Journal of Guidance, Control, and Dynamics
Vol. 32, No. 1, January–February 2009

Shape-Based Approach to Low-Thrust Rendezvous Trajectory Design

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DOI: 10.2514/1.36848

Mission design using low-thrust propulsion requires a method for approximating the spacecraft's trajectory and its cost. This can be for the purpose of evaluating a large search space or for providing a suitable initial guess to be used with more accurate trajectory optimizers. Such optimizers (for example, direct transcription methods in which the continuous problem is converted into a nonlinear programming problem) are more likely to converge when given an initial guess satisfying the equation of motion constraints and the terminal boundary conditions. A shape-based method is derived here that is capable of determining such trajectories for both time-free and time-fixed cases. When combined with a genetic algorithm, it is possible to find solutions by adjusting the free parameters of the problem within a few percent of optimal solutions found with a very exact optimizer. The method is intended for the rendezvous case but, with small modifications, it is able to solve for the less constrained cases of interception, orbit transfer, and escape.

Introduction

Mission design using low-thrust propulsion requires a method for quickly approximating the spacecraft's trajectory and its cost. This is because there is no analytical solution for the general case and for problems of interest there may be a large feasible range of departure and arrival dates, resulting in numerous local minima. There may also be a very large number of ways to accomplish the mission and it may be necessary to determine the cost for all, or a significant fraction, of them. For example, for the second Global Trajectory Optimisation Competition (GTOC2) problem, a spacecraft using low-thrust propulsion was required to rendezvous with one asteroid from each of four groups of asteroids, each group containing hundreds of asteroids, resulting in over 41 billion possible asteroid tour sequences [1]. This is a succession of interplanetary transfers qualitatively similar to trajectories that have been flown to asteroids such as Deep Impact, Deep Space 1, and Dawn. One approach to this problem is to prune the number of possible missions by hand to a tractable number of candidate missions that are then evaluated using traditional optimizers. In that case, an approximation method is needed for two purposes: to provide a cost estimate for a given mission that can be used to prune a large search space of possible missions and to provide a suitable initial guess to be used with more accurate trajectory optimizers. Such optimizers (for example, direct transcription methods [2,3], in which the continuous problem is converted into a nonlinear programming problem) are more likely to converge when given an initial guess satisfying the equation of motion constraints and the terminal boundary conditions.

Many approaches have been taken to approximate low-thrust trajectories; only a small number are referenced here [4–17]. If an assumption of constant radial thrust is made, as by Boltz [6] and by Prussing and Coverstone-Carroll [9], analytic expressions may be derived for the spacecraft's motion. However, this is most useful for escape trajectories and some Earth-orbit applications. An assumption of a constant tangential thrust can also be made [7,8]; however, this is most useful for escape trajectories or very low-thrust orbit-raising problems.

Edelbaum [11,12] derived an analytic approximation for the ΔV required for the special case of low-thrust transfer between inclined circular orbits. However, this was accomplished with the assumption that the in-plane (pitch) thrust pointing angle was always zero and the out-of-plane (yaw) thrust angle was constant over each revolution. Wiesel and Alfano [13] improved Edelbaum's solution. However, in all of these analyses, the transfer orbit is never allowed to accumulate any significant eccentricity (or is assumed to return to circularity each revolution). Kechichian [14] revisited the Edelbaum problem and derived an improved solution allowing continuous variation of the pitch and yaw angles.

None of these (suboptimal) analytic solutions is, however, appropriate to the general case of transfer between orbits of arbitrary eccentricity: for example, transfer from Earth to the orbit of an asteroid or transfer from one asteroid orbit to another, both of which were required for the GTOC2 problem. One method that in principle applies to such cases is that of Petropoulos and Longuski [15], who approximated the low-thrust trajectory with a shape having a specified form [i.e., via a parametric equation $r(\theta)$] and then solved for the thrust magnitude and thrust pointing-angle time histories such that the boundary conditions and the equations of motion are satisfied. Petropoulos and Longuski used an exponential sinusoid

$$r(\theta) = k_0 e^{k_1 \sin(k_2\theta+\phi)} \tag{1}$$

for low-thrust interception trajectories, but other shapes such as the logarithmic spiral, Cartesian oval, and Cassini oval are also feasible [16]. De Pascale and Vasile [18] and Vasile et al. [19] successfully created a shape-based method for 3-D rendezvous trajectories by incorporating shapes for each of the pseudoequinoctial elements (not including true longitude). This allows for all of the rendezvous boundary conditions to be satisfied, however, it does have two disadvantages: the satisfaction of the boundary conditions requires the solution to a sixth-order system of nonlinear equations and the system of shaping parameters and boundary conditions is overdetermined (i.e., given two points in space and a transfer time, the solution is not unique). The cost of such trajectories may be a

Received 25 January 2008; revision received 21 July 2008; accepted for publication 7 August 2008.

