Geodesics in Linear Dilaton Spacetime

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Abstract

Here we derive the solutions to the geodesic equation for the linear dilaton metric.

1 Introduction

The linear dilaton metric can be written like this:

$$ds^2 = -y^2 dt^2 + dy^2 + y^2 \sum_{i=1}^{d-1} (dx^i)^2. \tag{1}$$

This metric can be express like a diagonal matrix:

$$g_{\mu\nu} = \mathrm{diag}(-y^2, 1, y^2, ..., y^2). \tag{2}$$

The inverse metric:

$$g^{\mu\nu} = \mathrm{diag}(-y^{-2}, 1, y^{-2}, ..., y^{-2}). \tag{3}$$

It's important to note that the metric $g_{\mu\nu}$ is independent of the coordinates $x^\mu \neq y$. Then, there are d Killing Vectors, one related to conserved energy:

$$K^\mu = (\partial_t)^\mu \tag{4}$$

and $d-1$ related to conserved momentum in x^i direction:

$$R^\mu = (\partial_i)^\mu. \tag{5}$$

The conserved quantities will be:

$$E = -K_\mu \frac{dx^\mu}{d\lambda} = y^2 \frac{dt}{d\lambda} \tag{6}$$

and

$$P^i = R_\mu \frac{dx^\mu}{d\lambda} = y^2 \frac{dx^i}{d\lambda}. \tag{7}$$

For massless particles, E is the energy and P^i is the momentum in x^i direction, while for massive particles is the energy or momentum per unit of mass of the particle. The trajectory of a free particle will be such that it conserves this quantities.

The four-velocity of the particle obeys the normalization:

$$U^\mu U_\mu = -\epsilon, \tag{8}$$

where $\epsilon = 1$ for massive particles and $\epsilon = 0$ for massless particles. From the normalization condition:

$$-y^2\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + y^2\sum_i\left(\frac{dx^i}{d\lambda}\right)^2 = -\epsilon. \tag{9}$$

Substituting the conserved quantities E and P^i:

$$\left(\frac{dy}{d\lambda}\right)^2 = -\epsilon + \frac{1}{y^2}\left[E^2 - \sum_i (P^i)^2\right], \tag{10}$$

which is the geodesic equation for y. The conserved quantity in square brackets will be recurrent, so we will call it K^2:

$$K^2 = E^2 - \sum_i (P^i)^2, \tag{11}$$

and the geodesic equation becomes:

$$\left(\frac{dy}{d\lambda}\right)^2 = -\epsilon + \frac{K^2}{y^2}. \tag{12}$$

2 Timelike Geodesics

A massive particle will follow a timelike path in spacetime where the normalization of the four-velocity is $\epsilon = 1$. Then, the geodesic equation for y is:

$$\frac{dy}{d\tau} = \pm\sqrt{\frac{K^2}{y^2} - 1}. \tag{13}$$

We put $\lambda = \tau$, being τ the proper time of the particle. The solution for this equation is:

$$\boxed{y(\tau) = \pm\sqrt{K^2 - \tau^2},} \tag{14}$$

where $-K \leq \tau \leq K$. The trajectory of the particle depends on the sign of $\dot{y}$. If $\dot{y} > 0$, then the particle will describe a semicircle in negative side of y axis that starts and ends at $y = 0$. But, if $\dot{y} < 0$, the semicircle will be in positive side of y axis, starting and ending at $y = 0$. The fact that the geodesic is incomplete (ends in a finite proper time) is a reflection of the fact that the metric has a singularity at $y = 0$.

It's easy to find the parametric equations for t and x^i. We get the equations 6 and 7, and substitute the solution for y. We find the differential equations:

$$\frac{dt}{d\tau} = \frac{E}{K^2 - \tau^2} \tag{15}$$

and

$$\frac{dx^i}{d\tau} = \frac{P^i}{K^2 - \tau^2}, \tag{16}$$

that have solutions:

$$\boxed{t(\tau) = \frac{E}{K} \tanh^{-1}\left(\frac{\tau}{K}\right)} \tag{17}$$

and

$$\boxed{x^i(\tau) = \frac{P^i}{K} \tanh^{-1}\left(\frac{\tau}{K}\right).} \tag{18}$$

The parametric equations are defined only for $-K \leq \tau \leq K$. For $\tau \to -K$, t and x^i tends to negative infinity. And, for $\tau \to K$, t and x^i tends to positive infinity.

