The underlying infinity category of a model category

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Introduction

Every model category gives rise to an ∞-category that describes the same homotopy theory. These notes aim to explain how one can start with a model category and arrive at such ∞-category. This process is made in three steps, each being done by applying a functor.

In synthesis, we have three functors

$$\mathbf{RelCat} \xrightarrow{L^H} \mathbf{Cat}_{\mathbf{sSet}} \xrightarrow{\mathbb{R}_B} \mathbf{Cat}_{\mathbf{sSet}} \xrightarrow{N^{\mathrm{hc}}} \mathbf{sSet}$$

whose composition has ∞-categories as output.

Before properly starting to talk about these three constructions, we refresh some central notions we will be using, like model and infinity categories.

We proceed to construct, after the refresher, the hammock localization L^H, which gives us a way to arrive at a simplicially enriched category, already a nice way to store homotopical data.

Subsequentially, we talk about a form of fibrant replacement in the category of simplicially enriched categories. This fibrant replacement, denoted by $\mathbb{R}_B$, will allow us to obtain categories enriched over Kan complexes.

However, we can do better: the homotopy coherent nerve is the last step in our construction, offering a bridge from simplicially enriched categories to simplicial sets. More than that, the fact that the fibrant replacement returns categories enriched over Kan complexes will ensure that after chaining the homotopy coherent nerve with the previous constructions, we end up with a quasi-category, which will be called the *underlying category* of our starting model category.

We end by discussing in what sense the underlying category depicts the same homotopy theory as its originating model category, touching on other aspects of homotopy theory, like the "homotopy theory of homotopy theories" and equivalences of different homotopy theories.

Notation and conventions

We will adopt the following notations.

- **Cat** will be the category of locally small categories;
- **sSet** will denote the category of simplicial sets;
- **RelCat** is the category of relative categories (categories with weak equivalences);
- $\mathbf{Cat}_{\mathbf{sSet}}$ will be the category of simplicially enriched categories[1].

1 A homotopy theory refresher

Nowadays, there are many ways to describe a homotopy theory. One of the first means people found to do so is with *model categories*. Model categories are categories with special morphisms representing weakened isomorphisms between objects. They also have other structures that mirror our past experiences with homotopy theories and help us to do more concrete work inside them. A more general notion is that of a *relative category*, which only contains the weakened isomorphisms and are around the topics we will discuss.

Apart from that low categorical approach to homotopy theory, there is the concept of $(\infty, 1)$-categories. These describes categories with higher dimensional morphisms, that is, categories where besides only objects and morphisms, there are morphisms between morphisms and morphisms between morphisms between morphisms, and so on until infinity. Ordinary morphisms are usually referred to as 1-morphisms, while morphisms between k-morphisms are said to be $(k+1)$-morphisms. In addition, the "1" in "$(\infty, 1)$" tells us that every morphism of dimension more than 1 is invertible. The theory of $(\infty, 1)$-categories, which we will be calling just ∞-categories, is not unique, in the sense that there are many different ways to define something that fits in our intuitive idea of what an infinity category should be.

We will be interested in two of these models for ∞-categories. One is that of *quasi-categories*, which are simplicial sets with some properties. The other one is described by *Kan complex enriched categories*, categories enriched over special simplicial sets.

1.1 Model categories

One of the ways to describe homotopy theories is by the means of *model categories*. We abstract the core of homotopy theory in some environments to arrive at a definition that captures the needed aspects to do homotopy theory.

[1] another common notation for this category is **sCat**, but that may rises some ambiguity between simplicially enriched categories and simplicial objects in **Cat**

Taking **Top** as an example, we often find ourselves concerned not about homeomorphisms, but (weak or strong) homotopy equivalences. Weak homotopy equivalences are not isomorphisms in **Top**, but we wish they were since it is the notion of equivalence we are interested in. The weak homotopy equivalences are the special morphisms we use to do homotopy theory in **Top**.

In general, model categories will be categories with some special class of morphisms that will work like weak homotopy equivalences. In reality, model categories have other special morphisms and some properties connecting these different types of arrows, while a category with only the weak equivalences will be called a *relative category.* These other special morphisms present in model categories frequently help in calculations, allowing one to work in a more algebraic manner.

We proceed with some definitions, following the presentation of [Hov99]:

Definition 1.1.1 (retract)**.** Given a category C, a map f in C is a *retract* of a map g in C if there is some diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd},$$

such that the composition of the horizontal arrows are identities.

Definition 1.1.2. A *functorial factorization* is a pair of functors (α, β) from Map$C$ to Map$C$ such that $f = \beta(f) \circ \alpha(f)$ for all maps in C.

Definition 1.1.3. If $i : A \to B$ and $p : X \to Y$ are maps in a category C, we say that i has the *left lifting property* (LLP) with respect to p and p has the *right lifting property* (RLP) with respect to i if every commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd} \quad \text{embeds into a diagram} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dotted, "h"] & Y \end{tikzcd}.$$

Definition 1.1.4 (model category)**.** A *model category* is a category C with three distinguished classes of morphisms, two functorial factorizations (α, β) and (γ, δ), and some axioms.

