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Supercritical Accretion Disks Around Black Holes

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ABSTRACT

When the accretion rate approaches the critical value $\dot{M}_{cr}$, the luminosity approaches the critical value L_{cr}, and the disk becomes geometrically thick. We consider here thick, supercritical ($\dot{M} \gg \dot{M}_{cr}$, $L > L_{cr}$) accretion disks in hydrostatic equilibrium. No explicit knowledge of the equation of state or of the viscosity is assumed, except that $\alpha \ll 1$ in the language of the "α disk theory". Our approach is based on the conservation laws and stability requirements only. We construct disk models in hydrostatic equilibrium and with the local surface flux of radiation below (or equal to) the critical value. There is no mass loss from the disk even for very high accretion rates, $\dot{M} \gg \dot{M}_{cr}$. In general, the higher the accretion rate, the closer the inner edge of the disk to the innermost bound orbit around a Kerr black hole. Although the efficiency of converting the accreted mass into radiation decreases as the location of the inner disk edge becomes closer to the black hole, the total disk luminosity may exceed the Eddington limit for a spherical object by more than an order of magnitude. Most of this luminosity is radiated from the surfaces of two funnels which form along the rotation axis. A wind from the disk surface may inject modest amount of matter into the funnels. This matter may be accelerated by the radiation pressure up to relativistic velocities and a double jet could be formed.

Our models may be relevant to active galactic nuclei, quasars, and some galactic X-ray sources like Sco X-1 and Cyg X-1, and also SS 433.

1. Introduction

A standard theory of accretion disks was developed under the assumption that disks are geometrically thin (Pringle and Rees 1972, Shakura and Sunyaev 1973, Novikov and Thorne 1973, Lynden-Bell and Pringle 1974). In that case it is possible to consider the vertical and radial disk structures separately, and it is reasonable to assume local heat balance and Keplerian rotation. For a thin, not selfgravitating disk the vertical component of gravitational acceleration may be written in the Newtonian approximation as

$$g_z = \frac{GM}{r^3} z, \tag{1.1}$$

where M is the mass of central object, r is the disk radius, and z is the veritical distance from the equatorial plane. If the disk accretion is stationary then the rate of heat generation can be calculated from the conservation laws of mass, momentum and energy, independently of the viscosity mechanism. The local heat balance requires that heat be radiated from the disk surface at a rate given by the flux

$$F = (-\dot{M}) \frac{3}{8\pi} \frac{GM}{r^3} \left[1 - \left(\frac{r_{in}}{r}\right)^{1/2}\right]. \tag{1.2}$$

Here $(-\dot{M})$ is the mass accretion rate; it is assumed that there is no torque applied at the inner disk radius, r_{in}. For the disk to remain in hydrostatic equilibrium the heat flux must be less than the critical value

$$F < F_{cr} = \frac{c}{\varkappa} g, \tag{1.3}$$

where c is the speed of light, $\varkappa$ is the opacity, and g is the surface gravity. Combining eqs. (1.1)-(1.3) we find that the disk thickness must be larger than the critical value,

$$z > z_{cr} = (-\dot{M}) \frac{3\varkappa}{8\pi c} \left[1 - \left(\frac{r_{in}}{r}\right)^{1/2}\right]. \tag{1.4}$$

The ratio z_{cr}/r reaches maximum at $r/r_{in} = 9/4$ (assuming $\varkappa$ = const.), and we have

$$\left(\frac{z}{r}\right)_{\max} > \left(\frac{z_{cr}}{r}\right)_{\max} = (-\dot{M}) \frac{\varkappa}{18\pi c r_{in}}. \tag{1.5}$$

It is clear that if the accretion rate is sufficiently high the disk has to be thick in the sense that $z \approx r$. It is frequently suggested that many astrophysically interesting objects (quasars, Cyg X-1, Sco X-1) are luminous accretion disks. High luminosity implies a high accretion rate, and through eq. (1.5) that the disk must be thick. The aim of this paper is to present a simple theory of thick, fully relativistic disks around black holes. We shall follow the approach developed by Abramowicz, Jaroszyński and Sikora (1978), Kozłowski, Jaroszyński and Abramowicz (1978), and Paczyński and Wiita (1980).

It will be convenient to define a critical accretion rate and a critical luminosity. In the case of a *spherical* star a concept of critical (Eddington) luminosity is introduced by integrating the critical radiation flux [eq. (1.3)] over the whole *spherical* surface

$$L_{cr} \equiv 4\pi R^2 F_{cr} = \frac{4\pi c G M}{\varkappa}, \tag{1.6}$$

where M is the mass and R is the radius of the spherical star. A total luminosity of a *thin disk* may be calculated by integrating the flux (eq. (1.2)) over the two disk surfaces,

$$L_d \equiv 2 \int\limits_{r_{in}}^{\infty} \mathrm{F} 2\pi r \, \mathrm{d}r = \frac{GM}{2r_{in}} (-\dot{M}). \tag{1.7}$$

Now we shall artificially combine eqs. (1.6) and (1.7) derived for two *different* objects, to define a quantity which we shall name a critical accretion rate

$$(-\dot{M}_{cr}) \equiv \frac{8\pi c r_{in}}{\varkappa}. \tag{1.8}$$

If a thin disk could accrete at the critical rate it would have the total luminosity equal to the critical value, *i.e.* the "Eddington limit" of a spherical star. However, such a disk cannot be thin. Combining eqs. (1.5) and (1.8) we find

$$\left(\frac{z}{r}\right)_{\max} > \frac{4}{9} \frac{\dot{M}}{\dot{M}_{cr}}. \tag{1.9}$$

Eq. (1.9) gives physical meaning of the critical accretion rate: when the accretion rate is comparable to, or larger than the critical value, the disk must be thick. Under these conditions the thin disk theory cannot be applied. It follows that a theory of thick disks is needed for objects with a critical or supercritical accretion.

We would like to make one more general remark before considering specific models. As the nature of disk viscosity is not known, it is customary to introduce a free parameter α, defined by the relation

$$\eta\left(-r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right) = \alpha p\,, \tag{1.10}$$

where η is the viscosity, Ω is the angular disk velocity at radius r, and p is the pressure (either gas pressure or total pressure, depending on the preference of the investigator). For physical reasons the dimensionless parameter α should satisfy the conditions

$$0 < \alpha < 1, \tag{1.11}$$

but there is no other general constraint on its value. Any physical situation may be described by treating α as a free function of space and time. In order to get any specific results at all it is usually assumed that α is constant. It is also customary to consider a rather small subrange of possible values, $0.1 \leqslant \alpha < 1$. In spite of these entirely *ad hoc* assumptions the so-called "alpha disk theory" is very popular, as there is no better theory available at this time.

There are some very general properties of thin disks which do not depend at all on the viscosity mechanism. The formula for the surface radiation flux (eq. (1.2)) is derived from the conservation laws, without any reference to a specific value of viscosity. A similar formula is available for thin disks around black holes (Novikov and Thorne, 1973). Dynamical stability requires the inner disk edge to coincide with the marginally stable orbit (Novikov and Thorne, 1973). This property is also viscosity independent.

The aim of this paper is to find those properties of thick disks which are viscosity independent. In this paper we restrict ourselves to thick disks which are in hydrostatic equilibrium. This requires the radial drift velocity of matter (*i.e.* the accretion velocity) to be much less than the sound velocity

$$|V_r| \ll V_s. \tag{1.12}$$

The "alpha disk theory" may be used to derive the following relation between these two velocities and the rotational velocity V:

$$|V_r| \approx \alpha\frac{V_s^2}{V} \approx \alpha V_s \frac{z}{r}. \tag{1.13}$$

In a thin disk we have $z \ll r$, and the condition (1.12) is fulfilled for any $\alpha < 1$. In a thick disk we have $z \approx r$, and the condition (1.12) requires

$$\alpha \ll 1. \tag{1.14}$$

The condition (1.14) is implicitly assumed throughout this paper. We may put it this way. Our theory does not depend on the unknown viscosity, but the viscosity has to be small.

The condition (1.14) has a very important consequence. The surface mass density of a disk is given as

$$\sum \equiv \int\limits_{-z}^{+z} \varrho \mathrm{d}z = \frac{\dot{M}}{2\pi r V_r} \approx \frac{1}{\alpha} \frac{(-\dot{M})r}{2\pi z^2 V}. \tag{1.15}$$

For given global disk parameters, $\dot{M}$, r, z, V, the surface mass density is proportional to α^{-1}, and small α implies large surface density. It is convenient to consider a disk which is like a star with a peculiar shape. Throughout this paper we assume that a disk, just like a star, has a well defined surface, a photosphere. That means that the pressure scale height at the photosphere is much smaller than disk thickness or disk radius. We assume that any atmospheric phenomena, like a chromosphere, a corona, or a wind, do not disturb the interior disk structure and therefore allow the photosphere to exist in hydrostatic equilibrium. There is plenty of evidence from studies of spherical stars that this is a reasonable assumption. There may be stars, like those of the Wolf-Rayet type, for which there may be no photosphere in hydrostatic equilibrium, and it is possible that there are such disks as well. In this paper we confine our interest to those disks in which a static photosphere exists, just like it does in most spherical stars.

We feel that this rather long introduction is necessary because so much work was published in the past for $\alpha \approx 1$ disk models, while our theory is developed for disks with $\alpha \ll 1$. It is our experience that it is very difficult for many readers to accept our convention: $\alpha \ll 1$. Our theory is developed with general relativity fully taken into account, but this aspect of our work does not introduce any conceptually difficult problems, and does not require any special introduction.

Although we employ the usual $c = 1 = G$ "geometrical" convention in the rest of the paper, some of our formulae for the convenience of the reader are given in c.g.s. units. They are distinguished from the others by asterisks: for example (2.32)*. Many of the formulae have the same form in both geometrical and c.g.s. units. If it is obvious that this is the case asterisk is dropped. By putting $c = 1 = G$ in the formulae with asterisk one can reduce *all* our formulae to the geometrical units.

2. The Model

In this section we outline our theory of thick relativistic accretion disks rotating around Kerr black holes. We emphasize the physical model and present only those few formulae which are necessary to understand the mathematical nature of our approach. All the technical details and derivations are presented in APPENDIX A, where the equations are labelled: (A.1)-(A.33).

We assume that the mass of the disk is so small that the gravitational field may be described by the Kerr geometry of the black hole with the mass M and the angular momentum per unit mass a (*cf.* eqs. (A.1)).

The disk and the hole rotate in the same direction, and their equatorial planes coincide. We use the Boyer-Lindquist (B-L) spherical coordinates, t, φ, r, θ throughout this paper. Far away from the hole, in the weak field regime, the B-L coordinates are just standard spherical coordinates.

