Hilbert's Third Problem and Dehn Invariants

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Hilbert's Third Problem statement:

If two Polyhedra P and Q have the same volume then is it possible to split P in smaller polyhedra and reorganize them in order to build Q?

Theorem (Wallace-Bolyai-Gerwien Theorem)

Given two polygons with P and Q with the same area, it is always possible to split P into smaller polygons and rearrange them into Q.

The proof also suggests an algorithm to do so.

Aiming to work properly with these concepts, we bring two definitions which will give a mathematically more precise idea of ideas like "split a polyhedra" and "reorganize its pieces".

Definition

A *Dissection* for a polyhedron P is a finite collection $M_P = \{P_1, P_2, ..., P_n\}$ of subsets of P such that

1. P_i is a polyhedron itself for all $i \in 1, 2, ..., n$;
2. $\cup_{i=1}^{n} P_i = P$;
3. The sets in M_P have pairwise disjoint interiors.

Our last needed definition, which reflects our intuition about reorganizing polyhedral pieces into other polyhedra:

Definition

Given two polyhedra P and Q, they are said to be *Scissors Congruent* if there are dissections M_P, M_Q for P and Q and a bijection $\rho : M_P \to M_Q$ such that $\rho(L)$ is isometric to L for all $L \in M_P$.

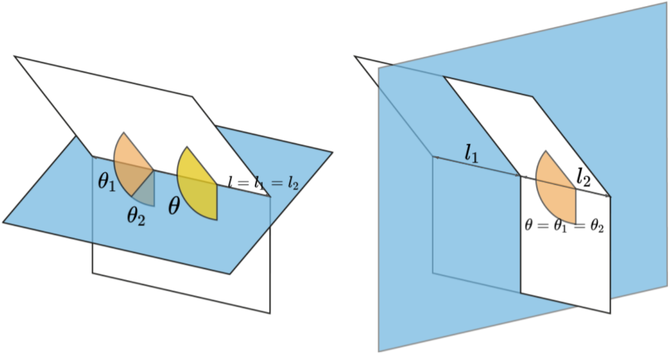
Note: Scissors Congruence is an equivalence relation.

We can now restate the problem.

Given two polyhedra P and Q, does Vol$(P) =$ Vol(Q) implies P is scissors congruent with Q?

We proceed to analyse one simple form of dissecting polyhedra: cutting them with a plane.

Hilbert's Third Problem and Scissors Congruence



We can see in the image above the two ways a plane can cross an edge.

1. Left (plane contains the edge): Dihedral angle divided into two parts and length of the edge preserved;
2. Right (edge crosses the plane): Dihedral angle preserved and length of the edge divided into two parts;

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We can see a structure of bilinearity there. Based on that analysis we define the Dehn Invariant as follows.

Definition

Given a polyhedron P with edges $A_1, A_2, ..., A_n$ where ℓ_i is the length of A_i and θ_i its dihedral angle, the *Dehn Invariant* of P is defined as

$$D(P) = \sum_{i=1}^{n} \ell_i \otimes \theta_i.$$

$D(P)$ is an element of the abelian group $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$.

$\mathbb{R}$ represents the possible lengths of the edges while $\mathbb{R}/\pi\mathbb{Z}$ is related with the dihedral angles, identifying angles differing by a half-turn.

Dehn Invariants and Scissors Congruence

Lemma

If P is a polyhedron and $M_P = \{P_1, P_2, ..., P_n\}$ a dissection for P, then

$$D(P) = D(P_1) + D(P_2) + ... + D(P_n).$$

Each edge of a polyhedron in P_i is of one of the three following types:

1. Edges of P_i contained in edges of P;
2. Edges of P_i contained in faces of P (save, perhaps, the edge extremities);
3. Edges of P_i contained in the interior of P (save, perhaps, the edge extremities).

We split the sum $D(P_1) + ... + D(P_n)$ in three terms $S_1 + S_2 + S_3$, such that S_i is the sum of the tensors related with edges of type i.

We calculate S_3 first and replicate the argument with the proper adaptations for S_2 and S_1.

Choose an edge of the type 3 and consider all the edges of polyhedra in M_P which are colinear with this edge. We denote these edges by $A_1, A_2, ..., A_m$.

We take the set $\{p_1, p_2, ..., p_k\}$ consisting of all extremity points of these edges, already ordered by distance to one of the ends of $\cup_{i=1}^{m} A_i$. This set determines a partition for $\cup_{i=1}^{m} A_i$.

Each edge-related tensor can be decomposed in a sum $|p_i p_{i+1}| \otimes \theta_j + |p_{i+1} p_{i+2}| \otimes \theta_j + ... + |p_{i+t} p_{i+t+1}| \otimes \theta_j$.

Since M_P is a dissection for P, each neighbourhood (contained in P) of an interior point of P is contained in the union of some subsets of M_P.

It translates to "There are polyhedra of the decomposition of P all around each segment $p_i p_{i+1}$", so that summing over j in $|p_i p_{i+1}| \otimes \theta_j$ gives you $|p_i p_{i+1}| \otimes 2\pi = 0$.

We conclude that $S_3 = 0$. Using the same argument, the sums of the tensors in S_2 give us $|p_i p_{i+1}| \otimes \pi = 0$.