The three distinguished classes of morphism are:

- **weak equivalences**;
- **fibrations**;
- **cofibrations**.

We require the following axioms:

1. C is complete and cocomplete;

2. If f, g are morphisms such that $f \circ g$ is defined, then every time two morphisms from $\{f, g, f \circ g\}$ are weak equivalences, also is the third;

3. if f is a retract of g and g is a distinguished morphism of some kind, then also is f, of the same kind;

4. we define *trivial fibrations* and *trivial cofibrations* to be weak equivalences that also are fibrations or cofibrations, respectively. Trivial cofibrations have the LLP with respect to fibrations and cofribrations have the LLP with respect to trivial fibrations;

5. if f is a morphism in C, then $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.

Remark 1.1.5. Trivial fibrations and trivial cofibrations are sometimes also called *acyclic fibrations* and *acyclic* cofibrations.

Remark 1.1.6. We defined a model category to be complete and cocomplete, implying that every model category has initial and final objects. Given a category C with initial and final objects 0 and 1, respectively, and an object $x \in C$ such that the (unique) morphism $x \to 1$ is a fibration, then x is said to be a *fibrant object.* The object $x \in C$ is a *cofibrant object* if $0 \to x$ is a cofibration.

Fibrant objects will play an important role in defining the functor $\mathbb{R}_B$, which will allow us to replace general simplicially enriched categories by ∞-categories.

Fixed an object x in a model category C, the map $x \to 1$ may not be a fibration, but by the functorial factorization, it can be written as a composition of a trivial cofibration and a fibration:

$$\begin{tikzcd} x \arrow[rr] \arrow[dr, "g"'] & & 1 \\ & y \arrow[ur, "f"'] & \end{tikzcd} .$$

Here, f is a fibration and g is a trivial cofibration. Thus, y is a fibrant object. Since g is a trivial cofibration and, in particular, a weak equivalence, this process of factorizing maps whose codomain is 1 finds fibrant objects which are weak equivalent to the domain of our maps. Replacing our objects with such fibrant objects is called a *fibrant replacement.*

Fibrant objects may have wanted properties not found in all objects of the category C, so that the fibrant replacement is a trick for working with fibrant objects instead of any objects.

There may be many ways to fibrantly replace an object or to fibrantly replace all objects in a category. We will be interested in doing so for $\mathbf{Cat}_{\mathbf{sSet}}$, with a suitable model structure.

One classic example of a model category is the one of topological spaces:

Example 1.1.7 ($\mathbf{Top}_{\text{Quillen}}$)**.** **Top** has model structure, called the *Quillen model structure*, where

- **weak equivalences** are the weak homotopy equivalences;
- **fibrations** are the *Serre fibrations* (maps that have the right lift property with respect to all inclusions of D^n, the n-disk, in $D^n \times [0,1]$);
- **cofibrations** are retracts of relative cell complexes (inclusions of subspaces into spaces that can be constructed by attaching cells in the subspace).

In general, the functorial factorizations are not made explicit since they can be recovered from the special morphisms once you know what they are. In this case, it tells us about factorizing continuous functions as special maps.

Remark 1.1.8. Two classes of special morphisms are enough to define a model category, two of them (no matter which ones) determine the third one.

The general philosophy of model categories is that the weak equivalences are the really important morphisms and we would like them to be isomorphisms. By sticking formal inverses in a model category we can turn weak equivalences into isomorphisms. This is done by localizing a category by the weak equivalences, leading us to a new category, the *localization of C by W* and that is the way we build the *homotopy category* of a model category.

Definition 1.1.9 (localization)**.** Let C be a category and W a collection of morphisms of C. The *localization of C by W* is a category $C[W^{-1}]$ and a functor $Q : C \to C[W^{-1}]$ such that

- if $w \in W$ then $Q(w)$ is an isomorphism in $C[W^{-1}]$;
- for every category A and functor $F : C \to A$ carrying W into isomorphisms, there is a functor $F_W : C[W^{-1}] \to A$ and a natural isomorphism $F \Rightarrow F_W \circ Q$:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "Q"'] \arrow[dr, phantom, "\Swarrow"] & A \\ C[W^{-1}] \arrow[ur, "F_W"'] & \end{tikzcd} ;$$

- the functor

$$(-) \circ Q : \mathbf{Fun}(C[W^{-1}], A) \to \mathbf{Fun}(C, A)$$

is full and faithful for every category A.

Remark 1.1.10. The localization is unique up to equivalence of categories.

Definition 1.1.11. Let C be a model category with weak equivalences W. The *homotopy category* of C is $\mathrm{Ho}(C) := C[W^{-1}]$, the localization of C by W.

Looking at the homotopy category definition, we don't see fibrations and cofibrations anywhere. That is because weak equivalences already contain the interesting information about homotopy. Indeed, we can discard fibrations and cofibrations completely and still have categories that describe homotopy theories, known as relative categories. However, having fibrations and cofibrations is a very useful property for homotopy theory, as illustrated by the fact that

$$\mathrm{Hom}_{\mathrm{Ho}\,C}(X,Y) = \mathrm{Hom}_C(X^c, Y^f),$$

where X^c means the cofibrant replacement of X and Y^f the fibrant replacement of Y. We see that fibrations and cofibrations help us to handle calculations in our model category.

