

# Value Function Methods

## Quantitative Macroeconomics

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The value function iteration algorithm (VFI) described in our previous set of slides [Dynamic Programming] is used here to solve for the value function in the neoclassical growth model. We will discuss first the deterministic model, then add a productivity shock to discuss the stochastic version.<sup>1</sup>

- We focus on the application of global methods to approximate for the value function using both discretization and continuous methods. For a detailed discussion of these methods see the set of slides [Function Approximation.pdf]
- We briefly note the existence of local methods to solve for the value function using linear-quadratic approximations.

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<sup>1</sup>These slides build on notes by Carlos Urrutia 'Metodos Numéricos para Resolver Modelos Macroeconómicos Dinámicos', and the courses by José-Víctor Ríos-Rull and Makoto Nakajima (check their respective websites). Read further textbook references in the syllabus.

# Workhorse: The Neoclassical Model

**Sequential formulation.** Consider a decentralized economy populated by a large number,  $N_t$ , of identical infinitely lived households. Each of them maximizes the discounted future stream of utility,

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1),$$



subject to a budget constraint,

$$c_t + k_{t+1} = w_t + (1 + r_t - \delta)k_t,$$



given  $k_0$  and  $z_0$ .

An aggregate firm maximizes profits (an static problem),

$$Y_t - w_t N_t - r_t K_t,$$

subject to aggregate technology

$$Y_t = F(K_t, N_t).$$

**Remark.** We will work with the recursive formulation of this problem (next). For an equivalence theorem between the sequential and recursive formulations see SLP.

# Recursive Competitive Equilibrium (RCE)

A RCE is a set of functions  $v(k, K)$ ,  $c(k, K)$ ,  $k'(k, K)$ ; prices  $w(K)$  and  $r(K)$ ; and an aggregate law of motion  $\Gamma(K)$  such that,

- 1 Given  $w$ ,  $r$  and  $\Gamma$ , the value function  $v(k, K)$  solves the Bellman equation,

$$v(k, K) = \max_{c \geq 0, k'} u(c) + \beta v(k', K') \quad (1)$$

such that

$$\begin{aligned} c + i &= w(K) + r(K)k \\ k' &= i + (1 - \delta)k \\ K' &= \Gamma(K). \end{aligned}$$

Associated to the value function we have the set of optimal decision rules  $c(k, K)$  and  $k'(k, K)$ .

- 2 Factor prices equate marginal products,  $r(K) = f'(K)$  and  $w(K) = f(K) - f'(K)K$ .
- 3 Market clearing,  $f(K) = c(k, K) + i(k, K)$ .
- 4 Aggregate consistency,  $\Gamma(K) = k'(K, K)$ .

# Pareto Optimal Equilibrium



**Remark.** Welfare theorems hold, that is, the RCE is Pareto Optimal. We define the recursive PO equilibrium as a set of functions  $v(k)$ ,  $c(k)$ ,  $k'(k)$  that solve

$$v(k) = \max_{c \geq 0, k'} u(c) + \beta v(k') \quad (2)$$

such that

$$\begin{aligned} c + i &= f(k) \\ k' &= i + (1 - \delta)k \end{aligned}$$

That is, the PO problem is

$$v(k) = \max_{k' \in [0, f(k) + (1 - \delta)k]} u(f(k) + (1 - \delta)k - k') + \beta v(k') \quad (3)$$



**Remark.** Remember that the Bellman principal of optimality ensures that we can recover the sequential solution using the recursive solution. To see this, note that starting with an initial  $k_0$  we can use the optimal decision rules of the recursive problem to, recursively, get the sequence

$$\begin{aligned}k_1 &= k'(k_0) \\k_2 &= k'(k_1) = k'(k'(k_0))\end{aligned}$$

and so on for all  $t \geq 3$ .

# VFI with Global Approximation of the Value Function



**Remark.** Remember that the value function algorithm (VFI) holds independently of what method we use to approximate  $V$ .

Here, we will start by discretizing  $V$  to approximate the value function. Later we will approximate  $V$  with finite element methods, as well as weighted residual methods that we have seen in our previous sessions.





# Discretization

- **STEP 1:** The continuous variable  $k$  is discretized by defining a grid for the values of  $k$ , that is,  $K = (k_1, k_2, \dots, k_i, \dots, k_p)$  with  $k_1 = k_{min}$ , and  $k_p = k_{max}$ . For example, an evenly spaced grid is

$$k_i = k_{min} + (i - 1) \eta$$

with  $\eta = \frac{k_{max} - k_{min}}{p - 1}$ . Obviously, with higher  $p$ , and smaller  $\eta$ , the approximation will be more precise, but more costly to compute.

For the  $k_{min}$  we should choose a value slightly higher than zero to avoid potential problems of having zero output, and perhaps zero consumption. For  $k_{max}$  there is no rule, but its choice may depend on what we are interested in. For instance, if we know the steady state  $k^*$ , and we are interested only in dynamics below the steady state, we can safely choose  $k_{max}$  to be slightly above the steady state.

Here we choose the discretization to be uniformly distributed over the state space. However, if we knew where a potential kink is, or the regions of higher curvature, we might be interested in clustering points in those regions. That is, we would not need much points in regions where the value function is known to be linear (or close to).





- **STEP 2:** Guess a solution  $V^{s=0} \in R^p$ , say the null vector, that is,  $V^{s=0}(k_i) = 0$  for all  $i = \{1, \dots, p\}$ .  $s$  keeps track of the number of iterations.


An alternative is to set  $V_i^{s=0} = \frac{u(f(k_i) + (1-\delta)k_i - k_i)}{1-\beta}$ . That is,  $k_j = k_i$ .



