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Black hole spacetimes with self-gravitating, massive accretion tori

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Abstract. We review the theory of stationary, axisymmetric spacetimes with black holes surrounded by a massive torus and present a numerical scheme for finding self-consistent solutions in a compactified grid that reaches to spatial infinity. Such numerical solutions have been utilized in studying nonaxisymmetric dynamical instabilities occurring in the torus.

1. Introduction

Thick relativistic accretion disks and tori around black holes (BHs) can form as transient structures in several astrophysical scenarios, including the core-collapse of massive stars [1, 2] and the merger of neutron star (NS) and NS-BH binaries [3, 4, 5]. Recent numerical simulations have demonstrated that the mass of the disk resulting from binary NS-NS or BH-NS mergers (which are candidates for the central engine of short GRBs), can be in the range of $\sim 0.01 - 0.2M_{\odot}$ [6, 7, 5, 4]. Early studies of the stability of accretion disks have revealed that they can be subject to several types of axisymmetric and/or non-axisymmetric instabilities in a number of circumstances [8, 9, 10, 11, 12, 13, 14], which can lead to highly variable and unstable accretion rates. It is thus important to create a comprehensive overall picture of the stability of accretion disks for a wide range of parameters.

We first review the theory of tori in a fixed spacetime, the so-called AJS disks [15]. This is then generalized to self-gravitating disks in quasi-isotropic coordinates, following [16]. We reformulate the equations in a compactified coordinate system, which allows for high accuracy and describe in detail a numerical scheme that leads to self-consistent solutions. First numerical results of this approach can be found in [17], where a detailed study of nonaxisymmetric instabilities or tori around black holes in a fully general-relativistic treatment is presented. A study of the parameter space of equilibrium solutions will appear elsewhere.

2. Disks in a stationary, axisymmetric spacetime

A stationary, axisymmetric spacetime is described by a metric of the general form

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2, \tag{1}$$

(we assume a signature of $-, +, +, +$ and set $c = G = 1$). Excluding meridional circulation, for a disk to be stationary its 4-velocity must have the form

$$u^{\alpha} = (u^t, 0, 0, u^{\phi}). \tag{2}$$

The angular velocity measured by an observer at infinity is

$$\Omega = \frac{u^\phi}{u^t}. \tag{3}$$

Along each fluid trajectory, the existence of a Killing vector t^α describing time symmetry implies the conservation of $\mathcal{E} = -hu_t$ (energy per unit rest mass), while the existence of a Killing vector ϕ^α describing ϕ-symmetry implies the conservation of $j = hu_\phi$ (angular momentum per unit rest mass), where $h = (\epsilon + p)/\rho$ is the specific enthalpy, ρ is rest mass density, p is pressure and ϵ is energy density. When both t^α and ϕ^α exist, then also the ratio

$$\frac{j}{\mathcal{E}} = -\frac{u_\phi}{u_t} := l, \tag{4}$$

is conserved (the *angular momentum per unit energy*). One can eliminate u^t, u^ϕ to write

$$l = -\frac{g_{t\phi} + \Omega g_{\phi\phi}}{g_{tt} + \Omega g_{t\phi}}, \tag{5}$$

or

$$\Omega = -\frac{g_{t\phi} + l g_{tt}}{g_{\phi\phi} + l g_{t\phi}}. \tag{6}$$

From the normalization of the 4-velocity, $u^\alpha u_\alpha = -1$, one can obtain u_t as

$$u_t = -\left(-g^{tt} + 2lg^{t\phi} - l^2 g^{\phi\phi}\right)^{-1/2}, \tag{7}$$

$$= -\left[-\frac{g_{\phi\phi} + 2lg_{t\phi} + l^2 g_{tt}}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2}\right]^{-1/2}, \tag{8}$$

where the components of the inverse metric were used

$$g^{tt} = \frac{1}{g_{tt} - (g_{t\phi})^2/g_{\phi\phi}}, \tag{9}$$

$$g^{\phi\phi} = \frac{1}{g_{\phi\phi} - (g_{t\phi})^2/g_{tt}}, \tag{10}$$

$$g^{t\phi} = -\frac{g_{t\phi}}{g_{tt}g_{\phi\phi} - (g_{t\phi})^2}. \tag{11}$$

From the same normalization it also follows that

$$u^t = \left(-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}\right)^{-1/2}, \tag{12}$$

and

$$u^t u_t = -\frac{1}{1 - \Omega l}, \tag{13}$$

while (4), (13) imply

$$u^t u_\phi = \frac{l}{1 - \Omega l}. \tag{14}$$

Notice that the choice of sign when taking the square root in (7), (8) and (12) is made in agreement with the flat-space limit of u^α for the chosen signature of the metric.