There are many different model categories and it is interesting to have a way to transit between them. We move between different model categories via *Quillen adjunctions*.

Definition 1.1.12 (Quillen adjunction)**.** Given two model categories C and D, a *Quillen adjunction* between them is a pair of adjoint functors

$$\begin{tikzcd} C \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & D \arrow[l, bend left, "L"] \end{tikzcd}$$

satisfying any of the following conditions.

- L preserves cofibrations and trivial cofibrations;
- R preserves fibrations and trivial fibrations;
- L preserves cofibrations and R preserves fibrations;
- L preserves trivial cofibrations and R preserves trivial cofibrations.

Remark 1.1.13. All the conditions above are equivalent.

The homotopy category allows us to define a *Quillen equivalence*, the notion of equivalence between model categories.

Definition 1.1.14 (Quillen equivalence)**.** Let $L : C \leftrightarrows D : R$ be a Quillen adjunction and $Q_C : C \to \mathrm{Ho}\,C$ and $Q_D :\to \mathrm{Ho}\,D$ be the universal functors from the definition of the localization (definition 1.1.9). We define

- $\mathbb{L}$, the *left derived functor* of $Q_D \circ L$ to be the right Kan extension of L along Q_C;
- $\mathbb{R}$, the *right derived functor* of $Q_C \circ R$ to be the left Kan extension of R along Q_D;

The Quillen adjunction $L : C \leftrightarrows D : R$ is said to be a *Quillen equivalence* if some of the two equivalent conditions below hold:

- $\mathbb{L}$ is an equivalence of categories;
- $\mathbb{R}$ is an equivalence of categories;

We can see why a Quillen equivalence is said to be the correct notion of equivalence between model categories: two model categories may not be equivalent, but they still can have equivalent homotopy categories, which means they describe the same homotopy theory. A Quillen equivalence doesn't care about whether two model categories are the same in a strict categorical sense, but if they are the same for homotopical purposes.

When working with the homotopy coherent nerve, we will make use of the following well known lemma:

Theorem 1.1.15 (Ken Brown's lemma)**.** *In a Quillen adjunction $L : C \leftrightarrows D : R$,*

- *L preserves weak equivalences between cofibrant objects;*
- *R preserves weak equivalences between fibrant objects;*

1.2 ∞-categories

1.2.1 Simplicially enriched categories

Simplicially enriched categories[2] are categories enriched over simplicial sets (definition 1.2.3). That means that for any simplicially enriched category $C \in \mathbf{Cat_{sSet}}$, instead of hom-sets $\mathrm{Hom}_C(x, y)$, we have simplicial sets $\mathrm{Hom}_C(x, y)_\bullet$.

When requiring further properties to the hom simplicial sets, we obtain a model for $(\infty, 1)$-categories. Categories enriched over Kan complexes (definition 1.2.7) are a model to ∞-categories as each component of the hom simplicial set describes morphisms of certain dimension. The 0th level of a hom simplicial set $\mathrm{Hom}_C(x, y)$ corresponds to the morphisms between x and y. The 1st level represents 2-dimensional arrows between the morphisms that go from x to y. That pattern is maintained as the k-th level of $\mathrm{Hom}_C(x, y)$ represents k-morphisms between the $(k-1)$-morphisms, the later being elements of the $(k-1)$-th level of the $\mathrm{Hom}_C(x, y)$.

The inverses of higher morphisms are determined by properties of Kan complexes which, as discussed later, allows one to find inverses for all higher dimensional morphisms, satisfying the requirements to be an $(\infty, 1)$-category.

Thus, the category of simplicially enriched categories, $\mathbf{Cat_{sSet}}$, has a subcategory, the one of Kan complex enriched category, of ∞-categories. By trying to apply other models for homotopy theory to the category $\mathbf{Cat_{sSet}}$, we end up with what we may call a homotopy theory of homotopy theories, since some elements of $\mathbf{Cat_{sSet}}$, the ones that are infinity categories, describe homotopy theories. This will be done by considering the *Bergner model structure* in $\mathbf{Cat_{sSet}}$.

[2]sometimes called *simplicial categories*, although we will not use the term here to avoid confusion with other things that also receive that name

However, to properly define this model structure, we need to talk about the homotopy category of a simplicially enriched category.

Definition 1.2.1 (homotopy category of a simplicially enriched category)**.** Given $C \in \mathbf{Cat_{sSet}}$, the homotopy category of C is the category $\mathrm{Ho}(C)$ which has as objects the same objects as C and as morphisms the connected components of the hom simplicial sets of C. That is, $\mathrm{Ob}(\mathrm{Ho}(C)) = \mathrm{Ob}(C)$ and $\mathrm{Hom}_{\mathrm{Ho}(C)}(X, Y)$
$= \pi_0(\mathrm{Hom}_C(X, Y))$.

