Introduction

Framework	Posible actions	State	Model
Bandits	Multiple	One	No model
Dynamic programming	Multiple	Sequential Structure	Given
Later	Multiple	Sequential Structure	Not given, obtained by interaction

Markov Decision Process

> Assumtion: The environment is fully observable

Note

Several problems can be formalised as an MDP:

- Optimal control problems deal with continuous MDPs
- Partially observable can be converted into MDP given the right history
- Bandits are MDP with one state

Definition:

A *Markov decision process* is a tuple (S, A, p, γ) where

- S ius the set of posible states
- A is the set of posible actions
- $p = p(r, s'|s, a)$
- γ is the discount factor

Observation:

- p denotes the *transition dynamics* of the problem
- I'ts useful to marginalise out the state transitions as follow

$$p(s'|s,a) = \sum_r p(s',r|s,a)$$
$$E[R|s,a] = \sum_r r \sum_{s'} p(r,s'|s,a)$$

Definition:

Consider a sequence of random variables $\{S_t\}_{t\in\mathbb{N}}$. A state s has the *markov property* when

$$p(S_{t+1} = s'|S_t = s) = p(S_{t+1} = s'|h_{t-1}, S_t = s)$$

For all the history.

Observation

> The state S "sumarizes" all the history that it has seen before it in terms of the prediction value
>
> - The state captures all the relevant information from the history.
> - The state is a sufficient statistic of the past.

Definition:

Given a MDP the return at the timestep t is given by:

$$G_t = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{T-t} R_T$$

Where:

- $\gamma = 1$ Implies an undiscounted return
- $\gamma < 1$ Implies a discounted return

- $\frac{G_t}{T-t-1}$ is the average return

Observation:

The horizon can be infite ($T \to \infty$)

$$G_t = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

Discussion:

Most MDP are discounted since:

- Sometimes it makes sense to have short term rewards
- Human and animals show preference for inmediate reward
- Matheematiocal conveniet
- Avoid infinite cycles

Definition

The *goal* of the agent is to find a behaviout poicy that maximises the expected return G_t

Definition

The *value function* is given by

$$v_\pi(s) = \mathbb{E}[G_t | S_t = s, \pi]$$

And the state-action values are given by

$$q_\pi(s,a) = \mathbb{E}[G_t | S_t = s, A_t = a, \pi]$$

Proposition

$$
\begin{aligned}
v_\pi(s) &= \sum_a \pi(a|s) q_\pi(s,a) \\
&= E[q_\pi(S_t, A_t)|S_t = s, \pi]
\end{aligned}
$$

Definition

The *optimal state-value function* is the maximum value function over all policies

$$v^*(s) = \max_\pi v_\pi(s)$$

The *optimal action-value function* is the maximum action-value function over all policies

$$q^*(s,a) = \max_\pi q_\pi(s,a)$$

Proposition

There exists a partial ordering in the policy space given by:

$$\pi \geq \pi' \iff v_\pi \geq v_{\pi'}, \forall s$$

Proposition (Optimal Policy)

$$\pi^*(s,a) = \begin{cases} 1 & a = argmax_{a \in A} q^*(s,a) \\ 0 & otherwise \end{cases}$$

Observation

> There is always a deterministic optimal policy for a given MDP

Bellman equations

Theorem (Bellman Expectation Equations)

The value function can be written recursively

$$\begin{aligned} v_\pi(s) &= \mathbb{E}[G_t | S_t = s, \pi] \\ &= E[R_{t+1} + \gamma G_{t+1} | S_t = s, A_t \sim \pi(S_t)] \\ &= \sum_a \pi(a|s) \sum_r \sum_{s'} p(r, s'|s, a)(r + \gamma v_\pi(s')) \end{aligned}$$

Similarly for the state-action values function can be written recursively

$$q_\pi(s, a) = \sum_r \sum_{s'} p(r, s'|s, a) \left(r + \gamma \sum_{a'} \pi(a'|s') q_\pi(s', a') \right)$$

Problems in RL

Prediction – Policy evaluation

Estimating the value and state-action value function.

> Given a policy what is my expected return under that behaviour

Control – Policy optimisation

Estimating the optimal valua and state-action value function.

> What is the optimal way of behaving

Solving the bellman optimality Equation:

- Using Models (dynamic promgramming)
- Using sampling
  - Monte carlo
  - Q-learning
  - Sarsa

Dynamic Programming

> A collection of algorithms that can be used to compute optimal policies given a perfect model of the environmet as an MDP

Note

All this methods consist in two important parts

- Policy evaluation
- Policy optimization

Algorithm (Policy evaluation)

