Quantitative Macroeconomics - PS2

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Question 1. Computing Transition in a Representative Agent Economy

Consider the following closed optimal growth economy populated by a large number of identical infinitely lived households that maximize:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}$$

over consumption and leisure $u(c_t) = lnc_t$, subject to:

$$c_t + i_t = y_t$$

$$y_t = k_t^{1-\theta}(zh_t)^{\theta}$$

$$i_t = k_{t+1} - (1-\delta)k_t$$

Set labor share to $\theta = 0.67$. Also, to start with, set $h_t = 0.31$ for all t. Population does not grow.

(a) Compute the steady-state. Choose z to match an annual capital-output ratio of 4, and an investment-output ratio of 0.25.

The problem is:

$$\max_{c_t, k_{t+1}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}$$

Subject to:

$$c_t + k_{t+1} = k_t^{1-\theta}(zh_t)^{\theta} + (1-\delta)k_t$$

Notice that since the constraint holds with equality, we can isolate consumption and introduce it in the objective function. So we have:

$$c_t = k_t^{1-\theta}(zh_t)^{\theta} + (1-\delta)k_t - k_{t+1}$$

And the problem becomes:

$$\max_{k_{t+1}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(k_t^{1-\theta}(zh_t)^{\theta} + (1-\delta)k_t - k_{t+1}) \right\}$$

The first order condition gives:

$$\frac{\partial L}{\partial k_{t+1}} = 0 \Longleftrightarrow$$

$$-\beta^t u'(k_t^{1-\theta}(zh_t)^\theta + (1-\delta)k_t - k_{t+1}) + \beta^{t+1}u'(k_{t+1}^{1-\theta}(zh_{t+1})^\theta + (1-\delta)k_{t+1} - k_{t+2})((1-\theta)k_{t+1}^{-\theta}(zh_{t+1})^\theta + (1-\delta)) = 0$$

Which is in fact the Euler equation and can be written as:

$$\beta u'(c_{t+1})((1-\theta)k_{t+1}^{-\theta}(zh_{t+1})^\theta + (1-\delta)) = u'(c_t)$$

And now we can impose steady state, which imply that $c_t = c_{t+1} = c^*$ and $k_t = k_{t+1} = k^*$, we get:

$$\beta u'(c^*)((1-\theta)k^{*-\theta}(zh^*)^\theta + (1-\delta)) = u'(c^*)$$

$$k^* = zh^* \left[\frac{(1-\theta)\beta}{1-(1-\delta)\beta}\right]^{\frac{1}{\theta}}$$

Now that we have characterized the steady state level of capital. Now we can impose that the capital-output ratio should be 4 and the investment-output 0.25. To do so, let's first normalize output equal to 1. By doing so, we get that $k = 4$ and $i = 0.25$. Now we can solve for the other parameters of the model:

$$i^* = \delta k^* \leftrightarrow \delta = \frac{0.25}{4} = 0.0625$$

$$c^* = y^* - i^* = 1 - 0.25 = 0.75$$

$$y^* = k^{*1-\theta}(zh^*)^\theta \leftrightarrow z = \left(\frac{y}{k^{*1-\theta}h^{*\theta}}\right)^{\frac{1}{\theta}} = \left(\frac{1}{4^{1-0.67}0.31^{0.67}}\right) = 1.629$$

$$\beta = \frac{1}{((1-\theta)k^{*-\theta}(zh^*)^\theta) + (1-\delta)} = 0.98$$

(b) Double permanently the productivity parameter z and solve for the new steady state

So if we assume that the new $z = 2z$ we get that the new steady state level of capital is:

$$k^*_{new} = 2zh^* \left[\frac{(1-\theta)\beta}{1-(1-\delta)\beta}\right]^{\frac{1}{\theta}}$$

Which imply that k^* has doubled and now it is $k^* = 8$. Furthermore, we get that:

$$y^*_{new} = (2k^*)^{1-\theta}(2h^*)^\theta = 2(k^{*1-\theta}(zh^*)^\theta) = 2$$

Since before it was normalized to 1 and now it has doubled. Also, regarding the new investment level we have:

$$i^*_{new} = \delta k^*_{new} = 0.5$$

And from this we get that the new consumption is:

$$c^*_{new} = y^*_{new} - i^*_{new} = 1.5$$

(c) Compute the transition from the first to the second steady state and report the time-path for savings, consumption, labor and output.

To move from one steady state to another we will need to use the law of motion of capital. This law of motion needs to satisfy the optimality condition (Euler equation), so for this reason our transition path will be characterized by n Euler equaitons, that will be written only as a function of capital. So take the Euler equation:

$$\beta u'(c_{t+1})((1-\theta)k_{t+1}^{-\theta}(zh_{t+1})^{\theta} + (1-\delta)) = u'(c_t)$$

And since the utility function is $u(c_t) = lnc_t$ we get that:

$$c_{t+1} = \beta c_t((1-\theta)k_{t+1}^{-\theta}(zh_{t+1})^{\theta} + (1-\delta))$$

Now we can substitute by consumption as a function of capital to get the Euler equation fully dependent on capital:

$$c_t = k_t^{1-\theta}(zh_t)^{\theta} + (1-\delta)k_t - k_{t+1}$$

So what we get is:

$$k_{t+1}^{1-\theta}(zh_{t+1})^{\theta} + (1-\delta)k_{t+1} - k_{t+2} = \beta(k_t^{1-\theta}(zh_t)^{\theta} + (1-\delta)k_t - k_{t+1})((1-\theta)k_{t+1}^{-\theta}(zh_{t+1})^{\theta} + (1-\delta))$$

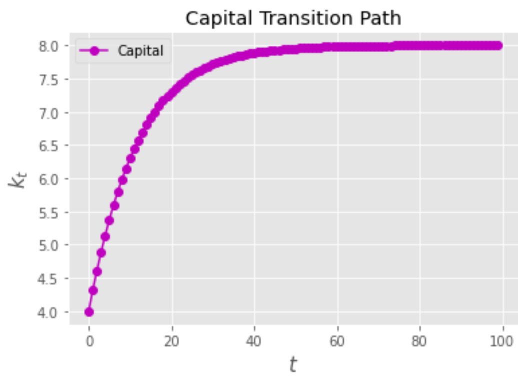
The procedure to characterize the transition path is the following:

1. We need to characterize n different periods Euler equations, so will will have n equations and $n+2$ unknowns
2. To have an identified system of equations, we can establish the initial capital level, which is the initial steady state and also the final capital level, which is the second steady state capital value.
3. Once we have a perfect identified system of n equations and n unknowns we can solve for it. Notice that our n unknowns are nothing else than the transition values of capital towards the steady state.

