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GRAVITATIONAL WAVES IN ALGEBRAICALLY SPECIAL SPACETIMES

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Abstract

Black holes are one of the most astonishing predictions of General Relativity. In the advent of Gravitational Wave astronomy, it has become possible to contrast observations from black hole mergers with our theoretical predictions. A crucial ingredient in constructing such predictions is the theory of gravitational fluctuations of Kerr's black hole developed in the 70's. The aim of this work is to provide an exhaustive introduction to such theory. We start by introducing the algebraic classification of spacetimes based on their Weyl tensor. Then, we provide a complete derivation of Teukolsky's equations and specialise them for Kerr. Finally, we conclude by discussing how these can be used to compute the quasinormal modes and tidal Love numbers of Kerr's black hole, which are some of the main observables characterising the waveforms of a black hole merger.

Contents

1 INTRODUCTION

Black holes (BHs) constitute a robust and astonishing prediction of Generel Relativity (GR), and play a central role in theoretical physics. At a fundamental level, they are theoretical laboratories in which one can test deep aspects about the strong regime of the gravitational interaction. From a more phenomenological point of view, the existence and structure of BHs can be probed nowadays. Indeed, Gravitational Wave (GW) astronomy has recently opened a new window to observe the universe, and the LIGO and Virgo collaboration have reported several events of BH and NS mergers [1], [2], [3].

It is crucial to develop robust theoretical predictions about BHs in order to meaningfully test our theories using GW signals. For example, a main goal of gravitational wave astronomy consists in testing the no-hair theorem, a prediction of GR which establishes that any vacuum, stationary black hole is uniquely determined by its mass and angular momentum, and its spacetime geometry is described by Kerr's metric [4].

The natural framework to study GWs propagating on BH spacetimes is perturbation theory. However, studying gravitational fluctuations on Kerr's spacetime (the most general BH, according to GR) is highly non-trivial. It was not until the 70's that scientists understood how to approach the problem [5], several years after the beginning of BH perturbation theory. Nowadays, those results play a central role in the study of GW emission by BHs.

The purpose of this work is to provide a comprehensive yet detailed introduction to the theory of gravitational perturbations of Kerr's BH. The framework is based on some rather sophisticated mathematics that are natural when studying the algebraic description of solutions of Einstein's theory. We begin by reviewing the Newman-Penrose formalism and some basic geometric notions which are necessary to introduce the Petrov's classification of the Weyl tensor. Then, we provide a detailed derivation of the master equations (the so-called Teukolsky equations) that govern the propagation of gravitational waves on any vacuum, Petrov type D spacetime. The Kerr BH belongs to that family, and focusing on that case we reduce the Teukolsky equations to a set of decoupled ordinary differential equations (ODE's).

The last section is devoted to the description of two of the most important applications of Teukolsky's ODE's. The first one concerns tidal Love numbers (LNs). These are quantities that measure the gravitational deformability of a system, and depend entirely on its internal structure. It is expected that tidal Love numbers leave an imprint in the (late stages) of the inspiral phase of a binary merger. The second application consists in studying the free oscilations emitted by a black hole spacetime, the so-called quasi-normal modes (QNMs). These govern the ring-down phase of a merger, and depend entirely on the internal structure of the final object.

In order to be in agreement with the standard literature in perturbation theory, we will be using the signature $(+,-,-,-)$. All the presented calculations have been carried out using *Mathematica notebooks*.

2 THE TEUKOLSKY EQUATION

In spherically symmetric black holes, one can obtain decoupled master equations by simply projecting in spherical harmonics [6], [7], [8]. In the case of rotating black holes such an approach is not possible. It took several years to physicists to unlock this, and the solution came from Teukolsky [5]. He could do so by following a very different approach, based on an elaborate description of four-dimensional spacetimes which turns out to be crucial in order to obtain the equations. The purpose of the chapter is to introduce such theory and provide a pedagogic derivation of the master equations. In order to that, we are going to present the tetrad formalism, and how adapting a tetrad to the light-cone structure of the spacetime sets a natural framework to study gravitational perturbations.

2.1 TETRAD FORMALISM

How do we determine the effects of gravitation in a physical system? The usual way to address this problem is by making use of the so-called *covariance principle*. The idea that lies behind this is that one only needs to find the special relativistic equations that are valid in the absence of gravity, then replace $\eta_{\mu\nu}$ by $g_{\mu\nu}$ and replace all the derivatives with covariant derivatives.

However, there are several issues with this approach. First, it is not enough to couple gravity to objects that are more general than tensors, such as spinors. And second, which is the one that concerns us, if we are interested in measuring e.g. the amplitude of a GW, the value of quantities referred to an arbitrary coordinate system may be physically meaningless, and we should introduce an inertial reference frame adapted to an observer, built in terms of covariant quantities related to the frame. In this context, the *tetrad formalism* provides a natural way of dealing with these problems.

It takes advantage of the *Einstein's equivalence principle*, which states that at every point of an arbitrary spacetime, it is always possible to choose a reference frame *locally* where the laws of physics take the same form as the ones in an inertial reference frame in the absence of gravity. In other words, there always exist the *Riemann normal coordinates* adapted to a point x, where at x, $g_{\mu\nu} = \eta_{\mu\nu}$ and the Christoffel symbols vanish.

A local inertial frame is called a **tetrad** and it defines a basis at each point in spacetime by a set of vector fields e_a. When we build an orthonormal frame (although other no orthonormal frames could be constructed) we create an inertial frame which only exists locally, at the neighbourhood of a point, not globally. Furthermore, in general, it is *not* associated to a coordinate system (that is, $e_a \neq \partial_a$ or $e^a \neq dx^a$ where x^a is a coordinate system). If a frame exists globally, then the spacetime is called *parallelisable* and the orthonormal frame can be associated to a coordinate system, but this only happens in Mikowski's spacetime.

If the vector fields have to define an orhtonormal frame in a neighbourhood of a point, then we want that $g(e_a, e_b) = \eta_{ab}$ where the letters a, b only enumerates our basis vectors. Therefore:

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu) = \eta_{ab} e^a_\mu(x) e^b_\nu(x)$$

where x is a point of the spacetime, and an expansion of the coordinate basis in terms of the new basis $e_a = e^\mu_a \partial_\mu$ (using the notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$, with x^μ coordinates of the background) has been done. Note that e_a could be interpreted as the *directional derivates* along the axes of the frame, but now with the difference that they are not necessarily associated to a coordinate system in general, so $[e_a, e_b] \neq 0$.

The $e^a{}_\mu(x)$ is the so-called **vielbein** (from the German, four legs) or frame fields. Note that it depends on the point, and they are basically a set of 4x4 matrices with 16 components. Introducing a tetrad increases the number of (of-shell) independent components (16 degrees of freedom from the vielbein compared with 10 degrees of freedom of the metric), but it also enlarges the gauge freedom: besides coordinate transformations, there are also the *local Lorentz transformations* $\Lambda^a{}_b(x)$. They transform a local inertial frame e^a into another physically equivalent local inertial frame e'^a in the usual way from special relativity: $e'^a = \Lambda^a{}_b e^b$ where $\Lambda^a{}_b(x)$ satisfies: $g_{\mu\nu} = \eta_{ab} e'^a_\mu e'^b_\nu = \eta_{ab} \Lambda^a{}_c e^c \Lambda^b{}_d e^d = \eta_{ab} e^a_\mu e^b_\nu$. Thus, in tetrad formalism the gauge freedom is enhanced from just diffeomorphisms (general coordinate transformations) to diffeomorphisms and local Lorentz transformations (with 6 degrees of freedom), resulting in a equivalent number of degrees of freedom to those of the metric.

So we have a system of four (dimension of the spacetime) independent vector fields: $e_a = e^\mu_a \partial_\mu$ and correspondingly their dual 1-forms: $e^a = e^a_\mu dx^\mu$, as our fundamental pieces instead of the metric field. Latin indices label vielbeins while greek characters are used as spacetime indices. This is because the vielbein ones are not tensor indices, they transform well under local Lorentz transformations of frames but do not transform under coordinate transformations. This two kinds of indices imply that to lower and raise the coordinate indices one has to use the metric $g_{\mu\nu}$, and for the tetrad indices use the Minkowski metric η_{ab}. The relation between spacetime coordinates and the Vielbein is given by the definitios of the vector fields and 1-forms, allowing us to switch between coordinate and tetrad frames.