- **STEP 3:** Define the return matrix  $M$  as follows,


$$M = \begin{bmatrix} m(k_1, k_1) & m(k_1, k_2) & \dots & m(k_1, k_p) \\ m(k_2, k_1) & m(k_2, k_2) & \dots & m(k_2, k_p) \\ \dots & \dots & \dots & \dots \\ m(k_p, k_1) & m(k_p, k_2) & \dots & m(k_p, k_p) \end{bmatrix}.$$


The generic element in matrix  $M$  contains the return function

$$M_{i,j} = m(k_i, k_j) = u(f(k_i) + (1 - \delta)k_i - k_j)$$


evaluated at all possible combinations  $k_i \in K$  and  $k_j \in K$ .




- **STEP 4:** However, we know that not all values for  $c$ , hence  $k'$ , are feasible. The nonnegativity restriction on consumption  $c_{ij} = f(k_i) + (1 - \delta)k_i - k_j \geq 0$ , implies that  $k_j \leq f(k_i) + (1 - \delta)k_i$ .

Therefore, we replace  $M_{ij} = m(k_i, k_j) = \omega$  if  $k_j > f(k_i) + (1 - \delta)k_i$ , where  $\omega$  is a very negative number.



- **STEP 5:** Given the vector  $V^s = (V_1^s, V_2^s, \dots, V_i^s, \dots, V_p^s)$  and the matrix  $M$ :

- **STEP 5.1:** Compute the matrix,  $\chi$ ,




$$\chi = \begin{bmatrix} M_{11} + \beta V_1^s & M_{12} + \beta V_2^s & \dots & M_{1p} + \beta V_p^s \\ M_{21} + \beta V_1^s & M_{22} + \beta V_2^s & \dots & M_{2p} + \beta V_p^s \\ \dots & \dots & \dots & \dots \\ M_{p1} + \beta V_1^s & M_{p2} + \beta V_2^s & \dots & M_{pp} + \beta V_p^s \end{bmatrix}.$$

where the generic element of  $\chi$  is

$$\chi_{i,j} = M_{i,j} + \beta V_j^s.$$

- **STEP 5.2:** Compute the updated value function  $V^{s+1}$  as the maximum element in each row of  $\chi$ ,

$$V^{s+1} = \begin{bmatrix} \max\{\chi_{11}, \chi_{12}, \dots, \chi_{1p}\} \\ \max\{\chi_{21}, \chi_{22}, \dots, \chi_{2p}\} \\ \dots\dots\dots \\ \max\{\chi_{p1}, \chi_{p2}, \dots, \chi_{pp}\} \end{bmatrix}.$$


- **STEP 6:** If  $\|V^{s+1} - V^s\| < \varepsilon$ , stop and report success. Otherwise, go back to the previous step, **STEP 5**, replacing  $s = s + 1$ .



- **STEP 7:** Being thorough.



- Check the bounds of the state space are not binding. Otherwise, relax them.
- Make sure  $\varepsilon$  is small enough. Reduce  $\varepsilon$ , redo the VFI, and check your answer does not change. Otherwise, keep reducing  $\varepsilon$ .
- Make sure the size of the grid,  $p$ , is large enough. Increase  $p$ , redo the VFI, and check your answer does not change. Otherwise, keep increasing  $p$ .

- **Remark.** Once we know  $V^s$ , the decision rule is obtained as

$$g^s = \begin{bmatrix} \arg \max\{\chi_{11}, \chi_{12}, \dots, \chi_{1p}\} \\ \arg \max\{\chi_{21}, \chi_{22}, \dots, \chi_{2p}\} \\ \dots\dots\dots \\ \arg \max\{\chi_{p1}, \chi_{p2}, \dots, \chi_{pp}\} \end{bmatrix}.$$

That is,  $g^s$  is a column vector attaining values  $g_i^s \in \{1, \dots, p\}$  that state the column number  $j$  that maximizes row  $i$ .

- **Remark.** It is going to be useful to store the decision rule at each iteration  $s$  for the purposes of implementing some of the speeding-up algorithms that we will see next.
- **Remark.** We can then find the optimal sequence of capital for a given initial capital.



# Speeding-up the VFI Algorithm

- We explore four ways of speeding up the VFI algorithm,
  - Using the monotonicity of the optimal decision rule.
  - Using the concavity of the value function.
  - Local search.
  - Howard's policy iteration step.

- Here we pursue to reduce the number of points to be searched on the grid by the VFI algorithm.
- To do so, we use the known property that the optimal decision rule increases in  $K$ . Therefore, if  $k_j > k_i$ , then  $g(k_j) \geq g(k_i)$ .<sup>2</sup>
- Hence, if we want to find  $g(k_j)$  when  $k_j > k_i$ , we can rule out searching for all grid values of capital that are smaller than  $g(k_i)$ .
- This means we can modify **STEP 5.1** in the VFI.

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<sup>2</sup>When strictness applies the equality arises if grids are too loose.

- **Replace STEP 5.1**

- For each  $k_i$ , set an optimal lower bound for  $g^s(k_i)$  as

$$\underline{g^s(k_i)} = g^s(k_{i-1}).$$

- Compute the matrix  $\chi$  only for the pairs  $(k_i, k_j)$  that satisfy the lower bound,  $k_j \geq \underline{g^s(k_i)}$ .
- That is, not all elements in  $\chi$  need to be computed.
- Note that you will have to store the decision rule at the end of each iteration to implement this improvement.

- We use the known property that the maximand in the Bellman equation,  $M_{i,j} + \beta V_j$ , is strictly concave in  $k'$ .
- Therefore, if  $M_{i,j} + \beta V_j > M_{i,j+1} + \beta V_{j+1}$ , then  $M_{i,j} + \beta V_j > M_{i,j+2} + \beta V_{j+2}$ .
- This means we can modify **STEP 5.1** in the VFI to improve the efficiency of the algorithm.