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sufficiently accurate approximation of the cost, in flight time or in ΔV, of the true optimal trajectory to suffice for preliminary mission-planning purposes. Alternatively, the approximate shape-based trajectory may be used as the initial guess for a more accurate method, such as direct collocation [2,3], with good likelihood of convergence of the nonlinear programming (NLP) problem solver to a solution.

A simpler shape-based method is derived here that is capable of determining low-thrust rendezvous trajectories for both time-free and time-fixed cases. When combined with a genetic algorithm to optimize free parameters (e.g., departure and arrival dates and the number of revolutions about the central body), it is possible to find solutions that are not just feasible, but approximately optimal, thereby improving the approximation and the initial guess for a direct optimizer. The method is intended for the rendezvous case but, with small modifications, it is used to solve the less constrained cases of interception and orbit transfers. For all of the example trajectories solved, the shape-based solution is compared with a direct solution that uses the shape-based solution as the initial guess.

Shape-Based Approximation Method

The shape-based approach approximates the shape of a low-thrust trajectory with a function that may be parameterized in terms of the polar coordinate θ [i.e., $r(\theta)$]. For a rendezvous trajectory, the spacecraft must match both position and velocity (both magnitude and direction) of the departure and arrival points and must satisfy the equations of motion. It will be shown that this requires that the parametric function $r(\theta)$ (and subsequently derived expressions for the flight-path angle γ and $\dot{\theta}$) must satisfy six scalar boundary conditions. The exponential sinusoid used by Petropoulos and Longuski [15] contains only four parameters; therefore, a different function is required. Adding two parameters to create new exponential sinusoids, for example,

$$r(\theta) = k_0 e^{k_3\theta^2 + q\theta + k_1 \sin(k_2\theta+\phi)} \quad \text{or} \qquad r(\theta) = (k_0 + k_3\theta) e^{q\theta + k_1 \sin(k_2\theta+\phi)} \tag{2}$$

was the obvious approach; however, after a wide variety of trajectories were tested, these new shape-based algorithms were computationally slow and resulted in spacecraft trajectories that were obviously far from being optimal. These poor results suggested that an entirely new parametric function was required.

To locate a new function, parametric data were taken from a known optimal low-thrust rendezvous trajectory. The trajectory chosen was a low-thrust rendezvous trajectory from a circular orbit with a radius of 1 distance unit (DU) and angular position of 0 rad to an asteroid with an initial angular position of 4.5 rad in a coplanar circular orbit with a radius of 3 DU in a fixed time of flight of 10 time units (TU) [2]. This is admittedly an ad hoc choice. Other optimal low-thrust spacecraft trajectories could have been chosen (e.g., an escape trajectory). However, the circular-orbit-to-circular-orbit rendezvous trajectory is qualitatively similar to the problem to be solved, that is, it is a low-thrust rendezvous in a fixed time.

The parametric data were given to a curve-fitting algorithm[‡] that tests hundreds of functions and returns the functions that best fit the data, in the least-squares sense. An inverse polynomial function was chosen by the algorithm:

$$r = \frac{1}{a + b\theta + c\theta^2 + d\theta^3 + e\theta^4 + f\theta^5} \tag{3}$$

The inverse polynomial in Eq. (3) has six parameters, a number chosen so that it can satisfy the six boundary conditions of the rendezvous problem. It is assumed, as in the Lambert problem, that the initial position and the transfer time are known quantities and that the final position is calculated from the given transfer time. The easiest constraints to represent are the constraints related to the initial and final positions. Assuming that the θ angle is referenced to the initial position vector of body 1, as shown in Fig. 1, evaluating Eq. (3) at $\theta = 0$ yields

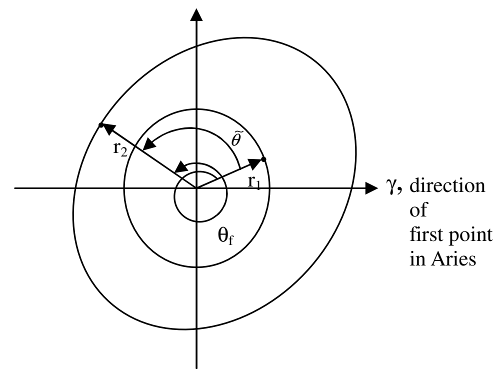

Fig. 1 Diagram of low-thrust orbit transfer.

$$r_1 = \frac{1}{a} \tag{4}$$

and at $\theta = \theta_f$ yields

$$r_2 = \frac{1}{a + b\theta_f + c\theta_f^2 + d\theta_f^3 + e\theta_f^4 + f\theta_f^5} \tag{5}$$

where the total transfer angle is represented by

$$\theta_f = \tilde{\theta} + 2\pi \cdot N\text{rev} \tag{6}$$

The angle $\tilde{\theta}$ is the angle between the position vectors of body 1 and body 2 measured counterclockwise, as shown in Fig. 1, and Nrev is the number of revolutions about the sun, an additional unknown that obviously applies to multiple revolution trajectories.