As usual, geometric quantities can be expressed in components relative to a given frame. Given a tetrad e^a, the *connection 1-forms* of the Levi-Civita connection ∇ relative to e^a can be defined as a collection of 1-forms $\omega^a{}_b$ acting on vector fields X as $\omega^a{}_b(X) = e^a\left(\nabla_X e_b\right)$. On the other hand, the connection components relative to e^a are defined by $\nabla_{e_a} e_b = \Gamma^c_{ba} e_c$. Hence, one has:

$$\omega^a{}_b = \omega^a{}_b(e_c) e^c = e^a\left(\nabla_{e_c} e_b\right) e^c = \Gamma^a_{bc} e^c$$

where $\omega^a{}_b \in \Lambda^1(U)$ (they are antisymmetric, which implies that $\omega^a{}_b = -\omega_b{}^a$) are the connection 1-forms, and the components $\Gamma^a_{bc} = e^a\left(\nabla_{e_c} e_b\right)$ are the *connection components* (or, in the Newman-Penrose formalism, *spin coefficients*), which will be relevant in our equations as we will see. The connection was a structure on the tangent bundle that stitched together neighbour tangent spaces in a partiuclar way, so now this coefficients will account for how the basis changes as we move to different points in the manifold.

Only with the connection, one can define the torsion (which GR assumes to vanish) as the commutator of the covariant derivative acting on a scalar. In the general case of a non-constant tetrad metric, and non-vanishing torsion, the tetrad connections are given in terms of the derivatives of them tetrad metric, the torsion and the vielbein covariant derivatives [9]. The Riemann curvature tensor is defined in the usual way by the commutator of the covariant derivative acting on a contravariant 4-vector. Recall that torsion (respectively, curvature) measures how covariant derivatives fail to commute when acting on functions (respectively, tensors).

The same way we have defined connection 1-forms, one can define *torsion forms* $\Theta^a \in \Lambda^2(U)$ and *curvature forms* $\Omega^a{}_b \in \Lambda^2(U)$:

$$\begin{aligned} T(X,Y) &= \Theta^a(X,Y)e_a \\ R(X,Y)e_b &= \Omega^a{}_b(X,Y)e_a \end{aligned}$$

where X and Y are 2 contravariant vector fields and T and R are the torsion and curvature respectively. These pieces are related by **Cartan's structure equations** [9]:

$$\begin{aligned} 0 &= de^a + \omega^a{}_b \wedge e^b \\ \Omega^a{}_b &= d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \\ \Omega^a{}_b &= \frac{1}{2!} R^a{}_{bcd} e^c \wedge e^d \end{aligned} \tag{1}$$

The Eqs.(1) are fundamental in the tetrad formalism since they relate the connection 1-forms, the curvature 2-forms, our local coordinate basis and the Riemann tensor. The first equation is only true if the torsion vanishes, which we will assume, and is called the first structure equation. The second is called the second structure equation and the third equation relates the curvature forms with the Riemann curvature tensor projected into the tetrad.

One can see now an evident advantage of the tetrad formalism: instead of computing e.g. 40 Christoffel symbols and then do the partial derivatives to obtain the Riemann tensor, we only have to compute 6 curvature forms to get the same result. The connection 1-forms do not need to be computed since they can be read off from Cartan's first equation in (1). On top of that, if our spacetime has symmetries, a lot of the components will vanish, and one will only have to calculate the right curvature forms (by doing exterior derivatives and taking wedge products $\wedge$ and differentials, which are always easier to compute than calculating the inverse metric, the derivatives...), avoiding computing several components of the Riemann that will turn out to be zero, or be related by symmetries with the other components.

To sum up, it is remarkable to notice that, as a formalism, it is an equivalent way to define the dynamics without changing the final results. The usual metric formulation is enough to completely map the physics of GR. However, as we will see later, some calculations are more involved in one formlism, whereas in other can be hugely simplified.

2.2 THE NEWMAN-PENROSE FORMALISM

When we use the tetrad formalism, we think about the frame of a local observer, where our frame fields are chosen to be three space-like vectors defining the coordinate axes of our local frame, and one time-like vector being tangent to the wordline of the observer (the "clock" of our laboratory frame). But, if we want to study massless perturbations of BHs or simply the behavior of light on the spacetime it is convenient to work in frames that are adapted to lightlike trajectories in the spacetime. In this way the **Newman-Penrose** (NP henceforth) formalism appears.

In the NP formalism we have a null tetrad formed by 4 null vector fields: ℓ, n, m and $\bar{m}$, where m and $\bar{m}$ are complex conjugates of each other, and the others real. They must satisfy the orthogonality conditions:

$$\ell \cdot m = \ell \cdot \bar{m} = n \cdot m = n \cdot \bar{m} = 0$$

the null conditions:

$$\ell \cdot \ell = n \cdot n = m \cdot m = \bar{m} \cdot \bar{m} = 0,$$

and the normalization conditions:

$$\ell \cdot n = 1 \text{ and } m \cdot \bar{m} = -1.$$

Now, note that the new "tetrad metric" (products of the null vectors) is not exactly Minkowski due to this normalization conditions.

Given an orthonormal frame $e_a = e_1, e_2, e_3, e_4$ a NP frame can be constructed as:

$$\ell = \frac{1}{\sqrt{2}}(e_1 + e_2), \qquad n = \frac{1}{\sqrt{2}}(e_1 - e_2), \qquad m = \frac{1}{\sqrt{2}}(e_3 + ie_4). \tag{2}$$

However, we will generally rotate this tetrad in a more convenient way as we will see (aligning ℓ and n with the tangent vectors of special null geodesics).

We are going to project all the geometric quantities as we have done with the orthonormal basis, but we are going to name each quantity with a special symbol, decomposing our tensor objects into scalars. First we give special names to $\ell, n, m, \bar{m}$ when acting as directional derivatives on functions (that is, along each null vector e.g. $l^\mu \partial/\partial x^\mu$):

$$\begin{aligned} e_1 = e^2 = D = l^\mu \partial_\mu \quad e_2 = e^1 = \Delta = n^\mu \partial_\mu \\ e_3 = -e^4 = \delta = m^\mu \partial_\mu \quad e_4 = -e^3 = \delta^* = m^\mu \partial_\mu \end{aligned}$$

where e_1, e_2, e_3, e_4 represents, in order, $\ell, n, m, \bar{m}$.

The spin coefficients (the connection components), defined in the previous section 2.1 are now expressed as:

$$\gamma_{abc} = e_a^\mu e_{b\mu;\nu} e_c^\nu$$

to be consistent with the usual notation [9] (the connection is represented now with ;). They are are antisymmetric in the first pair of indices and we also designate them special symbols:

$$\left.\begin{array}{lll} \kappa = \gamma_{311}; & \rho = \gamma_{314}; & \varepsilon = \frac{1}{2}(\gamma_{211} + \gamma_{341}); \\ \sigma = \gamma_{313}; & \mu = \gamma_{243}; & \gamma = \frac{1}{2}(\gamma_{212} + \gamma_{342}); \\ \lambda = \gamma_{244}; & \tau = \gamma_{312}; & \alpha = \frac{1}{2}(\gamma_{214} + \gamma_{344}); \\ \nu = \gamma_{242}; & \pi = \gamma_{241}; & \beta = \frac{1}{2}(\gamma_{213} + \gamma_{343}); \end{array}\right\}$$

An important fact of the spin coefficients is that when ℓ and n are taken as tangent to null geodesics, one can attribute physical meanings to some of these quantities, that is, looking at those coefficients one can obtain geometrical properties of the null congruences in that space time [9]. It can be shown that when one considers the change at first order of one of our basis vectors $\delta e_a(c)$ when we displace them along a direction c, they can be express as:

$$\delta e_a(c) = -\gamma_{abc} e^b.$$

A relevant case is the change in ℓ per unit displacement along ℓ:

$$\delta \ell(1) = -\gamma_{1b1} e^b = (\varepsilon + \varepsilon^*)\,\ell - \kappa \bar{m} - \kappa^* m. \tag{3}$$

If the right hand side vanishes, then ℓ is parallely transported along itself, that is, it generates a geodesic congruence. This equation could be seen as the geodesic deviation equation for ℓ. Therefore κ measures how ℓ fails to be parallely transported along itself and it can be interpreted as refraction (bending) of the light rays. If $\kappa = 0$, ℓ is geodesic and if, additionally, $\varepsilon = 0$ then it is affinely parametrised. The shear of the congruence generated by ℓ is measured by the spin coefficient σ. Therefore if we want ℓ to be a generator of null shear-free and affinely parametrised geodesics, then $\sigma = \kappa = \varepsilon = 0$. For more interpretations of the other spin coefficients, check the optical scalars section of [9]. Apart from that, when one uses a null tetrad adapted to a spacetime, it will give us information not only about the trajectories of the light rays but also about the properties of the background itself [10].

Instead of using the Riemann tensor, for convenient reasons is better to work with the *Weyl tensor* C_{pqrs}, which is the trace-free part of the Riemann tensor, and coincides with the Riemann in the vacuum (that is, when the Ricci tensor vanishes).