Using the results above, we find that the motion in the $x - t$ plane will be a straight line just like in flat space:

$$x^i = \frac{P^i}{E} t. \tag{19}$$

We see that the motion in the $x - t$ plane is not bounded, unlike motion in the $y - t$ plane that obeys:

$$y = \pm K \mathrm{sech}\left(\frac{K}{E} t\right). \tag{20}$$

When $t \to \infty$ or $t \to -\infty$, $y \to 0$. The particle that starts on the positive side of the y axis will stay on that side forever, and the same goes for the particle that starts on the negative side of the y axis. The two regions, $y > 0$ and $y < 0$ are separate regions and a particle following a geodesic path cannot cross $y = 0$ from one side to the other.

3 Null Geodesics

A massless particle will follow a null geodesic where $\epsilon = 0$. The equation 12 becomes:

$$\frac{dy}{d\lambda} = \pm \frac{K}{y}, \tag{21}$$

that has the solution:

$$y(\lambda) = \sqrt{\pm 2K\lambda}. \tag{22}$$

For a particle going forward ($\dot{y} > 0$), $\lambda \geq 0$, the particle starts in $y = 0$ and and goes to negative or positive infinity. But, for a particle going inward ($\dot{y} < 0$), $\lambda \leq 0$, and the particle goes to $y = 0$ when $\lambda = 0$.

Using the equations 6 and 7, we can find the other parametric equations, which are solutions to the differential equations:

$$\frac{dt}{d\lambda} = \frac{E}{y^2} = \pm \frac{E}{2K\lambda} \tag{23}$$

and

$$\frac{dx^i}{d\lambda} = \frac{P^i}{y^2} = \pm \frac{P^i}{2K\lambda}. \tag{24}$$

We have the solutions:

$$t(\lambda) = \frac{E}{2K}\ln\lambda, \text{for}\lambda > 0 \tag{25}$$
$$t(\lambda) = \frac{-E}{2K}\ln(-\lambda), \text{for}\lambda < 0 \tag{26}$$

and

$$x^i(\lambda) = \frac{P^i}{2K}\ln\lambda, \text{for}\lambda > 0 \tag{27}$$
$$x^i(\lambda) = \frac{-P^i}{2K}\ln(-\lambda), \text{for}\lambda < 0 \tag{28}$$

The motion in the $x-t$ plane:

$$x^i = \left(\frac{P^i}{E}\right)t. \tag{29}$$

The motion in $y-t$ plane:

$$y = \sqrt{2K}e^{\pm\frac{K}{E}t}, \tag{30}$$

where the positive sign is for $\lambda > 0$ and the negative sign for $\lambda < 0$.

4 Geodesic Distance

For a timelike geodesic, the spacetime distance between two points in a geodesic is $\Delta\tau = \tau_2 - \tau_1$. The idea is write an expression to $\Delta\tau$ using t_1, t_2, y_1, y_2, x_1^i and x_2^i. First, note that

$$\Delta t^2 - \sum_i(\Delta x^i)^2 = \left[\tanh^{-1}\left(\frac{\tau_2}{K}\right) - \tanh^{-1}\left(\frac{\tau_1}{K}\right)\right]^2. \tag{31}$$

Using the logarithmic definition of arctanh, we can prove that

$$\tanh^{-1}\left(\frac{\tau_2}{K}\right) - \tanh^{-1}\left(\frac{\tau_1}{K}\right) = \tanh^{-1}\left(\frac{K\Delta\tau}{K^2 - \tau_2\tau_1}\right). \tag{32}$$

Thus,

$$\frac{K\Delta\tau}{K^2 - \tau_2\tau_1} = \tanh\left(\sqrt{\Delta t^2 - \sum_i(\Delta x^i)^2}\right). \tag{33}$$