We assume that accretion is stationary and axially symmetric in the sense that no physical quantity depends on time, t, or on the azimuthal angle, φ. The disk has two parts. The outer disk, which extends beyond the radial coordinate r_{out}, is thin and may be described by a standard thin disk theory. The inner part, located between r_{out} and r_{in}, is thick. The whole disk is in a hydrostatic equilibrium. That means that only rotational (*i.e.* azimuthal) velocity is dynamically important, while the accretion flow is subsonic and has no dynamical effect. At r_{in} located between the marginally stable and marginally bound orbits there is a cusp similar to that in the L_1 Lagrange point on the Roche lobe in a close binary system (Abramowicz *et al.*, 1978, Kozłowski *et al.*, 1978). The gas flows from the disk through the cusp into the black hole, forming a flat axially symmetric stream. The matter in the stream is in free fall, so neither pressure nor viscosity is dynamically important, and the streamlines may be approximated with free particle trajectories. In this model the cusp separates the disk, where the accretion flow is highly subsonic, from the stream where the flow is highly supersonic. The flow through the cusp is transonic, and no simple approximation can be made. Fortunately, since the cusp region is small, its poorly known structure cannot have an appreciable effect on the disk and the stream flow. The whole object is schematically shown in Fig. 1.

Figure 1 deserves a comment. In this figure the absolute thickness of the thin outer disk, z, is smaller than the absolute thickness of the thick inner disk. In general this does not have to be so. It is sufficient for the relative disk thickness, z/r, to decrease between the thick and thin parts.

The surface of the outer thin disk is close to the equatorial plane,

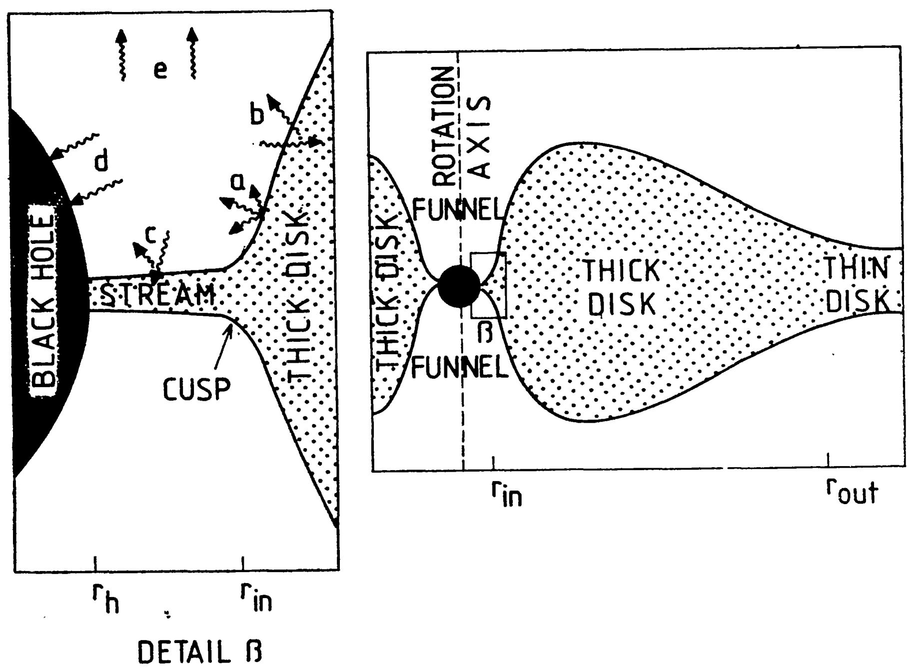

Fig. 1. The main properties of our model are pictured in this meridional section (not to scale). Outside r_{out} the disk is *thin* in the sense that its height $h(r)$ is small in comparison with $r: h \ll r$. This means that for $r > r_{out}$ all kinematical characteristics of the fluid motion (*e. g.*, angular velocity, angular momentum, *etc.*) are equal to their Keplerian values. Between r_{out} and r_{in} the disk is *thick*. Its shape follows from assumed, non-Keplerian, angular momentum distribution. It is also assumed that between r_{out} and r_{in} the radial velocity is so small that it has no dynamical importance and that on the surface of the disk the effective gravity is balanced by the radiation pressure. Between r_{in} and the horizon ($r = r_H$) the flow of the *stream* can be approximated by free-fall. Close to the hole the walls of the disk are very steep and form *funnels* along the rotation axis. Radiation emitted here (a) can be re-absorbed (b), or reflected (c), or absorbed by the optically thick stream (c), or swallowed by the hole (d). We assume that the net result of these processes is high collimation of the radiation flux (the Lynden-Bell (1978) mechanism) in the direction of the axis (e).

and it may be described in the B-L coordinates by the simple equation

$$\theta = \theta_t(r) \approx \frac{\pi}{2} \quad \text{for } r_{out} < r < \infty. \tag{2.1}$$

The thin disk dynamics is not affected by either accretion flow or pressure gradients. The streamlines are well approximated by circular orbits of free particles. That means that angular velocity, Ω, and specific angular momentum, $\mathscr{L}$, are very nearly Keplerian

$$\Omega = \Omega_K(r), \quad \mathscr{L} = \mathscr{L}_K(r) \quad \text{for } r_{out} < r < \infty, \tag{2.2}$$

where the subscript K stands for Keplerian.

Only the azimuthal component of the velocity is dynamically important. Its value, measured by the zero-angular-momentum-observer (Bardeen, 1973) is equal:

$$V_\varphi = \frac{r^2+a^2+2a^2M/r}{(r^2-2Mr+a^2)^{1/2}}\left[\frac{M^{1/2}}{r^{3/2}+aM^{1/2}} - \frac{2aM}{r(r^2+a^2+2a^2M/r)}\right] \quad \text{for } r_{out} < r < \infty. \tag{2.3}$$

The surface of the inner, thick disk may be described as

$$\theta = \theta_d(r) \qquad \text{for } r_{in} < r < r_{out}, \tag{2.4}$$

with the boundary conditions

$$\theta_d(r_{in}) \approx \frac{\pi}{2}, \qquad \theta_d(r_{out}) \approx \frac{\pi}{2} \tag{2.5}$$

and

$$r_{mb} < r_{in} < r_{ms}, \tag{2.6}$$

where r_{mb} and r_{ms} are the radii of marginally bound and marginally stable orbits, respectively. The cusp is located at r_{in}. As the thick disk is assumed to be in a hydrostatic equilibrium, the accretion flow is not important for its dynamics, but the pressure gradients are important. The disk surface must be perpendicular to the "effective gravity" vector, a_i (*cf.* eqs. (A.12)). The angular velocity, Ω and the specific angular momentum, $\mathscr{L}$, may be found at every point of the disk *surface* (eqs. (A.13) and (A.9a)). For a given disk shape, $\theta_d(r)$, these are unique functions

$$\Omega = \Omega_d(r), \qquad \mathscr{L} = \mathscr{L}_d(r) \qquad \text{for } r_{in} \leqslant r \leqslant r_{out}. \tag{2.7}$$

The thick disk is thin at the two boundaries, r_{in} and r_{out}, and pressure gradients are small at these boundaries. Therefore, we have the boundary conditions

$$\Omega_d(r_{in}) = \Omega_K(r_{in}), \qquad \Omega_d(r_{out}) = \Omega_K(r_{out}), \tag{2.8a}$$

$$\mathscr{L}_d(r_{in}) = \mathscr{L}_K(r_{in}), \qquad \mathscr{L}_d(r_{out}) = \mathscr{L}_K(r_{out}). \tag{2.8b}$$

Any one of the three equations: (2.4), (2.7), with the corresponding boundary conditions: (2.5), (2.8a), or (2.8b) may be used to describe the shape of the thick disk surface. Given any one of these relations, the other two may be obtained using the eqs. (A.15) and (A.9a). This is possible because the disk surface is assumed to be in a hydrostatic equilibrium; we do not have to know *anything* about the disk interior such as the equation of state or the distribution of rotational velocity there. Only the azimuthal component of the velocity is dynamically important. Its value, given in the terms of the metric components g_{tt}, $g_{t\varphi}$, $g_{\varphi\varphi}$, (*cf.* eq. (A.1)) and the angular velocity of the disk surface, Ω, and measured by the zero-angular-momentum-observer located at the surface (note that such an observer *is not* at rest with respect to the surface of the disk) is given by

$$v_\varphi = \frac{|g_{\varphi\varphi}|(\Omega + g_{t\varphi}/g_{\varphi\varphi})}{(g^2_{t\varphi} - g_{tt}g_{\varphi\varphi})} \quad \text{for } r_{in} \leqslant r \leqslant r_{out}. \tag{2.9}$$

Although this physical quantity plays an important role in the description of the Keplerian motion and therefore is also important in the theory of the *thin*, relativistic accretion disks (Bardeen, 1973), it never appears explicitly in our theory of *thick* accretion disks. Therefore we shall not discuss its physical meaning here.

We ignore at the moment the structure of the cusp at r_{in}, but we assume that it is thin (*cf.* eq. (2.5)). The stream flows supersonically from the cusp, and we approximate the streamlines with the trajectories of freely falling particles. The pressure gradients are ignored within the stream, and therefore its surface may be approximately described by the relation

$$\theta = \theta \quad (r) \approx \frac{\pi}{2} \quad \text{for } r_h < r < r_{in}, \tag{2.10}$$

where r_h is the radius of the black hole horizon. Within the stream the specific angular momentum, $\mathscr{L}$, and the specific energy, U, are conserved, and we have

$$\mathscr{L} \approx \mathscr{L}_K(r_{in}) \equiv \mathscr{L}_* \quad \text{for } r_h < r < r_{in}, \tag{2.11a}$$

$$U \approx U_K(r_{in}) \equiv U_* \quad \text{for } r_h < r < r_{in}. \tag{2.11b}$$

Since the quantities U_* and $\mathscr{L}_*$ describe the conditions on the Keplerian orbit at r_{in} they are related by (Bardeen, 1973)

$$U_*^2 = \frac{(r_{in}^2 - 2Mr_{in} + a^2)r_{in}}{(r_{in}^2 + a^2)r_{in} + 2a^2M - 4\mathscr{L}_* aM - \mathscr{L}_*^2(r_{in} - 2M)}. \tag{2.12}$$

Both the azimuthal component of the stream velocity measured by the zero-angular-momentum-observer, v_φ and the radial component of the stream velocity, u^r, may be easily calculated from the conservation laws of energy and angular momentum, (eq. (2.11))

$$v_\varphi(r) = \frac{r(r^2 - 2Mr + a^2)^{1/2}\mathscr{L}_*}{(r^2 + a^2)r + 2a^2M - 2aM\mathscr{L}_*} \quad \text{for } r_h < r < r_{in}, \tag{2.13a}$$

$$\begin{aligned} -u^r(r) = \{ & r^{-3}[(r^2 + a^2)r + 2a^2M - 2aM\mathscr{L}_* - (r - 2M)\mathscr{L}_*^2]U_*^2 - \\ & - r^{-2}(r^2 - 2Mr + a^2)\}^{1/2} \quad \text{for } r_h < r < r_{in}. \end{aligned} \tag{2.13b}$$

The quantity u^r is just the r-component of the velocity of matter expressed in the B-L coordinates. Finally, the surface rest mass density measured by an observer comoving with matter equals

$$\sum = \frac{(-\dot{M})}{2\pi r(-u^r)c}. \qquad (2.14)^*$$

The variation of surface mass density in the stream with the radial coordinate is shown for a critical accretion rate $(-\dot{M}_{cr})$ for a Schwarzschild black hole in Fig. 2 and for a Kerr black hole ($a = 0.998$) in Fig. 3. The critical accretion rate is defined by eqs. (3.6), (3.7).

