TA Structural Empirical Methods for Labour Economics

Tutorial 3: Chapter 3

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Preliminaries

- Remarks on problem set 2
- Problem set to come
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Step 2.2: Fixed Point Algorithm

1. Make an initial guess for g
2. Compute $h = \frac{1}{1-g}$
3. With this h compute $g_{new} = \left(\frac{h_R - h}{h^{\psi}}\right)^{\frac{1}{\eta}}$
4. Check for tolerance and iterate until convergence.

Problems and hints:

- Remember you have to do this for each education level, so you will get 2 different values of g depending on the education choice.
- The algorithm might fail to converge. If that is the case include an intermediate step between step 3 and 4:

$$g_{new} = \frac{mg - g_{g\,new}}{m - 1} \tag{1}$$

where m is a negative number. See paper for understanding.

Step 2.3 & 2.4: Estimate the parameters

Using the previous fixed point algorithm we will now estimate the parameters. We will minimize

$$\sum_i \sum_a (W_{i,a}^{data} - W_a(\eta_e, \phi_e, r_{et}, h_R))^2$$

The procedure is the following:

1. Make initial guess for η and ψ.
2. Solve for the fixed point in g.
3. Simulate data using the wage equation.
4. Move in the direction that minimizes the difference between observed and simulated wages to obtain a new guess.
5. Update the guess and repeat until convergence.

Model preliminaries

- Maintenance of biking company in Barcelona
- Bikes can be either replaced or maintained
- State variable: a_t - age of the bike
- $a_{t+1} = 1$ in case of replacement
- $a_{t+1} = a_t + 1$ otherwise
- If the bike is A years old, it is replaced no matter what

Utility

$$u(a_t, d_t, \epsilon_t) = \begin{cases} -\theta_R + \epsilon_{1t} & \mathit{if}\ \ d_t = 1 \\ -\theta_{M1} a_t - \theta_{M2} a_t^2 + \epsilon_{0t} & \mathit{if}\ \ d_t = 0 \end{cases}$$

where e_{0t} and e_{1t} are unobserved by the econometrician, iid with all Rust assumptions

Conditional Value Functions

- conditional value function given our assumptions:

$$v_{jt}(a) = u_{jt}(a_t) + \beta \int \ln(\textstyle\sum_{h \in D} exp\{v_{ht+1(a)}\}) F_a(a_{t+1}|a_t, j)$$

- in our model, given 5 possible states ($A = 5$) we have:

$$v_{1t} = -\theta_R + \beta V_t(1)$$

$$v_{0t}(1) = -\theta_{M1} - \theta_{M2} + \beta V_t(2)$$

$$v_{0t}(2) = -2\theta_{M1} - 4\theta_{M2} + \beta V_t(3)$$

$$v_{0t}(3) = -3\theta_{M1} - 9\theta_{M2} + \beta V_t(4)$$

$$v_{0t}(4) = -4\theta_{M1} - 16\theta_{M2} + \beta V_t(5)$$

Value Functions

- given Type-I Extreme Value assumption *Emax* can be written as a function of $v_{jt}(at)$:

  $$V_t(a) = \ln(\textstyle\sum_{j \in D} exp\{v_{jt}(a)\}$$

- in our case, depending on the state a_t:

  $$V_t(1) = \ln(exp\{v_{1t}\} + exp\{v_{0t}(1)\})$$

  $$V_t(2) = \ln(exp\{v_{1t}\} + exp\{v_{0t}(2)\})$$

  $$V_t(3) = \ln(exp\{v_{1t}\} + exp\{v_{0t}(3)\})$$

  $$V_t(4) = \ln(exp\{v_{1t}\} + exp\{v_{0t}(4)\})$$

  $$V_t(5) = -\theta_R + \beta V_{t+1}(1)$$

Probabilities

- given Type-I Extreme Value assumption, the conditional choice probabilities $p_{jt}(at)$ are defined as:

$$p_{jt}(a_t) = E[d^*_{jt}|a_t] = \frac{e^{v_{jt}(a_t)}}{\sum_{h \in D} e^{v_{ht}(a_t)}}$$

- in considered model, depending on decision j we get:

$$p_{1t}(a_t) = \frac{e^{v_{1t}(a_t)}}{e^{v_{1t}(a_t)} + e^{v_{0t}(a_t)}} = \frac{e^{v_{1t} - v_{0t}(a_t)}}{1 + e^{v_{1t} - v_{0t}(a_t)}}$$

$$p_{0t}(a_t) = \frac{e^{v_{0t}(a_t)}}{e^{v_{1t}(a_t)} + e^{v_{0t}(a_t)}} = \frac{1}{1 + e^{v_{1t} - v_{0t}(a_t)}}$$

Log-Likelihood

- in the general framework, a full information maximum likelihood function is:

$$\mathcal{L}_N(\theta) = \sum_{i=1}^{N} lnPr(d_{it}, ..., d_{iT}, a_{it}, ..., a_{iT}; \theta) = \sum_{i=1}^{N} l_i(\theta)$$

- what, given the assumptions, can be factorized as:

$$l_i(\theta) = \sum_{t=1}^{T} lnPr(d_{it}|a_{it}; \theta) + lnPr(ait|a_{it} - 1, d_{it} - 1; \theta) + lnPr(a_{i1}; \theta)$$

- in out example, the log-likelihood function takes form:

$$\mathcal{L}_N = \sum_{t=1}^{T} \sum_{i=1}^{N} d_{it} ln(p_{1t}(a_t) + (1 - d_{it}) ln(p_{0t}(a_t))$$

Infinite Horizon. Steps

Remember that we have a **finite number of states** $at = 1, ..., 5$, however an **infinite time horizon** ($T = \text{inf}$).

