Euclidean topology, determine the interior, closure and boundary of the following sets:

(i) $A = \left\{(x,y) \in \mathbb{R}^2 : y = 0\right\}$.

(ii) $B = \left\{(x,y) \in \mathbb{R}^2 : x > 0, y \neq 0\right\}$.

(iii) $C = A \cup B$.

(iv) $D = \{(x, y) \in \mathbb{R}^2 : y \in \mathbb{Q}\}$.

(v) $E = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\}$.

(vi) $F = \{(x, y) \in \mathbb{R}^2 : x \neq 0, y \leq 1/x\}$.

51. Determine which of the following families are bases or subbases of a Euclidean topology on $\mathbb{R}^2$. In any affirmative case, decide whether it is precisely the Euclidean one.

(i) $\mathcal{B}_1 = \{\text{interiors of rectangles}\} \cup \{\varnothing\}$.

(ii) $\mathcal{B}_2 = \{B_{a,b,r} : a, b, r \in \mathbb{R}\} \cup \{\varnothing\}$ where $B_{a,b,r} = \left\{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 > r^2\right\}$.

(iii) $\mathcal{B}_3 = \{B_{a,b,c,d,e,f} : a, b, c, d, e, f \text{ such that } ad + be = 0\} \cup \{\varnothing\}$ where

$$B_{a,b,c,d,e,f} = \left\{(x, y) \in \mathbb{R}^2 : ax + by + c > 0, dx + ey + f > 0\right\}.$$

52. Give an example of a topological space (X, τ) not associated to any metric.

53. Prove Lemma 2.5.9.

54. Let X be a topological space and $A \subset X$ a subset. Prove that all open subsets of A (in the subspace topology) are open in X if, and only if, A is an open subset of X.

Study the same statement replacing "open" by "closed".

55. Prove Lemma 2.4.9.

56. Let (X, τ) be a topological space and β a basis thereof. Then $\beta \cup V$ is also a basis for every $V \in \tau \setminus \beta$. Ditto for any subset $\sigma \subset \tau$ instead of V.

57.

(i) Prove that for a given metric space (X, d), the set of all limit points of a subset $S \subset X$ is closed.

(ii) Is this true for general topological spaces?

58. Let $X = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. Define $\beta = \{\varnothing\} \cup \{V((x, y), r) : (x, y) \in X\}$, where

$$V((x, y), r) = \begin{cases} B_{d_2}((x, y), r), & y > r, \\ B_{d_2}((x, y), y) \cup \{\text{tangency point}\}, & \text{otherwise} \end{cases}$$

(i) Prove β is a basis for a topology on X.

(ii) Can the resulting topology be compared to the subspace Euclidean topology on X? Answer in detail.

59. A subset S of a topological space X is **nowhere dense** if the interior of the closure of S is equal to $\varnothing$.

(i) Prove that if S is open ($S = \mathring{S}$), then ∂S is a nowhere dense set.

(ii) Is the previous statement true if we replace "open" by "closed"?

(iii) Is it true for general subsets S of a topological space, which are not necessarily open or closed?

(iv) Is the following statement true? *A subset S is nowhere dense iff the complement of its closure is dense in X.*

(1) $X = \mathcal{C}([a,b])$ $a<b$

Exercise 41

* $d(f,g) = \max_{x\in[a,b]} |f(x)-g(x)|$ well-defined on account of Weierstrass theorem: $h(x) = |f(x)-g(x)|$ continuous function (being a composition of continuous functions) defined on a closed & bounded interval

$\Rightarrow \exists \max_{x\in[a,b]}, \min_{x\in[a,b]} h(x) \in \mathbb{R}$.

* $D(f,g) = \int_a^b |f(x)-g(x)|\,dx$ well-defined

because any continuous function defined on a closed & bounded interval is integrable.

Both are easily proven to be metrics on X.

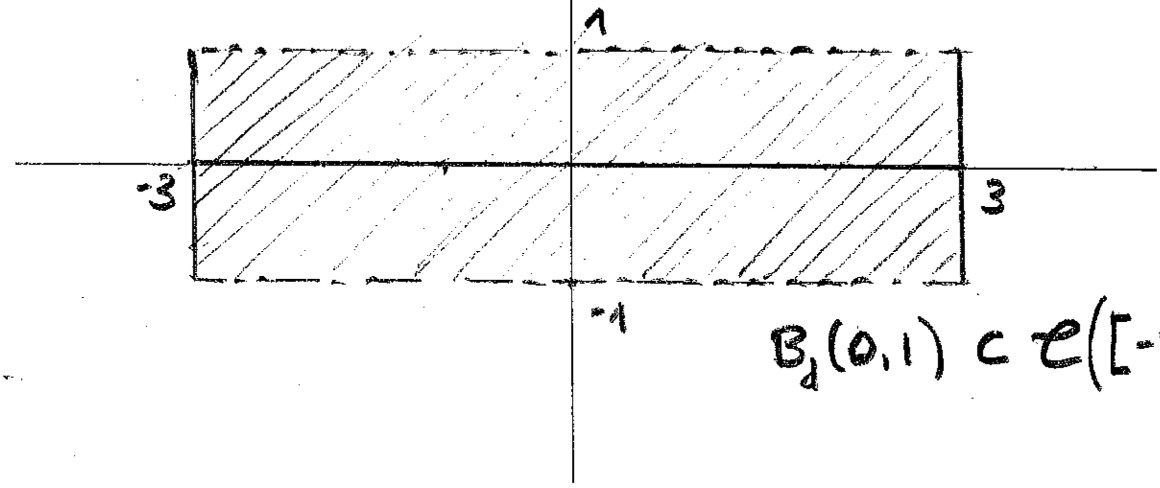
EXERCISE: prove it.

The MIN-MAX INEQUALITY FOR DEFINITE INTEGRALS will come handy: $\forall h \in \mathcal{C}[a,b]$,

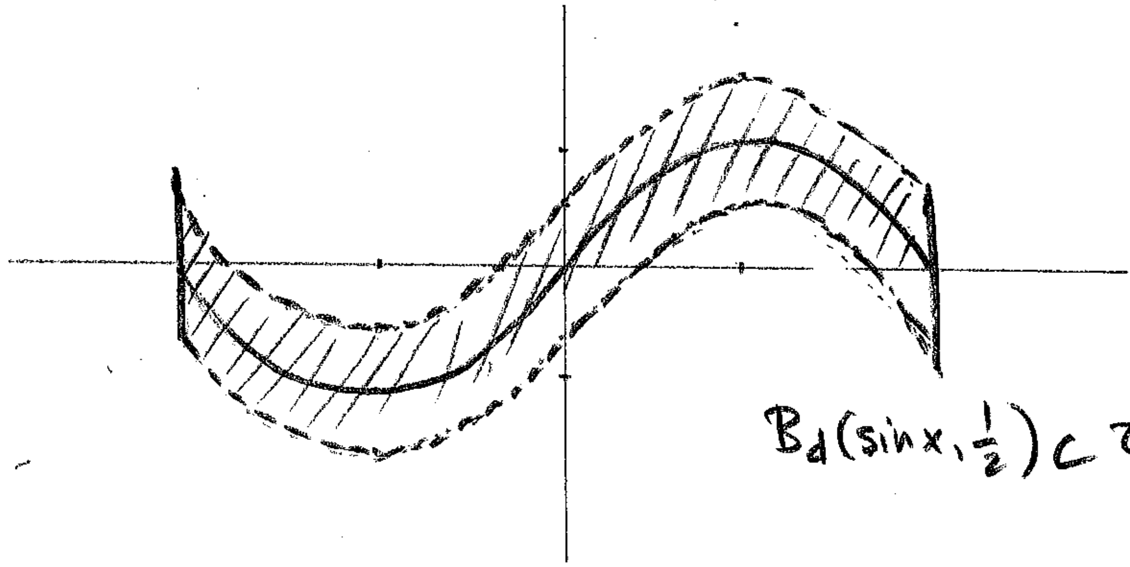
$$(b-a)\min_{x\in[a,b]} h(x) \leq \int_a^b h(x)\,dx \leq (b-a)\max_{x\in[a,b]} h(x)$$

hence $\boxed{D(f,g) \leq (b-a)\, d(f,g) \quad \forall f,g\in\mathcal{C}([a,b])}$ (*)

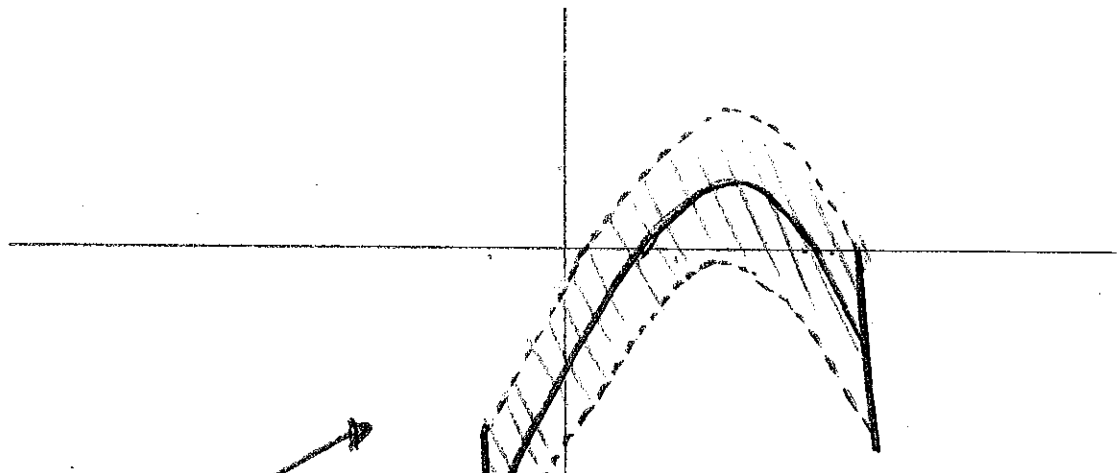
i) Open balls for d: graph $\pm$ vertical displacement



$B_d(0,1) \subset \mathcal{C}([-3,3])$.

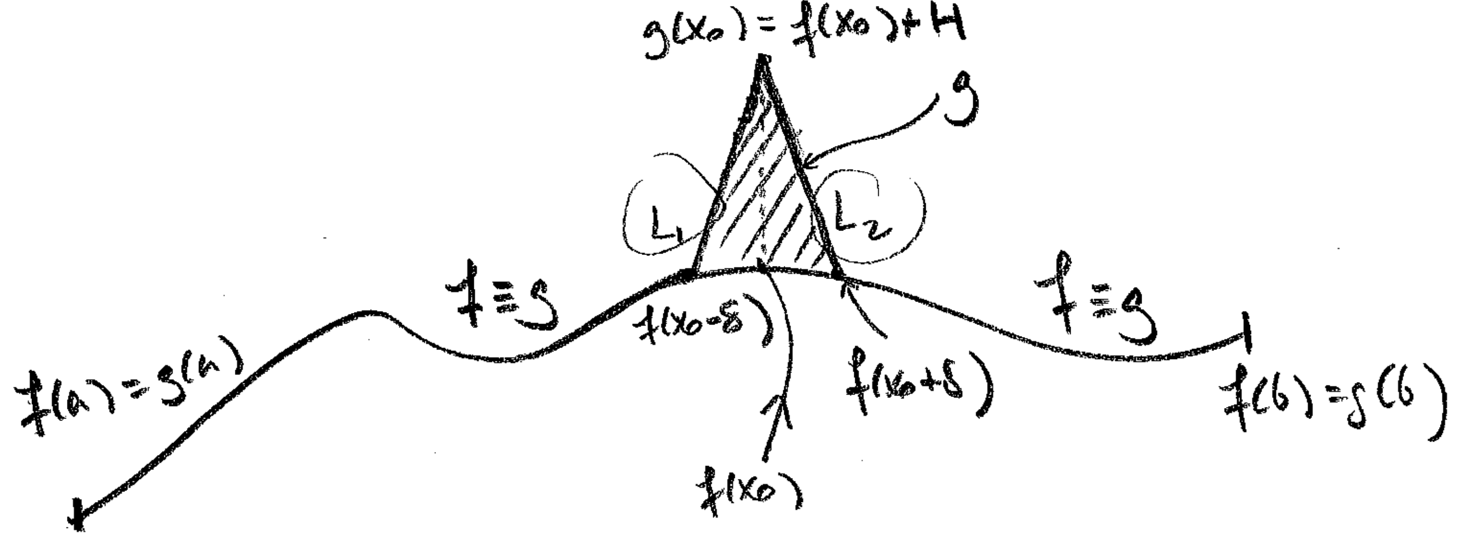


$B_d(\sin x, \frac{1}{2}) \subset \mathcal{C}[-\pi,\pi]$



$B(-(x-1)(x-3), 1)$
$\subset \mathcal{C}[-1,4]$

All open balls for D are equal (fixing $a<b$) to $[a,b]\times\mathbb{R}$. Indeed, let $f\in\mathcal{C}[a,b]$ and consider $g\in\mathcal{C}[a,b]$ defined by "pinching" f in a vicinity of $x_0\in[a,b]$, and equal to f elsewhere:



The integral $\int_a^b |f(x)-g(x)|\,dx$ equals the area of the shaded "triangle", which is

$$A_{H,\delta,x_0}=\int_{x_0-h}^{x_0}\left[L_1(x)-f(x)\right]dx+\int_{x_0}^{x_0+h}\left[L_2(x)-f(x)\right]dx,$$

where L_1, L_2 are the linear "tent" functions above

$$L_1(x)=\frac{(x-x_0+\delta)\left[f(x_0)+H\right]-(x-x_0)f(x_0-\delta)}{\delta}$$

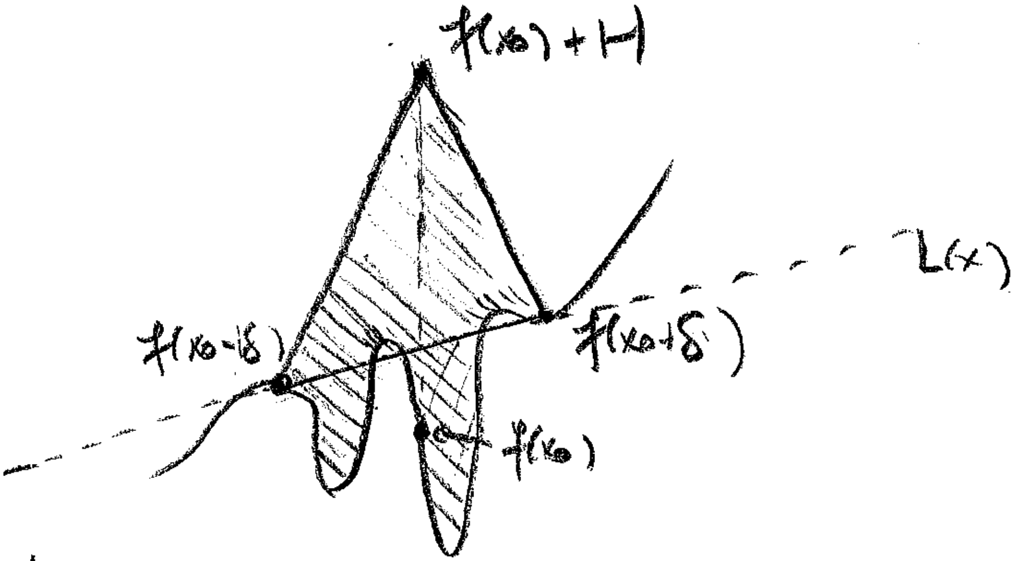
$$L_2(x)=\frac{(-x+x_0+\delta)\left[f(x_0)+H\right]+(x-x_0)f(x_0+\delta)}{\delta}$$

(Note: we are assuming $x_0\in(a,b)$ and $H>0$, for x_0 in one of the extremes or $H<0$ the result is similar and is left as an ~~EXERCISE~~ EXERCISE)

We want to prove: $\forall x_0 \in [a,b]$, $\forall H > 0$, $\forall \varepsilon > 0$
$\exists \delta > 0$: $A_{H,\delta,x_0} \leq \varepsilon$

In other words: regardless of height H, we can always adjust width 2δ (or δ if $x_0 = a$ or $x_0 = b$) so that the area remains bounded by ε.

this is obvious for real triangles, needs just a bit more work for this one.



it is clear that

$$A_{H,\delta,x_0} \leq A_1 + A_2$$

where $A_1 = \frac{\delta}{2} \left| f(x_0+\delta) - 2f(x_0) + f(x_0-\delta) - 2H \right|$

is the area of the triangle formed by $f(x_0)$, $f(x_0-h)$, $f(x_0+h)$,

and $A_2 = \displaystyle\int_{x_0-\delta}^{x_0+\delta} |L(x) - f(x)|\, dx$

is the area between $f(x)$ $(x \in [x_0-\delta, x_0+\delta])$ and the line joining $f(x_0-\delta)$ and $f(x_0+\delta)$

We can ignore A_2 because

$$\int_{x_0-\delta}^{x_0+\delta} |L(x) - f(x)|\,dx \underset{\uparrow}{\leq} 2\delta \max_{x\in[x_0-\delta,x_0+\delta]} |L(x)-f(x)|$$

(*) first page

and the latter quantity can be bounded by Weierstrass th.

$$2\delta \max |L(x) - f(x)| =: \underline{2\delta m} \in \mathbb{R}_+$$

hence $$A_{H,\delta,x_0} \leq \underbrace{A_1}_{\substack{\text{we need}\\ \text{to bound this}}} + 2\delta m$$

A crude, yet effective bound would be, defining $\underline{M := \max_{x\in[a,b]} |f(x)|}$,

$$\begin{aligned} A_1 &= \frac{\delta}{2} \left| f(x_0+\delta) - 2f(x_0) + f(x_0-\delta) - 2H \right| \\ &\leq \frac{\delta}{2} \left[|f(x_0+\delta)| + 2|f(x_0)| + |f(x_0-\delta)| + 2H \right] \\ &\leq \frac{\delta}{2} \left[4M + 2H \right] = 2\delta M + H\delta \end{aligned}$$

$$\Rightarrow \quad A_{H,\delta,x_0} \leq 2\delta M + H\delta + 2\delta m$$

and if we major $\underline{m}$ by the uniform bound M (over all $[a,b]$), then

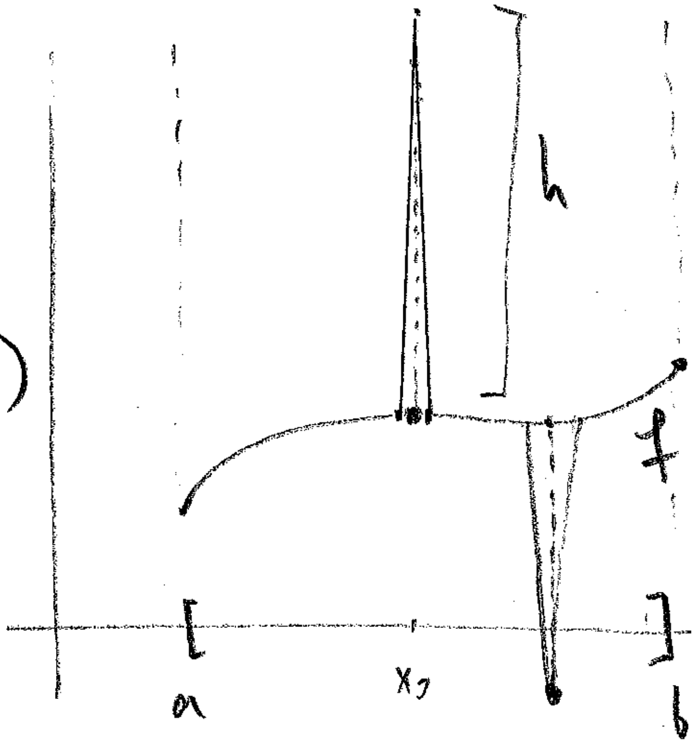
$A_{H,\delta,x_0} \leq \delta(4M+H)$ Hence choosing

$$\boxed{\delta := \frac{\varepsilon}{4M+H}}$$

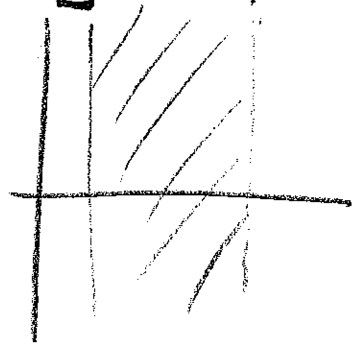
we can ensure the area of the pinched region is $\leq \varepsilon$.

Since $H>0$ can be <u>any</u> positive number and x_0 can be <u>any</u> element in $[a,b]$, there is no vertical bound on the pinched graphs which will always belong to $\underline{B_D(f,\varepsilon)}$

Same applies pinching from below $(H<0)$



This means $B_D(f,\varepsilon) = [a,b] \times \mathbb{R}$, the entire vertical region:



(iii) Judging from the previous item, it is clear $\tau_D \subset \tau_d$ but $\tau_d \not\subset \tau_D$.

* $\underline{\tau_D \subset \tau_d}$: intuitively we have the previous open balls for d, D.

<u>Another way</u>: if we ignore the figures, let $f \in \mathcal{C}[a,b]$ and $\varepsilon > 0$. Then defining $\underline{\delta = \frac{\varepsilon}{b-a}}$,

let us prove $\underline{B_d(f,\delta) \subset B_D(f,\varepsilon)}$.

Indeed, $\forall g \in B_d(f,\delta)$, $\max_{x\in[a,b]} |f(x)-g(x)| < \delta = \frac{\varepsilon}{b-a}$,

hence

$$D(f,g) = \int_a^b |f(x)-g(x)|\,dx \underset{(*)}{\leq} (b-a) \max_{[a,b]} |f-g|$$

$$< (b-a)\frac{\varepsilon}{b-a} = \underline{\varepsilon}$$

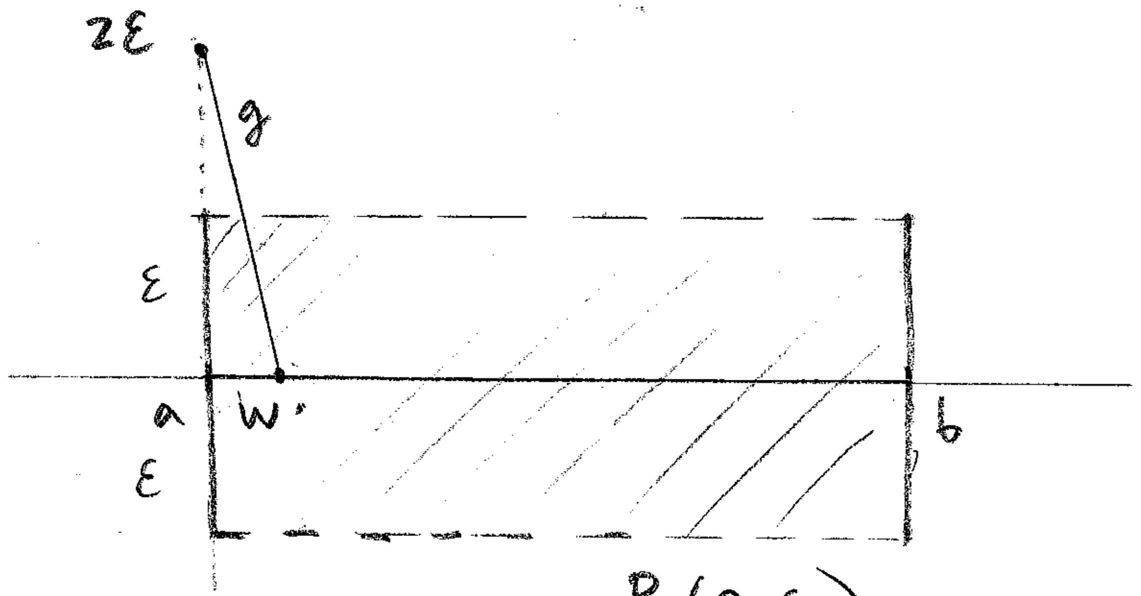
$\Rightarrow g \in B_D(f,\delta)$

To prove $\tau_d \not\subset \tau_D$ we may either resort to the general form of the open balls in (i) or use the following example. Let $f \equiv 0$ be defined on $[a,b]$. Let $\varepsilon > 0$. Our purpose is to prove that no D-ball is contained in the d-ball of radius ε:

$$B_D(f,\delta) \not\subset B_d(f,\varepsilon) \quad \forall \delta > 0$$

Indeed, let us build $g \in \mathcal{C}[a,b]$ such that

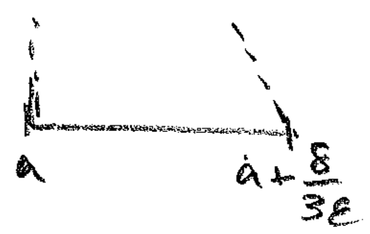
$$g \in B_D(0,\delta) \setminus B_d(0,\varepsilon):$$



$B_d(0,\varepsilon)$

Let us fix w so that $g \in B_D(0,\delta)$: the area of the triangle is $\frac{3\varepsilon w}{2}$, hence we need e.g. $\frac{3\varepsilon w}{2} < \frac{\delta}{2} < \delta \Leftrightarrow w < \frac{\delta}{3\varepsilon}$

hence we can choose this width



and the function

$$g(x) = \begin{cases} \frac{2\varepsilon}{\delta}\left(\delta - 3\varepsilon(a-x)\right), & x \in [a, \min(a+\delta, b)] \\ 0 & x > \min(a+\delta, b) \end{cases}$$

belongs to $B_D(0, \delta)$ but does not belong to $B_d(0, \varepsilon)$.

Hence both topology can be compared but are not equivalent :

$$\begin{cases} \tau_D \subset \tau_d \quad (\tau_d \text{ finer}) \\ \tau_d \not\subset \tau_D \end{cases}$$

1) i) Easiest choice: basis of open balls for the metric:

$$\beta = \{ B_d(n, \varepsilon) : n \in \mathbb{N}, \varepsilon > 0 \} \cup \{ \emptyset \}.$$

Exercise 49

For example:

$$B\left(1, \tfrac{1}{10}\right) = \{ n \in \mathbb{N} : d(n,1) < \tfrac{1}{10} \}$$
$$= \{ n \in \mathbb{N} : d(n,1) \leq \tfrac{1}{3^3} \}$$
$$= \{ n \in \mathbb{N} : 3^3 \mid n-1 \}$$
$$= \{ 27k + 1 : k \in \mathbb{N} \}$$
$$= 27\mathbb{N} + 1 \quad \left(= \{1, 28, 55, \dots\} \right)$$

easy source of mistakes

$$B\left(4, \tfrac{1}{75}\right) = \{ n \in \mathbb{N} : d(n,4) < \tfrac{1}{75} \}$$
$$= \{ n \in \mathbb{N} : d(n,4) = 0 \text{ or } \in \{ \dots, \tfrac{1}{3^5}, \tfrac{1}{3^4} \} \}$$
$$= \{ n \in \mathbb{N} : 81 \mid n-4 \}$$
$$= 81\mathbb{N} + 4 \quad \left(= \{4, 85, 166, \dots\} \right)$$

The pattern is clear:

$$B(n, \varepsilon) = 3^k \mathbb{N} + n \quad \text{where}$$
$$k = \min\{ m : 3^{-m} \leq \varepsilon \}.$$

Not only that: for any $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$\left.\begin{array}{l} B_d(n, \frac{\varepsilon}{2}) \subset B_d(n, \varepsilon) \\ \frac{1}{2\cdot 3^j} \leq \frac{\varepsilon}{2} \Rightarrow \frac{1}{3^{j+1}} \leq \frac{\varepsilon}{2} \end{array}\right\} \Rightarrow \begin{array}{l} B_d(n, \frac{\varepsilon}{2}) = 3^{j+1}\mathbb{N} + n \\ \text{if} \\ B_d(n, \varepsilon) = 3^j \mathbb{N} + n \end{array}$$

<u>Hence</u>: if $B_d(n, \varepsilon) = 3^j \mathbb{N} + n$ for some $j \geq 0$,

then $3^\ell \mathbb{N} + n \subset B_d(n, \varepsilon) \quad \forall \ell \geq j$.

(unsurprisingly)

ii) The interior for S will be $\emptyset$. Indeed, elements of $\mathring{S}$ must be those $2k \in S$ for which $\exists \varepsilon > 0$: $B_d(2k, \varepsilon) \subset S$.

But $B_d(2k, \varepsilon) = 3^j \mathbb{N} + n$ for some $j \geq 0$ and asking for this set to be totally contained in S leads to a contradiction, e.g.

$3^j + 2k$ is an odd number,

hence $\notin S$.

---

thus $\mathring{S} = \emptyset$

$\mathring{T} = \emptyset$ as well: open balls are <u>infinite</u> sets of natural numbers and thus there is no way that any ball containing 1, 2, 3 or 4 can be contained in the finite set $\{1,2,3,4\}$.

Let us find $\overline{S}$. Remember:

$$n \in \overline{S} \iff \forall \underbrace{U}_{} \in \tau_{\mathbb{N}}^{d} : n \in U, \quad U \cap S \neq \emptyset.$$

which can be chosen to be a ball for d

Let $n \in \mathbb{N}$. Our purpose is to prove $n \in \overline{S}$, i.e. for every ball $B_d(n, \varepsilon)$, $B_d(n, \varepsilon) \cap S \neq \emptyset$.
In other words: every ball $B_d(n, \varepsilon)$ contains at least an even number.

But this is evident from the fact that 3-adic balls are $3^{\ell}\mathbb{N} + n$ whenever centered at n, and

$$B = 3^{\ell}\mathbb{N} + n = \{3^{\ell}k + n : k \in \mathbb{N}\} \quad (\ell \geq 0)$$

$$\left.\begin{array}{l} \text{- if } n \text{ is odd then } 3^{\ell} + n \text{ is even} \\ \text{- if } n \text{ is even, so is } 2 \cdot 3^{\ell} + n \end{array}\right\} \Rightarrow B \cap S \neq \emptyset$$

Hence: $\underline{\overline{S} = \mathbb{N}}$.

And $\underline{\partial S = \overline{S} \setminus \mathring{S} = \mathbb{N}}$

Let us prove $\overline{T} = T$. Obviously $T \subset \overline{T}$.

Let $n \notin T$. Then all we have to do is find an open ball $B_d(n, \varepsilon)$ not intersecting T.

This open ball will be of the form

$$B = 3^j \mathbb{N} + n \quad \text{for some } j \geq 0.$$

All we have to find is j large enough so that

$$3^j k + n \notin \{1, 2, 3, 4\} \quad \forall k \in \mathbb{N}.$$

And that is easy: $j = 2$, for instance, yields

$$3^j \mathbb{N} + n = \{9 + n, 18 + n, \dots\}$$

whose minimum is already larger than 4 regardless of n.

Hence: $\mathbb{N} \setminus T$ open $\Rightarrow$ T closed

$\Rightarrow$ $\overline{T} = T$

And $\partial T = T$

iii) Can there be a homeomorphism

$$f : (\mathbb{N}, \tau^d_{\mathbb{N}}) \longrightarrow (\mathbb{N}, P(\mathbb{N})) \quad ?$$

It would have to be a continuous function:

$$\forall U \in P(\mathbb{N}) \Rightarrow f^{-1}(U) \in \tau^d_{\mathbb{N}} .$$

But in $P(\mathbb{N})$ everything is open, e.g. single-element sets $\{n\}$. Hence $f^{-1}(\{n\})$ must be open $\forall n \in \mathbb{N}$.

But f must also be <u>bijective</u> $\Rightarrow f^{-1}(\{n\}) = \{m\}$ for some $m \in \mathbb{N}$.

And $\{m\}$ can never be open in $\tau^d_{\mathbb{N}}$. We have seen that open balls are infinite sets,

$$\underline{3^{\circ}\mathbb{N} + m}$$

and none of those can ever fit into the finite set $\{m\}$.

Thus $(\mathbb{N}, \tau^d_{\mathbb{N}}) \not\cong (\mathbb{N}, P(\mathbb{N}))$

Chapter 3

Continuous maps

In Chapter 1 we introduced the notion of continuity for maps between metric spaces, and characterised said notion in Proposition 1.4.2 as equivalent to the following property: the pre-image of an open set in the second metric space is an open set in the first metric space. This alternative definition is the driving force behind the generalisation of the concept to the wider setting of topological spaces.

3.1 Introduction

Definition 3.1.1. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function between topological spaces. We say f is **continuous** if for every open set $U \in \tau_Y$ of Y the inverse image of U is an open set of X, $f^{-1}(U) \in \tau_X$.*

We can also define continuity at a single point without having to resort to open sets explicitly:

Definition 3.1.2. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function. We say f is **continuous at** $x \in X$ if for every $N \in \mathcal{N}_{f(x)}$, $f^{-1}(N) \in \mathcal{N}_x$. Equivalently, if for every $N \in \mathcal{N}_{f(x)}$ there exists $M \in \mathcal{N}_x$ such that $f(M) \subset N$.*

Examples 3.1.3.

1. Let τ_1 and τ_2 be two topologies on a set X. Function

$$\mathrm{Id} : (X, \tau_1) \to (X, \tau_2), \qquad x \mapsto x,$$

is continuous if, and only if, τ_1 is finer than τ_2. In particular, the inclusion of a subset $Y \subset X$ into X,

$$i : Y \hookrightarrow X, \qquad y \mapsto y,$$

the topology on Y being the subspace topology induced by X, is continuous.

2. If (X, τ_X) is a topological space and τ_Y is the coarse topology on a set Y, then any function $f : X \to Y$ will be continuous.

3. If (Y, τ_Y) is a topological space and the topology considered on a set X is the discrete topology, then any function $X \to Y$ will be continuous as well.

4. If τ is the finite complement topology on $\mathbb{R}$, then $\mathrm{id} : \left(\mathbb{R}, \tau_{\mathbb{R}}^{\mathrm{Eucl}}\right) \to (\mathbb{R}, \tau)$ is continuous but $\mathrm{id} : (\mathbb{R}, \tau) \to \left(\mathbb{R}, \tau_{\mathbb{R}}^{\mathrm{Eucl}}\right)$ is not.

5. Constant functions $X \to Y$, $x \mapsto a \in Y$, are continuous. Indeed, the pre-image of any element $y \in Y$ is either X or $\varnothing$ depending on whether or not $y = a$, hence the pre-image of any open subset $U \subset Y$ is equal to the union of pre-images of elements $y \in Y$ and thus equal to X or $\varnothing$ once again, both of them open sets of X.

6. Affine and linear functions

$$f : \mathbb{R}^n \to \mathbb{R}^k, \qquad \boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b},$$

(where A is a matrix, or a number if $n = k = 1$, and $\boldsymbol{b}$ is a vector, or a number if $k = 1$) are trivially continuous. See Exercise 70.

7. Products and sums of continuous functions $X \to \mathbb{R}$ are continuous. Quotients of such functions are continuous wherever the denominator does not vanish. Hence polynomial functions and rational functions $p(x)/q(x)$ (domain adjusted to nonvanishing denominators) are continuous.

3.2 Properties

Finding out whether or not a function is continuous can be done with the aid of bases.

Proposition 3.2.1. *Let $f : X \to Y$ be a function between topological spaces and β a basis for the topology on Y. Then f is continuous if, and only if, for every basis element $V \in \beta$, $f^{-1}(V)$ is an open set in X.*

Proof. The direct implication is immediate from the definition. Let us prove the converse implication. Assume the pre-image of every basis open set V in Y is an open set in X. Let U be an open set in Y. The definition of basis entails $U = \bigcup_{i \in I} V_i$ for some collection $\{V_i : i \in I\}$ of elements in β. Hence set theory implies $f^{-1}(U) = \bigcup_{i \in I} f^{-1}(V_i)$, a union of open sets in X and thus an open set in and of itself. □

Another useful way of characterising continuous functions is using closed sets instead of open sets.

Proposition 3.2.2. *A map between topological spaces is continuous if, and only if, the pre-image of a* closed *subset is* closed.

Proof. We have the basic set-theoretical property $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ whenever $C \subset Y$ is closed. □

Proposition 3.2.3. *The composition of continuous maps is continuous.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between topological spaces. Let $U \subset Z$ be an open set in Z. Then $g^{-1}(U)$ is an open set of Y in virtue of the continuity of g, hence $(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right)$ is an open set of X in virtue of the continuity of f. □

Proposition 3.2.4. *Let $f : X \to Y$ be a function between topological spaces. Let $Z := f(X) \subset Y$ with the subspace topology induced by Y. We define $\tilde{f} : X \to Z$ by restricting the range on f: $\tilde{f}(x) := f(x)$ for every $x \in X$. Then:*

a) f is continuous if, and only if, $\tilde{f}$ is.

b) If f is continuous and $A \subset X$ is a subset, then considering A with its subspace topology induced by X, $f|_A : A \to Y$ is continuous.

Proof. a) Assume $\tilde{f}$ is continuous. So is the inclusion $i_Z : Z \hookrightarrow Y$, hence $f = i \circ \tilde{f}$ is continuous.

Conversely, assume f is continuous. Let $U \subset Z$ be an open set of Z. In virtue of the definition of the subspace topology, there exists an open set $V \subset Y$ such that $U = V \cap Z$. Hence $\tilde{f}^{-1}(U) = \tilde{f}^{-1}(V \cap Z) = f^{-1}(V \cap Z) = f^{-1}(Z)$ which is open on account of f being continuous.

b) Restriction $f|_A$ is obtained by means of the composition of inclusion $i_A : A \hookrightarrow X$ and f, both of them continuous. Proposition 3.2.3 provides the rest.

□

3.3 Homeomorphisms

We may now decide, within the context (read: category) of topological spaces, what it means for two such spaces to be "equivalent".

Definition 3.3.1. *A function between two topological spaces X and Y is called a* ***homeomorphism*** *if it is bijective, continuous and has a continuous inverse. X and Y are called* ***homeomorphic*** *if there exists at least a homeomorphism between them, and we will write $X \cong Y$ in such case.*

Remark 3.3.2. The continuous inverse condition is a necessary part of the definition if we want spaces to be equivalent in the sense of "deformable into one another". For instance, defining $X = (0,1]$, $Y = \mathbb{S}^1$, $f : X \to Y$ defined by $t \mapsto e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t))$ is continuous and bijective. The latter fact implies $f(U) = \left(f^{-1}\right)^{-1}(U)$ for every open set U of X, yet $\left(f^{-1}\right)^{-1}\left(\left(\frac{1}{2}, 1\right]\right)$ is not an open set of $\mathbb{S}^1$, hence f is not a homeomorphism. See Solved Exercise 64 for more details.

Definition 3.3.3. *Let $f : X \to Y$ be a function between topological spaces. We say f is* ***open*** *(resp.* ***closed****) if the image by f of every open (resp. closed) subset of X is an open (resp. closed) subset of Y.*

Proposition 3.3.4. *Let $f : X \to Y$ be a map between topological spaces. The following are equivalent:*

(i) *f is a homeomorphism.*

(ii) *f is bijective, continuous and open.*

(iii) *f is bijective, continuous and closed.*

Proof.

(i) ⇒ (ii) Let U be an open set of X. The fact f is bijective implies $f(U) = \left(f^{-1}\right)^{-1}(U)$. The fact f^{-1} is continuous implies $f(U)$ is an open set in virtue of the previous identity. Hence f is open.

(ii) ⇒ (iii) Assume f is bijective, continuous and open. Let C be a closed set of X. The fact f is bijective implies $f(C) = Y \setminus f(X \setminus C)$. The fact f is open implies $f(X \setminus C)$ is an open set of Y, hence $f(C)$ is closed.

(iii) ⇒ (i) Assume f is bijective, continuous and closed. Let C be a closed set of X. We have $\left(f^{-1}\right)^{-1}(C) = f(C)$ which is a closed set by hypothesis, hence f^{-1} is continuous.

□

A way of checking openness or closedness for maps which is easier than Definition 3.3.3 is the following:

Proposition 3.3.5. *A function $f : (X.\tau_X) \to (Y, \tau_Y)$ is open if, and only if, it maps basis open sets to open sets: for every basis $\beta \subset \tau_X$ of X and every $V \in \beta$, $f(V) \in \tau_Y$.*

Proof. Exercise 81. □

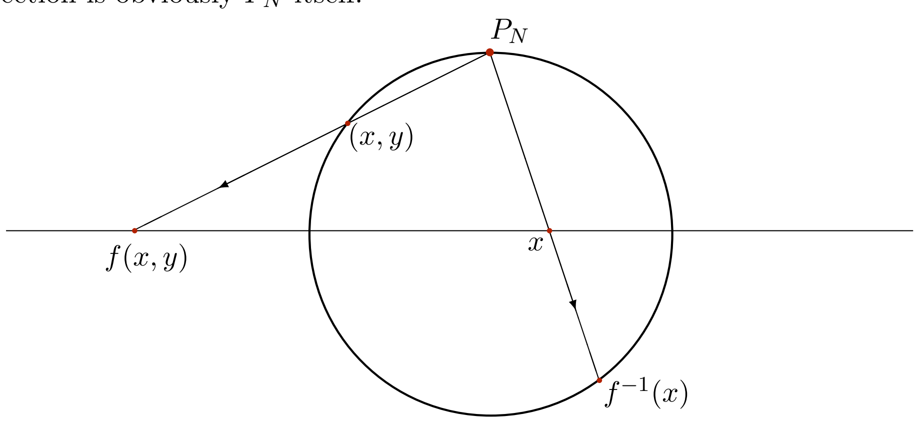
Examples 3.3.6.

1. The identity between a topological space and itself $\mathrm{Id} : (X, \tau) \to (X, \tau)$ is a homeomorphism.

2. Every open interval $(a, b) \subset \mathbb{R}$, $a < b$ with the usual induced Euclidean topology is homeomorphic to the interval $(0, 1)$. Indeed, define
$$f : (0,1) \to (a,b), \qquad x \mapsto bx + (1-x)\,a.$$
It is immediate to check this is a homeomorphism. See Solved Exercise 66 for more details.

3. Let $X = \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\}$ and $Y = \left\{(x, y) \in \mathbb{R}^2 : y = x^2\right\}$. Function
$$f : X \to Y, \qquad (x, 0) \mapsto \left(x, x^2\right),$$
is a homeomorphism.

4. Let $\mathbb{S}^1 \subset \mathbb{R}^2$ be the usual unit circle of centre $(0, 0)$. The **stereographic projection on** $\mathbb{S}^1$ is given by projecting points of $\mathbb{S}^1$ from one of its poles, say the north pole $P_N : (0, 1)$, and intersecting the resulting lines with the x-axis. The only point in $\mathbb{S}^1$ excluded from this projection is obviously P_N itself.



The projection induces a homeomorphism
$$f : \mathbb{S}^1 \setminus P_N \to \mathbb{R}, \qquad (x, y) \mapsto \frac{x}{1-y},$$
having inverse
$$f^{-1} : \mathbb{R} \to \mathbb{S}^1 \setminus P_N, \qquad x \mapsto \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right).$$
Exercise: verify this.

5. There is also a stereographic projection for arbitrary dimension, which is a homeomorphism between $\mathbb{S}^n \setminus (0, \ldots, 0, 1)$, where $\mathbb{S}^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\right\}$ and $\mathbb{R}^n$. Exercise: find it explicitly. See Solved Exercise 82.

The following are easy to prove and are left as an exercise:

Remarks 3.3.7.

1. The countability axioms are invariant by homeomorphism.

2. If $f : X \to Y$ is a homeomorphism and $A \subset X$, then $A \cong f(A)$ and $X \setminus A \cong f(X \setminus A) = Y \setminus f(A)$.

3.4 Continuous functions and coverings

Piecewise definition of continuous functions is a usual necessity in practical examples. This has a topological formulation.

Definition 3.4.1. *Let X be a set. A **covering** of X is a collection $\{X_i\}_{i\in I}$ of subsets of X such that $\bigcup_{i\in I} X_i = X$.*

*If X is a topological space, we call the covering **open** (resp. **closed**) if $X_i = \mathring{X}_I$ (resp. $X_i = \overline{X_i}$) for every $i \in I$.*

Example 3.4.2. A basis for X is an open covering. Taking closures of all elements in a given covering we have a closed covering.

In basic set theory, the problem arises on when to extend the definition of a function from subsets to the whole sets. Given two sets X, Y and a function $f : X \to Y$, for any covering $\{X_i\}_{i\in I}$ of X, define restrictions

$$f_i := f_i|_{X_i} : X_i \to Y, \qquad x \mapsto f(x).$$

The following is a necessary and sufficient condition on the definition of f as a function (continuity issues still not addressed): restrictions should coincide on the intersections,

$$f_i|_{X_i\cap X_j} = f_j|_{X_i\cap X_j}, \qquad \text{for every } i, j \in I. \tag{3.1}$$

In other words, there exists a well-defined function $f : X \to Y$ if and only if (3.1) holds on every covering of X, and in such case f is unique satisfying (3.1).