We can work with just half of the components and obtain the others by complex-conjugate relations (which, in the NP formalism, it is equivalent to swapping the indices 3 and 4). Therefore, the ten independent components of the Weyl tensor are represented by the five complex scalars:

$$\left.\begin{array}{l}
\Psi_0 = -C_{1313} = -C_{pqrs} l^p m^q l^r m^s. \\
\Psi_1 = -C_{1213} = -C_{pqrs} l^p n^q l^r m^s, \\
\Psi_2 = -C_{1342} = -C_{pqrs} l^p m^q \bar{m}^r n^s, \\
\Psi_3 = -C_{1242} = -C_{pqrs} l^p n^q \bar{m}^r n^s \\
\Psi_4 = -C_{2424} = -C_{pqrs} n^p \bar{m}^q n^r \bar{m}^s.
\end{array}\right\}$$

When pertubed, one can associate them physicial meanings, being Ψ_4 and Ψ_0 representations of ingoing and outgoing transverse GWs [11].

The same way, the Ricci tensor and the Ricci scalar can be decomposed into nine traceless components Φ_{ab} and one curvature scalar Λ:

$$\left.\begin{array}{rl}
\Phi_{00} & = -\frac{1}{2} R_{11};\ \Phi_{22} = -\frac{1}{2} R_{22};\ \Phi_{02} = -\frac{1}{2} R_{33};\ \Phi_{20} = -\frac{1}{2} R_{44} \\
\Phi_{11} & = -\frac{1}{4}\left(R_{12} + R_{34}\right);\ \Phi_{01} = -\frac{1}{2} R_{13};\ \Phi_{10} = -\frac{1}{2} R_{14} \\
\Lambda & = \frac{1}{24} R = \frac{1}{12}\left(R_{12} - R_{34}\right);\ \Phi_{12} = -\frac{1}{2} R_{23};\ \Phi_{21} = -\frac{1}{2} R_{24}
\end{array}\right\}$$

The Riemann can always be constructed from the Weyl and the Ricci tensors. One evident useful application of these interpretations is the fact that we can classify e.g. GWs in any metric theory [11].

Now the final step would be to write down in these tetrad formalism the different equations we had in GR. We can expand the Lie bracket $[e_a, e_b]$ in terms of our basis (since it is a tangent vector itself), giving rise to the 24 *commutation relations*, which relates the structure constants and the spin coefficients. Then we can project the 36 *Ricci identities* and the 20 *Bianchi identities* (obtained as usual from the symmetries of the Riemann) onto the tetrad frame, with the advantage that we can omit half of them since we can obtain them by complex conjugating our equations. Moreover, we can get rid of some of the Ricci identities thanks to additional symmetries in this formalism with the *eliminant relations*.

To illustrate this, we can take one commutation relation:

$$[e_a, e_b] = (\gamma_{cba} - \gamma_{cab})\, e^c = C^c_{ab} e_c.$$

where C^c_{ab} are the structure constants. For example, let us compute for:

$$\begin{aligned}
&[\delta, D] = [m, l] = [e_3, e_1] = (\gamma_{c13} - \gamma_{c31})\, e^c \\
&= (\gamma_{113} - \gamma_{131})e^1 + (\gamma_{213} - \gamma_{231})e^2 + (\gamma_{313} - \gamma_{331})e^3 + (\gamma_{413} - \gamma_{431})e^4 \\
&\xrightarrow[\text{in the } 1^{st} \text{ 2 indices}]{\text{Using the antisymmetry}} \\
&= -\gamma_{131}\Delta + (\gamma_{213} - \gamma_{231})D - (\gamma_{313})\delta^* - (\gamma_{413} - \gamma_{431})\,\delta
\end{aligned}$$

Now, given the spin coefficients their designated symbols, we get

$$\delta D - D\delta = +\kappa\Delta + (\alpha^* + \beta - \pi^*)\, D - \sigma\delta^* - (\rho^* + \varepsilon - \varepsilon^*)\,\delta$$

In [9] all these equations are listed. They are commonly known as the **Newman-Penrose equations**. They are simply the commutation relations, the Cartan's equations etc., written in terms of the NP variables. They are a system of coupled second order PDE's (partial differential equations) linking the tetrad, the spin coefficients, Weyl tensor, Ricci tensor and the scalar curvature.

We still do not understand how the decomposition into these equations relates with the Einstein's equations or subtitute them. Indeed, there are way more variables in the NP formalism than in the metric approach, so it is not clear at all whether these equations are simpler to solve or not. This elaborate approach will become tremendously useful for particular relevant solutions, where several NP quantities will vanish and the equations will simplify radically.

Finally, it is necessary to know how to go from a null tetrad to another equivalent one. Recalling there was a local Lorentz invariance (which can be extended continuosly through all the space-time), Penrose and Janis realised that one can cast this transformations into 3 abelian subgroups:

- **Type I**: rotates n leaving ℓ invariant (affecting m and $\bar{m}$ too).
- **Type II**: rotates ℓ leaving n invariant (affecting m and $\bar{m}$ too).
- **Type III**: the directions of ℓ and n remains the same and rotate m and $\bar{m}$. Intuitively, apart from the other two, there is still residual freedom in the choice of the scaling of each tetrad leg, and also in the relative orientation of the remaining two tetrad legs m and $\bar{m}$.

These three rotations encode the six parameters of the Lorentz group and affects all the NP quantities [9], a fact that will be used to simplify the NP equations.

2.3 PETROV CLASSIFICATION

Let us stop here for a moment. Note that all the quantities including the Weyl tensor are being projected onto the tetrad, therefore they depend on the election of the tetrad. Some reasonable questions would be: How a tetrad can be chosen in order to get rid of the most possible quantities? Would it be some of them invariant quantities? The second question can be adress easily by just varying our null tetrad using the three rotations introduced (which evidently preserve the ortogonality and null conditions), and seeing how our quantities vary. It turns out that one can classify a spacetime looking at the algebraic properties of its Weyl tensor, with the so-called **Petrov Classification**.

Starting from a generic NP frame, one can always type-II rotate it to a frame where $\Psi_0' = 0$ by solving:

$$\Psi_4 b^4 + 4\Psi_3 b^3 + 6\Psi_2 b^2 + 4\Psi_1 b + \Psi_0 = 0 \tag{4}$$

for the rotation parameter b. In this case the direction defined by the new ℓ is called **Principal Null Direction** (PND henceforth).

Since Eq.(4) is of fourth order in b there are, in general, four solutions and hence four PNDs. However, if two roots coincide, then two of the PNDs coincide and we say that such PND is *repeated* (if more roots coincide, then more PNDs coincide accordingly). It is precisely the PND structure of the Weyl tensor what gives raise to Petrov's classification. If all four PNDs are different then the Weyl tensor is **algebraically general** or type I, and it is **algebraically special** otherwise. Within the class of algebraically special Weyl tensors there are several possibilities:

-Type II: one double and two simple principal null directions.
-Type D: two double principal null directions.
-Type III: one triple and one simple principal null directions.
-Type N: one quadruple principal null directions.

Choosing ℓ and n aligned with PNDs, several NP quantities vanish. In particular:

-Type I: four simple principal null directions. $\rightarrow \Psi_0 = 0$
-Type Il: one double and two simple principal null directions. $\rightarrow \Psi_0 = \Psi_1 = 0$
-Type D: two double principal null directions. $\rightarrow \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$
-Type III: one triple and one simple principal null directions. $\rightarrow \Psi_0 = \Psi_1 = \Psi_2 = 0$
-Type N: one quadruple principal null directions. $\rightarrow \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$
-Type 0: the Weyl tensor vanishes.

Based on the Weyl scalars properties with respect to the Lorentz rotations, Petrov came up with this classification which is independent of the chosen tetrad, namely, it is a property of the spacetime itself. This classification can be used even to classify GWs or other alternative theories of gravity [11]. Petrov also realised that only Ψ_2 is *invariant* [9].

Now let us tackle the first question of how one should choose the NP tetrad. On the one hand, the property of being algebraically special translates into the vanishing of the Ψ's. On the other hand, if e.g. ℓ or n are geodesics, this translates into the vanishing of several spin coefficients. To show this, let us recall the displacement of ℓ when moving along ℓ (2.2). If we have $\kappa = 0$, we can Type III rotate our basis, rescaling ℓ a doing $\varepsilon = 0$, having ℓ tangent to null geodesics. The Weyl scalars will be modified (except Ψ_2). The Lorentz rotations modifies all the NP quantities, including the Weyl scalars as we displayed, but also the spin coefficients.

Remarkably, there is a *fundamental* theorem which links these two worlds, the algebraic properties of the Weyl tensor and the geometrical properties of geodesics in spaces that satisfy Einstein's equations. This is the **Goldbergs-Sachs** theorem, which states (in 4D) that:
In a vacuum spacetime (allowing for a cosmological constant), a null vector field is a repeated PND if, and only if, it is geodesic and shearfree.