Example 1.2.2 ($(\mathbf{Cat_{sSet}})_{\mathrm{Bergner}}$)**.** The *Bergner model structure* in simplicially enriched categories is defined by:

- **weak equivalences**: functions whose induced maps on the homotopy categories (definition 1.2.1) are equivalences of categories and induced maps on the homs are weak equivalences in $\mathbf{sSet}_{\mathrm{Quillen}}$ (example 1.2.12);
- **fibrations**: morphisms whose induced maps on the homs have the RLP with respect to horn (definition 1.2.6) inclusions (fibrations in $\mathbf{sSet}_{\mathrm{Quillen}}$) and induces isofibrations[3] in the homotopy categories;
- **cofibrations**: morphisms with the LLP with respect to trivial fibrations.

1.2.2 Quasi-categories

Quasi-categories are simplicial sets - presheaves on the simplicial category[4] Δ - satisfying a lifting property. Quasi-categories are particularly interesting because they model ∞-categories. We first define what a simplicial set is.

Definition 1.2.3 (simplicial set)**.** A simplicial set is a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$.

For each object $[n] \in \Delta$, $X[n] \subset \mathbf{Set}$, also denoted X_n, is thought as the collection of n-simplices of X. The images by X of the degeneracy and face maps of Δ describes how these n-simplices of X are glued together. By the Yoneda lemma, there is a correspondence between X_n and $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X)$, where $\Delta^n = \Delta[n] = \mathrm{Hom}_\Delta(-, [n])$.

Remark 1.2.4. The simplicial category Δ is generated by two classes of maps: the face and degeneracy. This implies that stating what a simplicial set $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$ does to objects and to face and degeneracy maps is enough to define the whole functor. The practical implication of this fact is that when defining a simplicial set, we usually only make explicit the n-simplices and the face and degeneracy maps inside the simplicial set, since everything else is induced from there.

[3] a functor $F : C \to D$ such that for any object $x \in C$ and any isomorphism $\phi : F(x) \to a$, there is an isomorphism $\psi : x \to y$ such that $F(\psi) = \phi$

[4] objects in the simplicial category are the natural numbers and arrows are order-preserving functions

As commented before, quasi-categories are simplicial sets with an extra property. We define *faces* of a simplex and then *horns*, which will make possible to talk about this extra property.

Definition 1.2.5 (face)**.** The i-th face of $\Delta[n]$ is the image of δ_i, the face map in $\Delta[n]$.

Intuitively, we are excluding the i-th vertex, keeping only the other ones, defining a simplex which is the face opposite to i.

Definition 1.2.6 (horn)**.** The *k-horn* of $\Delta[n]$ is the simplicial set obtained by the the union of all faces of $[n]$ but the k-th one.

A horn will be called an *inner horn* if $0 < k < n$ and an *outer horn* otherwise. Horns will be denoted by $\Lambda^k[n]$.

Definition 1.2.7 (Kan complex)**.** A Kan complex is a simplicial set $X : \Delta^{\mathrm{op}} \to$ **Set** such that every horn $\Lambda^k[n]$ in X, for $n > 1$ can be filled:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \quad , \quad \text{for } 0 \leq k \leq n.$$

Definition 1.2.8 (quasi-category)**.** A quasi-category is a simplicial set X : $\Delta^{\mathrm{op}} \to$ **Set** such that every inner horn $\Lambda^k[n]$ in X, for $n > 1$ can be filled:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \quad , \quad \text{for } 0 < k < n.$$

Notice the subtle difference between general quasi-categories and Kan complexes: the horn filling condition for quasi-categories is only required for inner horns, while for Kan complexes it is required for every horn. This situates the collection of Kan complexes inside the collection of quasi-categories, since filling all horns implies filling the inner ones.

Like simplicially enriched categories, quasi-categories are also models for ∞-categories. The 0th level of a quasi-category X represents objects and the k-th one represents the k-morphisms. The horn filling condition is used to mimic ordinary categories in all levels, like composition, associativity and identities. In the case of Kan complexes, it also provides inverses for all morphisms, including 1-morphisms. A nice discussion of how the horn filling conditions describe some of these things can be found in [Rie09].

Remark 1.2.9. The term ∞-category is largely used as a synonym of quasi-category. That is because quasi-categories are one of the most simple and easy models of infinity categories to work with and there are certain notions of equivalence between other models of ∞-categories and quasi-categories.

We use the case study of a 2-simplex to picture how the horn filling condition contains the data about composition of morphisms and how being a Kan complex gives you also a way to invert morphisms.

Example 1.2.10. A typical inner horn $\Lambda^1[2]$ inside X looks like

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] & & c \end{tikzcd}.$$

To fill it is to find a 2-simplex in X that contains the horn:

$$\begin{tikzcd} & b \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow"] & \\ a \arrow[ur, "g"] \arrow[rr, "h"'] & {} & c \end{tikzcd}.$$

So we found a 2-morphism $f \circ g \Rightarrow h$ or, in intuitive terms, we found a map which is (in a weak sense) the composition of f and g.

As for outer horns, it is not always true that given a diagram

$$\begin{tikzcd} & b & \\ a \arrow[ur, "f"] \arrow[rr, "g"] & & c \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & b \arrow[dr, "f"] & \\ a \arrow[rr, "g"] & & c \end{tikzcd},$$

we can find a map h to fill the gap between b and c. More than that, there would also need to be a 2-morphism between g and the composition of f with h. However, this map does exist if X is a Kan complex and we interpret this as inverting the morphism g, converting, morally, the horn into an inner horn and allowing the filling.