Necessary and sufficient condition (3.1) becomes only necessary in a topological setting if we want f to be continuous:

Proposition 3.4.3. *Let X, Y be topological spaces and $f : X \to Y$ such that (3.1) holds for a given covering $\{X_i\}_{i\in I}$, and assume f_i is continuous for every $i \in I$. Then,*

a) if the covering is closed and finite, f is continuous;

b) if the covering is open, f is continuous.

Proof. a) Let C be a closed set of Y. We have

$$f^{-1}(C) = \bigcup_{i\in I} \left(f^{-1}(C) \cap X_i\right) = \bigcup_{i\in I} f_i^{-1}(C),$$

which is closed in virtue of the continuity of each f_i, Proposition 3.2.2, the finiteness of I and property F_3 of closed sets.

b) Let U be an open set of Y. Then

$$f^{-1}(U) = \bigcup_{i \in I} \left(f^{-1}(U) \cap X_i \right) = \bigcup_{i \in I} f_i^{-1}(U),$$

an arbitrary union of open sets of X, hence an open set of X.

$\square$

Remark 3.4.4. The finiteness condition on closed coverings is not superfluous. For instance, let $X = [0,1] \subset \mathbb{R}$ with the induced Euclidean topology and $Y = \mathbb{R}$ with the Euclidean topology. Define

$$X_n := \left[\frac{1}{n+1}, \frac{1}{n} \right], \qquad \text{for every } n \in \mathbb{N}, \qquad X_\infty := \{0\}.$$

Define function

$$f : X \to Y, \qquad x \mapsto f(x) := \begin{cases} x & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0. \end{cases}$$

This function is continuous on each one and all of the subsets belonging to the covering, i.e. $f_n := f|_{X_n}$, $n \in \mathbb{N}$ and $f_\infty := f|_{X_\infty}$ are continuous, and $f_n|_{X_n \cap X_m} = f_m|_{X_n \cap X_m}$ for every $n, m \in \mathbb{N} \cup \{\infty\}$. However, f is not continuous.

3.5 Solved Exercises

60. Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a non-constant polynomial and

$$P : \mathbb{R}^n \to \mathbb{R}, \qquad (x_1, \ldots, x_n) \mapsto p(x_1, \ldots, x_n),$$

the corresponding polynomial function. Consider both sets with the Euclidean topology. Prove the set $P^{-1}(0)$ is closed and has an empty interior.

Solution. P is continuous (it is left as an exercise to check this) and $\{0\}$ is closed in $\mathbb{R}$, as are all subsets consisting of only one point. Hence $P^{-1}(0)$ is closed.

Let us now prove $\overset{\circ}{\overbrace{P^{-1}(0)}}$ is empty. This is equivalent to proving that for every $\boldsymbol{a} = (a_1, \ldots, a_n) \in P^{-1}(0)$, i.e. such that $P(a_1, \ldots, a_n) = 0$, and every $\varepsilon > 0$, $B(\boldsymbol{a}, \varepsilon)$ contains points not in $P^{-1}(0)$.

Let us assume $P^{-1}(0) \neq \varnothing$; otherwise $\overset{\circ}{\overbrace{P^{-1}(0)}} \subset P^{-1}(0)$ is also empty and we are finished.

We will prove this by induction over the amount n of variables of the polynomial. For $n = 1$, we have a function $P : \mathbb{R} \to \mathbb{R}$ and we assume $P^{-1}(0) \neq \varnothing$. The amount of elements in $P^{-1}(0)$ will be bounded by $\deg p$, since we know the amount of roots of a polynomial is upwardly bounded by the polynomial's degree. Hence for every element $a \in \mathbb{R}$ such that $P(a) = 0$, and for every $\varepsilon > 0$, there will exist not only one, but infinitely many elements $x \in (a - \varepsilon, a + \varepsilon)$ such that $P(x) \neq 0$: bear in mind $(a - \varepsilon, a + \varepsilon)$ is an infinite set, whereas $P^{-1}(0)$ is finite as said above, hence $P^{-1}(0) \cap (a - \varepsilon, a + \varepsilon) \neq (a - \varepsilon, a + \varepsilon)$.

Assume this is true for polynomials in $n - 1$ or less variables. Let $\varepsilon > 0$ and assume $P : \mathbb{R}^n \to \mathbb{R}$ has a root $\boldsymbol{a} \in P^{-1}(0)$ and expand P as a (finite) Taylor polynomial along a_n, as a function of x_n:

$$P(x_1, \ldots, x_n) = P_0(x_1, \ldots, x_{n-1}) + P_1(x_1, \ldots, x_{n-1})(x_n - a_n) + \cdots + P_k(x_1, \ldots, x_{n-1})(x_n - a_n)^k. \quad (3.2)$$

Assume $P(x_1, \ldots, x_n) \neq 0$. This means there exists $i = 1, \ldots, k$ such that $P_i(x_1, \ldots, x_{n-1}) \neq 0$. Fixing this index i, if $P_i(a_1, \ldots, a_{n-1}) \neq 0$, define $b_1 = a_1, b_2 = a_2, \ldots, b_{n-1} = a_{n-1}$. Otherwise, if $P_i(a_1, \ldots, a_{n-1}) = 0$, using our hypothesis of induction, there exists $\boldsymbol{b} = (b_1, \ldots, b_{n-1}) \in B\left(\boldsymbol{a}, \frac{\varepsilon}{\sqrt{2}}\right)$ such that $P(\boldsymbol{b}) \neq 0$.

$\boldsymbol{b} = (b_1, \ldots, b_{n-1})$ defined as above (depending on whether or not $P_i(a_1, \ldots, a_{n-1}) = 0$), we consider the expression (3.2) of our polynomial, with all its variables fixed except for the last one:

$$P(b_1, \ldots, b_{n-1}, x_n) = \sum_{i=0}^{k} P_i(b_1, \ldots, b_{n-1})(x_n - a_n)^n,$$

where by construction there will be at least one coefficient $P_i(b_1, \ldots, b_n) \neq 0$. Using the same reasoning as in $n = 1$, there must exist a $b_n \in \left(a_n - \frac{\varepsilon}{\sqrt{2}}, a_n + \frac{\varepsilon}{\sqrt{2}}\right)$ such that $P(b_1, \ldots, b_n) \neq 0$.

We have thus built an element $(b_1, \ldots, b_n) \in \mathbb{R}^n$ such that $P(b_1, \ldots, b_n) \neq 0$. We only need to check this element is in the ball $B(\boldsymbol{a}, \varepsilon)$ and we are finished. Indeed,

$$d_{\mathbb{R}^{n-1}}((b_1, \ldots, b_{n-1}), (a_1, \ldots, a_{n-1})) < \frac{\varepsilon}{\sqrt{2}},$$

and $|b_n - a_n| < \frac{\varepsilon}{\sqrt{2}}$, hence

$$d_{\mathbb{R}^n}((b_1, \ldots, b_n), (a_1, \ldots, a_n))^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + \cdots + (b_{n-1} - a_{n-1})^2 + (b_n - a_n)^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2,$$

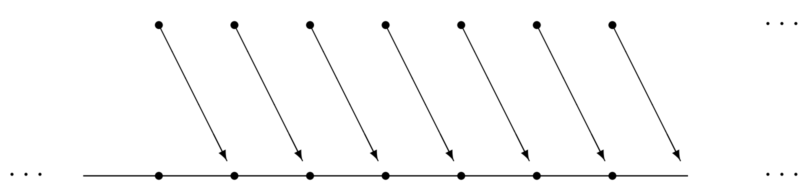
and this implies $d_{\mathbb{R}^n}((b_1, \ldots, b_n), (a_1, \ldots, a_n)) < \varepsilon$.

61. (*Sample Coursework November 2013*) Consider $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ defined by $f(n) := n + 1$ and let τ, τ_1, τ_2 be the topologies defined by

$$\begin{aligned} \tau &:= \{\varnothing, \mathbb{R}, [1, \infty)\} \cup \{[1, x) : x \geq 1\}, \\ \tau_1 &:= \{\varnothing, \mathbb{N} \cup \{0\}\} \cup \{\{0, \ldots, n\} : n \geq 0\}, \\ \tau_2 &:= \{\varnothing, \mathbb{N} \cup \{0\}\} \cup \{\{0, \ldots, n\} : n \geq 1\}, \end{aligned}$$

Study the continuity of $f : (\mathbb{N} \setminus \{0\}, \tau_i) \to (\mathbb{R}, \tau)$ for $i = 1, 2$.

Solution. This is nothing but the shift function with its image embedded in $\mathbb{R}$:



Let us first study the pre-image of an open set in $\mathbb{R}$, since we are only considering one topology in $\mathbb{R}$. We have $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(\mathbb{R}) = \mathbb{N} \cup \{0\}$. In order to find $f^{-1}([1, \infty))$, let us first find the pre-images of the rest of the sets: for every $x \in \mathbb{R}$ such that $x > 1$,

$$f^{-1}([1, x)) = \{n \in \mathbb{N} \cup \{0\} : 1 \leq n + 1 \leq x\} = \{0, \ldots, k\}, \quad \text{where} \quad k = \begin{cases} [x] - 1 & \text{if} \quad x \notin \mathbb{N}, \\ x - 2 & \text{if} \quad x \in \mathbb{N}, \end{cases}$$

hence taking this to the limit $x \to \infty$, $f^{-1}([1, \infty)) = \mathbb{N} \cup \{0\}$.

Let us now study the continuity of the function for the two topologies on $\mathbb{N} \cup \{0\}$:

For τ_1: We have $f : (\mathbb{N} \cup \{0\}, \tau_1) \to (\mathbb{R}, \tau)$. The pre-image of an open subset of $(\mathbb{R}, \tau)$ can only be, as we have seen, of the following types:

$$f^{-1}(U) \in \{\varnothing, \mathbb{N} \cup \{0\}\} \cup \{\{1, \ldots, k\} : k \geq 1\}, \qquad \text{for every } U \in \tau.$$

It is clear all three types of subsets of $\mathbb{N} \cup \{0\}$ belong to τ_1. Hence $f^{-1}(U) \in \tau_1$ for every $U \in \tau$ which means f is continuous.

For τ_2: Now we are considering $f:(\mathbb{N}\cup\{0\},\tau_2)\to(\mathbb{R},\tau)$. Take $U=[1,2)$. The pre-image of $U:=[1,2)$ is $f^{-1}(U)=\{0\}$, which does not correspond to any of the subsets that make up τ_2. Hence f cannot be continuous.

62. In $\mathbb{R}$ with the Euclidean topology we define maps

$$f_1:\mathbb{R}\to\mathbb{R},\qquad x\mapsto\begin{cases} n, & x\in[n,n+1)\cap\mathbb{Q},\\ -(n+1), & x\in[n,n+1)\cap(\mathbb{R}\setminus\mathbb{Q}),\end{cases}$$

and

$$f_2:\mathbb{R}\to\mathbb{R},\qquad x\mapsto\begin{cases} 0, & x\le a \text{ or } x\ge b,\\ (x-a)^2(x-b), & a<x<b,\end{cases}$$

where a,b are fixed real numbers such that $b>a$. Are they continuous?

Solution.

- Let us prove f_1 is not continuous. $(-1/2,1/2)$ is an open subset of $\mathbb{R}$, and its pre-image is

$$f^{-1}((-1/2,1/2))=\{x\in\mathbb{R}:f(x)=0\}=([0,1)\cap\mathbb{Q})\cup([-1,0)\cap(\mathbb{R}\setminus\mathbb{Q})),$$

which is not an open set of $\mathbb{R}$ (you may use the fact that between any two rational numbers there are infinitely many irrational numbers, hence points in $[0,1)\cap\mathbb{Q}$ cannot have open balls centred in them and totally contained in $[0,1)\cap\mathbb{Q}$).

- Let us prove f_2 is continuous. Consider the finite covering by closed sets

$$\mathbb{R}=(-\infty,a]\cup[a,b]\cup[b,\infty).$$

Restricted functions $f|_{(-\infty,a]}$ and $f|_{[b,\infty)}$ are trivially continuous (they are the constant function $\equiv 0$). Function $f|_{[a,b]}$ is continuous as well, being the product of linear (hence continuous) functions $(x-a)\cdot(x-a)\cdot(x-b)$. All we need to check is an agreement of these restrictions on the pairwise intersections of the covering: $(-\infty,a]\cap[a,b]=\{a\}$ and $[b,\infty)\cap[a,b]=\{b\}$. We have $f|_{(-\infty,a]}(a)=0=f|_{[a,b]}(a)$ and $f|_{[b,\infty)}(b)=f|_{[a,b]}(b)=0$, hence restrictions do coincide in the intersections. We are clearly in the hypotheses of Proposition 3.4.3 (a) and the function is thus continuous.

63. Let $X=\mathbb{R}^n$ and d,d' two distances associated to two norms $\|\cdot\|,\|\cdot\|'$ on X. Prove that if d and d' are equivalent distances, then balls $B_d(\mathbf{0},1)$ and $B_{d'}(\mathbf{0},1)$ are homeomorphic.

Solution. We need to build a homeomorphism

$$f:B_d(\mathbf{0},1)=\{\boldsymbol{x}:\|\boldsymbol{x}\|<1\}\xrightarrow{\cong}B_{d'}(\mathbf{0},1)=\left\{\boldsymbol{x}:\|\boldsymbol{x}\|'<1\right\}.$$

Define $f(\boldsymbol{x}):=\frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|'}\boldsymbol{x}$ whenever $\boldsymbol{x}\neq\mathbf{0}$; it is well defined, using one of the basic properties of a norm (i.e. $\|\lambda\boldsymbol{x}\|=|\lambda|\,\|\boldsymbol{x}\|$):

$$\left\|\frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|'}\boldsymbol{x}\right\|'=\frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|'}\|\boldsymbol{x}\|'=\|\boldsymbol{x}\|<1,\quad\text{for every }\boldsymbol{x}\in B_d(\mathbf{0},1)\setminus\{\mathbf{0}\},$$

and its inverse is $g(\boldsymbol{y}):=\frac{\|\boldsymbol{y}\|'}{\|\boldsymbol{y}\|}\boldsymbol{y}$ for $\boldsymbol{y}\neq\mathbf{0}$ which renders both f and g bijective. Both functions are continuous for vectors different from $\mathbf{0}$. Extend both of them to $\mathbf{0}$ by $f(\mathbf{0})=\mathbf{0}$ and $g(\mathbf{0})=\mathbf{0}$ and we obtain continuous functions. Indeed, let $\varepsilon>0$ and $\boldsymbol{x}\in\mathbb{R}^n\setminus\{0\}$. We want to find $\delta>0$ such that for every $\boldsymbol{y}\in\mathbb{R}^n$ such that $||\boldsymbol{x}-\boldsymbol{y}||<\delta$, $||f(\boldsymbol{x})-f(\boldsymbol{y})||'<\varepsilon$. But

$$||f(\boldsymbol{x})-f(\boldsymbol{y})||'=\left\|\frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|'}\boldsymbol{x}-\frac{\|\boldsymbol{y}\|}{\|\boldsymbol{y}\|'}\boldsymbol{y}\right\|'=\left\|\frac{\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|'\boldsymbol{x}-\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|\,\boldsymbol{y}}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'}\right\|'=\frac{\left\|\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|'\boldsymbol{x}-\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|\,\boldsymbol{y}\right\|'}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'}$$

which is smaller than or equal to (using triangle inequality)

$$\frac{\left|\left|\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|'\,\boldsymbol{x}-\|\boldsymbol{y}\|'\,\|\boldsymbol{y}\|\,\boldsymbol{x}\right|\right|'+\left|\left|\|\boldsymbol{y}\|\,\|\boldsymbol{y}\|'\,\boldsymbol{x}-\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|\,\boldsymbol{y}\right|\right|'}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'}=\|\boldsymbol{y}\|'\,\frac{||(\|\boldsymbol{x}\|-\|\boldsymbol{y}\|)\,\boldsymbol{x}||'}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'}+\frac{\left|\left|\|\boldsymbol{y}\|\,\|\boldsymbol{y}\|'\,\boldsymbol{x}-\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|\,\boldsymbol{y}\right|\right|'}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'},$$

itself smaller than or equal to

$$\|\boldsymbol{y}\|'\,\|\boldsymbol{x}\|'\,\frac{|\|\boldsymbol{x}\|-\|\boldsymbol{y}\||}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'}+\|\boldsymbol{y}\|\,\frac{\left|\left|\|\boldsymbol{y}\|'\,\boldsymbol{x}-\|\boldsymbol{x}\|'\boldsymbol{y}\right|\right|'}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'}=|\|\boldsymbol{x}\|-\|\boldsymbol{y}\||+\|\boldsymbol{y}\|\,\frac{\left|\left|\|\boldsymbol{y}\|'\,\boldsymbol{x}-\|\boldsymbol{x}\|'\boldsymbol{y}\right|\right|'}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'}\leq||\boldsymbol{x}-\boldsymbol{y}||+\|\boldsymbol{y}\|\,\frac{\left|\left|\|\boldsymbol{y}\|'\,\boldsymbol{x}-\|\boldsymbol{x}\|'\boldsymbol{y}\right|\right|'}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'},$$

which is smaller than or equal to

$$||\boldsymbol{x}-\boldsymbol{y}||+\frac{\|\boldsymbol{y}\|\left(\left|\left|\|\boldsymbol{y}\|'\,\boldsymbol{x}-\|\boldsymbol{x}\|'\boldsymbol{x}\right|\right|'+\left|\left|\|\boldsymbol{x}\|'\,\boldsymbol{x}-\|\boldsymbol{x}\|'\boldsymbol{y}\right|\right|'\right)}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'}\leq||\boldsymbol{x}-\boldsymbol{y}||+\frac{\|\boldsymbol{y}\|\left(\|\boldsymbol{x}\|'\,||\boldsymbol{y}-\boldsymbol{x}||'+\|\boldsymbol{x}\|'\,||\boldsymbol{x}-\boldsymbol{y}||'\right)}{\|\boldsymbol{x}\|'\,\|\boldsymbol{y}\|'},$$

for the same reasons as the previous inequality. Hence we have

$$||f(\boldsymbol{x})-f(\boldsymbol{y})||'\leq||\boldsymbol{x}-\boldsymbol{y}||+2\frac{\|\boldsymbol{y}\|}{\|\boldsymbol{y}\|'}\,||\boldsymbol{x}-\boldsymbol{y}||$$

We are told d and d' are equivalent: there exist C_1, C_2 such that $C_1 d(\boldsymbol{x},\boldsymbol{y})\leq d'(\boldsymbol{x},\boldsymbol{y})\leq C_2 d(\boldsymbol{x},\boldsymbol{y})$. This entails $C_1\,||\boldsymbol{y}||\leq||\boldsymbol{y}||'\leq C_2\,||\boldsymbol{y}||$ hence $\frac{||\boldsymbol{y}||}{||\boldsymbol{y}||'}\leq\frac{1}{C_1}$. Thus

$$||f(\boldsymbol{x})-f(\boldsymbol{y})||'\leq||\boldsymbol{x}-\boldsymbol{y}||+2\frac{||\boldsymbol{y}||}{||\boldsymbol{y}||'}\,||\boldsymbol{x}-\boldsymbol{y}||'\leq||\boldsymbol{x}-\boldsymbol{y}||+\frac{2C_2}{C_1}\,||\boldsymbol{x}-\boldsymbol{y}||=\left(1+\frac{2C_2}{C_1}\right)||\boldsymbol{x}-\boldsymbol{y}||$$

which will be smaller than ε, for instance, defining $\delta=\frac{\varepsilon}{1+\frac{2C_2}{C_1}}$.

For $\boldsymbol{x}=\boldsymbol{0}$,

$$||f(\boldsymbol{x})-f(\boldsymbol{y})||'=||f(\boldsymbol{y})||'=\left|\left|\frac{||\boldsymbol{y}||}{||\boldsymbol{y}||'}\cdot\boldsymbol{y}\right|\right|=\frac{||\boldsymbol{y}||}{||\boldsymbol{y}||'}\,||\boldsymbol{y}||'=||\boldsymbol{y}||=||\boldsymbol{x}-\boldsymbol{y}||\,,$$

hence we can choose $\delta=\varepsilon$.

64. Prove that the function

$$f:[0,1)\to\mathbb{S}^1,\qquad t\mapsto(\cos 2\pi t,\sin 2\pi t)\,,$$

is continuous and bijective, but is not a homeomorphism.

Solution. Let us prove it is continuous. *We will use results 4.1.5 and 4.1.11 from the next Chapter*. We can compose f with the inclusion of $\mathbb{S}^1$ into $\mathbb{R}^2$,

$$i\circ f:[0,1)\to\mathbb{S}^1\hookrightarrow\mathbb{R}^2,\qquad t\mapsto(\cos 2\pi t,\sin 2\pi t)\mapsto(\cos 2\pi t,\sin 2\pi t)\,,$$

which in turn can be composed with each of the projections

$$p_1:\mathbb{R}^2\to\mathbb{R},\qquad(x,y)\mapsto x,$$

and

$$p_2:\mathbb{R}^2\to\mathbb{R},\qquad(x,y)\mapsto y.$$

The topology on $\mathbb{R}^2$ is the product topology of the topology on $\mathbb{S}^1$ by itself, hence the initial topology with respect to p_1 and p_2. The fact functions

$$p_1\circ(i\circ f):[0,1)\to\mathbb{R},\qquad t\mapsto\cos 2\pi t,$$

and

$$p_2 \circ (i \circ f) : [0,1) \to \mathbb{R}, \qquad t \mapsto \sin 2\pi t,$$

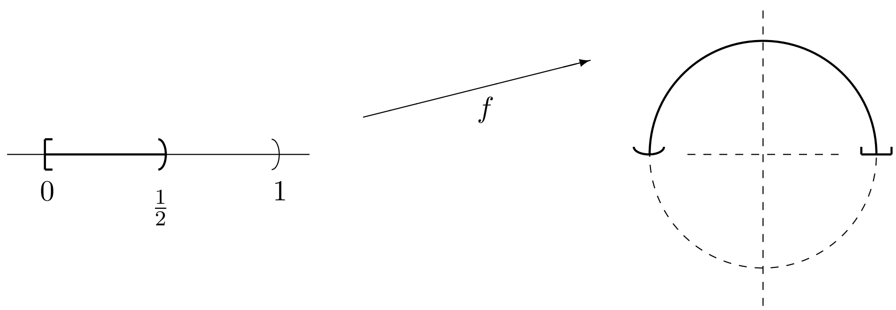
are continuous (this is immediate and does not need any checking) implies, according to Proposition 4.1.5 (or 4.1.11) the continuity of function $i \circ f$. On the other hand, the topology on $\mathbb{S}^1$ is nothing but the subspace topology with respect to $\mathbb{R}$, hence the initial topology respect to $i : \mathbb{S}^1 \hookrightarrow \mathbb{R}^2$. The fact $i \circ f$ is continuous, and again Proposition 4.1.5, imply the continuity of f.

The rest of the Exercise is easier and does not need any results from Chapter 4.

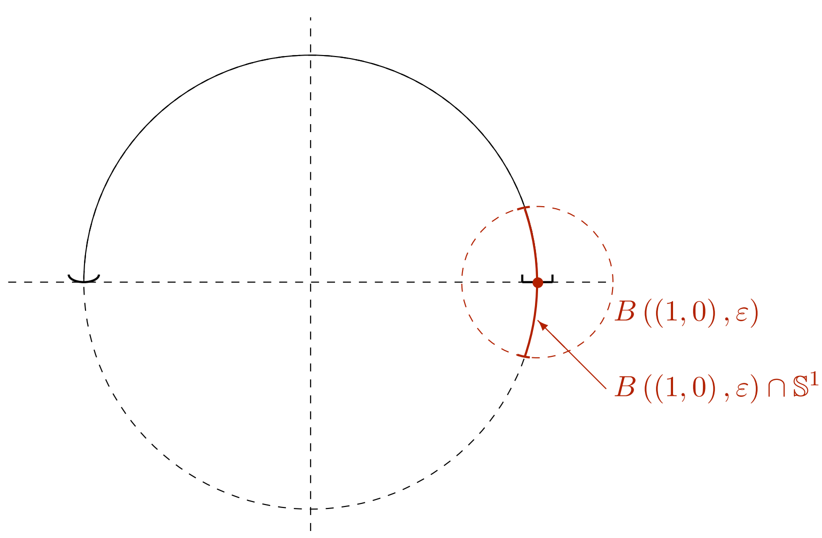
- f is injective; indeed, $f(t) = f(s)$ if, and only if, $t - s \in \mathbb{Z}$. The only possibility is $t - s = 0$, given the fact that there are no two such elements $t, s \in [0,1)$ such that $|t - s| \geq 1$.

- f is surjective; let $\boldsymbol{p} = (x, y) \in \mathbb{S}^1$. Define $\alpha := \widehat{\boldsymbol{p}, (1,0)}$ be the angle described by $\boldsymbol{p}$ and the x-axis. There is a determination of α belonging to $[0, 2\pi)$ (adding or subtracting an adequate amount of copies of 2π) which means $t := \frac{\alpha}{2\pi} \in [0,1)$, and it is immediate that $f(t) = (x, y)$.

- f is not a homeomorphism. Indeed, we have proven it continuous and bijective so it suffices to prove, according to Proposition 3.3.4, that it is not open. Assume it is. We know $[0, 1/2) = [0,1) \cap (-1/2, 1/2)$ is an open set of $[0,1)$.



If $f([0, 1/2))$ were an open set of $\mathbb{S}^1$, for every point (x, y) in $f([0, 1/2))$ (for instance, $(x, y) = f(0) = (1, 0)$ marked in the picture below) there should be a ball $B((x, y), \varepsilon) \cap \mathbb{S}^1$ centred at (x, y) and totally contained in $f([0, 1/2))$. Taking $(x, y) = (1, 0)$, we see this is not possible since every intersection $B((1,0), \varepsilon) \cap \mathbb{S}^1$ will contain points not in $f([0, 1/2))$ although $(1, 0) \in f([0, 1/2))$.



Hence $f([0, 1/2))$ is not open, which means f is not open and therefore is not a homeomorphism.

65. Prove the interiors of a square, a circle and an ellipse are homeomorphic.

Solution. We are obviously working with the Euclidean topology on $\mathbb{R}^2$, having a basis consisting of all open balls $\left\{(x-a)^2+(y-b)^2<r^2\right\}$ of arbitrary centres (x,y) and arbitrary radii r.

First of all, we can always assume all three figures are centred at the origin and the axes of the ellipse coincide with the x and y axes. Indeed, given any subset $Y\subset\mathbb{R}^2$:

- the translation $Y\mapsto Y+\boldsymbol{v}=\{\boldsymbol{y}+\boldsymbol{v}:\boldsymbol{y}\in Y\}$ for any fixed $\boldsymbol{v}=(v_1,v_2)$ is a homeomorphism. In order to check this, all we need to do is verify that it is the restriction $f|_Y:Y\to\mathbb{R}^2$ of a function

$$f:\mathbb{R}^2\to\mathbb{R}^2,\qquad \boldsymbol{y}\mapsto\boldsymbol{y}+\boldsymbol{v},$$

  which is a homeomorphism.

  - f is bijective: indeed, it is immediate to check that the function $g:\mathbb{R}^2\to\mathbb{R}^2$ defined by $g(\boldsymbol{y}):=\boldsymbol{y}-\boldsymbol{v}$ is its inverse.
  - f is continuous: indeed, for every basis element

$$B\left((a,b),r\right)=\left\{(x-a)^2+(y-b)^2<r^2\right\}\subset\mathbb{R}^2,$$

    we have

$$f^{-1}\left(B\left((a,b),r\right)\right)=g\left(B\left((a,b),r\right)\right)=\left\{(x,y)+(v_1,v_2):(x-a)^2+(y-b)^2<r^2\right\},$$

    and the reader may check this is nothing but $B\left((a+v_1,b+v_2),r\right)$ which is also an open set (because it is also an open ball). Applying Proposition 3.2.1 this implies f is continuous.
  - g is continuous for the same reason f is (replace $+$ by $-$ in adequate places in the previous item).
  - Finally, homeomorphism $f:\mathbb{R}^2\to\mathbb{R}^2$ induces a homeomorphism $f|_Y:Y\to Y+\boldsymbol{v}$. This is true in virtue of Remark 3.3.7 (**2**).

- It is left an Exercise to check that rotations

$$r_\alpha|_Y:Y\to MY:=\{M\boldsymbol{y}:\boldsymbol{y}\in Y\},\qquad \boldsymbol{y}=\begin{pmatrix}y_1\\y_2\end{pmatrix}\mapsto M\boldsymbol{y}:=\begin{pmatrix}\cos\alpha&\sin\alpha\\-\sin\alpha&\cos\alpha\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix},$$

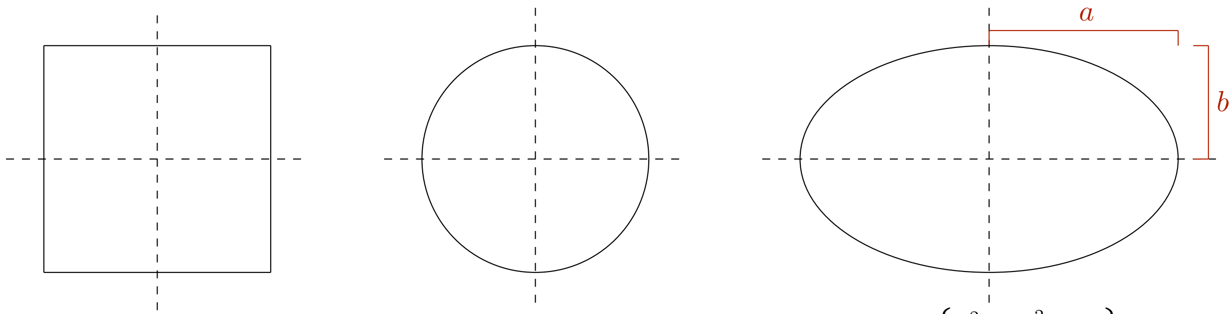
  are also homeomorphisms from every subset Y to the rotated set MY, for every rotation M as above, and their inverse is $r_{-\alpha}:\boldsymbol{y}\mapsto M^{-1}\boldsymbol{y}$.

- It is also left as an Exercise to check that a homothety

$$h:Y\to\alpha Y:=\{\alpha\boldsymbol{y}:\boldsymbol{y}\in Y\},\qquad \boldsymbol{y}=\begin{pmatrix}y_1\\y_2\end{pmatrix}\mapsto\alpha\boldsymbol{y}=\begin{pmatrix}\alpha y_1\\\alpha y_2\end{pmatrix},$$

  for any $\alpha>0$, is a homeomorphism as well.

Hence, we may assume our square S, our circle C and our ellipse E are centred at the origin and have the adequate size to make our calculations simpler:



$S := \{|x| \leq 1, |y| \leq 1\}$ $\qquad$ $C := \left\{x^2 + y^2 \leq 1\right\}$ $\qquad$ $E := \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$

Let us prove $\mathring{C} \cong \mathring{E}$. The other homeomorphism is left as Exercise 71. We have

$$\mathring{C} = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}, \qquad \mathring{E} = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\},$$

and can parametrize points in each of these subspaces of $\mathbb{R}^2$ in the following manner:

$$(x, y) = (r\cos\theta, r\sin\theta) \quad \text{for every } (x, y) \in \mathring{C}, \quad \text{where} \quad 0 \leq r < 1 \quad \text{and} \quad 0 \leq \theta < 2\pi, \tag{3.3}$$

and

$$(x, y) = (ra\cos\varphi, rb\sin\varphi) \quad \text{for every } (x, y) \in \mathring{E}, \quad \text{where} \quad 0 \leq r < 1 \quad \text{and} \quad 0 \leq \varphi < 2\pi; \tag{3.4}$$

define

$$f : \mathring{C} \to \mathring{E}, \qquad (x, y) = (r\cos\theta, r\sin\theta) \mapsto (ax, by) = (ar\cos\theta, br\sin\theta);$$

it is a well-defined function (that is, $(x, y) \in \mathring{C}$ implies $f(x, y) \in \mathring{E}$) in virtue of (3.3) and (3.4)

- Proving f is continuous can be done in a number of ways. Let us do it, once again, using Proposition 4.1.5. Define function $F : \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x, y) := (ax, by)$. This is a continuous function (Exercise: use Proposition 3.2.1 and prove the pre-image of every basis element $B((x, y), \rho) \subset \mathbb{R}^2$ is an open set). We have the following commutative diagram

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "F"] & \mathbb{R}^2 \\ \mathring{C} \arrow[u, "i_{\mathring{C}}"] \arrow[r, "f"] & \mathring{E} \arrow[u, "i_{\mathring{E}}"'] \end{tikzcd} \tag{3.5}$$

where

$$i_{\mathring{C}} : \mathring{C} \hookrightarrow \mathbb{R}^2, \qquad (x, y) \mapsto (x, y),$$

and

$$i_{\mathring{E}} : \mathring{E} \hookrightarrow \mathbb{R}^2, \qquad (x, y) \mapsto (x, y),$$

are the inclusions of these subsets into $\mathbb{R}^2$. The Euclidean induced topology on $\mathring{E}$ coincides with the initial topology with respect to $i_{\mathring{E}}$; hence applying Proposition 4.1.5 function f is continuous if, and only if,

$$i_{\mathring{E}} \circ f : \mathring{C} \to \mathring{E} \to \mathbb{R}^2$$

is continuous. In view of commutative diagram (3.5), $F \circ i_{\mathring{C}} \equiv i_{\mathring{E}} \circ f$. Hence the latter function $i_{\mathring{E}} \circ f$ is continuous if, and only if, $F \circ i_{\mathring{C}}$ is. But we know $F \circ i_{\mathring{C}} = F|_{\mathring{C}}$, which is continuous due to the continuity of F and Proposition 3.2.4 (b). Hence $i_{\mathring{E}} \circ f$ is continuous, and in virtue of the previous reasoning so is f.

- f is bijective. Indeed, its inverse is

$$f^{-1} : \mathring{E} \to \mathring{C}, \qquad (X,Y) = (ar\cos\theta, br\sin\theta) \mapsto \left(\frac{X}{a}, \frac{Y}{b}\right) = (r\cos\theta, r\sin\theta)\,;$$

  it is also well-defined and it is trivial to ascertain $f^{-1} \circ f = \mathrm{id}_{\mathring{C}}$ and $f \circ f^{-1} = \mathrm{id}_{\mathring{E}}$.

- Finally, f^{-1} is continuous. This works exactly along the same lines as the first item in this proof.

66. Consider $\mathbb{R}$ with the Euclidean topology, induce this topology on any of the subspaces considered in this Exercise and let $a, b \in \mathbb{R}$ such that $a < b$.

a) Prove that (a,b) is homeomorphic to $(0,1)$. Prove $[a,b]$ is homeomorphic to $[0,1]$.

b) Prove (a,b) is homeomorphic to $\mathbb{R}$.

Solution.

a) Define

$$f : [a,b] \to [0,1]\,, \qquad t \mapsto f(t) := \frac{a-t}{a-b}.$$

The function satisfies $f(a) = 0$, $f(b) = 1$ and is continuous since the denominator will always be non-zero ($a < b$). It is also bijective: $\frac{a-t}{a-b} = \frac{a-t'}{a-b}$ implies, multiplying both sides by $a-b$ and then subtracting a, $t = t'$, and in order to prove it is surjective we can actually construct a well-defined inverse explicitly:

$$f^{-1} : [0,1] \to [a,b]\,, \qquad s \mapsto f^{-1}(s) := a + s(b-a)\,,$$

trivially continuous since it is actually an affine function (see Example 3.1.3 (**5**)).

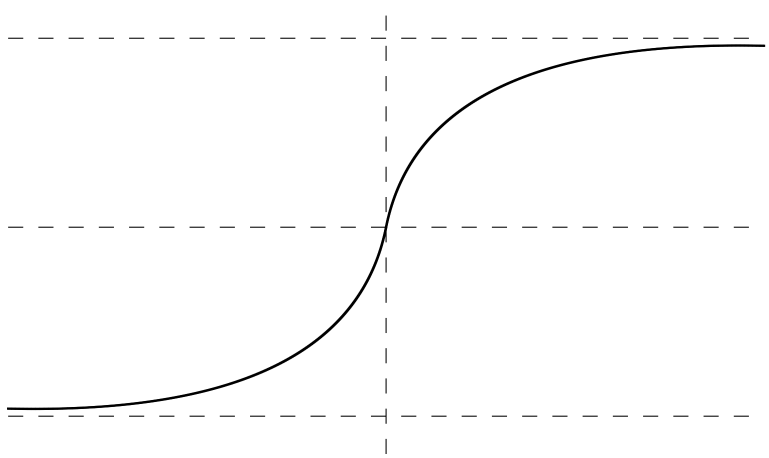
In order to prove $(a,b) \cong (0,1)$ all we need to use is the previous homeomorphism between the closed intervals. We have $(a,b) \subset [a,b]$ and $f((a,b)) = (0,1) \subset [0,1]$ with the induced subspace topology, hence the restriction of the homeomorphism to (a,b), $f|_{(a,b)}$, is a homeomorphism between (a,b) and its image $(0,1)$ in virtue of Remark 3.3.7 (**2**).

b) Using the previous item with $a = -1$ and $b = 1$, $(-1,1) \cong (0,1)$. Hence let us prove $(-1,1) \cong \mathbb{R}$. Define

$$f : \mathbb{R} \to (-1,1) \subset \mathbb{R}, \qquad x \mapsto \frac{x}{1+|x|} = \begin{cases} \frac{x}{x+1} & \text{if} \quad x \geq 0, \\ \frac{x}{1-x} & \text{if} \quad x < 0. \end{cases}$$

- It is well-defined:

$$\lim_{x\to+\infty} f(x) = 1, \qquad \lim_{x\to-\infty} f(x) = -1.$$



- Let us prove f is continuous. It is continuous at each side of 0: it is the quotient of continuous functions, and the denominator vanishes nowhere. The fact it is piecewise defined requires checking limits on both sides at 0:

$$\lim_{x \to 0^+} f(x) = f(0) = 0 = \lim_{x \to 0^-} f(x).$$

This is the same as verifying continuity on each of the closed sets $(-\infty, 0]$, $[0, \infty)$ and applying Proposition 3.4.3.

- Its inverse,

$$f^{-1} : (-1,1) \to \mathbb{R}, \qquad x \mapsto f^{-1}(x) = \begin{cases} \frac{x}{1-x}, & x \in [0,1), \\ \frac{x}{1+x}, & x \in (-1,0], \end{cases} \tag{3.6}$$

is well-defined:

$$\lim_{x \to -1} \frac{x}{1+x} = -\infty, \qquad \lim_{x \to 1} \frac{x}{1-x} = \infty.$$

- f is bijective: this is clear from the well-defined inverse we just constructed.
- f^{-1} is continuous: same procedure as the one used above for f, proving $f^{-1}(0) = 0$ equal to the left and right limits.

Hence using the homeomorphism proven in item (b) and the one proven in the previous item (a), $\mathbb{R} \cong (-1,1) \cong (0,1) \cong (a,b)$ for every $a, b \in \mathbb{R}$ such that $a < b$.

67. Prove the following spaces are homeomorphic among them. Consider all of them subspaces of $\mathbb{R}^3$ or $\mathbb{R}^2$ with the induced subspace Euclidean topology.

a) $\mathbb{R}^2 \setminus \{(0,0)\}$.

b) $\mathbb{R} \times \mathbb{S}^1$.

c) $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, z > 0\}$.

d) $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, -1 < z < 1\}$.

e) $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$.

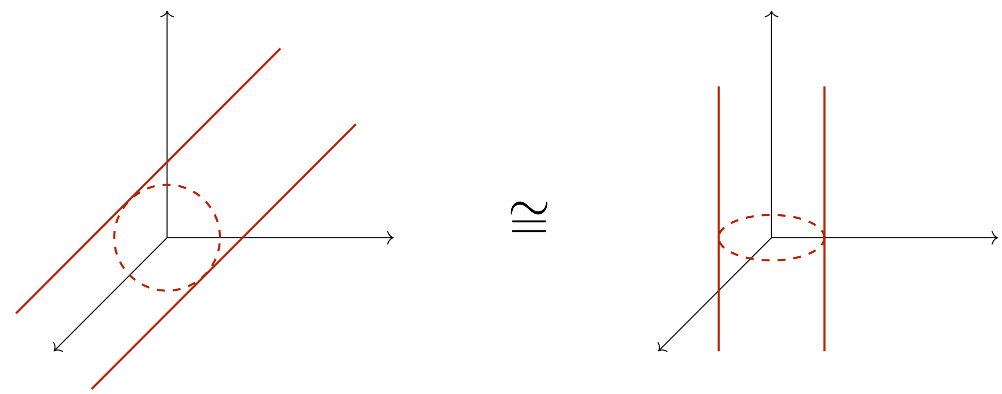
f) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$.

Solution. We are going to prove the following chain of homeomorphisms:

$$\begin{aligned} \mathbb{R} \times \mathbb{S}^1 &\cong \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \\ &\cong \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, -1 < z < 1\} \\ &\cong \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\} \\ &\cong \mathbb{R}^2 \setminus \{(0,0)\} \end{aligned}$$

and leave $\mathbb{R}^2 \setminus \{(0,0)\} \cong \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, z > 0\}$ as Exercise 74.

- Let us first prove $\mathbb{R} \times \mathbb{S}^1 \cong \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. We can obviously write the first space as $\mathbb{R} \times \mathbb{S}^1 = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1\}$.



A clear candidate for a homeomorphism is

$$f : \mathbb{R} \times \mathbb{S}^1 \to \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\right\}, \qquad (x, y, z) \mapsto (y, z, x),$$

having inverse

$$g : \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\right\} \to \mathbb{R} \times \mathbb{S}^1, \qquad (x, y, z) \mapsto (z, x, y),$$

These two functions are indeed inverses of one another: you may check yourselves that they are well-defined and $f \circ g = \mathrm{id}_{\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=1\}}$ and $g \circ f = \mathrm{id}_{\mathbb{R}\times\mathbb{S}^1}$. Hence both functions are bijective. They are continuous because their composition with projections are simply other projections: defining

$$\mathrm{pr}_1 : \mathbb{R}^3 \to \mathbb{R}, \qquad (x, y, z) \to x,$$

and

$$\mathrm{pr}_2 : \mathbb{R}^3 \to \mathbb{R}, \qquad (x, y, z) \to y,$$

and

$$\mathrm{pr}_3 : \mathbb{R}^3 \to \mathbb{R}, \qquad (x, y, z) \to z,$$

and recalling that both topological spaces $\mathbb{R} \times \mathbb{S}^1$ and $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ have the Euclidean induced topology, which is the same as the initial topology with respect to projections pr_i, $i = 1, 2, 3$, these three projections are continuous when restricted to all three spaces, hence so are

$$\mathrm{pr}_1 \circ f = \mathrm{pr}_2, \quad \mathrm{pr}_2 \circ f = \mathrm{pr}_3, \quad \mathrm{pr}_3 \circ f = \mathrm{pr}_1,$$

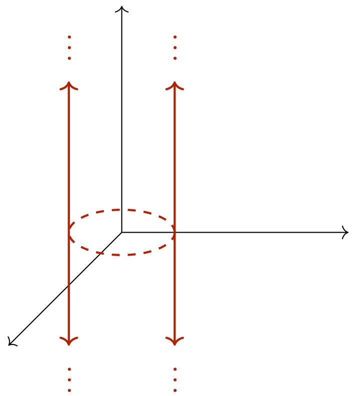
and

$$\mathrm{pr}_1 \circ g = \mathrm{pr}_3, \quad \mathrm{pr}_2 \circ g = \mathrm{pr}_1, \quad \mathrm{pr}_3 \circ g = \mathrm{pr}_2,$$

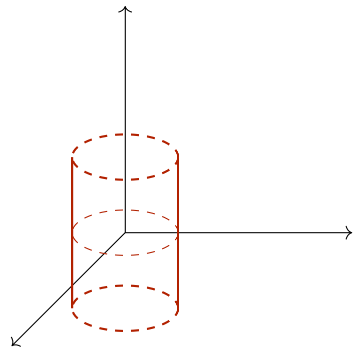
thereby implying (in virtue of Proposition 4.1.5, or 4.1.11) the continuity of f and g.

We have just proven f (and thus its inverse g) is a homeomorphism.

- Let us prove $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=1\}\cong\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=1, -1<z<1\}$.