- **Replace STEP 5.1**

- When computing the elements of the matrix  $\chi$  apply the following loop to each row  $i$ ,

do  $j = \{1, p\}$

$$\chi_{i,j} = M_{i,j} + \beta V_j$$

if  $\chi_{i,j-1} > \chi_{i,j}$ , stop & report the optimal value is  $\chi_{i,j-1}$ .  
end do

- That is, not all elements in  $\chi$  need to be computed.

- We exploit the property that the optimal decision rule is continuous.
- Hence, if we know that  $k_j = g(k_i)$ , and we have reasonable fine grids, then we hope that  $g(k_{i+1})$  is in a small neighborhood of  $k_j$ . This restricts the elements of  $\chi$  that need to be computed.
- Local search is followed by practitioners as it has been proven useful. But there is no theorem behind this.
- We can make sure that our solution is the one that is not limited to search locally by checking that the bounds that define the neighborhood are not binding at each iteration.
- Note that you will have to store the decision rule at the end of each iteration to implement this improvement.

- This method does not rely on the properties of the optimal value function, or decision rule.
- We can apply a guessed decision rule many times to update the value function many times, without solving for the optimal rule at each iteration.
- We will need to decide how many times,  $n_h$ , to update the value function without updating the decision rule.
- If we have a decision rule that is not far from the optimal one, higher  $n_h$  implies faster convergence, but iterating on the value function while keeping a decision rule obtained in early stages of the VFI might not be that useful.
- Note that you will have to store the decision rule at the end of each iteration to implement this improvement.

**Remark.** Note that the speeding-up algorithms are not exclusive. We can use any combination of them to speed up the VFI.



# An Extension: Adding Labor Supply

**[Your homework]**

# An Extension: The Stochastic Model



- Productivity is subject to a stochastic process. At each period, output is

$$Y_t = z_t f(K_t)$$

and decisions are taken before the realization of  $z_t$  is observed.



- To apply the VFI we assume that  $z_t$  takes a finite number of  $q$  values,  $Z \in (z_1, z_2, \dots, z_q)$ , and follows a first order Markov process that has a square transition matrix  $\Pi_{q \times q}$  with generic element

$$\pi_{l,l'} = \text{Prob}[z_{t+1} = z_{l'} | z_t = z_l],$$

with  $\sum_{l'=1}^q \pi_{l,l'} = 1$ .



A stochastic RCE is a set of functions  $v(k, K, z)$ ,  $c(k, K, z)$ ,  $k'(k, K, z)$ ; prices  $w(K, z)$  and  $r(K, z)$ ; and an aggregate law of motion  $\Gamma(K, z)$  such that,

- 1 Given  $w$ ,  $r$  and  $\Gamma$ , the value function  $v(k, K, Z)$  solves the Bellman equation,

$$v(k, K, z) = \max_{c \geq 0, k'} u(c) + \beta E_Z v(k', K', z') = \max_{c \geq 0, k'} u(c) + \beta \sum_{l'=1}^q \pi_{l, l'} v(k', K', z')$$

such that

$$\begin{aligned} c + i &= w(K, z) + r(K, z)k \\ k' &= i + (1 - \delta)k \\ K' &= \Gamma(K, z) \end{aligned}$$

Associated to the value function we have the set of optimal decision rules  $c(k, K, z)$  and  $k'(k, K, z)$ .

- 2 Factor prices equate marginal products,  $r(K, z) = zf'(K)$  and  $w(K, z) = zf(K) - zf'(K)K$ .
- 3 Market clearing,  $zf(K) = c(k, K, z) + i(k, K, z)$ .
- 4 Aggregate consistency,  $\Gamma(K, z) = k'(K, K, z)$ .



**Remark.** To obtain the optimal time paths for we need to simulate a path for the productivity shocks.

- **Remark.** Welfare theorems hold. The stochastic recursive PO equilibrium as a set of functions  $v(k, z)$ ,  $c(k, z)$ ,  $k'(k, z)$  that solve

$$v(k, z) = \max_{c \geq 0, k'} u(c) + \beta E_Z v(k', z') = \max_{c \geq 0, k'} u(c) + \beta \sum_{l'=1}^q \pi_{l, l'} v(k', z') \quad (4)$$

such that

$$\begin{aligned} c + i &= zf(k) \\ k' &= i + (1 - \delta)k \end{aligned}$$

That is, the PO problem is


$$v(k) = \max_{k' \in [0, zf(k) + (1 - \delta)k]} u(zf(k) + (1 - \delta)k - k') + \beta \sum_{l'=1}^q \pi_{l, l'} v(k', z') \quad (5)$$



- **STEP 1:** Define a grid for the values of capital  $k_i$ , that is,  $K = (k_1, k_2, \dots, k_i, \dots, k_p)$  and a grid for the values of the productivity shock  $z_i$ ,  $Z = (z_1, z_2, \dots, z_q)$ .



- **STEP 2:** Define the matrix  $M$  as follows,



$$M_{pq \times p} = \begin{bmatrix} m(k_1, k_1, z_1) & m(k_1, k_2, z_1) & \dots & m(k_1, k_p, z_1) \\ m(k_2, k_1, z_1) & m(k_2, k_2, z_1) & \dots & m(k_2, k_p, z_1) \\ \dots & \dots & \dots & \dots \\ m(k_p, k_1, z_1) & m(k_p, k_2, z_1) & \dots & m(k_p, k_p, z_1) \\ m(k_1, k_1, z_2) & m(k_1, k_2, z_2) & \dots & m(k_1, k_p, z_2) \\ m(k_2, k_1, z_2) & m(k_2, k_2, z_2) & \dots & m(k_2, k_p, z_2) \\ \dots & \dots & \dots & \dots \\ m(k_p, k_1, z_2) & m(k_p, k_2, z_2) & \dots & m(k_p, k_p, z_2) \\ \dots & \dots & \dots & \dots \\ m(k_1, k_1, z_q) & m(k_1, k_2, z_q) & \dots & m(k_1, k_p, z_q) \\ m(k_2, k_1, z_q) & m(k_2, k_2, z_q) & \dots & m(k_2, k_p, z_q) \\ \dots & \dots & \dots & \dots \\ m(k_p, k_1, z_q) & m(k_p, k_2, z_q) & \dots & m(k_p, k_p, z_q) \end{bmatrix}.$$

The generic element in matrix  $M$  contains the return function

$$M_{p(l-1)+i,j} = m(k_i, k_j, z_l) = u(z_l f(k_i) + (1 - \delta)k_i - k_j)$$

evaluated at all possible combinations  $k_i \in K$ ,  $k_j \in K$ , and  $z_l \in Z$ .