The hydrostationary equilibrium equation can be written in the following equivalent forms

$$\frac{\nabla p}{\epsilon+p} = \nabla \ln u^t - u^t u_\phi \nabla\Omega, \tag{15}$$

$$= \nabla \ln u^t - \frac{l\nabla\Omega}{1-\Omega l}, \tag{16}$$

$$= -\nabla \ln(-u_t) + \frac{\Omega\nabla l}{1-\Omega l}, \tag{17}$$

and has a first integral only under specific conditions. For a barotropic equation of state of the form $p = p(\rho)$ one can define the log-enthalpy

$$H(p) := \int_0^p \frac{dp'}{\epsilon(p')+p'}. \tag{18}$$

Then, $dH/dp = (\epsilon+p)^{-1}$ and the l.h.s. of (15) becomes $(\epsilon+p)^{-1}\nabla p = \nabla H$. The r.h.s. of (15) can be written in the form of the gradient of a potential, only if $u^t u_\phi$ is a function of Ω, or equivalently, one can write the r.h.s. of (17) as

$$-\nabla \ln(-u_t) + \frac{\Omega\nabla l}{1-\Omega l} := -\nabla W, \tag{19}$$

only if $\Omega = \Omega(l)$. In (19), $-\nabla W$ is the *effective gravity*, given by the gradient of the *effective potential*

$$W = \ln(-u_t) - \int^l \frac{\Omega}{1-\Omega l} dl + \text{const.} \tag{20}$$

Notice that at zero pressure $H(0) = 0$, while W is only defined up to a constant, which can be set by requiring that W vanishes at infinity. Under these conditions, it follows that

$$\nabla H = -\nabla W, \tag{21}$$

and *the surfaces of constant pressure coincide with the equipotential surfaces.* At the location of maximum density $\nabla H = \nabla W = 0$.

The first integral of (17) is

$$H + \ln(-u_t) - \int^l \frac{\Omega}{1-\Omega l} dl = \text{const.}, \tag{22}$$

or, equivalently, the first integral of (16) is

$$H - \ln(u^t) + \int^\Omega \frac{l}{1-\Omega l} d\Omega = \text{const.}. \tag{23}$$

For a *homentropic flow* (uniform entropy throughout the fluid), the log-enthalpy becomes simply

$$H = \ln\left(\frac{h}{h_{\rm min}}\right), \tag{24}$$

where $h_{\rm min} = 1$ is the specific enthalpy at the limit of zero pressure[1]. At the disk surface, $h = h_{\rm min}$ and $H = 0$.

[1] Notice that this differs from the Newtonian definition of specific enthalpy $h_{\rm Newt} = e + p/\rho$, where e is the specific internal energy and the relation $h \to h_{\rm Newt} + 1$ in the nonrelativistic, zero-pressure limit connects the two definitions.

3. Constant specific angular momentum

A specific choice of the distribution of the specific angular momentum in the disk is to simply assume that $l = \text{const.}$, when (23) becomes

$$H - \ln[u^t(1 - l\Omega)] = \text{const.}, \tag{25}$$

or (22) becomes

$$H + \ln(-u_t) = \text{const.}. \tag{26}$$

If the disks terminates at some $r = r_{\text{in}}$, then the first integral becomes

$$H = \ln\left(\frac{u^t}{u^t_{\text{in}}}\right) + \ln\left(\frac{1 - l\Omega}{1 - l\Omega_{\text{in}}}\right), \tag{27}$$

$$= \ln\left(\frac{u_{t,\text{in}}}{u_t}\right), \tag{28}$$

and a similar relation holds when an outer termination radius $r = r_{\text{out}}$ is used. Notice that $\ln(-u_{t,\text{in}}) = \ln(-u_{t,\text{out}})$ and thus the the inner and outer radii of a disk are not independent, but are located on the same constant-pressure (or equipotential) surface. If the location $r = r_{\text{max}}$ of the density maximum ρ_{max} is used, then

$$H - H_{\text{max}} = \ln\left(\frac{u^t}{u^t_{\text{max}}}\right) + \ln\left(\frac{1 - l\Omega}{1 - l\Omega_{\text{max}}}\right), \tag{29}$$

$$= \ln\left(\frac{u_{t,\text{max}}}{u_t}\right), \tag{30}$$

where the subscripts $\text{in}, \text{out}, \text{max}$ always indicate values at the corresponding radii.

For the homentropic, polytropic equation of state (EOS)

$$p = K\rho^\Gamma, \tag{31}$$

$$\epsilon = \rho + \frac{p}{\Gamma - 1}, \tag{32}$$

where K is the polytropic constant and Γ is the polytropic exponent, one obtains

$$H = \ln[1 + \Gamma/(\Gamma - 1)K\rho^{\Gamma-1}], \tag{33}$$

and from (28) one can solve algebraically for the density distribution, as an implicit function of Ω (for given EOS, l_0, r_{in} and stationary, axisymmetric spacetime):

$$\rho = \left\{\frac{\Gamma - 1}{K\Gamma}\left[\frac{u^t}{u^t_{\text{in}}}\left(\frac{1 - l\Omega}{1 - l\Omega_{\text{in}}}\right) - 1\right]\right\}^{\frac{1}{\Gamma-1}}, \tag{34}$$

$$= \left\{\frac{\Gamma - 1}{K\Gamma}\left[\frac{u_{t,\text{in}}}{u_t} - 1\right]\right\}^{\frac{1}{\Gamma-1}}. \tag{35}$$