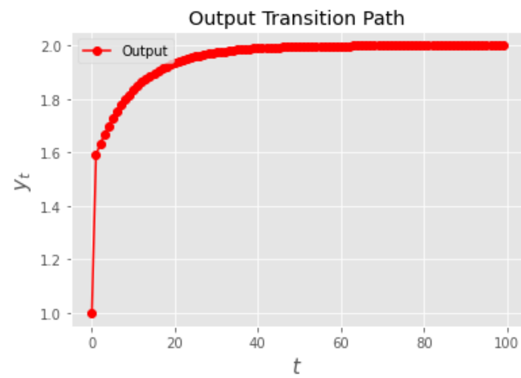
Results:

The results are displayed in Figure 1 and Figure 2. Both figures show the overall transition from the initial steady state, towards the second steady state after having introduced a shock in labor productivity. The most important things to mention are:

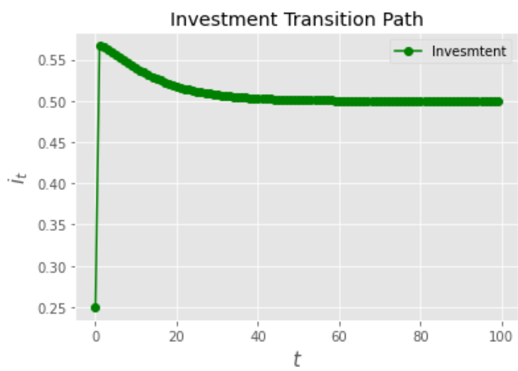
1. Capital converges steadily towards the steady state but the increases in the level of capital are bigger at initial periods, then they become smaller.
2. Output experiences a big increase directly created by the shock and then increases towards the steady state driven by the increase in k. This is because output fully depends on labor productivity.
3. Investment more than doubles after the shock and then stabilizes. This is because individuals have the shock and save a big amount to future consumption, and once the shock is there forever they go towards the optimal level of savings.
4. Once the shock is introduced consumption increases as it directly depends on output. Then it grows steadily towards the steady state value.



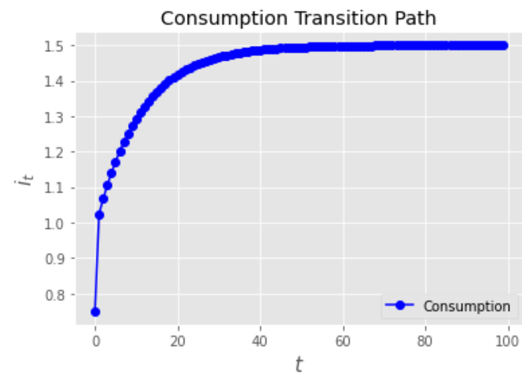
(a) Capital Path



(b) Output Path



(c) Investment Path



(d) Consumption Path

Figure 1: Transitions when first shock of z

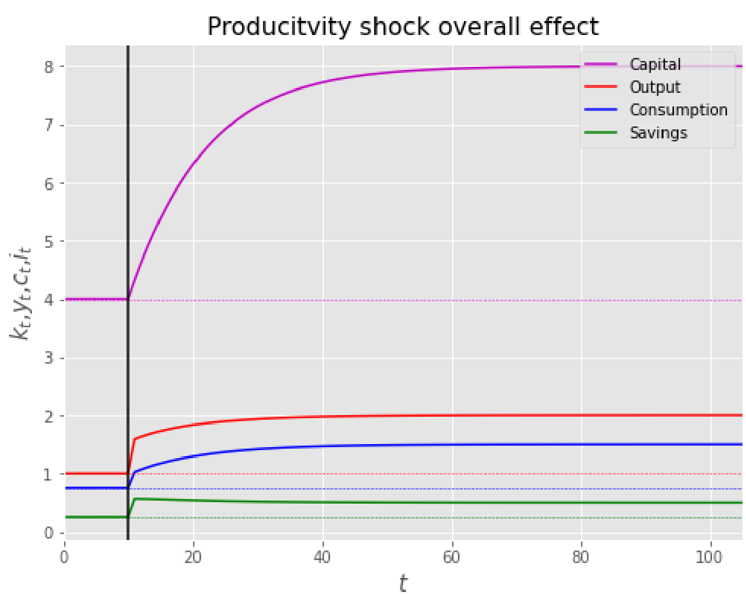


Figure 2: Overall Effect First Shock

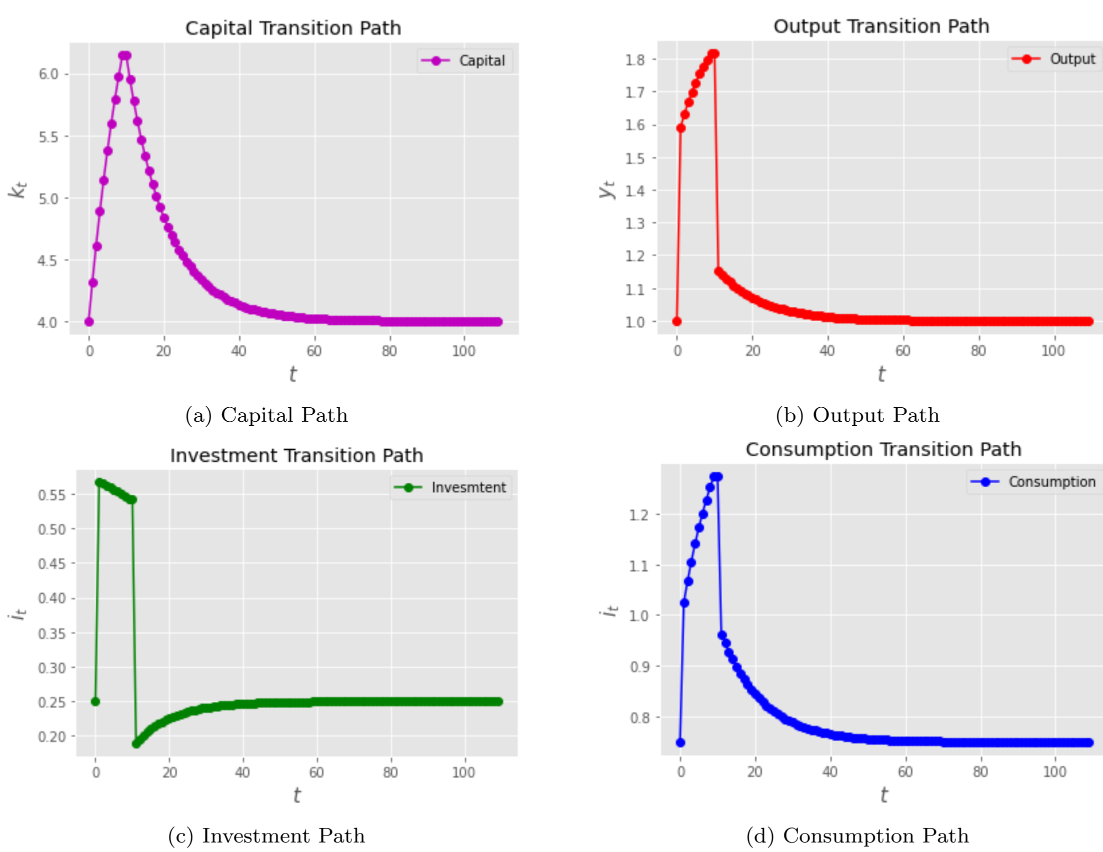
(d) Unexpected shocks. Let the agents believe productivity z_t doubles once and for all periods. However, after 10 periods, surprise the economy by cutting the productivity z_t back to its original value. Compute the transition for savings, consumption, labor and output.