- **STEP 3:** However, we know that not all values for  $c$ , hence  $k'$ , are feasible. The nonnegativity restriction on consumption implies,

$$M_{p(l-1)+i,j} = m(k_i, k_j, z_l) = \omega$$

if  $k_j > z_l f(k_i) + (1 - \delta)k_i$ , where  $\omega$  is a very negative number.



- **STEP 4:** Guess a solution  $V^{s=0} \in R^{pq}$ , say the null vector, that is,  $V^{s=0}(k_i, z_l) = 0$  for all  $i = \{1, \dots, p\}$  and  $l = \{1, \dots, q\}$ .  $s$  keeps track of the number of iterations.



- **STEP 5:** Given the vector  $V^s = (V_1^s, V_2^s, \dots, V_{pq}^s)$ , with generic element

$$V_{p(l'-1)+j}^s = V^s(k_j, z_{l'}),$$

compute the following in-steps:

- **Step 5.0:** Compute the matrix  $W_{q \times p}^s$ ,

$$\begin{bmatrix} \sum_{l'=1}^q \pi_{1,l'} V_{p(l'-1)+1}^s & \sum_{l'=1}^q \pi_{1,l'} V_{p(l'-1)+2}^s & \dots & \sum_{l'=1}^q \pi_{1,l'} V_{p(l'-1)+p}^s \\ \sum_{l'=1}^q \pi_{2,l'} V_{p(l'-1)+1}^s & \sum_{l'=1}^q \pi_{2,l'} V_{p(l'-1)+2}^s & \dots & \sum_{l'=1}^q \pi_{2,l'} V_{p(l'-1)+p}^s \\ \dots & \dots & \dots & \dots \\ \sum_{l'=1}^q \pi_{q,l'} V_{p(l'-1)+1}^s & \sum_{l'=1}^q \pi_{q,l'} V_{p(l'-1)+2}^s & \dots & \sum_{l'=1}^q \pi_{q,l'} V_{p(l'-1)+p}^s \end{bmatrix},$$

with generic element

$$W_{l,j}^s = \sum_{l'=1}^q \pi_{l,l'} V_{p(l'-1)+j}^s$$



- ... (continued)

- **Step 5.1:** Compute the matrix  $\chi_{pq \times p}$ ,

Notice we have block structure

$$\begin{bmatrix} M_{11} + \beta W_{11}^s, & M_{12} + \beta W_{12}^s, & \dots, & M_{1p} + \beta W_{1p}^s \\ M_{21} + \beta W_{11}^s, & M_{22} + \beta W_{12}^s, & \dots, & M_{2p} + \beta W_{1p}^s \\ \dots & \dots & \dots & \dots \\ M_{p1} + \beta W_{11}^s, & M_{p2} + \beta W_{12}^s, & \dots, & M_{pp} + \beta W_{1p}^s \\ M_{p+1,1} + \beta W_{21}^s, & M_{p+1,2} + \beta W_{22}^s, & \dots, & M_{p+1,p} + \beta W_{2p}^s \\ M_{p+2,1} + \beta W_{21}^s, & M_{p+2,2} + \beta W_{22}^s, & \dots, & M_{p+2,p} + \beta W_{2p}^s \\ \dots & \dots & \dots & \dots \\ M_{p+p,1} + \beta W_{21}^s, & M_{p+p,2} + \beta W_{22}^s, & \dots, & M_{p+p,p} + \beta W_{2p}^s \\ \dots & \dots & \dots & \dots \\ M_{p(q-1)+1,1} + \beta W_{q1}^s, & M_{p(q-1)+1,2} + \beta W_{q2}^s, & \dots, & M_{p(q-1)+1,p} + \beta W_{qp}^s \\ M_{p(q-1)+2,1} + \beta W_{q1}^s, & M_{p(q-1)+2,2} + \beta W_{q2}^s, & \dots, & M_{p(q-1)+2,p} + \beta W_{qp}^s \\ \dots & \dots & \dots & \dots \\ M_{p(q-1)+p,1} + \beta W_{q1}^s, & M_{p(q-1)+p,2} + \beta W_{q2}^s, & \dots, & M_{p(q-1)+p,p} + \beta W_{qp}^s \end{bmatrix},$$

with the generic element

$$\chi_{p(l-1)+i,j} = M_{p(l-1)+i,j} + \beta W_{l,j}^s$$

- ... (continued)