The constant in the definition of the effective potential can be set by requiring that $W(r = \infty) = 0$. Then, the effective potential is

$$W = \ln\left(\frac{u_t}{u_{t,\infty}}\right). \tag{36}$$

4. AJS disks in a Schwarzschild background

In a spherically symmetric spacetime, the metric in Schwarzschild coordinates $(t, \mathfrak{r}, \theta, \phi)$ is

$$ds^2 = -\left(1 - \frac{2M}{\mathfrak{r}}\right) dt^2 + \left(1 - \frac{2M}{\mathfrak{r}}\right)^{-1} d\mathfrak{r}^2 + \mathfrak{r}^2 d\theta^2 + \mathfrak{r}^2 \sin^2\theta d\phi^2, \tag{37}$$

where M is the total mass-energy. In this metric

$$u_t = -\left[\left(1 - \frac{2M}{\mathfrak{r}}\right)^{-1} - \frac{l^2}{\mathfrak{r}^2 \sin^2\theta}\right]^{-1/2}, \tag{38}$$

and $u_{t,\infty} = -1$, so that the effective potential for $l = \text{const.}$ disks is

$$W = \ln(-u_t). \tag{39}$$

A disk of finite size extends between $\mathfrak{r} = \mathfrak{r}_{\rm in}$ and $\mathfrak{r} = \mathfrak{r}_{\rm out}$ in the equatorial plane. At the density maximum, $\nabla_{\mathfrak{r}} p = 0$ and the fluid elements there move as free test particles with Keplerian angular momentum, which restricts $\mathfrak{r}_{\rm max} > \mathfrak{r}_{\rm ms}$, where $\mathfrak{r}_{\rm ms} = 6M$ is the radius of the marginally stable circular orbit for test particles. The Keplerian angular momentum is $\Omega_{\rm K} = \sqrt{M/\mathfrak{r}^3}$ and the corresponding specific angular momentum is

$$l_{\rm K} = \frac{\sqrt{M\mathfrak{r}^3}}{\mathfrak{r} - 2M}, \tag{40}$$

which has a minimum at $\mathfrak{r} = \mathfrak{r}_{\rm ms}$ so that $\mathfrak{r}_{\rm max} > \mathfrak{r}_{\rm ms} \Rightarrow l > l_{\rm ms} = 3.674M$. For a given value of l, the largest of three roots of $l = l_{\rm K}$ corresponds to $\mathfrak{r} = \mathfrak{r}_{\rm max}$. The intermediate root, $\mathfrak{r} = \mathfrak{r}_{\rm cusp}$, corresponds to the existence of a cusp, where again $\nabla_{\mathfrak{r}} p = 0$ and fluid elements move as free test particles with Keplerian angular momentum. Thus, for given EOS and black-hole mass M and for a given value of l, there exists a one-parameter family of different finite-size disks with $\mathfrak{r}_{\rm cusp} < \mathfrak{r}_{\rm in} < \mathfrak{r}_{\rm max}$, which all have the same $\mathfrak{r}_{\rm max}$. For a chosen $\mathfrak{r}_{\rm in}$ in this range, one obtains the density distribution from (35). Members of this one parameter family differ in the maximum density, $\rho_{\rm max}$ and total mass.

Another factor that limits the parameter space of possible equilibrium configurations is the existence of the marginally bound orbit, which for nonrotating black holes is at $\mathfrak{r}_{\rm mb} = 4M$. For $l = l_{\rm mb} = 4M$, the location of the cusp is at $\mathfrak{r} = \mathfrak{r}_{\rm mb}$. At the same time, the location of $W = 0$ in the equatorial plane is relevant. For $l < l_{\rm mb}$, the effective potential W is always negative in the equatorial plane. For $l = l_{\rm mb}$, $W = 0$ at $\mathfrak{r}_{\rm mb} = 4M$ (an inflection point), while for $l > l_{\rm mb}$ there exists a region with $W > 0$. Thus, for any $l > l_{\rm mb}$, only disks without a cusp are possible, with the inner radius for this family of disks being limited by the location of $W = 0$.

Within each family of disks with same specific angular momentum l, the disk with the smallest $\mathfrak{r}_{\rm in}$ (i.e. the disk with $\mathfrak{r}_{\rm in} = \mathfrak{r}_{\rm cusp}$, for $l_{\rm ms} < l_{\rm mb}$, or the marginally stable disk for $l > l_{\rm mb}$) has the largest $\rho_{\rm max}$. Among all models with $\mathfrak{r}_{\rm in} = \mathfrak{r}_{\rm cusp}$, $\rho_{\rm max}$ *increases* with increasing l. For example, for $\Gamma = 4/3$ polytropes $\rho_{\rm max}$ reaches up to 1.341×10^{-6} (in units where $c = G = K = 1$) for the model with $\mathfrak{r}_{\rm in} = \mathfrak{r}_{\rm mb}$. For $l > l_{\rm mb}$ the largest $\rho_{\rm max}$ *decreases* with increasing l. Thus, among all finite size disks, the largest $\rho_{\rm max}$ is attained for the model with $\mathfrak{r}_{\rm in} = \mathfrak{r}_{\rm mb}$.