Given two simplicial sets X, Y, we can form another simplicial set $\mathrm{Fun}(X, Y)$, whose components are given by

$$\mathrm{Fun}(X, Y)_k = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n \times X, Y)$$

and the face and degeneracy maps are

$$\sigma_i : (\Delta^n \times X \xrightarrow{f} Y) \mapsto (\Delta^{n+1} \times X \xrightarrow{\sigma_i \times \mathrm{id}} \Delta^n \times X \xrightarrow{f} Y)$$
$$\delta_i : (\Delta^n \times X \xrightarrow{f} Y) \mapsto (\Delta^{n-1} \times X \xrightarrow{\delta_i \times \mathrm{id}} \Delta^n \times X \xrightarrow{f} Y).$$

Remark 1.2.11. If Y is an ∞-category, then the simplicial set $\mathrm{Fun}(X, Y)$ is an ∞-category.

As for simplicially enriched categories, one can form the category of simplicial sets and try to attach homotopical structure in there. This introduces the homotopy theory of simplicial sets, which will be given in terms of a model structure in **sSet**, the *Quillen model structure*.

Example 1.2.12 ($\mathbf{sSet}_{\text{Quillen}}$)**.** The *Quillen model structure in simplicial sets* is given by:

- **weak equivalences**: weak homotopy equivalences of simplicial sets. That is, maps whose geometric realization is a weak equivalence in $\mathbf{Top}_{\text{Quillen}}$ (example 1.1.7);
- **fibrations**: maps with the RLP with respect to all horn inclusions (definition 1.2.6);
- **cofibrations**: component-wise injective maps.

We can also define the homotopy category of a quasi-category, which is important for the *Joyal model structure* in simplicial sets.

Definition 1.2.13 (homotopy category of a quasi-category)**.** The *homotopy category* of a quasi-category X is the category hX whose

- objects are the elements of X_0;
- morphisms are the equivalence classes of X_1 under the relation

$$f \sim g \text{ if and only if there is a 2-simplex in } X \text{ of the form}$$

$$\begin{tikzcd} & y \arrow[dr, equal] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, equal] \arrow[rr, "g"'] & & y \end{tikzcd} .$$

The Quillen model structure places an homotopy theory on simplicial sets, but there is another model structure in **sSet** that model a homotopy theory of quasi-categories, the *Joyal model structure*.

Definition 1.2.14 ($\mathbf{sSet}_{\text{Joyal}}$)**.** There is a model structure in **sSet** where:

- **weak equivalences** are maps $f : X \to Y$ inducing equivalences between the categories

$$h\,\mathrm{Fun}(Y, C) \to h\,\mathrm{Fun}(X, C)$$

for any quasi-category C;
- **cofibrations** are the monomorphisms;
- **fibrations** are the morphism having the RLP wih respect to trivial cofibrations.

A very important difference between the Quillen and the Joyal model structure is that fibrant objects in the Quillen structure are only Kan complexes, while fibrant objects in Joyal structure are all quasi-categories.

2 The underlying construction

2.1 Hammock localization

The hammock localization is a way to obtain simplicially enriched categories from model categories (or relative categories). It is presented as a better way to store homotopical data by upgrading hom-sets to simplicial sets. Dwyer and Kan first described the The *simplicial localization* in [DK80c] as an assignment of simplicial objects in **Cat** to categories. Later, in [DK80a], they defined the *hammock localization*, which returns simplicially enriched categories. The two localizations share some important properties and are, for our purposes, the same, but the hammock one is largely more used since it has a clearer definition and, for this reason, we focus on the hammock localization here.

Definition 2.1.1. Given a category C and a subcollection W of morphisms of C, we define the *hammock localization* of C by W as the simplicially enriched category $L^H(C,W)$, that has as objects the same objects as C and the k-th component of a home simplicial set $L^H(C,W)(X,Y)$ will be the collection of hammocks

$$\begin{tikzcd}
& C_{0,1} \arrow[r, dash] \arrow[d] & C_{0,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{0,n} \arrow[d] \arrow[ddr, dash] & \\
& C_{1,1} \arrow[r, dash] \arrow[d] & C_{1,1} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{1,n} \arrow[d] \arrow[dr, dash] & \\
X \arrow[uur, dash] \arrow[ur, dash] \arrow[dr, dash] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & Y \arrow[dl, dash] \\
& C_{k,1} \arrow[r, dash] & C_{k,2} \arrow[r, dash] & \cdots \arrow[r, dash] & C_{k,n} &
\end{tikzcd}$$

of width k and any length n, satisfying a few properties:

- all vertical maps are in W;
- maps of the same column have the same direction;
- if a map goes from the right to left, then it is in W;
- adjacent columns have different directions;
- every column has a non-identity map.

The face map δ_i deletes the i-th row and the degeneracy map σ_i repeats the i-th row. One may step into hammocks that do not satisfy the two last properties listed. However, these problems can be solved by deleting columns with only identity maps or composing columns that have the same direction.

Remark 2.1.2. The hammock localization is functorial (See 3.4 in [DK80a]).