Therefore the most convenient way to choose our tetrad is to try to set ℓ tangent to geodesic, shear-free congruences and then Type II rotate to set n also tangent to geodesic, shearfree congruences. This implies that $\kappa = \sigma = \nu = \lambda = 0$ when $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and viceversa. The most notable fact is that *all BH solutions of GR are Petrov type D* or algebraically specials spacetimes. So assuming type D spacetimes, these several variables vanish (both curvature pieces and spin coefficients), hugely simplifying the NP equations, turning this NP formalism into a more than suitable approach to calculate quantities in GR.

2.4 DERIVATION OF THE TEUKOLSKY EQUATION

GWs are perturbations of the spacetime itself. Since they are weak by definition, one expects perturbative methods to be accurate. The most common way of studying them is to consider a small perturbation of the metric: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is a solution (the background), and then solving Einstein's equation to first order in $h_{\mu\nu}$. The general equation for $h_{\mu\nu}$ is [12] [13]:

$$\begin{aligned}
&\nabla^2 h_{ab} + \nabla_a \nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} \\
&+ 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c\nabla^d h_{cd} - \nabla^2 h) \\
&- 2R^c{}_{(a} h_{b)c} + R h_{ab} - g_{ab} R^{cd} h_{cd} = 0
\end{aligned}$$

where ∇, R are associated to the background metric and $h = h^\mu{}_\mu$, where indices are raised and lowered using the background metric $\bar{g}$.

As a matter of fact, in Kerr, an analytical solution has not yet been found through this method.

The equation (2.4) governing $h_{\mu\nu}$ is a large set of linear, coupled PDE's and it is in general really complicated to construct solutions. However, not all of the components of $h_{\mu\nu}$ are independent due to gauge freedom. As a consequence of general covariance of GR, two metric perturbations $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ are physically equivalent if $\tilde{h}_{\mu\nu} = h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}$ for any ξ_μ. Such freedom can be used to constrain the form of $h_{\mu\nu}$ and simplify its equation of motion. In a vacuum spacetime, where $R_{\mu\nu} = 0$, a popular choice is *de Donder*'s gauge $\nabla^\mu h_{\mu\nu} = 0 = h$. However, even after these simplifications, the equations may still be very involved. Surprisingly, on spherically symmetric backgrounds it is possible to reduce the equations to a pair of ODE's. One can decompose the perturbation in spherical harmonics because equations governing different modes are completely separable, arriving to two master wave equations for odd and even parity sectors, as we will derive in Sec.3.2.1.

Unfortunately, in the case of Kerr, rotation breaks spherical symmetry and the former approach is not useful. It took several years until Teukolsky understood how, taking advantage of the NP formalism (a completely different approach from the one followed in the past to study fluctuations of black holes), one can derive decoupled ("master") equations that govern the propagation of waves in more general spacetimes [5]. In particular, Teukolsky considered any vacuum space of Petrov type D, which includes the Kerr black hole (his approach, however, can be extended to the Kerr–Newman BHs and type D spaces allowing for a cosmological constant). The purpose of this section is to provide a detailed derivation of Teukolsky's master equations: *a pair of decoupled PDE's governing the propagation of gravitational waves on any Ricci-flat, Petrov type D spacetime.*

Instead of handling with metric pertubations, we are going to work with pertubations of the NP quantities (since a perturbation in the metric will infer a perturbation in the scalars).

A gravitational perturbation in the NP formalism is encoded in a small change in the NP frame:

$$\ell = \ell^u + \ell^p, \quad n = n^u + n^p, \quad m = m^u + m^p,$$

where u and p means "unperturbed" and "perturbed", respectively. The perturbation of the frame propagates accordingly to the rest of NP quantities. Since we are in vacuum, from the linearised Einstein's equations we know that all the Ricci scalars must vanish: Φ^p_{ab} and Λ^p. Assuming that the background spacetime is, furthermore, of Petrov type D, one can construct a NP frame where ℓ and n are PNDs of the Weyl tensor. As we have shown, this implies

$$\Psi_0^u = \Psi_1^u = \Psi_3^u = \Psi_4^u = 0 \tag{5}$$

together with:

$$\kappa^u = \sigma^u = \nu^u = \lambda^u = 0 \tag{6}$$

To alleviate the notation, we shall drop the superscript "p" in the perturbed values associated to quantities that vanish in the background, e.g. $\Psi_3^p = \Psi_3$, and the superscript u for quantities that do not vanish in the background, too.

Thanks to this choice of frame, several background quantities vanish. This has the tremendous advantage that NP equations which are homogeneous in quantities that vanish in the background, when perturbed them, *they are already linearised.* For example, let us see these Bianchi identity:

$$-\delta^*\Psi_0 + D\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \varepsilon)\Psi_1 + 3\kappa\Psi_2 + [\text{Ricci}]{=}0$$

Each term contains a quantity that vanishes in the background. When linearised, the Ricci and its perturbation still vanish, all the quantities that vanish promote to the perturbed quantities, and the ones which did not vanish, retain their background value.

Amongst all NP equations, six of them are homogeneous in quantities that vanish in the background. These can be found within the Bianchi and Ricci identities of [9] pages 47–50, and read:

$$\begin{cases} (-4\alpha+\pi+\delta^*)\Psi_0-(D-2\varepsilon-4\rho)\Psi_1=3\kappa\Psi_2; \\ (\Delta-4\gamma+\mu)\Psi_0-(\delta-4\tau-2\beta)\Psi_1=3\sigma\Psi_2; \\ (D-\rho-\rho^*-3\varepsilon+\varepsilon^*)\sigma-(\delta-\tau+\pi^*-\alpha^*-3\beta)\kappa=\Psi_0; \end{cases} \tag{7}$$

$$\begin{cases} (D+4\varepsilon-\rho)\Psi_4-(\delta^*+2\alpha+4\pi)\Psi_3=-3\lambda\Psi_2; \\ (\delta-\tau-4\beta)\Psi_4-(\Delta+2\gamma+4\mu)\Psi_3=-3\nu\Psi_2; \\ (\Delta+\mu+\mu^*+3\gamma-\gamma^*)\lambda-(\delta^*+3\alpha+\beta^*+\pi-\tau^*)\nu=-\Psi_4 \end{cases} \tag{8}$$

All the variables that vanish in the background have been factorised, so inside the brackets there are background quantities which can be calculated from their usual definitions. If one has additional symmetries, more spin coefficients will vanish.

It can be shown that, quite crucially, Ψ_0 and Ψ_4 are fully gauge invariant [5]: they are invariant under the linear action of diffeomorphisms, but also under local, first order frame rotations, so they are measurable physical quantities. It is thereby very appropiate to obtain equations for these quantities from the equations above.

Note that the three equations in (7) have Ψ_0 and Ψ_1 as perturbed variables together with σ and κ. We must eliminate Ψ_1. Since they are differential equations, we can derive them, combine them and try to eliminate one variable. Let us call $\mathcal{D}_1$ and $\mathcal{D}_2$ two undetermined differential operators, acting on the second and third equations of (7). Subtracting them one arrives at:

$$\begin{aligned} &\mathcal{D}_2(\Delta-4\gamma+\mu)\Psi_0-\mathcal{D}_1(-4\alpha+\pi+\delta^*)\Psi_0 \\ &-\overbrace{(\mathcal{D}_2(\delta-4\tau-2\beta)-\mathcal{D}_1(D-2\varepsilon-4\rho))}\Psi_1 \\ &=3\left(\mathcal{D}_2\sigma\Psi_2-\mathcal{D}_1\kappa\Psi_2\right) \end{aligned} \tag{9}$$

The operators $\mathcal{D}_1$ and $\mathcal{D}_2$ need to be chosen in such a way that the terms containing Ψ_1 cancel (the ones in the brace). Following Teukolsky [5], this is achieved by taking:

$$\mathcal{D}_1=\delta+\pi^*-\alpha^*-3\beta-4\tau, \quad \mathcal{D}_2=D-3\varepsilon+\varepsilon^*-4\rho-\rho^*$$

a fact that can be checked by using some of the remaining (background and linearised) NP equations.