Steps:

1. Download the data and **formulate the dynamic programming problem** (conditional value functions $v_{jt}(xt)$ and $Emax\ V_t(x)$)
2. Find **fixed points** of each value function - Value Function Iteration
3. Formulate conditional choice probabilities
4. Construct **log-likelihood** function
5. Solve the maximization problem

Conditional choice probabilities

For each state a and decision $j = 0, 1$ formulate the conditional choice probabilities.

Take first period $t = 1$:

$\mathcal{L}_N(t = 1) = \sum_{i=1}^{N} d_{i1} ln(p_{11}(1) + (1 - d_{i1}) ln(p_{01}(1))$

where:

$p_{11}(1) = \frac{e^{v_{11} - v_{01}(1)}}{1 + e^{v_{11} - v_{01}(1)}} = 1 - \frac{1}{1 + e^{v_{11} - v_{01}(1)}}$

$p_{01}(1) = \frac{1}{1 + e^{v_{11} - v_{01}(1)}}$

$v_{11} - v_{01}(1) = [-\theta_r + \beta V_2(1)] - [-\theta_{M1} - \theta_{M2} + \beta V_2(2)]$

Finite Horizon. Steps

We have a **finite number of states** $at = 1, 2, 3$, and a **finite time horizon** ($T = 3$).

Steps:

1. Download the data and **formulate the dynamic programming problem** (conditional value functions $v_{jt}(xt)$ and $Emax\ V_t(x)$)
2. Use backwards induction
3. Formulate conditional choice probabilities
4. Construct **log-likelihood** function
5. Solve the maximization problem

Backwards Induction

- imagine a **finite horizon** version of our example ($T = 3$, $A = 3$)
- given 3 possible states and $t = 1, 2, 3$ we have:

$$v_{13}(3) = v_{1t} = v_1 = -\theta_R$$

$$v_{02}(2) = -2\theta_{M1} - 4\theta_{M2} + \beta v_{13}(3) = -2\theta_{M1} - 4\theta_{M2} + \beta(\theta_R)$$

$$v_{01}(1) = -\theta_{M1} - \theta_{M2} + \beta ln(exp\{v_{02}(2)\} + exp\{v_{12}\} =$$
$$-\theta_{M1} - \theta_{M2} + \beta(exp\{-2\theta_{M1} - 4\theta_{M2} + \beta(-\theta_R)\} + exp\{-\theta_R\})$$

Conditional choice probabilities

For each state a and decision $j = 0, 1$ formulate the conditional choice probabilities.

Take

second period $t = 2$:

$\mathcal{L}_N(t = 2) = \sum_{i=1}^{N} d_{i2} ln(p_{12}(1) + (1 - d_{i2}) ln(p_{02}(2))$

where:

$p_{12}(2) = \frac{e^{v_{12} - v_{02}(2)}}{1 + e^{v_{12} - v_{02}(2)}} = 1 - \frac{1}{1 + e^{v_{12} - v_{02}(2)}}$

$p_{02}(1) = \frac{1}{1 + e^{v_{12} - v_{02}(1)}}$

$v_{12} - v_{02}(2) = [-\theta_R] - [-2\theta_{M1} - 4\theta_{M2} + \beta(-\theta_r)]$

Nested Fixed Point Algorithm

The algorithm is composed by an inner loop and an outer loop.

- **Inner loop**: For every value of $\boldsymbol{\theta}$ solves the fixed point of the dynamic problem.
- **Outer loop**: Iterates over $\hat{\theta}$ to maximize the log-likelihood of the sample.

For the PS the outer loop could be done by just using *fminunc*.

Inner loop

Given a guess of $\boldsymbol{\theta}$ solves the fixed point of the dynamic problem.

Steps:

1. Make an initial guess for EV
2. Given θ and EV compute the conditional value functions $v_j(x)$ at all possible states.
3. Using the following formula update your guess for EV:

$$EV(x) = \ln \sum_{j \in D} exp\{v_j(x)\}$$

4. Check for tolerance and repeat until convergence.

How to code the algorithm

The main part is to write a function that does the following:

- Takes θ and the data as inputs.
- For every value of θ solves the fixed point using the inner loop.
- Computes the conditional choice probabilities.
- Returns the likelihood value.

Once we have such a function we can maximize it using *Matlab fminuc* function for example.

Problem Set

Steps of NFPA in Problem Set

1. Estimate $\hat{\varphi}_{\mathbf{x}} = (\hat{\varphi}_0, \hat{\varphi}_1, \hat{\varphi}_2)$ - parameters of the Markov transition probabilities for mileage x
   $\hat{\varphi}_j = Pr(x_{t+1} = x_t + j | x_t, d_t = 0), \; j \in \{0, 1, 2\}$
   in Rust (1987) the parameters are $(\theta_{30}, \theta_{31}, \theta_{32})$
2. Estimate $(\hat{\theta_{\mathbf{U}}}) = (\hat{\theta_R}, \hat{\theta_M})$ using transition matrix $F^0_{x_t, x_{t+1}}$ constructed from $\hat{\varphi}_j$ and log-likelihood function :

   $l_i(\hat{\theta_U}, \hat{\varphi}_{\mathbf{x}}) = \sum_{t=1}^{T} d_{it} ln(p_{1t}(a_t) + (1 - d_{it}) ln(p_{0t}(a_t))$
3. * Single iteration for the full likelihood optimization (N-R or BHHH) using $(\theta_{\mathbf{U}}', \hat{\varphi}_{\mathbf{x}}')$'

Problem set 3

- Problem set 3 handed out today
- Easy replication and understandable code by comments is key here!
- Deadline: **15.02.2021** (see problem set on how to submit files)
- Optimize your time spend w.r.t. my email responsiveness on weekends