- **Step 5.2:** Compute the updated value function  $V^{s+1}$  as the maximum element in each row of  $\chi$ ,


$$V^{s+1} = \begin{bmatrix} \max\{\chi_{1,1}, & \chi_{1,2}, & \dots, & \chi_{1,p}\} \\ \max\{\chi_{2,1}, & \chi_{2,2}, & \dots, & \chi_{2,p}\} \\ \dots, & \dots, & \dots, & \dots \\ \max\{\chi_{p,1}, & \chi_{p,2}, & \dots, & \chi_{p,p}\} \\ \max\{\chi_{p+1,1}, & \chi_{p+1,2}, & \dots, & \chi_{p+1,p}\} \\ \max\{\chi_{p+2,1}, & \chi_{p+2,2}, & \dots, & \chi_{p+2,p}\} \\ \dots, & \dots, & \dots, & \dots \\ \max\{\chi_{p+p,1}, & \chi_{p+p,2}, & \dots, & \chi_{p+p,p}\} \\ \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \max\{\chi_{p(q-1)+1,1}, & \chi_{p(q-1)+1,2}, & \dots, & \chi_{p(q-1)+1,p}\} \\ \max\{\chi_{p(q-1)+2,1}, & \chi_{p(q-1)+2,2}, & \dots, & \chi_{p(q-1)+2,p}\} \\ \dots, & \dots, & \dots, & \dots \\ \max\{\chi_{p(q-1)+p,1}, & \chi_{p(q-1)+p,2}, & \dots, & \chi_{p(q-1)+p,p}\} \end{bmatrix}.$$

- **STEP 6:** If  $\|V^{s+1} - V^s\| < \epsilon$ , stop and report success. Otherwise, go back to the previous step, **STEP 5**, replacing  $s = s + 1$ .

- **Remark.** We can store the decision rule as

$$g^s = \begin{bmatrix} \arg \max\{\chi_{1,1}, & \chi_{1,2}, & \dots, & \chi_{1,p}\} \\ \arg \max\{\chi_{2,1}, & \chi_{2,2}, & \dots, & \chi_{2,p}\} \\ \dots, & \dots, & \dots, & \dots \\ \arg \max\{\chi_{p,1}, & \chi_{p,2}, & \dots, & \chi_{p,p}\} \\ \arg \max\{\chi_{p+1,1}, & \chi_{p+1,2}, & \dots, & \chi_{p+1,p}\} \\ \arg \max\{\chi_{p+2,1}, & \chi_{p+2,2}, & \dots, & \chi_{p+2,p}\} \\ \dots, & \dots, & \dots, & \dots \\ \arg \max\{\chi_{p+p,1}, & \chi_{p+p,2}, & \dots, & \chi_{p+p,p}\} \\ \dots, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \arg \max\{\chi_{p(q-1)+1,1}, & \chi_{p(q-1)+1,2}, & \dots, & \chi_{p(q-1)+1,p}\} \\ \arg \max\{\chi_{p(q-1)+2,1}, & \chi_{p(q-1)+2,2}, & \dots, & \chi_{p(q-1)+2,p}\} \\ \dots, & \dots, & \dots, & \dots \\ \arg \max\{\chi_{p(q-1)+p,1}, & \chi_{p(q-1)+p,2}, & \dots, & \chi_{p(q-1)+p,p}\} \end{bmatrix}.$$

- Explicitly, the optimal value function  $V(k, z)$  and the decision rule  $g(k, z)$  take the form of vectors,

$$V = \begin{bmatrix} v(k_1, z_1) \\ v(k_2, z_1) \\ \dots \\ v(k_p, z_1) \\ v(k_1, z_2) \\ v(k_2, z_2) \\ \dots \\ v(k_p, z_2) \\ \dots \\ v(k_1, z_q) \\ v(k_2, z_q) \\ \dots \\ v(k_p, z_q) \end{bmatrix} \quad g = \begin{bmatrix} g(k_1, z_1) \\ g(k_2, z_1) \\ \dots \\ g(k_p, z_1) \\ g(k_1, z_2) \\ g(k_2, z_2) \\ \dots \\ g(k_p, z_2) \\ \dots \\ g(k_1, z_q) \\ g(k_2, z_q) \\ \dots \\ g(k_p, z_q) \end{bmatrix}$$


- To find the optimal sequence  $k_0, k_1, \dots, k_T$  first we need to simulate a history of realizations of productivity shocks  $z_0, z_1, \dots, z_T$ . To simulate these productivity shocks we use the transition matrix  $\Pi$ .
- Using the simulated series  $z_0, z_1, \dots, z_T$ , and an initial  $k_0$ , we can find the optimal sequence  $k_0, k_1, \dots, k_T$  using the decision rule  $g$ .
- Given this time-series for the productivity shock and capital, we can recover the all time-series in the model, consumption, output and factor prices.
- Therefore, we can compute statistics on variance, correlations, and joint dynamics. Before doing so, we usually filter them using some of the techniques in our previous slides [Data: Some Tools and Sources.pdf].

# Solving the competitive equilibrium directly using VFI

- VFI is fairly standard as a solution method for pareto optimal economies, but not so commonly used for decentralized economies.
- The fact that we have two states,  $k$  and  $K$ , is not the reason. We have seen how to deal with two states earlier in the economy with stochastic productivity shocks.
- The problem of using VFI in decentralized economies is that agents need to know (at least, in expected terms) the aggregate law of motion (i.e., future prices) before optimizing, while out of the VFI we only know this object after solving for the equilibrium.
- One way to circumvent that problem is to add one STEP to the VFI algorithm. What this STEP does is provide a guess  $a$  for the aggregate law of motion (i.e., this is again a function approximation problem). Given that guess for the aggregate law of motion, solve for the decentralized economy using VFI. Once the optimal decision rule is obtained, we compare it to the aggregate law of motion. If they are close, we stop. Otherwise we update the aggregate law of motion to be the last optimal decision rule and start again the VFI.
- There is no guarantee of convergence.



# Some limitations

- While highly applied to economies in which the welfare theorem holds, it is not that easy to solve directly for the competitive equilibrium using value function iteration.
- Increasing the state variables (or the number of points in the grid) implies the dimension of the matrix  $\chi$  will grow exponentially making the algorithm very time-consuming.
- We can only introduce shocks of a particular structure, primarily, Markov processes of order one.

# Discretization of exogenous processes

- 1 Tauchen (1986) procedure to discretize an AR(1) stochastic process. It has a multivariate version, i.e., a VAR (Tauchen and Hussey, 1991).
- 2 Rouwenhorst (1995) for the case where the persistence parameter is close to 1. See Kopecky and Suen (2010).