Substituting these operators again in (9) leads to:

$$\begin{aligned} &(D-3\varepsilon+\varepsilon^*-4\rho-\rho^*)(\Delta-4\gamma+\mu)\Psi_0-(\delta+\pi^*-\alpha^*-3\beta-4\tau)(-4\alpha+\pi+\delta^*)\Psi_0 \\ &=3\left((D-3\varepsilon+\varepsilon^*-4\rho-\rho^*)\sigma\Psi_2-(\delta+\pi^*-\alpha^*-3\beta-4\tau)\kappa\Psi_2\right) \end{aligned}$$

One should expect a coupled equation for the remaining perturbed variables: Ψ_0, σ and κ. Staggeringly, it turns out that they appear in a combination that they either cancel out or vanish by themselves when using the first equation of (7) and Bianchi identities, obtaining a decoupled differential equation for Ψ_0:

$$\begin{aligned} &[(D-4\rho-\rho^*-3\varepsilon+\varepsilon^*)(\Delta-4\gamma+\mu)-(\delta-3\beta+\pi^*-\alpha^*-4\tau)(-4\alpha+\pi+\delta^*) \\ &-3\Psi_2]\Psi_0=0 \end{aligned} \tag{10}$$

One could repeat exactly the same procedure with the equations of (8), but instead one can use the fact that the NP equations are invariant under GHP transformations, a symmetry consisting in the interchange $l \leftrightarrow n$, $m \leftrightarrow \bar{m}$ which preserves the vanishing (5) and (6) conditions. Applying these transformation to (10), one obtains a second equation for Ψ_4 (since $\Psi_0 = -C_{1313} \to -C_{2424} = \Psi_4$):

$$[(\Delta + 4\mu + \mu^* + 3\gamma - \gamma^*)(D + 4\varepsilon - \rho) - (\delta^* + 3\alpha - \tau^* + \beta^* + 4\pi)(4\beta + \delta - \tau) \\ - 3\Psi_2]\Psi_4 = 0 \tag{11}$$

These two equations (10) and (11) are known as **Teukolsky's equations**. They are decoupled PDE's for the perturbed Weyl scalars Ψ_0 and Ψ_4, a notorious advantage compared with the usual metric pertubation approach. Moreover, let us remark that all the quantities except Ψ_0 and Ψ_4 in these equations are background quantities and one does not need to calculate their perturbations.

2.5 APPLICATION TO KERR'S BLACK HOLE

After having obtained the equations describing gravitational perturbations on general Ricci-flat Petrov type D spacetimes, one can start specifying for spacetimes of interest. The Kerr metric [4] was obtained looking for algebraically special solutions of Einstein's vacuum field equation, and it is of Petrov type D. No-hair theorems establish that this is the unique metric describing a vacuum, stationary, asymptotically flat black hole in GR. Thus, it describes a stationary state of all astrophysical BHs of the universe (in astrophysical scenarios, a charged BH will quickly be neutralized, so only mass and angular momentum are relevant). In the following we specialise the Teukolsky equation to the Kerr metric.

The Kerr metric in Boyer-Lindquist coordinates in units $G = c = 1$, reads [9]:

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mar\sin^2\theta}{\Sigma} dt d\varphi - \frac{\Sigma}{\Delta} dr^2 \\ - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\Sigma}\right) \sin^2\theta d\varphi^2$$

where M is the mass, a is the angular momentum per unit mass of the black hole, $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. We have to align our tetrad with the PND's. Thanks to the Goldberg-Sachs theorem, we can do it by obtaining tangents to families of null, shear-free geodesics. In Schwarzschild is easier since intuitively those are ingoing and outgoing radial null geodesics. In Kerr, even the natural approach of knowing how the spin coefficients transform under local Lorentz transformations and try to set the suitable ones (6) to zero are very involved. By other approaches, [9] obtains the frame that fullfills our requirements:

$$l^\mu = \left[\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right], \quad n^\mu = \frac{2}{2\Sigma}\left[r^2 + a^2, -\Delta, 0, a\right] \\ m^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}[ia\sin\theta, 0, 1, \frac{i}{\sin\theta}] \tag{12}$$

with the nonvanishing spin coefficients:

$$\rho = 1/(ia\cos\theta - r), \quad \beta = -\rho^* \cot\theta/(2\sqrt{2}), \quad \pi = ia\rho^2 \sin\theta/\sqrt{2},$$
$$\tau = -ia\rho\rho^* \sin\theta/\sqrt{2}, \quad \mu = \rho^2\rho^*\Delta/2, \quad \gamma = \mu + \rho\rho^*(r-M)/2, \quad \alpha = \pi - \beta^*$$

and the only nonvanishing Weyl scalar:

$$\Psi_2 = M\rho^3$$

Now we can replace all these background quantities and the directional derivatives generated by our frame (12) into the master equations (10) and (11). Moreover, the corresponding wave equation for perturbations of other spins on Kerr's space, turns out that all of them have a very similar form [5], so we can arrive at a single master equation valid equally for a test scalar field ($s=0$), a test electromagnetic field ($s=\pm 1$), or a gravitational perturbation ($s=\pm 2$) in the Kerr background:

$$\begin{aligned}
&\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2} \\
&- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta]}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi} \\
&- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + \left(s^2\cot^2\theta - s\right)\psi = 4\pi\Sigma T
\end{aligned} \tag{13}$$

with T the energy–momentum (EM) tensor of the perturbation source projected into the tetrad.

Remarkably, in the vacuum case ($T=0$), trying an ansatz of the form:

$$\psi = e^{-i\omega t}e^{im\varphi}S(\theta)R(r).$$

allows to separate the angular and radial factors of the perturbation:

$$\begin{aligned}
\Delta\frac{d^2R}{dr^2} + (s+1)\frac{dR}{dr}\frac{d\Delta}{dr} + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right)R = 0 \\
\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \\
\left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + A\right)S = 0
\end{aligned} \tag{14}$$

where $K \equiv (r^2+a^2)\,\omega - am$ and $\lambda \equiv A + a^2\omega^2 - 2am\omega$. A is the separation constant and it is determined by imposing boundary conditions of regularity at $\theta = 0$ and π, which for vanishing $a\omega$, one can see that $A = l(l+1) - s(s+1)$. This is a *Sturm-Liouville eigenvalue problem* for A, and the eigenfunctions ${}_sS^m{}_l$ are complete and orthogonal on $0 \leq \theta \leq \pi$ for each m, s, and $a\omega$, and they are called **spin-weighted spheroidal harmonics** ${}_sY^m{}_l = {}_sS^m{}_l(\theta)e^{im\varphi}$ (for aω=0, eliminating the rotation, and s=0 we recover the spherical harmonics, so it gives an intuition about the vibration modes of the BH). Adding sources is not a problem, since we can use the Green's function to obtain the inhomogeneous solution.

Now, How do we extract information from solutions of (14)? For scalar and EM waves, we have a well-defined E-M tensor at every point of our spacetime, but this is more subtle in the case of GWs. There is an interesting theorem called the **Peeling theorem**, which, based on the asymptotic behavior of the radial variable, predicts that the scalars go as:

$$\Psi_i \propto \frac{e^{-i\omega t+i\omega r_*}}{r^{5-i}} \tag{15}$$

with r_* the tortoise coordinate $\left(\frac{dr_*}{dr} = \frac{(r^2+a^2)}{\Delta}\right)$. This means that we will detect at infinity only Ψ_4, and we can interpret it as the outgoing waves. Conversely, we can enunciate the theorem on the other way around and interpret Ψ_0 as the ingoing waves. This will be very useful for boundary conditions, since we only expect ingoing waves at the horizon. With that in mind, one can use the standard equations of linearized theory and check that the total energy flux per unit of solid angle is [5]:

$$\frac{d^2E^{(\text{out})}}{dtd\Omega} = \lim_{r\to\infty} \frac{r^2}{4\pi\omega^2} |\psi_4|^2 \,; \qquad \frac{d^2E^{(\text{In})}}{dtd\Omega} = \lim_{r\to\infty} \frac{r^2}{64\pi\omega^2} |\psi_0|^2 \,.$$

If one wanted to recover the metric perturbations, this can done by using the corresponding Hertz potential [14], where one could relate the Weyl scalars with the strain measured by our detector: $\Psi_4 \propto |h_+ + ih_x|^2$.

We have obtained a set of uncoupled ODE's which govern the behavior of perturbations of Kerr BHs. In which context can we make use of Eqs.(14)? There are countless applications: stability of the BH, spin down due to perturbations, superradiant scatterings, interaction with accreting test matter... But the most important one concerns processes of emission of gravitational waves.

The typical GW signals measured by the LIGO and Virgo collaborations [2] correspond to the merger of two compact objects, typically BHs, but also neutron stars (NSs henceforth). We are going to study two phases of the signal where perturbation theory can be used to extract physical information from the source system: the inspiral and the ring-down. We shall focus on the two main observables of each phase: the **Tidal Love Numbers** (TLNs), or simply Love Numbers, in the inspiral, and the **Quasi-normal Modes** (QNMs) in the ringdown.

3 GRAVITATIONAL WAVES IN BINARY MERGERS

We are living a revolutionary era where thanks to earth-based interferometers [2], we can contemplate and analyse a part of the Universe that had been hidden until now: the Gravitational Waves. The study of these waves encode unique and valuable information about the source that emitted them.

Compact binary systems are the main sources of GWs (although future missions [15] will be able to detect others such as Primordial GWs). One can see [16] that the radiated power in those systems goes as $P \sim (M/d)^5$, with M the mass and d the separation. To get a large P we need the system to have M/d as large as possible, so we want objects as compact as possible. BHs appear naturally as one of the most appropriate objects to test GR in the strong field regime through the study of these ripples. NSs are also incredibly compact, so the type of binaries we expect to detect more frequently are: NS/NS, NS/BH or BH/BH systems. We will focus at a fusion merger of BHs, the most detected sytems by GWs.