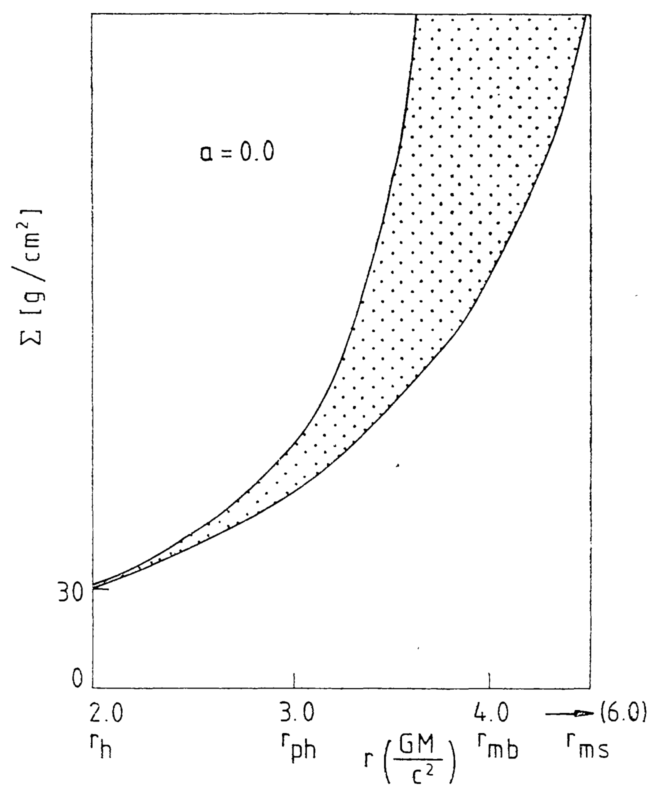

Fig. 2. The surface density in the stream for the critical accretion rate. With the cusp structure not well-determined, the initial values of the velocity and the surface density are not known very accurately. Therefore, we have a stripe in this Figure rather than a single line. Note that, because $\sum > 30\,\mathrm{g/cm^2}$, the stream is optically thick.

Let us consider now the restrictions that limit the freedom of the shape of the thick disk. For rotation to be dynamically stable the specific angular momentum, $\mathscr{L}$, should not decrease outwards from the rotation axis along any surface of constant entropy (Seguin, 1975). We assume

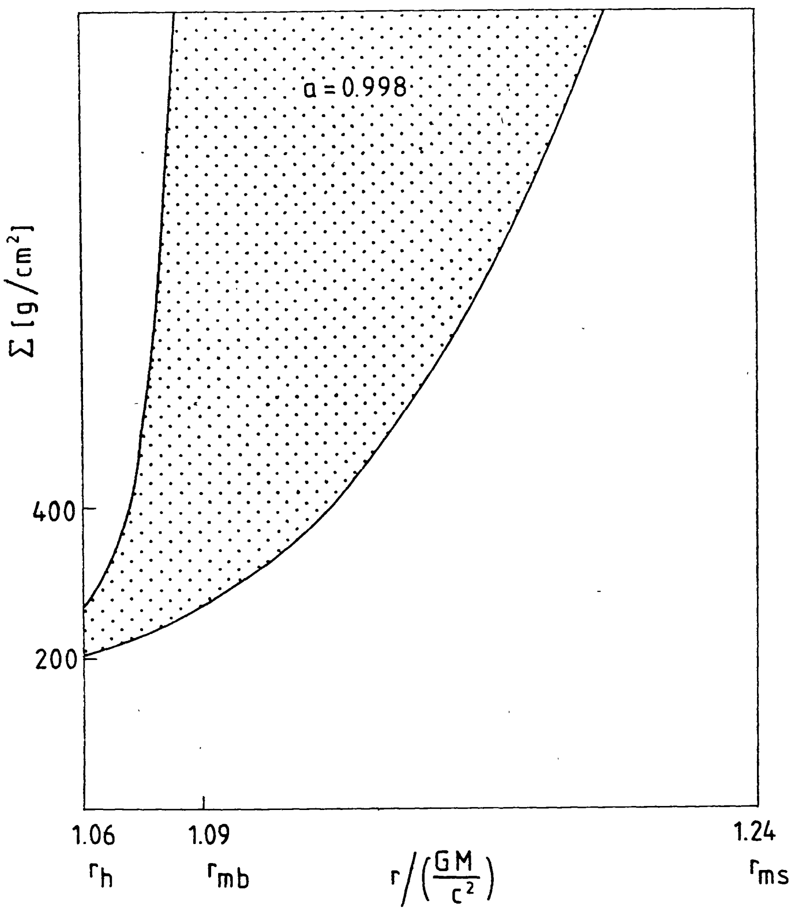

Fig. 3. The same as in Fig. 2 but for the canonical black hole, $a = 0.998\ M$. The stream is optically thick.

that this condition is satisfied along the thick disk surface. Therefore, the function $\mathscr{L}_d(r)$ (*cf.* eq. (2.7b)) must satisfy the condition

$$\frac{d\mathscr{L}_d(r)}{dr} \geqslant 0 \qquad \text{for } r_{in} \leqslant r \leqslant r_{out}. \tag{2.15}$$

For a stationary accretion and positive viscosity the surfaces of constant angular velocity, Ω, have a topology of cylinders, with Ω decreasing outwards from the rotation axis. We assume that the disk surface topology is sufficiently simple that the function $\Omega_d(r)$ (*cf.* eq. (2.7)) must satisfy the condition

$$\frac{d\Omega_d(r)}{dr} \leqslant 0 \qquad \text{for } r_{in} \leqslant r \leqslant r_{out}. \tag{2.16}$$

If the disk surface is specified with the function $\theta_d(r)$ (eqs. (2.4), (2.5)), then the corresponding functions $\Omega_d(r)$ and $\mathscr{L}_d(r)$ (eqs. (2.7)) have to be evaluated, and it should be checked whether the boundary conditions (2.8) are satisfied. Next it should be checked whether the conditions (2.15) and (2.16) are satisfied over the whole disk surface, before the function $\theta_d(r)$ can be accepted as a valid description of a thick disk shape. It may be more convenient to specify initially the functions $\Omega_d(r)$ or $\mathscr{L}_d(r)$, or a combination of the two, which would automatically satisfy the conditions (2.15), (2.16), and (2.8), and to evaluate the corresponding shape of the disk surface, $\theta_d(r)$, later. In this case we have to check whether the boundary conditions (2.3) are satisfied. It may be shown (*cf.* the derivation of the eq. (A. 19)) that this requirement is equivalent to the condition

$$\int_{r_{in}}^{r_{out}} \frac{\Omega_d(r)\,\mathrm{d}\mathscr{L}_d(r)}{1-\Omega_d(r)\mathscr{L}_d(r)} = \ln\frac{U_K(r_{out})}{U_K(r_{in})}, \tag{2.17}$$

where U_K is the energy per unit mass in a Keplerian orbit.

The models to be presented in this paper were constructed in the following way. The distribution of angular velocity $\Omega_d(r)$ or angular momentum $\mathscr{L}_d(r)$ was specified along the disk surface for $r_{in} \leqslant r \leqslant r_{out}$. The distribution was chosen so that the boundary conditions (2.8) and the conditions (2.15) and (2.16) were satisfied, and the only free parameter was varied so as to satisfy the condition (2.17). At this point the shape of the thick disk was considered to be dynamically acceptable.

Let us now consider radiative processes at the disk surface. In the outer thin disk we have a local heat balance. That means that heat generated within the thin disk at a radial coordinate r is radiated from the disk surface at the same value of r.

Using the energy conservation law (see Novikov and Thorne, 1973) one can write the following expression for the flux of radiation emitted locally from the surface of the thin disk:

$$F = -\frac{A_K}{4\pi r}\left(\frac{\mathrm{d}\Omega_K}{\mathrm{d}r}\right)T \quad \text{for } r_{out} < r < \infty, \tag{2.18}$$

where A_K is the redshift factor (*cf.* eqs. (A.7) and (A.9)) of a particle in the Keplerian orbit, Ω_K is the angular velocity in this orbit, and T is the total torque acting through the cylinder $r = \mathrm{const}$. Let T_K be an analogous quantity related to a disk which is *everywhere* thin. For $r_{out} < r < \infty$ both T and T_K obey the same differential equation valid for thin disks,

$$\frac{\mathrm{d}T}{\mathrm{d}r} - \frac{\mathscr{L}_K}{1-\Omega_K\mathscr{L}_K}\left(\frac{\mathrm{d}\Omega_K}{\mathrm{d}r}\right)T = \dot{M}\frac{U_K}{1-\Omega_K\mathscr{L}_K}\left(\frac{\mathrm{d}\mathscr{L}_K}{\mathrm{d}r}\right), \tag{2.19}$$

where U_K is the energy per unit rest mass of a particle in the Keplerian orbit. Therefore, for any $r_{out} < r < \infty$ one has

$$T(r) - T(r_{out}) = T_K(r) - T_K(r_{out}). \tag{2.20a}$$

The value of the torque on the outer edge of the thick disk is not, in general, equal to $T_K(r_{out})$. Let us denote by F_K the flux emitted from the surface of a disk which is *everywhere* thin. The radiation flux, F, emitted from the surface of the thin part of our disk will be modified by the presence of the thick part:

$$F = F_K - \frac{A_K}{4\pi r}\left(\frac{\mathrm{d}\Omega_K}{\mathrm{d}r}\right)[T(r_{out}) - T_K(r_{out})] \tag{2.20b}$$

for $r_{out} < r < \infty$.

Note that the correspondence between the quantity T_K introduced here for convenience and the integrated stress, W, of Thorne and Novikov (1973) is given by

$$T_K \equiv 2\pi A_K(r^2 - 2Mr + a^2)\, W \quad \text{for } r_{out} < r < \infty \tag{2.21}$$

and thus can be easily calculated.