5. AJS disks in isotropic coordinates

The metric of the Schwarzschild spacetime in isotropic coordinates (t, r, θ, ϕ) is

$$ds^2 = -\left[\frac{1 - M/2r}{1 + M/2r}\right]^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{41}$$

The relation between the isotropic radial coordinate r and the Schwarzschild radial coordinate $\mathfrak{r}$ is

$$r = \frac{1}{2}\left[\mathfrak{r} - M + \sqrt{\mathfrak{r}^2 - 2M\mathfrak{r}}\right], \tag{42}$$

or

$$\mathfrak{r} = \frac{M^2}{4r} + M + r. \tag{43}$$

The Keplerian angular momentum is

$$l_{\mathrm{K}} = \frac{\sqrt{M}\,(1 + M/2r)^6\, r^2}{(1 - M/2r)^2\,(M + M^2/4r + r)^{3/2}}, \tag{44}$$

while

$$u_t = -\left[\left(\frac{M + 2r}{M - 2r}\right)^2 - \frac{16 l^2 r^2}{(M + 2r)^4 \sin^2\theta}\right]^{-1/2}. \tag{45}$$

The radius of the marginally stable orbit is at

$$r_{\mathrm{ms}} = \frac{1}{2}(5 + 2\sqrt{6}), \tag{46}$$

or, $r_{\mathrm{ms}} \simeq 4.950$. The radius of the marginally bound orbit is the location where $W = 0$ for $l = 4$, i.e. $r_{\mathrm{mb}} \simeq 2.914$.

6. Self-gravitating disks in quasi-isotropic coordinates

In quasi-isotropic coordinates, the metric of a stationary, axisymmetric spacetime is written as

$$ds^2 = -e^{2\nu}dt^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2) + e^{2(\gamma-\nu)} r^2 \sin^2\theta (d\phi - \omega dt)^2, \tag{47}$$

where ν, γ, α and ω are metric functions that depend only on the coordinates r and θ. Because γ and ρ are divergent at the event horizon, we will use, instead, the functions

$$B := e^{\gamma}, \tag{48}$$

$$\lambda := e^{\nu}. \tag{49}$$

Then, the following conditions hold on the event horizon

$$B = 0, \tag{50}$$

$$\lambda = 0, \tag{51}$$

$$\omega = \omega_h, \tag{52}$$

where ω_h is the *constant* angular velocity of the horizon. Within the choice of metric (47), the coordinate r can be defined such that the horizon is a sphere of constant radius h_0 (see, e.g. Carter 1973). Along the rotation axis, the condition $\alpha = \gamma - \nu$ ensures local flatness.

In the angular direction we use the coordinate $\mu \equiv \cos\theta$ instead of θ, which distributes grid points more evenly near the rotation axis. In the radial direction, we use a compactified, dimensionless coordinate s, defined through

$$r := r_e \frac{s}{1 - s}, \tag{53}$$

where r_e is a suitably chosen radius (such as the outer radius of the disk). In this way, the infinite domain $r \to [0, +\infty)$ is mapped onto the finite domain $s \to [0, 1]$. In transforming the field equations to this new coordinate system, the following relations are useful

$$\frac{\partial}{\partial r} = \frac{(1-s)^2}{r_e}\frac{\partial}{\partial s}, \tag{54}$$

$$\frac{\partial}{\partial \theta} = -\sqrt{1-\mu^2}\frac{\partial}{\partial \mu}, \tag{55}$$

$$\vec{\nabla} a = \frac{(1-s)^2}{r_e}\frac{\partial a}{\partial s}\hat{r} - \frac{1-s}{r_e s}\sqrt{1-\mu^2}\frac{\partial a}{\partial \mu}\hat{\theta}, \tag{56}$$

$$\vec{\nabla} a \vec{\nabla} b = \frac{(1-s)^4}{r_e^2}\frac{\partial a}{\partial s}\frac{\partial b}{\partial s} - \frac{(1-s)^2}{r_e^2 s^2}(1-\mu^2)\frac{\partial a}{\partial \mu}\frac{\partial b}{\partial \mu}, \tag{57}$$

where a, b are some scalar functions.

For the 4-velocity, one can easily show that

$$u^t = \frac{e^{-\nu}}{\sqrt{1-v^2}}, \tag{58}$$

where

$$v = (\Omega - \omega) B e^{-2\nu} r \sin\theta, \tag{59}$$

is the 3-velocity, as measured by a zero-angular-momentum observer (ZAMO).

The field equations for the three metric functions λ, B and ω are of elliptic type

$$\nabla^2 \lambda = S_\lambda(s, \mu), \tag{60}$$

$$\left(\nabla^2 + \frac{(1-s)^3}{r_e^2 s}\frac{\partial}{\partial_s} - \frac{(1-s)^2}{r_e^2 s^2}\mu\frac{\partial}{\partial \mu}\right) B = S_B(s, \mu), \tag{61}$$

$$\left(\nabla^2 + \frac{2(1-s)^3}{r_e^2 s}\frac{\partial}{\partial_s} - \frac{2(1-s)^2}{r_e^2 s^2}\mu\frac{\partial}{\partial \mu}\right) \omega = S_\omega(s, \mu), \tag{62}$$