# Discretization of exogenous processes

## Tauchen Method

Solving the growth model with a continuous shock can be done, but it is very costly, computationally.

So it's useful to learn how to approximate a continuous process through a finite-state Markov chain that will mimic closely the underlying process.

Goal: Discretize an AR(1) stochastic process

$$z_t = \rho z_{t-1} + \varepsilon_t$$

with stationary distribution  $N(0, \sigma_z^2)$ , where  $\sigma_z = \frac{\sigma_\varepsilon}{\sqrt{1-\rho^2}}$

This is a typical example for income processes that are assumed  $Y_t = \exp(y_t)$  where  $y_t$  follows the AR(1) of the class posed above.

To approximate the AR(1) process above we need two ingredients: the points on the state space and the transition probabilities.

**Example:** If the Markov Chain we want to use is a simple 2-state chain then a simple method of moments can be used. Symmetry in the original process implies that the Markov chain for log-income can be written as the vector  $[-z, +z]$  and that the transition probability matrix has the form,

$$\begin{bmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{bmatrix}$$

so we simply need to find the value for parameters  $z$  and  $\pi$  in the MC.

These can be found by equating the variance and the serial correlation in the AR(1) and in the Markov chain (MC), i.e.,

$$\sigma_{AR(1)}^2 \equiv \frac{\sigma_\varepsilon^2}{1-\rho^2} = z^2 \equiv \sigma_{MC}^2$$

$$E_{AR(1)}(y_t y_{t-1}) \equiv \rho \frac{\sigma_\varepsilon^2}{1-\rho^2} = (2\pi - 1)z^2 \equiv E_{MC}(y_t y_t - 1)$$

which implies

$$z = \sqrt{\frac{\sigma_\varepsilon^2}{1-\rho^2}} \quad \text{and} \quad \pi = \frac{1+\rho}{2}$$

The Tauchen method is more general.

## Tauchen Method (1986)

- Step 1 Set  $n$ , the number of potential realizations of the process  $z$  (usually 5 or 6 works well).
- Step 2 Set the upper  $z_{max}$  and lower  $z_{min}$  bounds for the process. An intuitive way, appealing to the symmetry of the normal distribution, is to pick  $m$  and set:

$$z_{max} = m\sigma_z$$

$$z_{min} = -m\sigma_z,$$

where usual values of  $m$  are 2 and 3.

- Step 3 Set  $\{z_i\}_{i=1}^n$  such that:  $z_i = z_{min} + \frac{z_{max} - z_{min}}{n-1}(i-1)$  and construct the midpoints  $\{\tilde{z}\}_{i=1}^{n-1}$ , which are given by  $\tilde{z}_i = \frac{z_{i+1} + z_i}{2}$ .
- Step 4 The transition probabilities  $\pi_{ij} \in \Pi_{z, z'}$ , i.e., the probability of going to state  $z_j$  conditional on being on state  $z_i$ , is computed according to:

$$\pi_{ij} = \Phi\left(\frac{\tilde{z}_j - \rho z_i}{\sigma}\right) - \Phi\left(\frac{\tilde{z}_{j-1} - \rho z_i}{\sigma}\right) \quad j = 2, 3, \dots, n-1$$

$$\pi_{i1} = \Phi\left(\frac{\tilde{z}_1 - \rho z_i}{\sigma}\right)$$

$$\pi_{in} = 1 - \Phi\left(\frac{\tilde{z}_{n-1} - \rho z_i}{\sigma}\right)$$

where  $\Phi(\cdot)$  denotes a CDF of a  $N(0, 1)$ .

For example, assume that we estimated a stochastic process to be:

$$z_t = 0.95z_{t-1} + \varepsilon_t \text{ with } \varepsilon_t \sim N(0, 0.0007^2)$$

and we want to approximate it using a 5-point Markov chain and  $m = 3$ .

Tauchen's procedure implies the following grid for  $z_t$ ,

$$z_t \in \{-0.0673, -0.0336, 0, 0.0336, 0.0673\}$$

and the following transition matrix:

$$\Pi_{z,z'} = \begin{bmatrix} 0.9727 & 0.0273 & 0 & 0 & 0 \\ 0.0041 & 0.9806 & 0.0153 & 0 & 0 \\ 0 & 0.0082 & 0.9837 & 0.0082 & 0 \\ 0 & 0 & 0.0153 & 0.9806 & 0.0041 \\ 0 & 0 & 0 & 0.0273 & 0.9727 \end{bmatrix}$$

Note how the entries in the diagonal are close to 1 (the persistence of the continuous stochastic process is high).

[Credit for this example goes to the HdBook chapter by Fernandez-Villaverde, Rubio-Ramirez and Schorfheide]

**[Homework:]** Assume that  $z_t$  is a stochastic process for household income that follows the Markov chain that we have computed in the previous example:

- How many periods does it take to go from the lowest quintile to the highest?
- Does the monotone mixing condition hold in this case?
- What is a stationary distribution? Does it exist in this case? If yes, compute it?
- What are useful statistics for income mobility?



# Discretization of exogenous processes

## Tauchen for Multivariate Processes

Tauchen describes how to approximate also a multivariate process of the form

$$y_t = Ay_{t-1} + \varepsilon_t$$

where  $y_t$  is now a  $n \times 1$  vector,  $A$  is a  $n \times n$  matrix and  $\varepsilon_t$  is a  $n \times 1$  vector of iid random variables with mean 0 and variance-covariance matrix  $\Sigma$ .