Next, constraints for the initial and final flight-path angles are derived. The time derivative of Eq. (3) yields

$$\dot{r} = -r^2 \cdot (b + 2c\theta + 3d\theta^2 + 4e\theta^3 + 5f\theta^4) \cdot \dot{\theta} \tag{7}$$

Thus, the flight-path angle γ may be found from

$$\tan(\gamma) = \frac{\dot{r}}{r\dot{\theta}} = -r \cdot (b + 2c\theta + 3d\theta^2 + 4e\theta^3 + 5f\theta^4) \tag{8}$$

The initial and final required flight-path angles, γ_1 and γ_2, are easily calculated given the orbital elements of the departure and arrival orbits and the departure and arrival times, which yields the following two constraints:

$$\tan(\gamma_1) = -r_1 \cdot b \tag{9a}$$

$$\tan(\gamma_2) = -r_2 \cdot (b + 2c\theta_f + 3d\theta_f^2 + 4e\theta_f^3 + 5f\theta_f^4) \tag{9b}$$

The inverse polynomial must also satisfy the equations of motion given in polar coordinates as

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = T_a \sin(\alpha) \tag{10a}$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = T_a \cos(\alpha) \tag{10b}$$

where T_a is the thrust acceleration of the low-thrust spacecraft and α is the thrust pointing angle.

Equation (10b) can be expanded and written as

$$r\ddot{\theta} + 2r\tan(\gamma)\dot{\theta}^2 = T_a \cos(\alpha) \tag{11}$$

Taking the derivative of Eq. (7), the acceleration of r is

[‡]Data available online at http://www.zunzun.com [retrieved 4 November 2008].

$$\ddot{r} = 2r(\tan^2\gamma)\dot{\theta}^2 - r^2(2c + 6d\theta + 12e\theta^2 + 20f\theta^3)\dot{\theta}^2 + r(\tan(\gamma))\ddot{\theta} \tag{12}$$

Equation (11) yields $\ddot{\theta}$:

$$\ddot{\theta} = \frac{T_a\cos(\alpha) - 2r\tan(\gamma)\dot{\theta}^2}{r} \tag{13}$$

Thus, Eq. (12) becomes

$$\ddot{r} = T_a\cos(\alpha)\tan(\gamma) - r^2(2c + 6d\theta + 12e\theta^2 + 20f\theta^3)\dot{\theta}^2 \tag{14}$$

Equating $\ddot{r}$ from Eqs. (10a) and (14) yields

$$r\dot{\theta}^2 + T_a\sin(\alpha) - \frac{\mu}{r^2} = T_a\cos(\alpha)\tan(\gamma) - r^2(2c + 6d\theta + 12e\theta^2 + 20f\theta^3)\dot{\theta}^2 \tag{15}$$

Equation (15) simplifies if it is assumed that the thrust pointing angle is either along or against the flight-path angle; that is,

$$\alpha = \gamma + n\pi \tag{16}$$

where n is 0 or 1 for along or against the flight-path angle, respectively. Tangential thrust is often assumed in approximations to optimal low-thrust trajectories [7,8,15,16]. This assumption is reasonable, as it corresponds to increasing (or decreasing) the kinetic energy of the spacecraft at the maximum rate. This assumption allows Eq. (15) to be solved for $\dot{\theta}$ purely in terms of the inverse polynomial coefficients,

$$\dot{\theta}^2 = \frac{\mu}{r^4}\frac{1}{[(1/r) + 2c + 6d\theta + 12e\theta^2 + 20f\theta^3]} \tag{17}$$

regardless of whether $n = 0$ or 1. Equation (9) guarantees that the flight-path angle of the spacecraft is equal to the flight-path angle of the initial and terminal orbits. To ensure that the velocity magnitude is also correct at the initial point and terminal point, only one velocity component must be constrained and the other may be found using Eq. (8). Equation (17) is therefore equivalent to a velocity-magnitude constraint. Evaluating Eq. (17) at the departure and arrival points yields

$$\dot{\theta}_1^2 = \left(\frac{\mu}{r_1^4}\right)\frac{1}{(1/r_1) + 2c} \tag{18a}$$

$$\dot{\theta}_2^2 = \left(\frac{\mu}{r_2^4}\right)\frac{1}{(1/r_2) + 2c + 6d\theta_f + 12e\theta_f^2 + 20f\theta_f^3} \tag{18b}$$

Equations (4), (5), (9), and (18) constitute the six scalar constraints that must be satisfied by the six inverse polynomial coefficients for a rendezvous trajectory. Three of the unknown inverse polynomial coefficients (a, b, and c) can be easily determined by rearranging Eqs. (4), (9a), and (18a), respectively;

$$a = \frac{1}{r_1}, \qquad b = -\frac{\tan\gamma_1}{r_1}, \qquad c = \frac{1}{2r_1}\left(\frac{\mu}{r_1^3\dot{\theta}_1^2} - 1\right) \tag{19}$$