To leave one variable less, we will always substitute τ_2 by $\Delta\tau + \tau_1$. For simplicity, we can also do $a = \tanh\left(\sqrt{\Delta t^2 - \sum_i(\Delta x^i)^2}\right)$. Therefore, the expression above becomes

$$\frac{K\Delta\tau}{K^2 - \tau_1\Delta\tau - \tau_1^2} = a, \tag{34}$$

where we have the variables K and τ_1 to be replaced by y_1 and y_2. The square root in 14 will be a problem. But we can calculate two equations, which is exactly what we need to substitute the two variables. First, we calculate

$$y_1^2 - y_2^2 = \Delta\tau^2 + 2\tau_1\Delta\tau. \tag{35}$$

Second, the equation

$$y_1^2 = K^2 - \tau_1^2. \tag{36}$$

From the first equation, we have

$$\tau_1 \Delta\tau = \frac{y_1^2 - y_2^2 - \Delta\tau^2}{2}. \tag{37}$$

And, from the second equation

$$K^2 = y_1^2 + \tau_1^2. \tag{38}$$

Substituting the first equation into the second, we find

$$K^2 = y_1^2 + \frac{(y_1^2 - y_2^2 - \Delta\tau^2)^2}{4\Delta\tau^2}. \tag{39}$$

Bringing back the equation 34, and replacing K and τ_1 by the expressions above

$$\sqrt{y_1^2 + \frac{(y_1^2 - y_2^2 - \Delta\tau^2)^2}{4\Delta\tau^2}}\Delta\tau = a\left(\frac{y_1^2 + y_2^2 + \Delta\tau^2}{2}\right). \tag{40}$$

Squaring both sides of the equation:

$$\left[y_1^2 + \frac{(y_1^2 - y_2^2 - \Delta\tau^2)^2}{4\Delta\tau^2}\right]\Delta\tau^2 = \frac{a^2}{4}\left(y_1^2 + y_2^2 + \Delta\tau^2\right)^2. \tag{41}$$

Developing this expression, we find a polynomial of fourth degree in $\Delta\tau$:

$$(1 - a^2)\Delta\tau^4 + 2\left[2y_1^2 - a^2(y_1^2 + y_2^2) - (y_1^2 - y_2^2)\right]\Delta\tau^2 + (y_1^2 - y_2^2)^2 - a^2(y_1^2 + y_2^2)^2 = 0. \tag{42}$$

We call the coefficients

$$A = 1 - a^2 = \operatorname{sech}^2 z \tag{43}$$

$$B = 2\left[2y_1^2 - a^2(y_1^2 + y_2^2) - (y_1^2 - y_2^2)\right] = 2(y_1^2 + y_2^2)\operatorname{sech}^2 z \tag{44}$$

$$C = (y_1^2 + y_2^2)^2 - a^2(y_1^2 + y_2^2)^2 = (y_1^2 - y_2^2)^2 - \tanh^2 z\left(y_1^2 + y_2^2\right), \tag{45}$$

where $z = \sqrt{\Delta t^2 - \sum_i (\Delta x^i)^2}$. Therefore, we must solve the following polynomial. We can find the roots of this polynomial by making the substitution $\chi = \Delta\tau^2$:

$$A\chi^2 + B\chi + C = 0, \tag{46}$$

has roots

$$\chi = \Delta\tau^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{47}$$

I.e,

$$\Delta\tau = \sqrt{\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}}. \tag{48}$$

Doing all the algebra, we find that:

$$\Delta\tau = \sqrt{\pm 2y_1y_2\cosh z - y_1^2 - y_2^2}. \tag{49}$$

As we have seen, equation 14 describes a particle that never crosses $y = 0$. Therefore, y_1 and y_2 always have the same sign. For the geodesic distance $\Delta\tau$ to be a real number, then the sign of the term $2y_1y_2\cosh z$ must be positive. So, the geodesic distance is

$$\boxed{\Delta\tau = \sqrt{2y_1y_2\cosh\left(\sqrt{\Delta t^2 - \sum_i(\Delta x^i)^2}\right) - y_1^2 - y_2^2}.} \tag{50}$$

References

[1] Carroll, Sean M. *Spacetime and geometry*. Cambridge University Press, 2019.