Within a thick disk there is no *local* heat balance. However, there must be *global* energy and angular momentum conservation. Radiation leaving the disk surface carries away a total flux of energy L_E and angular momentum L_j which may be calculated as

$$L_E = \int F_i u_t \mathrm{d}S^i, \tag{2.22a}$$

$$L_j = -\int F_i u_\varphi \mathrm{d}S^i, \tag{2.22b}$$

where the integral is taken over the whole thick disk surface, $\mathrm{d}S^i$ is the oriented surface element (*cf.* eq. (A. 32)), F_i is the radiative heat flux through a unit area, and u_t and u_φ are defined by eqs. (A.5)-(A.9). At the inner edge of the thick disk, r_{in}, matter is freely flowing out, carrying the rest mass at the rate $(-\dot{M})$, but *there is no torque* at this boundary. At r_{out} matter is carrying rest mass into the thick disk at the same rate $(-\dot{M})$, but *there is a torque* at this boundary. Combining the integrals over the disk surface (2.19) with the boundary conditions at r_{in} and r_{out} we may write two global conservation laws (*cf.* eqs. (A.28)). These equations may be used with the integrals (2.22) to express the accretion rate and torque at r_{out} as

$$\dot{M} = (L_E - \Omega_{out} L_j)/\mathscr{E}, \tag{2.23a}$$

$$T_{out} \equiv T(r_{out}) = [(\mathscr{L}_{out} U_{out} - \mathscr{L}_{in} U_{in}) L_E - (U_{out} - U_{in}) L_j]/\mathscr{E}, \tag{2.23b}$$

where $\mathscr{E}$ is the efficiency of converting rest mass into radiation within the thick disk. It does not depend on the disk structure but only on r_{in} and r_{out} (*cf.* eq. (A. 31)).

Given the accretion rate and torque at r_{out} we may calculate the total radiative flux from the thin disk as

$$(L_E)_{thin} = 2 \int\limits_{r_{out}}^{\infty} F^i u_t \mathrm{d}S_i = T_{out}\Omega_{out} + \dot{M}(1-U_{out}), \tag{2.24}$$

and the total luminosity of the whole disk as

$$(L_E)_{total} \equiv L_d = L_E + (L_E)_{thin} = (-\dot{M})(1-U_{in}). \tag{2.25}$$

Equation (2.25) has the same form as that of Novikov and Thorne's (1973) equation relating the accretion rate and total luminosity of a thin disk. The main difference is that in our model the inner disk edge is in general closer to the horizon than r_{ms}, and therefore the efficiency of conversion of rest mass into radiation is *lower*. Of course we do not know what fraction of $(L_E)_{total}$ will reach infinity, what fraction will be swallowed by the hole, and what fraction will be recaptured and reprocessed by the disk surface (the "reflection effect").

Now we shall relate the radiative flux F_i to the shape of the thick disk, $\theta_d(r)$. A well defined upper limit to F_i follows from the assumption of hydrostatic equilibrium. The limit has the name "critical flux" and is given as

$$(F^i)_{cr} = -\frac{1}{\varkappa} a^i, \tag{2.26}$$

where a^i is the acceleration or *effective gravity* on the disk surface (*cf.* eq. (A. 12)) and $\varkappa$ is the coefficient of opacity per gram of rest mass. Note that in c.g.s. units one has to use $c/\varkappa$ instead of $1/\varkappa$ in eq. (2.26). Eq. (2.26) is a relativistic analogue of eq. (1.3). In general, for any specific disk-stream model we may write

$$F^i = C(r)(F^i)_{cr} \quad \text{for } r_h < r < \infty, \tag{2.27}$$

where $C(r)$ is a function of r which must satisfy the condition

$$0 \leqslant C(r) < 1 \quad \text{for } r_h < r < \infty. \tag{2.28}$$

If we know the viscosity and equation of state, then in principle we should be able to calculate $C(r)$, just as we should be able to calculate the disk shape, $\theta_d(r)$. However, as we do not know the viscosity, we are forced to treat $C(r)$ as a free function, subject to the conditions (2.28).

In this paper we limit ourselves to the simple case

$$C(r) \equiv C \equiv \text{const}, \quad F^i = -\frac{C}{\varkappa} a^i \quad \text{for } r_{in} < r < \infty, \tag{2.29a}$$

$$C(r) = 0, \quad F^i = 0 \quad \text{for } r_h < r < r_{in}, \tag{2.29b}$$

with

$$0 < C < 1, \tag{2.30a}$$

$$\varkappa = \varkappa_{el} = \text{const}, \tag{2.30b}$$

where $\varkappa_{el}$ is the electron scattering opacity. The condition (2.29a) is frequently referred to as the von Zeipel theorem. The condition (2.29b) describes the simple fact that within our model matter in the stream has no internal energy to be radiated away. With the approximations listed above we may write eqs. (2.22) and (2.23) as

$$L_E = \frac{C}{\varkappa} L_E^* \equiv -\frac{C}{\varkappa} \int a_i u_t \mathrm{d}S^i, \tag{2.31a}$$

$$L_j = \frac{C}{\varkappa} L_j^* \equiv \frac{C}{\varkappa} \int a_i u_\varphi \mathrm{d}S^i, \tag{2.31b}$$

$$(-\dot{M}) = \frac{C}{\varkappa} \left[\frac{L_E^* - L_j^* \Omega_K(r_{out})}{\mathscr{E}(r_{in}, r_{out})} \right], \tag{2.31c}$$

$$T_{out} = \frac{C}{\varkappa} \frac{(\mathscr{L}_{out} U_{out} - \mathscr{L}_{in} U_{in}) L_E^* - (U_{out} - U_{in}) L_j^*}{\mathscr{E}(r_{in}, r_{out})}. \tag{2.31d}$$

Given eqs. (2.31) and the thick disk shape $\theta_d(r)$ we may calculate the luminosity radiated by the thick disk, L_E, the accretion rate, $\dot{M}$, and the torque at r_{out}, T_{out}.

In the Newtonian limit eq. (2.18) gives

$$F_{out} \approx \frac{3}{8\pi} \frac{T_{out} \Omega_{out}}{r_{out}^2}. \tag{2.32}$$

This is, therefore, an approximated value of the flux radiated from the outer edge of the thick disk. We use this value in order to make the last consistency check for our models. It is obvious that the flux (2.32) should be less than the critical, F_{cr}. In the Newtonian approximation

$$F_{cr} \approx \frac{C}{\varkappa} \frac{GM}{r^2} \left[\frac{\pi}{2} - \theta_d(r_{out}) \right]. \tag{2.33}$$

Combining eqs. (2.32), (2.33) and (1.6) we obtain

$$\frac{3}{2}\frac{T_{out}\,\Omega_{out}}{L_{cr}} \leqslant \left[\frac{\pi}{2} - \theta_d(r_{out})\right]. \tag{2.34}$$

Therefore, for the thin disk at and beyond r_{out} the condition

$$\frac{3}{2}\frac{T_{out}\,\Omega_{out}}{L_{cr}} \ll 1 \tag{2.35}$$

must be satisfied. This is the last consistency check we use for our models.

We do not take into account the "reflection effect", *i.e.* recapturing, reprocessing and reemission of radiation by the disk and stream. A theory of this effect was studied by Cunningham (1975, 1976) for thin disks, and it is studied by Sikora (1979) for our thick disk models. The effect may be very important in the "funnels" which are present near the rotation axis of our disks (*cf.* Fig. 1).

3. Results

The disk model described in the previous section is fairly general. There is a lot of freedom in choosing its shape or the distribution of angular momentum on its surface. In this section we present some specific models as an example of our general approach.

In the thick disks most of the outgoing radiation is emitted through the funnels along the rotation axis (*cf.* Fig. 1). We shall consider, therefore, two sets of models which are as different as possible within the funnels region. In the first set we assume that the disk surface rotates rigidly (*i.e.* Ω = const.) in the region close to the cusp, $r \geqslant r_{in}$. In the second set we assume that the specific angular momentum is constant $\mathscr{L}$ = const. in this region. Rigid rotation, Ω = const., and vorticity-free rotation, $\mathscr{L}$ = const, are two limiting cases (*cf.* eqs. (2.15), (2.16)) and a real disk should have its properties near the cusp intermediate between those of these two sets. Because the angular momentum distribution of the thick disk has to satisfy the outer boundary condition (2.8) and the condition (2.17), neither Ω = const. nor $\mathscr{L}$ = const can be continued all the way from r_{in} to r_{out}. There are, of course, many possible ways to specify a surface distribution of Ω and $\mathscr{L}$ in the remaining part of the thick disk which agrees with Ω = const. or $\mathscr{L}$ = const. in the funnel region and with all the conditions adopted in our model. For the purpose of our examples we use the following distributions consistent with these conditions:

Case 1. The thick disk surface consists of the three sections (*cf.* Fig. 4)

a) $\Omega = \Omega_{in} = \text{const.} \quad \text{for } r_{in} \leqslant r \leqslant r_{c1},$ (3.1a)

b) $\mathscr{L} = \mathscr{L}_c = \text{const.} \quad \text{for } r_{c1} \leqslant r \leqslant r_{c2},$ (3.1b)

c) $\Omega = \Omega_{out} = \text{const.} \quad \text{for } r_{c2} \leqslant r \leqslant r_{out}.$ (3.1c)

Abramowicz *et al.* (1979) have shown in the Newtonian theory that this particular distribution maximizes the luminosity for the given r_{in} and r_{out}.

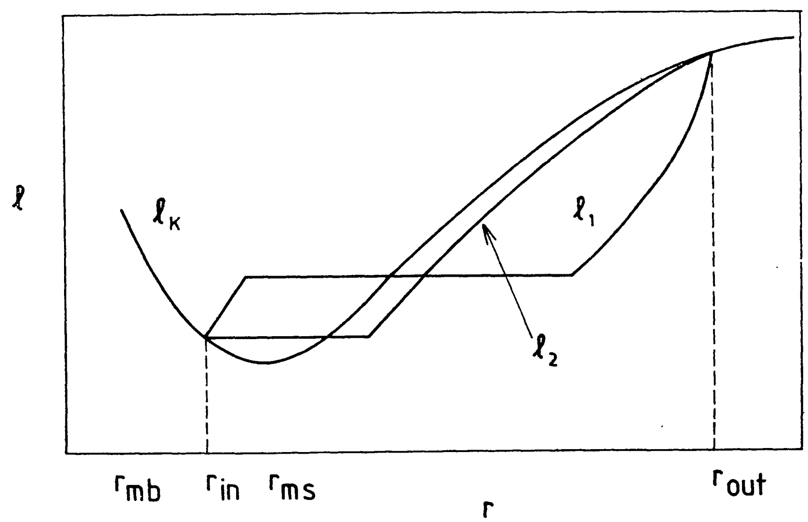

Fig. 4. Typical angular momentum distributions for the two families of our models (schematically).