The remaining three coefficients d, e, and f can be found using Eqs. (5), (9b), and (18b):

$$\begin{bmatrix} \theta_f^3 & \theta_f^4 & \theta_f^5 \\ 3\theta_f^2 & 4\theta_f^3 & 5\theta_f^4 \\ 6\theta_f & 12\theta_f^2 & 20\theta_f^3 \end{bmatrix}\begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{r_2} - (a + b\theta_f + c\theta_f^2) \\ -\frac{\tan\gamma_2}{r_2} - (b + 2c\theta_f) \\ \frac{\mu}{r_2^4\dot{\theta}_2^2} - (\frac{1}{r_2 + 2c}) \end{bmatrix} \tag{20}$$

The solution for coefficients d, e, and f is then

$$\begin{bmatrix} d \\ e \\ f \end{bmatrix} = \frac{1}{2\theta_f^5}\begin{bmatrix} 20\theta_f^2 & -8\theta_f^3 & \theta_f^4 \\ -30\theta_f & 14\theta_f^2 & -2\theta_f^3 \\ 12 & -6\theta_f & \theta_f^2 \end{bmatrix}\begin{bmatrix} \frac{1}{r_2} - (a + b\theta_f + c\theta_f^2) \\ -\frac{\tan\gamma_2}{r_2} - (b + 2c\theta_f) \\ \frac{\mu}{r_2^4\dot{\theta}_2^2} - (\frac{1}{r_2} + 2c) \end{bmatrix} \tag{21}$$

Once all of the inverse polynomial coefficients are known, Eq. (17) yields

$$\int_0^{t_f} \mathrm{d}t = \int_0^{\theta_f} \sqrt{\frac{r(\theta)^4}{\mu}[(1/r(\theta)) + 2c + 6d\theta + 12e\theta^2 + 20f\theta^3]}\mathrm{d}\theta \tag{22}$$

where $r(\theta)$ is given by Eq. (3). This equation may be integrated to determine the transfer time t_f (i.e., the required transfer time for the inverse polynomial that satisfies the position and velocity boundary conditions). Unfortunately, the transfer time determined from Eq. (22) is not constrained in this analysis to equal the transfer time used to calculate the position of body 2. Therefore, the fifth-degree inverse polynomial is best suited for time-free problems (e.g., orbit-transfer problems), in which the rendezvous point is not fixed.

To be able to specify the transfer time, an additional free parameter is required in the inverse polynomial shape; that is, a sixth-degree inverse polynomial is required:

$$r = \frac{1}{a + b\theta + c\theta^2 + d\theta^3 + e\theta^4 + f\theta^5 + g\theta^6} \tag{23}$$

The method in which the values of the sixth-degree inverse polynomial coefficients are found is nearly identical to that used for the fifth-degree inverse polynomial coefficients. To satisfy the position and velocity constraints at body 1, coefficients a, b, and c are still determined from Eq. (19). The remaining four coefficients must satisfy the final position, final velocity, and the given transfer time constraints. Note that there is a unique solution to this problem; thus, any three of the remaining four coefficients can be chosen to satisfy the final position and velocity with the remaining coefficient varied to solve for the given transfer time. The coefficient chosen to solve for the transfer time is d, because it has the smallest sensitivity to change in transfer time, giving a root-finding function such as `fzero` in MATLAB the best conditions for convergence to a solution [20]. This leads to an equation analogous to Eq. (21) that allows solution for coefficients e, f, and g:

$$\begin{bmatrix} e \\ f \\ g \end{bmatrix} = \frac{1}{2\theta_f^6}\begin{bmatrix} 30\theta_f^2 & -10\theta_f^3 & \theta_f^4 \\ -48\theta_f & 18\theta_f^2 & -2\theta_f^3 \\ 20 & -8\theta_f & \theta_f^2 \end{bmatrix}\begin{bmatrix} \frac{1}{r_2} - (a + b\theta_f + c\theta_f^2 + d\theta_f^3) \\ -\frac{\tan\gamma_2}{r_2} - (b + 2c\theta_f + 3d\theta_f^2) \\ \frac{\mu}{r_2^4\dot{\theta}_2^2} - (\frac{1}{r_2} + 2c + 6d\theta_f) \end{bmatrix} \tag{24}$$

By inspection, $\dot{\theta}^2$ for the sixth-degree inverse polynomial case can be determined in the same manner as it was in Eq. (17), and it is clear that the only difference is the addition of a term multiplied by g:

$$\dot{\theta}^2 = \left(\frac{\mu}{r^4}\right)\frac{1}{[(1/r) + 2c + 6d\theta + 12e\theta^2 + 20f\theta^3 + 30g\theta^4]} \tag{25}$$