Theorem 2.1.3 ([DK80a] Prop. 3.1)**.**

$$Ho(L^H(C,W)) \simeq C[W^{-1}].$$

Results from [DK80b] (4.4 and 5.4) led to the following conclusion, pointed out in [Maz15].

Theorem 2.1.4. *If $L : C \rightleftarrows D : R$ is a Quillen equivalence, then $L^H C$ and $L^H D$ are weak equivalent in the Bergner model structure.*

2.2 The $\mathbb{R}_B$ functor

As we have seen, the hammock localization is a functor from **RelCat** to $\mathbf{Cat_{sSet}}$, so that after applying it to a model category, one ends with an object in $\mathbf{Cat_{sSet}}$. By this time, we have a rich storage of homotopical data in simplicially enriched categories, but they still are not ∞-categories. We are going to replace simplicially enriched categories by fibrant objects in $(\mathbf{Cat_{sSet}})_{\text{Bergner}}$ in order to obtain, in fact, infinity categories. Let us see that the fibrant objects in $(\mathbf{Cat_{sSet}})_{\text{Bergner}}$ are indeed ∞-categories.

Theorem 2.2.1. *$C \in (\mathbf{Cat_{sSet}})_{\text{Bergner}}$ is a fibrant object if and only if C is enriched over Kan complexes.*

Proof. Suppose C is a fibrant object, we want to prove that it is enriched over Kan complexes. C being a fibrant object, by definition, implies that the map $f : C \to \Delta[0] = 1$ to the final object is a fibration, meaning it must induce fibrations (in the Quillen model structure) between the simplicial hom sets of C and 1. Remember that fibrations in the Quillen model structure are the maps that have the RLP with respect to horn inclusions.

So let $g : \Lambda^k[n] \to \mathrm{Hom}_C(x,y)$ be any map between simplicial sets and $\overline{f} : \mathrm{Hom}_C(x,y) \to \mathrm{Hom}_1(fx,fy) = \Delta[0]$ the induced map of f on the hom simplicial sets. Since for each n the n-th set $\Delta[0]_n$ is composed only by the constant map

$$\begin{aligned} \mathrm{const}_0^n : [n] &\to [0] \\ k &\mapsto 0, \end{aligned}$$

the function $\overline{f}$ is then forced to be constant, mapping every n-simplex of $\mathrm{Hom}_C(x,y)$ into const_0^n. Take the following square:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "g"] \arrow[d, hook] & \mathrm{Hom}_C(x,y) \arrow[d, "\overline{f}"] \\ \Delta[n] \arrow[r, "\mathrm{tr}"] & \Delta[0] \end{tikzcd},$$

where tr is the trivial map (there is no other one). To see that it commutes, just observe that whatever g is, $\overline{f} \circ g$ will always return the constant maps independently of the input, because of $\overline{f}$. The same occurs for the other path, since the trivial map collapses everything into the constant map. We conclude that the square is commutative for any g.

Since the square commutes and $\overline{f}$ is a fibration in $\mathbf{sSet}_{\text{Quillen}}$, there is a lift from $\Delta[n]$ to $\text{Hom}_C(x, y)$:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "g"] \arrow[d, hook] & \text{Hom}_C(x,y) \arrow[d, "\overline{f}"] \\ \Delta^n \arrow[r, "\text{tr}"] \arrow[ur, dashed] & \Delta^0 \end{tikzcd} .$$

The upper triangle is precisely the horn filling condition for the simplicial set $\text{Hom}_C(x, y)$.

As for the other direction, the unique morphism from a Kan complex to $\Delta[0]$ must return the constant maps and the only map from $\Delta[n]$ to $\Delta[0]$ is tr, the trivial one. Hence, to check the RLP for the unique map $f : \text{Hom}_C(x, y) \to \Delta[0]$, one just needs to check the existence of lifts for the squares

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & \text{Hom}_C(x,y) \arrow[d, "f"] \\ \Delta[n] \arrow[r, "\text{tr}"] & \Delta[0] \end{tikzcd} ,$$

(that commute because of f and tr) which is given by the hypothesis that $\text{Hom}_C(x, y)$ is a Kan complex. The only left thing to verify is that f is an isofibration, which is immediate from the fact that f is constant in each component. □

There may be many ways to fibrantly replace the objects in $(\mathbf{Cat}_{\mathbf{sSet}})_{\text{Bergner}}$. Indeed, this category comes equipped with a natural fibrant replacement functor. Instead of using this natural functor, we are going to do it by the means of the Ex^∞ functor. We are not going to discuss it in details here, but a more complete description of this functor is given in [Gui]. In summary, the Ex^∞ functor works by subdividing our simplicial sets in a limiting process in order to create fillings for some horns. The Ex^∞ functor is a fibrant replacement in $\mathbf{sSet}_{\text{Quillen}}$, where the fibrant objects are Kan complexes.

Given a simplicially enriched category $C \in \mathbf{Cat}_{\mathbf{sSet}}$, for any $x, y \in C$, $\text{Hom}_C(x, y)$ is a simplicial set and we can apply the Ex^∞ functor to it. We define $\mathbb{R}_B$ to be the functor that applies Ex^∞ to the hom simplicial sets of simplicially enriched categories. Hence, $\mathbb{R}_B$ gives us Kan complex-enriched categories.