1. Initialize v_0
2. Iterate and update

$$v_{k+1} = E[R_{t+1} + \gamma v_k(S_{t+1})|s, \pi] \quad \forall s$$

3. Stop whenever

$$v_{k+1}(s) = v_k(s)$$

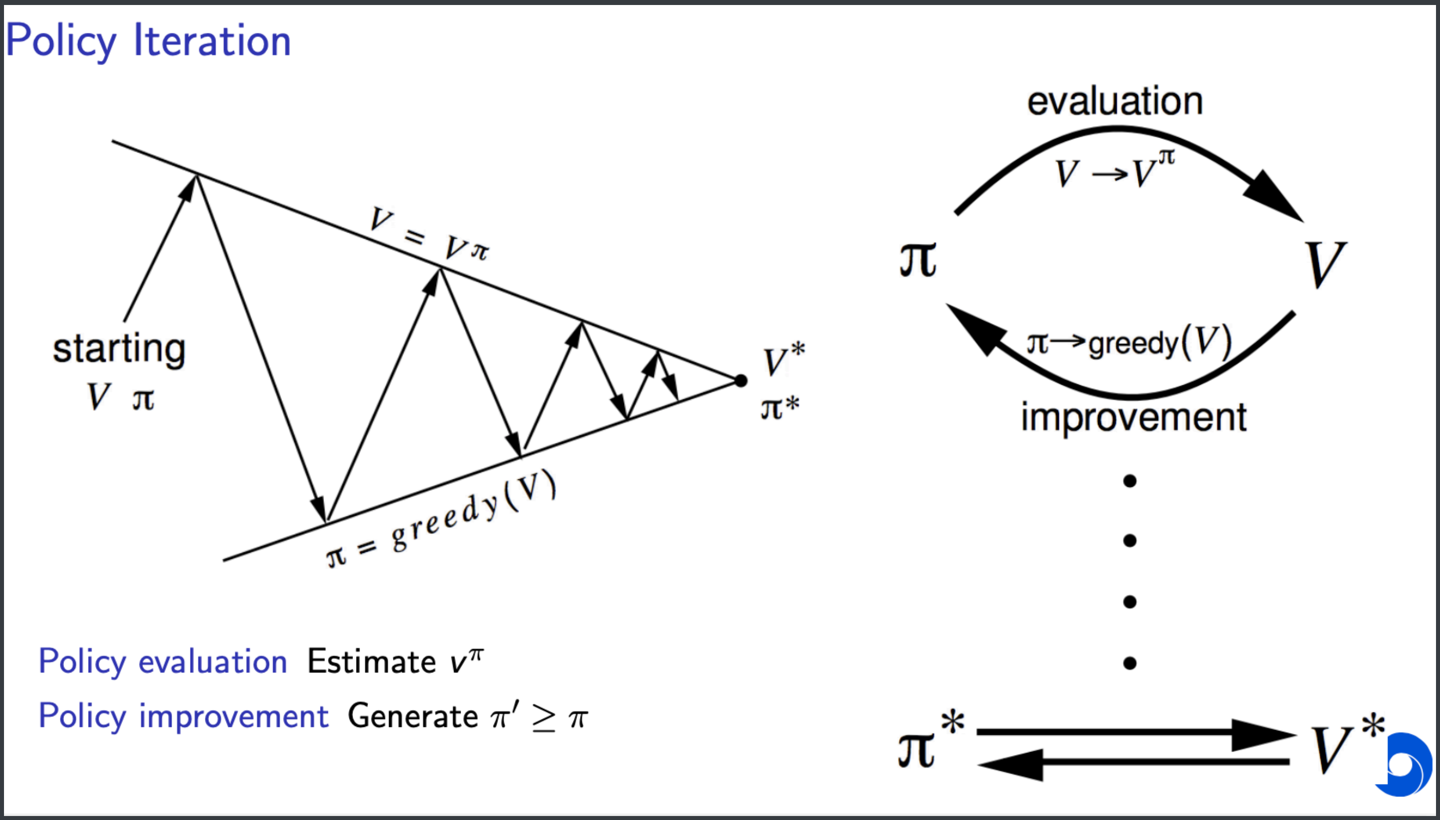
Algorithm (Policy Iteration)

Algorithm (Policy iteration)

1. Iterate using the policy

$$\pi_{new} = argmax_a q_\pi(s, a)$$

2. Evaluate and repeat



Algorithm (value iteration)

1. Initialize v_0
2. Iterate and update

$$v_{k+1}(s) = \max_a [E[R_{t+1} + \gamma v_k(S_{t+1})|s, \pi]] \quad \forall s$$

3. Stop whenever

$$v_{k+1}(s) = v_k(s)$$

Dynamic programming algorithms summary

Problem	Bellman Equation	Algorithm
Prediction	Bellman Expectation Equation	Iterative Policy Evaluation
Control	Bellman Expectation Equation + (Greedy) Policy Improvement	Policy Iteration
Control	Bellman Optimality Equation	Value Iteration

Observation

- Algorithms are based on the state-value function and have a complexity of $O(|A||S|^2)$ per iteration
- Could also apply to action-value function and have a complexity of $O(|A|^2|S|^2)$ per iteration

Prioritised Swepping

> Used to converge faster

- Use magnitud of bbellman error to guide state selection
- Update bellman error of affected states aftear each backup