To determine the transfer time t_f for this inverse polynomial, Eq. (25) yields

$$\int_0^{t_f} \mathrm{d}t = \int_0^{\theta_f} \sqrt{\frac{r(\theta)^4}{\mu}[(1/r(\theta)) + 2c + 6d\theta + 12e\theta^2 + 20f\theta^3 + 30g\theta^4]}\mathrm{d}\theta \tag{26}$$

where $r(\theta)$ is given by Eq. (23). This equation may be integrated to determine the transfer time t_f. This calculated transfer time is then compared with the specified transfer time used to calculate the position and velocity of body 2. Using a root-finding finding function such as `fzero` in MATLAB, the coefficient d is varied until the transfer time determined from Eq. (26) is equal to the specified transfer time used to calculate the position and velocity of body 2 (i.e., the spacecraft and body 2 arrive at the rendezvous point at the same time). Using a simple initial guess of zero for coefficient d, the `fzero` function in MATLAB is robust in finding a solution that drives the transfer time difference to zero.

The thrust magnitude is required such that the spacecraft following the inverse polynomial trajectory satisfies the equations of motion. Recall that the thrust direction is assumed to be along or against the velocity vector. An expression for $\ddot{\theta}$ is determined by taking the time derivative of Eq. (25):

$$\ddot{\theta} = -\frac{\mu}{2r^4}\left[\frac{4\tan\gamma}{[(1/r)+2c+6d\theta+12e\theta^2+20f\theta^3+30g\theta^4]} + \frac{6d+24e\theta+60f\theta^2+120g\theta^3-(\tan\gamma)/r}{[(1/r)+2c+6d\theta+12e\theta^2+20f\theta^3+30g\theta^4]^2}\right] \tag{27}$$

This expression and Eq. (25) may then be substituted into Eq. (11) and the thrust magnitude T_a can be determined:

$$T_a = -\frac{\mu}{2r^3\cos\gamma}\cdot\frac{6d+24e\theta+60f\theta^2+120g\theta^3-(\tan\gamma)/r}{[(1/r)+2c+6d\theta+12e\theta^2+20f\theta^3+30g\theta^4]^2} \tag{28}$$

where r is from Eq. (23) and the flight-path angle γ is determined by the analogue of Eq. (8):

$$\tan(\gamma) = \frac{\dot{r}}{r\dot{\theta}} = -r\cdot(b+2c\theta+3d\theta^2+4e\theta^3+5f\theta^4+6g\theta^5) \tag{29}$$

Assuming that the flight-path angle γ is always between $-\pi/2$ and $\pi/2$ (i.e., that there are no retrograde trajectories), the $\cos\gamma$ term in Eq. (28) is easily determined.

It is often required to maximize the final spacecraft mass (i.e., to minimize propellant consumption for the mission); therefore, an expression for final mass is also derived. Assuming that the spacecraft has an engine with constant I_{sp}, the thrust acceleration T_a is simply

$$T_a = \frac{T}{m} = \frac{-\dot{m}\cdot c_{exh}}{m} \tag{30}$$

where T is the thrust magnitude and c_{exh} is the constant exhaust velocity. Thus,

$$\dot{m} = -\frac{T_a\cdot m}{c_{exh}} \tag{31}$$

so that

$$\int_{m_o}^{m_f}\frac{dm}{m} = -\int_0^{\theta_f}\frac{T_a}{c_{exh}\cdot\dot{\theta}}d\theta \tag{32}$$

where T_a is from Eq. (28) and $\dot{\theta}$ is from Eq. (25). By integrating Eq. (32), the final mass m_f may be determined.

The minimization of fuel consumption is equivalent to the minimization of the low-thrust ΔV given by Eq. (33):

$$\Delta V = c_{exh}\cdot\ln\left(\frac{m_o}{m_f}\right) \tag{33}$$

Integrating the left-hand side of Eq. (32) yields

$$\ln\left(\frac{m_f}{m_o}\right) = \frac{1}{c_{exh}}\int_0^{\theta_f}\frac{T_a(\theta)}{\dot{\theta}(\theta)}d\theta \tag{34}$$

where $T_a(\theta)$ and $\dot{\theta}(\theta)$ are from Eqs. (25) and (28), respectively. Substituting this expression into Eq. (33) yields

$$\Delta V = \int_0^{\theta_f}\frac{T_a(\theta)}{\dot{\theta}(\theta)}d\theta \tag{35}$$

This expression for the ΔV of the shape-based trajectory is independent of the exhaust velocity c_{exh}. This means that for the objective of minimizing fuel consumption or, equivalently, minimizing the ΔV, an optimizer implementing this shape-based method will converge to the same trajectory, independently of the chosen engine parameters.

Note that this analysis, which can find a transfer orbit between two specified points in a fixed time, is the low-thrust analogue of the solution to Lambert's problem.