$\cong$



In essence, we need a function compressing the real vertical infinite axis into a vertical interval of the same length as $(-1,1)$. One possible such function is

$$f:\{x^2+y^2=1\}\to\{x^2+y^2=1,-1<z<1\},\quad (x,y,z)\to\left(x,y,\frac{2}{\pi}\arctan z\right);$$

indeed, if we choose the $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ determination of the arctangent, $\arctan:\mathbb{R}\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is bijective, continuous and has continuous inverse $\tan:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$, hence it is a homeomorphism. So is, therefore, the same function multiplied by a constant, $\frac{2}{\pi}\arctan:\mathbb{R}\to(-1,1)$, which simply changes the range of the original one. Thus (again using Proposition 4.1.5 or 4.1.11 to characterise continuity of functions defined on products in terms of each of their coordinate functions) function f defined above is continuous because each of its 3 coordinate functions $x, y, \frac{2}{\pi}\arctan z$ is, and

$$g:\{x^2+y^2=1,-1<z<1\}\to\{x^2+y^2=1\},\quad (x,y,z)\mapsto\left(x,y,\tan\left(\frac{\pi}{2}z\right)\right),$$

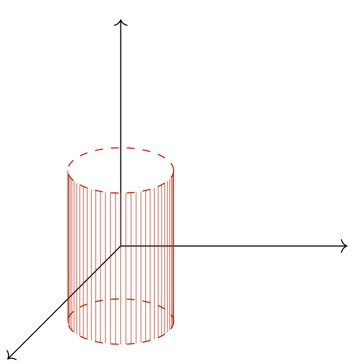
which you may check is its inverse, is also continuous following a similar argument to that used for f. We have a bijective, continuous function with a continuous inverse between the two spaces. We are finished with these two.

Remark: another possible homeomorphism between the two sets can be given by

$$f:\{x^2+y^2=1\}\to\{x^2+y^2=1,-1<z<1\},\quad (x,y,z)\to\left(x,y,\frac{z}{1+|z|}\right);$$

which benefits from what we proved in Solved Exercise 66: the third coordinate function is nothing but the homeomorphism we found in that Exercise between $\mathbb{R}$ and $(-1,1)$. This is the same as saying $x\mapsto\frac{2}{\pi}\arctan x$ is also a useful homeomorphism solving Exercise 66.

- Let us prove $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=1,-1<z<1\}\cong\{(x,y)\in\mathbb{R}^2 : x^2+y^2>1\}$.



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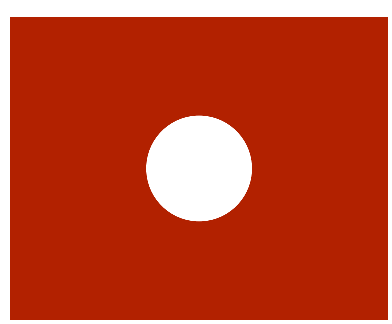
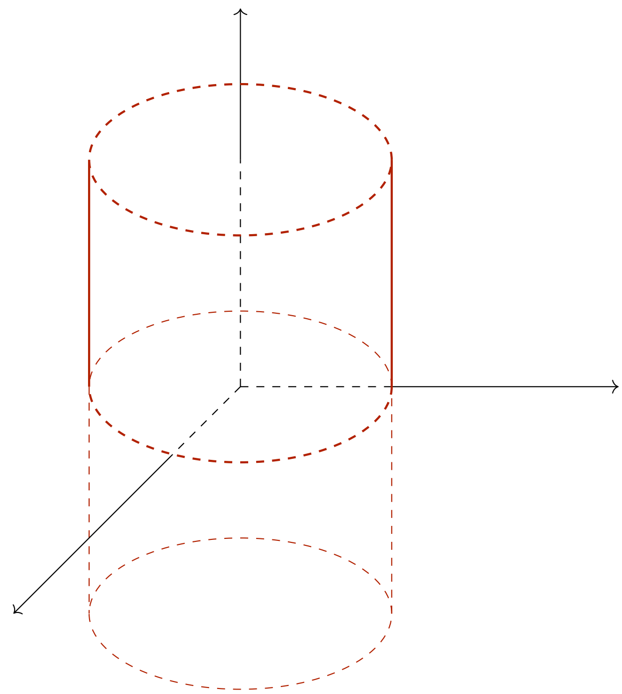
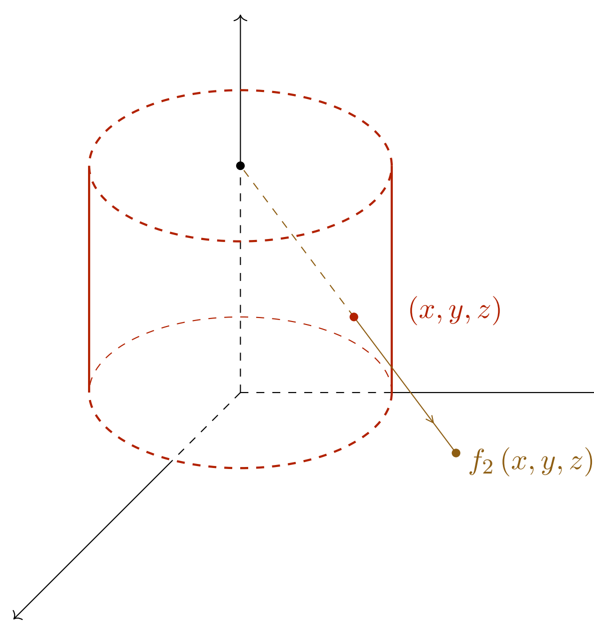


It is clear that one possible way forward is "squashing" the cylinder in such a way that it becomes outwardly infinite (i.e. one of the two boundaries is sent to infinity); a way of doing this, for instance, is moving all points upwards in order to rest the cylinder's base on the plane $\pi = \{z = 0\}$ and dividing the third coordinate by two:

$$f_1 : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x, y, z) \mapsto \left(x, y, \frac{z+1}{2}\right).$$



followed by projecting all points of the cylinder from an adequate point in $\mathbb{R}^3$, for instance $(0, 0, 1)$, and intersecting the projected lines with π:



In order to find an explicit expression for f_2, we only need to use basic analytic geometry. The line joining $(0, 0, 1)$ and (x, y, z) is

$$r = \{(0, 0, 1) + \lambda\,(x, y, z - 1) = (\lambda x, \lambda y, 1 + \lambda\,(z - 1)) : \lambda \in \mathbb{R}\}.$$

The point in r intersecting π corresponds to $1+\lambda(z-1)=0$, hence $\lambda=-\frac{1}{z-1}$ and identifying $\pi=\{z=0\}$ with $\mathbb{R}^2$ we have

$$f_2:\left\{(x,y,z)\in\mathbb{R}^3: x^2+y^2=1, 0<z<1\right\}\to\left\{(x,y)\in\mathbb{R}^2: x^2+y^2>1\right\},\quad (x,y,z)\mapsto\left(-\frac{x}{z-1},-\frac{y}{z-1}\right).$$

The composition of f_1 with f_2 is

$$f:\left\{(x,y,z)\in\mathbb{R}^3: x^2+y^2=1, -1<z<1\right\}\to\left\{(x,y)\in\mathbb{R}^2: x^2+y^2>1\right\},\quad (x,y,z)\mapsto\left(-\frac{2x}{z-1},-\frac{2y}{z-1}\right).$$

It is left as an Exercise to check that the inverse of f is

$$f^{-1}:(x,y)\mapsto\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}},1-\frac{2}{\sqrt{x^2+y^2}}\right),$$

a well-defined function:

$$f^{-1}\left(\left\{(x,y)\in\mathbb{R}^2: x^2+y^2>1\right\}\right)\subset\left\{(x,y,z)\in\mathbb{R}^3: x^2+y^2=1,-1<z<1\right\}.$$

Both functions are continuous since they are compositions of continuous functions, as the reader may check as well.

- Let us prove $\left\{(x,y)\in\mathbb{R}^2: x^2+y^2>1\right\}\cong\mathbb{R}^2\setminus\{(0,0)\}$. We can define a function $f:\mathbb{R}^2\setminus\{(0,0)\}\to\left\{(x,y)\in\mathbb{R}^2: x^2+y^2>1\right\}$ by $f(x,y)=(\lambda x,\lambda y)$ in such a way that $\|(\lambda x,\lambda y)\|=\|(x,y)\|+1$ (same angle, norm plus one): defining $r:=\|(x,y)\|=\sqrt{x^2+y^2}$ we have $\|(\lambda x,\lambda y)\|=\sqrt{\lambda^2(x^2+y^2)}=\lambda r$ and equation $\lambda r=r+1$ implies $\lambda=\frac{r+1}{r}$:

$$f:\mathbb{R}^2\setminus\{(0,0)\}\to\left\{(x,y)\in\mathbb{R}^2: x^2+y^2>1\right\},\qquad (x,y)\mapsto\frac{\sqrt{x^2+y^2}+1}{\sqrt{x^2+y^2}}(x,y),$$

and it is an Exercise to check that the inverse for this is

$$f^{-1}:\left\{(x,y)\in\mathbb{R}^2: x^2+y^2>1\right\}\to\mathbb{R}^2\setminus\{(0,0)\},\qquad (x,y)\mapsto\frac{\sqrt{x^2+y^2}-1}{\sqrt{x^2+y^2}}(x,y),$$

and that both functions are continuous and well-defined.

- The reader can check
$$(x,y,z)\mapsto\left(\ln z,\frac{x}{z},\frac{y}{z}\right)$$
is a homeomorphism between two of the spaces (Exercise: find which) with inverse $(x,y,z)\mapsto(ye^x,ze^x,e^x)$.

68. (*Coursework January 2014*)

(i) (15 marks) Prove a function $f:X\to Y$ is continuous if, and only if, for every subset $S\subset X$, $f\left(\overline{S}\right)\subset\overline{f(S)}$. Hint: you may use the fact that closures and pre-images preserve inclusions.

(ii) (5 marks) Use the above result to prove that the function defined below is not continuous:

$$f:\mathbb{R}\to\mathbb{R},\qquad x\mapsto f(x):=\begin{cases}x+1 & \text{if}\quad x>0,\\ 0 & \text{if}\quad x=0,\\ x-1 & \text{if}\quad x<0.\end{cases}$$

Solution.

(i) Assume f is continuous. We have $S \subset f^{-1}(f(S)) \subset f^{-1}\left(\overline{f(S)}\right)$, the latter inclusion due to the fact that $f(S) \subset \overline{f(S)}$. The fact $\overline{f(S)}$ is closed in Y, coupled with the continuity of f, implies $f^{-1}\left(\overline{f(S)}\right)$ is closed in X.

$\overline{S}$ is the smallest closed set containing S, which implies $S \subset \overline{S} \subset f^{-1}\left(\overline{f(S)}\right)$ and thus

$$f(S) \subset f\left(\overline{S}\right) \subset f\left(f^{-1}\left(\overline{f(S)}\right)\right). \tag{3.7}$$

But

$$f\left(f^{-1}\left(\overline{f(S)}\right)\right) = f\left(\left\{x \in X : f(x) \in \overline{f(S)}\right\}\right) \subset \overline{f(S)},$$

by which (3.7) becomes $f\left(\overline{S}\right) \subset \overline{f(S)}$.

Conversely assume $f\left(\overline{S}\right) \subset \overline{f(S)}$ for every $S \subset X$. Let $C \subset Y$ be any closed set. We intend to prove $S := f^{-1}(C)$ is a closed set of X, thereby proving f continuous.

We have $f\left(\overline{S}\right) = f\left(\overline{f^{-1}(C)}\right) \subset \overline{f(f^{-1}(C))}$, the latter inclusion by hypothesis. And

$$f\left(f^{-1}(C)\right) = f(\{x \in X : f(x) \in C\}) \subset C,$$

hence the closure of one set is contained in the closure of the other: $\overline{f(f^{-1}(C))} \subset \overline{C} = C$, which reprising the original chain of inclusions implies $f\left(\overline{S}\right) \subset f\left(f^{-1}(C)\right) \subset C$. Preimages also preserve inclusions:

$$\overline{S} \subset f^{-1}\left(f\left(\overline{S}\right)\right) \subset f^{-1}(C) = S.$$

This, along with the other obvious inclusion $S \subset \overline{S}$, implies $S = \overline{S}$. Hence f is continuous.

(ii) We can use, for instance, $S = (0,1) \subset \mathbb{R}$. We have $f(S) = (1,2)$ and thus $f\left(\overline{S}\right) = f([0,1]) = \{0\} \cup (1,2]$, whereas $\overline{f(S)} = [1,2]$. Since $\{0\} \cup (1,2] \not\subseteq [1,2]$, we have $f\left(\overline{S}\right) \not\subseteq \overline{f(S)}$ which implies f is not continuous.

69. (*Coursework January 2014*) Prove $\left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\right\} \cong \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = z^2, z > 0\right\}$.

Solution. Let $X_1 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1, 0 < z < 1\right\}$ be the bounded open cylinder and $Y = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = z^2, z > 0\right\}$ be the open infinite half cone. We are going to first find a homeomorphism $f : X_1 \to Y$. Let us consider variables (x,y) separately from z. The third coordinate function (corresponding to variable z) will be a homeomorphism between $(0,1)$ and $(0,\infty)$:

$$f_1 : (0,1) \to (0,\infty), \qquad z \mapsto \frac{1}{z} - 1,$$

which is continuous and has a continuous inverse

$$f_1^{-1} : (0,\infty) \to (0,1), \qquad t \mapsto \frac{1}{t+1}.$$

We still need to know the action of $f : X_1 \to Y$ on variables x, y. This function must be defined by $(x,y,z) \mapsto (\overline{x},\overline{y},\overline{z}) := \left(\overline{x},\overline{y},\frac{1}{z}\right)$ in such a way that $\overline{x}^2+\overline{y}^2 = \overline{z}^2$, i.e. $\overline{x}^2+\overline{y}^2 = \frac{1}{z^2}$. If we choose $\overline{x} := \frac{x}{z}$ and $\overline{y} := \frac{y}{z}$, we have $\overline{x}^2+\overline{y}^2 = \frac{x^2}{z^2}+\frac{y^2}{z^2} = \frac{1}{z^2}$, the latter equality a consequence of $x^2+y^2 = 1$ due to $(x,y,z) \in X_1$. Hence we have a function

$$f : X_1 \to Y, \qquad (x,y,z) \mapsto \left(\frac{x}{z}, \frac{y}{z}, \frac{1}{z}\right),$$

which is well-defined, continuous (on account of all its coordinate functions being continuous) and has a continuous inverse

$$g : Y \to X_1, \qquad (a, b, c) \mapsto \left(\frac{a}{c}, \frac{b}{c}, \frac{1}{c}\right).$$

It is easy to check $g \circ f = \mathrm{id}_{X_1}$ and $f \circ g = \mathrm{id}_Y$. Hence $X_1 \cong Y$.

It remains to show $X \cong X_1$, where $X = \left\{x^2 + y^2 = 1\right\}$ is the unbounded cylinder. The only change needed is stretching the domain of the third variable z, $(0,1)$, into all of $\mathbb{R}$. This can be achieved, for instance, by composing homeo $z \to 2z - 1$ between $(0,1)$ and $(-1,1)$ with homeomorphism (3.6) between $(-1,1)$ and $\mathbb{R}$:

$$f_3 : (0,1) \to \mathbb{R}, \qquad z \mapsto \frac{2z-1}{1-|2z-1|}.$$

Function $F : X_1 \to X$ defined by $(x, y, z) \mapsto (x, y, f_3(z))$ is the required homeomorphism, having inverse $(a, b, c) \mapsto \left(a, b, f_3^{-1}(c)\right)$. Finally, $f \circ F^{-1} : X \xrightarrow{\cong} Y$.

70. (*Sample Coursework November 2013*) Prove affine functions $\mathbb{R}^n \to \mathbb{R}^m$, $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ are continuous.

Solution. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $f(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$, for every $\boldsymbol{x} = (x_1, \ldots, x_n)$ where $A = (a_{ij})_{i=1,\ldots,m, j=1,\ldots,n}$. We will use Proposition 3.2.1 and the equivalence of metrics in $\mathbb{R}^n, \mathbb{R}^m$; let $V = B_{d_\infty}(\boldsymbol{y}, r) \subset \mathbb{R}^m$ be an open ball of the arrival set. We need to show $f^{-1}(V)$ is an open set of $\mathbb{R}^n$. Indeed, assume $f^{-1}(V) \neq \varnothing$ and let $\boldsymbol{x} \in f^{-1}(V)$. This means $d_\infty(f(\boldsymbol{x}), \boldsymbol{y}) < r$, hence $d_\infty(f(\boldsymbol{x}), \boldsymbol{y}) < r$ using (1.5). In other words,

$$|a_{11}x_1 + \cdots + a_{1n}x_n + b_1 - y_1| < r, \qquad \ldots, \qquad |a_{m1}x_1 + \cdots + a_{mn}x_n + b_m - y_m| < r. \tag{3.8}$$

We will find $\varepsilon > 0$ such that $B_{d_\infty}(\boldsymbol{x}, \varepsilon) \subset V$, i.e. such that for every $\boldsymbol{z}$ such that $d_\infty(\boldsymbol{z}, \boldsymbol{x}) < \varepsilon$, then $d_\infty(f(\boldsymbol{z}), \boldsymbol{y}) < r$, i.e. the above sequence of inequalities (3.8) holds with x_i replaced by z_i.

In other words: we need ε such that whenever

$$|x_1 - z_1| < \varepsilon, \quad |x_2 - z_2| < \varepsilon, \quad \ldots \quad |x_n - z_n| < \varepsilon,$$

the following holds: $|a_{i1}z_1 + \cdots + a_{in}z_n + b_i - y_i| < r$, $i = 1, \ldots, m$.

The summary of our proof will be as follows: let $\tilde{r} = d_\infty(f(\boldsymbol{x}), \boldsymbol{y}) < r$; if we can find a number $\|A\|_\infty$ such that $d_\infty(f(\boldsymbol{x}), f(\boldsymbol{z})) \leq \|A\|_\infty d_\infty(\boldsymbol{x}, \boldsymbol{z})$, then using triangular inequality

$$d_\infty(f(\boldsymbol{z}), \boldsymbol{y}) \leq d_\infty(f(\boldsymbol{z}), f(\boldsymbol{x})) + d_\infty(f(\boldsymbol{x}), \boldsymbol{y}) < \|A\|_\infty d_\infty(\boldsymbol{x}, \boldsymbol{z}) + \tilde{r} < \|A\|_\infty \varepsilon + \tilde{r} \tag{3.9}$$

we can solve inequality $\|A\|_\infty \varepsilon + \tilde{r} < r$ for ε. For every $i = 1, \ldots, m$,

$$\begin{aligned} |a_{i1}(z_1 - x_1) + \cdots + a_{in}(z_n - x_n)| &\leq |a_{i1}|\,|z_1 - x_1| + \cdots + |a_{in}|\,|z_n - x_n| \\ &\leq (|a_{i1}| + \cdots + |a_{in}|) \max_{j=1,\ldots,n} |z_j - x_j| \\ &= (|a_{i1}| + \cdots + |a_{in}|)\, d_\infty(\boldsymbol{z}, \boldsymbol{x}), \end{aligned}$$

hence such $\|A\|_\infty$ exists and is equal to $\max_{i=1,\ldots,m} |a_{i1}| + \cdots + |a_{in}|$. We can always assume $\|A\|_\infty \neq 0$, otherwise this would entail $a_{ij} = 0$ for every i, j and the function would be constant $f(\boldsymbol{x}) = \boldsymbol{b}$, hence trivially continuous.

If $\|A\|_\infty > 0$, therefore, (3.9) prevails and we obtain $\|A\|_\infty \varepsilon + \tilde{r} < r$ which means $\varepsilon = \varepsilon_{r,x} := \frac{r - d_\infty(A\boldsymbol{x} + \boldsymbol{b}, \boldsymbol{y})}{\|A\|_\infty} > 0$ places a $< r$ at the end of (3.9) and solves the Exercise.

71. (*Sample Coursework November 2013*) Prove the interiors of a square, a circle and any triangle are homeomorphic.

Solution. We will prove the interiors of a triangle and a circle are homeomorphic, and leave the remainder as an Exercise using similar techniques.

Firstly, translations by a fixed vector (i.e. special case $A = \mathrm{Id}_2$ in Exercise 70),

$$T_{v_1,v_2} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (x, y) + (v_1, v_2)\,,$$

rotations of a fixed angle (A equal to a rotation matrix in Exercise 70),

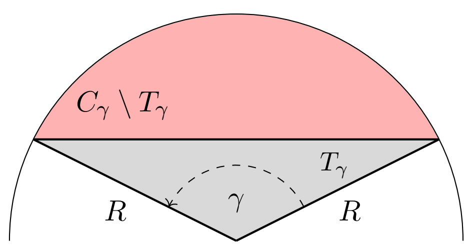
$$R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto ((\cos\alpha)\, x + (\sin\alpha)\, y, -(\sin\alpha)\, x + (\cos\alpha)\, y)\,,$$

and homotheties with a given ratio ($A = r\mathrm{Id}_2$ and $\boldsymbol{b} = 0$ in Exercise 70)

$$H_r : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (rx, ry)\,,$$

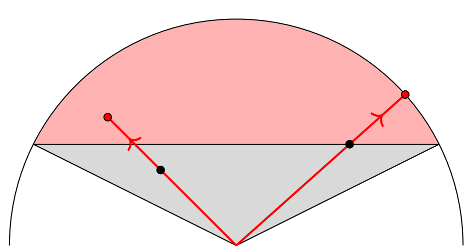
are homeomorphisms (Exercise: check by finding explicit inverses, which will also be of the form in Exercise 70, hence continuous). Hence we can assume, without loss of generality, that C is the circumscribed circle of T (i.e. the three vertices belong to C) and has its centre at the origin.

Tracing line segments from $(0, 0)$ to each vertex we divide the triangle in three different isosceles triangles; applying a certain rotation to every one of these triangles we have the following scenario:

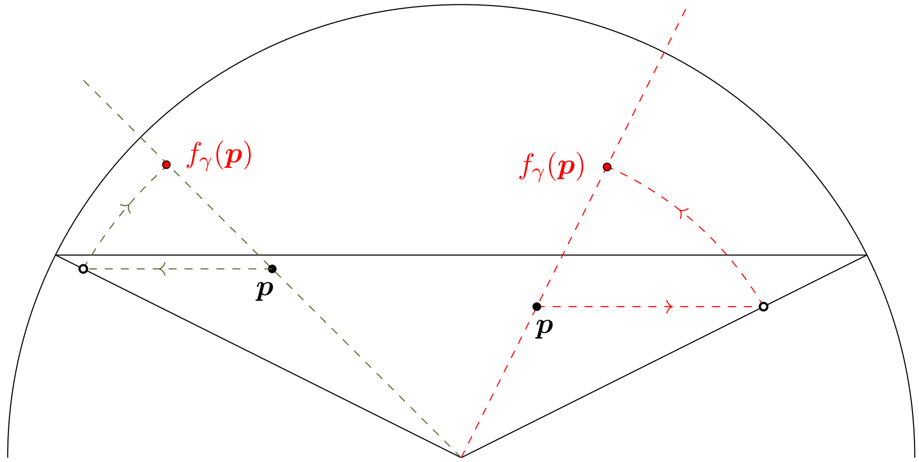


γ obviously varying depending on which isosceles triangle we choose.

Our intention is to find a homeo f_γ between the grey area (i.e. the isosceles triangle) and the union of the grey and red areas (i.e. the whole shaded region). In other words, we want to "expand" the isosceles triangle continuously into a region with one curved side and two straight sides. The easiest way to achieve this is by means of a certain radial expansion from the origin: $f(0, 0) = (0, 0)$ for every $(x, y) \neq (0, 0)$ belonging to the isosceles triangle T_γ, $f_\gamma(x, y)$ will be in the same line containing $(0, 0)$ and (x, y), albeit closer to the outer boundary of C_γ.



We describe the steps for a good candidate for f_γ, first with a diagram and then enumerating them one by one.



Let $\boldsymbol{p}$ be a point in T_γ. Let r the line through $\boldsymbol{p}$ and the origin. Consider the following steps:

a) Let s be the horizontal line containing $\boldsymbol{p}$.

b) Let $\overline{\boldsymbol{p}}$ be the point of intersection of s with any of the two equal (non-horizontal) sides of T_γ.

c) Let $\overline{C}$ the circle centered at the origin and containing $\overline{\boldsymbol{p}}$.

d) Let $\tilde{\boldsymbol{q}}$ be the point of intersection of $\overline{C}$ with the original line r.

We thereby define $f_\gamma(\boldsymbol{p}) := \tilde{\boldsymbol{p}}$. Needless to say, f_γ^{-1} is defined by performing the process in reverse order. EXERCISE: find the explicit expression of these two functions analytically, i.e. using coordinates.

Finally, since $T = T_{\gamma_1} \cup T_{\gamma_2} \cup T_{\gamma_3}$, $C = C_{\gamma_1} \cup C_{\gamma_2} \cup C_{\gamma_3}$, $T_{\gamma_i} \cong C_{\gamma_i}$ for $i = 1, 2, 3$ and the corresponding homeomorphisms are consistent on the intersections $T_{\gamma_i} \cap T_{\gamma_j}$ (you must check this), we can apply Proposition 3.4.3 (a) and obtain a continuous function defined globally on T, whose inverse is also defined piecewise on C_{γ_i} as the inverse of $f_{\gamma_i}^{-1}$. Thus f is bicontinuous and bijective. $T \cong C$ implies $\mathring{T} \cong \mathring{C}$.

72.

a) Let $f : X \to Y$ be a function between two topological spaces. Prove f is continuous if, and only if, $\overline{f^{-1}(C)} \subset f^{-1}\left(\overline{C}\right)$ for every subset $C \subset Y$.

b) Use the above item to prove that the following function is not continuous:

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \boldsymbol{x} \mapsto f(\boldsymbol{x}) := \begin{cases} 2\boldsymbol{x}, & \|\boldsymbol{x}\|_2 \geq 1, \\ \boldsymbol{x}, & \|\boldsymbol{x}\|_2 < 1. \end{cases}$$

73. (*May 2014*)

a) Let $f : X \to Y$ be a function between two topological spaces. Prove f is continuous if, and only if, $\overline{f^{-1}(C)} \subset f^{-1}\left(\overline{C}\right)$ for every subset $C \subset Y$.

b) Use the above item to prove that the following function is not continuous:

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \boldsymbol{x} \mapsto f(\boldsymbol{x}) := \begin{cases} 2\boldsymbol{x}, & \|\boldsymbol{x}\|_2 \geq 1, \\ \boldsymbol{x}, & \|\boldsymbol{x}\|_2 < 1. \end{cases}$$

Solution.

a) $\Rightarrow$) For any $C \subset Y$, $\overline{C}$ is a closed set and the pre-image of a closed set by a continuous function is also closed, hence

$$f^{-1}\left(\overline{C}\right) = \overline{f^{-1}\left(\overline{C}\right)}. \tag{3.10}$$

We know $C \subset \overline{C}$ and pre-images and closures preserve inclusions, hence $f^{-1}(C) \subset f^{-1}\left(\overline{C}\right)$ and thus

$$\overline{f^{-1}(C)} \subset \overline{f^{-1}\left(\overline{C}\right)}. \tag{3.11}$$

(3.10) and (3.11) finally imply $\overline{f^{-1}(C)} \subset f^{-1}\left(\overline{C}\right)$.

$\Leftarrow$) Any closed set of Y can be written $Y \setminus U$ where U is open in Y; hence in virtue of our hypothesis and the fact $\overline{Y \setminus U} = Y \setminus U$, we have

$$\overline{f^{-1}(Y \setminus U)} \subset f^{-1}\left(\overline{Y \setminus U}\right) = f^{-1}(Y \setminus U),$$

which implies $f^{-1}(Y \setminus U)$ is closed. We have thus proven that the pre-image of any closed set is closed, which is the same as proving f is continuous.

b) Let $C = B((0,0),1)$. We have $f^{-1}(C) = B((0,0),1) = f^{-1}\left(\overline{C}\right)$, whereas

$$\overline{f^{-1}(C)} = \overline{B((0,0),1)} = \left\{x^2 + y^2 \leq 1\right\} \not\subset B((0,0),1),$$

hence $\overline{f^{-1}(C)} \not\subset f^{-1}\left(\overline{C}\right)$.

74. (*July 2013*) Prove $\mathbb{R}^2 \setminus \{(0,0)\} \cong \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, z > 0\right\}$, the last remaining homeomorphism left unfinished in Solved Exercise 67.

Solution. Define

$$f : \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, z > 0\right\} \to \mathbb{R}^2 \setminus \{(0,0)\}, \qquad (x,y,z) \mapsto (x,y),$$

and

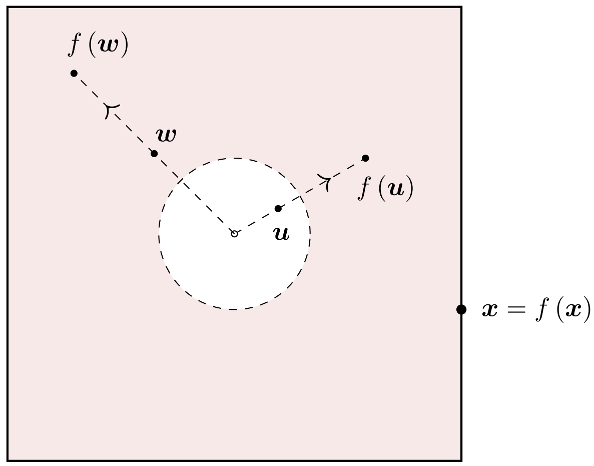
$$g : \mathbb{R}^2 \setminus \{(0,0)\} \to \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, z > 0\right\}, \qquad (x,y) \mapsto \left(x, y, \sqrt{x^2 + y^2}\right).$$

They are inverses of each other trivially. They both are continuous as well.

75. (*May 2014*) Prove $[-1,1] \times [-1,1] \setminus \{(0,0)\}$ is homeomorphic to $[-1,1] \times [-1,1] \setminus \overline{B((0,0),r)}$ for any $r \in (0,1)$. Hint: place these spaces in two parallel planes of $\mathbb{R}^3$ and use a cone having its vertex in one of them.

SOLUTION. We will do this for $r=\frac{1}{3}$ and leave the general case as an EXERCISE.

We need a function carrying every element $\boldsymbol{x}=(x,y)\in[-1,1]\times[-1,1]\setminus\{(0,0)\}$ to a multiple of itself, $f(x,y):=(\alpha x,\alpha y)$ lying closer to the boundary $\partial([-1,1]\times[-1,1])$. Said boundary must be left invariant for the function to be continuous:



Let $(x,y)\in[-1,1]^2\setminus\{\mathbf{0}\}$. Place $[-1,1]^2\setminus\{\mathbf{0}\}$ on plane $z=0$ and $[-1,1]^2\setminus\overline{B\left(\mathbf{0},\frac{1}{3}\right)}$ on plane $z=1$ in the three-dimensional Euclidean space. The "walls" of the parallelepiped thus defined are given by $x=\pm1$ and $y=\pm1$. Thus $\boldsymbol{z}=(\overline{x},\overline{y},1):=\left(\frac{x}{||(x,y)||_\infty},\frac{y}{||(x,y)||_\infty},1\right)$ lies on the intersection of one of these walls with the plane containing $(x,y,1)$. Let $\pi_{x,y}$ be the plane containing both $(\overline{x},\overline{y},1)$, $(x,y,1)$ and the z axis. The intersection of this plane with the wall containing $(\overline{x},\overline{y},1)$ is a vertical line containing that same point: $L_{x,y}=\left\{X=\frac{x}{||(x,y)||_\infty},Y=\frac{y}{||(x,y)||_\infty}\right\}$.

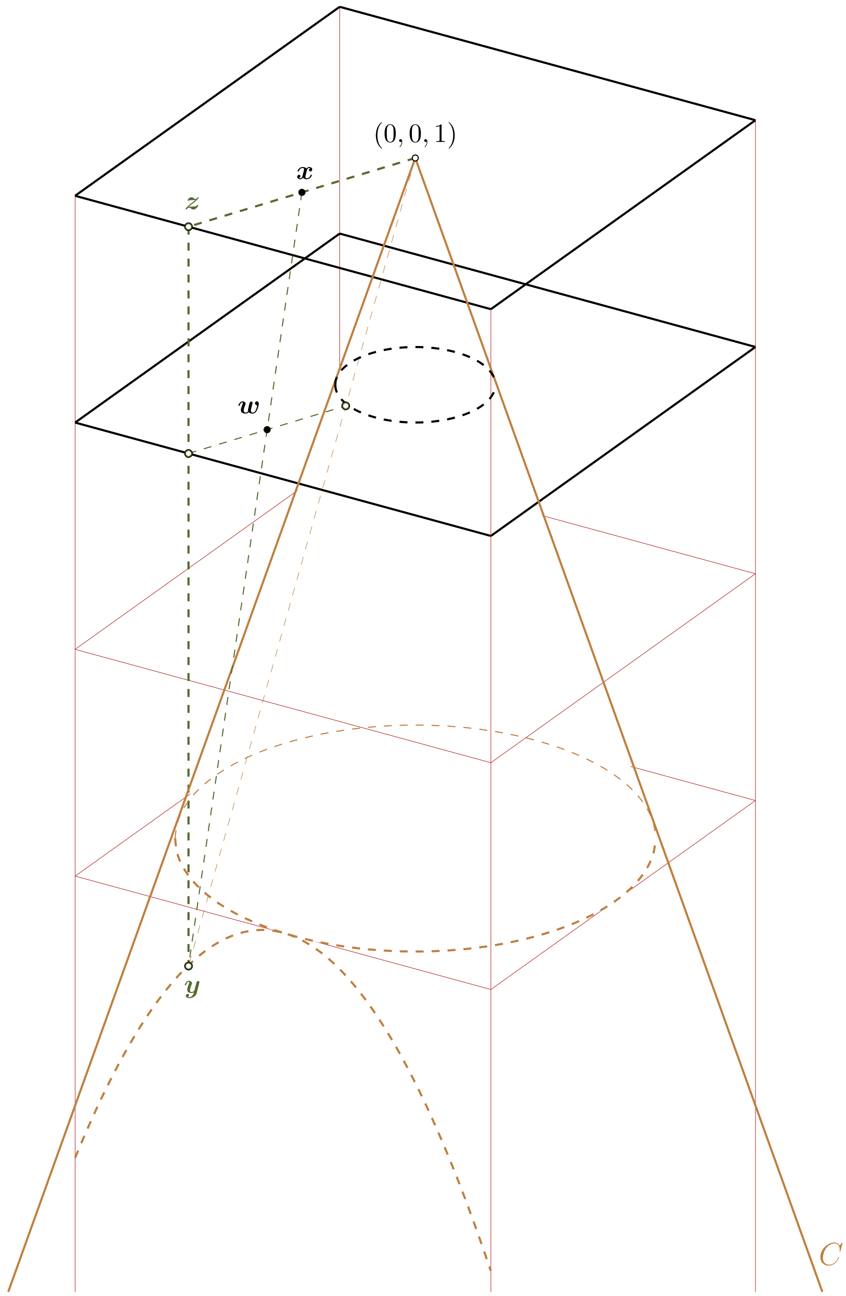
This line intersects cone $x^2+y^2=\frac{1}{9}(1-z)^2$ at $\boldsymbol{y}=\left(\frac{x}{||(x,y)||_\infty},\frac{y}{||(x,y)||_\infty},1-3\frac{||(x,y)||_2}{||(x,y)||_\infty}\right)$ and the line joining $\boldsymbol{y}$ and $\boldsymbol{x}=(x,y,1)$ intersects plane $z=0$ at point

$$\boldsymbol{w}=\left(\frac{x\left(1+3\,||(x,y)||_2-||(x,y)||_\infty\right)}{3\,||(x,y)||_2},\frac{y\left(1+3\,||(x,y)||_2-||(x,y)||_\infty\right)}{3\,||(x,y)||_2},0\right),$$

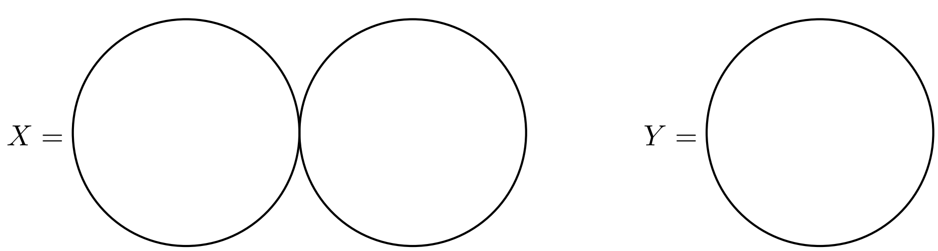
thereby defining our function as $f:[-1,1]^2\setminus\{\mathbf{0}\}\to[-1,1]^2\setminus\overline{B\left(\mathbf{0},\frac{1}{3}\right)}$ by $(x,y)\mapsto(\alpha x,\alpha y)$ with $\alpha=\frac{1+3||(x,y)||_2-||(x,y)||_\infty}{3||(x,y)||_2}$. The function is well-defined on account of $||(x,y)||_2\neq0$ and its inverse is built by following the path described above (and in the figure below) in reverse order: $g:(x,y)\mapsto(\beta x,\beta y)$, where $\beta=\frac{3||(x,y)||_2-1}{3||(x,y)||_2-||(x,y)||_\infty}$. Checking $f\circ g=\mathrm{id}_{[0,1]^2\setminus\overline{B}}$ and

$g \circ f = \mathrm{id}_{[0,1]^2 \setminus \{\mathbf{0}\}}$ as well as the continuity of these functions is left as an Exercise.

$$
\begin{aligned}
\boldsymbol{x} &= (x, y, 1) \\
\boldsymbol{z} &= \left(\frac{x}{||(x,y)||_\infty}, \frac{y}{||(x,y)||_\infty}, 1 \right) \\
L_{x,y} &= \overline{\boldsymbol{xz}} \cap \{x = \pm 1, y = \pm 1\} \\
\boldsymbol{y} &= L_{x,y} \cap C \\
\boldsymbol{w} &= (f(x,y), 1) \\
&= \overline{\boldsymbol{y}\,(0,0,1)} \cap \{z = 0\}
\end{aligned}
$$



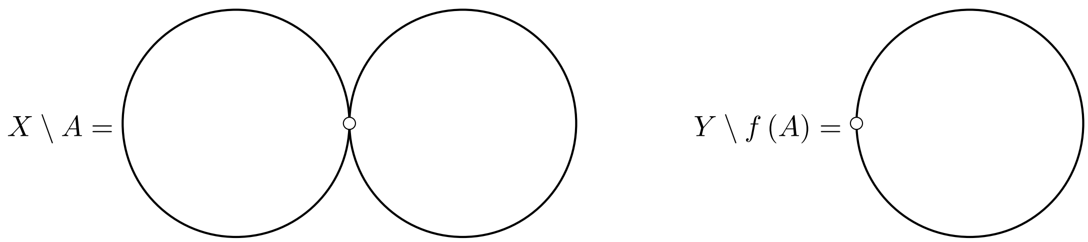
76. (May 2015) Prove the following subspaces of $(\mathbb{R}^2, \tau^{\mathrm{Eucl}})$ are not homeomorphic using the tools seen hitherto:



Hint: you may use Remark 3.3.7 and ancillary homeomorphisms if you want to.

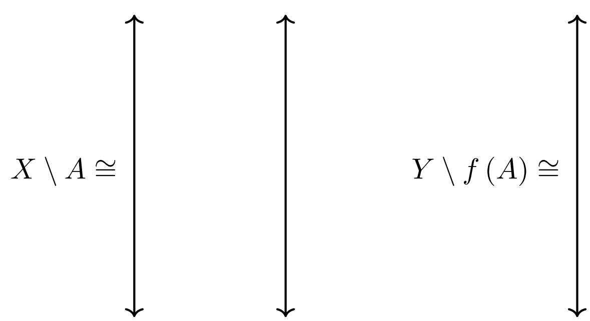
Solution. A remark in the Lecture Notes asserts the following: If $f : X \to Y$ is a homeomorphism and $A \subset X$, then $A \cong f(A)$ and $X \setminus A \cong f(X \setminus A) = Y \setminus f(A)$.

Hence taking just one point (because homeomorphisms transform single points into single points) from each space should keep the homeomorphism; this should be true regardless of the points chosen, hence if A is the point of tangency between the two circles, the two spaces described below should be homeomorphic,



and the choice of the missing point in the second set is irrelevant (rotations are homeomorphisms).

$X \setminus A$ is homeomorphic to two infinite parallel lines, e.g. by means of two simultaneous stereographic projections taken from the line tangent to both circles at the missing point. A similar stereographic projection is a homeomorphism between $Y \setminus f(A)$ and a single line:



And it all boils down to proving that two parallel lines $L_1 \amalg L_2$ can never be homeomorphic to a single line L_3.

If they were homeomorphic, there should exist a homeomorphism $g : L_1 \amalg L_2 \to L_3$.

L_1 and L_2 are open in $L_1 \amalg L_2$, and their intersection is empty. Thus $g(L_1)$ and $g(L_2)$ are open in L_3 as well if we assume g is a homeomorphism, and $g(L_1 \amalg L_2) = g(L_1) \uplus g(L_2) = L_3$. In other words, L_3 (which is homeomorphic to the real line) would have to be the disjoint union of two open sets $g(L_1) \uplus g(L_2)$. Thus same should apply to $\mathbb{R}$: $\mathbb{R} = V_1 \uplus V_2$. Then the function

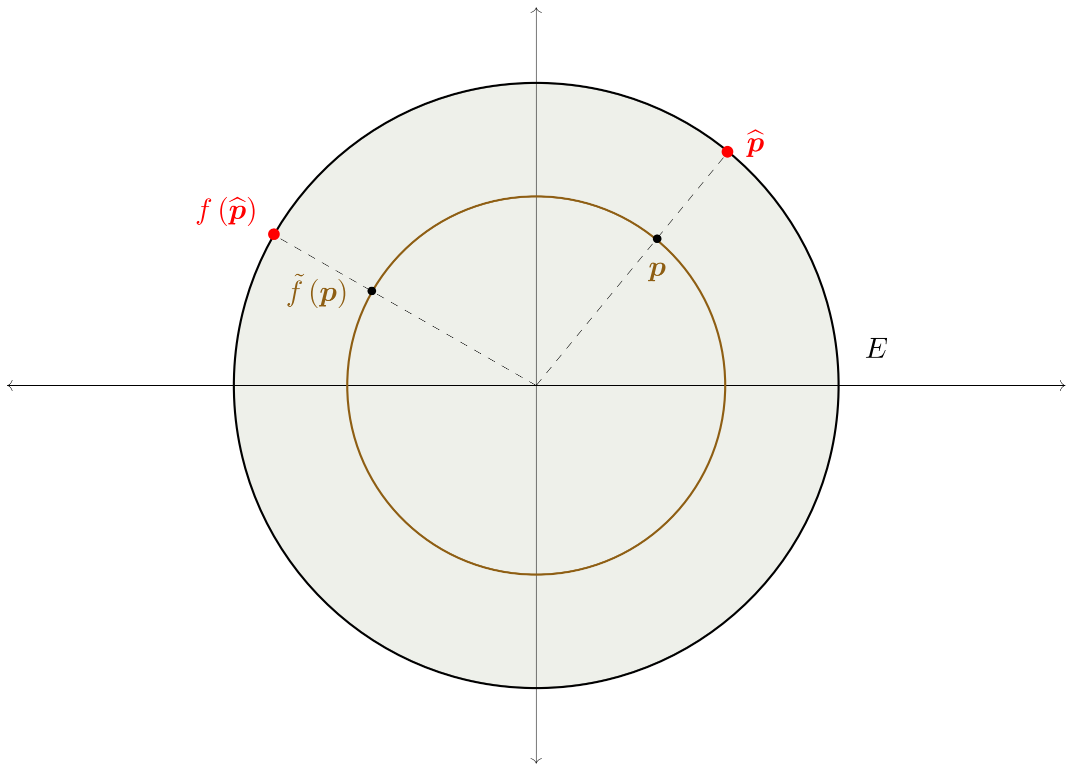
$$f : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} 1 & \text{if } x \in V_1, \\ -1 & \text{if } x \in V_2, \end{cases}$$

would be continuous, contradicting Bolzano's Theorem.

77. Let $E := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the unit disk and, consequently, $\mathbb{S}^1$ its boundary. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be any homeomorphism. Prove f can be extended to a homeomorphism on the entire disk, i.e. there exists $\tilde{f} : E \xrightarrow{\cong} E$ such that $\tilde{f}\big|_{\mathbb{S}^1} = f$.

Solution. For example, we can define the function $\tilde{f} : E \to E$ which reproduces the behaviour of f (on the boundary ∂E) on a smaller scale, on any circle of radius $0 \leq r < 1$

centered at the origin: given $\boldsymbol{p}$ belonging to that smaller circle, proceed as follows:



The expression for $\widehat{\boldsymbol{p}}$, given $\boldsymbol{p} = (x, y)$, is $\frac{1}{\|\boldsymbol{p}\|}(x, y)$ (unless $\boldsymbol{p} = (0, 0)$ in which case we will have to define $\tilde{f}(\boldsymbol{p}) = (0, 0)$). Hence

$$\tilde{f} : E \to E, \qquad (x, y) \mapsto \begin{cases} \sqrt{x^2 + y^2} f\left(\frac{1}{\sqrt{x^2+y^2}}(x, y)\right), & (x, y) \neq (0, 0), \\ (0, 0), & (x, y) = (0, 0) \end{cases}$$

is a continuous function: it is certainly continuous outside of the origin, and for every $\varepsilon > 0$, let us find $\delta > 0$ such that $\tilde{f}(B_{d_2}((0, 0), \delta)) \subset B_{d_2}((0, 0), \varepsilon)$. We can choose $\delta = \varepsilon$, since $\tilde{f}$ does not alter the norm of points outside of 0 and thus $\tilde{f}(B_{d_2}((0, 0), \delta)) \subset B_{d_2}((0, 0), \delta)$.