From the boundary conditions (2.8a) and from the condition of hydrostatic equilibrium (2.17) it follows that

$$\Omega_{in} = \Omega_K(r_{in}), \tag{3.1d}$$

$$\Omega_{out} = \Omega_K(r_{out}), \tag{3.1e}$$

$$\mathscr{L}_c = \frac{A_{in} - A_{out}}{A_{in}\Omega_{in} - A_{out}\Omega_{out}}, \tag{3.1f}$$

where A_{in} and A_{out} are given by eq. (A. 9c) with Ω equal to either Ω_{in} or Ω_{out}.

The outer and inner edges of the thick disk were chosen to be

$$r_{out} = 3 \times 10^3 M, \tag{3.2a}$$

$$r_{mb} < r_{in} < r_{ms}. \tag{3.2b}$$

The choice of the outer edge is rather arbitrary but the choice of the inner edge is well motivated physically (see Kozłowski *et al.*, 1978). The thick disk luminosity L_E is given by eq. (2.31a) and the accretion rate $\dot{M}$ could be calculated from (2.31c). Note that both L_E and $\dot{M}$ depend only on r_{in} in this set of models because this set forms a one-parameter family with r_{in} being the parameter. Finally, the total disk luminosity L_d is given by eq. (2.25) The total disk luminosity depends also only on r_{in}. Solving equations $L_d = L_d(r_{in})$ and $\dot{M} = \dot{M}(r_{in})$, one gets the function $L_d = L_d(\dot{M})$ which describes the most important relation among the disk's global characteristics. This function is shown in Fig. 5.

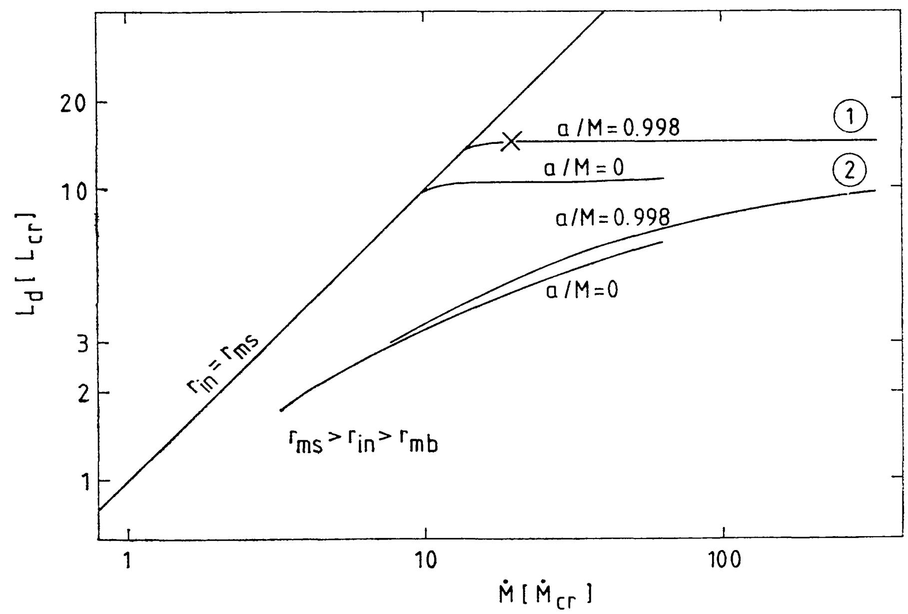

Fig. 5. The relation between L_d and $\dot{M}$ expressed relative to the critical values of L and $\dot{M}$ (see text for details). The straight line corresponds to the maximum efficiency of converting accreted mass into outgoing radiation, *i. e.*, to the case $r_{in} = r_{ms}$.

Case 2. The thick disk surface consist of the two sections (see Fig. 4)

a) $\mathscr{L} = \mathscr{L}_{in} = \text{const.} \quad \text{for } r_{in} \leqslant r \leqslant r_c,$ (3.3a)

b) $\Omega \mathscr{L}^{\beta} = \Omega_{out} \mathscr{L}^{\beta}_{out} = \text{const.} \quad \text{for } r_c \leqslant r \leqslant r_{out},$ (3.3b)

for some $\beta = \text{const.}$ Again, from the boundary conditions it follows that

$$\mathscr{L}_{in} = \mathscr{L}_K(r_{in}), \tag{3.3c}$$

$$\mathscr{L}_{out} = \mathscr{L}_K(r_{out}), \tag{3.3d}$$

$$\Omega_{out} = \Omega_K(r_{out}). \tag{3.3e}$$

The exponent β is chosen in such a way that the condition (2.17), *i.e.* the condition of hydrostatic equilibrium, is satisfied. It may be shown that

$$0 < \beta < 3. \tag{3.4}$$

The models are constructed again for outer and inner edges given by

$$r_{out} = 3 \times 10^3 M, \tag{3.5a}$$

$$r_{mb} < r_{in} < r_{ms}, \tag{3.5b}$$

and again they form a one-parameter family with r_{in} being the parameter. The function $L_d = L_d(\dot{M})$ for this family is again shown in Fig. 5.

Note that of course some fraction of the total luminosity does not reach an observer at infinity because it is swallowed by the hole.

In Fig. 5 the results are presented in dimensionless units. The total disk luminosity L_d is expressed in units of the critical luminosity L_{cr}, connected *by definition* (eq. (1.6)) with any mass M (*i.e.*, also corresponding to a black hole with mass M). The accretion rate $\dot{M}$ is expressed in units of the critical accretion rate $\dot{M}_{cr}$, defined with the relationship

$$\frac{4\pi c G}{\varkappa} M \equiv L_{cr} \equiv \dot{M}_{cr} c^2 (1 - U_{ms}), \tag{3.6)*$$

where $c^2(1-U_{ms})$ is the binding energy of a free particle on the marigi-nally stable circular orbit. Note that eq. (3.6) follows from eq. (2.25) and that it is the relativistic analogue of eqs. (1.6)-(1.8). Note also that, while our definition of the critical luminosity depends on the black hole mass only, the definition of the critical accretion rate depends on both the mass and the angular momentum of the hole.

We have calculated models in the case of the Schwarzschild black hole ($a = 0$) and the "canonical" Kerr black hole ($a = 0.998\, M$). For these two cases one has:

$$1 - U_{ms} = \begin{cases} 0.0572 & \text{for } a = 0, \\ 0.324 & \text{for } a = 0.998\text{M}. \end{cases} \tag{3.7}$$

The results presented in Fig. 5 correspond to the case of disks radiating locally the critical flux, *i.e.*, $C(r) = 1$ (*cf.* eqs. (2.27)-(2.30)). The case $C < 1$ may be obtained easily by changing the units of each axis according to

$$\dot{M}(C) = C\dot{M}, \quad L_d(C) = CL_d. \tag{3.8}$$

For example, a disk with $C = 0.5$ may reach a total luminosity of $7L_{cr}$ (for $a = 0.998\, M$ and in the case 1.). This means that even a disk which locally radiates "subcritically" *can* produce a global luminosity which is highly "supercritical". Although a large (and at present unknown) fraction of the outgoing radiation is swallowed by the black hole, one may still expect that the luminosity seen by a distant observer located on the axis of symmetry may be highly supercritical due to the high collimation caused by the funnels.

4. Discussion

In the previous sections we have shown that it is possible to construct models of accretion disks in hydrostatic equilibrium *and* with highly supercritical accretion rates, $\dot{M} \gg \dot{M}_{cr}$. Such disks are geometrically thick and have their inner edges close to the marginally bound circular orbit. The efficiency of converting the accreted mass into the outgoing radiation is therefore *less* than in the thin disk case. Nevertheless, the total disk's luminosity may exceed considerably the critical (Eddington) luminosity, even though locally the flux of radiation is subcritical at every point on the disk surface. This is a consequence of the complicated geometry of the surface of the disk, which is very far from spherical. Because most of the radiation is emitted through the walls of the two funnels close to the rotation axis, we expect the disk brightness as seen by a distant observer to depend very strongly on the aspect. An observer located on the rotation axis could see the luminosity orders of magnitude larger than the observer close to the equatorial plane. This high anisotropy of the radiation field may provide an opportunity for mass outflow along the rotation axis.

In this paper we did not study the stability of the disk surface or any possible disk winds. Let us speculate now that, as a results of some activity of the surface, a modest amount of gas can enter the funnels. As there is no centrifugal support within the funnels, gas may either fall into the black hole or be pushed out by the radiation pressure. This very complicated problem is being studied numerically now by Sikora (1979). Here we would like only to present a very crude but very simple model of the gas outflow from the funnels. Let us start with a spherical star with a radius somewhat larger than the Schwarzschild radius. Let the total luminosity of the star be well above the critical value, $L \gg L_{cr}$. Gas placed in the vicinity of such a star will be pushed out by the radiation pressure. We have calculated the velocity reached by that gas at infinity as a function of two parameters:

$$\Gamma = L/L_{cr}, \qquad x = r_0/(GM/c^2), \tag{4.1)*}$$

where r_0 is the "initial" radial position of the gas introduced at rest into the vicinity of the star. M and L are the stellar mass and luminosity. L_{cr} is the critical luminosity given by (1.6). The details of the calculations are given in Appendix B and the results are shown in Fig. 6. We believe that these results represent crude estimates of the jet velocities in supercritical accretion disks. For example, gas injected into the funnel regions at about 10 gravitational radii may be expelled as a relativistic jet with half the speed of light in the case $\Gamma = 10$ and about 0.8 of the speed of light in the case $\Gamma = 100$. One may expect that the jet activity in the two funnels is independent because the funnels are separated by the optically thick stream and the hole.

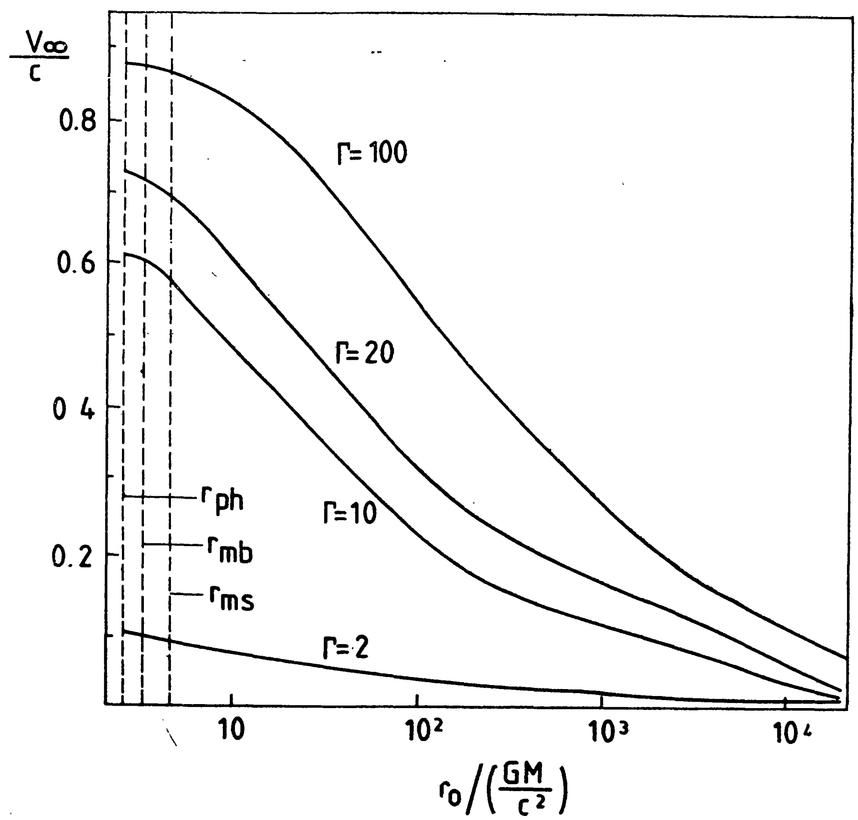

Fig. 6. The asymptotic velocity of the matter accelerated by the radiation flux in the funnels.