Notice we already know the values for the initial steady state level, so there is no need to analytically compute them. I am not sure whether the shock is during the transition to the steady state or once you are in the second steady state. For this reason, I will compute both transitions.

Figure 3 shows the following results:

1. Once the first shock is introduced capital behavior changes towards a higher level, but after the second shock the consumer rectifies its behavior and it converges steadily towards its previous level. The first periods after the shock are always the more sensitive to changes.
2. Output experiences a big increase directly created by the first shock and once productivity is reduced there is also a big reduction on output level. Then output converges towards its steady state level.
3. Investment more than doubles after the first shock and then once the second shock is introduced decreases more than its initial level, to then converge again to the first steady state level.It is the most sensitive variable.
4. Once the shock is introduced consumption increases as it directly depends on output. As the other variables, when the second shock is introduced the level has a important decrease to then decrease steadily towards its steady state value.

Results:



(a) Capital Path

(b) Output Path

(c) Investment Path

(d) Consumption Path

Figure 3: Transitions when second shock is introduced ten periods after first shock

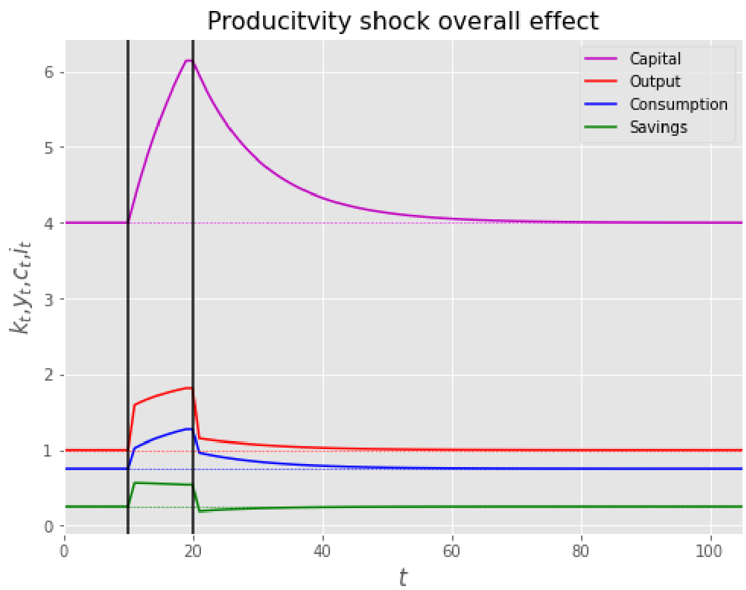
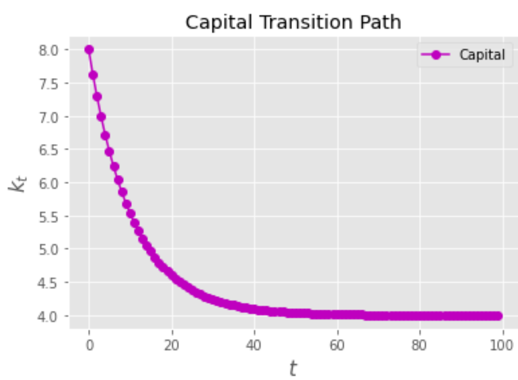
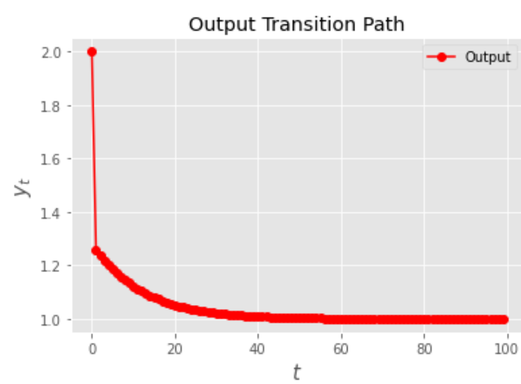


Figure 4: Overall Effect when second shock is introduced ten periods after first shock

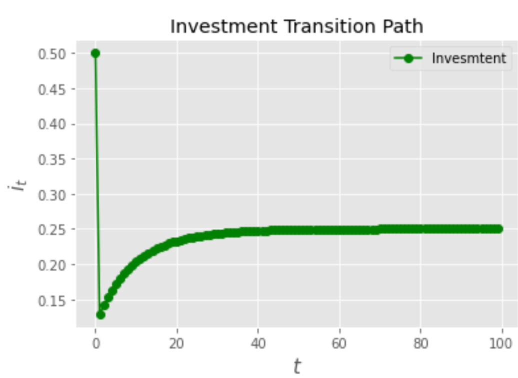
I will also include the results from the transition from the second steady state towards the first. There is no need to comment as everything has already been said.