The inverse of $\tilde{f}$ will be

$$\tilde{g} : E \to E, \qquad (x, y) \mapsto \begin{cases} \sqrt{x^2 + y^2} g\left(\frac{1}{\sqrt{x^2+y^2}}(x, y)\right), & (x, y) \neq (0, 0), \\ (0, 0), & (x, y) = (0, 0) \end{cases}$$

where g is the inverse of homeomorphism f. $\tilde{g}$ is continuous for the same reasons $\tilde{f}$ is, and the fact g is continuous (being the inverse of a homeomorphism). $\tilde{g}$ is the inverse of $\tilde{f}$: for every non-trivial (x, y),

$$\tilde{g} \circ \tilde{f} : (x, y) \mapsto (X, Y) = \sqrt{x^2 + y^2} f\left(\frac{1}{\sqrt{x^2 + y^2}}(x, y)\right) \mapsto \sqrt{X^2 + Y^2} g\left(\frac{1}{\sqrt{X^2 + Y^2}}(X, Y)\right),$$

which equals ($\sqrt{x^2 + y^2} = \sqrt{X^2 + Y^2}$, $g \circ f = \mathrm{id}_{\mathbb{S}^1}$)

$$\sqrt{x^2 + y^2} g\left(\frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} f\left(\frac{(x, y)}{\sqrt{x^2 + y^2}}\right)\right) = \sqrt{x^2 + y^2} g\left(f\left(\frac{(x, y)}{\sqrt{x^2 + y^2}}\right)\right) = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}(x, y) = (x, y),$$

hence $\tilde{g} \circ \tilde{f} = \mathrm{id}_E$. The other identity $\tilde{f} \circ \tilde{g} = \mathrm{id}_E$ follows along the same lines.

We have two continuous functions $E \to E$ which are inverses of one another, hence homeomorphisms. Both restrict to homeomorphisms on $\mathbb{S}^1$ if (x, y) has norm 1.

78. Let $f : X \to Y$ and $g : Y \to Z$ be two functions between topological spaces. Prove the following:

a) If f and g are open functions, so is $g \circ f$.

b) If $g \circ f$ is open and f is surjective and continuous, then g is open.

c) If $g \circ f$ is open and g is injective and continuous, then f is open.

Solution

a) For any open set $U \subset X$, $f(U)$ is open in Y (f being open), hence $g(f(U))$ is open in Z since g is open.

b) Let U be an open set of Y. The fact f is surjective implies $U = f\left(f^{-1}(U)\right)$ (in general only one of the inclusions would hold, think which one). The fact f is continuous implies $f^{-1}(U)$ is open in X. Hence $g(U) = g\left(f\left(f^{-1}(U)\right)\right)$, the image of an open set $\left(f^{-1}(U)\right)$ by an open function $(g \circ f)$, hence open in Z.

c) Let U be an open set in X. We want to prove $f(U)$ is open. We know $g \circ f$ is an open map, hence $g(f(U))$ is open in Z. g is continuous, hence $g^{-1}(g(f(U)))$ is open in Y. g is injective, hence $g^{-1}(g(f(U))) = f(U)$ (again, Exercise think which inclusion would not necessarily hold, in general, were the function not injective). Hence $f(U)$ is open in Y.

79. (*July 2013*) Let $\mathbb{R}_+ := [0, \infty)$,

$$f : \mathbb{Z} \to \mathbb{R}_+, \qquad n \mapsto f(n) := n^2,$$

and τ, τ_1, τ_2 be the topologies defined as follows:

$$\begin{aligned}
\tau &:= \{\varnothing, \mathbb{R}_+\} \cup \{[0, x) : x \geq 0\}, \\
\tau_1 &:= \{\varnothing, \mathbb{Z}\} \cup \{\{-n, -n+1, \ldots, n-1, n\} : n \in \mathbb{Z}\}, \\
\tau_2 &:= \{\varnothing, \mathbb{Z}\} \cup \{\{-n, -n+1, \ldots, n-1, n\} \setminus \{0\} : n \in \mathbb{Z}\}.
\end{aligned}$$

Study the continuity of $f : (\mathbb{Z}, \tau_i) \to (\mathbb{R}_+, \tau)$ for $i = 1, 2$.

Solution. Handwritten at the end of this section.

3.6 Exercises

80. Prove Remarks 3.3.7.

81. Prove Proposition 3.3.5.

82. (*Sample Coursework November 2013*) Find an explicit expression of the stereographic projection of the unit n-sphere on the space $\mathbb{R}^n$ and prove $\mathbb{S}^n \setminus (0, \ldots, 0, 1) \cong \mathbb{R}^n$.

83. Prove $\left(\mathbb{R}, \tau_{\mathbb{R}}^{\mathrm{Eucl}}\right) \not\cong \left(\mathbb{R}^2, \tau_{\mathbb{R}^2}^{\mathrm{Eucl}}\right)$.

84. Let $E := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the unit disk and, consequently, $\mathbb{S}^1$ its boundary. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be any homeomorphism.

Prove f can be extended to a homeomorphism on the entire disk, i.e. there exists $\tilde{f} : E \xrightarrow{\cong} E$ such that $\tilde{f}\Big|_{\mathbb{S}^1} = f$.

85. A **retract** of a topological space X is any subspace $A \subset X$ for which there exists a continuous function $r : X \to A$ such that $r \circ i = \mathrm{id}_A$, where $i : A \hookrightarrow X$, $x \mapsto x$ is the usual inclusion. Prove:

(i) $[0,1]$ is a retract of $\mathbb{R}$;

(ii) $\{\boldsymbol{x} \in \mathbb{R}^n : ||\boldsymbol{x}|| \leq 1\}$ is a retract of $\mathbb{R}^n$;

(iii) $\mathbb{S}^{n-1}$ is a retract of $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

86. Prove that the torus is not homeomorphic to the 2-sphere.

4) $\mathbb{R}_+ = [0, +\infty)$

$$f : \mathbb{Z} \longrightarrow \mathbb{R}_+$$
$$n \longmapsto f(n) = n^2$$

$$\tau = \{\emptyset, \mathbb{R}\} \cup \{[0, x) : x \geq 0\}$$
$$\tau_1 = \{\emptyset, \mathbb{R}\} \cup \{[-n, n] \cap \mathbb{Z} : n \in \mathbb{Z}\}$$
$$\tau_2 = \{\emptyset, \mathbb{R}\} \cup \{[-n, n] \cap \mathbb{Z} \setminus \{0\} : n \in \mathbb{Z}\}.$$

* The student may prove τ, τ_1, τ_2 are topologies on $\mathbb{Z}$, $\mathbb{R}_+$ but this is not required.

* Continuity of $f : (\mathbb{Z}, \tau_1) \to (\mathbb{R}_+, \tau)$:

f is continuous iff for every open set $U \in \tau$, $f^{-1}(U) \in \tau_1$.

Let $U \in \tau$. U can be of the following types:

- $U = \emptyset \Rightarrow f^{-1}(U) = \emptyset \in \tau_1$ ✓
- $U = \mathbb{R}_+ \Rightarrow f^{-1}(U) = \{n \in \mathbb{Z} : n^2 \geq 0\}$ $= \mathbb{Z} \in \tau_1$ ✓

- $U = [0, x)$ for some $x \geq 0$.

In such case,

$$
\begin{aligned}
f^{-1}(U) &= f^{-1}([0,x)) \\
&= \{n \in \mathbb{Z} : n^2 < x\} \\
&= \begin{cases} \{0\} & \text{if } x \leq 1 \\ \{-[\sqrt{x}]+1, \dots, [\sqrt{x}]-1\}, & \text{if } x > 1 \end{cases} \\
&\in \tau_1 \quad \checkmark
\end{aligned}
$$

Hence the pre-image of every open set in τ is an open set in τ_1, which means

f is continuous.

* Let us study the continuity of

$$f : (\mathbb{Z}, \tau_2) \longrightarrow (\mathbb{R}_+, \tau).$$

Let $U \in \tau$. Then

- $U = \emptyset \Rightarrow f^{-1}(U) = \emptyset \in \tau_2$ ✓
- $U = \mathbb{R}_+ \Rightarrow f^{-1}(U) = \{n \in \mathbb{Z} : n^2 \geq 0\}$
  $= \mathbb{Z} \in \tau_2$ ✓

- $U = [0,x) \Rightarrow f^{-1}([0,x)) \notin \tau_2$.

  Indeed, $f^{-1}([0,x))$ contains 0 (see above).

and open sets of τ_2 do not contain 0 unless they are all of $\mathbb{Z}$.

Therefore, f is not continuous if considered on $(\mathbb{Z}, \tau_2)$.

Chapter 4

Construction of topological spaces

We will now introduce the usual procedures in which to obtain new topological spaces from already existing ones: products, identifications, wedge sums and similar examples. All constructions adjust to one of two models: initial and final topologies with respect to a family of functions.

4.1 Initial topology

4.1.1 Introduction

Motivated by Examples 3.1.3, we know any function $X \to Y$ will always be continuous whenever X is endowed with the discrete topology but would like coarser topologies retaining this continuity condition–at the expense of having to focus on a fixed set of given functions $X \to Y$ rather than all of them.

In Section 2.3 we introduced a way of defining a default topology on every subset of a known topological space; this topology on $Y \subset X$ could be seen as "induced" by the inclusion $i_Y : Y \hookrightarrow X$ in a way that made i_Y automatically continuous.

Changing Y and i_Y by arbitrary set Z and function $f : Z \to X$, define

$$\tau_Z := \left\{ f^{-1}(U) : U \text{ open in } X \right\}. \tag{4.1}$$

This is a topology (Exercise 101) and is the coarsest topology rendering f continuous. Let us generalise this to any finite amount of functions:

$$f_i : Z \to X_i, \qquad i = 1, \ldots, n,$$

where (X_i, τ_i) are topological spaces.

Definition 4.1.1. *The* ***initial topology*** *on Z with respect to functions $f_1, \ldots, f_n$ is the topology having basis*

$$\beta_Z := \left\{ f_1^{-1}(U_1) \cap \cdots \cap f_n^{-1}(U_n) : U_i \in \tau_i,\ i = 1, \ldots, n \right\}.$$

Let us prove β_Z indeed satisfies the necessary and sufficient properties for a basis given in Proposition 2.5.4:

- B_1: $\varnothing \in \beta_Z$ by definition and $Z = f_i^{-1}(X_i)$ for every $i = 1, \ldots, n$.
- B_2: The intersection of any two elements of β_Z belongs to β_Z, hence it is a union of elements (namely one) of β_Z.

Example 4.1.2. The induced topology on any subset Y of a topological space X is the initial topology with respect to the inclusion $i_Y : Y \hookrightarrow X$.

By construction of β_Z, all functions f_i in the above Definition are continuous on Z. We could have done this endowing Z with the discrete topology from the outset (see Example 3.1.3 (**3**), but it would have implied taking more open sets than was necessary. Let us prove that the "smallest" amount of sets necessary to render functions $f_1, \ldots, f_n$ continuous is precisely the one spanned by basis β_Z:

Proposition 4.1.3. *The initial topology is the coarsest topology on which $f_1, \ldots, f_n$ are continuous.*

Proof. let τ_Z be a topology on Z for which $f_1, \ldots, f_n$ are continuous. Then, for every open set $U_i \in \tau_i$, $f^{-1}(U_i) \in \tau_Z$. Using property T_3 for topologies every finite intersection $f_1^{-1}(U_1) \cap \cdots \cap f_n^{-1}(U_n)$ must belong to τ_Z, and using property T_2 for topologies, every union of such open intersections must be an open set as well. Hence τ_Z must be finer than or equal to the initial topology on Z. $\square$

Remark 4.1.4. The construction of the initial topology can also be made for an arbitrary set of maps $f_i : Z \to (X_i, \tau_i)$, $i \in I$, so long as intersections remain finite:

$$\beta_Z := \left\{ f_{i_1}^{-1}(U_{i_1}) \cap \cdots \cap f_{i_n}^{-1}(U_{i_n}) : U_{i_j} \in \tau_{i_j},\, j = 1, \ldots, n,\, i_1, \ldots, i_n \in I \right\}.$$

In other words: the set of pre-images

$$\mu_Z = \left\{ f_i^{-1}(U) : U \in \tau_i, i \in I \right\},$$

is a sub-basis for a unique topology on X in virtue of Proposition 2.5.11 (its elements covering all of X).

Proposition 4.1.5. *Let τ_Z be the initial topology on Z with respect to maps $f_i : Z \to X_i$, $i \in I$. A function $\varphi : Y \to Z$ is continuous if, and only if, all compositions $f_i \circ \varphi$, $i \in I$, are continuous.*

Proof. If φ is continuous, so are $f_i \circ \varphi$ for every $i \in I$. Conversely, assume $f_i \circ \varphi$, $i \in I$ are continuous. We have

$$\varphi^{-1}\left(f_{i_1}^{-1}(U_{i_1}) \cap \cdots \cap f_{i_n}^{-1}(U_{i_n})\right) = (f_{i_1} \circ \varphi)^{-1}(U_{i_1}) \cap \cdots \cap (f_{i_n} \circ \varphi)^{-1}(U_{i_n})$$

for any open sets $U_{i_k} \in \tau_k$, $k = 1, \ldots, n$, hence using Proposition 3.2.1 φ must be continuous. $\square$

Proposition 4.1.6. *The initial topology is the only topology on X, up to homeomorphism, for which the property in Proposition 4.1.5 holds.*

4.1.2 Product topology

Let (X, τ_X) and (Y, τ_Y) be two topological spaces. We would like to define a topology on the Cartesian product $X \times Y$. A natural request we should be entitled to is the continuity of projections

$$\mathrm{pr}_X : X \times Y \to X, \qquad (x, y) \mapsto x,$$

and

$$\mathrm{pr}_Y : X \times Y \to X, \qquad (x, y) \mapsto y.$$

Definition 4.1.7. *The **product topology** on $X \times Y$ is the initial topology with respect to pr_X and pr_Y. Hence, a basis for the topology is*

$$\beta_{X \times Y} = \left\{ \mathrm{pr}_X^{-1}(U) \cap \mathrm{pr}_Y^{-1}(V) = U \times V : U \in \tau_X, V \in \tau_Y \right\}.$$

Remark 4.1.8. The product topology on any finite amount of topological spaces $X_1, \dots, X_n$ can be defined analogously.

Examples 4.1.9.

1. Let $X = \mathbb{R}$ with the Euclidean topology. The product topology on $\mathbb{R} \times \mathbb{R}$ is the one generated by a basis of open sets of the form $(a, b) \times (c, d)$, $a, b, c, d \in \mathbb{R}$ with $a \leq b$ and $c \leq d$. Hence it coincides with the Euclidean topology on $\mathbb{R}^2$: rectangles can be contained in Euclidean balls, and viceversa. Bear in mind *not all open subsets of* $\mathbb{R}^n$ *are of the form* $U \times V$.

2. In a similar way the product topology on $\mathbb{R}^n = \left(\mathbb{R}, \tau_{\mathbb{R}}^{\text{Eucl}}\right) \times \cdots \times \left(\mathbb{R}, \tau_{\mathbb{R}}^{\text{Eucl}}\right)$ is exactly the Euclidean topology on this set.

3. There is an analog definition for infinite copies of $\mathbb{R}$ in **2.**. Exercise: think how.

4. If (X, d_X) and (Y, d_Y) are metric spaces, the topology associated to the *product metric* introduced in Example 1.1.4 (**5**),
$$d_{X \times Y}\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) := \max\left\{d_X\left(x_1, x_2\right), d_Y\left(y_1, y_2\right)\right\},$$
is the product topology.

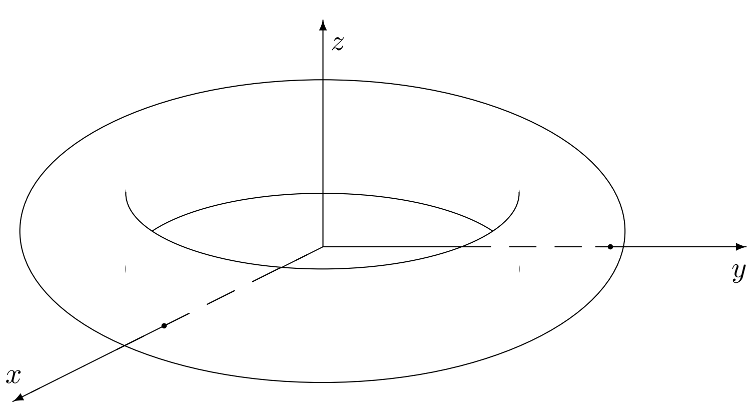
5. The product of induced topologies on $S_1 \subset X_1$ and $S_2 \subset X_2$ is equal to induced subspace topology on $S_1 \times S_2 \subset X_1 \times X_2$. Exercise.

Propositions 4.1.3 and 4.1.5 become the following in this setting.

Proposition 4.1.10. *Projections* pr_{X_i} *are continuous whenever* $X_1 \times \cdots \times X_n$ *is endowed with the product topology, and the latter is the coarsest in which the former are continuous.* □

Proposition 4.1.11. *Let* X, Y, Z *be topological spaces and* $f : Z \to X$, $g : Z \to Y$ *two functions. Map*
$$\varphi : Z \to X \times Y, \qquad z \mapsto \left(f\left(z\right), g\left(z\right)\right),$$
is continuous if, and only if, coordinate functions f *and* g *are continuous.* □

Example 4.1.12. We define $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$ as the **(2-dimensional) torus**. Let us prove $\mathbb{T}^2$ is homeomorphic to the following subspace of $\mathbb{R}^3$ (bear in mind the original space is defined in $\mathbb{R}^4$):
$$X := \left\{(x, y, z) \in \mathbb{R}^3 : \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 = 1\right\},$$
obtained rotating a circle of centre $(2, 0)$ and radius 1 on the (x, z) plane around the z axis:



Indeed, functions

$$f : \mathbb{T}^2 \to X, \qquad ((x, y), (z, t)) \mapsto (x(z+2), y(z+2), t)$$

and

$$g : X \to \mathbb{T}^2, \qquad (x, y, z) \mapsto \left(\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right), \left(\sqrt{x^2+y^2} - 2, z\right)\right),$$

are mutual inverses of one another (Exercise: verify this) and thus bijective. Using the previous Proposition we conclude g is continuous. In order to prove g is also closed, use Corollary 6.3.7 in Section 6.3. Hence g is a homeomorphism.

4.2 Final topology

Same as above, and still motivated by Examples 3.1.3, we know any function $X \to Y$ will always be continuous whenever Y is endowed with the coarse topology but would like finer topologies on Y retaining this continuity condition–at the expense of having to focus on a fixed set of given functions $X \to Y$.

4.2.1 Definition and first examples

Let X be a topological space and $f : X \to Y$ a map from X to an arbitrary set Y. We would like to endow Y with a topology rendering f continuous. In virtue of Example 3.1.3 (**2**), endowing Y with the coarse topology would suffice, but we would like to do this in an *optimal* way, the operative word implying "as many open sets in Y as possible" in this context. Hence we would like the topology on Y to be as fine as possible making f continuous.

Definition 4.2.1. *The* ***final topology*** *on Y with respect to f is equal to*

$$\tau_Y := \left\{U \subset Y : f^{-1}(U) \text{ is open in } X\right\}.$$

Checking it is a topology is trivial: $\varnothing = f^{-1}(\varnothing)$, $X = f^{-1}(Y)$ and given any collection of subspaces $\{U_i\}_{i\in I}$ whose pre-images are open in X, $f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i)$ which is an open set in X; finally, $f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i)$ which is open as well if I is finite.

Proposition 4.2.2. *The final topology on Y with respect to f is the finest topology for which f is continuous.*

Proof. f is obviously continuous by construction. Let τ be any other topology on Y for which $f : X \to Y$ is continuous. Then for every $U \in \tau$ we have $f^{-1}(U)$ is an open set of X, hence $U \in \tau_Y$ as well. Therefore $\tau \subset \tau_Y$. □

We may also generalise this to any finite number of functions,

$$f_i : X_i \to Y, \qquad i = 1, \ldots, n,$$

where X_i are topological spaces.

Definition 4.2.3. *We define the* ***final topology with respect to*** *$f_1, \ldots, f_n$ on Y in the following way:*

$$\tau_Y := \left\{U \subset Y : f_i^{-1}(U) \text{ is open in } X_i \text{ for every } i = 1, \ldots, n\right\}.$$

This is the finest topology rendering all $f_1, \ldots, f_n$ continuous (see Exercise 102).

Proposition 4.2.4. *With the above notations let $g : Y \to Z$ be a function between topological spaces. Then g is continuous if, and only if, $g \circ f_i$ are continuous for every $i = 1, \ldots, n$.*

Proof. If g is continuous, $g \circ f_i$ will be continuous for every i. Conversely, let U be an open set of Z. The definition of the final topology entails $g^{-1}(U)$ is open if and only if $f_i^{-1}\left(g^{-1}(U)\right)$ is open in X_i for every i, which implies the continuity of $g \circ f_i$ for $i = 1, \ldots, n$. □

4.2.2 Identifications

Definition 4.2.5. *An* ***identification*** *is a continuous and surjective function $f : X \to Y$ such that Y has the final topology with respect to f.*

Proposition 4.2.6. *Let $f : X \to Y$ be a continuous, surjective and open (resp. closed) function. Then f is an identification.*

Proof. All we need to prove is that Y has the final topology with respect to f, i.e. for every subset $U \subset Y$ such that $f^{-1}(U)$ is open in X, U is open in Y.

Assume f is open; the fact f is surjective implies $U = f\left(f^{-1}(U)\right)$, hence U is open.

Assume f is closed. Again due to the fact f is surjective we have $f\left(X \setminus f^{-1}(U)\right) = Y \setminus U$. Hence $Y \setminus U$ is closed which implies U is open. □

Examples 4.2.7.

1. $X = [0, 1]$, $Y = \mathbb{S}^1$ and $f(t) = (\cos 2\pi t, \sin 2\pi t)$ fulfil the conditions: f is continuous and surjective and applying Corollary 6.3.7 in Section 6.3 once again, it is closed. Hence it is an identification.

2. Let X be a topological space and $\sim$ a relation of equivalence on X. Let $X/\sim$ be the set of all classes according to this relation of equivalence and

$$\pi : X \to X/\sim, \qquad x \mapsto [x],$$

the class map. Endowing $X/\sim$ with the final topology with respect to π we have an identification. We call this final topology the **quotient topology**.

We can obtain equivalence relations on a set X using maps $f : X \to Y$ to other sets. More specifically, the relation

$$x_1 \sim_f x_2 \text{ if and only if } f(x_1) = f(x_2), \quad \text{for every } x_1, x_2 \in X, \tag{4.2}$$

is a relation of equivalence (Exercise 103).

Lemma 4.2.8. *Let $f : X \to Y$ be an identification. Then Y is homeomorphic to $X/\sim_f$.*

Proof. The definition of an equivalence relation implies the existence of an injective set map $\varphi : X/\sim_f \to Y$ such that $f = \varphi \circ \pi$, where

$$\pi : X \to X/\sim_f, \qquad x \mapsto [x],$$

is the class projection. The fact f is surjective implies the same property for φ, which is therefore bijective. The fact Y and $X/\sim_f$ both have the final topologies implies that for any subset U of Y, U is open if, and only if, $f^{-1}(U) = \pi^{-1}\left(\varphi^{-1}(U)\right)$ is open, which holds if and only if $\varphi^{-1}(U)$ is open. Hence, φ is a homeomorphism. □

Examples 4.2.9.

1. Let X be a topological space and $A \subset X$ a subset. We define the equivalence relation

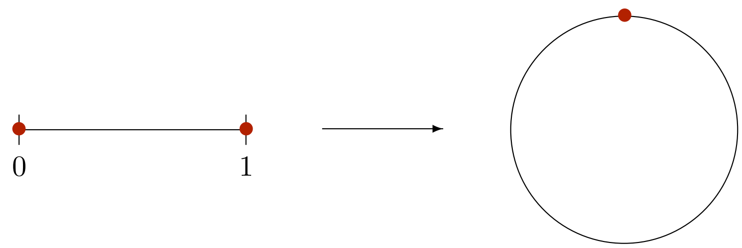
$$x \sim_A x, \text{ for every } x \in X, \qquad x \sim_A y, \text{ for every } x, y \in A.$$

We write set $X/\sim_A$ as X/A and endow it with the final topology with respect to the projection $\pi : X \to X/\sim_A$. We say we have **identified** A **into a point** to obtain this quotient topology.

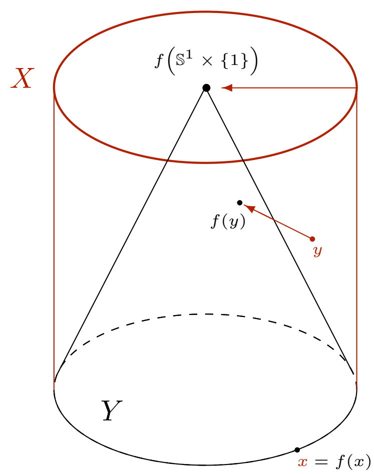
2. An example of the above. Let $X = [0,1]$ and $A = \{0,1\}$. The relation of equivalence $\sim_A$ coincides with $\sim_f$ where

$$f : [0,1] \to \mathbb{S}^1, \qquad t \mapsto (\cos 2\pi t, \sin 2\pi t)\,,$$

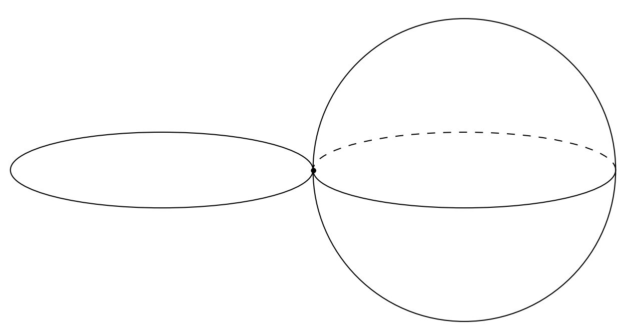
and according to Lemma 4.2.8 X/A is homeomorphic to $\mathbb{S}^1$:



3. Let $X = \mathbb{S}^1 \times [0,1]$ and $A = \mathbb{S}^1 \times \{1\}$. Let $\sim_A$ be the relation defined in the previous example. Consider the continuous function f defined from the cylinder X to the cone Y as shown in the figure below. The fact f is surjective, along with Corollary 6.3.7 in Section 6.3 once again, means f is an identification. Furthermore, the relation of equivalence $\sim_f$ linked to this map coincides with relation $\sim_A$. Hence $Y \cong X/\sim_f = X/\sim_A$.



4. Let X be the disjoint union of two topological spaces X_1 and X_2, and let $x_1 \in X_1$ and $x_2 \in X_2$. Let $A = \{x_1, x_2\}$. We say $X/\sim_A$ is the **wedge sum** of X_1 and X_2. For instance,

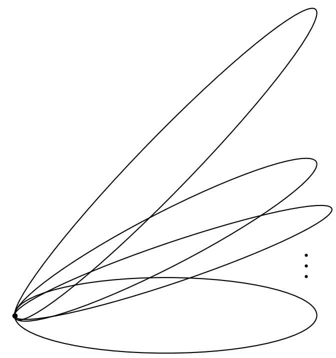
$\mathbb{S}^1 \vee \mathbb{S}^2 =$ 

5. We may generalise the above operation using several topological spaces, e.g. if

$$X = \mathbb{S}^1 \sqcup \overset{n}{\cdots} \sqcup \mathbb{S}^1, \quad A = \{1\} \cup \overset{n}{\cdots} \cup \{1\}\,,$$

then

$$X/A = \mathbb{S}^1 \vee \mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1 =$$



which is usually called a *bouquet of circles*.

4.3 Locally Euclidean spaces and manifolds

Among all topological spaces, let us pinpoint those whose local behaviour is completely analogous to that of the Euclidean space $\mathbb{R}^n$.

Definition 4.3.1. *A topological space X is called **locally Euclidean of dimension** n if every point x has an open neighbourhood homeomorphic to the open ball $B(\mathbf{0};1)$ of centre $(0,\ldots,0)$ and radius 1 in $\mathbb{R}^n$.*

The following is immediate:

Lemma 4.3.2. *Being locally Euclidean is a property invariant under homeomorphism. In other words: if $X \cong Y$ then X is locally Euclidean if, and only if, Y is.*

Proof. Exercise 104. $\square$

Definition 4.3.3. *A topological space X is **Hausdorff** (or T_2) if for every two different elements $x, y \in X$ there exist two disjoint open sets U, V such that $x \in U$ and $y \in V$.*

Definition 4.3.4. *A topological space X is a **(topological) manifold** if it is locally Euclidean and Hausdorff.*

Examples 4.3.5. Some locally Euclidean spaces of dimension 1.

1. $\mathbb{R}$ or any open set in $\mathbb{R}$.

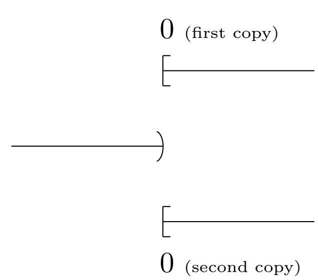
2. $\mathbb{S}^1$.

3. Let $X = \mathbb{R} \sqcup \mathbb{R}$. We define the relation of equivalence on X: $x \sim x$ for every $x \in X$, and

$$x \sim y \quad \text{if} \quad \begin{cases} x \in \text{first copy of } \mathbb{R} \text{ and } x < 0, \\ y \in \text{second copy of } \mathbb{R} \text{ and } y < 0, \\ |x| = |y| \end{cases}$$

Hence

$$X/\sim\, = \tag{4.3}$$



This space is locally Euclidean. An interesting exercise (105) is finding neighbourhoods of 0 (first copy) and 0 (second copy) homeomorphic to $(-1,1)$. Use this to prove the space is locally Euclidean, but not a manifold.

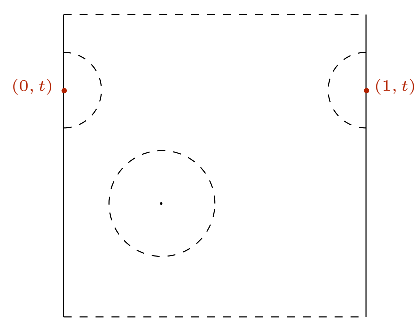
4. Any curve in $\mathbb{R}^n$, i.e. the image of any continuous function $\gamma : I \to \mathbb{R}^n$, is a one-dimensional manifold.

Examples 4.3.6. Some manifolds of dimension 2.

1. $\mathbb{S}^2$ is locally Euclidean. Indeed, for every point $\boldsymbol{p} = (x, y, z) \in \mathbb{S}^2$ the stereographic projection based at $-\boldsymbol{p} \in \mathbb{S}^2$ provides a homeomorphism of $\mathbb{S}^2 \setminus \{-\boldsymbol{p}\}$ with $\mathbb{R}^2$.

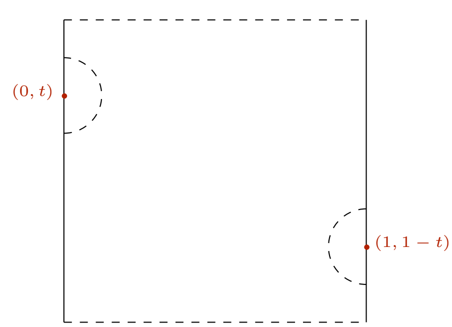
2. The same reasoning as above proves $\mathbb{S}^n$ is a locally Euclidean space.

3. Let $Y = (0,1) \times \mathbb{S}^1$ the open cylinder. The fact $\mathbb{S}^1$ is the identification of points 0 and 1 in $[0,1]$ implies we can also describe the cylinder by identifying points of the form $(0,t)$ and $(1,t)$, $t \in (0,1)$, in the square $X = [0,1] \times (0,1)$:



A neighbourhood (homeomorphic to a 2-ball) of such identified point is the union of the two open semi-balls drawn in the picture. A similar neighbourhood of any other point (meaning: any point in the interior of the square) is shown in the picture as well. Hence the cylinder is locally Euclidean of dimension 2

4. The **Möbius band** is obtained by identifying, in $X = [0,1] \times [0,1]$, points of the form $(0,t)$ with those of the form $(0, 1-t)$:



A similar reasoning to that of the cylinder proves this is a dimension-two locally Euclidean space.

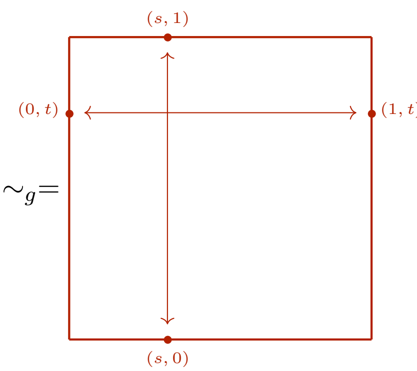
5. The torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ is also a locally Euclidean topological space of dimension 2. Indeed, function

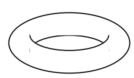
$$g : X = [0,1] \times [0,1] \to \mathbb{S}^1 \times \mathbb{S}^1, \qquad (s,t) \mapsto ((\cos 2\pi s, \sin 2\pi s), (\cos 2\pi t, \sin 2\pi t)),$$

is continuous and surjective and, again using Corollary 6.3.7 in Section 6.3, it is closed, hence g is an identification and thus induces a homeomorphism

$$\begin{tikzcd}
{[0,1]^2} \arrow[rr, "g"] \arrow[dr, "\pi"'] & & \mathbb{S}^1 \times \mathbb{S}^1 \\
& {[0,1]^2 / \sim_g} \arrow[ur, dashed, "\cong", "\tilde{g}"'] &
\end{tikzcd}$$

What about the subsets being identified by this process? They are boundary points as shown in the picture (think which points have the same image by g):

$[0,1] \times [0,1] / \sim_g =$ 

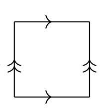
On the other hand, if we denote $D =$  by the "donut-shaped" space we commonly associate torus with, we have the following function:

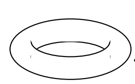
$$f : [0,1]\times[0,1] \to D \subset \mathbb{R}^3, \qquad (\theta, \varphi) \mapsto ((a + b\cos 2\pi\theta)\cos 2\pi\varphi, (a + b\cos 2\pi\theta)\sin 2\pi\varphi, b \sin 2\pi\theta),$$

where a, b are any two numbers determining the shape of the donut (e.g. $a = 2, b = 1$ in Example 4.1.12), we have an identification as well, for the same reasons as g

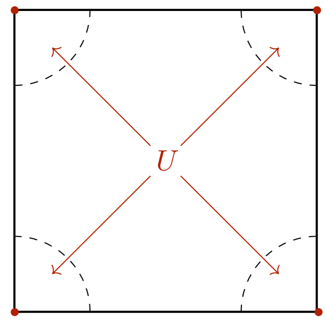
$$\begin{tikzcd}
{[0,1]^2} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & D \\
& {[0,1]^2 / \sim_f} \arrow[ur, dashed, "\cong", "\tilde{f}"'] &
\end{tikzcd}$$

and the subsets identified (opposite boundaries) are the same as well. Which means $\sim_f = \sim_g$ and

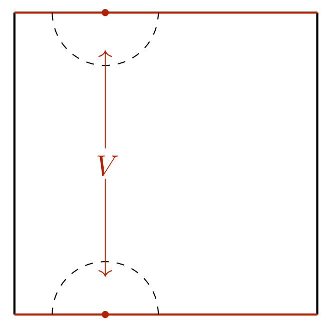
$\mathbb{S}^1 \times \mathbb{S}^1 \xrightarrow[\cong]{\tilde{g}^{-1}}$  $\xrightarrow[\cong]{\tilde{f}} D$

hence we have yet another proof of $\mathbb{S}^1 \times \mathbb{S}^1 \cong$ . Open neighbourhoods homeomorphic to $B(0,0,1) \subset \mathbb{R}^2$ can be represented on the identified square for every point $(x,y) \in \mathbb{T}^2$ according to the position of (x,y):

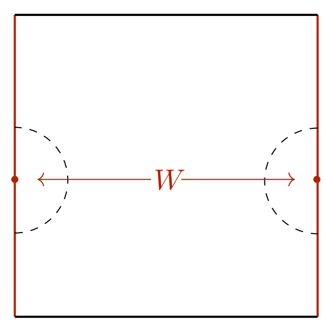
(a) All points $(x,y) \in \{(0,0), (0,1), (1,0), (1,1)\}$ are identified into a single point of $\mathbb{T}^2$. The neighbourhood, accordingly, will be $\varphi(U)$ where U is represented by the four open circle quadrants below:



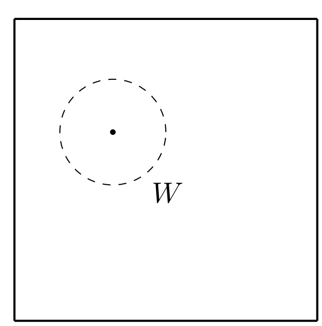
(b) Segments $(0,1)\times\{0\}$ and $(0,1)\times\{1\}$ are identified into a single curve on $\mathbb{T}^2$. Accordingly, neighbourhoods of points belonging to $\varphi([0,1]\times\{0\}) = \varphi([0,1]\times\{1\})$ will be of the form $\varphi(V)$, where V is represented below.



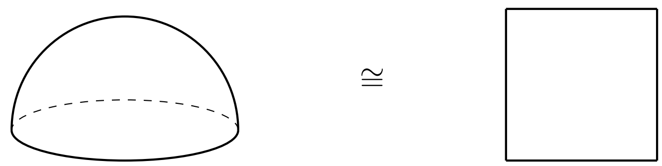
(c) Same applies to points belonging to $\varphi(\{0\}\times(0,1)) = \varphi(\{1\}\times(0,1))$: they possess neighbourhoods of the form $\varphi(W)$, W shown below, which are homeomorphic to the Euclidean unit 2-ball:



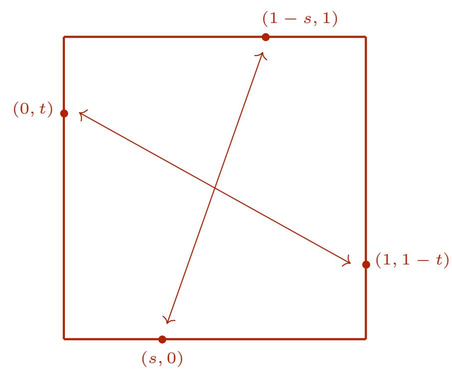
(d) Finally, adequate neighbourhoods for points not belonging to $\partial\,[0,1]^2$ are as shown:



6. The **real projective plane** $\mathbb{P}^2_{\mathbb{R}}$ appears from identifying the antipodal points in $\mathbb{S}^2$. This means we only need half a sphere, say the upper closed half (call it S^+) to build our identification $S^+ \to \mathbb{P}^2_{\mathbb{R}}$. Using the fact that H^+ and $X = [0.1] \times [0,1]$ are homeomorphic,



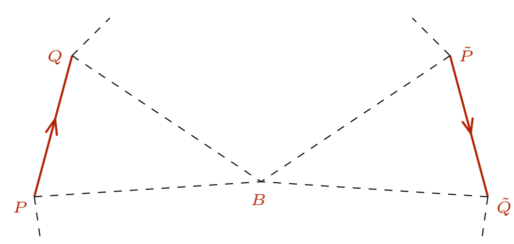
we can actually represent the real projective plane by identifying the points in the boundary of X as shown below:



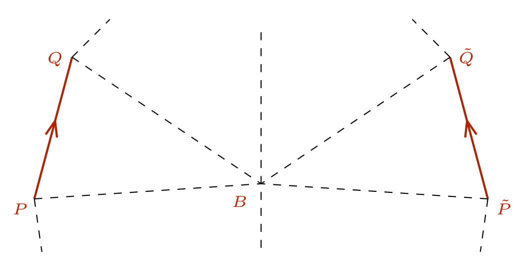
It should not be difficult to guess, in view of the previous examples, what neighbourhoods can be found for each point in the projective plane which are homeomorphic to an open ball in $\mathbb{R}^2$.

7. The previous examples can be generalised in the following way. Let X be a regular polygon of $2n$ sides. $n \geq 2$. Identify every side with one and only one of the others in the following way:

(a) applying a rotation (always centred at polygon's barycentre B), sending P to $\tilde{P}$ and Q to $\tilde{Q}$:

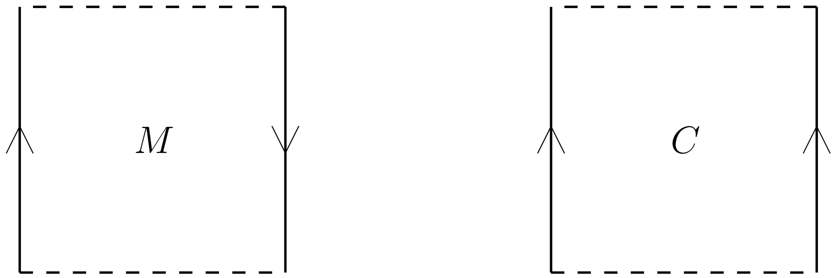


(b) Applying an axial symmetry around a line passing through barycentre B:

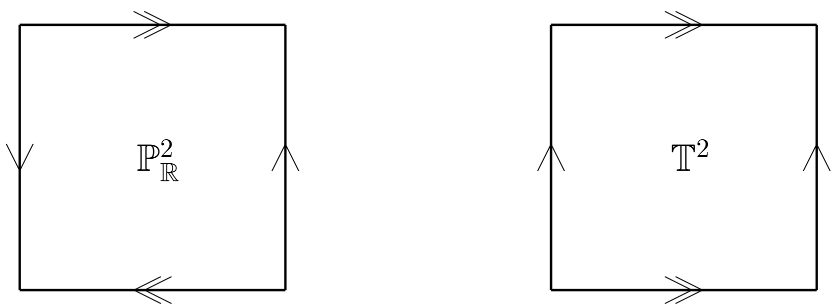


In the first case we say we identify *maintaining the orientation of the side*, whereas in the second case we identify sides *inverting orientation*. The resulting topological space is locally Euclidean of dimension 2. We leave it to the reader to check what neighbourhoods can be built for every point which are homeomorphic to open balls of $\mathbb{R}^2$.

Remark 4.3.7. It is customary to denote identified spaces obtained from the unit square $[0,1]^2$ using arrows, e.g. the Möbius strip and the open cylinder would have the representation



whereas the projective plane and the torus would look like this:



4.4 Solved Exercises

87. Justify the following propositions:

a) Projections pr_X, pr_Y in the product topology on $X \times Y$ are open, but in general not closed maps.

b) For any given topological space (X, τ) and any relation of equivalence $\sim$ on X, identification $\pi : (X, \tau) \to (X/\sim, \tau/\sim)$ is, in general, not open.

SOLUTION.

a) Let $\mathrm{pr}_X : (x, y) \mapsto x$ and $\mathrm{pr}_Y : (x, y) \mapsto y$ be the projections for which the product topology, having basis
$$\beta_{X\times Y} = \{U \times V : U \in \tau_X, V \in \tau_Y\},$$
is the initial topology. Every basis element can be represented as $W = U \times V$, U being open in X and V being open in V. For every such W, therefore, $\mathrm{pr}_X(W) = U$ which is open in X, and $\mathrm{pr}_Y(W) = V$ which is open in Y. Hence the image by pr_X (resp. pr_Y) of every basis element in $X \times Y$ is an open set in X (resp. Y). Exercise 100 does the rest.

   Let us find a counterexample to a similar claim for closed functions. Let $X = \mathbb{R}$ with the Euclidean topology. The product topology on $X \times X$ is exactly the Euclidean topology on $\mathbb{R}^2$ (Examples 4.1.9 (**1**)) and the set $C = \{(x, y) : xy = 1\}$ is thus closed in $\mathbb{R}^2$ with this topology (EXERCISE: why?). However, the projection on the first variable $\mathrm{pr}_1(C) = \mathbb{R}\backslash\{0\}$ is not closed in $\mathbb{R}$.

b) The function referred to is the class map
$$\pi : X \to X/\sim, \qquad x \mapsto [x],$$

mapping every element to its class of equivalence. It is obviously surjective and by calling it an identification we are implicitly assuming the topology on $X/\sim$ is the final topology with respect to π, hence rendering it automatically continuous.