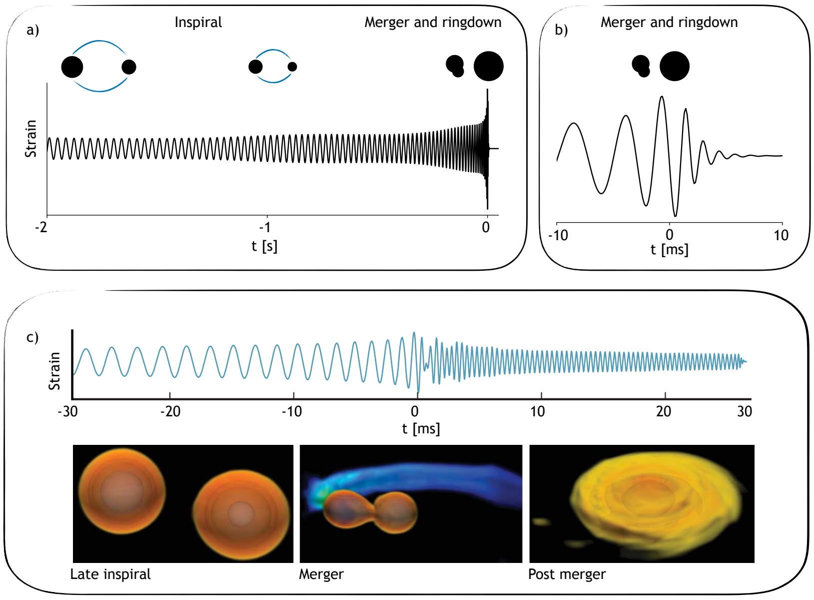

Figure 1: Extracted from [1] **Predicted gravitational waveforms from compact binaries.** a) Final two seconds GW emission produced by a binary BH. b) The same signal in the 10 ms around the merger. The resulting BH rings down to an equilibrium state. c) GW signal in the 30 ms around the merger of a binary NS merger. GW emission continues after the merger. The NSs are firstly tidally deformed and then disrupted in the late inspiral. A disk of matter forms at the end.

How the equations (14) can be applied to these events? During the whole process of emission, 3 stages can be identified, as shown in Fig. 1:

- The orbital separation between the BH gradually decreases because of the loss in energy and angular momentum emitted in the form of GWs. As the frequency increases, the approximation of point massses for BHs is no longer valid and we start noticing the tidal interactions. The BHs should acquire a tidal deformation, affecting both the GW and the orbital motion along with the gravitational field. This phase is called the **inspiral**, and the numbers that encode the induced multipole moments due to the companion's gravitational field are called **Love numbers**. We can model this situation with a Kerr metric of one object that is perturbed by the other one, represented as a stationary tidal field (at first order), so our equations hold.

- When they are close enough, perturbation theory starts to fail, the objects are hugely deform, the signal becomes very noisy and we have to use Numerical Relativity. This is the **merger**.

- However, in the **ring-down**, the last stage of the GW emission, our equations are valid again since we end up with a Kerr BH that has been perturbed. It is noticeable that in this phase the BH starts to vibrate in a superposition of modes that are independent of the way it was excited. These are the **quasi-normal modes**. The first three QNMs can be used to extract information about the resulting BH, but the fourth and successive are uniquely determined by GR, presenting an accurate way to test the theory.

The following sections focus on the TLNs and QNMs of Kerr.

3.1 LOVE NUMBERS

The newtonian study of gravitational deformation and dissipation of a body due to tidal fields where studied by A.E.H. Love in 1900s [17]. The same way a rotation around a certain axes yields a dense bulge at the equator of an object (due to the centrifugal force), an external field caused by a companion will also deformed their structure [18]. The classical example is the tides created by the Moon, which also produces dissipative effects due to the tidal acceleration of the Moon by gradually spinning down the Earth, affecting their orbital motion. In this cases, the quotient between the typical size of our object R and the distance to the object producing the tidal field b, is small enough so a Taylor expansion of the external potential can be done. This is the usual *multipolar expansion* used in countless areas of physics, from electrostatic fields, to magnetic moments in nuclear physics. Each term, characterised by a multipolar polar index l, tries to capture the shape, the angular distribution of the source, e.g. $l = 2$ would account for the 4-fold symmetry (rotation of $\frac{\pi}{2}$).

This tidal moments $\mathcal{E}_L(t)$, which can be described as symmetric trace-free tensors (STF), perturbe our body, exciting internal multipole moments, again described by STF tensors. If we choose e.g. the center of mass (CM) frame (to vanish the dipole moment), the cuadrupole moment would be describe as: $\mathcal{E}_{ab}(t) := -\partial_{ab}U_{\text{ext}}$, where the external potential is generated by the companion in the body. The deformation destroys the spherical symmetry, and the body acquires a quadrupole moment $I^{ab} := \int \rho(x^a x^b - \frac{1}{3}\delta^{ab}r^2)\, d^3x$, where ρ is the mass density, r the distance to the CM and x^a cartesian coordinates in our frame. When the tidal field is *weak*, the quadrupole moment is proportional to the tidal field, and by dimensional analysis [19] one gets:

$$I_{ab} = -\frac{2}{3}k_2 R^5 \mathcal{E}_{ab}$$

where $G = c = 1$ and R is the body's radius. k_2 is the **Love number** related to a quadrupolar deformation, and is a dimensionless constant. Taking into account everything, the Newtonian potential in the neighbourhood outside the body is:

$$U = \frac{M}{r} - \frac{1}{2}\big[1 + 2k_2\frac{R}{r^5}\big]\mathcal{E}_{ab}(t)x^a x^b. \tag{16}$$

We observe the usual monopolar term of the potential (the potential that would generate a mass M located at our origin), and then two more terms. The first one represents the applied quadruople tidal field moment, which scales as r^l, and the second term is the induced quadrupole moment, the body's response, which contains the LN k_2 and goes like r^{-l+1}.

This analysis can be seen as the gravity analogue of of electric polarisabilities and magnetic susceptibilities. When we applied an electric field $\vec{E}$ to a dielectric material the dipole moments of the atoms will reorganise and acquire an induced polarization $\vec{P}$. This linear response is quantify by the electric susceptibility χ: $\vec{P} = \chi\vec{E}$, so one would measure $\vec{E}$ plus the response $\vec{P}$. One can study this in a broader setup where $\vec{E}$ varies with time, appearing dissipative effects since the material does not polarize instantaneously: $\vec{P(t)} = \chi^0\vec{E} - \tau_0\chi^1\dot{\vec{E}}(t) + ...$ with τ_0 the typical response time of the material. In Fourier space $P(\omega) = \chi(\omega)E(\omega)$, where $\chi(\omega) = \chi^0 + i\omega\tau_0\chi^1$ with the real part accounting for the conservative response and the imaginary part the dissipation [20]. The same concepts apply for magnetic fields $\vec{B}$ and the induced magnetization M.

Clearly our analysis not only can be extended to an arbitrary l-moment but to a sum of multipole moments (decomposition of the external potential), although higher orders are smaller and smaller contributions. If the tidal field is just a pure multipole of order l we can generalize (16) to [19]:

$$U = \frac{M}{r} - \frac{1}{(l-1)l}\left[1 + 2k_l\left(\frac{R}{r}\right)^{2l+1}\right]\mathcal{E}_L(t)x^L. \tag{17}$$

where now k_l is the LN associtaed with this multipolar configuration. L is a multi-index which have l individual indices $L \equiv a_1a_2\cdots a_l$.

Once seen the classical approach, we should promote all these definitions to GR, specially to our binary systems. First we require a weak interaction between the body and its companion, that is, a weak tidal field, so linear pertubation theory can be used. It does not matter if our body creates strong gravitational fields, so it will be valid for BHs. Secondly, we assume t to be an *adiabatic parameter*, which means that, albeit the tidal moments depends on time, its variation is small enough so the perturbation is stationary and the body's response displays only a parametric dependence, throwing off time derivatives in the equations.

The common way to proceed is to construct a metric in a neighbourhood around the body, which is being pertubed by a static tidal field. For objects with spherical symmetry is convenient to break down the perturbation depicted e.g. by a multipole of order l into scalar, vectorial and tensorial spherical harmonics [13], classifying them according to their properties under parity transformations [8] (more details in 3.2.1). Solving the linearised Einstein's equations, one can extract from the metric components an effective Newtonian potential which reads [19]:

$$U_{\text{effective}} = -\frac{M}{r} - \frac{1}{(l-1)l}\left[A_1 + 2k_{el}\left(\frac{R}{r}\right)^{2l+1}B_1\right]\mathcal{E}_L(v)x^L. \tag{18}$$

In the nonrelativistic limit A_1 and B_1 goes to one and we recover Eq. (17). k_{el} are the **electric Love Numbers**, and they are the relativistic version of the Newtonian LNs. However, in the solution of the metric perturbation [19] one observes another type of LNs with no analog in Newtonian physics, the **magnetic Love Numbers** k_{mag}. They are the analogue of magnetic susceptibilities.