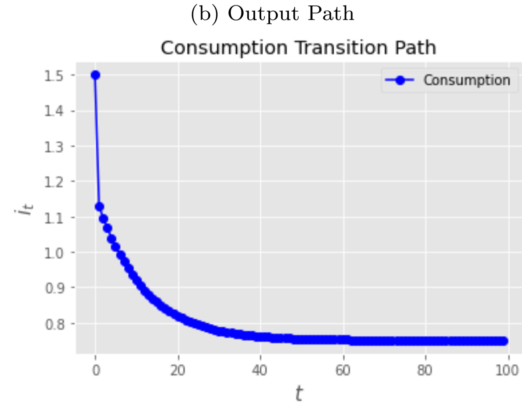
(a) Capital Path



(b) Output Path



(c) Investment Path



(d) Consumption Path

Figure 5: Transitions when second shock is introduced at second steady state

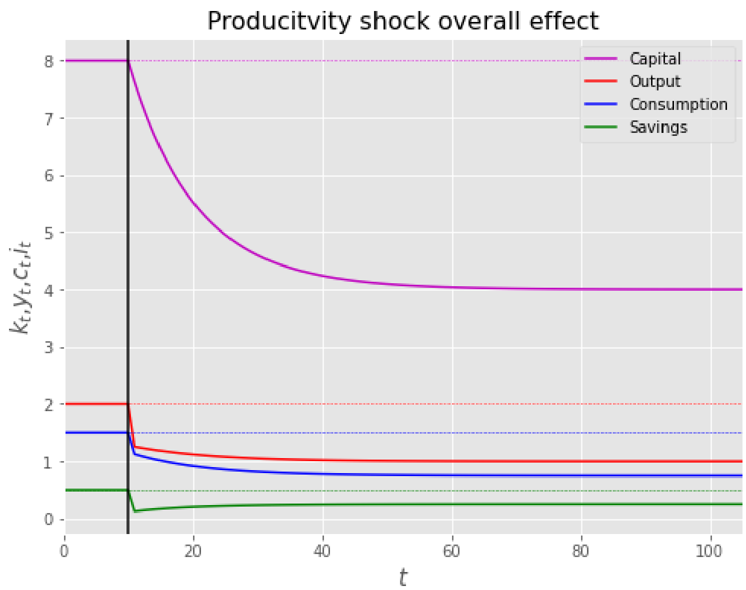


Figure 6: Overall Effect when second shock is introduced at second steady state

(e) Bonus Question: Labor Choice Allow for elastic labor supply.

I could not solve it, but I show my work.

If we now endogenize the labor supply decisions the new optimization problem of the social planner becomes:

$$\max_{\{c_t, k_{t+1}, h_t\}} \quad E_0 \Bigg\{ \sum_t \beta^t \left(lnc_t - \kappa \frac{h_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right) \Bigg\}$$

subject to:

$$c_t + k_{t+1} = k_t^{1-\theta}(zh_t)^\theta + (1-\delta)k_t$$

So the problem can be written as:

$$\max_{\{k_{t+1}, h_t\}} \quad E_0 \Bigg\{ \sum_t \beta^t \left(ln(k_t^{1-\theta}(zh_t)^\theta + (1-\delta)k_t - k_{t+1}) - \kappa \frac{h_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right) \Bigg\}$$

And the First Order Conditions give:

$$\frac{\partial}{\partial k_{t+1}} = 0 \leftrightarrow (k_{t+1}^{1-\theta}(zh_{t+1})^\theta + (1-\delta)k_{t+1} - k_{t+2}) = \beta((1-\theta)k_{t+1}^{-\theta}(zh_{t+1})^\theta + (1-\delta))(k_t^{1-\theta}(zh_t)^\theta + (1-\delta)k_t - k_{t+1})$$

$$\frac{\partial}{\partial h_t} = 0 \leftrightarrow \frac{\beta^t}{(k_t^{1-\theta}(zh_t)^\theta + (1-\delta)k_t - k_{t+1})}(\theta k_t^{1-\theta} z^\theta h_t^{\theta-1}) = \kappa h_t^{\frac{1}{\nu}}$$

With this two equations we just need to isolate h_t and plug it in the foc of k_{t+1} so we have the Euler only as a function of capital, then repeat the same process as in question 1.c. The problem is that the expression is analytically complicated and I do not know how to isolate h. Therefore, I will normalize output to 1 to get:

$$1 = k_t^{1-\theta}(zh_t)^\theta \leftrightarrow h_t = \left(\frac{1}{k_t^{1-\theta} z^\theta} \right)^{\frac{1}{\theta}}$$

And now we can plug the value of h_t in the Euler equation to repeat the same process as before. Notice that the value of capital in the steady state is the same as before which was:

$$k^* = zh^* \left[\frac{(1-\theta)\beta}{1-(1-\delta)\beta} \right]^{\frac{1}{\theta}}$$

And using the value of h found before we get:

$$k^* = z \left(\frac{1}{k^{*(1-\theta)} z^{\theta}} \right)^{\frac{1}{\theta}} \left[\frac{(1-\theta)\beta}{1-(1-\delta)\beta} \right]^{\frac{1}{\theta}} \leftrightarrow k^* = \left[\frac{(1-\theta)\beta}{1-(1-\delta)\beta} \right]$$

we can then impose that the capital/output ratio is equal to 0.5 to get: (assuming that the invesment/ output ratio is 0.03125 so that $\delta = 0.0625$)

$$\left[\frac{(1-\theta)\beta}{1-(1-\delta)\beta} \right] = 0.5 \leftrightarrow \beta = \frac{0.5}{(1-\theta) + 0.5(1-\delta)} = 0.966$$

Basically I am trying to get similar parameter values as before changing the values of the ratios at the initial steady state. And now I just need to solve for z and h^* and I will try to do it in python. From Python I get that $h^* = 0.81737266$ and $z = 1.72126762$.

I have noticed that my k^* does not depend on z, therefore no matter the shocks there is no change, since it only depends on parameters. Probably I have to isolate h_t at the foc of h_t and then introduce it in the Euler to overcome this problem but I cannot see how.

Question 2. Solve for the optimal COVID-19 lockdown model posed in the slides.

(a) Show your results for a continuum of combinations of the $\beta \in [0,1]$ parameter (vertical axis) and the $c(TW) \in [0,1]$ parameter (hztal axis). That is, plot for each pair of β and $c(WT)$ the optimal allocations of $H, H_f, H_{nf}, H_f = H$, output, welfare, amount of infections and deaths. Note that if $H = N$ there is no lockdown, so pay attention to the potential non-binding constraint $H < N$. Discuss your results. You may want to use the following parameters: $A_f = A_{nf} = 1$; $\rho = 1.1$, $k_f = k_{nf} = 0.2$, $\omega = 20$, $\gamma = 0.9$, $i_o = 0.2$ and $N = 1$.