The radiation of the thick disk is thermal but we expect the spectrum to be much broader than that given by the Planck formula. The reflection effect can raise the temperature of the inner parts of the funnels. The highest temperature of the supercritical disk may be estimated roughly to be

$$T_{\max} \simeq 3 \times 10^7 (M/M_\odot)^{-1/4}. \quad (4.2)^*$$

Therefore, an optically thick disk could give rise to the soft X-ray flux in Cyg X-1 or Sco X-1 or to the visual, infrared and ultraviolet flux of active galactic nuclei, but it cannot explain the hard X-ray and γ-ray fluxes from any of these objects. However, this hard radiation hopefully can be accounted for by the relativistic jets that flow out of the funnels. These jets, upon the collision with an ambient medium (circumstellar, stellar, interstellar, intergalactic), would produce a relativistic shock in which the electrons and nuclei could be accelerated to relativistic random velocities. Discussion of the interesting processes that may follow is beyond the scope of this paper. We would like to note here, however, that a fair fraction of the total radiation of the disk may be used for the jet acceleration and for the subsequent production of nonthermal emission in the shocked medium. Perhaps more than 10 percent of the total flux could be used for this purpose. The problem of relativistic jets was recently reviewed by Rees (1978).

Thorne (1974) calculated the asymptotic value of the black hole angular momentum ($a/M = 0.998$ "canonical" Kerr black hole) for thin disk accretion, *i.e.*, for the case in which $r_{in} = r_{ms}$. Note that in our accretion disks $r_{in} < r_{ms}$. Each gram of matter brings *more* angular momentum into the hole and produces *less* radiation. For these two reasons we may expect a thick accretion disk to produce a more rapidly rotating black hole (radiation counteracts the spinning of the hole).

Finally, we would like to discuss a fundamental limitation to our disk models. We assumed that the selfgravity of the disk may be neglected. This means, roughly, that the density in the disk cannot be too large:

$$\varepsilon_{\max} \leqslant \frac{M}{r^3}, \tag{4.3)*$$

(*cf.* Paczyński, 1978). We assumed also that the cusp is so thin that the flux of the internal energy (or enthalpy) carried by the stream into the black hole is negligible in the global energy balance. Let us require that the flux of the enthalpy, L_{ent}, is less than 10% of the total radiative disk's luminosity, L_d. For the purpose of this discussion we shall adopt a polytropic equation of state

$$p = K\varepsilon^{4/3}, \quad h = 4K\varepsilon^{1/3}, \quad n = 3, \tag{4.4)*$$

where p is the pressure, ε is the rest mass density, h is the specific enthalpy, and n is the polytropic index. The maximum disk density is given by

$$\varepsilon_{\max} = \left(\frac{\Delta W_{\max}}{4K}\right)^3, \tag{4.5)*$$

where ΔW_{max} is the potential difference between the disk surface and the disk "center" and can be computed for any disk's model (*cf.* eq. (A. 12) with $\partial_i W = -\partial_i p/\varepsilon$). The rate of accretion through the cusp has been calculated for the polytropic equation of state by Kozłowski *et al.* (1978). Using eq. (58) of their paper we find

$$\dot{M} = 2\pi r \int_{-Z_0}^{+Z_0} \varepsilon v_r dz \approx 5\times 10^{-2} \frac{G^2 M^2}{c^5} \frac{\Delta W_{cusp}^4}{K^3}, \tag{4.6)*}$$

where ΔW_{cusp} is the potential difference between the surface of the disk and the equatorial plane in the cusp (the left-hand side of this equation represents only a *symbolic* Newtonian depiction of the fully relativistic calculations of Kozłowski *et al.*). The numerical coefficient was taken from one of the case 1 model: $a/M = 0.998$, $\dot{M} = 20\dot{M}_{cr}$, $L_d = 14\, L_{cr}$ (the cross in Fig. 5). The flux of enthalpy may be estimated in the same way:

$$L_{ent} = 2\pi r \int_{-Z_0}^{+Z_0} h\varepsilon v_r dz \approx 0.9 \Delta W_{cusp} \dot{M}. \tag{4.7a)*}$$

One can use the general relation (2.25) in order to write

$$L_d = \dot{M}c^2(1-U_{in}) = 0.231 c^2 \dot{M} \tag{4.7b)*}$$

and therefore our condition becomes

$$\frac{L_{ent}}{L_d} \approx 4\frac{\Delta W_{cusp}}{c^2} < 0.1. \tag{4.8)*}$$

Combining eqs. (4.3)* and (4.5)* we find

$$\varepsilon_{max} = \left(\frac{\Delta W_{max}}{4K}\right)^3 < \frac{M}{r^3} < \frac{M}{r_{in}^3} \approx 0.8\frac{c^6}{G^3 M^2}. \tag{4.9)*}$$

For our specific model we have

$$\Delta W_{max} = 0.32 c^2. \tag{4.10)*}$$

The last two formulae together with (4.6)* give

$$\dot{M} < 8\times 10^5 \frac{\Delta W_{cusp}^4}{G c^5}, \tag{4.11)*}$$

and finally, combining this with (4.8)*, we get

$$\dot{M} < 3\times 10^{-5}\frac{c^3}{G} \approx 10^{34}\frac{\mathrm{g}}{\mathrm{sec}}. \tag{4.12)*}$$

This corresponds to the luminosity limit

$$L_d < 7\times 10^{-6}\frac{c^5}{G} \approx 2\times 10^{54}\,\frac{\text{erg}}{\text{sec}}. \tag{4.13)*$$

The last number is so much above the most luminous quasars ($\sim 10^{48}$ erg/sec) that it is unlikely that our models are limited by the self-gravity of the disk or by the flux of the enthalpy competing with the radiative disk's flux. A formula similar to (4.13)* has been derived in the Newtonian theory also by Abramowicz *et al.* (1979).

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APPENDIX A

Here we give all the formulae which are needed to determine models of super-critical accretion disks described in the body of the paper. Some of them have been published elsewhere and some are even well-known. We decided to collect them together here because in the literature many different conventions are used and it is not easy to locate the necessary expressions and make a complete, self-consistent set.

We choose the $+ - - -$ signature and employ the $c = 1 = G$ convention. The symbol ∂_i denotes the partial derivative; for example, $\partial_r \equiv \partial/\partial r$.

In the Boyer-Lindquist spherical coordinates the gravitational field of the Kerr black hole is given by

$$g_{tt} = 1 - 2Mr/(r^2 + a^2\cos^2\theta), \tag{A.1a}$$

$$g_{t\varphi} = 2Mar\sin^2\theta/(r^2 + a^2\cos^2\theta), \tag{A.1b}$$

$$g_{\varphi\varphi} = -[r^2 + a^2 + 2Ma^2r\sin^2\theta/(r^2 + a^2\cos^2\theta)]\sin^2\theta, \tag{A.1c}$$

$$g_{rr} = -(r^2 + a^2\cos^2\theta)/(r^2 - 2Mr + a^2), \tag{A.1d}$$

$$g_{\theta\theta} = -(r^2 + a^2\cos^2\theta), \tag{A.1e}$$

where M is mass and $0 < a < M$ is the specific angular momentum of the black hole. Let us introduce the "canonical" Weyl cylindrical coordinate ϱ by writing

$$\varrho^2 \equiv g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi} = (r^2 - 2Mr + a^2)\sin^2\theta. \tag{A.2}$$

Because in the B-L coordinates the horizon is given by

$$r^2 - 2Mr + a^2 = 0$$

and the rotation axis by $\sin^2\theta = 0$, we have $\varrho = 0$ on the horizon and the rotation axis. The Schwarzschild black hole does not rotate and therefore $a = 0 = g_{t\varphi}$ for it.

Let us denote the four-velocity of matter by u_i, and its enthalpy per particle by h. The definitions of the energy per particle, E, and the angular momentum per particle, j, are

$$E = hu_t, \tag{A.3}$$

$$j = -hu_\varphi. \tag{A.4}$$

The specific energy U and the specific angular momentum are given by the following formulae:

$$U = u_t = E/h, \tag{A.5}$$

$$\mathscr{L} = -u_\varphi/u_t = j/E, \tag{A.6}$$

while the redshift factor A and the angular velocity are introduced by

$$A = u^t, \tag{A.7}$$

$$\Omega = u^\varphi/u^t. \tag{A.8}$$

If the four-velocity has neither a radial nor a meridional component *i.e.*, if $u_r = 0 = u_\theta$, then one has

$$\Omega = -(\mathscr{L}g_{tt} + g_{t\varphi})/(\mathscr{L}g_{t\varphi} + g_{\varphi\varphi}), \tag{A.9a}$$

$$\mathscr{L} = -(g_{t\varphi} + \Omega g_{\varphi\varphi})/(g_{tt} + \Omega g_{t\varphi}), \tag{A.9b}$$

$$A = (g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi})^{-1/2}, \tag{A.9c}$$

$$U = [-(\mathscr{L}^2 g_{tt} + 2\mathscr{L}g_{t\varphi} + g_{\varphi\varphi})/\varrho^2]^{-1/2}, \tag{A.9d}$$

$$AU(1 - \Omega\mathscr{L}) = 1. \tag{A.9e}$$

In the perfect fluid case (no dissipation) the energy and the angular momentum per particle do not change along the trajectory of a small element which consists of N particles:

$$E = \text{const.}, \quad j = \text{const. along the trajectory.} \tag{A.10}$$

Therefore also $\mathscr{L} = \text{const.}$ along the trajectory.

a) *Mechanical equilibrium.*

The stress energy tensor $T^i{}_k$ of a perfect fluid has the form

$$T^i{}_k = (p+\varepsilon)\, u^i u_k - p\, \delta^i{}_k, \tag{A.11}$$

where ε is the total energy density and p is the pressure.