Hence, $S_2 = 0$. Again, applying this argument for tensors in S_1, we do not get multiples of π, instead, we get dihedral angles of P, showing that $S_1 = D(P)$.

Dehn Invariants and Scissors Congruence

Theorem

If P and Q are scissors congruent polyhedron, then $D(P) = D(Q)$.

We take $\{\mathrm{P}_1, ..., P_n\}$ and $\{\mathrm{Q}_1, ..., Q_n\}$ the dissections that make P and Q scissors congruent with P_i isometric with Q_i.

$D(P_i) = D(Q_i)$ since isometries preserves angles and lengths. Then by previous lemma,

$$D(P) = D(P_1) + ... + D(P_n) = D(Q_1) + ... + D(Q_n) = D(Q).$$

This way, the Dehn Invariant is indeed an invariant.

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Answering the Problem

We now give an answer to Hilbert's Third Problem, by finding two polyhedra with the same volume and distinct Dehn Invariants.

Consider the unitary cube and the regular tetrahedron with unitary volume.

The cube has 12 edges each with length 1 and dihedral angle $\frac{\pi}{2}$. Hence, its Dehn Invariant is $D(\text{Cube}) = 12 \cdot (1 \otimes \pi/2) = 1 \otimes 6\pi = 0$.

The calculation for the tetrahedron is a bit more complicated. With some geometry, one can show that the length of the edges of the tetrahedron is $6^{1/3}2^{1/6}$ and its dihedral angles are $\arccos 1/3$.

We just use the fact that $\arccos 1/3$ is not a rational multiple of π to conclude that $6 \cdot (6^{1/3}2^{1/6} \otimes \arccos 1/3) = 6^{4/3}2^{1/6} \otimes \arccos 1/3 \neq 0$.

Answering the Problem

We evoke the last theorem to observe that if two polyhedra have distinct Dehn Invariants, then they can't be scissors congruent.

Hence, we found two polyhedra with the same volume which are not scissors congruent.

We proceed now by making some comments on generalizations of Hilbert's Third Problem, some open questions, and other areas of mathematics related to the theory of Dehn Invariants.

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Generalizations of the Problem

In 1967, SYDLER shows the reciprocal of Dehn's Theorem.

Theorem (Dehn-Sydler Theorem)

Two polyhedra P and Q with the same volume are scissors congruent if, and only if, they have the same Dehn Invariant.

It shows how the volume and the Dehn Invariant form a complete set of invariants for scissors congruence and they classify all of its congruence classes.

Similarly, the area is a complete invariant for scissors congruence in two dimensions.

Generalizations of the Problem

Similarly to Hilbert's Third Problem being an upper dimensional analogous to questions concerning plane geometry, we can go higher and ask the analogous question for $\mathbb{R}^n$.

If two polytopes in $\mathbb{R}^n$ have the same n-volume, do they necessarily are scissors congruent?

We observe the need for some adaptation of the notions of dissection and scissors congruence to work for polytopes in $\mathbb{R}^n$.

DUPONT and SAH reinterpreted the Dehn Invariant in terms of homology theory.

Generalizations of the Problem

DUPONT and SAH proves the following sequence is exact:

$$
\begin{tikzcd}
0 \arrow[r] & H_2(SO(3), \mathbb{R}^3) \arrow[r] & \mathscr{P}(\mathbb{R}^3)/\mathscr{L}_2(\mathbb{R}^3) \arrow[dll, "D"] \\
\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z} \arrow[r] & H_1(SO(3), \mathbb{R}^3) \arrow[r] & 0
\end{tikzcd}
$$

$\mathscr{P}(\mathbb{R}^3)$ is the set of classes of polyhedra module the group of isometries of $\mathbb{R}^3$. $\mathscr{L}_2(\mathbb{R}^3)$ is the subgroup generated by prisms, over the operation of union of disjoint elements of classes.

Dehn's Theorem is then equivalent to $H_2(SO(3), \mathbb{R}^3) = 0$.

Generalizations of the Problem

DUPONT and SAH shows that The Hadwiger Invariant (analogous of the Dehn Invariant for $\mathbb{R}^n$) gives the Dehn-Sydler theorem for $\mathbb{R}^3$.

The theorem is also true for S^2 and $\mathbb{H}^3$.

In S^3 and $\mathbb{H}^3$, it is true that if two polyhedra are scissors congruent, then they have the same Dehn Invariant, but we still don't know if the reciprocal is true.

Relation with other areas

We have already seen some relation between Dehn Invariants and Homology Theory. We can go further by noticing some other aspects of Hilbert's Third Problem.

There are exact sequences (similar to the previous one) relating some K-theories with groups of polyhedra in $\mathbb{R}^n$.

Furthermore, each class of isometric polyhedra can be seen as a K-theory of certain categories (in $\mathbb{R}^n$, S^n and $\mathbb{H}^n$ too).

DEBRUNNER showed in 1980 that every polyhedra that tiles $\mathbb{R}^3$ periodicly has null Dehn Invariant. LAGARIAS and MOEWS proved lately that any polytope that tiles $\mathbb{R}^n$ periodically has null Hadwiger Invariant.

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