where ∇^2 is the flat-space Laplacian, and

$$\begin{aligned} S_\lambda := & 4\pi\lambda e^{2\alpha}\left[(\epsilon + p)\frac{1+v^2}{1-v^2} + 2p\right] + \frac{1}{2}(-\mu^2)B^2\lambda^{-3}[s^2(1-s)^2(\omega_{,s})^2 + (1-\mu^2)(\omega_{,\mu})^2] \\ & - \frac{(1-s)^2}{r_e^2}\left[(1-s)^2(\gamma_{,s} - \nu_{,s})\lambda_{,s} - \frac{1-\mu^2}{s^2}(\gamma_{,\mu} - \nu_{,\mu})\lambda_{,\mu}\right], \end{aligned} \tag{63}$$

$$S_B := 16\pi p B e^{2\alpha}, \tag{64}$$

$$S_\omega := \frac{(1-s)^2}{r_e^2}\left[(1-s)^2(4\nu_{,s} - 3\gamma_{,s})\omega_{,s} + \frac{1-\mu^2}{s^2}(4\nu_{,\mu} - 3\gamma_{,\mu})\omega_{,\mu}\right] - 16\pi e^{2\alpha}(\epsilon + p)\frac{\Omega - \omega}{1-v^2}, \tag{65}$$

are source terms and *a comma in the subscript* denotes partial differentiation with respect to that coordinate.

The three elliptic-type equations are inverted using appropriate Green's functions

$$\begin{aligned}\lambda =&1-\frac{h_0}{r}-\sum_{n=0}^{\infty}P_{2n}(\mu)\\ &\int_{h_0}^{\infty}dr'\int_0^1 d\mu' r'^2 f_{2n}^{(2)}(r,r')P_{2n}(\mu')S_\lambda(r',\mu'),\end{aligned} \tag{66}$$

$$\begin{aligned}B =&\left(1-\frac{h_0^2}{r^2}\right)-\frac{2\pi}{r\sin\theta}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\theta}{2n-1}\\ &\int_{h_0}^{\infty}dr'\int_0^1 d\mu' r'^2 f_{2n-1}^{(1)}(r,r')\sin(2n-1)\theta' S_B(r',\mu'),\end{aligned} \tag{67}$$

$$\begin{aligned}\omega =&\frac{\omega_h h_0^3}{r^3}-\frac{1}{r\sin\theta}\sum_{n=1}^{\infty}\frac{P_{2n-1}^1(\mu)}{2n(2n-1)}\\ &\int_{h_0}^{\infty}dr'\int_0^1 d\mu' r'^3\sin\theta' f_{2n-1}^{(2)}(r,r')P_{2n-1}^1(\mu')S_\omega(r',\mu'),\end{aligned} \tag{68}$$

where

$$f_n^{(1)}(r,r') := \begin{cases}\left(\frac{r'}{r}\right)^n-\frac{h_0^{2n}}{(rr')^n} & \frac{r'}{r}\le 1,\\ \left(\frac{r}{r'}\right)^n-\frac{h_0^{2n}}{(rr')^n} & \frac{r'}{r}>1,\end{cases} \tag{69}$$

and

$$f_n^{(2)}(r,r') := \begin{cases}\frac{1}{r}\left(\frac{r'}{r}\right)^n-\frac{h_0^{2n+1}}{(rr')^{n+1}} & \frac{r'}{r}\le 1,\\ \frac{1}{r'}\left(\frac{r}{r'}\right)^n-\frac{h_0^{2n+1}}{(rr')^{n+1}} & \frac{r'}{r}>1.\end{cases} \tag{70}$$

If we define the location of the horizon in terms of the coordinate s through

$$h_0 = r_e\frac{s_0}{1-s_0}, \tag{71}$$

then the metric function λ is given by

$$\lambda(s,\mu) = 1-\frac{s_0(1-s)}{s(1-s_0)}-\sum_{n=0}^{\infty}P_{2n}(\mu)D_n^{\lambda(2)}(s), \tag{72}$$

where

$$D_n^{\lambda(2)}(s) := \int_{s_0}^1 ds'\tilde{f}_{2n}^{(2)}(s,s')D_n^{\lambda(1)}(s'), \tag{73}$$

$$D_n^{\lambda(1)}(s') := \int_0^1 d\mu' P_{2n}(\mu')\tilde{S}_\lambda(s',\mu'), \tag{74}$$

$$\tilde{f}_n^{(2)}(s,s') := \frac{r_e}{(1-s')^2}f_n^{(2)}(s,s'), \tag{75}$$

$$\tilde{S}_\lambda := r'^2 S_\lambda. \tag{76}$$

The metric function B is given by

$$B(s,\mu) = 1-\left[\frac{s_0(1-s)}{s(1-s_0)}\right]^2-2\pi r_e^2\sum_{n=1}^{\infty}\Pi_{2n-1}^1(\mu)D_n^{B(2)}(s), \tag{77}$$