Let us find a counterexample disproving the claim "π is in general open". Let $X = \mathbb{R}$ with the Euclidean topology and define the following relation of equivalence on X: $x \sim y$ if, and only if, $x, y \leq 0$ or $x, y > 0$. $\mathbb{R}/\sim$ only has two elements, which we can write as equivalence classes $[0]$ and $[1]$. The final topology on $\mathbb{R}/\sim$ is defined by

$$\tau/\sim = \left\{U : \pi^{-1}(U) \text{ open in } \mathbb{R}\right\}.$$

$(-\infty, 0)$ is an open set of $\mathbb{R}$, and $\pi((-\infty, 0)) = [0]$ which is not open in $(\mathbb{R}/\sim, \tau/\sim)$ since $\pi^{-1}([0]) = (-\infty, 0]$ which is not open in $\mathbb{R}$.

88. Consider on $\mathbb{R}$ the relation of equivalence: $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Prove $\mathbb{R}/\sim \cong \mathbb{S}^1$.

Solution. We will look for a continuous, surjective and open function $f : \mathbb{R} \to \mathbb{S}^1$ such that $f(x) = f(y)$ if, and only if, $x \sim y$. Lemmae 4.2.8 and 4.2.6 will automatically do the rest.

Let

$$f : \mathbb{R} \to \mathbb{S}^1, \qquad x \mapsto (\cos 2\pi x, \sin 2\pi x).$$

This is trivially continuous because its coordinate functions are (Proposition 4.1.11) and is surjective since every element $(a, b) \in \mathbb{S}^1$ can be written as $(a, b) = f(x)$ where

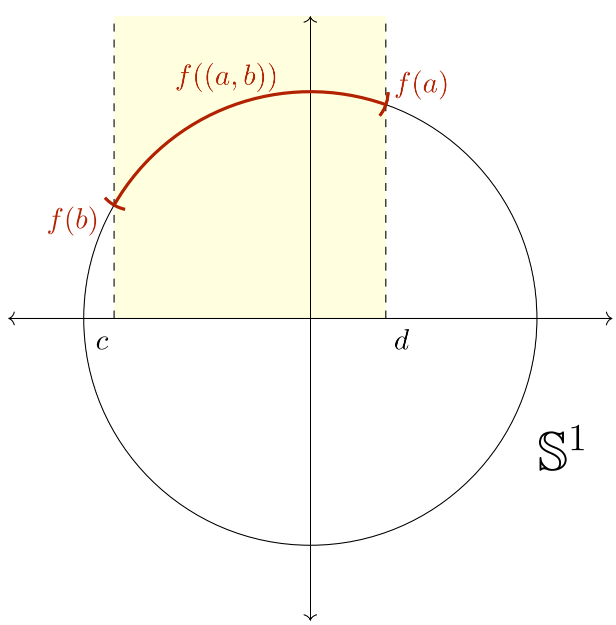
$$x = \begin{cases} 0, & \text{if} \quad a = 1, b = 0, \\ \pi, & \text{if} \quad a = -1, b = 0, \\ \frac{1}{2\pi}\arctan(y/x), & \text{if} \quad b \neq 0. \end{cases}$$

We have, for every $x, y \in \mathbb{R}$, $f(x) = f(y)$ if and only if $\exp 2\pi i x = \exp 2\pi i y$, which is equivalent to $2\pi x = 2\pi y + 2\pi k$ for some $k \in \mathbb{Z}$, hence to $x \sim y$. And for every basis element $(a, b) \subset \mathbb{R}$ its image

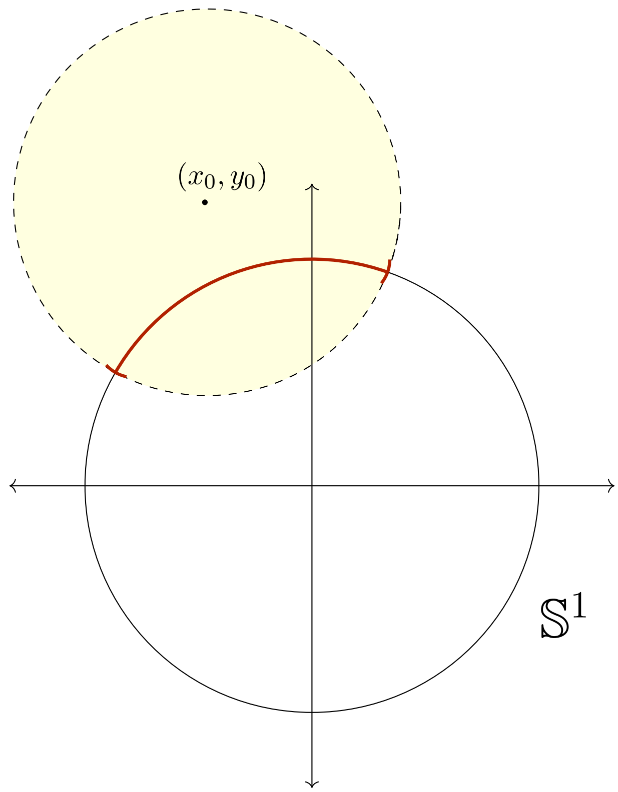
$$f((a, b)) = \{(\cos 2\pi x, \sin 2\pi y) : x \in (a, b)\}$$

is an open set of $\mathbb{S}^1$, since it is the intersection of $\mathbb{S}^1$ with an open set of $\mathbb{R}^2$: for instance, among the infinite ways to check this, let us point out two (remember the topology on $\mathbb{S}^1$ is the subspace topology induced by $\mathbb{R}^2$ as in Definition 2.3.1):

- we can express $f((a, b))$ as the intersection of $\mathbb{S}^1$ with an open vertical band $(c, d) \times (e, +\infty) \subset \mathbb{R}^2$ (think what should c and d and e be!):



or as the intersection of $\mathbb{S}^1$ with a horizontal open band (again, find it) or we can express $f((a,b))$ as the intersection of $\mathbb{S}^1$ with an open ball $B((x_0,y_0),r) \subset \mathbb{R}^2$ (again, give examples of such (x_0,y_0) and r):



Either way, $f((a,b))$ is an open subset of $\mathbb{S}^1$ for any basis element $(a,b) \subset \mathbb{R}$. Hence using Exercise 100 f is an open map.

Therefore, using Proposition 4.2.6, f is an identification. Using Proposition 4.2.8, $Y = \mathbb{S}^1 \cong \mathbb{R}/\sim$.

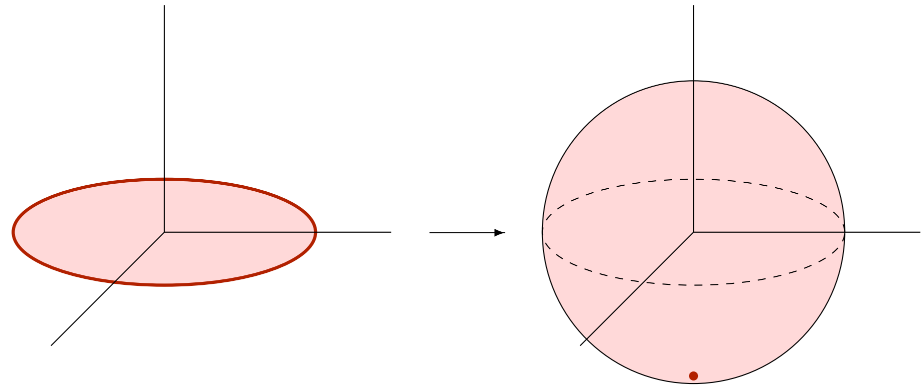
89. Let $E^2 := \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\right\}$ be the unit disk in $\mathbb{R}^2$. Prove $E^2/\mathbb{S}^1 \cong \mathbb{S}^2$.

Solution. It is obvious we are considering $\mathbb{S}^1 = \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\right\}$ as a subset of E^2 and are being asked to take quotient modulo this subset (which means, in the topological sense, identifying all points of $\mathbb{S}^1$). The relation of equivalence we need to consider in E^2 is

$$(x,y) \sim (X,Y) \qquad \text{if, and only if,} \quad \begin{cases} (x,y) = (X,Y), \\ \text{or} \\ (x,y) \text{ and } (X,Y) \in \mathbb{S}^1. \end{cases} \tag{4.4}$$

In other words every point in E^2 is equivalent to itself and points lying in its boundary $\mathbb{S}^1$ are all equivalent to one another.

What we are doing, essentially, is grabbing a planar disk and "wrapping" it up to become a sphere.



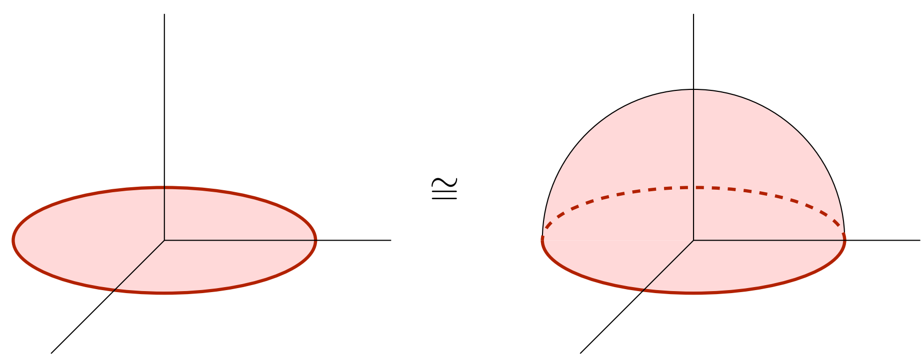
Let us formalise this. Pictures show E^2 embedded in $\mathbb{R}^3$ for more clarity, but this poses no problem since function

$$E^2 \to \left\{(x, y, 0) \in \mathbb{R}^3 : (x, y) \in E^2\right\}, \qquad (x, y, 0) \mapsto (x, y),$$

is a homeomorphism.

We need a surjective, continuous and open function $f : E^2 \to \mathbb{S}^2$ such that $f(x, y) = f(X, Y)$ if and only if $(x, y) \sim (X, Y)$, $\sim$ as in (4.4), we have finished the Exercise.

Building f will be done in several steps. Firstly, E^2 is homeomorphic to the sphere's upper half:



(4.5)

This is done pushing interior points upward adequately:

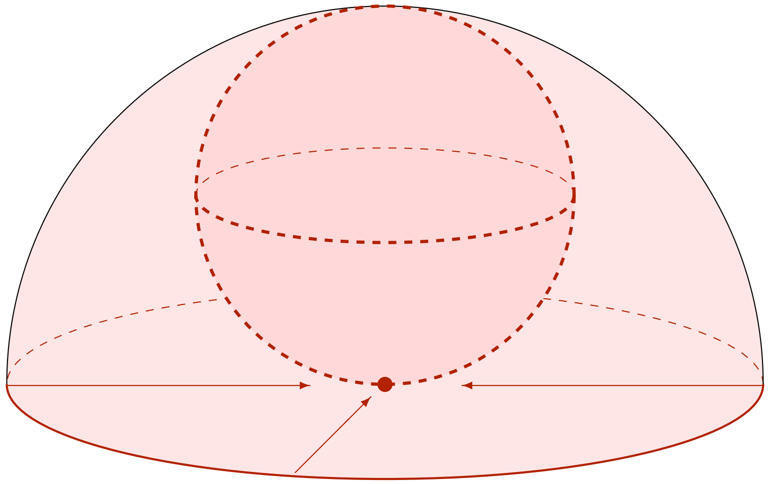
$$f_1 : E^2 \to S^+ := \left\{(x, y, z) \in \mathbb{S}^2 : z \geq 0\right\}, \qquad (x, y) \mapsto \left(x, y, \sqrt{1 - (x^2 + y^2)}\right).$$

The inverse of this function is

$$f_1^{-1} : S^+ \to E^2, \qquad (x, y, z) \mapsto (x, y),$$

and both f_1 and f_1^{-1} are well-defined and continuous (Exercise).

Next step in this process we have chosen is identifying all points in the boundary of S^+, thereby closing it into a complete (smaller) sphere:



(4.6)

This is a function $f_2 : S^+ \to \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}\right\}$; finding f_2 explicitly only needs, for instance, carrying out projections from every point $(x,y,z) \in S^+$ towards the z axis (which yields line $\{(\lambda x, \lambda y, z) : \lambda \in \mathbb{R}\}$) and intersecting the projected line with the smaller sphere, which is the same as adding requirement $(\lambda x)^2 + (\lambda y)^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$ to the already-existing $x^2 + y^2 + z^2 = 1$. Solving this yields $\lambda = \sqrt{\frac{z}{z+1}}$, hence f_2 is defined by $(x,y,z) \mapsto \left(x\sqrt{\frac{z}{z+1}}, y\sqrt{\frac{z}{z+1}}, z\right)$. This is a well-defined, surjective, continuous function as the reader may check as an Exercise.

It is trivial to check that the last function needed, namely the one transforming the smaller sphere into $\mathbb{S}^2$, is

$$f_3 : \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}\right\} \to \mathbb{S}^2, \qquad (x,y,z) \mapsto (2x, 2y, 2z - 1)\,.$$

This is a homeomorphism (Exercise: find its inverse and check both functions are continuous and well-defined).

We have a function

$$f := f_3 \circ f_2 \circ f_1 : E^2 \to \mathbb{S}^2, \qquad (x,y) \mapsto \left(\frac{x\left(1 - x^2 - y^2\right)^{1/4}}{\sqrt{1 + \sqrt{1 - x^2 - y^2}}}, \frac{y\left(1 - x^2 - y^2\right)^{1/4}}{\sqrt{1 + \sqrt{1 - x^2 - y^2}}}, \sqrt{1 - x^2 - y^2}\right),$$

which is the composition of three continuous and surjective functions, hence surjective and continuous itself. f is closed in virtue of Corollary 6.3.7 in Section 6.3, hence it is an identification.

So far we have proven $f : E^2 \to \mathbb{S}^2$ is an identification. Now all we need to do, before applying Proposition 4.2.8, is check is that identifying points in E^2 according to their belonging to $\mathbb{S}^1$ (that is, according to $\sim$) is *exactly the same* as identifying them according to their image by f. In other words: that $f(x,y) = f(X,Y)$ if, and only if, $(x,y) = (X,Y)$ or $(x,y), (X,Y) \in \mathbb{S}^1$:

$$\left(\frac{x\left(1 - x^2 - y^2\right)^{1/4}}{\sqrt{1 + \sqrt{1 - x^2 - y^2}}}, \frac{y\left(1 - x^2 - y^2\right)^{1/4}}{\sqrt{1 + \sqrt{1 - x^2 - y^2}}}, \sqrt{1 - x^2 - y^2}\right) = \left(\frac{X\left(1 - X^2 - Y^2\right)^{1/4}}{\sqrt{1 + \sqrt{1 - X^2 - Y^2}}}, \frac{Y\left(1 - X^2 - Y^2\right)^{1/4}}{\sqrt{1 + \sqrt{1 - X^2 - Y^2}}}, \sqrt{1 - X^2 - Y^2}$$

implies the equality of the last coordinate, which means $x^2 + y^2 = X^2 + Y^2$. Substituting this into the first two coordinates we have $x\left(1 - x^2 - y^2\right)^{1/4} = X\left(1 - X^2 - Y^2\right)^{1/4}$ and $y\left(1 - x^2 - y^2\right)^{1/4} = Y\left(1 - X^2 - Y^2\right)^{1/4}$, both of which hold if and only if either $x = X, y = Y$ or the term under the fourth root vanishes.

90. We define the *cone* and the *suspension* of the unit circle $\mathbb{S}^1$ as follows:

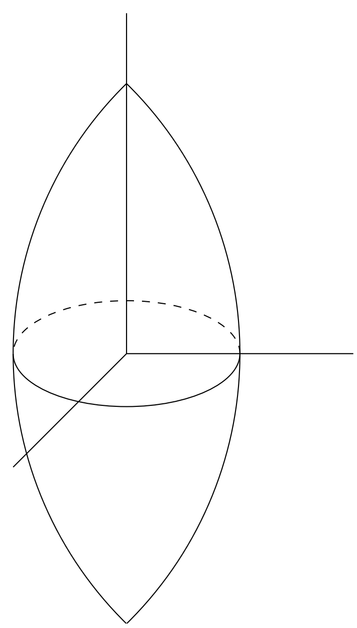
$$C\mathbb{S}^1 := \mathbb{S}^1 \times [0,1] / \mathbb{S}^1 \times \{0\}, \qquad \Sigma\mathbb{S}^1 := C\mathbb{S}^1 / \mathbb{S}^1 \times \{1\}.$$

Prove $C\mathbb{S}^1 \cong E^2 = \left\{(x,y) : x^2 + y^2 \leq 1\right\}$ and $\Sigma\mathbb{S}^1 \cong \mathbb{S}^2$.

Solution. This Exercise and the previous one share part of their essential procedure.

The first homeomorphism consists on building a cone from a cylinder (as in Example 4.2.9 (**3**)) and proving it deformable into or from the closed disk E^2 into a cone (the latter part of this process is analogous to what was done in (4.5)).

The second homeomorphism amounts to "closing" the cone:



and then proving it homeomorphic to a sphere – which, in a nutshell, we did in (4.6).

We will prove the second homeomorphism, $\Sigma\mathbb{S}^1 \cong \mathbb{S}^2$, which is easier once the first one is proven. The first one, which we will assume true by hypothesis, is left as Exercise 108.

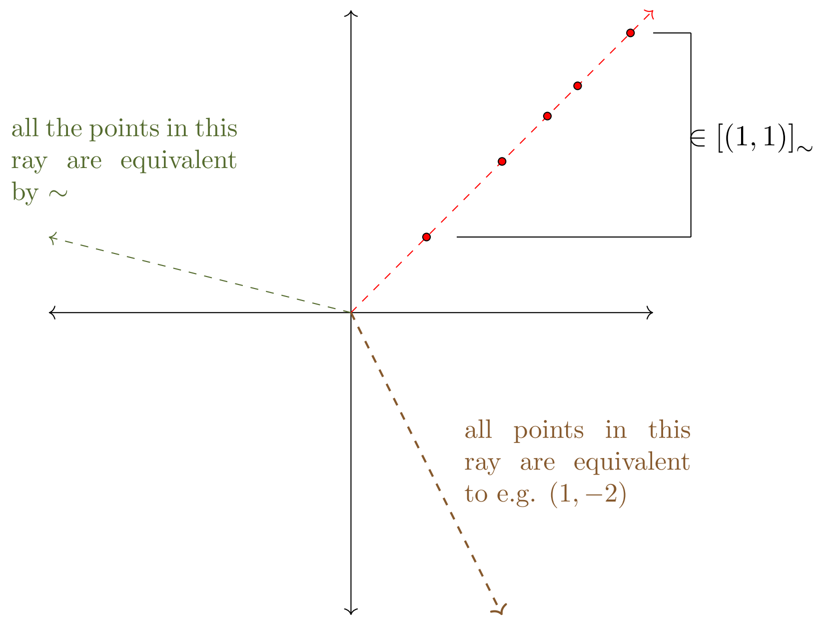
Assume, therefore, $C\mathbb{S}^1 \cong E^2$. Use Exercise 106: if $f : X \to Y$ is a homeomorphism and $A \subset X$, then $X/A \cong Y/f(A)$. In our case, $X = C\mathbb{S}^1$ and $Y = E^2$. Once you've done Exercise 108 you will know the adequate homeomorphism between X and Y carries $\mathbb{S}^1 \times \{1\}$ to $\mathbb{S}^1 \subset E^2$, hence $\Sigma\mathbb{S}^1 = C\mathbb{S}^1/\mathbb{S}^1 \times \{1\} \cong E^2/\mathbb{S}^1$. Solved Exercise 89 finishes the proof.

91. (*Coursework January 2014*) Let $X := \mathbb{R}^2 \setminus \{(0,0)\}$. Define the following relation of equivalence on X:

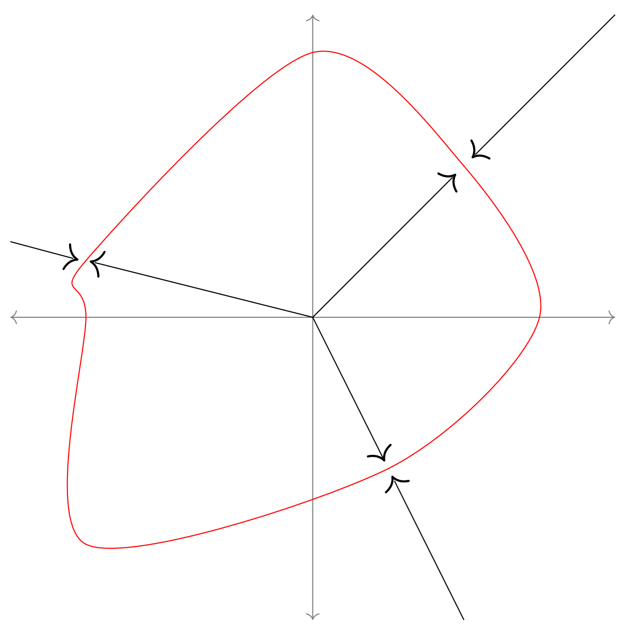
$$(x,y) \sim (X,Y), \qquad \text{if, and only if,} \qquad \det\begin{pmatrix} x & y \\ X & Y \end{pmatrix} = 0 \text{ and } xX + yY > 0. \tag{4.7}$$

Prove $X/\sim\, \cong \mathbb{S}^1$.

Solution. The identification, geometrically, consists on setting all collinear vectors in the same ray from the origin equivalent:



This is the same as collapsing all of $\mathbb{R}^2 \setminus \{(0,0)\}$ onto a closed curve containing the origin:



Obviously, the easiest way to represent this identification analytically is to choose a regular closed curve, e.g. a circle centered at the origin. Define for instance

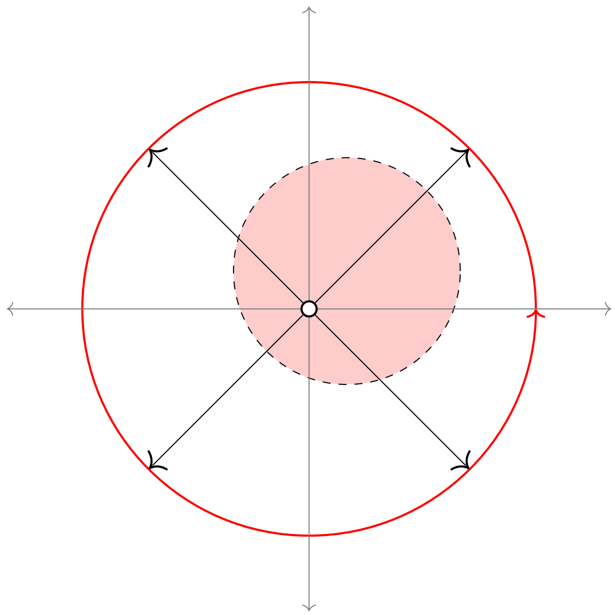
$$f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{S}^1 = \left\{x^2 + y^2 = 1\right\}, \qquad (x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right).$$

a) It is well-defined: for every $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 = \frac{x^2+y^2}{x^2+y^2} = 1$.
It is continuous: using Proposition 4.1.11 and the fact that the Euclidean topology is equal to the product topology $(\mathbb{R}, |\cdot|) \times (\mathbb{R}, |\cdot|)$, f is continuous if and only if its coordinate functions are. And they are indeed, being quotients of continuous functions with non-vanishing denominators.

b) f is surjective: every element of $\mathbb{S}^1$ is of the form (x, y) where $x^2 + y^2 = 1$. Hence $(x, y) = f(x, y)$. Think of $\mathbb{S}^1$ as the only subset of $\mathbb{R}^2 \setminus \{(0,0)\}$ left invariant by f.

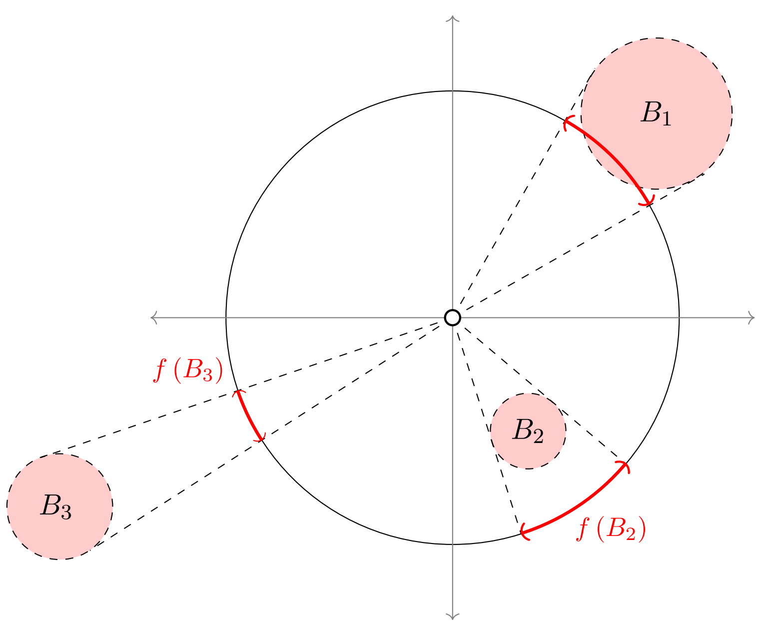
c) f is open. Indeed, Proposition 3.3.5 implies we only need to check basis elements of $\mathbb{R}^2 \setminus \{(0,0)\}$ (e.g. open d_2-balls) are mapped to open sets of $\mathbb{S}^1$. There are two possible cases for any ball $B_{d_2}((x, y), r)$:

   (a) $\partial B_{d_2}((x, y), r)$ is a curve containing $(0,0)$:



   The image of $B_{d_2}((x, y), r)$ is all of $\mathbb{S}^1$, which is open in $\mathbb{S}^1$.

   (b) $\partial B_{d_2}((x, y), r)$ does not surround $(0,0)$:



   Images of balls B_i are open arc intervals in $\mathbb{S}^1$, themselves equal to intersections of $\mathbb{S}^1$ with open balls, hence open sets in the subspace topology. EXERCISE: fill in the details.

We have a continuous, surjective and open map $f : X := \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{S}^1$. This implies, in virtue of Proposition 4.2.6, that f is an identification. Hence Lemma 4.2.8 implies f factors through a homeomorphism $\tilde{f}$ in the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathbb{S}^1 \\ & X/\sim_f \arrow[ur, dashed, "\cong", "\tilde{f}"'] & \end{tikzcd} \qquad \begin{tikzcd} (x,y) \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\pi"'] & & f(x,y) \\ & {[(x,y)]_f} \arrow[ur, mapsto, "\cong", "\tilde{f}"'] & \end{tikzcd}$$

And all we need to do is check that the relation $\sim$ defined in (4.7) coincides (i.e. defines the same classes of equivalence) with relation $\sim_f$. Indeed, $(x,y) \sim (X,Y)$ if, and only if, $(x,y) = a\,(X,Y)$ for some $a > 0$. This is the same as

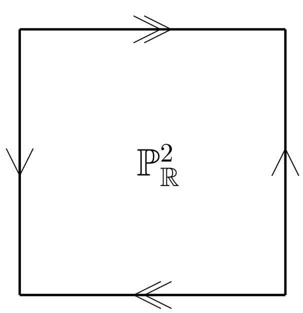
$$\sqrt{x^2+y^2} = a\sqrt{X^2+Y^2}, \qquad (x,y) = a\,(X,Y)\,,$$

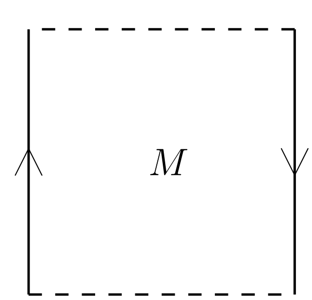
which takes us to

$$f\,(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) = \left(\frac{x}{a\,(X^2+Y^2)}, \frac{y}{a\,(X^2+Y^2)}\right) = \left(\frac{aX}{a\,(X^2+Y^2)}, \frac{aY}{a\,(X^2+Y^2)}\right) = f\,(X,Y)\,.$$

Therefore $(x,y) \sim (X,Y)$ if, and only if, $(x,y) \sim_f (X,Y)$. Hence $[(x,y)]_f = [(x,y)]_\sim$ for every (x,y) implying $X/\sim_f = X/\sim$, and thus $X/\sim \cong \mathbb{S}^1$.

92. (*May 2014*) Let $\mathbb{P}^2_{\mathbb{R}}$ and M be the (real) projective plane and the open Möbius band, respectively:





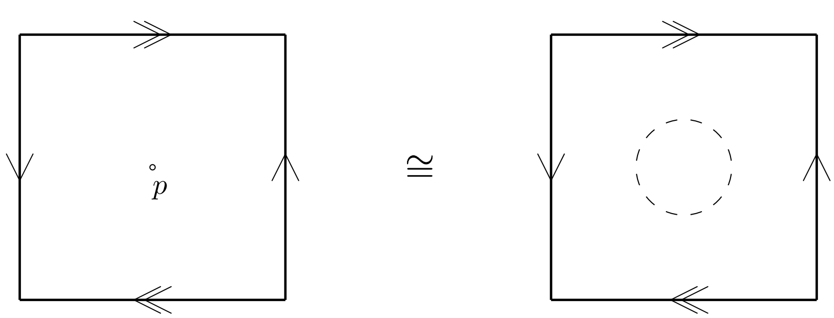
Find a point $p \in \mathbb{P}^2_{\mathbb{R}}$ such that $\mathbb{P}^2_{\mathbb{R}} \setminus \{p\} \cong M$. *Justify every step.* Hint: you may

a) express your identification as a sequence of homeomorphisms and identifications,

b) use Exercise 75 at some stage.

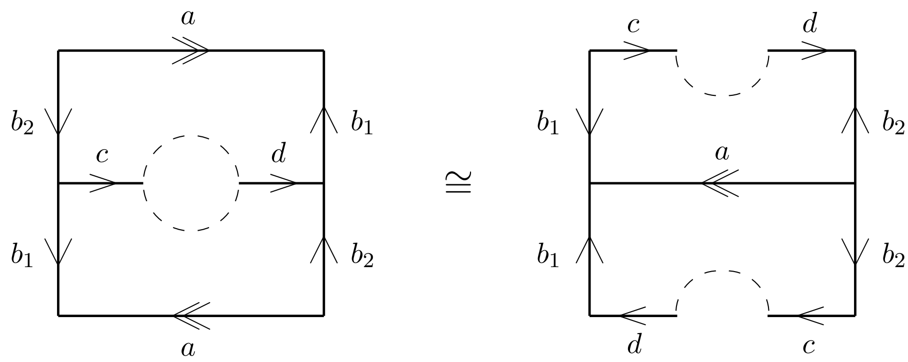
c) and use the fact that any homeomorphism $f : X \xrightarrow{\cong} Y$ yields a homeomorphism $X/S \cong Y/f\,(S)$ for any subset $S \subset X$.

Solution. The required sequence of homeomorphisms is:



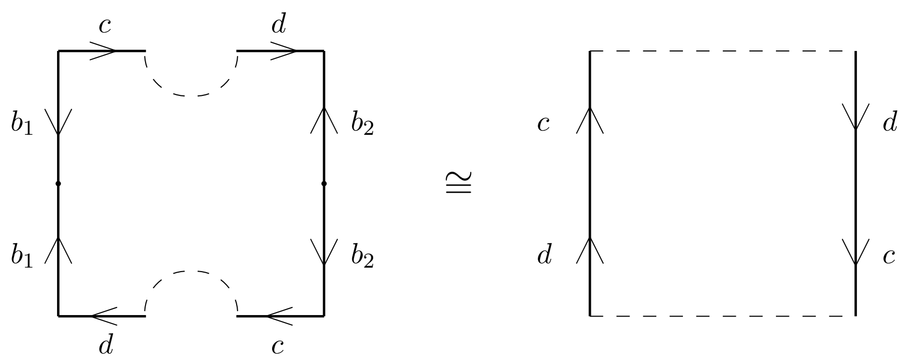
(4.8)

followed by



(4.9)

and finally

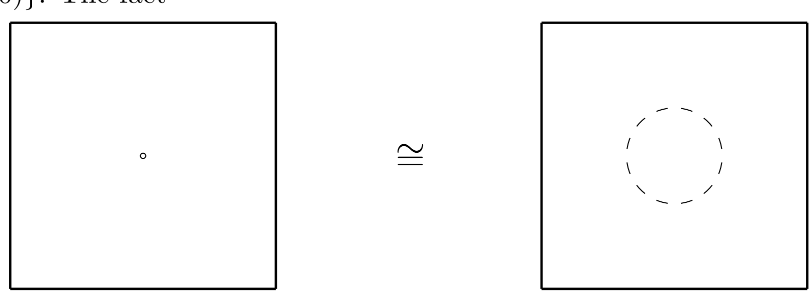


(4.10)

the latter of which is obviously nothing but M.

Isomorphism (4.9) is immediate from its graphic incarnation and one of the hints and needs no more attention; finding the explicit identifications analytically would require an amount of writing that adds nothing to the general solution of the exercise. Let us focus on isomorphisms (4.8) and (4.10).

Let $Y_1 = [0,1] \times [0,1]$ be the unit square. For practical purposes we may take $Y_2 = [-1,1] \times [-1,1] \cong Y_1$ instead. Choose any point in the interior of Y_2, for instance the origin. Define $X_1 := Y_2 \setminus \{(0,0)\}$. The fact



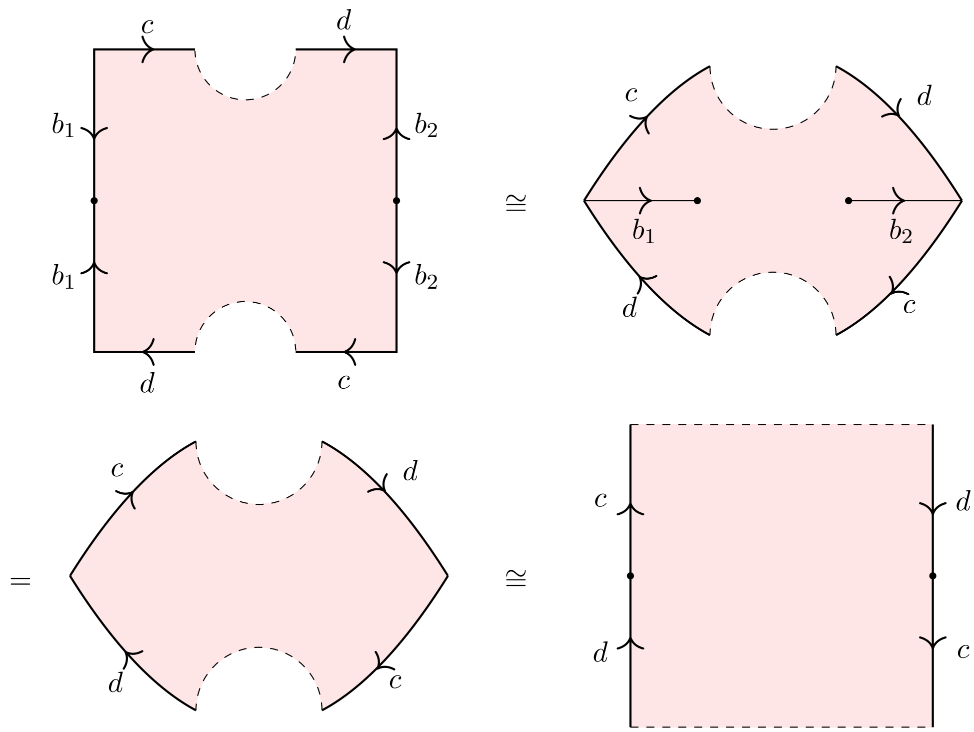
is a consequence of Exercise 75 and in turn implies (4.8) on account of Hint (c), since

$$\partial X_1 = \partial X_2 = (\{\pm 1\} \times [-1,1]) \cup ([-1,1] \times \{\pm 1\}),$$

and both boundaries are left invariant by homeomorphism f_1 and are affected by the same identification in both X_1 and X_2, which is

$$\{1\} \times [-1,1] \sim \{-1\} \times [-1,1], \quad [-1,1] \times \{1\} \sim [-1,1] \times \{-1\}.$$

As for homeomorphism (4.10), it is easier to represent graphically than to find explicit homeomorphisms (Exercise):



93. (*July 2014*) Let τ_2 be the topology on $\mathbb{R}$ having basis $\beta_2 := \{[s,t) : s < t\} \cup \{\varnothing\}$ and consider the family $\mathcal{F} := \{f_{a,b,c} : \mathbb{R} \to \mathbb{R} : a,b,c \in \mathbb{R}\}$ of quadratic functions of x,

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f_{a,b,c}} & \mathbb{R} \\ x & \longmapsto & ax^2 + bx + c \end{array}$$

Prove the initial topology τ_1 with respect to all functions in $\mathcal{F}$ is the discrete topology on $\mathbb{R}$.

Solution. All it takes is proving that simpletons $\{x_0\}$ are elements of τ_1. The initial topology τ_1 is defined as the one having all finite intersections of pre-images of open sets as a basis:

$$\beta_1 := \left\{ f^{-1}_{a_1,b_1,c_1}(U_1) \cap \cdots \cap f^{-1}_{a_n,b_n,c_n}(U_n) : U_i \in \tau_2 \right\}. \tag{4.11}$$

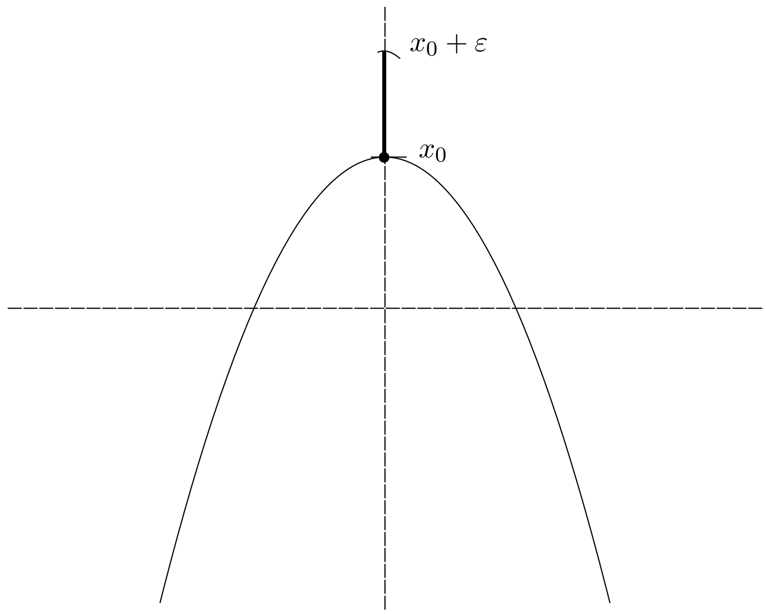
The only apparent change with respect to the definition given in class is that all $f_{a,b,c}$ are functions to the same topological space $(\mathbb{R}, \tau_2)$ – bear in mind there was no need for the topologies in the arrival sets to be different.

Hence if we are able to express any simpleton $\{x_0\}$ as a finite intersection of pre-images as in (4.11), we will obtain $\{x_0\} \in \beta_1$ for any $x_0 \in \mathbb{R}$ and thus $\tau_1 = \mathcal{P}(\mathbb{R})$.

We actually obtain something stronger than that: the simpleton of a *single* pre-image $f^{-1}_{a_1,b_1,c_1}([s_1,t_1))$:

$$\{x_0\} = f^{-1}_{-1,2x_0,x_0-x_0^2}([x_0, x_0+\varepsilon)), \qquad \text{for every } \varepsilon > 0,$$

as can be seen graphically:



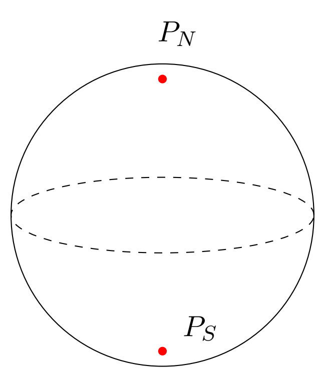
There is an alternative way of proving the Exercise with pre-images by *linear* functions $f_{a_1,b_1,c_1}, f_{a_2,b_2,c_2}$ (i.e. $a_1 = a_2 = 0$):

$$\{x_0\} = (x_0 - 1, x_0] \cap [x_0, x_0 + 1) = f^{-1}_{0,-1,x_0}([0,1)) \cap f^{-1}_{0,1,-x_0}([0,1)),$$

as the reader may check graphically. □

94. (*May 2015*) Find subsets $S_1 \subset \mathbb{S}^2$ and $S_2 \subset \mathbb{T}^2$ such that $\mathbb{S}^2/S_1 \cong \mathbb{T}^2/S_2$.

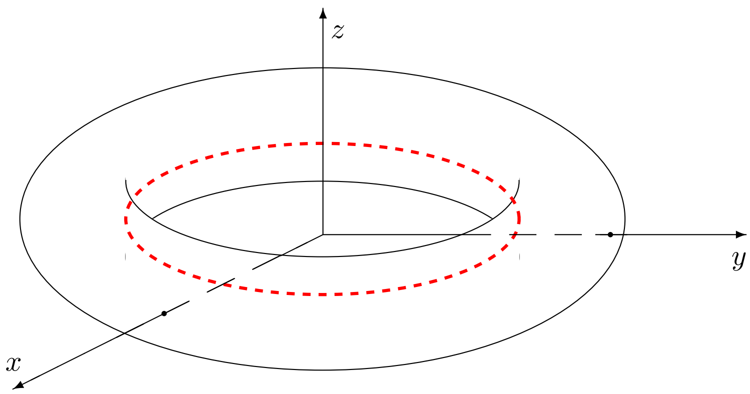
Solution. There is more than one possible way of solving this. For instance, let $S_1 := \{P_N, P_S\} = \{(0,0,1), (0,0,-1)\}$



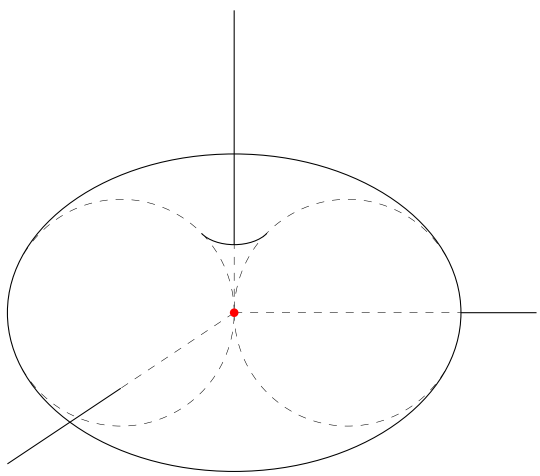
and $S_2 := \left\{x^2 + y^2 = 1\right\}$ the inner circle in the torus, assuming $\mathbb{T}^2$ is defined as

$$\left\{(x, y, z) \in \mathbb{R}^3 : \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 = 1\right\},$$

obtained rotating a circle of centre $(2,0)$ and radius 1 on the (x,z) plane around the z axis:



Each space modulo its subspace is homeomorphic to the *horn torus*

$X =$  $= \left\{(x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2}-1\right)^2 + z^2 = 1\right\}$

Identifications $f : \mathbb{T}^2 \to X$ and $g : \mathbb{S}^2 \to X$ can be hard to find in an exam, hence a description such as the above will suffice.

95. (Coursework 2015) Consider the functions

$$\begin{aligned} \mathbb{R} &\xrightarrow{\quad f,g \quad} \mathbb{R} \\ x &\longmapsto f(x) := \begin{cases} x^2, & x \le 1, \\ x+1, & x > 1, \end{cases} \\ x &\longmapsto g(x) := x^3 \end{aligned}$$

(i) (10 Marks) Find a topology τ_1 for which $f : \left(\mathbb{R}^2, \tau_1\right) \to \left(\mathbb{R}^2, \tau_1\right)$ is not continuous.

(ii) (15 Marks) Find a topology τ_0 for which $f, g : \left(\mathbb{R}^2, \tau_0\right) \to \left(\mathbb{R}^2, \tau_{\mathbb{R}}^{\mathrm{Eucl}}\right)$ are continuous.

Needless to say, you must justify both items.

Solution: handwritten at the end.

96. (Coursework 2015) $X_1 := \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\right\} / \left(\mathbb{S}^1 \times \{0\}\right)$ is homeomorphic to the union X_2 of two tangent spheres minus two points.

(i) (20 Marks) Prove the above statement.

(ii) (5 Marks) Give a heuristic explanation of why neither space is locally Euclidean.

Solution: handwritten at the end.

97. (Coursework 2016) Consider function

$$\begin{array}{rcl} \mathbb{R} & \xrightarrow{\quad f \quad} & \mathbb{R} \\ x & \longmapsto & f(x) := \begin{cases} |x|, & |x| \leq 1, \\ x, & |x| > 1. \end{cases} \end{array}$$

(i) (10 Marks) Prove $f : \left(\mathbb{R}, \tau_{\mathbb{R}}^{\mathrm{Eucl}}\right) \to \left(\mathbb{R}, \tau_{\mathbb{R}}^{\mathrm{Eucl}}\right)$ is not continuous. Hint: you can either use the direct definition of (metric) continuity, or some exercise from previous courseworks.