Let's first present the model of the slides. The probelm of the social planner is :

$$\max_{\{c,h_f,h_{nf}\}} \sum_i (c_i - k_f h_f - k_{nf} h_{nf}) - \omega D$$

where ω denotes how much the planner cares about the number of deaths $D = D_f + D_{nf}$. Also we can use the aggregate resource constraint:

$$Y = \sum_i c_i = C$$

And the labor market clearing condition is :

$$\sum_{i \in j} h_i = H_j \quad \text{for} \quad j = f, nf$$

So we get that the problem of the social planner is :

$$\max_{\{H_j \in [0,N]\}_{j=\{f,nf\}}} \quad Y(H_f, H_{nf}) - k_f H_f - k_{nf} H_{nf} - \omega D$$

subject to:

$$H = H_f + H_{nf} \leq N$$

And now, we can use the fact that the production function has the form:

$$Y = \left(A_f H_f^{\frac{\rho-1}{\rho}} + c(TW) A_f H_{nf}^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}}$$

Where $c(TW) \in [0,1]$ is the productivity loss due to teleworking. Furthermore, we have that the probability of dying is:

$$D = (1-\gamma) I$$

And I is the total number of infections that has the form:

$$I = i H_f$$

Notice that i is the unconditional infection rate at which a worker gets infected, and has the form:

$$i = \beta(HC) m(H_f)$$

Where $m(H_f)$ is the probability than an employed individual meets with a contagious individual at work and has the form:

$$m(H_f) = \frac{i_0 H_f}{N}$$

with i_0 being the initial share of infections at work. Also $\beta(HC) \in [0,1]$ is the conditional infection rate which depends on the extent of human contact. So after having considered all the previous equations we can rewrite the problem of the social planner as:

$$\max_{\{H_j \in [0,N]\}_{j=\{f,nf\}}} \quad \left(A_f H_f^{\frac{\rho-1}{\rho}} + c(TW) A_f H_{nf}^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}} - k_f H_f - k_{nf} H_{nf} - \omega((1-\gamma)\beta(HC)\frac{i_0 H_f^2}{N})$$

subject to:

$$H = H_f + H_{nf} \leq N$$

So the Lagrangian of the problem is:

$$L = \left(A_f H_f^{\frac{\rho-1}{\rho}} + c(TW) A_f H_{nf}^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}} - k_f H_f - k_{nf} H_{nf} - \omega \left((1-\gamma)\beta(HC)\frac{i_0 H_f^2}{N} \right) - \lambda[H_f + H_{nf} - N]$$

The first order conditions give:

$$\frac{\partial L}{\partial H_f} = 0 \leftrightarrow \left(A_f H_f^{\frac{\rho-1}{\rho}} + c(TW)A_f H_{nf}^{\frac{\rho-1}{\rho}}\right)^{\frac{-1}{\rho-1}} A_f H_f^{\frac{-1}{\rho}} - k_f - 2\omega\left((1-\gamma)\beta(HC)\frac{i_0 H_f}{N}\right) - \lambda = 0$$

$$\frac{\partial L}{\partial H_f} = 0 \leftrightarrow \left(A_f H_f^{\frac{\rho-1}{\rho}} + c(TW)A_f H_{nf}^{\frac{\rho-1}{\rho}}\right)^{\frac{-1}{\rho-1}} c(TW)A_f H_{nf}^{\frac{-1}{\rho}} - k_{nf} - \lambda = 0$$

Together with the slackness conditions :

$$\lambda[H_f + H_{nf} - N] = 0 \qquad \lambda \leq 0$$

And now to solve it we just need to solve this system of equations. Notice I will not do this analytically, since I have to solve for a continous range of values for two parameters. So I will use Python to compute the solution. Results can be seen in Figure 7.

(b) What happens to your result when you increase(decrease) ρ or ω?

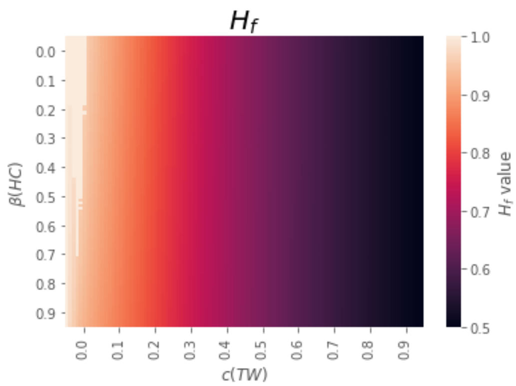
I will consider the following values : $(\omega, \rho) = (300, 1.1)$ and $(\omega, \rho) = (20, 10)$ and then $(\omega, \rho) = (2, 0.5)$

Results to both 2.1 and 2.2 can be shown below. Notice that in the first parametrization the constraint always binds, in fact, the constraint is always binding except for the situation characterized in Figure 8, when there is a lot of concert about the deaths of individuals. Overall the main effect is:

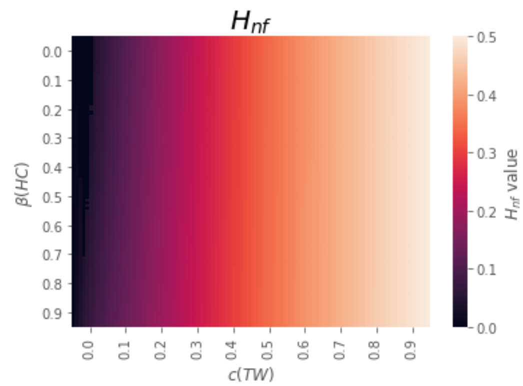
1. **Increase** ω. When we increase ω we are giving more concern about the deaths caused by COVID-19 to the social planner. Therefore, consistenly with the beliefs, the reaction to the social planner is to increase H_{nf} and decrease H_f, to avoid having more deaths. He arrives to a situation at which $H_f + H_{nf} \leq N = 1$ which imply total employment is not chosen. Notice that by increasing ω the number of infected and dead individuals decrease.
2. **Increase in** ρ By increasing ρ the amount of hours worked at the work place increase, and the hours teleworking decrease. This is because ρ measures the elasticity of substitution and now the planner prefers to allocate more workers at H_f since those teleworking have a productivity decline of size $c(TW)$. Notice that in Figure 9, the higher $c(TW)$ the more hours H_{nf}, since the cost of teleworking is lower.