Observe that we still need to impose suitable boundary conditions to our problem to determine the LNs. Once perturbation equations are solved, one needs to solve for the unperturbed equations in the interior of the object (generally for a given *equation of state* as in Tolman–Oppenheimer–Volkoff equations for hydrostatic equilibrium [21]), and then solve the perturbed ones. Matching the interior and exterior solutions fixes the LNs. Hence, LNs only depend on the internal structure of the body.

3.1.1 LOVE NUMBERS IN STATIC BLACK HOLES

Specifying for Schwarzschild BHs, it turns out that when r tends to $2M$, the functions B_n diverges logarithmically. We must get rid of these terms when imposing boundary conditions at the horizon. This removal entails that $k_{el} = k_{mag} = 0$ (forces the decaying term in Eq. (18) to vanish identically). So for Schwarzschild BHs, **all LNs vanish**. This means that whatever static perturbation that comes from infinity, it does not affect the BH. They remain *rigid* in the face of tidal forces. Coloquially, it is said that Schwarzschild BHs **do not fall in love**.

The effects of charge on the static response of BHs are carried out in [22], where they solve gauge-invariant master equations for vector and tensor modes in an arbitrary number of spacetime dimensions ($D \geq 4$). When a BH is charged, if electromagnetic fields perturbe the BH (e.g. the ones generated by the acceleration of accreting matter), they can produce GWs since the energy-momentum tensor is coupled to the metric. Besides, it turns out that **Reissner–Nordström BHs also have vanishing LNs.**

It gives the impression that all BHs do not deform when they are pertubed by a static external tidal field. Sadly, we cannot play the game of metric pertubations for stationary BHs. But we can take advantage of the formalism we have developed.

3.1.2 LOVE NUMBERS IN KERR

Finally, let us deal with the problem of computing the linear response of a Kerr BH due to a weak and slowly varying tidal field. This broader setup of the static response of a spinning BH can be adress with the equations (14) developed in Sec.2.5, the Teukolsky equation specialise for Kerr.

The deformability of a Kerr BH can again be obtained through the computation of LNs but now it is not possible to calculate them analytically from metric perturbations. Nevertheless, since we deal with a weak tidal field, which can be treated as a perturbation of the spacetime, the equations (14) hold. The slowly varying field implies that at first order $\omega = 0$, which are the static modes. Moreover, we are interested in $s = +2$ perturbations, so the solution to the angular part are spin-2 weighted spherical harmonics ${}_2Y_{lm}(\theta, \phi)$ and $A = l(l+1) - 2(2+1) + O(a\omega)$. We also want to compute for an arbitrary multipole moment, so decomposing the perturbed Weyl scalar Ψ_4 (which, under the Peeling theorem (15), describes the two transverse polarizations of GWs propagating towards ∞, and since it is gauge invariant, the LNs will also be) into (l, m) modes, yields:

$$\Psi_4 = \sum_{lm} Q_{lm}(t)\, R_{lm}(r)\, {}_2Y_{lm}(\theta, \phi) \equiv \sum_{lm} \Psi_4^{lm} \tag{19}$$

where $Q_{lm}(t)$ is a slowly varying (complex) function of time. LNs are defined at ∞, so we need to solve the radial equation at ∞. The physical meanings are always in the imposition of suitable boundary conditions. In this case, apart from solving at ∞ it is necessary that the solutions do not diverge in the Event Horizon. A comprehensive study of the solutions of the radial equation of (14) must be done.

Dividing by $\Delta(r)$, the radial part takes the form:

$$R'' + p(r)R' + q(r)R = 0 \tag{20}$$

From linear ODE's theory, the general solution to (20) is a linear combination of 2 particular linearly independent solutions $R(r) = C_1R_1(r)+C_2R_2(r)$. The coefficients will be fixed by imposing boundary conditions. To adequately study this type of equations, it is necessary to consider (20) in the complex plane $\mathbb{C}$, so $R = R(z)$.

To solve (20) one first has to classify it based on the number of regular and irregular singular points of $p(r)$ and $q(r)$. Recall that a point $z_0 \in \mathbb{C}, z_0 \neq \infty$ is a regular singular point when $(z - z_0)\, a(z)$ and $(z - z_0)^2\, b(z)$ are analytical in z_0, so they admit a power series expansion convergent in some disk centered at z_0. Basically z_0 is a maximum order one pole for $p(r)$ and a maximum order two for $q(r)$. In a regular point $p(r)$ and $q(r)$ are analytical. An irregular point is the other cases.

For $\omega \neq 0$, looking at the singularities of $p(r)$ and $q(r)$, one finds two poles of order 1, one at the Event Horizon (r_+) and other at the Cauchy Horizon (r_-) (basically when $\Delta = 0$); and an irregular singular point of Poincaré rank one at $z = \infty$ (to study the infinity, a change of variable $z \to \frac{1}{z'}$ must be done [23]). Such differential equation is known in the mathematics literature as the **confluent Heun equation**. Surprisingly, when taking the static limit $\omega = 0$ the infinity turns into a regular singular point, having the well-known Fuchsian differential equation: the **hypergeometric equation**.

The standard form to present a hypergeometric differential equation is by placing the poles at $z = 0$ and 1. To move the poles without changing its structure one can do a change of coordinates or a redefinition of the R variable. However, in order to avoid confusion, we will stick to the choice made by [14] the poles are at $z = -1$ and 0. Performing a Möbius transformation $u = \frac{r-r_+}{r_+-r_-}$ we can achieve it.

With this change of coordinates and defining $\gamma \equiv a/(r_+ - r_-)$ the Radial Teukolsky equation takes the form:

$$u(u+1)R''_{lm}(u) + (6u+2im\gamma+3)R'_{lm}(u) + \left[4im\gamma\frac{2u+1}{u(u+1)} - (l+3)(l-2)\right]R_{lm}(u) = 0$$

where the chain rule has been applied and $'$ denotes derivative with respect to u. Reorganising terms one can compare it with the form of the hypergeometric equation:

$$z(1-z)u''(z) + \left[c - (a+b+1)z\right]u'(z) - abu(z) = 0$$

ending up with $a = -l-2$,$b = l-1$ and $c = -1+2im\gamma$. It has a symmetry under the interchange $a \leftrightarrow b$ (so the choose is arbitrary).

The *hypergeometric function* solves the differential equation around the singular points by the series [24]:

$$u(z) = F(a,b,c;z) \equiv \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n) n!} z^n \tag{21}$$

where $(\cdot)_n$ is the Pochhammer symbol. This series converges for $|z| < 1$, and is normalized so that $u(0) = 1$.

For the general solution it is necessary another linearly independent solution. The Fuchs theorem states that, having a second order ODE, the general solution around a point z_0 in terms of 2 **Frobenius series** takes the form [23]:

$$u(z) = A\,(z - z_0)^{\alpha_1} \sum_{n\geq 0} c_n\,(z - z_0)^n \left[1 + C \log{(z - z_0)}\right] + B\,(z - z_0)^{\alpha_2} \sum_{n \geq 0} c_n'\,(z - z_0)^n$$

where α_1,α_2 are not integers in general, C is a constant that can be 0 and A and B are constants that can be determined with boundary conditions.[1] In our case the Frobenius series are the hypergeometric solutions, and the α's are tabulated around each point. The conditions for C to be 0 are enconded in the value of $\alpha_1 - \alpha_2$, which translates in the values of a,b and c. In our case, we want the solutions around infinity, and the local exponents (the α's) are $-a$ and $-b$. In that case the discriminant is just $2l + 1$, and since $l \in \mathbb{N}$, so does the discrimant. In that case, it is not possible to tell if C is zero or not. However, we are going to assume that $l \in \mathbb{R}$, so the case is degenerate and $C = 0$. Therefore, the fundamental solutions in the neighborhood of $z = \infty$ are finally given by:

$$\begin{aligned} R(u) &= A(-u)^{-a} F(a, 1 + a - c, 1 + a - b; 1/u) \\ &+ B(-u)^{-b} F(b, b - c + 1, b - a + 1; 1/u) \end{aligned} \tag{22}$$

Lastly, we have to impose the smoothness of our solutions at $u = 0$ (the Event Horizon). In [14] this has been carried out using analytic continuation, which means that $l \in \mathbb{R}$ (as we have done), and then taking the limit to the positive integers. This can seem, although mathematically correct, an unnatural way to proceed. Sadly, the expressions that appear in the asymptotic expansions are not well defined for $l \in \mathbb{N}$ so it is inevitable to use this approach and in these calculations we will be implicitly assuming analytic continuation.