(ii) (15 Marks) Find a basis for a topology $\tau_0 \neq \mathcal{P}(\mathbb{R})$ such that $f : (\mathbb{R}, \tau_0) \to \left(\mathbb{R}, \tau_{\mathbb{R}}^{\mathrm{Eucl}}\right)$ is continuous.

Solution: handwritten at the end.

98. (Coursework 2016) For every $n \in \mathbb{N}$, let $Z_n = \left\{ e^{\frac{2\pi \mathrm{i}}{n} k} : k = 0, \ldots, n-1 \right\}$ be the set of n^{th} roots of unity. Prove $\mathbb{S}^1 / Z_n \cong \mathbb{S}^1 \vee \overset{n)}{\cdots} \vee \mathbb{S}^1$ using figures and an intuitive explanation. You do not need to find the analytical (coordinate) expression of the functions involved, but your sets should be properly defined and any results you use must be stated explicitly.

Solution: handwritten at the end.

99. (*May 2013*) We define the following relations of equivalence on $\mathbb{R}^2$:

- $(x, y) \sim (X, Y)$ if, and only if, $y = Y$;
- $(x, y) \bowtie (X, Y)$ if, and only if, $x^2 + y^2 = X^2 + Y^2$.

Prove the following:

a) $\mathbb{R}^2 / \sim \, \cong \mathbb{R}$;

b) $\mathbb{R}^2 / \bowtie \, \cong [0, \infty)$.

Solution: handwritten at the end of the section.

4.5 Exercises

100. Prove $f : X \to Y$ is open if, and only if, the image of every element of a basis for the topology in X is an open set in Y.

101. Prove τ_Z defined in (4.1) is indeed a topology.

102. Prove the final topology with respect to a set of functions is the finest possible rendering all of them continuous.

103. Prove $\sim_f$ in (4.2) is a relation of equivalence.

104. Prove Lemma 4.3.2.

105. Find open neighbourhoods of both copies of 0 in (4.3) homeomorphic to the unit ball in $\mathbb{R}$.

106. If $f : X \to Y$ is a homeomorphism and $A \subset X$, prove $X/A \cong Y/f(A)$.

107. Define the following relation of equivalence on $\mathbb{R}^3$: $(x, y, z) \sim (X, Y, Z)$ iff $x^2+y^2 = X^2+Y^2$. Describe $\mathbb{T}^2/\sim$. We obviously consider the three-dimensional incarnation of $\mathbb{T}^2$, not the original definition $\mathbb{S}^1 \times \mathbb{S}^1$.

108. (*July 2013*) Prove the remaining homeomorphism in Solved Exercise 90.

109. Find analytical expressions for the functions involved in Exercise 94.

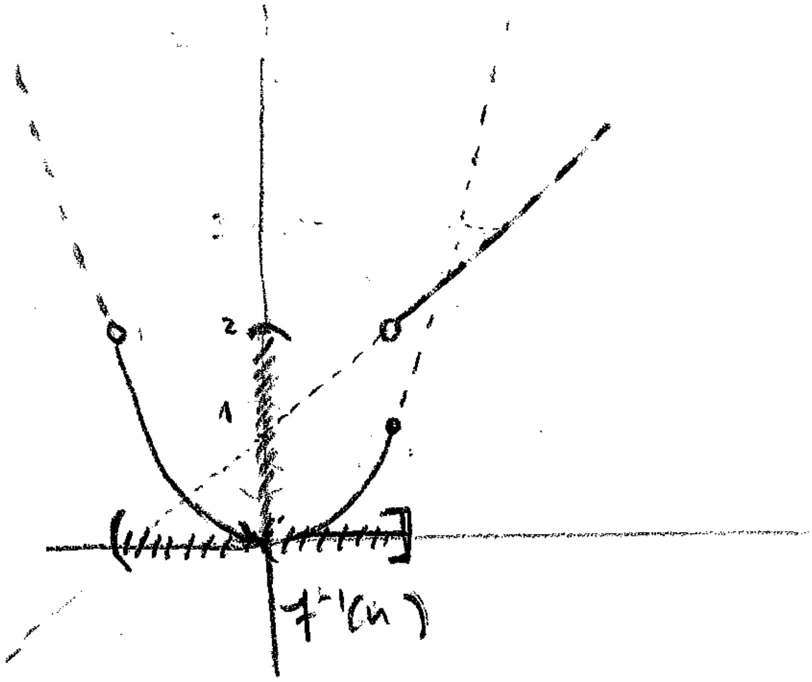
110. Find subsets $S_1 \subset X := \mathbb{R}^2$ and $S_2 \subset Y := \left\{(x, y, z) : x^2 + y^2 = z^2, z \geq 0\right\}$ such that $X/S_1 \cong Y/S_2$. Find explicit analytical expressions of all functions involved.

Exercise 95

③
$$\begin{cases} f(x) = \begin{cases} x^2 & x \leq 1 \\ x+1 & x > 1 \end{cases} \\[2ex] g(x) = x^3 \end{cases}$$

i) An easy choice would be the Euclidean topology $\tau_{\mathbb{R}}^{Eucl}$. Indeed, all it takes is choosing an open set containing 1, e.g. $U = (0,2) \in \tau_{\mathbb{R}}^{Eucl}$.

$$f^{-1}(U) = \{x \in \mathbb{R} : f(x) \in (0,2)\}$$



$$= \{x \in (1,\infty) : x+1 < 2\} \cup \{x \in (-\infty,1] : x^2 < 2\}$$

$$= \underbrace{\{x \in (1,\infty) : x < 1\}}_{= \emptyset} \cup (-\sqrt{2}, 1]$$

$$= (-\sqrt{2}, 1] \notin \tau_{\mathbb{R}}^{Eucl}$$

$\Longrightarrow$ f not continuous.

iii) the obvious choice: the <u>initial topology</u> with respect to f, g, i.e. the topology τ_0 having basis

$$\beta = \{ f^{-1}(U) \cap g^{-1}(V) : U, V \in \tau_{\mathbb{R}}^{\text{Eucl}} \}.$$

* let us first find the generic form for $f^{-1}(U)$ if $U = (a,b)$ $a<b$ is a basis open set of $\tau_{\mathbb{R}}^{\text{Eucl}}$.

$$f^{-1}((a,b)) = \begin{cases} \emptyset & \text{if } a<b\leq 0 & (1) \\ (-\sqrt{b}, \sqrt{b}) & a<0<b\leq 1 & (2) \\ (-\sqrt{b}, 1] & a<0<1<b\leq 2 & (3) \\ (-\sqrt{b}, b-1) & a<0<2<b & (4) \\ (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b}) & 0\leq a<b\leq 1 & (5) \\ (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, 1] & 0\leq a<1<b\leq 2 & (6) \\ (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, b-1) & 0\leq a<1<2<b & (7) \\ (-\sqrt{b}, -\sqrt{a}) & 1\leq a<b\leq 2 & (8) \\ (-\sqrt{b}, \sqrt{a}) \cup [1, b-1) & 1\leq a<2<b & (9) \\ (-\sqrt{b}, -\sqrt{a}) \cup (a-1, b-1) & 2\leq a<b & (10) \end{cases}$$

* and the fact $g(x) = x^3$ is increasing and bijective implies

$$g^{-1}((c,d)) = (\underbrace{c^{1/3}}_{\substack{\text{same}\\ \text{sign as}\\ c}}, \underbrace{d^{1/3}}_{\substack{\text{same}\\ \text{sign as } d}}) \qquad (11)$$

Hence $f^{-1}((a,b)) \cap g^{-1}((c,d))$ can take any of the following forms:

1. $\emptyset$ $\quad$ (1) $\cap$ (11)
2. (α, β), $\quad -1 \leq \alpha < \beta \leq 1$ $\quad$ (2) $\cap$ (11)
3. $(\alpha, 1]$, $\quad -\sqrt{2} \leq \alpha \leq 1$ $\quad$ (3) $\cap$ (11)
4. $\underline{(\gamma, \delta)}$ $\quad \forall\ \underline{\gamma < \delta}$ $\quad$ (comes from intersecting (4) with any interval of the form (11))

since all intervals (1) - (10) save for (3), (6), (9) are unions of these intervals $\overline{(\gamma, \delta)}$, we only need separate attention for (3), (6), (9) to finish the exercise:

5. $\underbrace{\left((-\beta, -\alpha) \cup (\alpha, 1]\right)}_{(6)} \cap \underbrace{(\gamma, \delta)}_{(11)}$ is equal

to either a union of intervals of the form (γ, δ) as well, or an interval of the form $(\alpha, 1]$ $0 \leq \alpha < 1$ which is a special case of 3 above.

6. $(-\beta, -\alpha) \cup [1, \beta^2 - 1)$, $\beta > \sqrt{2}$, hence intersecting with any (11) the only new basis open sets we obtain are $\underline{[1, \beta), \beta > 1}$

Thus a basis for τ_0 will be

$$\beta = \underbrace{\{\emptyset\} \cup \{(a,b) : a < b\}}_{\text{this is a basis for } \tau_{\mathbb{R}}^{\mathrm{Eucl}}} \cup \{(a,1] : -\sqrt{2} \leq a \leq 1\}$$

$$\cup \{[1,b) : b > 1\}.$$

Arbitrary unions yield

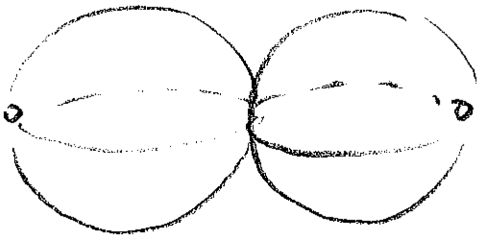
$$\tau_0 = \{\emptyset\} \cup \{(a,b) : -\infty \leq a < b \leq +\infty\}$$
$$\cup \{(a,1] : -\infty \leq a \leq 1\}$$
$$\cup \{[1,b) : 1 < b \leq +\infty\}$$

$\cup$ {all unions of the previous three generic intervals}

Note $\tau_{\mathbb{R}}^{\mathrm{Eucl}} \underset{\text{strictly coarser}}{\subsetneq} \tau_0$, i.e.

the coarsest topology rendering $f : (\mathbb{R}, \tau_0) \to (\mathbb{R}, \tau^{\mathrm{Eucl}})$ continuous is still finer than Euclidean; couldn't be otherwise since <u>new</u> open sets have to be added to solve discontinuity at $\underline{x = 1}$

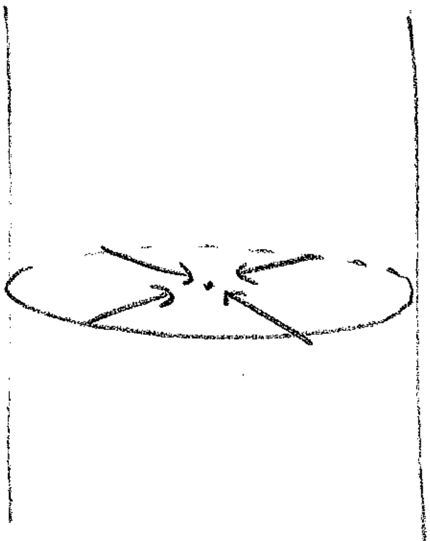
④ $X_1 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=1\}/(S^1\times\{0\})$

$X_2 =$ 

Exercise 96

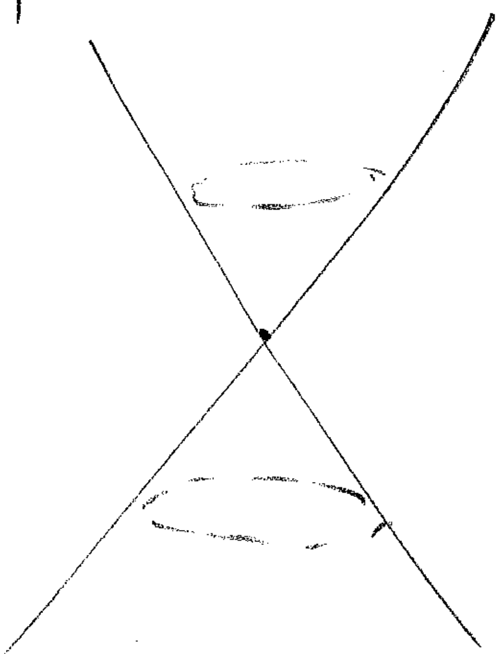
i) $\underline{X_1 \cong X_2}$

a) Let us find a topological space $X_0 \cong X_1$ which admits a simple analytical expression.



<u>identify</u> (collapse into a point)

This is ostensibly a <u>cone</u>



For instance let us choose

$$\bar{X}_0 = \{(x,y,z) : x^2+y^2 = z^2\}.$$

The identification f must yield a comm. diagram

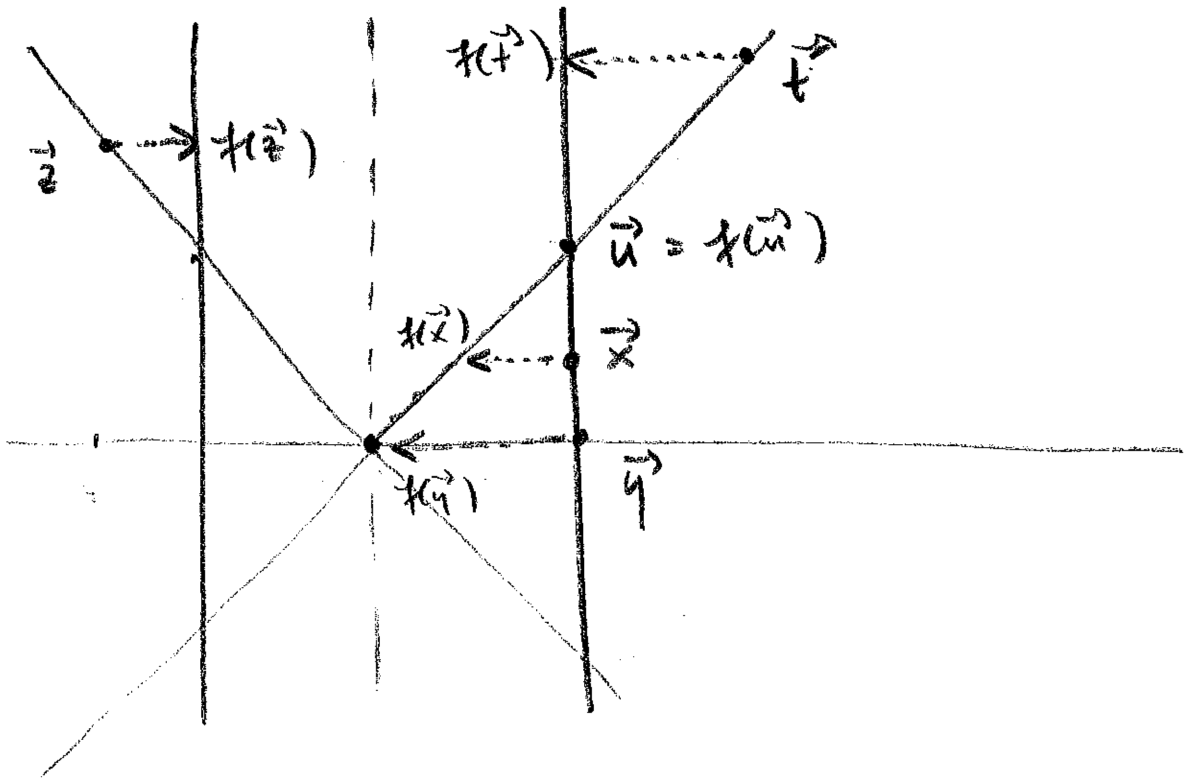
$$\begin{tikzcd}
C = \{x^2+y^2=1\} \arrow[rr, two heads, "f"] \arrow[dr] \arrow[drr, phantom, "|||"] & & \bar{X}_0 = \{x^2+y^2=z^2\} \\
& C/\sim \arrow[ur, dashed, "\cong"'] &
\end{tikzcd}$$

def. of $\sim$

and $(x,y,z) \sim (\bar{x},\bar{y},\bar{z}) \iff \begin{cases} (x,y,z) = (\bar{x},\bar{y},\bar{z}) \\ \text{or} \\ \text{both belong to } S^1 \times \{0\}. \end{cases}$

$\iff f(x,y,z) = f(\bar{x},\bar{y},\bar{z})$

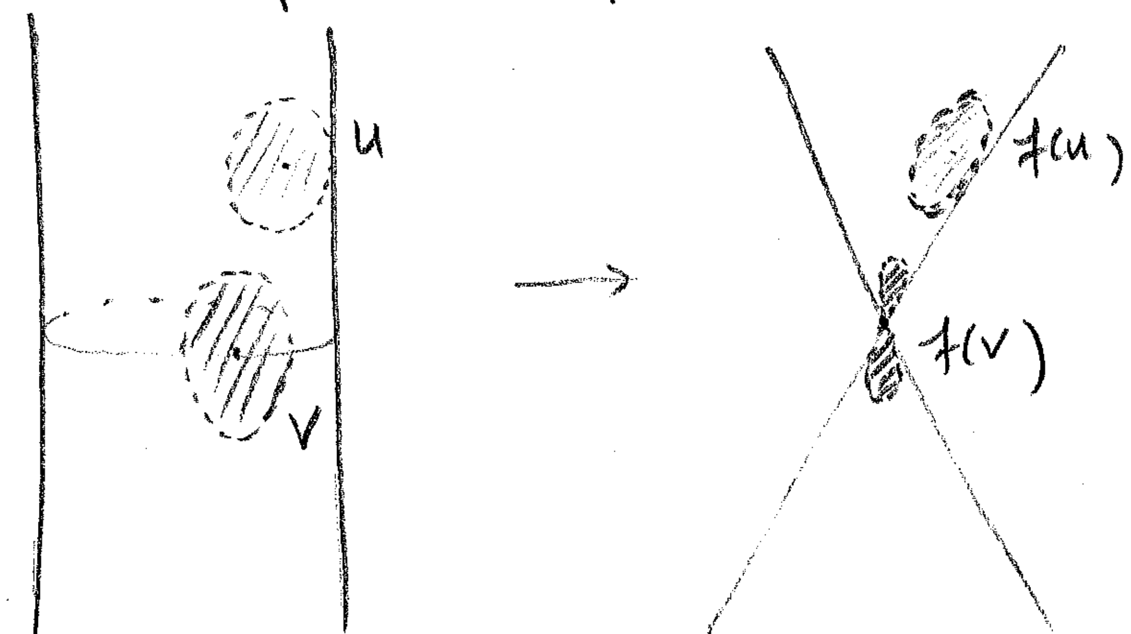
We can use axial symmetry to focus on a section:



We can choose, for instance,

$$f : C \longrightarrow \{x^2+y^2=z^2\} = X_0$$
$$(x,y,z) \longmapsto (xz, yz, z)$$

- well defined: $x^2+y^2=1 \Rightarrow (xz)^2+(yz)^2=z^2$ ✓
- surjective: $\forall (x,y,z) \in \{x^2+y^2=z^2\}$,
  - if $z=0 \Rightarrow (x,y,z)=(0,0,0) = f(\bar{x},\bar{y},0) \quad \forall \bar{x},\bar{y}$.
  - if $z \neq 0 \Rightarrow (x,y,z) = \left(\frac{x}{z}\cdot z, \frac{y}{z}\cdot z, z\right) = f\left(\frac{x}{z}, \frac{y}{z}, z\right)$
- continuous: every coordinate function is continuous
- open: any basis open set in C with the induced Euclidean topology, e.g. $B((x,y,z),\varepsilon)\cap C$, will be mapped to an open set in X_0, i.e.



We will accept a simple picture such as above as proof of this fact.

Thus f is an identification.

Plus, $f(x,y,z) = f(\bar{x},\bar{y},\bar{z})$

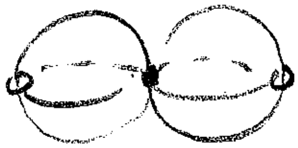
$\iff (xz, yz, z) = (\bar{x}\bar{z}, \bar{y}\bar{z}, \bar{z})$

$\iff \bar{z} = z$ and $\begin{cases} xz = \bar{x}\bar{z} \\ yz = \bar{y}\bar{z} \end{cases}$

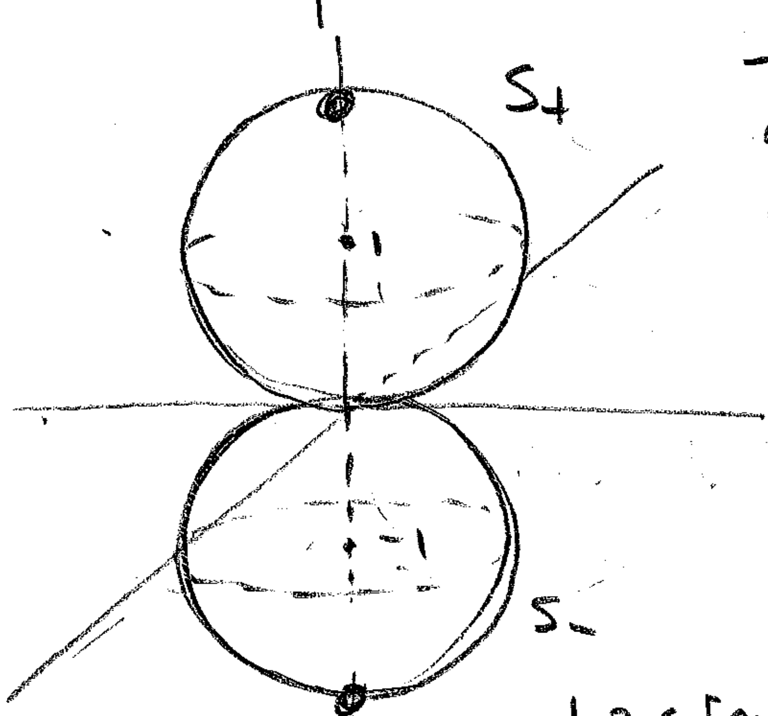
$\iff$ either $z = \bar{z} = 0$ or $(x,y,z) = (\bar{x},\bar{y},\bar{z})$

$\iff \underline{(x,y,z) \sim (\bar{x},\bar{y},\bar{z})}$

Thus : $\underline{X_1 \cong X_0}$.

We still need to prove $X_0 \cong$  $= X_2$

Let us fix an analytic expression for X_2.



Two spheres of radius 1, having centres $(0,0,1)$ and $(0,0,-1)$

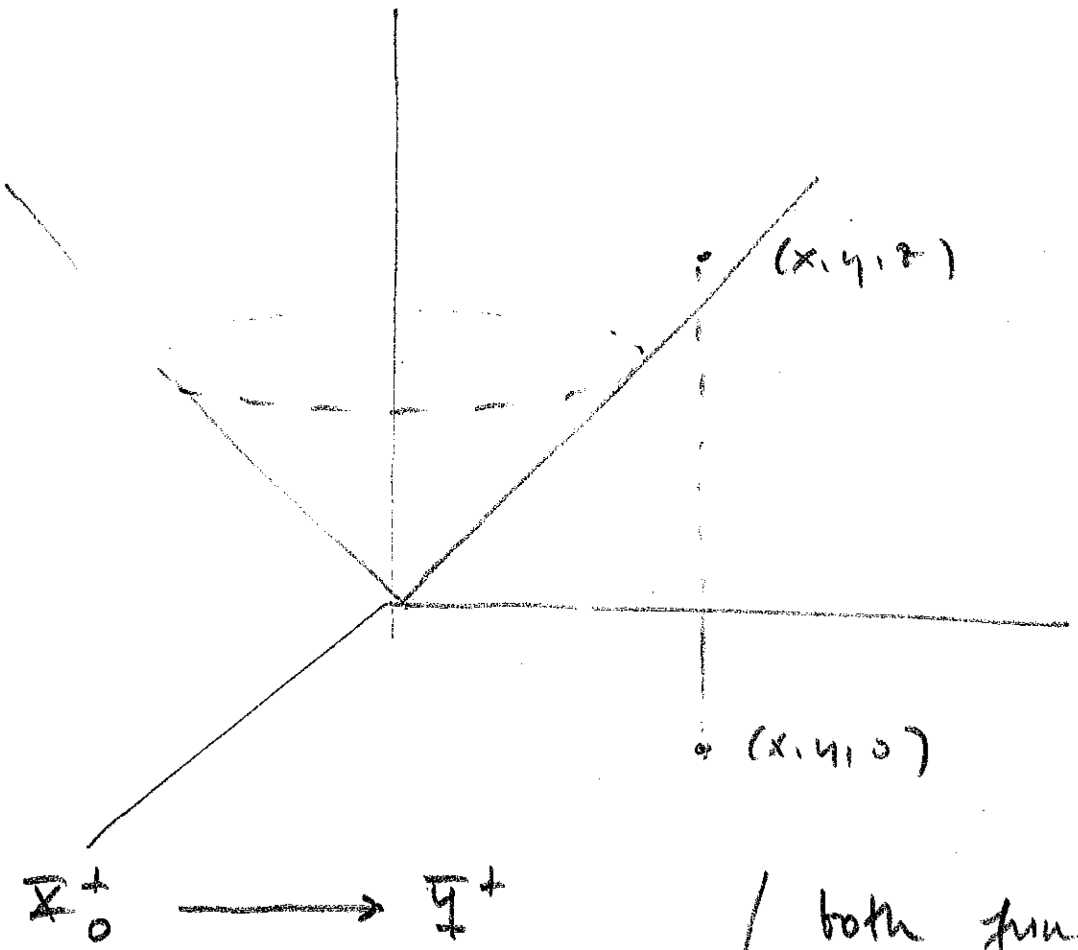
$$X_2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+(z-1)^2 = 1 \text{ and } z \in [0,1)\} \cup \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+(z+1)^2 = 1 \text{ and } z \in (-1,0]\} = S_+ \cup S_-$$

We can focus on $S_+ = \{x^2+y^2+(z-1)^2=1,\ z \neq 1\}$

and $\overline{X}_0^+ =$  $= \{x^2+y^2 = z^2,\ z \geq 0\}$

and find a homeo $h_+ : \overline{X}_0^+ \longrightarrow S_+$; by extension using symmetry, the final homeo $h : \overline{X}_0 \longrightarrow \overline{X}_2 = S_+ \cup S_-$ will be $h(x,y,z) = -h_+(x,y,-z) \quad \forall z < 0.$

* First let us prove $\overline{X}_0^+ \cong \underbrace{\overline{Y}^+ = \{z=0\}}_{\text{horizontal plane}}$



$$h_+^1 : \overline{X}_0^+ \longrightarrow \overline{Y}^+$$
$$(x,y,z) \longmapsto (x,y,0)$$
$$(\bar{x}, \bar{y}, \sqrt{\bar{x}^2+\bar{y}^2}) \longleftarrow\!\shortmid (\bar{x}, \bar{y}, 0)$$

both function and its inverse are continuous and obviously well-defined (because $z \geq 0$)

* let us prove $\Psi^+ = \{z=0\} \cong S_+$. This is nothing but the stereographic projection's inverse:

$$\boxed{\begin{aligned} h_+^2 : \Psi^+ &\longrightarrow S_+ \\ (x,y,0) &\longmapsto \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{x^2+y^2+1}+1\right)\end{aligned}}$$

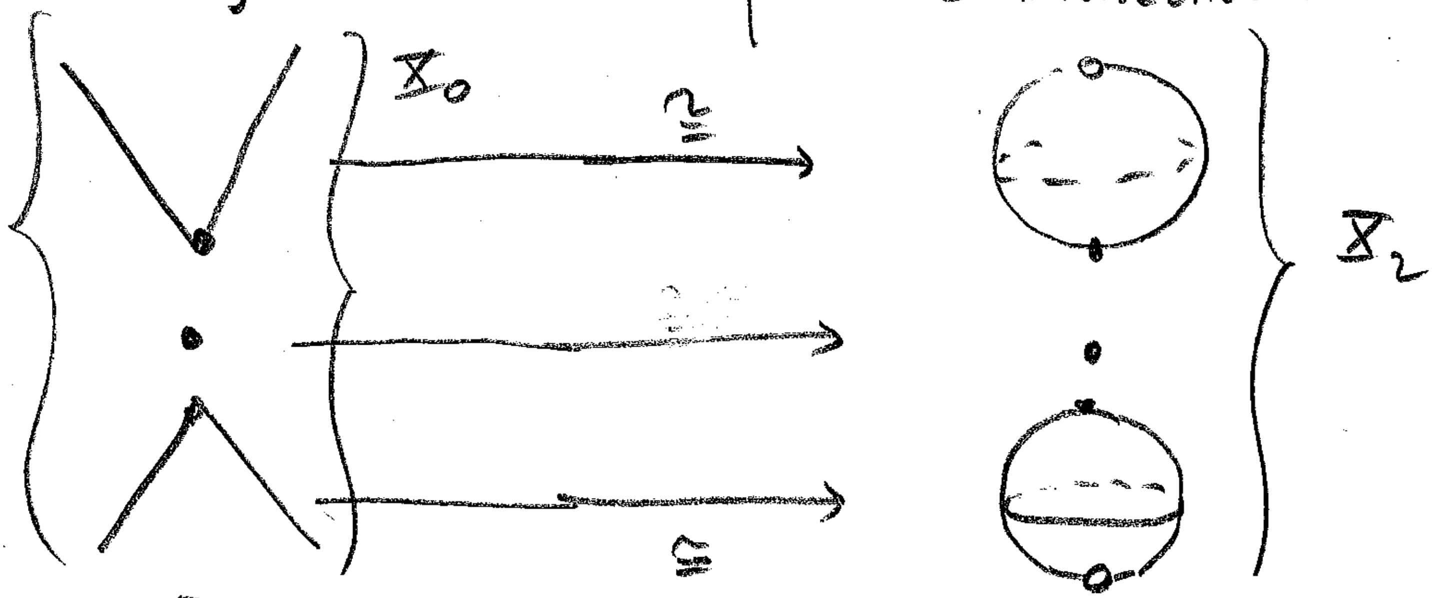
Thus $h_+ = h_+^2 \circ h_+^1 : X_0^+ \xrightarrow{\cong} S_+$

$$(x,y,z) \longmapsto (- , - , -)$$

and defining

$$\begin{aligned} h : X_0 &\longrightarrow X_2 \\ (x,y,z) &\longmapsto \begin{cases} h_+(x,y,z), & z \geq 0 \\ -h_+(x,y,-z), & z < 0 \end{cases}\end{aligned}$$

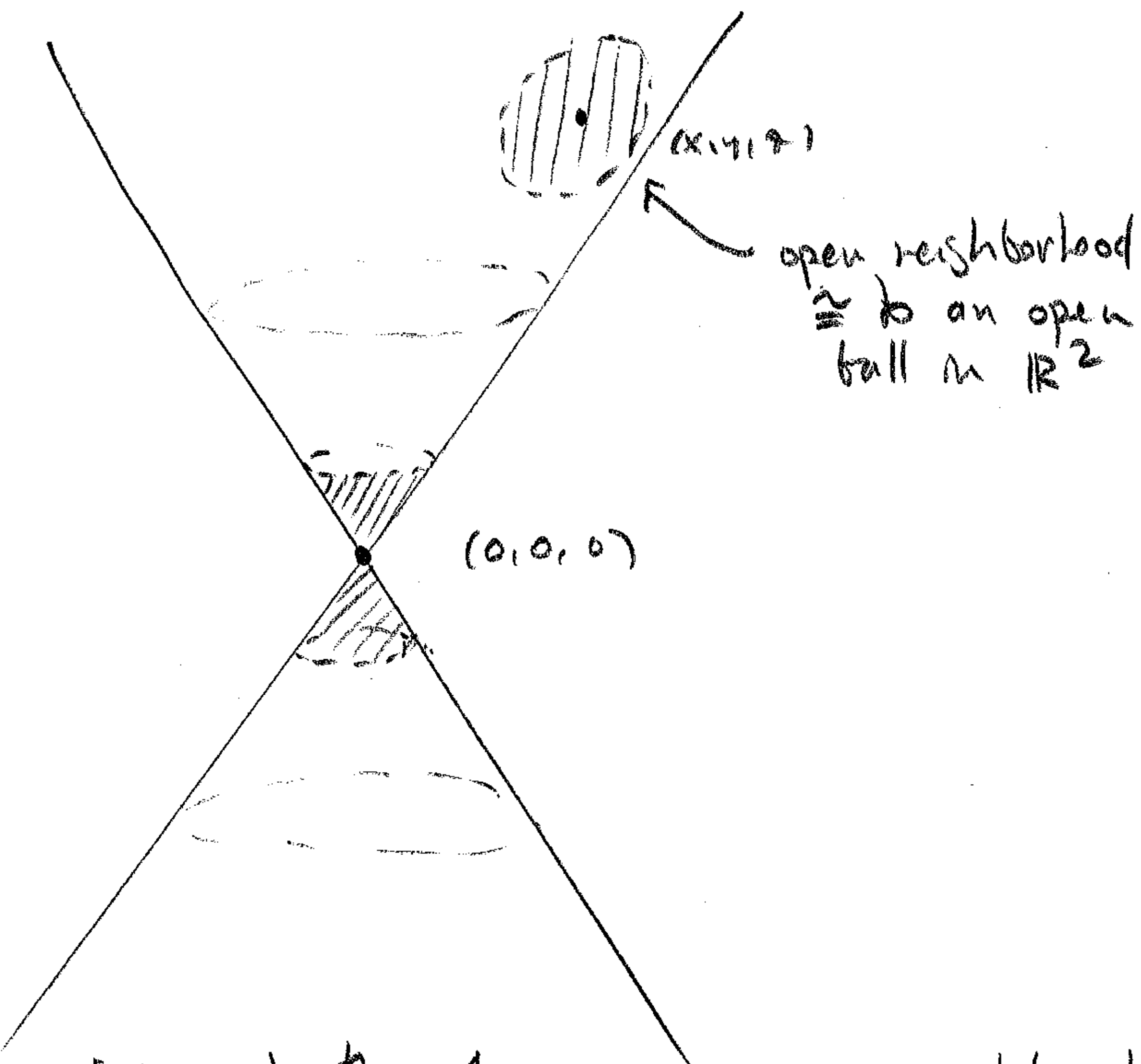
we obtain a function defined on a finite closed covering with consistency on the intersection:



$\Rightarrow$ $\boxed{\text{h is a continuous, bijective, bicontinuous function on all of } X_0}$

iii) - First of all, being locally Euclidean is a property invariant by $\cong$ so we only need to check this for, say, X_0.

- X_0 is not locally Euclidean of any dimension. If there were any such dimension it would have to be $\underline{2}$ as applies to nearly all of its points (e.g. (x,y,z) below)



But as seen in the figure, no open neighbourhood of $(0,0,0)$ will ever be $\cong$ to an open ball in $\mathbb{R}^2$ (as I said, a heuristic explanation such as this suffices).

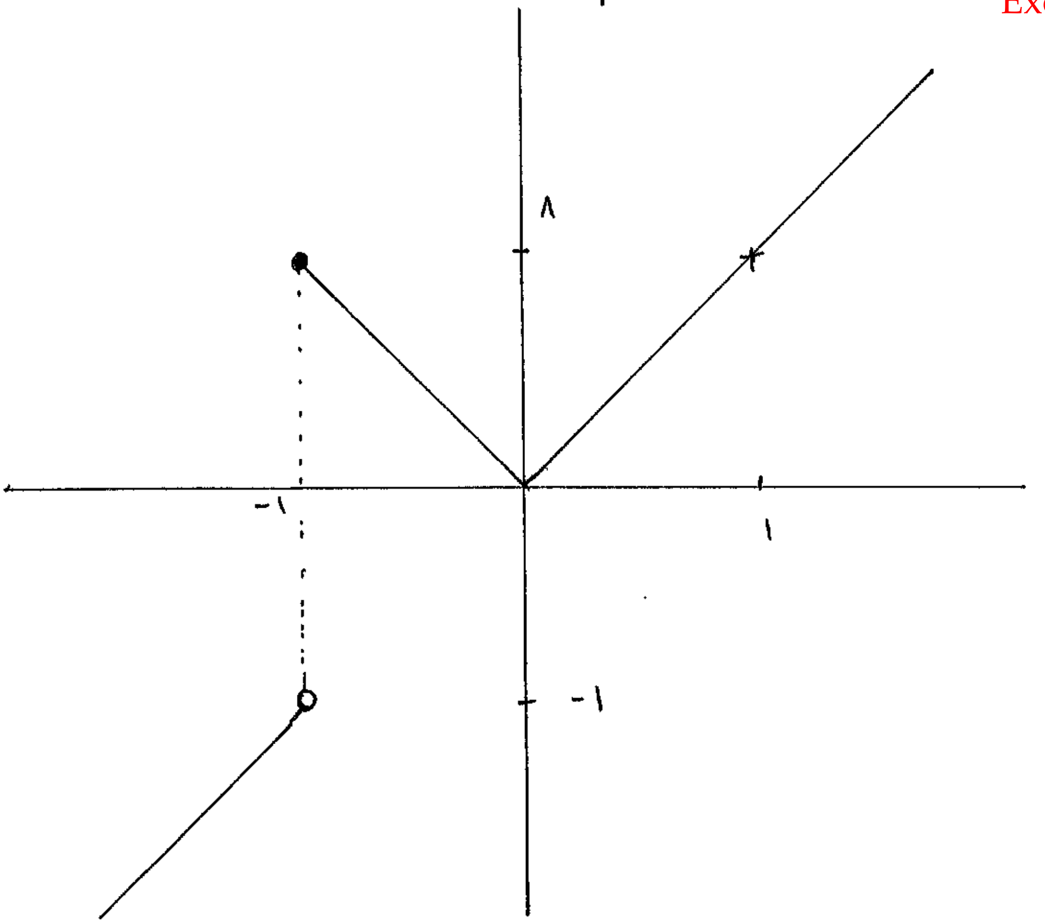
2) i) We know (Coursework January 2014):

$f : X \to Y$ continuous $\iff \forall S \subset X,\ f(\overline{S}) \subset \overline{f(S)}$.

Hence all we need is a subset $S \subset \mathbb{R}$: $f(\overline{S}) \not\subset \overline{f(S)}$.

If we take a look at the graph of the function:

Exercise 97



Our set S will need to involve $x = -1$ in a "critical" way.

Namely, as a part of ∂S (and $S \neq \overline{S}$).

For instance: choose $S = (-2,-1)$.

We have: $\begin{cases} \overline{S} = [-2,-1] \text{ in } \tau_{\mathbb{R}}^{Eucl} \\ f(S) = (-2,-1) \Rightarrow \overline{f(S)} = [-2,-1] \\ f(\overline{S}) = [-2,-1) \cup \{1\} \end{cases}$

$\Rightarrow f(\overline{S}) = [-2,-1) \cup \{1\} \not\subset [-2,-1] = \overline{f(S)}$.

An ε-δ approach (i.e. using the original definition of continuity between metric topologies) is also easy:

All we need to prove is that f is not continuous at $x=-1$:

$$\exists \varepsilon > 0 : \forall \delta > 0, \quad \exists x \in \mathbb{R} :$$
$$|x+1| < \delta \text{ and } |f(x) - f(1)| \geq \varepsilon .$$

Which means:
$$\left.\begin{array}{l} \exists \varepsilon > 0 : \forall \delta > 0, \exists x \in \mathbb{R} : \\ |x+1| < \delta \text{ and } |f(x) - 1| \geq \varepsilon . \end{array}\right\} (*)$$

Looking at the graph in the previous page all it takes is choosing points x to the left of -1 ($\underbrace{B_{d_2}(-1, \delta)}_{= (-1-\delta, -1+\delta)}$ must, after all, contain points on both sides).

Hence choosing $x \in (-1-\delta, -1)$ it becomes clear that if $\varepsilon = 2$ (for instance), then $(*)$ is satisfied $\forall \delta > 0$.

---

* Finally we can prove it using the original definition of continuity between topological spaces:

$$f^{-1}(U) \text{ open } \quad \forall \text{ open set in } U .$$

But given the fact that $\tau = \tau_{\mathbb{R}}^{Eucl}$ is a metric topology, this proof would be essentially the same as the one in the previous paragraphs (U would be conditioned by $f^{-1}(U)$ containing -1, and the rest would follow from the first half of this page).

ii) τ_0 will be the initial topology with respect to f, i.e. the topology having basis:

$$\beta = \{ f^{-1}(U) : U \in \tau_{\mathbb{R}}^{Eucl} \}.$$

* Every open set of $\tau_{\mathbb{R}}^{Eucl}$ can be obtained as a union of intervals (a,b), $a \leq b$.

Hence β can be expressed in the form

$$\beta = \left\{ f^{-1}\Big(\bigcup_{i \in I} (a_i, b_i) \Big) \overset{(*)}{=} \bigcup_{i \in I} f^{-1}\big((a_i, b_i)\big) : a_i \leq b_i \ \forall i \in I \right\}$$

basic Set theory (will need a HINT)

* Hence (this is not trivial and will require a HINT to the students): a basis (not equal to the above) for τ_0 can be

$$\beta^* = \{ f^{-1}\big((a,b)\big) : a \leq b \}.$$

Indeed:
- every subset in τ_0 (which the exercise does not ask you to find explicitly) is a union of subsets in β
- every subset in β can be expressed as a union of subsets in β^* (obvious from identity $(*)$)

$\Rightarrow$ every subset in τ_0 can be expressed as a union of elements of β^*

$\Rightarrow$ β^* BASIS for τ_0

We can find β^*, or apply the same reasoning after refining our choice of intervals (a,b) so that any interval in $\mathbb{R}$ can be obtained as a union thereof. This will yield a <u>simpler</u> basis.

* Check the graph two pages ago. Let $(a,b) \subset \mathbb{R}$. We assume $a < b$ (otherwise $(a,b) = \emptyset$ trivial).

We will choose (a,b) depending on whether or not it contains $1, 0, -1$. <u>ANY</u> other interval in $\mathbb{R}$ will be a union of intervals of this family.

<u>Case 1</u>: $1 \leq a \Rightarrow f^{-1}((a,b)) = (a,b)$

<u>Case 2</u>: $0 \leq a < 1 < b \Rightarrow f^{-1}((a,b)) = [-1,a) \cup (a,b)$

<u>Case 3</u>: $0 \leq a < b \leq 1 \Rightarrow f^{-1}((a,b)) = (-b,-a) \cup (a,b)$

<u>Case 4</u>: $-1 \leq a < 0 < b \leq 1 \Rightarrow f^{-1}((a,b)) = (-b,b)$

<u>Case 5</u>: $-1 \leq a < b \leq 0 \Rightarrow f^{-1}((a,b)) = \emptyset$

<u>Case 6</u>: $a < -1 < b \leq 0 \Rightarrow f^{-1}((a,b)) = (a,-1)$

<u>Case 7</u>: $a < b \leq -1 \Rightarrow f^{-1}((a,b)) = (a,b)$.

Any interval $(a,b) \subset \mathbb{R}$ either fits one of Cases 1-7 or is a union thereof, e.g. $(-\frac{1}{2},3) = \underbrace{(-\frac{1}{2},1)}_{\text{Case 4}} \cup \underbrace{(0,3)}_{\text{Case 2}}$

Thus the following is a basis of τ_0 as well:

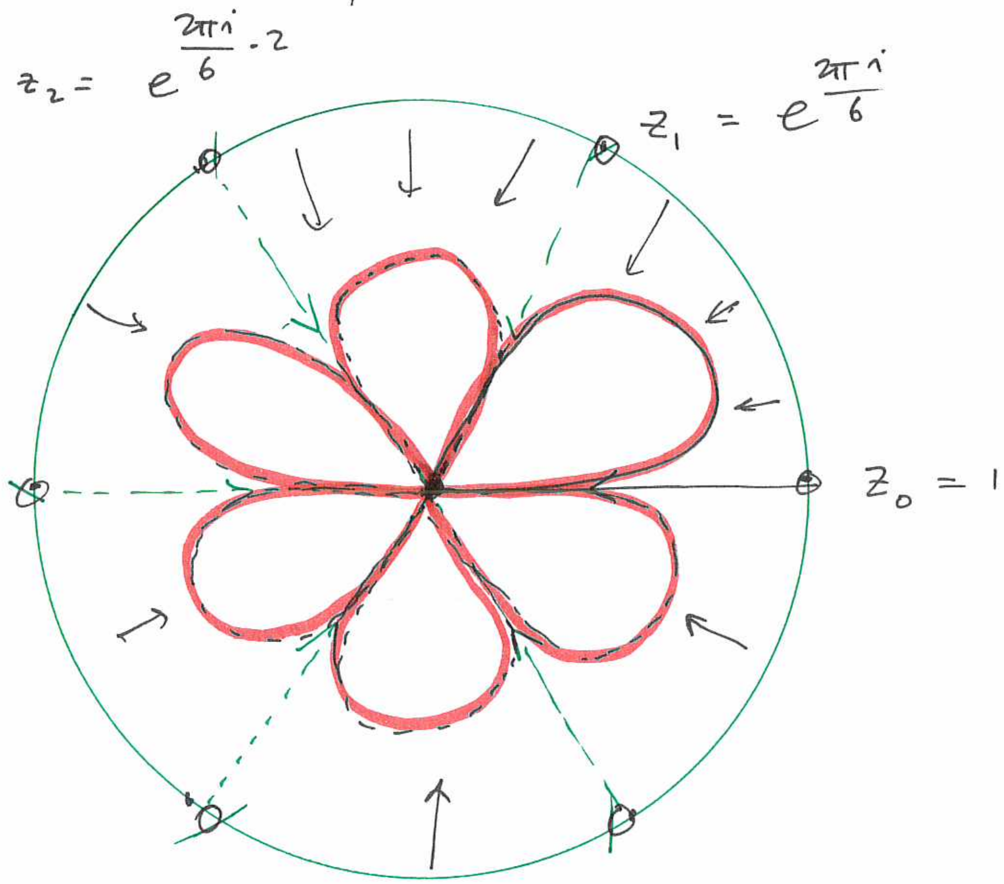
$$\beta^{**} = \{ f^{-1}((a,b)) : (a,b) \text{ fits one of Cases 1-7} \}$$

$$
\begin{aligned}
= &\{ (a,b) : 1 \leq a \} \\
\cup &\{ [-1,a) \cup (a,b) : 0 \leq a < 1 < b \} \\
\cup &\{ (-b,-a) \cup (a,b) : 0 \leq a < b \leq 1 \} \\
\cup &\{ (-b,b) : 0 < b \leq 1 \} \\
\cup &\{ (a,b) : b \leq -1 \}
\end{aligned}
$$

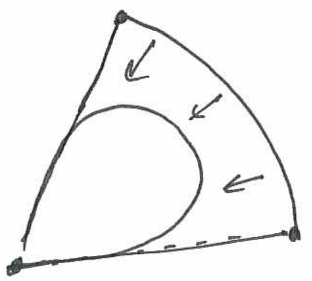
It is very easy to make mistakes in this process. What matters is that the process itself is described correctly rather than the actual output.