Shape-Based Method in Combination with a Genetic Algorithm

Using this approach, the only unknowns for this problem are the launch date, the arrival date, and the number of revolutions Nrev for the transfer. To locate the optimal dates and number of revolutions, a genetic algorithm (GA) is used because it does not require any a priori information about the optimal solution [21]. Because all of the rendezvous boundary conditions are satisfied, a multi-objective GA or the use of a penalty function method in the GA are not required [22], although penalty methods may be used if it is desired to find trajectories for which the thrust level of the shape-based solution is less than the maximum-allowable thrust level for a given problem. If the shape-based algorithm fails to converge to a feasible rendezvous trajectory (e.g., the inverse polynomial shape cannot satisfy all of the boundary conditions), an arbitrarily high fitness value is assigned to the individual, removing it from the next generation via selection.

The efficiency of the shape-based algorithm is extremely important for large problems in which millions of evaluations may be required. For this reason, both Eqs. (26) and (32) or Eq. (35), depending on the desired performance index, are integrated via quadrature using 100 equal segments and the trapezoidal rule. Although little time is saved when using quadrature rather than the more accurate numerical integration routines available in MATLAB when computing good trajectories (e.g., those that require little fuel), the computational savings is immense when calculating the poor trajectories. The integration error when using a quadrature rule is smaller for the good trajectories because the state variables vary more smoothly than in the poor trajectories; thus, using a simple quadrature rule rather than a higher accuracy numerical integrator is a good combination of overall speed and accuracy for the trajectories of interest. The search space, which includes the launch date, arrival date, and number of complete orbits Nrev, can thus be searched quickly and efficiently.

Additional Applications of the Inverse Polynomial Shape-Based Method

The shape-based algorithm using the sixth-degree inverse polynomial has been developed for the specific case of a rendezvous trajectory in fixed time. However, it can be used for other spacecraft trajectories (e.g., orbit transfers and interception trajectories). The fifth-degree inverse polynomial may be used to solve these same spacecraft trajectory problems where the transfer time is free (unspecified).

Example: Circular-Orbit-to-Circular-Orbit Transfer

In this case, a time-free orbit transfer between specified coplanar initial and final orbits is performed. The departure date and the total transfer angle θ_f may then be used to maximize the objective function: the final spacecraft mass. A lower bound on the ΔV that is required is that corresponding to a two-impulse Hohmann transfer. If a low-thrust spacecraft is used for the transfer, a larger effective ΔV is required. The inefficiency is described quantitatively by the gravity

Table 1 Gravity loss data for fifth-degree inverse polynomial shape-based approximation

Final transfer angle, rad	$\Delta V_{\text{low thrust}}$, DU/TU	Gravity loss, %
$3\pi/4$	2.143	444.0
π	0.727	84.5
3.9984	0.3996	1.4536
3π	0.419	6.4
5π	0.421	7.0
7π	0.422	7.2
9π	0.422	7.2

loss, determined from Eq. (36), because the velocity change is no longer instantaneous:

$$\text{gravity loss} = \left(\frac{\Delta V_{\text{low thrust}}}{\Delta V_{\text{impulse}}} - 1\right) \cdot 100\% \tag{36}$$

Using the fifth-degree inverse polynomial to solve a series of orbit-transfer problems, the gravity loss can be determined as a function of the total transfer angle θ_f. The specific orbit transfer chosen is a transfer in a free final time from a circular orbit with a radius of 1 DU to a coplanar circular orbit with a radius of 3 DU. Canonical units are used such that 1 DU is 1 AU and 2π TU is 1 year. The ΔV required for a two-impulse Hohmann transfer is 0.394 DU/TU. The gravity loss is given for several total transfer angles in Table 1. The optimal shape-based trajectory has a total transfer angle of 3.9984 rad and a gravity loss of 1.4536%.

Recall that the shape-based trajectory represents a feasible trajectory, but by inspection of the gravity loss listed in Table 1, the trajectories with transfer angles greater than approximately 3.9 rad are near optimal, because these trajectories have a gravity loss of less than 7.3% with respect to the Hohmann transfer ΔV. Note that the true optimal-gravity-loss curve for a low-thrust spacecraft orbit transfer goes to 0% as the total transfer angle increases. Thus, for transfer angles of greater than approximately 4 rad, the shape-based approximation method provides a solution no worse than 7.2% from optimal.

The thrust acceleration time histories of the four transfer trajectories corresponding to total transfers angles of 4.0, 3π, 5π, and 9π rad are shown in Fig. 2.

Notice that the thrust acceleration time history for the optimal shape-based trajectory most closely resembles a two-impulse transfer with two relatively short thrust arcs, one at departure and one at arrival. The other thrust acceleration time histories show much less variation in amplitude; that is, they are more representative of a practical low-thrust spacecraft trajectory. Also, it can be shown that by adding a penalty function proportional to the maximum thrust level available for a particular mission to the optimal performance index, an optimizer will result in a feasible optimal shape-based solution that can be used as an initial guess for a more accurate optimizer. For example, if a maximum thrust level for this transfer is set at 0.02 $\mathrm{DU/TU^2}$, it can be seen in Fig. 2 that all transfers with a transfer angle of 5π or less are infeasible, and therefore, with a properly selected penalty function, the optimizer will converge to a shape-based solution requiring a transfer angle of more than 5π.