One may ask why we don't use the natural fibrant replacement functor instead of $\mathbb{R}_B$. It happens tha $\mathbb{R}_B$ has some advantages: for example, $\mathbb{R}_B(C)$ has the same objects as C and it doesn't changes the 0-simplices of the hom simplicial sets. This way, we will keep carrying the original objects with us all the way, since the hammock localization itself also doesn't change the objects.

Remark 2.2.2. Notice that since categories enriched over Kan complexes are already ∞-categories, we could stop here, say that the underlying infinity category of C is just $\mathbb{R}_B(L^H(C))$ and call it a day. However, quasi-categories are much easier models for ∞-categories to work with, and we are encouraged to take one more step to arrive at them.

2.3 Homotopy coherent nerve

Finally, we turn our eyes to the homotopy coherent nerve, a functor

$$N^{\mathrm{hc}} : \mathbf{Cat}_{\mathbf{sSet}} \to \mathbf{sSet}$$

which will complete our toolkit to find the underlying category. Working with quasi-categories is, in general, better than simplicially enriched ones, since they require less data to be remembered. While quasi-categories are just simplicial sets, simplicially enriched categories have a simplicial set for each hom. That is why we take the homotopy coherent nerve, to get quasi-categories.

One may ask if by reducing the stored data we are not losing precious homotopical information. The answer is actually *no*. As we will see, using simplicially enriched categories or quasi-categories is a matter of choice, since they are equivalent as models of ∞-categories.

We start with a simplicially enriched category C and we want a simplicial set that has as 0-simplices (points) the objects of C, 1-simplices shall be the 0-cells of the hom simplicial sets in C, 2-simplices of our quasi-category should be the 1-cells of the hom simplicial sets of C and so on. Formally, to do that we follow the idea of considering the identity that characterizes the ordinary nerve,

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, N(C)) = \mathrm{Hom}_{\mathbf{Cat}}([n], C),$$

and replace $[n]$ by a category $\mathfrak{C}[\Delta^n]$.

Before stating the definition of $\mathfrak{C}[\Delta^n]$, we fix the following notation.

Given a natural number n and two other numbers $i, j < n$, $P_{i,j}$ is the poset, ordered by inclusion,

$$\{I \subseteq n : i, j \in I \text{ and } k \in I \Rightarrow i \leq k \leq j\}.$$

Definition 2.3.1. Given a natural number n, we define $\mathfrak{C}[\Delta^n]$ to be the category whose

- objects are the naturals $\{0, 1, ..., n\}$ not greater than n;
- maps from i to j are

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i, j) = \begin{cases} \emptyset & \text{if } j < i \\ N(P_{i,j}) & \text{if } i \leq j; \end{cases}$$

- composition of maps is induced by union in the $P_{i,j}$ sets

$$\begin{aligned} P_{i_0,i_1} \times P_{i_1,i_2} \times ... \times P_{i_{m-1},i_m} &\to P_{i_0,i_m} \\ (I_1, I_2, ..., I_m) &\mapsto I_1 \cup I_2 \cup ... \cup I_m. \end{aligned}$$

N denotes the ordinary *nerve functor*. Notice that $P_{i,j}$ is a poset and thus a category, so that it makes sense to take its nerve.

Definition 2.3.2 (homotopy coherent nerve)**.** Given a category C, the *homotopy coherent nerve*, also called the *simplicial nerve*, of C is the simplicial set $N^{\mathrm{hc}}(C)$ whose components are given by

$$N^{\mathrm{hc}}_k(C) = \mathrm{Hom}_{\mathbf{Cat}_{\mathbf{sSet}}}(\mathfrak{C}[\Delta^k], C)$$

Theorem 2.3.3. *The nerve is characterized by the equation*

$$\mathrm{Hom}_{\boldsymbol{sSet}}(\Delta^n, N^{\mathrm{hc}}(C)) \cong \mathrm{Hom}_{\boldsymbol{Cat}_{\boldsymbol{sSet}}}(\mathfrak{C}[\Delta^n], C). \tag{1}$$

This is a direct consequence of our definition and the Yoneda lemma.

Equation 1 looks very much like an adjunction between N^{hc} and $\mathfrak{C}[\Delta^{(-)}]$. That is because $\mathfrak{C}[\Delta^{(-)}]$ extends to a functor $\mathfrak{C} : \mathbf{sSet} \to \mathbf{Cat}_{\mathbf{sSet}}$ and that functor is a left adjoint of N^{hc}. This adjunction is actually way stronger, as the following theorem indicates.

Theorem 2.3.4. *The adjunction $\mathfrak{C} : (\boldsymbol{Cat}_{\boldsymbol{sSet}})_{\mathrm{Bergner}} \rightleftarrows \boldsymbol{sSet}_{\mathrm{Joyal}} : N^{\mathrm{hc}}$ is a Quillen equivalence.*

Both the Bergner and Joyal model structures are models for a homotopy theory of homotopy theories. This Quillen equivalence actually means that quasi-categories and simplicially enriched categories are somewhat equivalent as models for ∞-categories.