Therefore the equations of mechanical equilibrium $\nabla_i T^i{}_k = 0$, can be written (for the $u_r = 0 = u_\theta$ case) *e.g.* in one of the following forms:

$$\partial_i p/(p+\varepsilon) = \begin{cases} -\frac{1}{2} A^2 (\partial_i g_{tt} + 2\Omega \partial_i g_{t\varphi} + \Omega^2 \partial_i g_{\varphi\varphi}), & \text{(A.12a)} \\ \partial_i \ln A - \mathscr{L} \partial_i \Omega / (1 - \Omega \mathscr{L}), & \text{(A.12b)} \\ -\partial_i \ln \varrho + \dfrac{1}{2} \dfrac{\mathscr{L}^2 \partial_i g_{tt} + 2\mathscr{L} \partial_i g_{t\varphi} + \partial_i g_{\varphi\varphi}}{\mathscr{L}^2 g_{tt} + 2\mathscr{L} g_{t\varphi} + g_{\varphi\varphi}}, & \text{(A.12c)} \\ -\partial_i \ln U + \Omega \partial_i \mathscr{L} / (1 - \Omega \mathscr{L}) = a_i. & \text{(A.12d)} \end{cases}$$

The vector a_i is the acceleration or "effective" gravity. It is orthogonal to the surfaces $p = p(r, \theta) = \text{const}$, in particular to the surface $p(r, \theta) = 0$, *i.e.*, to the surface of the disk. One can write the equation $p(r, \theta) = 0$ in the form $\theta = \theta(r)$. The function $\theta(r)$ is given (*cf.* A. 12c) by

$$\frac{d\theta}{dr} = \frac{-[\mathscr{L}^2 \partial_r (g_{tt}/\varrho^2) + 2\mathscr{L} \partial_r (g_{t\varphi}/\varrho^2) + \partial_r (g_{\varphi\varphi}/\varrho^2)]}{\mathscr{L}^2 \partial_\theta (g_{tt}/\varrho^2) + 2\mathscr{L} \partial_\theta (g_{t\varphi}/\varrho^2) + \partial_\theta (g_{\varphi\varphi}/\varrho^2)}. \tag{A.13}$$

Because we assume the explicit form of the function $\mathscr{L} = \mathscr{L}(r)$ on the surface of the disk, equation (A. 13) has the form

$$\frac{d\theta}{dr} = F(r, \theta), \tag{A.14}$$

with $F(r, \theta)$ being an explicitly known function which therefore can be integrated.

We have assumed that at both the edges, r_{in} and r_{out}, the thick disk is *thin* in the following sense: mathematically the surface of the disk approaches the equatorial plane, $\cos\theta = 0$. Therefore, *formally* the thickness of the disk goes to zero at both r_{in} and r_{out}. Real disk of course, have finite thickness at the edges. The thickness near the inner cusp, r_{in}, can be evaluated by a method given in Kozłowski *et al.* (1978), while the thickness near r_{out} can be found from the condition that there is a thin disk in the region from r_{out} up to infinity which can be matched to the thick disk. Therefore, at both edges the angular momentum is *very close* to the Keplerian values at r_{in}, r_{out}. *Formally* one uses the condition:

$$\mathscr{L}(r_{in}) = \mathscr{L}_K(r_{in}), \qquad \mathscr{L}(r_{out}) = \mathscr{L}_K(r_{out}). \tag{A.15}$$

Of course, the same is true for other kinematical characteristic of the motion: $\Omega(r_{in}) = \Omega_K(r_{in})$, $U(r_{in}) = U_K(r_{in})$, *etc.* In the gravitational field of the Kerr black hole one has:

$$U_K(r) = \frac{r^{3/2} - 2Mr^{1/2} + aM^{1/2}}{(r^3 - 3Mr^2 + 2aM^{1/2}r^{3/2})^{1/2}}, \tag{A.16}$$

$$\mathscr{L}_K(r) = \frac{M^{1/2}(r^2 - 2aM^{1/2}r^{1/2} + a^2)}{r^{3/2} - 2Mr^{1/2} + aM^{1/2}}, \tag{A.17}$$

$$\Omega_K(r) = \frac{M^{1/2}}{r^{3/2} + aM^{1/2}}. \tag{A.18}$$

From (A. 12d) and (A. 15) it follows that:

$$\ln \frac{U_K(r_{out})}{U_K(r_{in})} = \int\limits_{r_{in}}^{r_{out}} \frac{\Omega(r)}{1 - \Omega(r)\mathscr{L}(r)} \frac{\mathrm{d}\mathscr{L}}{\mathrm{d}r} \mathrm{d}r, \tag{A.19}$$

where $\mathscr{L} = \mathscr{L}(r)$ is assumed *surface* angular momentum distribution which obeys (A. 15). Equation (A. 19) gives r_{out} for the assumed $\mathscr{L}(r)$ and r_{in}. In the general case one does not *a priori* know $\Omega(r)$ for the assumed $\mathscr{L}(r)$ on the (*a priori* unknown) surface of the disk. Therefore the numerical problem consists of the three equations (A. 9) (A.13) and (A.19) which have to be solved simultaneously. In the special case where $\mathscr{L}(r)$ is given implicitly by the relation between Ω and $\mathscr{L}$ the problem is much simpler, as one can compute the integral (A. 19) at the begining.

In the case of the Schwarzschild black hole the numerical problem is trivial because there is the analytic solution

$$\frac{1}{\sin^2\theta} = \left(\frac{r}{r_{in}}\right)^2 \left(1 - 2r_{in}^2 \int\limits_{r_{in}}^{r} \frac{\mathscr{L}_K^2(r)}{\mathscr{L}^2(r)r^3} \mathrm{d}r\right), \tag{A.20}$$

$$\int\limits_{r_{ii}}^{r_{out}} \frac{\mathscr{L}^2(r) - \mathscr{L}_K^2(r)}{\mathscr{L}^2(r)r^3} \mathrm{d}r = 0, \tag{A.21}$$

$$\mathscr{L}(r_{in}) = \mathscr{L}_K(r_{in}), \quad \mathscr{L}(r_{out}) = \mathscr{L}_K(r_{out}). \tag{A.22}$$

b) *Conservation laws.*

The stress-energy tensor which describes dissipation and transport phenomena in a viscous fluid has the form

$$T^i{}_k = (p+\varepsilon)u^i u_k - p\delta^i_k + 2\eta\sigma^i{}_k + F^i u_k + u^i F_k. \tag{A.23}$$

Here η is the dynamical viscosity of the fluid, $\sigma^i{}_k$ the shear tensor, and F^i the heat flux vector (see Misner *et al.* 1974). All these quantities are measured by an observer comoving with the fluid. We assume that the expression (A. 23) differs only little from that given by eq. (A. 11) so the mechanical equilibrium is a good approximation for the kinematic of the fluid. Let us consider three vectors, given in B-L coordinates by

$$E^i \equiv T^i{}_t = (p+\varepsilon)u^i u_t - p\delta^i{}_t + 2\eta\sigma^i{}_t + F^i u_t + u^i F_t, \qquad \text{(A.24a)}$$

$$j^i \equiv T^i{}_\varphi = (p+\varepsilon)u^i u_\varphi - p\delta^i{}_\varphi + 2\eta\sigma^i{}_\varphi + F^i u_\varphi + u^i F_\varphi, \qquad \text{(A.24b)}$$

$$n^i = nu^i,$$

where n is baryon number density. They describe the energy flux (measured at infinity) the angular momentum flux, and the flux of baryons.

These vectors are divergence-free according to the conservation laws

$$\nabla_i E^i = 0 = \nabla_i j^i = \nabla_i n^i. \qquad \text{(A.25)}$$

Now we shall integrate Eqs. (A. 25) over the whole world tube of the thick disk between the two hypersurfaces $t = t_0$ and $t = t_0 + \Delta t$. Using the Gauss theorem we can turn into surface integrals. The domain of integration consists of the history (from t_0 to $t_0 + \Delta t$) of the following two-dimensional spacelike surfaces:

S_1: located inside the disk, infinitesimally close to r_{in} and orthogonal to the equatorial plane,

S_2: as S_1 but close to r_{out},

S_3: the "upper" and "lower" faces of the disk.

We treat S_1 and S_2 as inner and outer edges of the thick disk, so various characteristics of the flow are constant on these surfaces and have Keplerian values.

We do not assume any outflow of matter from the disk surface. The disk is thin at its edges so one may neglect internal energy of the fluid here, obtaining

$$\int_{S_1} (p+\varepsilon)u^i \mathrm{d}S_i \approx -\int_{S_2} (p+\varepsilon)u^i \mathrm{d}S_i \approx \dot{M}, \qquad \text{(A. 26)}$$

due to the law of baryon conservation.

The integrals of $p\delta^i{}_t$, $p\delta^i{}_\varphi$ over the surfaces S_1, S_2, S_3 automatically vanish.

There are no stresses operating through the surface S_3. The stress through S_1 (inner edge of the disk) also vanishes because the matter falls freely from there. Nonvanishing torque is directed outward:

$$T_{out} = -\int_{S_2} 2\eta\sigma^i{}_\varphi \mathrm{d}S_i. \qquad \text{(A.27a)}$$

The identity $\sigma^i_{\ t}+\Omega\sigma^i_{\ \varphi}=O$ (note, that $u^i\sigma^k_i=0$) enables us to name the similar integral which arises in the energy conservation law:

$$\Omega_{out}T_{out}=\int_{S_2}2\eta\sigma^i_{\ t}\mathrm{d}S_i. \tag{A.27b}$$

On the surface S_3 one has $u^i\mathrm{d}S_i=0$ (no outflow of matter). One can also neglect the transport of energy and angular momentum by the heat flux going through the edges S_1, S_2. Therefore all the remaining terms in the integral form of conservation laws can be grouped into the expressions

$$L_E\equiv\int_{S_3}F^iu_t\mathrm{d}S_i\equiv\int_{S_3}F^iU\mathrm{d}S_i, \tag{A.28a}$$

$$L_j\equiv-\int_{S_3}F^iu_\varphi\mathrm{d}S_i\equiv\int_{S_3}F^i\mathscr{L}U\mathrm{d}S_i. \tag{A.28b}$$

They represent energy and angular momentum radiated from the disk surface. We have chosen signs in Eqs. (A. 26-28) in such a manner that quantities $\dot{M}$, T_{out}, L_E, L_j are positively defined. Now conservation of energy and angular momentum leads to the equations

$$L_E+\Omega_{out}T_{out}=\dot{M}(U_{out}-U_{in}), \tag{A.29a}$$

$$L_j+T_{out}=\dot{M}(\mathscr{L}_{out}U_{out}-\mathscr{L}_{in}U_{in}). \tag{A.29b}$$

Eliminating T_{out} one obtains

$$L_E-\Omega_{out}L_j=\dot{M}[U_{out}-U_{in}-\Omega_{out}(\mathscr{L}_{out}U_{out}-\mathscr{L}_{in}U_{in})]\equiv\dot{M}\mathscr{E}(r_{in},r_{out}). \tag{A.30}$$

The function $\mathscr{E}(r_{in},r_{out})$ is the efficiency of converting the accreted rest energy of matter into the energy radiated from the thick disk. It depends only on the locations of the thick disk's edges. (This approach is valid if the disk is thin on both its edges).

