Dynamic Structural Models for Policy Evaluation

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Practical Sessions Outline

1. Solving Value Functions
2. **Estimating the model**
3. Conditional Choice Probability (CCP) estimation
4. CCP estimation - finite dependence

Session 2: Estimating the model

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Session Outline

Introduction

Maximum Likelihood

Infinite Horizon

Nested Fixed Point Algorithm

Introduction

Empirical example

- Maintenance of **electric bikes' batteries** in Barcelona
- Bikes can be either **replaced** (R) or **maintained** (M)
- a_t - age of the bike battery
- $a_{t+1} = \begin{cases} 1 & \text{if replaced} \\ a_t + 1 & \text{if maintained} \end{cases}$
- Utility:

$$u(a_t, d_t, \epsilon_t) = \begin{cases} -\theta_R + \epsilon_{1t} & \text{if } d_t = 1 \\ -\theta_{M1} a_t - \theta_{M2} a_t^2 + \epsilon_{0t} & \text{if } d_t = 0 \end{cases}$$

  where ϵ_{0t} and ϵ_{1t} are unobserved by econometrician, iid with all Rust assumptions

- A is a number of states (if $A = 5$ then $a_t = 1, ..., 5$) and T is time horizon

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Maximum Likelihood

Log-Likelihood

- In general framework, a full information maximum likelihood (FIML) function is:

$$\mathcal{L}_N(\theta) = \sum_{i=1}^{N} \ln Pr(d_{i1}, ..., d_{iT}, a_{i1}, ..., a_{iT}; \theta) \equiv \sum_{i=1}^{N} l_i(\theta)$$

- what, given the assumptions, can be factorized as:

$$l_i(\theta) = \sum_{t=1}^{T} \ln Pr(d_{it}|a_{it}; \theta) + \sum_{t=2}^{T} \ln Pr(a_{it}|a_{it-1}, d_{it-1}; \theta) + \ln Pr(a_{i1}; \theta)$$

- In our example only the first element of the sum matters, the log-likelihood function takes form:

$$\mathcal{L}_N(\theta) = \sum_{t=1}^{5} \sum_{i=1}^{N} d_t \ln(p_{1t}(a_t)) + (1 - d_t) \ln(p_{01}(a_t))$$

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Estimation Procedure

Estimation

- $\boldsymbol{\theta} = (\boldsymbol{\theta}'_U, \boldsymbol{\theta}'_a)$ - a vector of parameters determining law of motion of a_t ($\boldsymbol{\theta}_a$) and remaining parameters of the model ($\boldsymbol{\theta}_U$)
- **Steps**:
  1. Estimate $\hat{\boldsymbol{\theta}}_a$:

$$\hat{\boldsymbol{\theta}}_a = arg \max_{\boldsymbol{\theta}_a} \sum_{i=1}^{N} \sum_{t=1}^{T} \ln Pr(a_{it} | a_{it-1}, d_{it-1}; \boldsymbol{\theta}_a)$$

  2. Estimate $\hat{\boldsymbol{\theta}}_U$:

$$\hat{\boldsymbol{\theta}}_U = arg \max_{\boldsymbol{\theta}_U} \sum_{i=1}^{N} \sum_{t=1}^{T} \ln Pr(d_{it} | a_{it}; \boldsymbol{\theta}_U, \hat{\boldsymbol{\theta}}_a)$$

  3. Single iteration for the full likelihood optimization (i.e. N-R) using $(\hat{\boldsymbol{\theta}}'_U, \hat{\boldsymbol{\theta}}'_a)$' **OPTIONAL**

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Maximum Likelihood

Probabilities

- Given Type-I Extreme Value assumption, the *conditional choice probabilities* $p_{jt}(a_t)$ are defined as:

$$p_{jt}(a_t) \equiv \mathbb{E}[d_{jt}^*|a_t] = \frac{e^{v_{jt}(a_t)}}{\sum_{h\in D} e^{v_{ht}(a_t)}}$$

- In considered model, depending on decision j we get:

$$p_{1t}(a_t) = \frac{e^{v_{1t}(a_t)}}{e^{v_{1t}(a_t)} + e^{v_{0t}(a_t)}} = \frac{e^{v_{1t}-v_{0t}(a_t)}}{1 + e^{v_{1t}-v_{0t}(a_t)}}$$

$$p_{0t}(a_t) = \frac{e^{v_{0t}(a_t)}}{e^{v_{1t}(a_t)} + e^{v_{0t}(a_t)}} = \frac{1}{1 + e^{v_{1t}-v_{0t}(a_t)}}$$

Infinite Horizon

Steps

Remember that we have **finite number of states** ($A = 5$), however **infinite time horizon** ($T = \infty$).

Steps:

1. **Formulate the dynamic programming problem** (conditional value functions $v_{jt}(x_t)$ and *Emax* $V_t(x)$).
2. Find **fixed points** of each value function - Value Function Iteration.
3. Formulate conditional choice probabilities $p_{jt}(x_t)$.
4. Construct **log-likelihood function**.
5. Solve the optimization problem.

Previous Concepts

Conditional value functions

- Conditional value function given our assumptions:

$$v_j(a_t) = u_j(a_t) + \beta V(a_{t+1})$$

- Given Type-I Extreme Value assumption *Emax* can be written as a function of $v_{jt}(a_t)$:

$$V(a) = \ln \sum_{j \in D} \exp\{v_j(a)\} + \gamma$$

where γ is the Euler's constant.