In addition, quasi-categories are the fibrant objects in $\mathbf{sSet}_{\mathrm{Joyal}}$, so that the homotopy coherent nerve has the property of mapping fibrant objects into fibrant objects and the fibrant objects of both categories are infinity categories. This result is summarized in the next theorem.

Theorem 2.3.5. *If $C \in \boldsymbol{Cat}_{\boldsymbol{sSet}}$ is a category enriched over Kan complexes, then $N^{\mathrm{hc}}(C)$ is a quasi-category.*

which follows from a combination of both results ([Cis19] Prop 3.7.2) and ([Lur09] 2.2.0.1).

Theorem 2.3.6. *If $C \in \boldsymbol{Cat}_{\boldsymbol{sSet}}$ is enriched over Kan complexes, then*

$$\mathrm{Ho}\, C \cong hN^{\mathrm{hc}}C.$$

Observe that the left hand side is the homotopy category of a simplicially enriched category, while the right hand side is the homotopy category of a quasi-category. Both are ordinary categories, but are built in slightly different manners.

3 The underlying category

We have now all the tools in order to define the underlying category of a category C. As explained in the introduction, one just apply the three defined functors in sequence and ends up with an ∞-category.

We shall explain how properties of these functors guarantee that the final output is indeed an ∞-category and how this infinity category describes the same homotopy theory than the starting model category. First, we define the ud functor.

Definition 3.0.1. The ud functor is the composition

$$\mathrm{ud} = N^{\mathrm{hc}} \circ \mathbb{R}_B \circ L^H.$$

The first functor in our composition is the hammock localization, which contains the important homotopical data of our category in form of simplicial hom sets. The homotopical data of the model category is carried with the functor.

Indeed, theorem 2.1.3 shows that all homotopical data contained in the original category can be recovered from the hammock localization.

After the hammock localization, we already get simplicially enriched categories. Then one could be tempted to apply the homotopy coherent nerve straight away. However, it would not necessarily return a quasi-category, but just general simplicial sets. The $\mathbb{R}_B$ functor will be a stepping stone which will lead us to infinity categories.

$\mathbb{R}_B$ return categories enriched over Kan complexes, which already are ∞-categories, so that this functor moves us closer to our destination. In addition, by theorem 2.3.5, the homotopy coherent nerve maps Kan complex enriched categories into quasi-categories. This way, $N^{\mathrm{hc}} \circ \mathbb{R}_B$ will always be a quasi-category.

We conclude that ud turns model categories into ∞-categories. We just need, now, to verify a last claim:

$$C \text{ and } \mathrm{ud}\, C \text{ describes the same homotopy theory.}$$

The above statement can be formalized in the following theorem.

Theorem 3.0.2. *For any model category C, $\mathrm{Ho}C \simeq h\,\mathrm{ud}(C)$.*

Proof. To prove this theorem, we show that every functor used in the definition of ud preserves the homotopy category. First, theorem 2.1.3 ensures that when hammock localizing C by its weak equivalences W, the homotopy category of $L^H(C, W)$ is equivalent to $C[W^{-1}]$, which is just $\mathrm{Ho}C$ since W is the set of weak equivalences of C.

Second, $\mathbb{R}_B$ preserves the homotopy category: remember that when replacing a category $X \in \mathbf{Cat_{sSet}}$ by a fibrant object, we are finding a weak equivalent substitute for it. In the Bergner model structure, weak equivalences are maps that induces equivalences on the homotopy categories. Hence, by definition, a category and its image by $\mathbb{R}_B$ will always have equivalent homotopy categories. Finally, theorem 2.3.6 ends our work straightforwardly. $\square$

An interesting remark is that all the functors in the construction of the underlying functor ud fix the objects. The hammock localization $L^H(C)$ has the same objects as C itself. $\mathbb{R}_B$ operates only on the higher morphisms and the nerve $N^{\mathrm{hc}}(D)$ has the objects of D as its 0th components. $\mathrm{ud}(C)$ is, then, very faithful to C.

We finish our discussion with the following theorem, which translates equivalences in the model categorical world to equivalences in the quasi-categorical one.

Theorem 3.0.3. *Every Quillen equivalence $L : C \rightleftarrows D : R$ rises a weak equivalence (in Joyal model structure) between $\mathrm{ud}\, C \to \mathrm{ud}\, D$.*

Proof. From theorem 2.1.4, the Quillen equivalence induces weak equivalences (in the Bergner model structure) between $C' = L^H C$ and $D' = L^H D$. $\mathbb{R}_B$ then replace C' and D' by fibrant objects which are weak equivalent to them, offering a sequence of weak equivalences $\mathbb{R}_B(C') \cong C' \cong D' \cong \mathbb{R}_B(D')$, so that $\mathbb{R}_B$ maintain the weak equivalences.

Now, since $\mathbb{R}_B(C')$ and $\mathbb{R}_B(D')$ are weak equivalent fibrant objects, theorem 1.1.15 together with the fact that the nerve is the right adjoint of a Quillen equivalence implies that $\mathbb{R}_B(C')$ and $\mathbb{R}_B(D')$ will land in weak equivalent objects (in the Joyal structure). $\square$

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