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where

$$D_n^{B(2)}(s) := \int_{s_0}^{1} ds' \tilde{f}_{2n-1}^{(1)}(s,s') D_n^{B(1)}(s'), \tag{78}$$

$$D_n^{B(1)}(s') := \int_{0}^{1} d\mu' \sin(2n-1)\theta' S_B(s',\mu'), \tag{79}$$

$$\tilde{f}_n^{(1)}(s,s') := \frac{s'^2}{(1-s')^4}\left(\frac{1-s}{s}\right) f_n^{(1)}(s,s'), \tag{80}$$

$$\Pi_n^1(\theta) := \frac{\sin(n\theta)}{n\sin\theta}. \tag{81}$$

And, the metric function ω is given by

$$\omega(s,\mu) = \left[\frac{s_0(1-s)}{s(1-s_0)}\right]^3 \omega_h - r_e^2 \sum_{n=1}^{\infty} \Pi_n^2(\mu) D_n^{\omega(2)}(s), \tag{82}$$

where

$$D_n^{\omega(2)}(s) := \int_{s_0}^{1} ds' \tilde{f}_{2n-1}^{(3)}(s,s') D_n^{\omega(1)}(s'), \tag{83}$$

$$D_n^{\omega(1)}(s') := \int_{0}^{1} d\mu' \sqrt{1-\mu'^2} P_{2n-1}^1(\mu') S_\omega(s',\mu'), \tag{84}$$

$$\tilde{f}_n^{(3)}(s,s') := \frac{s'^3(1-s)}{s(1-s')^3} \tilde{f}_n^{(2)}(s,s'), \tag{85}$$

$$\Pi_n^2(\mu) := \frac{P_{2n-1}^1(\mu)}{2n(2n-1)\sqrt{1-\mu^2}}. \tag{86}$$

In practice, we use the auxiliary variables

$$\bar{\rho} := \ln\frac{\lambda^2}{B}, \tag{87}$$

$$\hat{\nu} := \nu/r_e^2, \tag{88}$$

$$\hat{\gamma} := \gamma/r_e^2, \tag{89}$$

$$\hat{\omega} := r_e\omega. \tag{90}$$

The summations in (72), (77) and (82) are truncated to a maximum number of terms $n_{\max} = 10$. Both radial and angular integrations are performed with the 4th-order Simpson's rule.

The remaining metric function α is given in terms of the partial differential equation

$$
\begin{aligned}
\alpha_{,\mu} = &-\frac{1}{2}(\bar{\rho}_{,\mu}+\gamma_{,\mu}) - \left\{(1-\mu^2)[1+s(1-s)\gamma_{,s}]^2 + [-\mu+(1-\mu^2)\gamma_{,\mu}]^2\right\}^{-1} \\
&\times \Bigg[\frac{1}{2}\{s(1-s)[s(1-s)\gamma_{,s}]_{,s} + s^2(1-s)^2\gamma_{,s}^2 - [(1-\mu^2)\gamma_{,\mu}]_{,\mu} - \gamma_{,\mu}[-\mu+(1-\mu^2)\gamma_{,\mu}]\} \\
&\qquad [-\mu+(1-\mu^2)\gamma_{,\mu}] + \frac{1}{4}[s^2(1-s)^2(\bar{\rho}_{,s}+\gamma_{,s})^2 - (1-\mu^2)(\bar{\rho}_{,\mu}+\gamma_{,\mu})^2] \\
&\qquad [-\mu+(1-\mu^2)\gamma_{,\mu}] - s(1-s)(1-\mu^2)\Big[\frac{1}{2}(\bar{\rho}_{,s}+\gamma_{,s})(\bar{\rho}_{,\mu}+\gamma_{,\mu}) \\
&\qquad + \gamma_{,s\mu} + \gamma_{,s}\gamma_{,\mu}\Big][1+s(1-s)\gamma_{,s}] + s(1-s)\mu\gamma_{,s}[1+s(1-s)\gamma_{,s}] \\
&\qquad + \frac{1}{4}(1-\mu^2)e^{-2\bar{\rho}}\Big\{2\frac{s^3}{1-s}(1-\mu^2)\hat{\omega}_{,s}\hat{\omega}_{,\mu}[1+s(1-s)\gamma_{,s}] \\
&\qquad - [s^4\hat{\omega}_{,s}^2 - \frac{s^2}{(1-s)^2}(1-\mu^2)\hat{\omega}_{,\mu}^2][-\mu+(1-\mu^2)\gamma_{,\mu}]\Big\}\Bigg].
\end{aligned} \tag{91}
$$

This equation is integrated as an ODE along lines of constant s using Adams' multiple step method of local fourth-order accuracy.

In the chosen coordinate system, some problems appear in the source terms of the equations for λ and ω. Specifically, even though the term $(\gamma-\nu)_{,s}$ is regular at the horizon, its individual components, $\gamma_{,s}$ and $\nu_{,s}$ diverge (but their difference does not). In a numerical method, $\gamma_{,s}$ and $\nu_{,s}$ are evaluated separately and due to numerical finite-differencing errors it is not possible to exactly cancel the two terms at the horizon. One must therefore accept some approximation to keep the source S_λ regular. Similarly, problems appear in the second term of (63) and (65).