Assumptions

Previous Concepts

Finding the fixed point

- First, given a guess of $V(a)$ we get:

$$v_1 = -\theta_R + \beta V(2) + \beta\gamma$$

$$v_0(1) = -\theta_{M1} - \theta_{M2} + \beta V(2) + \beta\gamma$$

$$v_0(2) = -2\theta_{M1} - 4\theta_{M2} + \beta V(3) + \beta\gamma$$

$$\dots$$

$$v_0(5) = -5\theta_{M1} - 25\theta_{M2} + \beta V(4) + \beta\gamma$$

- And we can re-update our guess using:

$$V(1) = \ln(\exp\{v_1\} + \exp\{v_0(1)\}) + \gamma$$

$$V(2) = \ln(\exp\{v_1\} + \exp\{v_0(2)\}) + \gamma$$

$$\dots$$

$$V(5) = \ln(\exp\{v_1\} + \exp\{v_0(5)\}) + \gamma$$

Infinite Horizon

Conditional Choice Probablities

For each state a and decision $j = 0, 1$ formulate the conditional choice probabilities.

Take first period $t = 1$:

$$\mathcal{L}_N(t=1) = \sum_{i=1}^{N} d_{i1} \ln(p_{11}(1)) + (1 - d_{i1}) \ln(p_{01}(1))$$

where:

$$p_{11}(1) = \frac{e^{v_{11} - v_{01}(1)}}{1 + e^{v_{11} - v_{01}(1)}} = 1 - \frac{1}{1 + e^{v_{11} - v_{01}(1)}}$$

$$p_{01}(1) = \frac{1}{1 + e^{v_{11} - v_{01}(1)}}$$

$$v_{11} - v_{01}(1) = \Big[-\theta_R + \beta V_2(1) \Big] - \big[-\theta_{M1} - \theta_{M2} + \beta V_2(2) \big]$$

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Estimation Procedure

Estimation

- $\boldsymbol{\theta} = (\boldsymbol{\theta}'_U, \boldsymbol{\theta}'_a)$ - a vector of parameters determining law of motion of a_t ($\boldsymbol{\theta}_a$) and remaining parameters of the model ($\boldsymbol{\theta}_U$)
- **Steps**:
  1. Estimate $\hat{\boldsymbol{\theta}}_a$:

$$\hat{\boldsymbol{\theta}}_a = arg \max_{\boldsymbol{\theta}_a} \sum_{i=1}^{N} \sum_{t=1}^{T} \ln Pr(a_{it} | a_{it-1}, d_{it-1}; \boldsymbol{\theta}_a)$$

  2. Estimate $\hat{\boldsymbol{\theta}}_U$:

$$\hat{\boldsymbol{\theta}}_U = arg \max_{\boldsymbol{\theta}_U} \sum_{i=1}^{N} \sum_{t=1}^{T} \ln Pr(d_{it} | a_{it}; \boldsymbol{\theta}_U, \hat{\boldsymbol{\theta}}_a)$$

  3. Single iteration for the full likelihood optimization (i.e. N-R) using $(\hat{\boldsymbol{\theta}}'_U, \hat{\boldsymbol{\theta}}'_a)$'

Estimation Procedure

Nested Fixed Point Algorithm

- The key algorithm is how to estimate the parameters of the choice probabilities.
- **Steps**:
  1. Guess $\boldsymbol{\theta}_U^{k=1}$.**(Outer Loop)**.
     - Solve the Value Function by finding the fixed point. **(Inner Loop)**.
     - Compute the likelihood contribution.
     - Move $\theta^{k=k+1}$ in the optimal direction using some optmiziation criteria.
  2. Repeat until the optimal of the log likelihood has been found.

Appendix

Assumptions

Assumption 1. Additive Separability. Utility function additively separable between observable and unobservable components:

$$U(d_t, \boldsymbol{x}_t, \boldsymbol{\varepsilon}_t) = u(d_t, \boldsymbol{x}_t) + \varepsilon_t(d_t)$$

Assumption 2. IID Unobservables. Unobserved variables $\boldsymbol{\varepsilon}_t$ are iid across agents and over time given $\boldsymbol{x}_t$.

Assumption 3. Conditional Independence. Conditional on current decision and observable state variables, next period observables do not depend on $\boldsymbol{\varepsilon}_t$:

$$F_x(\boldsymbol{x}_{t+1}|d_t, \boldsymbol{x}_t, \boldsymbol{\varepsilon}_t) = F_x(\boldsymbol{x}_{t+1}|d_t, \boldsymbol{x}_t)$$
$$F(\boldsymbol{x}_{t+1}, \boldsymbol{\varepsilon}_{t+1}|d_t, \boldsymbol{x}_t, \boldsymbol{\varepsilon}_t) = F_x(\boldsymbol{x}_{t+1}|d_t, \boldsymbol{x}_t)F_\varepsilon(\boldsymbol{\varepsilon}_{t+1})$$

Assumption 4. Conditional Logit. Unobservables ε_{jt} are independent across alternatives and type-I extreme value distributed.

◂ Conditional value functions