Ур. в с.ч. произ. 1 порядка

Лекция 10 Октября 2021 [illegible]

$F(x_1, x_2 \dots x_n, u, \frac{\partial u}{\partial x_1} \dots \frac{\partial u}{\partial x_n}) = 0$

$u = u(x_1 \dots x_n)$

$F(x, u, \nabla u) = 0 \qquad \nabla u = (\frac{\partial u}{\partial x_1} \dots \frac{\partial u}{\partial x_n})$

$u = u(x)$

$\boxed{\frac{\partial u}{\partial x} = f(x, u)}$ ← дифференциальное ур-е; $u = u(x, y)$

$\int \frac{\partial u}{\partial x} = \int f(x, u)\, \partial x$

$\frac{\partial}{\partial x} F(x) = 0$

$u = F(x, y) + C_1(y)$

$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial C_1}{\partial x}$

$\to f(x, y)$

Линейное ур. с частн. произв.

$\boxed{\sum_{k=1}^{n} \frac{\partial u}{\partial x_k} A_k(x) = B(x) u + f(x)}$

иф. функция

$\underline{A}(x) \cdot \nabla u = B(x) u + f(x)$

1) $B(x) = 0$ – линейное ур-е $f \neq 0$ – полное

2) $f(x) = 0$ – однородное $f \neq 0$ – неоднородное

3) $A_k(x) = a_k = \text{const} \to$ с пост. коэф-тами

$\forall k$

$\exists k: A_k(x) = x\, \text{const} \to$ ур-е с перем. коэф-тами.

4) $\exists k: A_k(x, u)$, $B(x, u)$ или $f(x, u)$ (если перем. коэф-ты нелинейны)

нелин. →

→ квазилинейное

$u = \sum_{k=1}^{n} u_k + u_0$ $\quad u_k$ – частное решение u_0 – частное решение

Решение ур-я хар-х функции

1) Метод Характеристик.

$\boxed{F(u, x) = C}$ – функция хар-ка

Если она удовл. ур. ЛО.

$\boxed{\frac{\partial}{\partial x_k} F} = \frac{\partial F}{\partial u} \frac{\partial u}{\partial x_k} + \frac{\partial F}{\partial x_k} = 0$

но может ар?.

$\frac{\partial u}{\partial x_k} = \boxed{\frac{\frac{\partial F}{\partial x_k}}{\frac{\partial F}{\partial u}}}$

$\boxed{\sum_{k=1}^{n} \frac{\partial u}{\partial x_k} A_k(x) = B(x) u + f}$

Для ур-я разрешаем и имеем n частных решений.

$u \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = 0$

$u = C \quad \boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0}$

$u(x, y)$

$\frac{\partial u}{\partial x} = \frac{x \partial u}{\partial y} \quad \left| \frac{\partial u}{\partial y} \right| \cdot y$

$\frac{\partial u}{\partial x} : \frac{\partial u}{\partial y} = \frac{x}{y}$

$\frac{\partial u}{\partial x} = \frac{x}{y}$

$\int y\, \partial y = \int x\, \partial x$

$y^2 = x^2 + c \quad \boxed{u = C}$

$u = \boxed{x^2 + y^2}$

$\frac{\partial u}{\partial x} = \partial x$

$\frac{\partial u}{\partial y} = \partial y$

$y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 0$

$\partial y\, y - \partial x\, y = 0$

$\boxed{u = x^2 + y^2}$ – решение.

$\frac{y}{dx} - \frac{x}{dy} = 0$

$\boxed{\frac{dx}{y} = \frac{dy}{x}} \quad | \cdot xy$

$\int x\, dx + y\, dy = 0$

$\boxed{x^2 + y^2 = C}$

Метод Характеристик:

$n \to d$

1) Функции φ_1, φ_2 и т.д. называются и интегралами Эйлера

2) $+, = \to =$

$- \to =$

$A_1(x) \frac{\partial u}{\partial x_1} + A_2(x) \frac{\partial u}{\partial x_2} + \dots + A_n(x) \frac{\partial u}{\partial x_n} = [B(x) u + f(x)] \frac{\partial u}{\partial x_n}$

Соотв. хар-е $\frac{dx}{dt}$:

$\boxed{\frac{dx_1}{A_1(x)} = \frac{dx_2}{A_2(x)} = \dots = \frac{dx_n}{A_n(x)}}$ $\quad u = x_{n+1}$ $\quad \frac{du}{B(x) u + f(x)}$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} - u = 0$

$\boxed{\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}}$

Интегр. ур-я

$\begin{cases} F_1(u, x) = C_1 \\ F_n(u, x) = C_n \end{cases}$ ← общее решение

$\varphi(C_1, C_2 \dots C_n) = 0$

↑ ↑ ↑ ↑

$\forall \varphi$ – произвольная ф-я.

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$

Для обыкн. полных ур. с.з.п. $\exists$ ф-я φ перем. коэф-тами.