7. Numerical scheme

We use non-dimensional units of $c = G = M_{\rm BH}^{\rm AJS} = 1$ (the solution can then be scaled to any desired black hole mass). First, an AJS disk is computed as a trial configuration. A unique model is obtained by fixing the specific angular momentum l, the outer radius of the disk r_e (which also fixes the maximum enthalpy $H_{\rm max}$) and either of Γ, K or $\rho_{\rm max}$ (the latter three fluid properties are related to each other). Thus, for a chosen polytropic Γ, the solution space of AJS models is three-dimensional: the parameter l controls the rotation of the torus, r_e controls its distance from the black hole and its size and K (or $\rho_{\rm max}$) its mass content.

Once the AJS trial solution is obtained, one can then derive the location of the inner edge of the disk, $r_{\rm in}$, the location of the density maximum, $r_{\rm max}$, and the value of the density maximum, $\rho_{\rm max}$ (or K). The configuration is constructed so that the horizon of the black hole coincides with a grid point, s_0, in terms of the compactified coordinate s. The outer edge of the disk, r_e, also coincides with the grid point for which $s = 0.5$.

The AJS configuration is used as an initial guess for starting the self-consistent iteration between the field equations and the hydrostationary equilibrium equation. During this process, one can choose to keep some properties fixed, while others will necessarily vary. Since we always want the black hole horizon to coincide with a grid point, we choose to keep fixed the location of the horizon s_0 in terms of the compactified radial coordinate s. We also keep fixed the location of the outer radius at $s = 0.5$, so that the ratio h_0/r_e remains fixed (as do l and the EOS parameters K and Γ). Since r_e itself is allowed to vary during the iteration, we have to fix another quantity, in order to specify a model uniquely. This quantity can be $H_{\rm max}$ (or, equivalently, $\rho_{\rm max}$, since K is also fixed).

All other parameters of the BH-torus system (mass of the black hole, radius scale r_e, radius of horizon h_0, maximum density, mass of torus etc.) adjust themselves to self-consistent values during the iteration process. The iteration is carried out in units of $M_{\rm BH}^{\rm AJS} = 1$. After convergence, the results are rescaled to the newly computed mass of the black hole in the self-consistent solution.

The iteration proceeds as follows:

(i) A new value for r_e is obtained by solving the enthalpy equation (29) at $r = r_{\rm out}$, using (88) and (89)

$$r_e = \left\{ \frac{H_{\rm max} + 1/2 \ln \left[\left(\frac{1-v_{\rm max}^2}{1-v_{\rm out}^2} \right) \left(\frac{1-l\Omega_{\rm out}}{1-l\Omega_{\rm max}} \right)^2 \right]}{\hat{\nu}_{\rm out} - \hat{\nu}_{\rm max}} \right\}^{1/2}, \tag{92}$$

(ii) New angular velocity and 3-velocity distributions are obtained through (6) and (59)

$$\hat{\Omega} := \frac{\Omega}{r_e} = \left\{ \hat{\omega} + \frac{l}{r_e} \left[\frac{(1-s)^2 e^{r_e^2(4\hat{\nu}-2\hat{\gamma})}}{s^2(1-\mu^2)} - \hat{\omega}^2 \right] \right\} (1 - l\hat{\omega}/r_e)^{-1}, \tag{93}$$

$$v = (\hat{\Omega} - \hat{\omega}) e^{r_e^2(\hat{\gamma}-2\hat{\nu})} \frac{s}{1-s} \sqrt{1-\mu^2}. \tag{94}$$

(iii) New values $\hat{\Omega}_{\rm max}$ and $v_{\rm max}$ are obtained at the location of the density maximum $s_{\rm max}$.

(iv) A new enthalpy distribution is obtained through (29) and (58)

$$H = H_{\rm max} + r_e^2(\hat{\nu}_{\rm max} - \hat{\nu}) + \frac{1}{2} \ln \left[\frac{(1-v_{\rm max}^2)(1-l\hat{\Omega}/r_e)^2}{(1-v^2)(1-l\hat{\Omega}_{\rm max}/r_e)^2} \right]. \tag{95}$$

(v) New density, ρ, pressure, p, and energy density, ϵ, distributions are obtained through (33), (31) and (32).

(vi) New distributions of the four metric potentials λ, B, ω and α are obtained through (72), (77),(82), and (91).

(vii) The process is repeated from step (i) until the value for r_e converges with a desired relative accuracy (usually chosen as 10^{-6}).

8. Derived properties

The gravitational mass of the torus is

$$\begin{aligned} M_{\rm T} =& \int (-2T_t{}^t + T_i{}^i) \sqrt{-g} d^3x \\ =& 4\pi r_e^3 \int_0^1 \frac{s^2 ds}{(1-s)^4} \int_0^1 d\mu B e^{2\alpha} \left\{ \frac{\epsilon + p}{1-v^2} \left[1 + v^2 + \frac{2sv}{1-s} \sqrt{1-\mu^2} \hat{\omega} \frac{B}{\lambda^2} \right] + 2p \right\}. \end{aligned} \tag{96}$$

The rest mass of the torus is

$$\begin{aligned} M_0 =& \int \rho u^t \sqrt{-g} d^3x \\ =& 4\pi r_e^3 \int_0^1 \frac{s^2 ds}{(1-s)^4} \int_0^1 d\mu \frac{B}{\lambda} e^{2\alpha} \frac{\rho}{\sqrt{1-v^2}}. \end{aligned} \tag{97}$$