Example: Low-Thrust Interception Trajectory

The inverse polynomial shape-based solution method may also be applied to interception trajectories. Although an interception trajectory has two fewer constraints than a rendezvous trajectory [i.e., the final velocity components are free (unspecified)], the sixth-degree inverse polynomial, originally developed to satisfy all of the constraints of a rendezvous trajectory, may be used by letting the final velocity components be optimization parameters, similar to the approach taken for the escape trajectory.

The trajectory chosen as an example of the application of the shape-based method to an interception problem is a minimum ΔV trajectory in a fixed final time, 10 TU, from a circular orbit with a radius of 1 DU to a coplanar circular orbit with a radius of 3 DU. Canonical units are used such that 1 DU is 1 AU and 2π TU is 1 year. The target begins from an angular position of 4.5 rad.

The only unknown quantities for this problem that are required for the sixth-degree inverse polynomial shape-based method are the final velocity and flight-path angle. Using a GA to solve for the unknown quantities, a sixth-degree inverse polynomial is used to locate an optimal shape-based interception trajectory. The GA was run for 300 generations with a population of 30. The ΔV for the optimal shape-based interception trajectory is 0.348 DU/TU. As has been described previously, the shape-based method necessarily provides a suboptimal trajectory, and thus a direct method (the direct transcription method used previously) is used to locate the true optimal solution. The total time is divided evenly into 80 segments, a number that is a priori believed to be capable of yielding an accurate solution. There are four state variables (r, θ, v_r, and v_t) and two control variables (T_a and β); therefore, this problem is discretized using a total of 1286 NLP parameters. The maximum-allowable thrust acceleration is specified as 0.1 $\mathrm{DU/TU^2}$ (5.86×10^{-5} g). This NLP problem is solved using the MATLAB [20] toolbox TOMLAB [23] that calls the sparse NLP solver SNOPT [24]. Using the optimal shape-based trajectory as the initial guess for the direct method, the true optimal trajectory (shown in Fig. 3 with an associated DV of 0.286 DU/TU) is found.

Although the ΔV required for the optimal shape-based trajectory is 22% greater than that of the true optimal solution, it is clear from the comparison of the shape-based and true optimal trajectories that the shape-based solution provides a good initial guess for the direct optimizer without any a priori knowledge of the optimal solution.

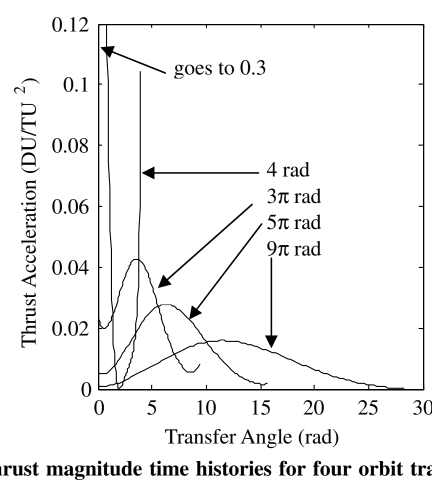

Fig. 2 Thrust magnitude time histories for four orbit transfers, with transfer angles of 4.0, 3π, 5π, and 9π.

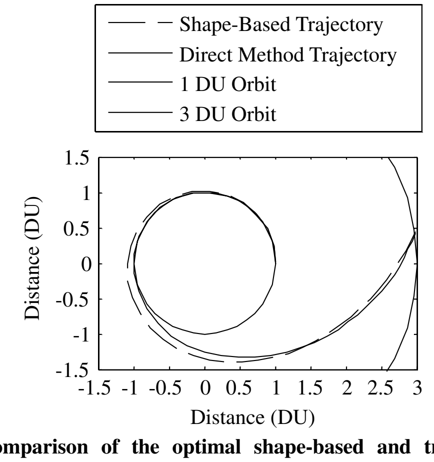

Fig. 3 Comparison of the optimal shape-based and true optimal interception trajectories.

Solution of an Asteroid Rendezvous Problem

To test the capability and accuracy of the shape-based approximation method for low-thrust rendezvous trajectories (that is to say, for the problem for which it was derived), a low-thrust multiple-asteroid rendezvous problem was solved. This problem was modeled after the GTOC2 problem [1]. Here, a low-thrust spacecraft begins at Earth and must rendezvous with one asteroid from each of four groups of asteroids, each group containing many asteroids, and stay with each asteroid for a minimum of 90 days. The objective was to complete a given sequence using minimum fuel. A 2-D version of the problem was solved, and so all asteroids were assumed to orbit in the ecliptic plane; then the only information required to be specified for each asteroid was the semimajor axis, eccentricity, and argument of perihelion on the epoch date.