Figure 7: Optimal lockdowns values for $\omega = 20$ and $\rho = 1.1$

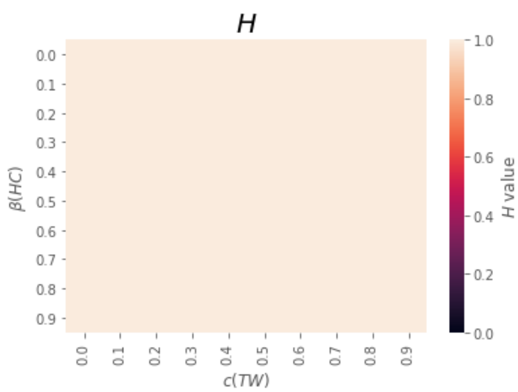
(a) H_f



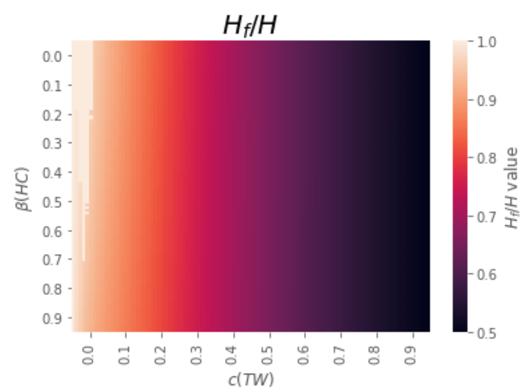
(b) H_{nf}



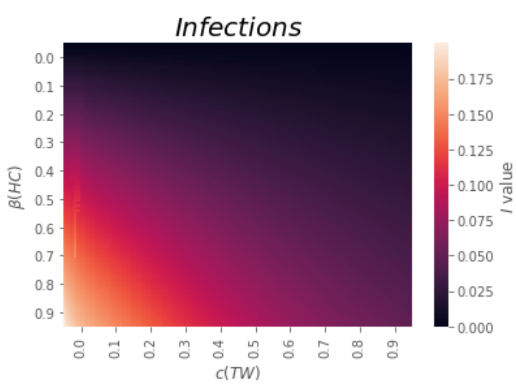
(c) H



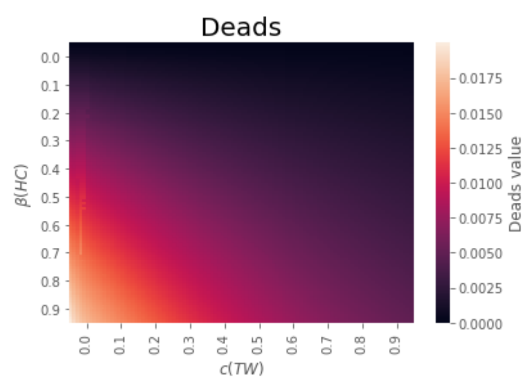
(d) H_{nf}/H



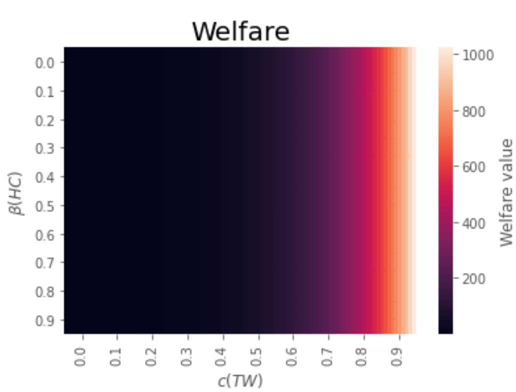
(e) Infections



(f) Deads



(g) Welfare



(h) Output

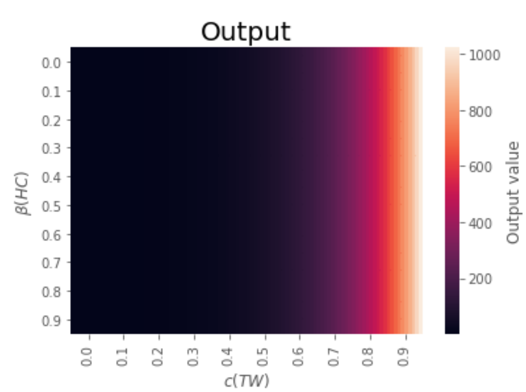
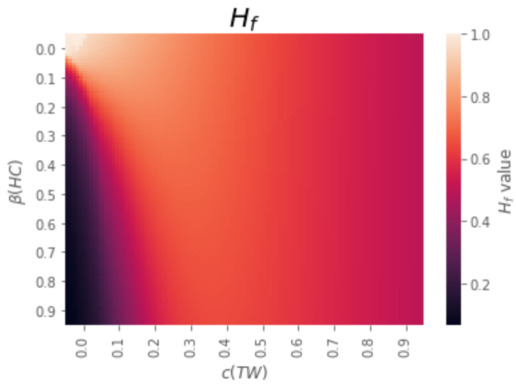
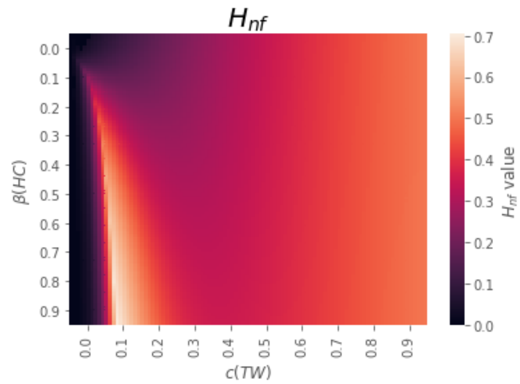