4) $S^1 / \mathbb{Z}_n \cong S^1 \vee \overset{n)}{\dots} \vee S^1$ Exercise 98

We can define our identification in such a way, that all of the roots of unity are carried to $(0,0)$



Hence all we need to do is define f on each portion

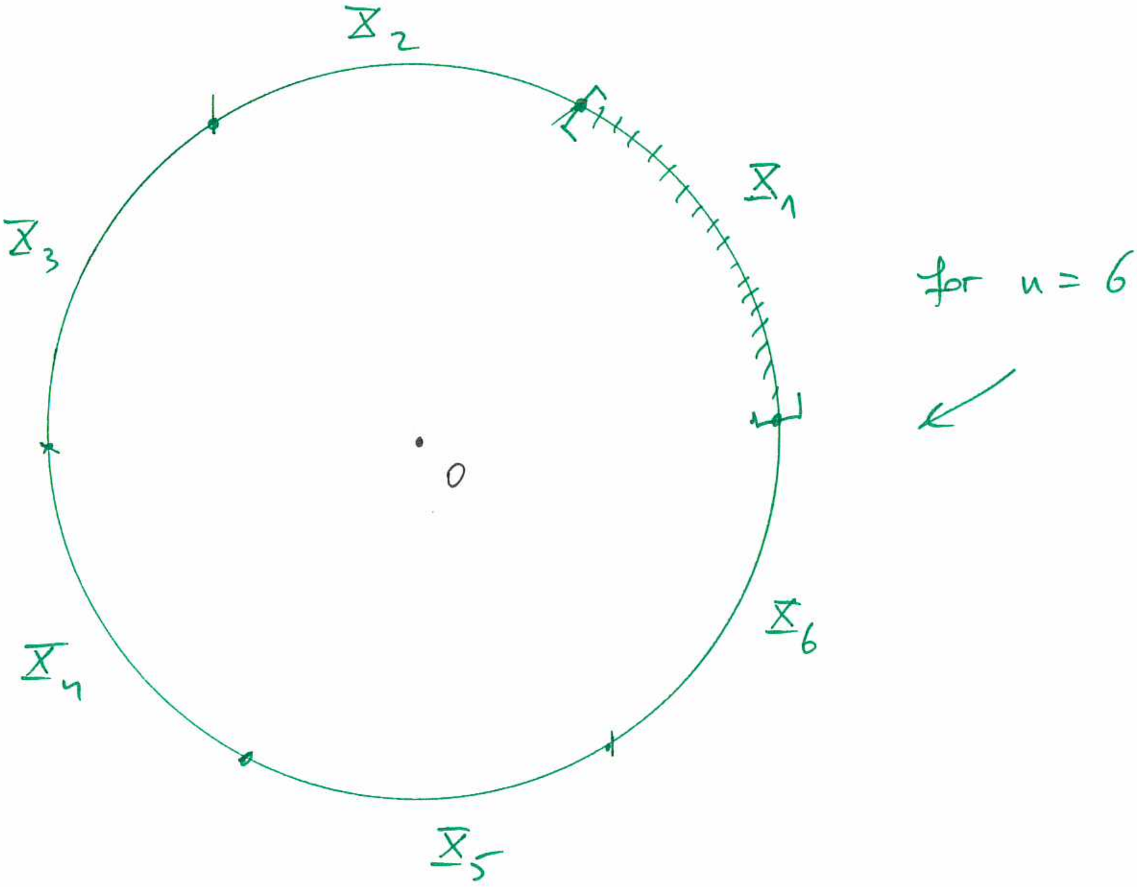


with the understanding that the restriction of f to the intersections must be consistent.

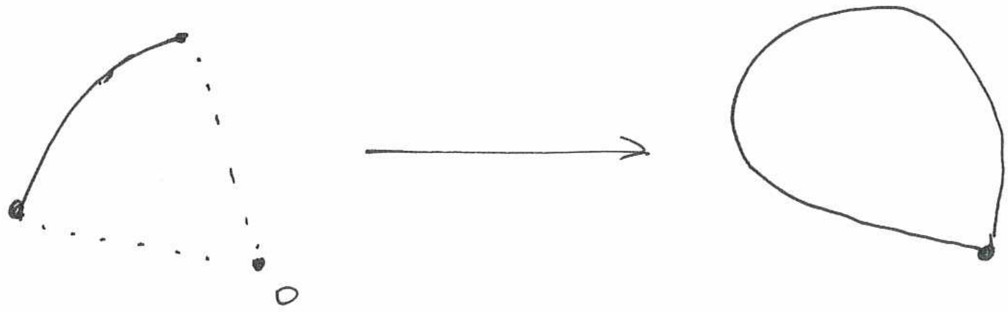
In other words: consider the closed, finite over of S^1:

$$S^1 = X_1 \cup \ldots \cup X_n,$$

$$X_i = \left\{ e^{\frac{2\pi i}{n} x} : x \in [i-1, i] \right\}$$



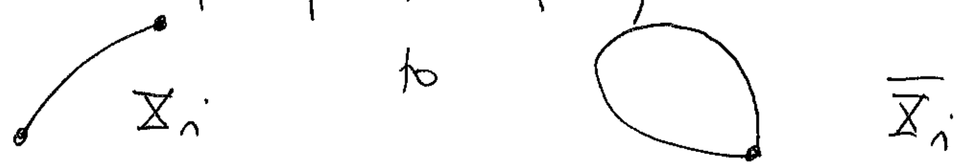
$f|X_i$:



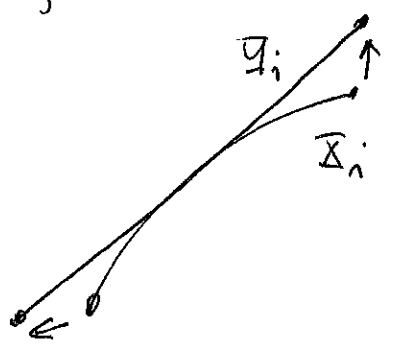
The intersections of $X_1, \ldots, X_n$ are the roots of unity and they are all mapped to 0 $\Rightarrow$ they all coincide.

All we need for f is to be continuous on each X_i.

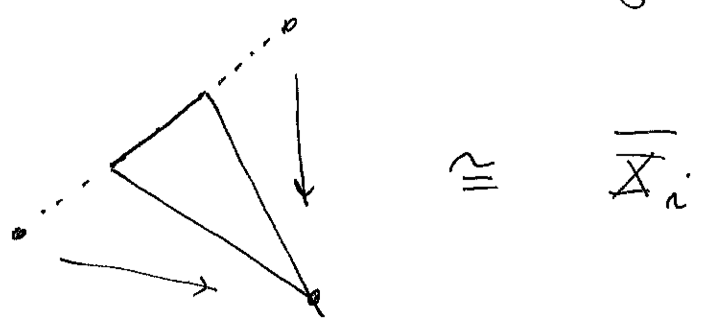
there are many ways of mapping a sector



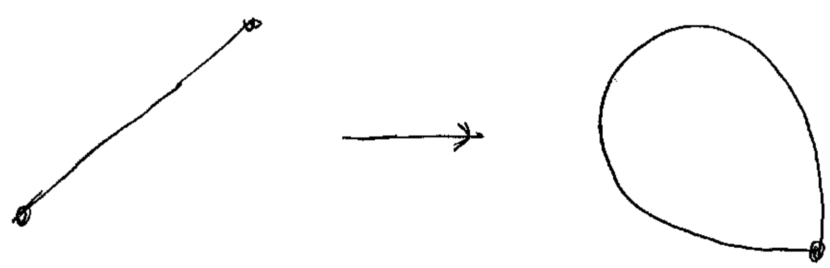
for example : - mapping the sector to a line segment Y_i tangent to it first,



and then divide the segment into three equal parts, taking the extremes of the outer two to the origin

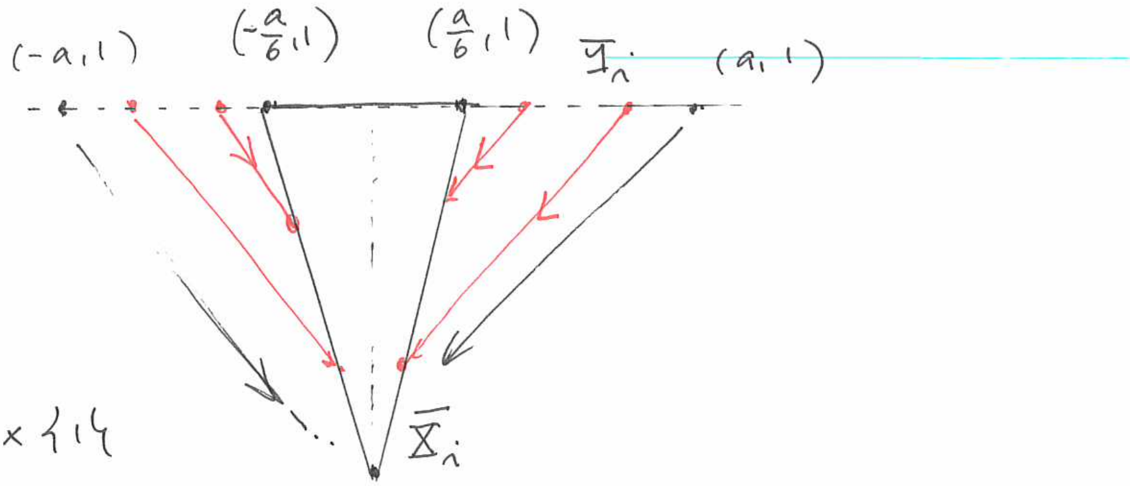


- another way would be using polar coordinates to transform Y_i (segment) into a continuous, smooth loop, directly



An explicit form for this function would add unnecessary complication and, thus, will not be required.

If, however, you want one such expression, it is



$$[-a,a]\times\{1\}$$
$$\|$$
$$f_i : Y_i \longrightarrow \bar{X}_i$$

$$(x,1) \longmapsto \begin{cases} (x,1) & \text{if } x\in\left[-\frac{a}{6},\frac{a}{6}\right] \\ \left(\frac{1}{5}-\frac{x}{5},\ \frac{6}{5}-\frac{6x}{5}\right) & \text{otherwise} \end{cases}$$

where the segment has been made horizontal (rotations are homeomorphisms).

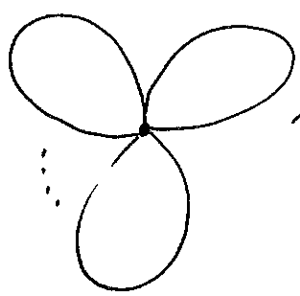
Proving

$\cong S^1$ is akin to proving any polygon is $\cong S^1$, along the lines of an exercise seen in class.

Hence: $\left\{\begin{array}{l} \text{- } f \text{ defined on each sector is continuous} \\ \text{- restrictions coincide on intersections (because} \\ \quad \text{all roots of unity are mapped to } 0) \\ \text{- } X_1, \ldots, X_n \text{ finite, closed covering of } \mathbb{S}^1 \end{array}\right.$

exact hypotheses of a Proposition seen in theory which ensures f (total function on $\mathbb{S}^1$) is continuous.

Hence

$f : \mathbb{S}^1 \longrightarrow$  $\cong \mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1$

this function f is :

- continuous (we just saw that)
- surjective (as seen from the drawings or the explicit expression)
- closed (result seen in class: continuous function between $\left.\begin{array}{l}\text{closed, bounded } X \subset \mathbb{R}^n \\ Y \subset \mathbb{R}^n\end{array}\right\}$ is closed)

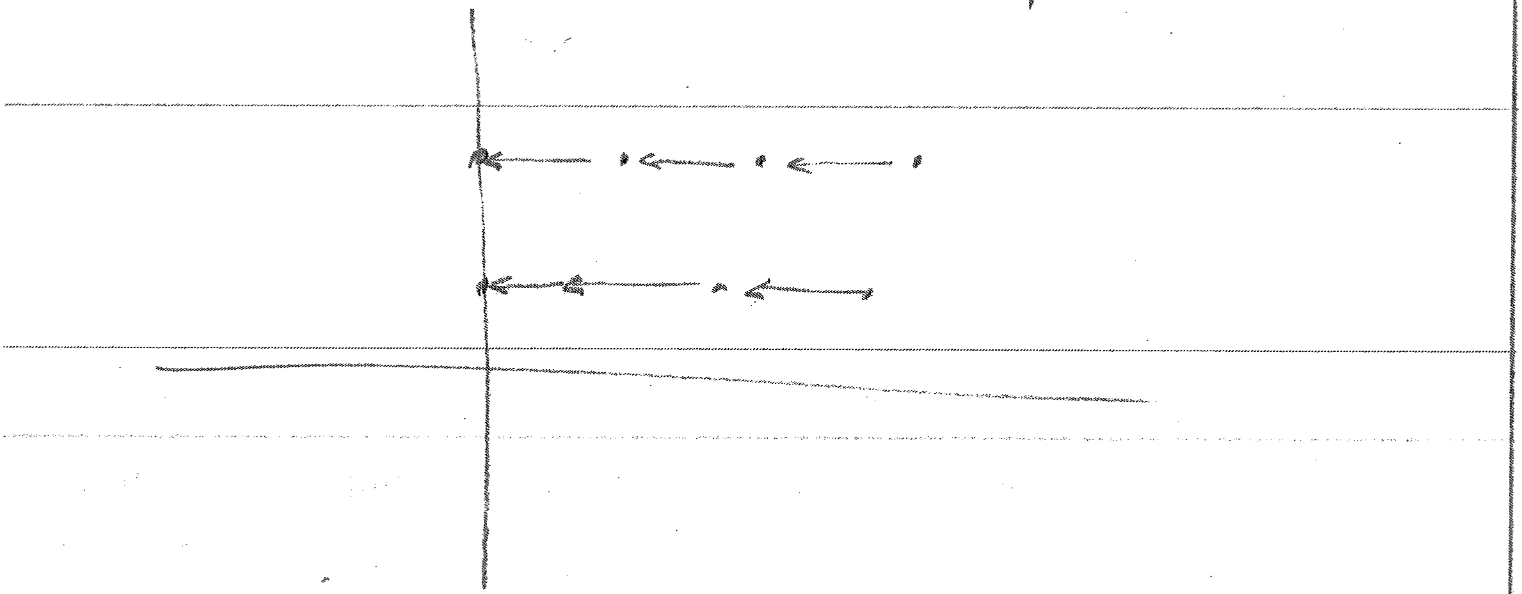
thus f is an identification. And two points are mapped to the same point $\Leftrightarrow$ they are equal or both are roots of unity. Hence $\mathbb{S}^1/\mathbb{Z}_n \cong \mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1$.

Solution to exercise 99

6)

i) $(x,y) \sim (\bar{x},\bar{y}) \iff y = \bar{y}$.

What we are doing, essentially, is identifying points with the same vertical component.



which means we are collapsing all of the plane onto the y-axis, which is obviously $\cong$ to $\mathbb{R}$.

Let us find an identification $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ summarizing this process:

$$\left.\begin{array}{rcl} f : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & y \end{array}\right\} \text{ projection on } y\text{-coordinate.}$$

This is <u>continuous</u> & <u>surjective</u>. It is an identification because it is <u>open</u>: projections are open,

$$\left.\begin{array}{rcl} pr_{\mathbb{Y}} : \mathbb{X} \times \mathbb{Y} & \longrightarrow & \mathbb{Y} \\ (x,y) & \longmapsto & y \end{array}\right\} \text{ ~~general result~~}$$

because they map basis elements $U \times V$ to open sets V.

Hence f is an identification and all we need to check is

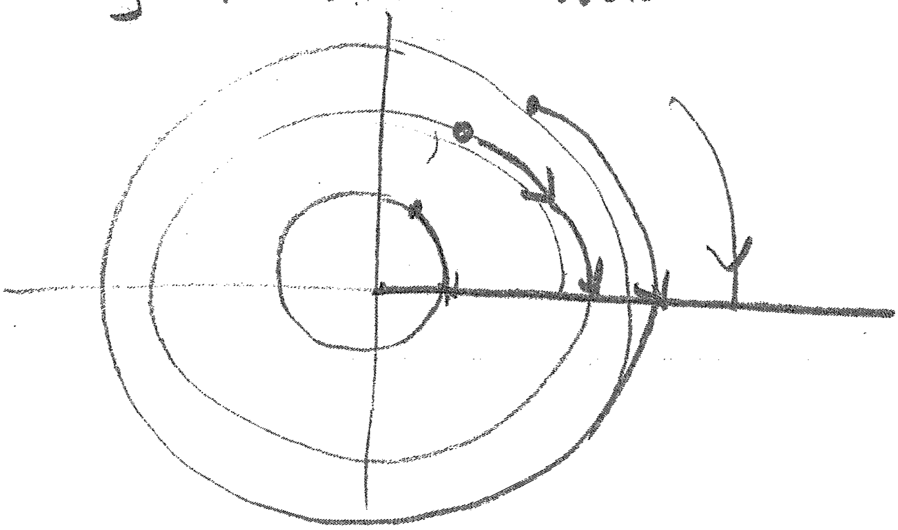
$$f(x,y) = f(\bar{x}, \bar{y})$$
$$\Leftrightarrow (x,y) \sim (\bar{x}, \bar{y})$$

but this is immediately TRUE by definition of f and $\sim$.

---

iii) $(x,y) \bowtie (\bar{x}, \bar{y}) \Leftrightarrow \|(x,y)\| = \|(\bar{x}, \bar{y})\|$.

We are essentially identifying points lying on the same circle, so we are "closing $\mathbb{R}^2$ like a book"



We need an identification

$$f : \mathbb{R}^2 \longrightarrow [0, +\infty)$$

to formalize this.

We also need f to be such, that

$$(x,y) \bowtie (\bar{x},\bar{y}) \iff f(x,y) = f(\bar{x},\bar{y}).$$

Immediate choice: $f(x,y) = \sqrt{x^2+y^2}$

$$f : \mathbb{R}^2 \longrightarrow [0,+\infty)$$
$$(x,y) \longmapsto \sqrt{x^2+y^2}$$

- continuous
- surjective (e.g. $\forall r \in [0,+\infty)$, $r = f(r,0)$)
- $(x,y) \bowtie (\bar{x},\bar{y}) \iff f(x,y) = f(\bar{x},\bar{y})$ <u>Trivial</u>
- f open, hence an identification.
  Indeed, $\forall \varepsilon > 0$, $\forall (x,y) \in \mathbb{R}^2$, $f(B((x,y),\varepsilon))$ is an open set of $[0,+\infty)$:

$$f(B((x,y),\varepsilon)) = \begin{cases} (f(x,y)-\varepsilon,\ f(x,y)+\varepsilon) & \text{if } f(x,y) \geq \varepsilon, \\ [0,\ f(x,y)+\varepsilon) & \text{if } f(x,y) < \varepsilon. \end{cases}$$

$\Rightarrow$ f open

$\Rightarrow$ f identification $\Rightarrow$ $[0,+\infty) \cong \mathbb{R}^2/\bowtie$.

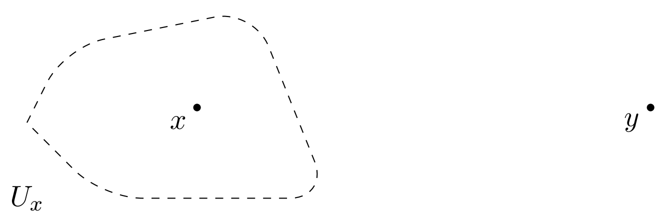
Chapter 5

Separability properties

Given any metric space, and two different points thereof, we can always find disjoint open balls centered on each. This property need not be shared by all topological spaces.

5.1 Fréchet spaces

Definition 5.1.1. *Let (X, τ) be a topological space. We call X* ***Fréchet*** *or* T_1 *if given any two different $x, y \in X$, there exists $U_x \in \tau$ such that $x \in U_x$ and $y \notin U_x$.*



Examples 5.1.2.

1. Given any metric space (X, d), and any $x, y \in X$, defining $r = d(x, y)$ then $y \notin B(x, r)$ and $x \in B(x, r)$.

2. Let $\mathbb{R}$ be with the following topology:

$$\tau = \{S \subset \mathbb{R} : \mathbb{R} \setminus S \text{ finite }\} \cup \{\varnothing\}.$$

Then for every $x \neq y \in \mathbb{R}$, $\mathbb{R} \setminus \{y\}$ is an open neighbourhood of x not containing y.

Proposition 5.1.3. *A topological space X is Fréchet if, and only if, every point $x \in X$ is closed.*

Proof. Assume X is Fréchet and let $x \in X$. Then for every $y \in X \setminus \{x\}$, there exists an open set U such that $x \notin U$ and $y \in U$. Hence $y \in U \subset X \setminus \{x\}$. Thus $X \setminus \{x\}$ is open, which means $\{x\}$ closed.

Assume all points are closed. Let $x, y \in X$ different. Then $U_x = X \setminus \{y\}$ satisfies $U_x \in \tau$, $x \in U_x$ and $y \notin U_x$. □

Lemma 5.1.4. *Any topological space homeomorphic to a Fréchet space is Fréchet.*

Proof. Any two different points x, y are mapped to different points $f(x), f(y)$ by a homeomorphism, and U_x is mapped to an open set by any open map. □

Proposition 5.1.5.

(i) A subspace of a Fréchet space is Fréchet.

(ii) The product of Fréchet spaces is Fréchet.

Proof. Exercise 113. Slight modification of Proposition 5.2.4 □

Remarks 5.1.6.

1. Not every topological space is T_1. For instance, if X has more than one point and τ is the coarse topology, (X, τ) is not Fréchet because its points are not closed: $x \in X$ has a non-open, non-closed complement.

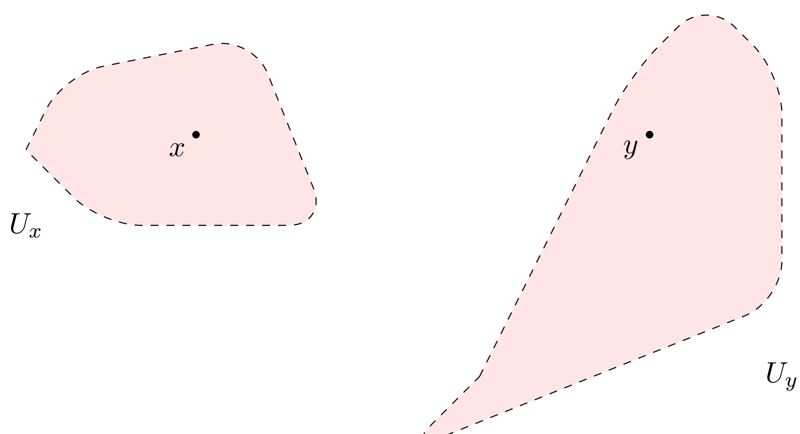
2. The image by a continuous map of a Fréchet space is, in general, not Fréchet. For instance if $\#X > 1$ and τ_d, τ_c are the discrete and coarse topologies on X, function

$$\mathrm{id} : (X, \tau_d) \to (X, \tau_c), \qquad x \mapsto x,$$

is continuous and surjective, yet the first space is Fréchet and the second one is not.

5.2 Hausdorff spaces

Definition 5.2.1. *A topological space (X, τ) is* ***Hausdorff, separated*** *or* T_2*if given different $x, y \in X$ there exist disjoint $U_x, U_y \in \tau$ such that $x \in U_x$ and $y \in U_y$.*



Lemma 5.2.2. *(i) Every Hausdorff space is Fréchet.*

(ii) Every space homeomorphic to a Hausdorff space is Hausdorff.

Examples 5.2.3.

1. Every metric space (X, d) is Hausdorff. Given $x, y \in X$ different points, define $r = \frac{1}{2} d(x, y)$. Then $x \in B(x, r)$, $y \in B(y, r)$ and $B(x, r) \cap B(y, r) = \varnothing$.

2. $\mathbb{R}$ with the finite complement topology is not Hausdorff. Exercise

Proposition 5.2.4.

(i) A subspace of a Hausdorff space is Hausdorff.

(ii) The product of Hausdorff spaces is Hausdorff.

Proof. (i) Let $A \subset X$. Given $x, y \in A$ different, let $U_x, U_y \in \tau_X$ disjoint open neighbourhoods separating x and y in X. Then $V_x = A \cap U_x$ and $V_y = A \cap U_y$ are disjoint, open in τ_A and contain x and y respectively.

(ii) Assume X and Y are Hausdorff. Let us prove $X \times Y$ is Hausdorff. If $(x_1, y_1), (x_2, y_2)$ are different, then one of the coordinates is different. Assume, w.l.o.g. that $x_1 \neq x_2$. Let U_1, U_2 be two open neighbourhoods separating x_1 and x_2. Then $V_1 = U_1 \times Y$ and $V_2 = U_2 \times Y$ are open sets in $X \times Y$, they are disjoint and $(x_i, y_i) \in V_i$ for $i = 1, 2$.

□

Proposition 5.2.5. *Let (X, τ) be a topological space and $\Delta = \{(x, y) \in X \times X : x = y\}$ its diagonal. Then X is Hausdorff if and only if Δ is closed in $X \times X$.*

Proof. Δ is closed if and only if every $(x, y) \in X \times X \setminus \Delta$ belongs to $\overset{\circ}{\overbrace{X \times X \setminus \Delta}}$. $\beta = \{U \times V : U, V \in \tau\}$ is a basis for the product topology on $X \times X$, hence (x, y) is interior to $X \times X \setminus \Delta$ if and only if there exist $U, V \in \tau$ such that $(x, y) \in U \times V \subset X \times X \setminus \Delta$. This is the same as calling X Hausdorff because $(U \times V) \cap \Delta = \{(x, x) : x \in U \cap V\}$. □

5.3 Normal spaces

Definition 5.3.1. *Let (X, τ) be a topological space. We say X is* ***regular*** *or* T_3*if for every point $x \in X$ and every closed set $C \subset X$ not containing x, there exist open sets $U_x, U_C \in \tau$ such that $x \in U_x$, $C \subset U_C$ and $U_x \cap U_C = \varnothing$.*

Definition 5.3.2. *A topological space (X, τ) is called* ***normal*** *or* T_4*if for any pair of disjoint closed sets $C_1, C_2 \subset X$, there exist disjoint open sets U_1, U_2 such that $U_i \subset C_i$ $i = 1, 2$.*

Lemma 5.3.3. *Every normal and Fréchet space is regular.*

Proposition 5.3.4. *Every metric space is normal and regular.*

Proof. Every metric space is Fréchet, hence it suffices to prove it normal. Let C_1, C_2 be closed subsets of X such that $C_1 \cap C_2 = \varnothing$. Let $x \in C_1$ and $r_x = \inf\{d(x, y) : y \in C_2\} \in [0, \infty)$. If $r_x = 0$, then for every $\varepsilon > 0$ there is $y \in C_2$ such that $d(x, y) < \varepsilon$, i.e. $C_2 \cap B(x, \varepsilon) \neq \varnothing$. This would mean $x \in \overline{C_2} = C_2$, absurd.

Thus $r_x > 0$. Let $U_1 = \bigcup_{x \in C_1} B\left(x, \frac{r_x}{2}\right)$. This is an open set containing C_1. Similarly $U_2 = \bigcup_{y \in C_2} B\left(y, \frac{r_y}{2}\right)$ is an open set containing C_2.

Let us prove $C_1 \cap C_2 = \varnothing$. Otherwise if there existed $z \in C_1 \cap C_2$, there would be $x_1 \in C_1$ and $x_2 \in C_2$ such that $z \in B\left(x_1, \frac{r_{x_1}}{2}\right) \cap B\left(x_2, \frac{r_{x_2}}{2}\right)$. By definition $d(x_1, x_2) \geq r_{x_1}$ and $d(x_1, x_2) \geq r_{x_2}$. Thus

$$\frac{r_{x_1}}{2} + \frac{r_{x_2}}{2} \leq d(x_1, x_2) \leq d(x_1, z) + d(z, x_2) < \frac{r_{x_1}}{2} + \frac{r_{x_2}}{2},$$

contradiction. □

Remark 5.3.5. A topological space with the coarse topology is normal and yet, if it has more than one point, is not Hausdorff. However if it is Fréchet as well as normal, then it is Hausdorff.

Exercise: A topological space homeomorphic to a normal topological space is normal. These spaces are interesting in their having "lots of" continuous functions, see the following Proposition.

Proposition 5.3.6 (Urysohn's Lemma, [1, pp. 207-211])**.** *Let (X, τ) be a topological space. Then X is normal if, and only if, for every pair C_1, C_2 of disjoint closed subsets of X there exists a continuous function $f : X \to [0, 1]$ such that $C_1 = f^{-1}(\{0\})$ and $C_2 = f^{-1}(\{1\})$.*

5.4 Solved Exercises

111. Prove the claim in Remark 5.3.5.

112. Let X be a set. Define the following topology:

$$\tau_X^c := \{S \subset X : X \setminus S \text{ is finite or countable}\} \cup \{\varnothing\} \subset \mathcal{P}(X).$$

(i) (15 Marks) A topological space (X, τ) is called *Hausdorff* if, given different $x, y \in X$, there exist open sets $U, V \in \tau$ such that $x \in U, y \in V$ and $U \cap V = \varnothing$. Prove (X, τ_X^c) is not Hausdorff whenever X is uncountably infinite.

(ii) (10 Marks) Find the closure and interior of intervals $[a, b], (a, b) \subset \mathbb{R}$ in $\tau_{\mathbb{R}}^c$ if $a < b$.

(iii) (Bonus 10 Marks) Is there a homeomorphism $\left(\mathbb{R}, \tau_{\mathbb{R}}^{\text{Eucl}}\right) \xrightarrow{\cong} (\mathbb{R}, \tau_{\mathbb{R}}^c)$? Justify your answer using one of the previous items.

Solution: handwritten at the end of this chapter

5.5 Exercises

113. Prove Proposition 5.1.5.

② $\tau_{\mathbb{X}}^{c} = \{ S \subset \mathbb{X} : \mathbb{X} \setminus S \text{ finite or countable} \} \cup \{\emptyset\}$

Exercise 112

(i) Let $x, y \in \mathbb{X}$ such that $x \neq y$.

Assume $\begin{cases} U \text{ open neighborhood of } x \\ V \text{ open neighborhood of } y \end{cases}$

Then $\begin{cases} U \text{ open} \Rightarrow \mathbb{X} \setminus U \text{ finite or countable} \\ V \text{ open} \Rightarrow \mathbb{X} \setminus V \quad \text{"} \quad \text{"} \quad \text{"} \end{cases}$

However: $\mathbb{X}$ uncountable $\Rightarrow \mathbb{X} \setminus \big((\mathbb{X} \setminus U) \cup (\mathbb{X} \setminus V) \big) \neq \emptyset$

$\Rightarrow \underline{U \cap V \neq \emptyset}$

$\Rightarrow$ the Hausdorff property can never hold.

(ii) $\overset{\circ}{\overbrace{[a,b]}}$ must be an open set contained in $[a,b]$. Thus

$$\underbrace{\mathbb{R} \setminus [a,b]}_{\text{uncountable}} \subset \mathbb{R} \setminus \overset{\circ}{\overbrace{[a,b]}} = \begin{cases} \boxed{\mathbb{R}} \\ \text{finite} \quad (\text{IMPOSSIBLE}) \\ \text{countable} \quad (\text{IMPOSSIBLE}) \end{cases}$$

$$\Rightarrow \mathbb{R} \setminus \overset{\circ}{\overbrace{[a,b]}} = \mathbb{R} \Rightarrow \underline{\overset{\circ}{\overbrace{[a,b]}} = \emptyset}$$

The same reasoning implies $\overset{\circ}{\overbrace{(a,b)}} = \emptyset$

$\overline{[a,b]}$ must be a closed set containing $[a,b]$.
Thus $\mathbb{R}\setminus\overline{[a,b]} \in \tau^c_{\mathbb{R}}$

$\Rightarrow \overline{[a,b]}$ is $\left\{\begin{array}{ll} \mathbb{R} & ✓ \\ \text{finite} & \times \\ \text{countable} & \times \end{array}\right\}$ containing $\underbrace{[a,b]}_{\text{uncountable}}$

$\Rightarrow \overline{[a,b]}^{\tau^c} = \mathbb{R}$.

Same argument mutatis mutandis $\overline{(a,b)}^{\tau^c} = \mathbb{R}$.

(viii) Easy method:

$\left.\begin{array}{l} (\mathbb{R}, \tau_{Eucl}) \text{ Hausdorff} \\ (\mathbb{R}, \tau^c_{\mathbb{R}}) \text{ not Hausdorff} \end{array}\right\} \Rightarrow \begin{array}{c} (\mathbb{R}, \tau_{Eucl}) \\ \not\cong \\ (\mathbb{R}, \tau^c_{\mathbb{R}}). \end{array}$

Indeed, in general, if $(X, \tau_1) \cong (Y, \tau_2)$ then one is Hausdorff $\Leftrightarrow$ the other is

Proof: Assume $\exists f : (X, \tau_1) \xrightarrow{\cong} (Y, \tau_2)$

* Given $y_1 \neq y_2 \in Y$, $\exists \begin{array}{l} f^{-1}(y_1) = x_1 \\ f^{-1}(y_2) = x_2 \end{array} \Big|\ x_1 \neq x_2$.

 And $\forall x_1 \neq x_2 \in X$, $f(x_1) \neq f(x_2)$.

* If U_1, U_2 are disjoint open neighbourhoods of y_1, y_2 (resp. x_1, x_2) then $f^{-1}(U_1), f^{-1}(U_2)$ (resp. $f(U_1), f(U_2)$) are open neighbourhoods of x_1, x_2 (resp. y_1, y_2) on account of f being continuous (resp. open) and they are disjoint because f is a function (resp. because it is an injective function).

We still need to prove $(\mathbb{R}, \tau_{Eucl})$ is Hausdorff.
But this is true for all metric spaces (X, d):

$$\forall x, y \in X, \quad x \neq y \implies d(x,y) = \delta > 0$$

And we can prove $B_d(x, \frac{\delta}{2}) \cap B_d(y, \frac{\delta}{2}) = \emptyset$;

any $z \in B_d(x, \frac{\delta}{2}) + B_d(y, \frac{\delta}{2})$ would fulfil

$$\delta = d(x,y) \leq d(x,z) + d(z,y) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

Absurd.

Chapter 6

Compact spaces

Working on an attempt to generalise the thesis of the well-known Heine-Borel theorem (which will prove at the end of the Chapter) let us define the following properties.

6.1 Quasi-compact spaces

Definition 6.1.1. *Let (X, τ) be a topological space. We say X is **quasi-compact** if for every open covering $\{U_i\}_{i \in I}$ there exists a* finite *index subset $J \subset I$ such that $\{U_i\}_{i \in J}$ is still a covering of X.*

Examples 6.1.2.

1. The well-known *Heine-Borel Theorem* stating that every open covering of a closed and bounded subset of $\mathbb{R}$ admits a finite subcovering is, within this context, equivalent to stating that every such subset (hence any interval $[a, b]$) is quasi-compact.

2. Every finite topological space X is compact. Indeed, the range of possible open subsets of X is finite as well, hence so is any open covering of X.

3. An infinite set X with the discrete topology $\tau = \mathcal{P}(X)$ is not quasi-compact. Indeed, there is at least one open covering, namely $\{\{x\}\}_{x \in X}$, does not admit a finite subcovering. Take any subset out of this covering and it ceases to be one.

4. An open interval $(a, b) \subset \mathbb{R}$ is not quasi-compact. Indeed, $(a, b) = \bigcup_{n \geq 1} U_n$ with e.g. $U_n := \left(a + \frac{1}{n}, b\right)$ and (a, b) cannot be expressed as a finite union of sets of the form U_n.

Taking complementaries we get the following equivalent:

Definition 6.1.3. *X is quasi-compact if for every family $\{C_i\}_{i \in I}$ of closed sets of X such that $\bigcap_{i \in I} C_i = \varnothing$, there exists finite $J \subset I$ such that $\bigcap_{i \in J} C_i = \varnothing$.*

6.2 Properties of quasi-compact spaces

Proposition 6.2.1. *a) Given a quasi-compact topological space and a closed subset $C \subset X$, then C is also quasi-compact.*

b) Let $f : X \to Y$ be a surjective, continuous function between topological spaces. If X is quasi-compact, then so is Y.

c) Quasi-compactness is homeomorphism invariant: if $X \cong Y$, then X is quasi-compact if, and only if, Y is.

Proof. a) Let $\{U_i\}_{i\in I}$ be an open covering of C. In accordance with the definition of subspace topology, $U_i = V_i \cap C$ for some open set V_i of X for every $i \in I$. Hence $\{V_i\}_{i\in I} \cup (X \setminus C)$ is an open covering of X. Quasi-compactness of X entails the existence of a finite subcovering $\{V_j\}_{j\in J} \cup (X \setminus C)$ for some $J \subset I$. Hence $C \subset \bigcup_{i\in J} V_j$, hence $C \subset \bigcup_{i\in J} U_j$.

b) Let $\{U_i\}_{i\in I}$ be an open covering of Y. Then $\left\{f^{-1}(U_i)\right\}_{i\in I}$ is an open covering of X, hence there exists finite $J \subset I$ such that $X = \bigcup_{j\in J} f^{-1}(U_j)$. The fact f is surjective implies $Y = \bigcup_{j\in J} U_j$.

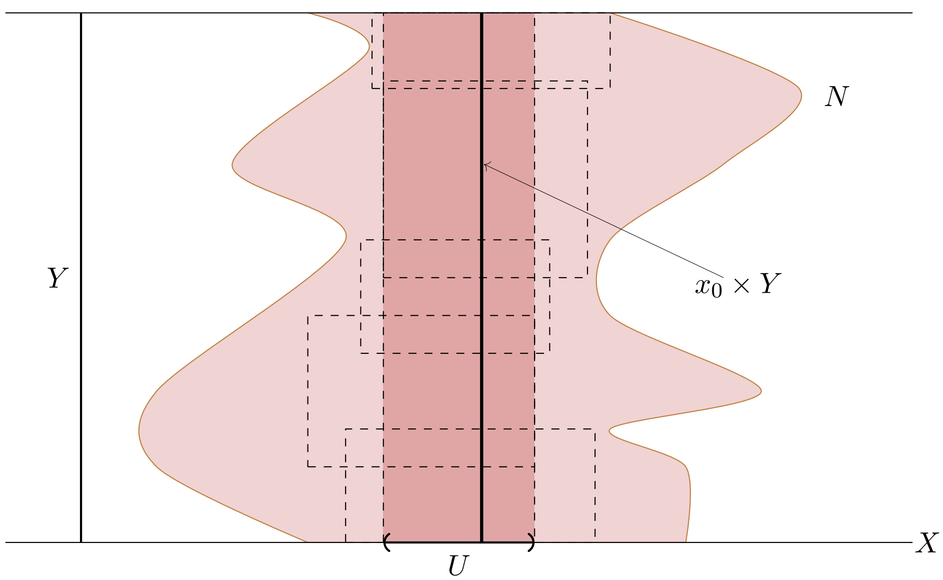
c) Immediate from the previous item.

□

We need an ancillary result to address products of quasi-compact spaces.

Lemma 6.2.2 (Tube Lemma)**.** *Let X be a topological space and Y a quasi-compact topological space. For every $x_0 \in X$ and every neighbourhood $N \subset X \times Y$ of $\{x_0\} \times Y$, there exists an open neighbourhood U of x_0 in X such that $U \times Y \subset E$.*

Proof. Open sets of the form $U \times V$ are a basis for $X \times Y$. Hence for every $y \in Y$ there exist open sets $U_y \in X$ and $V_y \in Y$ such that $(x_0, y) \in U_y \times V_y \subset N$ which means $\{V_y\}_{y\in Y}$ are an open covering of Y. Quasi-compactness implies the existence of $y_1, \ldots, y_n \in Y$ such that $V_{y_1} \cup \cdots \cup V_{y_n} = Y$. Let $U = U_{y_1} \cap \cdots \cap U_{y_n}$ which is an open neighbourhood of x_0. Then for every $(x, y) \in U \times U \times Y$ there exists $i \in \{1, \ldots, n\}$ such that $(x, y) \in U \times V_i \subset U_i \times V_i \subset N$. Hence $U \times V \subset N$.



□

Remark 6.2.3. If Y is not compact the result is false. For instance let $X = Y = \mathbb{R}$ and $x_0 = 0$. Let N be $\mathbb{R}^2 \setminus \{xy = 1\}$ be the complementary of a hyperbola. There exists no open neighbourhood U of 0 in $\mathbb{R}$ such as the one described in the Lemma.

Corollary 6.2.4. *In the hypotheses of the Tube Lemma, projection* $\mathrm{pr}_X : X \times Y \to X$ *is closed.*

Proof. Let C be a closed set of $X \times Y$ and $x_0 \in X \setminus \mathrm{pr}_X(C)$. $X \times Y \setminus C$ is an open set containing $\{x_0\} \times Y$ and, applying the Tube Lemma, there exists an open set U such that $x_0 \in U$ and $U \times Y \subset X \times Y \setminus C$. Projecting on X, $U \subset X \setminus \mathrm{pr}_X(C)$. □

Theorem 6.2.5. *Let $X_1, \dots, X_n$ be topological spaces. The product $X_1 \times \cdots \times X_n$ is quasi-compact if, and only if, each of the spaces $X_1, \dots, X_n$ is.*

Proof. If $X_1 \times \cdots \times X_n$ is quasi-compact, every factor is by applying Proposition 6.2.1.

Let us prove the other implication by induction. We only need to deal with $n = 2$. Let $\{U_i\}_{i\in I}$ be an open covering of $X \times Y$ and $x_0 \in X$. $\{U_i\}_{i\in I}$ is also a covering for $\{x_0\} \times Y$, hence there exist $i_1, \dots, i_n \in I$ such that $\{x_0\} \times Y \subset U_{i_1} \cup \cdots \cup U_{i_n}$. Applying the Tube Lemma to $N = U_{i_1} \cap \cdots \cap U_{i_n}$, for every point $x_0 \in X$ there exists an open neighbourhood W_{x_0} of x_0 in X such that $W_{x_0} \times Y$ is covered by finitely many elements in $\{U_i\}_{i\in I}$. Since $\{W_{x_0}\}_{x_0\in X}$ is an open covering of the quasi-compact space X, then $X = W_{x_1} \cup \cdots \cup W_{x_m}$ for some $x_1, \dots, x_m \in X$. Taking union of the open sets which, for every $i = 1, \dots, m$, yield a finite covering of $W_{x_0} \times Y$ by elements of $\{U_i\}_{i\in I}$, we obtain a finite subcovering. □

Examples 6.2.6.

1. $[a, b] \subset \mathbb{R}$ is quasi-compact.

2. $[a, b]^n = [a, b] \times \cdots \times [a, b] \subset \mathbb{R}^n$ is quasi-compact.

3. Any closed and *bounded* subset X of $\mathbb{R}^n$, i.e. such that there exists a real number $r > 0$ such that $X \subset B((0, \dots, 0); r)$, is quasi-compact. Indeed, if $C \subset \mathbb{R}^n$ is bounded, there exists an interval $I := [-r, r]$ such that $C \subset [-r, r]^n$. Hence C is a closed subset of a quasi-compact set, which renders it quasi-compact.