Physical units are obtained by multiplying with c^2L/G, where L is the fundamental length scale (set by $M_\odot = 1$). The internal energy of the torus is

$$\begin{aligned} U_{\rm T} =& \int (\epsilon-\rho) u^t \sqrt{-g} d^3x \\ =& 4\pi r_e^3 \int_0^1 \frac{s^2 ds}{(1-s)^4} \int_0^1 d\mu \frac{B}{\lambda} e^{2\alpha} \frac{\epsilon-\rho}{\sqrt{1-v^2}}, \end{aligned} \tag{98}$$

with physical units obtained by multiplying with c^4L/G. The angular momentum of the torus is

$$\begin{aligned} J_{\rm T} =& \int T^t{}_\phi \sqrt{-g} d^3x \\ =& 4\pi r_e^4 \int_0^1 \frac{s^3 ds}{(1-s)^5} \int_0^1 d\mu \sqrt{1-\mu^2} \frac{B^2 e^{2\alpha}}{\lambda^2} (\epsilon+p) \frac{v}{1-v^2}, \end{aligned} \tag{99}$$

with physical units obtained by multiplying with c^3L^2/G. The rotational kinetic energy of the torus is

$$\begin{aligned} T_{\rm T} =& \frac{1}{2} \int \Omega T^t{}_\phi \sqrt{-g} d^3x \\ =& 2\pi r_e^3 \int_0^1 \frac{s^3 ds}{(1-s)^5} \int_0^1 d\mu \sqrt{1-\mu^2} \frac{B^2 e^{2\alpha}}{\lambda^2} (\epsilon+p) \frac{v\hat{\Omega}}{1-v^2}, \end{aligned} \tag{100}$$

with physical units obtained by multiplying with c^4L/G.

The total mass of the spacetime can be obtained in two ways. From the asymptotic form of the metric potential λ, which becomes, to lowest order, $1 - M/r$ for $r \to \infty$, we find the ADM mass

$$M_{\rm ADM} = \lim_{r\to\infty} r^2 \frac{\partial \lambda}{\partial r} = \lim_{s\to 1} r_e s^2 \frac{\partial \lambda}{\partial s}. \tag{101}$$

On the other hand, the Komar mass, $M_{\rm K}$ is obtained from the integral

$$M_{\rm K} = \lim_{r\to\infty} \frac{1}{2} \int_0^\pi d\theta \sin\theta \left(r^2 B \frac{\partial \nu}{\partial r} - \frac{1}{2} \sin^2\theta r^4 B^3 e^{-4\nu} \omega \frac{\partial \omega}{\partial r} \right). \tag{102}$$

For stationary spacetimes, the total mass-energy is $M = M_{\rm ADM} = M_{\rm K}$ and since we use a compactified grid, we can evaluate M accurately from (101).

The Komar-charge of the black hole, $M_{\rm H}$, is equal to

$$M_{\rm H} = M - M_{\rm T}, \tag{103}$$

and does not coincide with the mass of the black hole, since it also includes part of the negative binding energy between the black hole and the torus. Our numerical results confirm that $M_{\rm H} < M - M_0$. The actual black hole mass should be larger than this difference, because of the negative binding energy. We compute the black hole mass from its circumference, i.e. from

$$M_{\rm BH} = \frac{C_{eq}}{4\pi}, \tag{104}$$

where

$$C_{eq} = \int_0^{2\pi} d\phi \sqrt{g_{\phi\phi}}|_{\theta=\pi/2} = 2\pi h_0 \frac{B}{\lambda}|_{\theta=\pi/2}, \tag{105}$$

is the proper equatorial circumference of the black hole. Since in our coordinate system both B and λ vanish on the horizon, we compute $M_{\rm BH}$ using a 3-point Lagrange extrapolation from near-by grid points. This procedure is highly accurate for vacuum black holes. Even for the most massive disks we examined, we estimate our extrapolation error to be less than 1%. Furthermore, our results confirm the finding of [18] that $M_{\rm BH}$ is larger than $M_{\rm H}$ by $\sim 0.1 M_0$.

The gravitational potential energy of the torus is

$$W_{\rm T} := M - M_{\rm BH} - M_0 - T_{\rm T} - U_{\rm T}, \tag{106}$$

and includes both self-energy and binding energy between the black hole and the torus.

The total angular momentum of the spacetime can be obtained from the Komar integral

$$J = \lim_{r\to\infty} -\frac{1}{8}\int_0^\pi d\theta \sin^3\theta r^4 B^3 e^{-4\nu}\frac{\partial\omega}{\partial r}, \tag{107}$$

which, for stationary spacetimes, agrees with the asymptotic expression

$$J = -\frac{1}{6}r^4\frac{\partial\omega}{\partial r} = -\frac{1}{6}r_e^3\frac{s^4}{(1-s)^2}\frac{\partial\omega}{\partial s}. \tag{108}$$

For the zero-angular-velocity black holes considered here, we find that the numerical determination of J and $J_{\rm T}$ agree to better than 1%. Part of this small difference is due to the fact that black holes with $\omega_h = 0$ have a small negative angular momentum in the presence of a disk.

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