For this example, the number of asteroids in each group was changed from what was assumed for the GTOC2 problem; it is convenient when formulating the problem for solution by the GA to have the number be a power of 2. Group 1 was assumed to have 64 asteroids, group 2 had 128 asteroids, group 3 had 256 asteroids, and group 4 had 256 asteroids, yielding approximately 12.9 billion possible asteroid tour sequences. The difficulty of the problem was not significantly changed, because the actual GTOC2 problem has 41 billion possible sequences. This is an impractically large number of possibilities for total enumeration. Using a nested-GA solution method, in which the outer-loop GA optimizes the asteroid tour sequence and the inner-loop GA evaluates each asteroid tour sequence using the shape-based approximation method and returns the cost to the outer-loop GA, a near-optimal (if not optimal) asteroid tour sequence may be found in far fewer evaluations than total enumeration would require. A complete description of the algorithm for the solution of this hybrid optimal control problem will be given in a companion paper [25]; the point here is to show that the shape-based method can be used to provide an approximately optimal trajectory sequence for this challenging mission, which may subsequently be improved using a more accurate optimizer.

A GA was used to locate the optimal departure time as well as the transfer time and the number of revolutions about the sun during each of the four transfers, yielding nine optimization parameters for each asteroid mission evaluated. The bounds on the GA optimization parameters are shown in Table 2, in which 2π TU is 1 year (1 TU $\approx$ 58 days and 40 TU $\approx$ 6.4 years). The inner-loop GA was run for 30 generations with a population of 250 individuals. The outer-loop GA was run for 50 generations with a population of 50 individuals.

Upon analysis of all of the asteroid tour sequences that were evaluated during the nested-GA optimization, the sequences (a period separates the group number from the asteroid number) 4.92–3.160–2.121–1.56 and 4.92–3.160–2.115–1.56 were found to have similar final masses: 967.8 and 967.7 kg, respectively. To determine which asteroid sequence is actually the best, these optimal shape-based trajectories were used as initial guesses for a more accurate solver using a direct method [2]. The optimal final spacecraft masses for the asteroid tour sequences were found to be 991.3 and 992.8 kg, respectively. Because the initial mass of the spacecraft is 1500 kg, it is clear that in both cases, the difference between the shape-based approximation and the true optimal trajectory, with regard to propellant consumed, is only a few percent. The execution time for the direct solver was 9 h on a 1.6 GHz processor, which is substantially longer than the 12.5 min required by the shape-based method. The optimal asteroid sequence trajectory is shown in Fig. 4.

Table 2 GA optimization parameters for the asteroid rendezvous problem

GA parameter	Lower bound	Upper bound
Departure time, TU	0	40
Transfer times (4), TU	1	40
*N*rev (4)	0	4

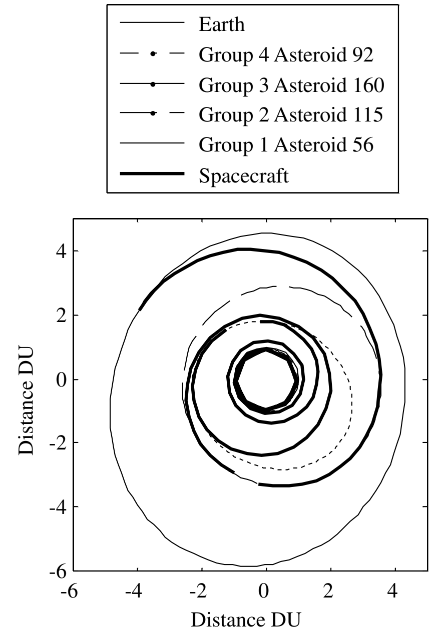

Fig. 4 Optimal trajectory for asteroid sequence 4.92–3.160–2.115–1.56.

Conclusions

The fifth- and sixth-degree inverse polynomial shape-based methods derived here are capable of providing near-optimal solutions to a variety of spacecraft trajectory problems. The time-fixed version is essentially a low-thrust analogue of the (impulsive) solution to Lambert's problem and shares the characteristic that the equations of motion are satisfied on the trajectory. Although the trajectories are useful in their own right for rapidly approximating the performance index (e.g., for the purpose of evaluating a large search space), they also make very good guesses for other optimization methods that require them (e.g., direct transcription methods that convert the problem into a nonlinear programming problem), in part because the shape-based trajectories satisfy the system equations of motion and boundary conditions. The shape-based method formulated in this work has been found to provide such good initial guesses for orbit-transfer, interception, and rendezvous trajectories. This shape-based method also provides significant computational savings over traditional optimizers, allowing for large mission-planning problems, such as the second global trajectory optimization competition problem, to be solved without prepruning the search space, which can be problematic. Although this shape-based method is a two-dimensional method, a three-dimensional shape-based method based on this research has been developed and will be the subject of a companion paper [26].

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