The key insight in representing such a multivariate process is to recognize that the variance covariance matrix (which is symmetric) can be represented as

$$\Sigma = Q\Lambda Q'$$

where  $\Lambda$  is a diagonal matrix and  $Q$  is the matrix of eigenvectors of  $\Sigma$  as columns (remember that since  $\Sigma$  is symmetric  $Q' = Q^{-1}$ ) so we can write the process as,

$$\begin{aligned}y_t &= Ay_{t-1} + \varepsilon_t \\Q'y_t &= Q'AQQ'y_{t-1} + Q'\varepsilon_t \\ \tilde{y}_t &= \tilde{A}\tilde{y}_{t-1} + \tilde{\varepsilon}_t\end{aligned}$$

where  $\tilde{y}_t = Q'y_t$  and  $\tilde{A} = Q'AQ$  and  $\tilde{\varepsilon}_t$  has a diagonal variance-covariance matrix  $Q'\Sigma Q = \Lambda$ .

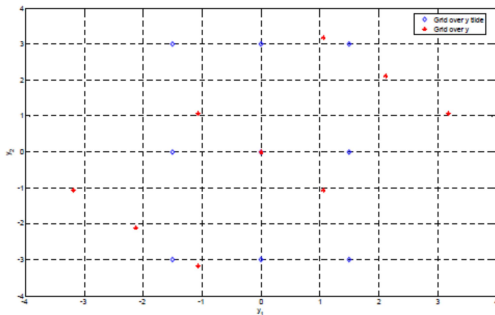
We can then approximate  $\tilde{y}_t$  with a Markov chain (since the  $\tilde{\varepsilon}_t$  are uncorrelated doing so is a straightforward extension of the procedure described for the univariate case) and let  $\tilde{Y}$  be the  $n \times m$  matrix containing the states of the Markov chain which approximate  $\tilde{y}_t$  (here we assume that the Markov chain has the same number of states ( $m$ ) for each of the  $n$  variables). Then, the states of the Markov chain which approximate  $y_t$  are simply given by the matrix  $Y = Q\tilde{Y}$ .

As an example consider a variance of a bivariate iid process  $y_t = \varepsilon_t$  where  $\varepsilon_t$  has a variance-covariance matrix given by:

$$\Sigma = \begin{bmatrix} 1.0 & 0.8 \\ 0.8 & 1.0 \end{bmatrix} = \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 0.2 & 0.0 \\ 0.0 & 0.8 \end{bmatrix} \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

[Credit for this example goes to Perri's lecture notes]

Using this transformation the figure plots the grids for  $y_t$  and for  $\tilde{y}_t$ :



Note that this strategy that can be used to approximate higher-order autoregressive processes as well. For example, consider the AR(2) process:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t$$

Define a column vector  $Z_t = [y_t \ y_{t-1}]'$ . Then one can write the AR(2) process as a multivariate process as:

$$Z_t = \begin{bmatrix} \rho_1 & \rho_2 \\ 0 & 1 \end{bmatrix} Z_{t-1} + [\varepsilon_t \ 0]'$$

# Discretization of exogenous processes

Rouwenhorst (1995)

Kopecky and Suen (2010) show that an alternative procedure proposed by Rouwenhorst (1995) is superior to Tauchens method when , the persistence of the stochastic process, is close to 1.

## Rouwenhorst Method (1995)

- Step 1 Set  $n$ , the number of potential realizations of the process  $z$  (usually 5 or 6 works well).
- Step 2 Set the upper  $z_{max}$  and lower  $z_{min}$  bounds for the process. Let  $z_{min} = -\lambda$  and  $z_{max} = \lambda$ , where  $\lambda$  can be set equal to  $\sqrt{n-1}\sigma_z$ .
- Step 3 Set  $\{z_i\}_{i=1}^n$  such that:  $z_i = z_{min} + \frac{z_{max}-z_{min}}{n-1}(i-1)$ .
- Step 4 When  $n = 2$ , let

$$\Pi_2 = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$

where  $p$  and  $q$  can be set to be  $p = q = \frac{1+p}{2}$ .

- Step 5 When  $n \geq 3$ , we recursively construct the transition matrix:

$$\Pi_n = p \begin{bmatrix} P_{n-1} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix} + (1-p) \begin{bmatrix} \mathbf{0} & P_{n-1} \\ 0 & \mathbf{0}' \end{bmatrix} + (1-q) \begin{bmatrix} \mathbf{0}' & 0 \\ P_{n-1} & \mathbf{0} \end{bmatrix} + q \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & P_{n-1} \end{bmatrix}$$

where  $\mathbf{0}$  is an  $(n-1) \times 1$  column vector of zeros. Divide all but the top and bottom rows by 2 so that the sum of the elements of each row is equal to one. The final outcome is  $\Pi_{z,z'}$ .





- Here we use a continuous method to solve the neoclassical growth model using, again, the value function algorithm (VFI).
- We approximate the value function using a linear combination of a set of  $m$  known linearly independent basis functions  $\psi_j(k)$ ,  $j = 0, \dots, m$


$$\tilde{V}(k) = \sum_{j=0}^m \theta_j \psi_j(k)$$
(6)

with  $m$ -basis coefficients  $\theta_j$  to be determined.

- That is, solving for  $V(k)$  reduces to find the  $\theta_j$  coefficients.

- In our working example, the continuous method that we use to approximate for  $\tilde{V}(k)$  is finite elements. That is, we are going to choose the basis functions to be nonzero over some subintervals of the state space, but not all.<sup>3</sup>
- Precisely, we use the B-spline formulation to approximate  $\tilde{V}(k)$ . This formulation subscribes to (6).<sup>4</sup>
- What follows applies to B-splines of any order.

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<sup>3</sup> Alternatively, we could have chosen the basis functions  $\psi_j(k)$  to be defined over the whole state space (i.e., spectral approximation methods). For example,  $\psi_j(k)$  could be monomial basis, or Chebyshev basis.

<sup>4</sup> Recall from our previous session [Function Approximation], Theorem: every spline function of a given degree, smoothness, and domain partition, can be represented as a linear combination of B-splines of that same degree and smoothness, and over that partition (de Boor 1978).