The torque acting through the outer edge of the thick disk can also be calculated from Eqs. (A. 29a, b):

$$T_{out}=[(\mathscr{L}_{out}U_{out}-\mathscr{L}_{in}U_{in})L_E-(U_{out}-U_{in})L_j]/\mathscr{E}. \tag{A.31}$$

In our approach the quantities L_E and L_j are obtained by the integration over surface S_3. The surface element, oriented outward, has the form

$$\mathrm{d}S_i=\frac{-a_i}{|a_ja^j|^{1/2}}(-g_{rr}\mathrm{d}r^2-g_{\theta\theta}\mathrm{d}\theta^2)^{1/2}\mathrm{d}\varphi. \tag{A.32}$$

We assume that heat flux on the disk surface is proportional to the effective gravity; $F^i=-C^*a^i$. Therefore L_E and L_j can be calculated with the formulae

$$L_E = 2\pi \int_{r_{in}}^{r_{out}} C^* U |a_j a^j|^{1/2} [-g_{rr} - g_{\theta\theta} (\mathrm{d}\theta/\mathrm{d}r)^2]^{1/2} \varrho \,\mathrm{d}r, \tag{A. 33a}$$

$$L_j = 2\pi \int_{r_{in}}^{out} C^* \mathscr{L} U |a_j a^j|^{1/2} [-g_{rr} - g_{\theta\theta} (\mathrm{d}\theta/\mathrm{d}r)^2]^{1/2} \varrho \,\mathrm{d}r. \tag{A. 33b}$$

The expressions standing under the integrals are calculable according to Eqs. (A. 1, 2, 9, 12, 13) providing the angular momentum distribution has been postulated. After the integrals (A. 33a, b) have been calculated it is possible to determine the accretion rate and the torque on the outer edge of the disk with the two formulae (A. 30) and (A. 31).

APPENDIX B

The amount of energy $\mathrm{d}E$ that is carried by photons across a unit surface area $\mathrm{d}A^*$ orthogonal to a spatial direction N^i during unit time $\mathrm{d}t$ and being directed into unit solid angle $\mathrm{d}\Omega^*$ about N^i is called the *total* intensity (Novikov and Thorne, 1973)

$$I = \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A^*\,\mathrm{d}\Omega^*}. \tag{B.1}$$

All the quantities $\mathrm{d}E$, $\mathrm{d}A^*$, N^i, $\mathrm{d}t$ and $\mathrm{d}\Omega^*$ are measured in the local rest-Lorentz frame of an observer. Therefore, the total intensity depends on the choice of the observer and the direction N^i. One can introduce the observer's orthonormal base, or *tetrad*, as a set of four othonormal vectors with the timelike vector being the observer's four velocity u^i:

$$e^i{}_{(a)} = \{e^i{}_{(t)} \equiv u^i,\ e^i{}_{(1)},\ e^i{}_{(2)},\ e^i{}_{(3)}\}. \tag{B.2}$$

The rule for calculating the tetrad components of a tensor (indicated by brackets) is

$$X_{(a)(b)} = X_{ik} e^i{}_{(a)} e^k{}_{(b)}, \tag{B.3a}$$

and, conversely,

$$X_{ik} = X_{(a)(b)} e_i{}^{(a)} e_k{}^{(b)}. \tag{B. 3b}$$

Usually one knows the total intensity only on the surface of the source of radiation. This is, however, enough to compute it along any particular light ray by using the red shift formula

$$I/(p^i u_i)^4 = \mathrm{const}. \tag{B.4}$$

Here p^i is the photon four-momentum. Note, also, that the energy of a photon measured in the observer's rest-frame is

$$E \equiv p^i u_i = p^{(t)} = h\nu, \tag{B.5}$$

where ν is the observed frequency of light. Therefore, $I/\nu^4 = \text{const}$ along the ray.

In general relativity any material field is described by its stress-energy tensor $T^i{}_k$. The stress-energy tensor for a radiation field is

$$T^{(a)(b)} = \int I n^{(a)} n^{(b)} \mathrm{d}\Omega^*, \tag{B.6}$$

where $n^{(a)} \equiv p^{(a)}/p^t$.

Let us now consider a particle with rest mass m and four-velocity v^i moving through the radiation field $T^i{}_k$. If there is no interaction between radiation and the particle, then the acceleration,

$$a^i \equiv V^k \nabla_k v^i$$

is equal to zero (geodesic motion). Suppose that there is an interaction and it can be characterised by the particle cross-section σ measured in its rest-frame. The equation of motion is in this case identical in form with the Newtonian formula, (acceleration) $\times$ (mass) $=$ (radiation force) $=$ (radiation flux) $\times$ (cross section):

$$ma^i = f^i = F^i \sigma. \tag{B.7}$$

The radiation flux F^i is defined by

$$F^i = h^i{}_j T^j{}_k v^k, \tag{B.8}$$

where

$$h^i{}_j \equiv \delta^i{}_j - v^i v_j$$

is the projection tensor. It projects perpendicularly to the velocity. Thus, the problem of particle motion in the radiation field of a given source can be solved by the following procedure:

1) Pick a function I on the source (*e. g.* from a detailed study of the structure of the accretion disk).

2) Compute trajectories of light rays and after that the total intensity along any particular ray, Eq. (B.4).

3) Compute the stress energy tensor and the radiation flux, Eqs. (B.6) and (B.7).

4) Solve the equation of motion, Eq. (B.7).

Such a procedure will be employed by Sikora (1979) to study the model for relativistic beams presented in this paper. Here we give only a simple example of its use in the case of a spherically symmetric source in Schwarzschild spacetime.

First, we shall compute the critical *Eddington* luminosity of a spherical star, and after that we shall assume that inside a narrow solid angle Ω_0^* the actual luminosity exceeds the critical by a factor $\Gamma \gg 1$. One can reasonable suppose that it will be a rather good model for Lynden-Bell's (1978) vortex-collimation-mechanism.

Let us assume that the intensity, measured by a stationary observer, does not depend on direction:

$$I_{\text{sufrace}} = I(R) = \text{const.} \tag{B.9}$$

The equation of the surface of the source is (in Schwarzschild coordinates) $r = r_{em} = R$. From the red shift formula (B. 4) it follows that, for any stationary observer located on the sphere $r_{obs} = r$

$$I(r) = \left(\frac{1-2M/R}{1-2M/r}\right)^2 I(R). \tag{B.10}$$

Thus, the intensity is constant and equal to $I(r)$ inside the solid angle centered about the direction to the source and with an angular opening $a = a(r)$ given by

$$\sin^2 a = \frac{1-2M/r}{1-2M/R}\frac{R^2}{r^2}, \tag{B.11}$$

and it is equal to zero outside this solid angle. The components of the stress-energy tensor which are needed to compute the radial component of the radiation force are

$$T^{tt} = 2\pi\frac{(1-2M/R)^2}{(1-2M/r)^3}(1-\cos a)I(R), \tag{B.12a}$$

$$T^{tr} = \pi\frac{R^2}{r^2}\frac{1-2M/R}{1-2M/r}I(R), \tag{B.12b}$$

$$T^{rr} = \frac{3}{2}\pi\frac{1-2M/R}{1-2M/r}(1-\cos^3 a)I(R). \tag{B.12c}$$

We have employed formulae (B. 3), (B. 4), (B. 6), (B. 10) and (B.11) and we have used the following expressions for the tetrad of the stationary observer:

$$\begin{aligned} e^t{}_{(t)} &= (1-2M/r)^{-1/2}, \quad & e^r{}_{(t)} &= 0 = e^\theta{}_{(t)} = e^\varphi{}_{(t)}, \\ e^r{}_{(r)} &= (1-2M/r)^{1/2}, \quad & e^t{}_{(r)} &= 0 = e^\theta{}_{(r)} = e^\varphi{}_{(r)}. \end{aligned} \tag{B.13}$$

The radial component of the radiation force (Eqs. (B.7) and (B.8)) is equal to

$$f^r = \sigma v_t(-T^{tt}v^r v_t + T^{rt}v^t v_t - T^{rt}v^r v_r + T^{rr}v^t v_r), \tag{B.14}$$

and the radial component of acceleration is

$$a^r = \frac{d^2r}{ds^2} + \frac{M}{r^2}, \tag{B.15}$$

where s is the spacetime length.

If particles on the surface of the source are in neutral equilibrium (gravity balanced by the radiation force) then they are at rest in the local frame of the stationary observer and $a^r = a^r_{stat.obs.}$. In this case $d^2r/ds^2 = 0$ and it follows from Eqs. (B. 7), (B. 14) and (B. 15) that

$$I(R) = \frac{Mm}{\pi\sigma R^2}(1-2M/R)^{-1/2}. \tag{B.16}$$

This corresponds to a total luminosity, measured by the stationary observer at the surface of the source, of

$$L(R) = \frac{4\pi Mm}{\sigma}(1-2M/R)^{-1/2}. \tag{B.17}$$

The total luminosity of a surface Σ is defined by

$$L = \int\limits_{\Sigma} T^{ik} v_k d \sum_i$$

thus it is an observer-dependent quantity. The stationary observer at infinity will observe the red-shifted luminosity

$$L(\infty) = \lim_{r\to\infty} \frac{1-2M/R}{1-2M/r} L(R) = L_{Edd} = \frac{4\pi Mm}{\sigma}(1-2M/R)^{1/2}. \tag{B.18}$$

This is the critical *Eddington* luminosity of a spherical relativistic star. If the whole radiation of such a critically radiating body is collimated in a narrow solid angle, in this angle luminosity will be super-critical:

$$L = \Gamma L_{Edd}, \quad \Gamma \gg 1. \tag{B.19}$$

The equation of motion reads:

$$\frac{d\chi}{d\tau} = -2\chi^2 v, \tag{B.20a}$$

$$\frac{dv}{d\tau} = \chi_0^2\left\{\left(\frac{\chi}{\chi_0}\right)^2 - \Gamma G(1-\chi_0)^{1/2}\left(\frac{\chi}{\chi_0}\right)^2\left(G^2+\frac{v^2}{(1-\chi)^2}\right)(1-\chi) - \frac{2}{3}\frac{1-\chi_0}{1-\chi}vGQ\right\}, \tag{B.20b}$$

where

$$G \equiv (1-\chi+v^2)^{1/2}/(1-\chi), \tag{B.20c}$$

$$Q \equiv (1-\cos a)(4+\cos a+\cos^2 a). \tag{B.20d}$$

We have used the notation

$$\tau = s/4M, \quad \chi = 2M/r, \quad \chi_0 = 2M/R, \quad v = \frac{dr}{ds}. \tag{B.20e}$$

Equation (B. 20) is valid for any value of Γ.

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