Figure 8: Optimal lockdowns values for $\omega = 300$ and $\rho = 1.1$

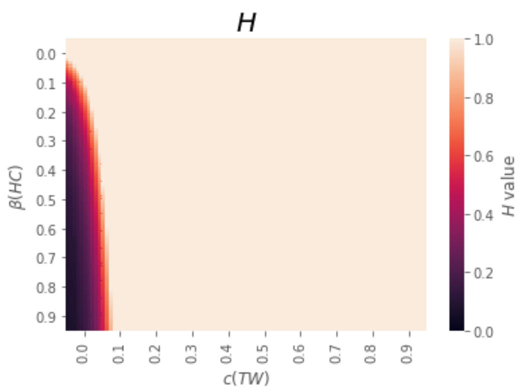
(a) H_f



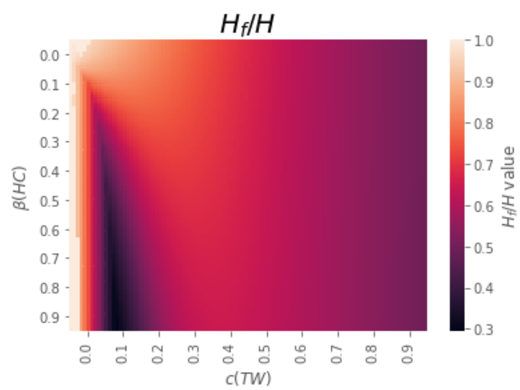
(b) H_{nf}



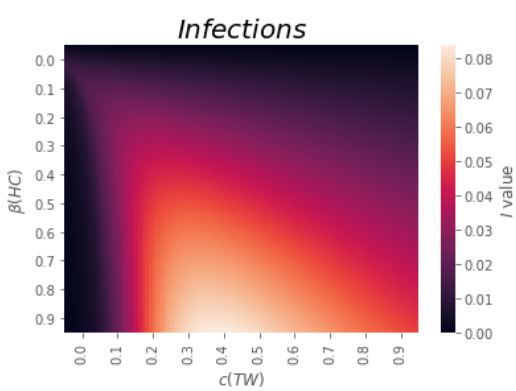
(c) H



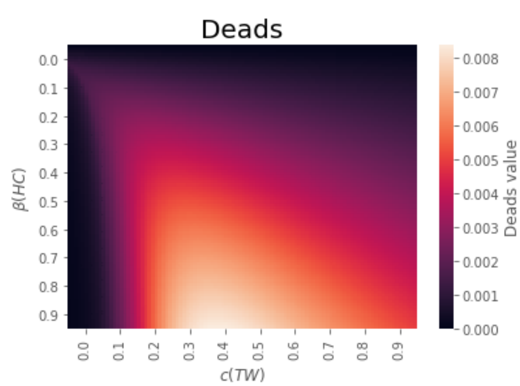
(d) H_{nf}/H



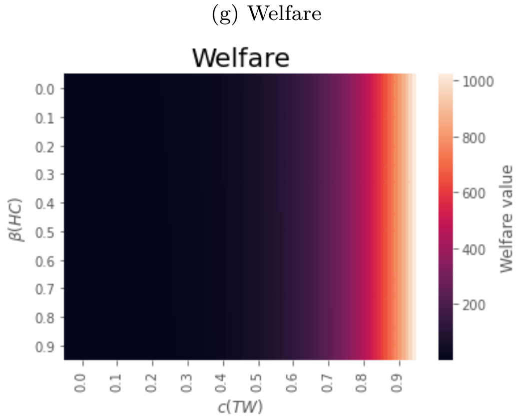
(e) Infections



(f) Deads



(g) Welfare



(h) Output

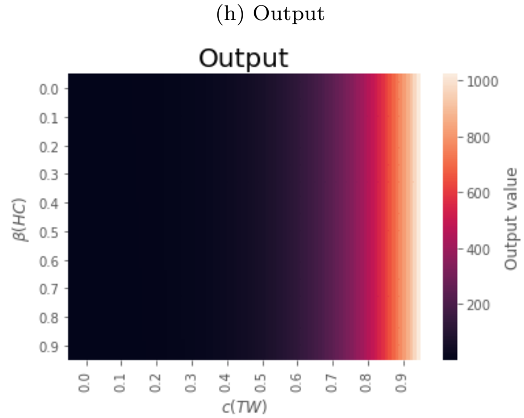
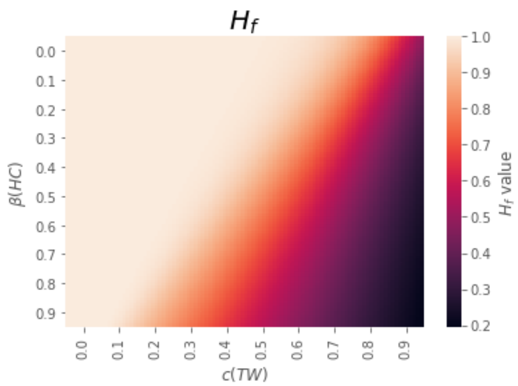
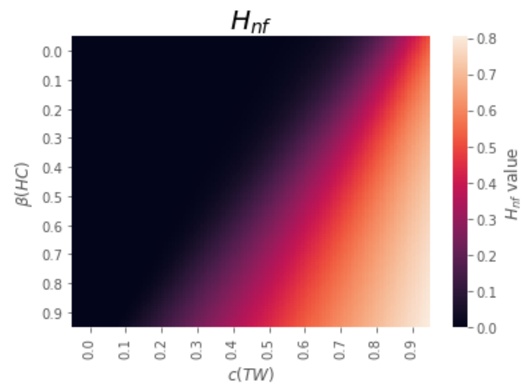