- For instance, an order-2 spline implements tent function interpolation (piecewise linear), and are spanned by the  $B^1$ -splines.

That is, the typical  $B^1$ -spline is defined by (6) with the following basis functions

$$\psi_j(k) = \begin{cases} \frac{k-k_{j-1}}{k_j-k_{j-1}}, & \text{if } k_{j-1} \leq k \leq k_j \\ \frac{k_{j+1}-k}{k_{j+1}-k_j}, & \text{if } k_j < k \leq k_{j+1} \\ 0, & \text{if } k_{j-1} < k, \text{ or } k > k_{j+1} \end{cases}$$

- Note that  $\psi_j(k)$  is the tent function with peak at  $k_j$ , and value zero below  $k_{j-1}$  and above  $k_{j+1}$ .





- **STEP 1.** Divide the state space into a finite number of  $n$  disjoint intervals. The points at which the spline pieces are connected are called knots, and there are  $n + 1$  of these,



$$k_{min} = k_0 < k_1 < k_2 \dots < k_i < \dots < k_n = k_{max}$$



- **STEP 2.** Guess a solution  $\tilde{V}^s(k)$ , that is, guess  $\{\tilde{\theta}_j^s\}_{j=1}^m$ , where

$$\tilde{V}^s(k) = \sum_{j=0}^m \tilde{\theta}_j^s \psi_j(k)$$

and  $s$  keeps track of the number of iterations. The first iteration is  $s = 0$ .

Note that, given  $\{\tilde{\theta}_j^s\}_{j=0}^m$ , we can evaluate  $\tilde{V}^s(k_i)$  at any value of capital in the state space,  $k \in (k_{min}, k_{max})$ , including the capital knots  $i = \{0, 1, \dots, n\}$ .



- **STEP 3.** For each capital knot  $i = \{0, 1, \dots, n\}$ , find

$$g^s(k_i) = \arg \max_{k' \in [k_{min}, k_{max}]} u(f(k_i) + (1 - \delta)k_i - k') + \beta \tilde{V}^s(k') \quad (7)$$

and store  $\{g^s(k_i)\}_{i=0}^n$ .




To solve for  $k'$  (i.e., find  $g^s(k_i)$ ) in (7) one can:

- implement a numerical optimization algorithm to (7),
- or one can compute the first order condition of (7)—note that we know the value function  $\tilde{V}^s(k')$ , and hence we can take numerical derivatives—and solve for  $k'$  in a univariate nonlinear equation.



- **STEP 4.** Given  $\{g^s(k_i)\}_{i=0}^n$ , update the value function at each  $i = 0, 1, \dots, n$ ,


$$\tilde{V}^{s+1}(k_i) = u(f(k_i) + (1 - \delta)k_i - g^s(k_i)) + \beta \tilde{V}^s(g^s(k_i))$$

and store  $\{\tilde{V}^{s+1}(k_i)\}_{i=0}^n$ .

- **STEP 5.** Given  $\{\tilde{V}^{s+1}(k_i)\}_{i=0}^n$ , solve for the  $m$ -basis coefficients  $\{\theta_j^{s+1}\}_{j=0}^m$  in

$$\tilde{V}^{s+1}(k_i) = \sum_{j=0}^m \theta_j^{s+1} \psi_j(k_i) \quad \forall i = \{0, \dots, n\} \quad (8)$$

That is, (8) is a nonlinear system with  $n + 1$  equations and  $m$  unknowns.

Two plausible options can arise:

- If  $m = n$ , solving for  $\{\theta_j^{s+1}\}_{j=0}^m$  in (8) implies that, in practice, we are implementing a weighted residuals method under collocation, and the solution to the nonlinear system (8) exists and it is hopefully unique.
- We can choose to have more knots than basis functions, that is,  $n > m$ , hence more equations,  $\{k_i\}_{i=0}^n$ , than unknowns,  $\{\theta_j^{s+1}\}_{j=0}^m$ . In this case we can solve for  $\{\theta_j^{s+1}\}_{j=0}^m$  in (8) by regression, that is, implementing the weighted residuals method under least squares.



- **STEP 6:** If  $\|\theta^{s+1} - \theta^s\| < \varepsilon$ , stop and report success. Otherwise, go back to the previous step, **STEP 3**, replacing  $s = s + 1$ .

- **STEP 7:** Being thorough.
  - Check the bounds of the state space are not binding. Otherwise, relax them.
  - Make sure  $\varepsilon$  is small enough. Reduce  $\varepsilon$ , redo the VFI, and check your answer does not change. Otherwise, keep reducing  $\varepsilon$ .
  - Make sure the number of knots for capital,  $n$ , is large enough. Increase  $n$ , redo the VFI, and check your answer does not change. Otherwise, keep increasing  $n$ .
  - Make sure the number of basis functions,  $m$ , is large enough. Increase  $m$ , redo the VFI, and check your answer does not change. Otherwise, keep increasing  $m$ .

Another nice example that uses continuous methods to solve the neoclassical growth model is in Section 5.6 of the Chapter by Fernández-Villaverde, Rubio-Ramírez and Schorfheide in the Handbook of Macroeconomics 2016.

In that case the authors approximate for both the value function and (inelastic) labor supply. With inelastic labor supply the Bellman equation is not enough to solve the problem, we need to use the intratemporal FOC to pin down the optimal labor function as well.

# More tricks: EGM and Precomputation of Integrals

- (1) We can implement the endogenous grid method (EGM) to value function iteration (Carroll 2006, Barillas and Fernandez-Villaverde, 2007, and Fella, 2014)
- (2) Precomputation of integrals (Judd, Maliar, Maliar and Tsener, 2016): Construct conditional expectations functions in the initial stage of a solution procedure.