6.3 Compact spaces

Definition 6.3.1. *Let (X, τ) be a topological space. We say X is **compact** if it is quasi-compact and Hausdorff.*

We recall from Proposition 6.2.1 that the closed subset of a quasi-compact space is quasi-compact itself. The converse is not true in general, e.g. if $X = \{a, b, c\}$ and $\tau = \{X, \varnothing\}$ is the coarse topology, then X is a quasi-compact space and so is $\{a, b\} \subset X$, however the latter is not closed. Let us prove, however, that the converse does hold with the additional separation property.

Lemma 6.3.2. *Let X be a Hausdorff space and $K \subset X$ a compact subspace. For every $x \in X\backslash K$ there exist disjoint open subsets U and V such that $x \in U$ and $K \subset V$*

Proof. X is Hausdorff, hence for every $y \in K$ there exist disjoint open sets U_y, V_y such that $y \in U_y$ and $x \in V_y$. Therefore $K \subset \bigcup_{y\in K} U_y$. Compactness of K implies there exist $y_1, \dots, y_n$ such that $K \subset U_{y_1} \cup \cdots \cup U_{y_n}$. Define $U = U_{y_1} \cup \cdots \cup U_{y_n}$ and $V = V_{y_1} \cap \cdots \cap V_{y_n}$. Then $K \subset U$ and $x \in V$, and the assumption $z \in U \cap V$ would entail the existence of an $i = 1, \dots, n$ such that $z \in U_{y_j} \cap V_{y_j} = \varnothing$, absurd. □

Proposition 6.3.3. *Every compact subset of a Hausdorff space is closed.*

Proof. Due to the Lemma, if K is a compact subspace of a Hausdorff space then every element $x \in X \setminus K$ belongs to the interior of $X \setminus K$, which means $X \setminus K$ is open, hence K is closed. □

Corollary 6.3.4. *$X \subset \mathbb{R}^n$ with the Euclidean subspace topology is compact if, and only if, X is closed and bounded.*

Proof. We have already seen that if X is closed and bounded then it is also quasi-compact, and the fact $\mathbb{R}^n$ (a metric space) is Hausdorff implies X is Hausdorff.

Conversely, if X is compact then it is closed in virtue of Proposition 6.3.3. It is also bounded; otherwise $\{B(\mathbf{0}, n) \cap X\}_{n\in\mathbb{N}}$ would be an open covering without a finite subcovering, absurd. □

Examples 6.3.5.

1. The sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, as well as the torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ and the projective space $\mathbb{P}^n$ are compact spaces.

2. $\mathbb{R}^n$, unbounded curves and hypersurfaces such as $\{z = 0\}$, parabolas or hyperbolas, etc are not compact because they are not bounded (even though they are closed).

Corollary 6.3.6. *Let $f : X \to Y$ be a continuous function between two topological spaces and assume X is quasi-compact and Y is Hausdorff. Then f is closed.*

Proof. Let C be a closed set of X. X is quasi-compact, hence C and $f(C)$ are closed. Y is Hausdorff, hence Proposition 6.3.3 implies $f(C)$ is closed. □

As a consequence of the above results, we obtain the result needed in Example 4.1.12 for the torus $\mathbb{T}^2$, among other examples:

Corollary 6.3.7. *Let $g : X \to Y$ be a continuous function between two subspaces of $\mathbb{R}^n$ and assume X is closed and bounded; then g is closed.*

Proof. Let C be a closed set of X. X is quasi-compact □

It is clear that compactness is invariant by homeomorphisms. This can be used as a tool to prove two spaces are not homeomorphic based, e.g. on the condition in Corollary 6.3.4:

Examples 6.3.8.

1. An ellipse and a parabola are not homeomorphic – one is bounded, the other is not.

2. The open cylinder and a sphere are not homeomorphic.

3. (a, b) and $[a, b]$ are not homeomorphic.

Proposition 6.3.9. *Every compact topological space is normal.*

Proof. Let C_1, C_2 be two closed subsets of a compact space X. The fact C_1 is compact implies, for every $x \in F_2$, due to Lemma 6.3.2, the existence of disjoint open sets U_x, V_x such that $C_1 \subset U_x$ and $x \in V_x$. Then $C_2 \subset \bigcup_{x\in C_2} V_x$ and, due to the compactness of C_1, there exist $x_1, \ldots, x_n$ such that $C_2 \subset V_{x_1} \cup \cdots \cup V_{x_n}$. Define $V = V_{x_1} \cup \cdots \cup V_{x_n}$ and $U = U_{x_1} \cap \cdots \cap U_{x_n}$. We have two open sets U, V such that $C_1 \subset U$ and $C_2 \subset V$. □

6.4 Metric compact spaces

Let (X, d) be a metric space throughout this Section. Let us introduce the notion of sequences and compare it to compactness.

Definition 6.4.1. *(X, d) is **sequentially compact** if for every sequence $\{x_n\}_{n\geq 1}$ of elements of X there exists a partial sequence $\{x_{n_k}\}_{k\geq 1}$ which is convergent.*

Our aim is to prove that sequential compactness and compactness are equivalent on metric spaces.

Definition 6.4.2. *Let $S \subset X$. The* ***diameter*** *of S is the element of* $\mathbb{R} \cup \{\infty\}$ $\operatorname{diam}(S) = \sup_{x,y\in S}\{d(x,y)\}$. *Alternatively, we can define it as* $\sup d(S \times S)$.

Examples 6.4.3.

1. $\operatorname{diam}(\mathbb{R}) = \infty$.

2. $\operatorname{diam}([a,b]) = \operatorname{diam}((a,b)) = b - a$.

3. The diameter of any ball of radius r is smaller than or equal to $2r$.

4. If S is compact, the fact d is continuous implies $d(S \times S)$ is a compact set of $\mathbb{R}$, hence $\operatorname{diam}(S) < \infty$.

Definition 6.4.4. *Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. We call $\lambda \in (0,\infty)$ a* ***Lebesgue number*** *of $\mathcal{U}$ if for every subset $S \subset X$ with* $\operatorname{diam}(S) < \lambda$ *there exists $i \in I$ such that $S \subset U_i$.*

Proposition 6.4.5. *If X is sequentially compact, every open covering of X has a Lebesgue number.*

Proof. Assume there exists no Lebesgue number for $\{U_i\}_{i\in I}$. This implies for every $n \in \mathbb{N}$ there will exist $S \subset X$ such that $\operatorname{diam}(S) < \frac{1}{n}$ and S is not contained in any U_i. If $x_n \in S$, then $S \subset B\left(x_n, \frac{1}{n}\right)$ and thus no open set U_i contains $B\left(x_n, \frac{1}{n}\right)$.

Let x be the limit of a partial convergent sequence of $\{x_n\}_{n\in\mathbb{N}}$. Let $j \in I$ such that $x \in U_j$. If p is large enough $B\left(x, \frac{1}{p}\right) \subset U_j$. An element x_{n_k} of the convergent partial sequence such that $n_k > 2p$ and $d(x_{n_k}, x) < \frac{1}{2^p}$ will satisfy, for every $y \in B\left(x_{n_k}, \frac{1}{n_k}\right)$,

$$d(y,x) \leq d(y, x_{n_k}) + d(x_{n_k}, x) \leq \frac{1}{n_k} + \frac{1}{2p} < \frac{1}{2p} + \frac{1}{2p} = \frac{1}{p},$$

hence $B\left(x_{n_k}, \frac{1}{n_k}\right) \subset B\left(x, \frac{1}{p}\right) \subset U_j$, absurd. □

Theorem 6.4.6. *Let (X,d) be a metric space. The following are equivalent:*

(i) X is compact.

(ii) X is sequentially compact.

Proof. Assume X is compact. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $C_n = \{x_n, x_{n+1}, \ldots\}$. Let us first prove $\bigcap_n \overline{C_n} \neq \varnothing$. Assume $\bigcap_n \overline{C_n} = \varnothing$. Then compactness implies $\overline{C_{n_1}} \cap \cdots \cap \overline{C_{n_k}} = \varnothing$. But if $m = \max\{n_1, \ldots, n_k\}$ then $\varnothing \neq C_m \subset C_{n_1} \cap \cdots \cap C_{n_k} \subset \overline{C_{n_1}} \cap \cdots \cap \overline{C_{n_k}}$, absurd.

Thus let $x \in \bigcap_n \overline{C_n}$. $x \in \overline{C_m}$ entails $B\left(x, \frac{1}{n}\right) \cap C_m \neq \varnothing$. Let x_{n_m} be any point of this intersection and consider $\{x_{n_m}\}_m$. For every m, $d(x_{n_m}, x) < \frac{1}{m}$. Thus $\lim_{m\to\infty} x_{n_m} = x$.

Let us prove the other implication. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. Proposition 6.4.5 implies the existence of Lebesgue number λ for $\mathcal{U}$. Let $\lambda_0 \in (0, \lambda/2)$. For every $x \in X$, there exists $j \in I$ such that $B(x, \lambda_0) \subset U_j$. Let $x_1 \in X$ and U_1 an open set of the covering such that $B(x_1, \lambda_0) \subset U_1$. Let $x_2 \in X \setminus U_1$ and choose U_2 similarly: $B(x_2, \lambda_0) \subset U_2$. If $\mathcal{U}$ does not admit a finite subcovering then we can repeat this process indefinitely. The resulting sequence $\{x_n\}_{n\in\mathbb{N}}$ is such that for every n, $x_m \in B(x_n, \lambda_0)$ iff $m = n$.

Hence for every $n \neq m$ $d(x_n, x_m) \geq \lambda_0$. A sequence with this property can never have a convergent partial sequence. Absurd. □

Definition 6.4.7. *A function $f : X \to Y$ between metric spaces is* ***uniformly continuous*** *if for every $\varepsilon > 0$ there exists $\delta > 0$ (note the fact that δ does not depend on x) such that for every $x \in X$, $f(B(x,\delta)) \subset B(f(x), \varepsilon)$.*

Proposition 6.4.8. *Let X be a compact metric space and $f : X \to Y$ a continuous function between metric spaces. Then it is uniformly continuous.*

Proof. Consider the open covering $\left\{f^{-1}\left(B\left(y, \varepsilon/2\right)\right)\right\}_{y \in Y}$ of X. The fact X is compact implies there exists a Lebesgue number λ for this covering. Hence for every $x \in X$ there exists $y \in Y$ such that $f\left(B\left(x, \lambda/2\right)\right) \subset B\left(y, \varepsilon/2\right)$. Thus $f\left(x\right) \in B\left(y, \varepsilon/2\right)$ and the triangle inequality implies $B\left(y, \varepsilon/2\right) \subset B\left(f\left(x\right), \varepsilon\right)$. Define $\delta = \lambda/2$ and $f\left(B\left(x, \delta\right)\right) \subset B\left(y, \varepsilon/2\right) \subset B\left(f\left(x\right), \varepsilon\right)$. □

6.5 The Heine-Borel Theorem

As an appendix let us prove the following:

Theorem 6.5.1 (Heine-Borel)**.** *Every closed and bounded subset of $\mathbb{R}$ is quasi-compact.*

Proof. Let $X \subset \mathbb{R}$ be a closed and bounded set. There exist a, b such that $a < b$ and $X \subset [a, b]$. If we prove $[a, b]$ is quasi-compact, the fact X is closed in a quasi-compact space implies its quasi-compactness. Let $\mathcal{U}$ be an open covering of $[a, b]$. Define

$$\mathcal{C} = \{x \in [a, b] : [a, x] \text{ can be covered by finitely many open sets in } \mathcal{U}\}.$$

$\mathcal{C}$ is not empty: given any $U \in \mathcal{U}$ such that $a \in U$, there exists $x' > a$ such that $[a, x') \subset U$ because closed-open intervals are a basis of the Euclidean topology induced on $[a, b]$. Thus for every $a < x < x'$ there exists $[a, x] \subset U$ and thus $x \in \mathcal{C}$. Let $c = \sup \mathcal{C}$. Let us prove $c \in \mathcal{C}$ and $c = b$.

Let us prove $c \in \mathcal{C}$, i.e. $[a, c]$ can be recovered by finitely many elements in $\mathcal{U}$. Let $V \in \mathcal{U}$ such that $c \in V$. The fact $c > a$ implies there exists $a < c' < c$ such that $(c', c] \subset V$. $c = \sup \mathcal{C}$ thus there exists $x \in \mathcal{C} \cap (c', c]$. Thus $[a, c] = [a, x] \cup [x, c]$. By definition of $\mathcal{C}$, $[a, x]$ can be covered by finitely many elements of $\mathcal{U}$ and $[x, c] \subset V \in \mathcal{U}$.

Let us prove $c = b$. If $c < b$ taking $V \in \mathcal{U}$ as before we have d such that $c < d \leq b$ and $[c, d] \subset V$. $[a, d] = [a, c] \cup [c, d]$ and $c \in \mathcal{C}$, hence $d \in \mathcal{C}$ contradicting $c = \sup \mathcal{C}$. □

Chapter 7

Connectedness properties

Let us study those spaces which can be defined as being "in one piece". This leads to two different definitions: not being able to split as a union of disjoint open subsets, or being such that any two points in them can be connected by a continuous curve. Both definitions are not equivalent but they are still intimately related.

7.1 Path-connected spaces

Definition 7.1.1. *Let (X, τ) be a topological space. A* ***path*** *in X is a continuous map $\alpha : I \to X$, where $I = [0, 1]$. We say $\alpha(0)$ is the initial point of α and $\alpha(1)$ is the terminal point. If $\alpha(0) = \alpha(1)$ we say the path is a* ***loop****.*

Definition 7.1.2. *A topological space X is called* ***path-connected*** *if for every $x, y \in X$ there exists a path in X having initial point x and terminal point y.*

Examples 7.1.3.

1. $\mathbb{R}^n$ is path-connected: for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, function $\alpha : I \to \mathbb{R}^n$ defined by

$$t \mapsto \alpha(t) = \boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x}),$$

   is continuous and $\alpha(0) = \boldsymbol{x}$, $\alpha(1) = \boldsymbol{y}$.

2. Bolzano's Theorem tells us every path $\alpha : I \to \mathbb{R}$ such that $\alpha(0) < 0$ and $\alpha(1) > 0$ must necessarily equal zero at least one. This is the same as saying $\mathbb{R} \setminus \{0\}$ is not path-connected.

3. However, $\mathbb{R}^n \setminus \{\mathbf{0}\}$, $n \geq 2$, is path-connected. Exercise: find a path connecting any two points in such spaces.

4. $\mathbb{S}^n$ is path-connected: if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^n$, choose a point $\boldsymbol{z} \neq \boldsymbol{x}, \boldsymbol{y}$ and perform stereographic projection (which as you know is a homeomorphism of $\mathbb{S}^n \setminus \boldsymbol{z} \cong \mathbb{R}^n$), hence $\mathbb{S}^n \setminus \{\boldsymbol{z}\}$ is path-connected and there exists a path in $\mathbb{S}^n \setminus \{\boldsymbol{z}\} \subset \mathbb{S}^n$ having initial point $\boldsymbol{x}$ and terminal point $\boldsymbol{y}$.

7.2 Properties of path-connected spaces

Proposition 7.2.1. *Let $f : X \to Y$ be a continuous, surjective function. If X is path-connected, so is Y. Hence path-connectedness is invariant by homeomorphisms.*

Proof. Let $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ such that $y_i = f(x_i)$. The fact X is path-connected implies the existence of a path $\alpha : I \to X$ having initial point in x_1 and terminal point in x_2. Hence $f \circ \alpha$ is a path in Y having initial point y_1 and terminal point y_2. $\square$

Corollary 7.2.2. *$\mathbb{R}$ and $\mathbb{R}^n$, $n \geq 2$, are not homeomorphic.*

Proof. If there existed a homeomorphism $f : \mathbb{R} \to \mathbb{R}^n$, then the restriction of this homeomorphism would still be a homeomorphism between $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^n \setminus \{f(0)\}$, but the first space is not path-connected whereas the second is. $\square$

Proposition 7.2.3. *Let X, Y be two topological spaces. The product $X \times Y$ is path-connected if, and only if, X and Y are.*

Proof. if $X \times Y$ is path-connected, the previous Proposition applied to the projections implies X and Y are path-connected.

Conversely if X and Y are path-connected, let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Let α_X, α_Y be respective paths in X, Y with initial points x_1, y_1 and final points x_2, y_2. Then

$$\alpha : I \to X \times Y, \qquad t \mapsto (\alpha_X(t), \alpha_Y(t))$$

is a path with initial point (x_1, y_1) and terminal point (x_2, y_2). $\square$

Examples 7.2.4.

1. Let $X = \mathbb{R}^2 \setminus \{y = 0\}$. We have $X = \mathbb{R} \times (\mathbb{R} \setminus 0)$, hence X is not path-connected.

2. $\mathbb{R}$ is path-connected, hence so is $\mathbb{R}^n$.

7.3 Path-connected components

Let (X, τ) be a topological space. We say two points $x, y \in X$ are **path-connected** if there exists a path α with initial point x and terminal point y.

Proposition 7.3.1. *The relation $x \sim y$ if x, y are path-connected is a relation of equivalence.*

Proof. It is reflexive due to the fact that constant functions are continuous. If $\alpha : I \to X$ is such that $\alpha(0) = x$ and $\alpha(1) = y$, then $\overline{\alpha} : I \to I \to X$ defined $\overline{\alpha}(t) = \alpha(1 - t)$ is continuous, $\overline{\alpha}(0) = y$ and $\overline{\alpha}(1) = x$. Hence we have symmetry.

Finally it is transitive because given x, y, z such that $x \sim y$ and $y \sim z$ that means there exist $\alpha, \beta : I \to X$ such that $\alpha(0) = x, \alpha(1) = y = \beta(0), \beta(1) = z$. Thus we can define the product of paths as follows:

$$I \xrightarrow{\alpha \cdot \beta} X$$

$$t \longmapsto \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}, \\ \beta(2t - 1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

The fact $\alpha\left(2 \cdot \frac{1}{2}\right) = \alpha(1) = y = \beta(0) = \beta\left(2\frac{1}{2} - 1\right)$ implies $\alpha \cdot \beta$ is continuous and $(\alpha \cdot \beta)(0) = \alpha(0) = x$, $(\alpha \cdot \beta)(1) = y$. $\square$

Definition 7.3.2. *Given $x \in X$ we call $P_x := \{y \in X : x \sim y\}$ the* ***path-connected component of*** *x.*

Proposition 7.3.3. *Let X be a topological space and $x \in X$. The path-connected component of x is the largest path-connected subspace of X containing x.*

Proof. Let us first prove P_x is path-connected. If $y \in P_x$, there exists a path $\alpha : I \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. We need to prove $\alpha(I) \subset P_x$. Indeed if $z = \alpha(t)$, then $\overline{\alpha} : s \mapsto \alpha(s \cdot t)$ is continuous with $\overline{\alpha}(0) = \alpha(0) = x$, $\overline{\alpha}(1) = \alpha(t) = z$ thus $z \in P_x$.

Let $S \subset X$ path-connected such that $x \in S$. For every point $y \in S$, y is path-connected to x in S, hence in X. Thus $y \in P_x$. $\square$

Examples 7.3.4.

1. $\mathbb{R}^2 \setminus \{y = 0\}$ has two path-connected components, hence is not path-connected.

2. If we endow $\mathbb{Q}$ with the induced topology in $\mathbb{R}$, for every $q \in \mathbb{Q}$ then $P_q = \{q\}$.

7.4 Connected spaces

Definition 7.4.1. *A topological space is called* ***connected*** *if it cannot be expressed as the union of two disjoint, non-empty open sets.*

Examples 7.4.2.

1. $\mathbb{R}$ is connected. Indeed, if $\mathbb{R} = U \uplus V$ with open $U, V \neq \varnothing$, then the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \begin{cases} 1 & \text{if} \quad x \in U, \\ -1 & \text{if} \quad x \in V, \end{cases}$$

would be continuous, contradicting Bolzano's Theorem.

We can use the same argument to conclude that all intervals of $\mathbb{R}$ are connected.

2. $X = \mathbb{R}^2 \setminus \{y = 0\}$ is not connected. Indeed $X = \{(x, y) : y > 0\} \uplus \{(x, y) : y < 0\}$, both open and non-empty.

Proposition 7.4.3. *A topological space is connected if, and only if, the only subsets in X that are both open and closed are X and $\varnothing$.*

Proof. Assume X is connected and let $S \subset X$ be open and closed. Then $X = S \uplus (X \setminus S)$, which necessarily would imply either S or $X \setminus S$ is empty.

Conversely, assume $X = U \uplus V$ with U, V non-empty open sets. Then U, being the complementary of an open set, is also closed, hence is equal to either X or $\varnothing$ – which implies, respectively, $V = \varnothing, X$. $\square$

7.5 Properties of connected spaces

Proposition 7.5.1. *Let $f : X \to Y$ be a continuous an surjective function. If X is connected, so is Y. Hence connectedness is invariant by homeomorphisms.*

Proof. If Y were the disjoint union of two open sets, then so would $X = f^{-1}(U) \uplus f^{-1}(V)$, contradicting connectedness. $\square$

Lemma 7.5.2. *Let X be a topological space and $\{X_i\}_{i \in I}$ a covering such that X_i is connected for every i, and there exists an index i_0 such that $X_i \cap X_{i_0} \neq \varnothing$ for every $i \in I$. Then X is connected.*

Proof. Assume U and V are two open disjoint subsets such that $X = U \cup V$. Then $X_i = (X_i \cap U) \cup (X_i \cap V)$. The fact X_i is connected implies $X_i \cap U = \varnothing$ or $X_i \cap V = \varnothing$. Hence $X_i \subset U$ or $X_i \subset V$. Assume $X_{i_0} \subset U$. Then if $X_i \subset V$ for any $i \in I$ that would imply $\varnothing \neq X_i \cap X_{i_0} \subset U \cap V = \varnothing$ which is a contradiction. Hence $X_i \subset U$ for every $i \in I$, hence $X = \bigcup_{i \in I} X_i = U$ and $V = \varnothing$. □

Proposition 7.5.3. *$X \times Y$ is connected if, and only if, X and Y are.*

Proof. If $X \times Y$ is connected, then since the projections pr_X and pr_Y are continuous and surjective, X and Y are connected.

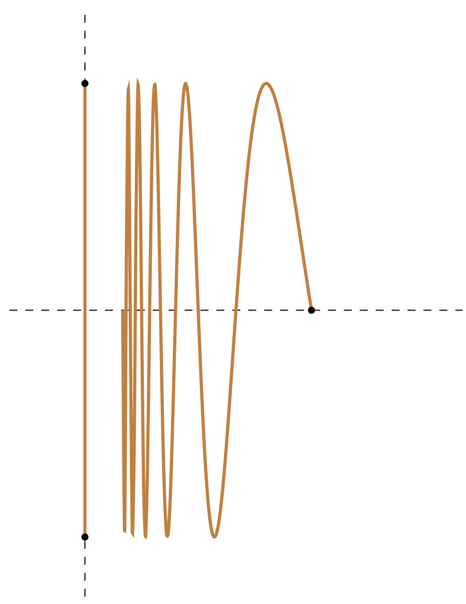
Conversely assume X and Y are connected. The fact $X \times Y = (\{x\} \times Y) \cup \bigcup_{y \in Y} (X \times \{y\})$ for any $x \in X$, and the fact $\{x\} \times Y$ and $X \times \{y\}$ are connected (being homeomorphic to X and Y respectively) implies, along with the previous Lemma, the connectedness of $X \times Y$. □

Proposition 7.5.4. *Every path-connected space is connected.*

Proof. Let X be path-connected and $x_0 \in X$. For every $x \in X$ choose a path $\alpha_x : I \to X$ with initial point x_0 and terminal point x. It is clear $X = \bigcup_{x \in I} \alpha_x(I)$. The fact I is connected implies each $\alpha_x(I)$ is. Furthermore, $x_0 \in \alpha_x(I) \cup \alpha_y(I)$ for every $x, y \in X$. Hence X is connected. □

Remark 7.5.5. Not every connected space is path-connected.

For instance define $X = (\{0\} \times [-1,1]) \cup \left\{\left(x, \sin \frac{\pi}{x}\right) : x \in (0,1]\right\}$.



This space is not path-connected because there is no path joining $(0,1)$ and $(1,0)$ which is totally contained in X. However, X is connected because it is the closure of the image of the function $f : (0,1] \to \mathbb{R}^2$ defined by $f(t) = \left(t, \sin \frac{\pi}{t}\right)$, which is connected. The rest follows from the result below.

Lemma 7.5.6. *Let Y be a topological space and $A \subset X$ a connected subspace. If $X \subset Y$ and $A \subset X \subset \overline{A}$, then X is connected.*

Proof. If $X = U \uplus V$ with U, V open and non-empty then $A = (U \cap A) \uplus (V \cap A)$ is a union of open sets of A, hence one of them is empty. If for instance $U \cap A = \varnothing$ then $A \subset X \setminus U$ which is closed, hence $\overline{A} \subset X \setminus U$ which means $\overline{A} \cap U = \varnothing$, hence $U = X \cap U = \varnothing$ which is absurd. □

7.6 Connected components

Definition 7.6.1. *Let (X, τ) be a topological space. We say two points $x, y \in X$ are* ***connected*** *if there exists a connected subset of X containing both.*

Proposition 7.6.2. *Let (X, τ) be a topological space. The relation on X $x \approx y$ iff x, y are connected is an equivalence relation.*

Proof. Reflexive and symmetric properties are immediate from the definition. Let us check transitivity. If $x \approx y$ and $y \approx z$, there must exist connected sets A, B such that $x, y \in A$, $y, z \in B$. $A \cup B$ is connected due to the fact that $y \in A \cap B \neq \varnothing$. Thus there is a connected set containing x, z, and $x \approx z$. □

Definition 7.6.3. *Let (X, τ) be a topological space and $x \in X$. The* ***connected component*** *of x is*

$$C_x = \{y \in X : x \approx y\}.$$

Needless to say, X is the disjoint union of its connected components.

Proposition 7.6.4. *Let X be a topological space and $x \in X$. Then C_x is the largest connected subspace of X containing x. In particular, $P_x \subset C_x$ and C_x is closed.*

Proof. For every $y \in C_x$, there exists a connected set A_y containing both x and y. Then $C_x = \bigcup_{y \in C_x} A_y$ is connected.

If $K \subset X$ connected such that $x \in K$, all points of K are connected with x and thus $K \subset C_x$.

The fact P_x is connected implies it is connected, and the fact $x \in P_x$ implies $P_x \subset C_x$. $\overline{C_x}$ is also connected containing x, hence $C_x = \overline{C_x}$. □

Remark 7.6.5. Connected components are generally not open, e.g. $X = \left\{x \in \mathbb{R} : x = 0 \text{ or } x = \frac{1}{n}, n > 0\right\}$. Then $C_0 = \{0\}$ which is not open in X.

Definition 7.6.6. *Let X be a topological space. We say X is* ***locally path-connected*** *if every point of X has a local basis whose neighbourhoods are path-connected.*

Examples 7.6.7.

1. $\mathbb{R}^n$ is locally path-connected.

2. $X = (\{0\} \times [-1, 1]) \cup \left\{\left(x, \sin \frac{\pi}{x}\right) : x \in (0, 1]\right\}$ in Remark 7.5.5 is not locally path-connected in points of the form $(0, y)$, $-1 \leq y \leq 1$.

Proposition 7.6.8. *Connected components of a locally path-connected space are open.*

Proof. Let $x \in X$ and $y \in C_x$. Let V be a path-connected (hence connected) neighbourhood of y. Then $C_x \cup V$ is the union of two connected non-disjoint sets since $y \in C_x \cap V$. Maximality of C_x entails $C_x \cup V \subset C_x$, thus $y \in V \subset C_x$. Thus $y \in \mathring{C}_x$. □

Proposition 7.6.9. *Let X be a connected, totally path-connected space. Then X is path-connected.*

Proof. Let $x \in X$. It will suffice to check every point $y \in X$ can be path-connected to x. In other words, $A = \{y \in X : y \sim x\}$ equals X.

The fact $x \in A$ implies $A \neq \varnothing$. Let $y \in A$ and V a path-connected neighbourhood of y. Every point in V can be path-connected to y, itself path-connected to x. Path-connectedness is a relation of equivalence, hence $V \subset A$ and thus $y \in \mathring{A}$. Hence A is open.

Let us prove A is also closed. If $z \in X \setminus A$ and W is a path-connected neighbourhood of z, then $W \cap A = \varnothing$. Otherwise there would exist points path-connected to z and x, contradicting $z \notin A$. Thus $W \subset X \setminus A$ and z is interior to $X \setminus A$. Thus A is closed.

A closed and open subset of a connected space can only be either $\varnothing$ or the total space. $X = A$. $\square$

7.7 Solved Exercises

7.8 Exercises

114. Prove that the union of two different lines in $\mathbb{R}^2$ is not homeomorphic to a single line.

Chapter 8

Construction of topological spaces from topological properties

Definition 8.0.1. *Let P be a property of topological spaces, e.g. Frenet or compact. We say X is **locally** P if every point of X has a local basis of neighbourhoods satisfying P.*

8.1 Locally compact spaces and compactifications

Let us study methods to construct compact spaces from non-compact ones. This process is not unique and can entail the adjunction of one or more points. If the space has some special local properties, this process is especially agreeable.

8.1.1 Locally compact spaces

Definition 8.1.1. *Let (X, τ) be a topological space. X is **locally compact** if it is Hausdorff and every point of X has a compact neighbourhood.*

Examples 8.1.2.

1. Every compact space is locally compact.

2. $\mathbb{R}^n$ is locally compact, yet it is not compact.

3. Every discrete space is locally compact.

4. Every open and closed set of $\mathbb{R}^n$ is locally compact. This can either be deduced directly from the definition or as a consequence of the following results.

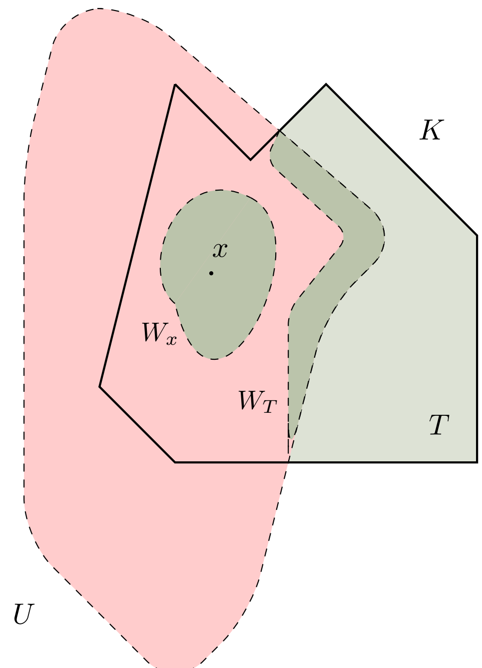
Proposition 8.1.3. *Let X be a locally compact space and $A \subset X$ a closed subset. Then A is locally compact.*

Proof. Let $x \in A$ and K a compact neighbourhood of x in X. Then $K \cap A$ is a neighbourhood of x in A and, due to the fact it is closed in compact K, it is compact itself. □

The same property is true for open subsets. We have this as a consequence of the following result:

Proposition 8.1.4. *Every point in a locally compact space has a local basis of compact neighbourhoods.*

Proof. Let $x \in X$ and U an open neighbourhood of x. By hypothesis there exists a compact neighbourhood K of x. Then $K \cap U$ is an open set of K and thus $T = K \setminus K \cap U$ is a closed set of K not containing x. The fact K is norma implies there exist open sets of K, W_x and W_T, such that $x \in W_x$, $T \subset W_T$ and $W_x \cap W_T = \varnothing$. Therefore $K \setminus W_T$ is a neighbourhood of x in K. The fact K is a neighbourhood of x implies $K \setminus W_T$ is also a neighbourhood of x in X. FInally $K \setminus W_T$ is closed in a compact space, hence compact and $K \setminus W_T \subset K \setminus T \subset U$.



□

Remark 8.1.5. The Proposition tells us that in Hausdorff spaces the above Definition for locally compact spaces coincides with the one we should have given following general Definition 8.0.1.

Corollary 8.1.6. *An open set of a locally compact space is locally compact.*

Proof. Let X be a locally compact space, U an open set and $x \in U$. The Proposition implies there exists a compact neighbourhood K of x such that $x \in K \subset U$. Therefore U is locally compact. □

Proposition 8.1.7. *The product of two locally compact spaces is locally compact.*

Proof. Let X and Y be locally compact. The product of Hausdorff spaces is Hausdorff, hence so is $X \times Y$. Let $(x, y) \in X \times Y$. Let K_x, K_y be compact neighbourhoods of x, y respectively. Then $K_x \times K_y$ is a compact neighbourhood of (x, y). □

8.1.2 Compactifications

Given a non-compact space, we would like to study compact spaces containing it as a dense subspace.

Definition 8.1.8. *Let X be a topological space. A* ***compactification*** *of X is a compact space $X^\star$ and a continuous map $h : X \times X^\star$ such that $X \to h(X)$ is a homeomorphism and $\overline{h(X)} = X^\star$.*

Examples 8.1.9.

1. $X = \{0,1\}$ and $X^\star = [0,1]$ with $h : X \hookrightarrow X^\star$ inclusion.

2. $X = \mathbb{R}^n$ and $X^\star = \mathbb{S}^n$ with $h : \mathbb{R}^n \to \mathbb{S}^n \setminus \{\boldsymbol{p}\} \hookrightarrow \mathbb{S}^n$ the inverse of the stereographic projection.

3. $(0,1)$ is homeomorphic to $\mathbb{R}$, hence the previous example yields a compactification of $(0,1)$ different from that of the first example.

4. $X = \mathbb{N}$, $X^\star = \{0\} \cup \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ and $h(n) = \frac{1}{n}$.

8.1.2.1 Alexandroff compactification

Assume X is a non-compact, locally compact (hence Hausdorff) space.

Definition 8.1.10. *A compactification $h : X \to X^\star$ is called* ***Alexandroff*** *or* ***one-point*** *if $X^\star \setminus X$ consists of a single point. We will denote this point by ∞.*

Let us prove the existence of such compactification. Define $X^\star = X \cup \{\infty\}$ where ∞ is an arbitrary point not belonging to X. Define the following topology:

$$\tau_{X^\star} = \{U = X^\star \setminus K : K \subset X \text{ compact }\} \cup \tau_X.$$

Lemma 8.1.11. *$\tau_{X^\star}$ is a topology on $X^\star$.*

Proof. $\varnothing \in \tau_X \subset \tau_{X^\star}$ and the fact $\varnothing$ is compact implies $X^\star = X^\star \setminus \varnothing \in \tau_{X^\star}$.

Let $\{U_i\}_{i\in I}$ be a family of elements of $\tau_{X^\star}$. Assume for every i $U_i \in \tau_X$, then $\bigcup_{i\in I} U_i \in \tau_X$. Otherwise write $I = I_1 \cup I_2$ where $U_i \in \tau_X$ for every $i \in I_1$ and $U_i = X^\star \setminus K_i$, K_i compact, for every $i \in I_2$. Then

$$\bigcup_{i\in I} U_i = \left(\bigcup_{i\in I_1} U_i\right) \cup \left(\bigcup_{i\in I_2} X^\star \setminus K_i\right) = \left(\bigcup_{i\in I_1} U_i\right) \cup \left(X^\star \setminus \bigcap_{i\in I_2} K_i\right) = X^\star \setminus \left(\bigcap_{i\in I_2} K_i \setminus \bigcup_{i\in I_1} U_i\right).$$

The fact $I_2 \neq \varnothing$ implies $\bigcap_{i\in I_2} K_i \setminus \bigcup_{i\in I_1} U_i$ is compact, hence $\bigcup_{i\in I_1} U_i \in \tau_{X^\star}$.

- It is sufficient for two elements in $\tau_{X^\star}$. If both U and V are open in τ_X so is their intersection, and $\tau_X \subset \tau_{X^\star}$. If $U = X^\star \setminus K$ with K compact and $V \in \tau_X$ then $U \cap V = (X^\star \setminus K) \cap V = (X \setminus K) \cap V \in \tau_X \subset \tau_{X^\star}$. Finally if $U = X^\star \setminus K_1$ and $V = X^\star \setminus K_2$ then $U \cap V = X^\star \setminus (K_1 \cup K_2) \in \tau_{X^\star}$ on account of the union of two compacts being compact. □

Needless to say, inclusion $h : (X, \tau_X) \to (X^\star, \tau_{X^\star})$ is continuous.

Theorem 8.1.12. *Couple $(X^\star, h)$ is a compactification of X.*

Proof. Let us first prove $X^\star$ is quasi-compact. Let $\{U_i\}_{i\in I}$ be an open covering of $X^\star$. Let i_0 such that $\infty \in U_{i_0}$. There exists a compact K such that $U_{i_0} = X^\star \setminus K$. Then $\{K \cap U_i\}_{i\in I\setminus\{i_0\}}$ is an open covering of K. Let $i_1, \ldots, i_n$ such that $K = (K \cap U_{i_1}) \cup \cdots \cup (K \cap U_{i_n})$. Then $X^\star = U_{i_0} \cup U_{i_1} \cup \cdots \cup U_{i_n}$.

Let us see $X^\star$ is Hausdorff. All we have to check is ∞ is separated from every point $x \in X$. Let K be a compact neighbourhood of x and $U \in \tau_X \subset \tau_{X^\star}$ such that $x \in U \subset K$. Then $V = X^\star \setminus K$ is an open neighbourhood of ∞ and $U \cap V = \varnothing$.

Finally let us prove $\overline{h(X)} = X^\star$. X is not compact, hence $h(X)$ is not closed. Therefore, $h(X) \subsetneq \overline{h(X)}$ which forces us to have $\overline{h(X)} = X^\star$. □

Let us prove one-point compactification is unique.

Theorem 8.1.13. *Let (X^+, g) be any other compactification of X such that $g(X)$ is open. Then there exists continuous $f : X^+ \to X^\star$ such that $f \circ g = h$ and $f(X^+ \setminus g(X)) = \{\infty\}$.*

Hence if X^+ is a one-point compactification, f is a homeomorphism.

Proof. f can only be defined as follows:

$$f(a) = \begin{cases} h(x) & a = g(x) \in g(X) \subset X^+ \\ \infty & a \in X^+ \setminus g(X) \end{cases}$$

Let us prove f is continuous. If $U \in \tau_X$ then $f^{-1}(U) = g\left(h^{-1}(U)\right)$. the fact $g : X \to g(X)$ is a homeomorphism and $g(X) \subset X^+$ is open implies $f^{-1}(U)$ is open. If $U = X^\star \setminus K$ where $K \subset X$ is a compact,

$$f^{-1}(U) = X^+ \setminus f^{-1}(K) = X^+ \setminus g\left(h^{-1}(K)\right).$$

$g\left(h^{-1}(K)\right)$ is compact in $X^\star$, hence closed, and its complementary is open. □

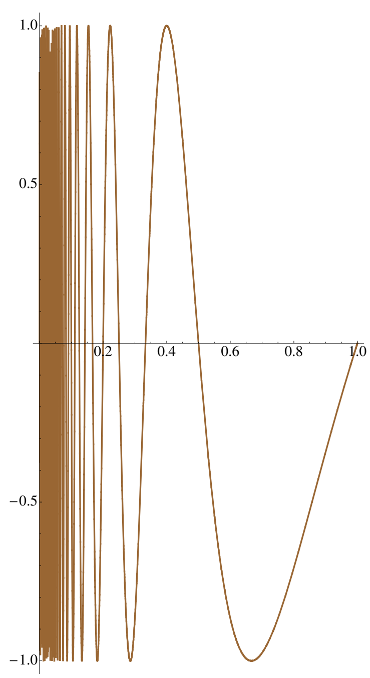
Examples 8.1.14. Examples 8.1.9(**2.**) and 8.1.9(**4.**) are one-point compactifications.

8.1.2.2 Stone - Čech compactification

8.2 Connectifications

8.3 Solved Exercises

115. Consider the following set of points in the plane:



$$X = (\{0\} \times [-1, 1]) \cup \left\{ \left(x, \sin \frac{\pi}{x}\right) : x \in (0, 1] \right\} \subset \mathbb{R}^2$$

(i) (10 Marks) Find a metric d on X such that $\left(X, \tau_X^d\right)$ is not connected.

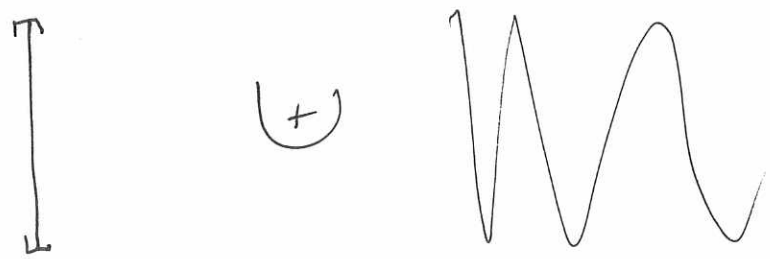
(ii) (15 Marks) Find a topology $\tau_1 \neq \tau_X^d$ on X such that (X, τ_1) is compact.

Solution: handwritten at the end of this section

3) i) The discrete metric $d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

Exercise 115

yields the discrete topology $\mathcal{P}(\mathbb{X})$ as the set of all the open sets for d. We can express $\mathbb{X}$ as a disjoint union of open sets in infinite ways, e.g.

$\mathbb{X} =$ 

$\{0\} \times [-1,1]$ $\qquad$ $\{(x, \sin\frac{\pi}{x}) : x \in (0,1]\}$

Both subsets of $\mathbb{X}$ $\Rightarrow$ both open in $(\mathbb{X}, \mathcal{P}(\mathbb{X}))$.

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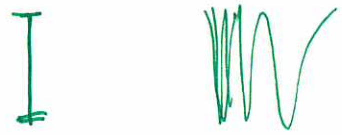
ii) Consider the Euclidean topology on $\mathbb{R}^2$ restricted to the subset $\mathbb{X}$ (subspace topology) $\tau^{Eucl}_{\mathbb{X}}$.

- In this topology, being compact equates to being closed and bounded.
- $\mathbb{X}$ is obviously bounded — it fits into a Euclidean ball, e.g. $B((0,0), 2)$.
- Let us prove $\mathbb{X}$ is closed: $\overline{\mathbb{X}} = \mathbb{X}$.
  Remember: $\overline{\mathbb{X}} = \mathbb{X} \cup L(\mathbb{X})$, where

  $$L(\mathbb{X}) = \{\text{limit points of } \mathbb{X}\}$$

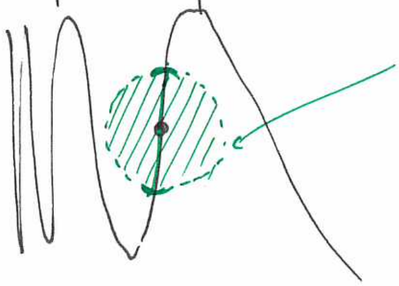
All we need to prove is that $L(X) = X$.
If we break up $X = X_1 \oplus X_2$ then



all we need to prove is
$$\begin{cases} L(X_1) = X_1 & (1) \\ L(X_2) = X_1 \cup X_2 = X. & (2) \end{cases}$$

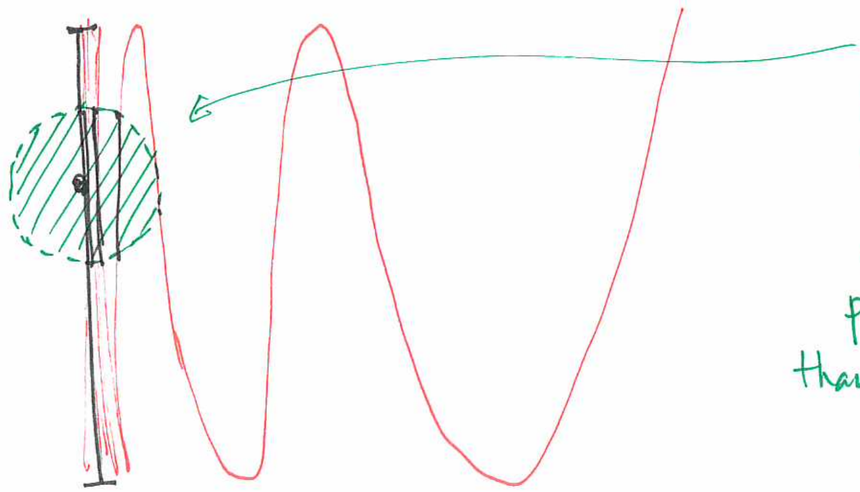
The first is obvious: any point $(0,y) \in$ | is the centre of infinitely many balls which contain points of | other than itself:

How about (2): obviously any point of X_2 is a limit point of X_2,



contains points of X_2 other than the point itself, no need to find an explicit expression

And any point in X_1 is not only a limit point of X_1 but also a limit point of X_2:



contains not only (infinitely many) points in X_1 other than itself, but also (infinitely many) points of $\underline{X_2}$ other than itself.

Part II

Algebraic topology

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