Figure 9: Optimal lockdowns values for $\omega = 20$ and $\rho = 10$

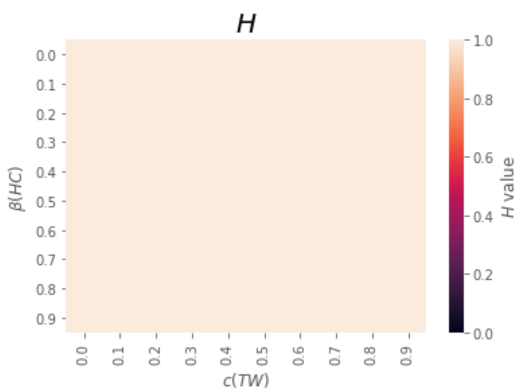
(a) H_f



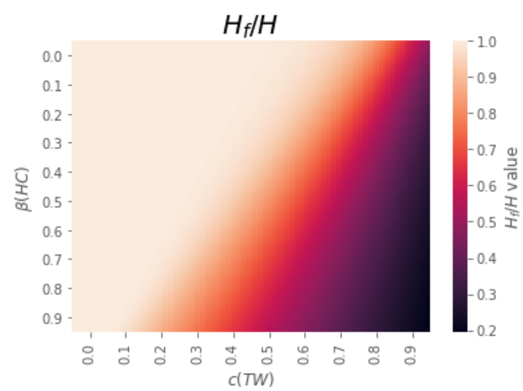
(b) H_{nf}



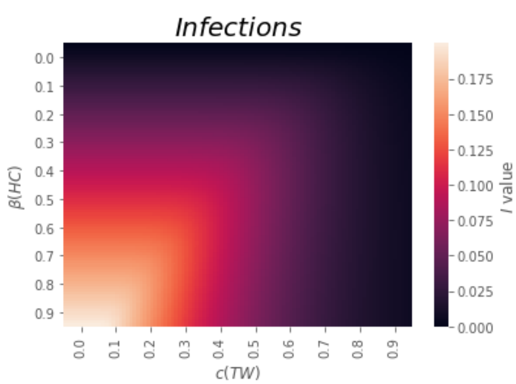
(c) H



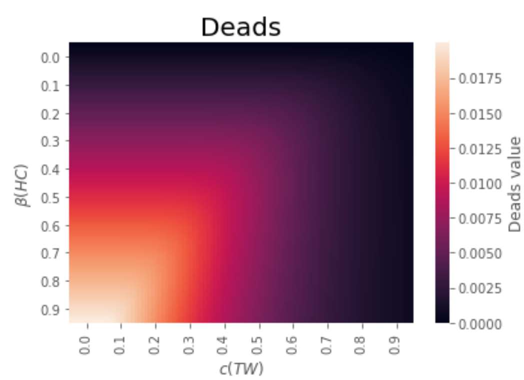
(d) H_{nf}/H



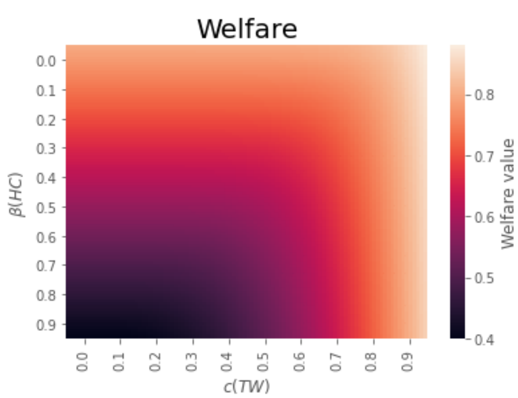
(e) Infections



(f) Deads



(g) Welfare



(h) Output

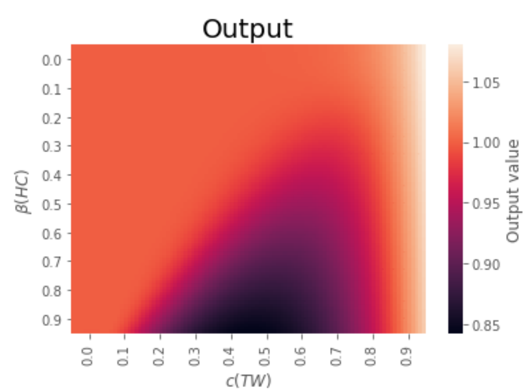
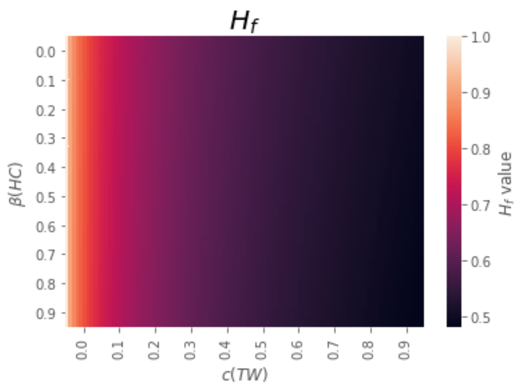
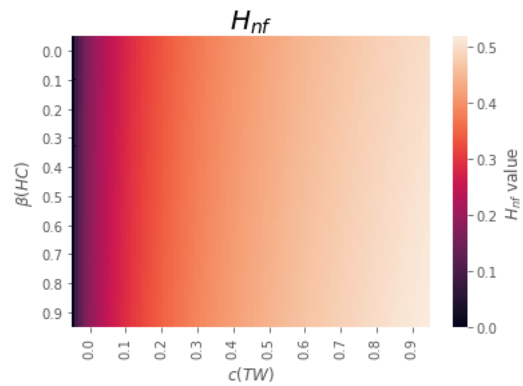


Figure 10: Optimal lockdowns values for $\omega = 2$ and $\rho = 0.5$

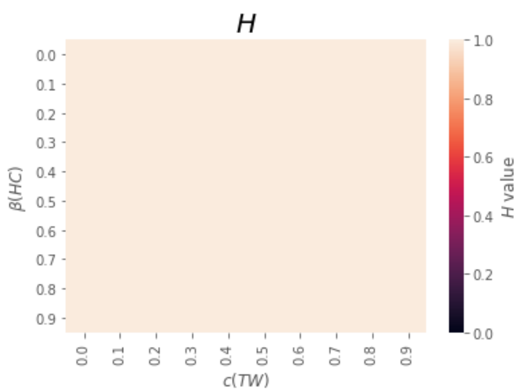
(a) H_f



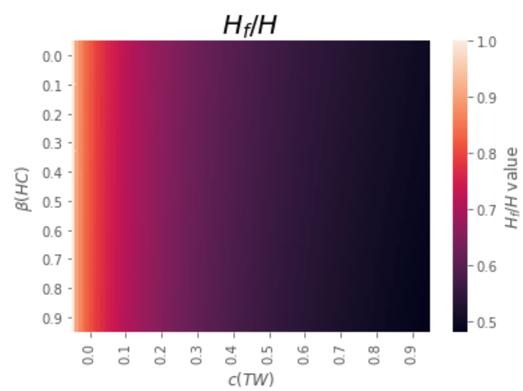
(b) H_{nf}



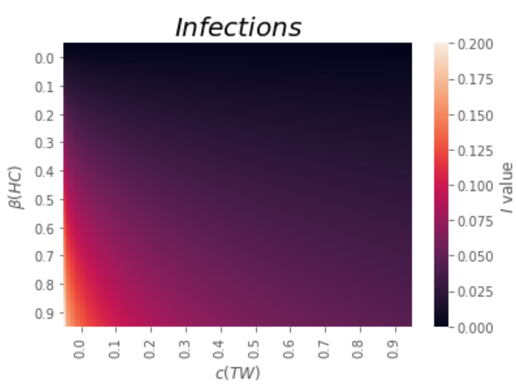
(c) H



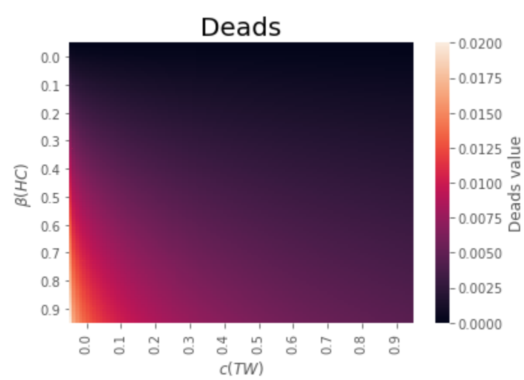
(d) H_{nf}/H



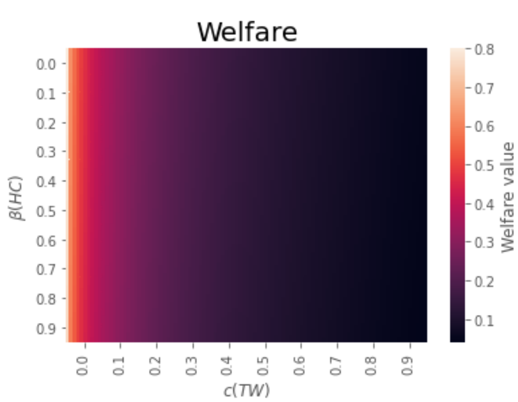
(e) Infections



(f) Deads



(g) Welfare



(h) Output