$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{u}$ $\qquad \frac{dx + dy}{y + x} = \frac{d(x+y)}{x+y}$

1) $\frac{dx}{x} = \frac{dy}{y}$

$\int x\, dx - y\, dy = 0$

$x^2 - y^2 = C_1^2 \to C_1 = \sqrt{x^2 - y^2}$

2) $\int \frac{du}{u} = \int \frac{dx}{\sqrt{x^2 - y^2}}$

$\ln|u| = \ln|x + \sqrt{x^2 - y^2}| + C_2$

$u = C_2 \left(x + \sqrt{x^2 - y^2}\right) = C_2(x + y)$

$\begin{cases} \sqrt{x^2 - y^2} = C_1 \\ \frac{u}{x+y} = C_2 \end{cases}$ – общий интеграл

$\varphi(C_1, C_2) = 0$

$\boxed{\varphi\left(\sqrt{x^2 - y^2}, \frac{u}{x+y}\right)} = 0$ – общ. решение

$0 = \frac{\partial}{\partial x} \varphi = \frac{\partial \varphi}{\partial C_1} \frac{\partial C_1}{\partial x} + \frac{\partial \varphi}{\partial C_2} \frac{\partial C_2}{\partial x} = \frac{\partial \varphi}{\partial C_1} \frac{x}{\sqrt{x^2 - y^2}} + \frac{\partial \varphi}{\partial C_2} \left(\frac{u}{(x+y)^2} \cdot 1 - \frac{1}{x+y}\right) \frac{\partial C_2}{\partial x}$

$0 = \frac{\partial}{\partial y} \varphi = \frac{\partial \varphi}{\partial C_1} \frac{\partial C_1}{\partial y} + \frac{\partial \varphi}{\partial C_2} \frac{\partial C_2}{\partial y} = \frac{\partial \varphi}{\partial C_1} \frac{-y}{\sqrt{x^2 - y^2}} + \frac{\partial \varphi}{\partial C_2} \left(\frac{u}{(x+y)^2} \cdot 1 - \frac{1}{x+y}\right) \frac{\partial C_2}{\partial y}$

$\to \frac{\partial \varphi}{\partial C_1} \frac{1}{\sqrt{x^2 - y^2}} = \frac{u}{(x+y)^2} - \frac{1}{x+y} \frac{\partial \varphi}{\partial C_2}$

$\boxed{\frac{\partial \varphi}{\partial C_1} \frac{1}{\sqrt{x^2 - y^2}} = \frac{u}{x+y} - \ln(x+y) \frac{\partial \varphi}{\partial C_2}}$

$\boxed{\frac{\partial \varphi}{\partial C_1} \frac{1}{\sqrt{x^2 - y^2}} = \frac{u}{(x+y)^2} - \ln(x+y) \frac{\partial \varphi}{\partial C_2}}$

$\boxed{\frac{\partial \varphi}{\partial C_1} \frac{-y}{\sqrt{x^2 - y^2}} = \frac{u}{x+y} + \ln(x+y) \frac{\partial \varphi}{\partial C_2}}$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$ $\qquad \forall \varphi$ удовл. ур-ю

$\boxed{\cancel{\frac{u x}{x+y}} - \cancel{\ln(x+y) \frac{\partial \varphi}{\partial C_2} x} + \cancel{\frac{u y}{x+y}} + \cancel{\ln(x+y) \frac{\partial \varphi}{\partial C_2} y} = u}$

$\frac{u(x+y)}{x+y} = u$

$\boxed{\frac{u}{x+y} = u}$

$\boxed{2}$ $\left(\frac{\partial u}{\partial x} - \ln \frac{\partial u}{\partial y}\right) - (\ln(x+y) + 1) \frac{\partial u}{\partial y} = u$ $\qquad$ общее ур-е

$\frac{u dx}{-dx + x dy} = \frac{u}{x}$

$u x \frac{\partial u}{\partial x} - \ln u x \frac{\partial u}{\partial y} - (\ln(x+y) + 1) \frac{\partial u}{\partial y} = \boxed{u}$

Свойств. ур-ия пост. коэф. $\frac{du}{du^2 - 4u^2 + 4u} = \frac{dx}{-4u^2} = \frac{du}{4u}$

$\to$ ур-е + пост

$\frac{u du + x du}{d(ux)}$ $\qquad \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

1) $\frac{dx}{dy} = \frac{dy}{dx}$

$\int x\, dx - y\, dy = 0$

$\boxed{x^2 - y^2 = C}$ – первый инт.

$\frac{u du + x du}{x u + x u} = \frac{x u + x dy}{x u + x u}$

$\frac{du}{u} + \frac{dx}{x} = \frac{x u + x dy}{x u + x u}$

$\frac{du}{u} + \frac{dx}{x} = \frac{1}{2} + \frac{1}{2}$

2) $\frac{dx}{u^2 - x^2} = \frac{dy}{2x}$

$\int 2x\, dx - u^2 dx = 0$

$\boxed{x^2 - u^2 x = C_2}$ – первый инт.

$\frac{u du + x du}{x^2 - u^2 x} = \frac{y dy}{u^2 x^2 - x^3}$

$u du + x du = y dy$

$\int d(ux) = \int d(y^2)$

$\boxed{ux - y^2 = C_2}$

$\begin{cases} x^2 = C_1 \\ u^2 x = C_2 \\ ux - y^2 = C_3 \end{cases}$ – общий интеграл

$\varphi(C_1, C_2, C_3) = 0$

$\boxed{\varphi(x^2 y, u^2 - x, ux - y^2)} = 0$ – общее решение.

Решение ур-я

$\boxed{3}$) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$

$\boxed{\frac{dx}{x} = \frac{dy}{2y^2}} \quad \frac{du = 0}{u = C_1}$

$\int 2y^2 dy - dx = 0$

$y^2 - x^2 = C_2$ $\qquad$ проверяем

$\varphi(u, y^2 - x^2) = 0$ $\qquad \forall \varphi$

$\boxed{u = F(y^2 - x^2)}$ – ответ $\qquad \varphi(C_1, C_2) = C_1 - F(C_2)$ $\quad \forall F$ – произв.