Firstly, observe that at infinity the argument $\frac{1}{u}$ in Eq.(22) tend to 0 so as we mentioned before both hypergeometric functions tend to $1(+O(\frac{1}{u}))$ (normalisaton of the first coefficient of the series). Hence, we have that $R(u) \propto Au^{+l+2} + Bu^{-l+1}$. The growing mode corresponds to the *external tidal field*, whereas the decaying mode is associated with the *response* of our object.

To establish no divergences of the solutions at $u = 0$ one needs to know how these solutions behave at the r_+. It is known that [23],[24]:

$$F(a,b,c;z) \xrightarrow{z \to 1} -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \log(1 - z) + \text{finite terms} \tag{23}$$

[1]Formally, α_1 and α_2 are the roots of the *indicial equation* arranged such that $\mathrm{Re}\,\alpha_1 \geq \mathrm{Re}\,\alpha_2$. The coefficients of the series are obtain recursively choosing $c_0 = 1, c_0' = 1$. This is where the tabulates solutions of the hypergeomtric functions come from [23].

where $\Gamma(a)$ is the Gamma function. We can interchange the points at 0 and 1 by a change of coordinates, which only does $a \to a$, $b \to b$ and conjugates c, but there is the symmetry $R_{lm}(x) \to \bar{R}_{l\,-m}(x)$ aso we can still use Eq. (22). Substituting our values in (23) for both solutions, we obtain 2 logaritmic divergences. In order to cancel them out we need that those terms cancel each other, so we can interchange the coefficents in front of them with a relative - sign. In that case the relative normalisation between A and B gets fixed and we end up with:

$$R(u) = A\Big[\frac{\Gamma(1+b-a)}{\Gamma(a)\Gamma(1+a-c)}u^{l+2}F(a,1+a-c,1+a-b;1/u)$$
$$-\frac{\Gamma(1+a-b)}{\Gamma(a)\Gamma(1+a-c)}u^{-l+1}F(b,b-c+1,b-a+1;1/u)\Big]$$

Since the LNs account for the fall-off induced by the tidal field, they will be the quotient between the static response and the tidal field. Therefore, replacing a,b and c by their values, we finally obtain the **Love Numbers for Kerr:**

$$k_{lm} = \frac{\Gamma(l-1)\Gamma(-2l-1)\Gamma(l+1+2im\gamma)}{2\Gamma(2l+1)\Gamma(-l-2)\Gamma(-l+2im\gamma)} \tag{24}$$

where $\Gamma(1+a) = a\Gamma(a)$ has been used. Eq.(24) admits a well-defined limit when $l \in \mathbb{N}$, which is proportional to the spin of the BH and vanishes for $a = 0$ (Schwarzschild) [14]. This result could be expected, since we are applying a static tidal field to a stationary Kerr BH, meaning there is a relative motion between the tidal field and the BH. This rotation should dissipate part of these perturbations (by losing energy and angular momentum), translating into non-vanishing imaginary LNs. Truly, the real part of Eq.(24) is the part that represents the concept that we have introduced of LNs. The imaginary part is called in the literature the dissipative response coefficient. Finally, notice that they do not depend on the amplitude of the tidal field (A is a global factor), meaning that *LNs are an intrinsic property of the BH.*

A sanity check would be to consider axisymmetric perturbations ($m = 0$) or nonspinning BHs ($a = 0$ recovering Schwarzschild). In both cases, $2im\gamma = 0$, and checking the roots of the LNs or playing with the properties of the Γ's, we obtain from Eq.(24) that the **LNs vanishes**, agreeing with previous results [19].

3.2 QUASI-NORMAL MODES

When a tight rope with its ends fixed is perturbed, it vibrates in a superposition of its *characterisitic vibration modes.* This characteristic oscillations not only appears when we play a string instrument or when we sing with our vocal cords, but in countless systems: a tenis ball when is hit, a played drum, the wave pressure in a clarinet, the vibrations in a wine glass or a bell... Indeed, they are used in almost every field of physics, from molecular physics till seismology or atmospheric dynamics. They are of paramount importance because they are inherent to the source, so their analysis allow us to obtain valuable information of the vibrating object.

This concepts can be applied to BHs. The vibration modes of BHs are called **quasinormal modes**. Mathematically, QNMs are the eigenfunctions of the equations governing BHs pertubations. Nonetheless, unlike the other systems mentioned, not only we have to impose a boundary condition at the Event Horizon but also an open boundary condition (at ∞). This is translated into dissipative effects and the eigenfunctions are not a complete set, which do not happen in other idealized systems. Moreover, this will overdetermine the problem, so only certain (real and imaginary) frequencies will be solution to the equations.

QNMs encompass a vast landscape of topics, going from holography to BH quantizations [25]. However, we will only examine their appearence in the ring-down of astrophysical BHs and their implications in the detection of GWs. The feautures that dominate this last phase of the coalescence are these QNMs, which appear as a superposition of exponentially damped sinusoids in the GW amplitude. The frequencies will depend **only** on the final BH parameters, whereas the amplitude will be given in terms of the stimulus that excited the BH. Furthermore, thanks to the no-hair theorems, since a BH is entirely described by the mass, charge and spin, if we perturbe it, the BH will wipe out all the additional structure in radiation form, leaving only the "three hairs". In other words, any event where BH pertubations appear is susceptible to end with GWs emission through quasinormal ringing. Additionally, not only BHs ring, other objects do so too, such as NSs, presenting another way to learn more about the internal structure of these objects [26].

3.2.1 QUASI-NORMAL MODES IN SCHWARZSCHILD

It starts with BH perturbation theory. As in Sec.3.1.1 one can make use of the gauge invariant formalism developed by Martel and Poisson [8]: take advantage of the symmetries of the background and decompose the metric perturbations into scalar, vectorial and tensorial spherical harmonics, converting the Einstein's pertubed equations into 2 decoupled master equations related with the even and odd parity sectors. Concretely, these equations take the form (in the *Regge-Wheeler gauge*) of a *Klein-Gordon* differential equation for the variables Ψ_{even} and Ψ_{odd} , which are combinations of the even and odd parity of the metric perturbations respectively:

$$(\Box - V_{\text{even;odd}})\Psi_{\text{even;odd}} = S_{\text{even;odd}} \tag{25}$$

with potentials:

$$V_{\text{even,l}}(r) = \frac{f(r)}{K(r)^2}\left[(l-1)^2(l+2)^2\left(\frac{l(l+1)}{r^2} + \frac{6M}{r^3}\right) + \frac{36M^2}{r^4}\left((l-1)(l+2) + \frac{2M}{r}\right)\right]$$

$$V_{\text{odd,l}}(r) = f(r)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right]$$

where $K(r) = (l-1)(l+2) + \frac{6M}{r}, f(r) = 1 - \frac{2M}{r}$ and $S_{\text{even;odd}}$ are the source term written in a convenient way. Solving these equation we can reconstruct all the perturbed metric. The potentials depend on the angular momentum but we will omit their labelling [13]. For QNMs we set the source term to 0, since we are interested only in the BH response.

It is remarkable to realise that the whole problem of the Schwarzschild pertubation solutions has been reduced to solving these two wave equations in (1+1) dimensions. To obtain the frequencies we can just go to the frequency domain. For example if we take $\Psi_{even}(t,r) = \int e^{-i\omega t}R(r)d\omega$ and using the tortoise coordinates: $dr_* \equiv \frac{dr}{f(r)}$, Eq. (25) turns into a *schrödinger type* equation:

$$\Big[\frac{d^2}{dr_*^2} + (\omega_{n,l}^2 - V_{even})\Big]R(r_*) = 0 \tag{26}$$

where $\omega_{n,l}$ are the **QNMs frequencies**, which can be decomposed into its real and imaginary part $\omega_{n,l} = \omega_R + i\omega_I$. The real part represents the tone, the physical vibration frequency whereas the imaginary part represents the dissipation, how the QNM exponentially decays with time since $e^{-i\omega t} = e^{-i\omega_R t}e^{-\omega_I t}$. Requiring only infalling solutions at the Event horizon and only outgoing at ∞ constraints the frequencies, so only a certain set of them are valid. For a given l one has a infinite number of QNMs, labeled by n, the overtone number. The fundamental mode $n = 0$ will be the least damped mode and therefore it will dominate the ringdown.

Thus, our problem can be treated as a wave with a frequency $\omega_{n,l}$ inmerse in a one-dimensional potential. For LNs, one would need to recover the source term defined in terms of the tidal moments, take the zero-frequency limit (we are interested in a long-wavelength external field) and then impose the regularity at the horizon [19].

For QNMs, our boundary conditions are defined so there are only ingoing waves at r_+ because nothing can escape the the event horizon, and only outgoing waves in the spatial ∞ since only the BH emits GWs and no waves should enter from ∞ in an isolate system. We can study the behaviour of the potentials near these zones. Luckily, for both potentials, they tend to 0 in both cases: $V \to 0$ when $r_* \to -\infty\,(r \to r_+)$ and when $r_* \to +\infty\,(r \to \infty)$. In these limits the solutions to Eq. (26) are simply wave solutions of the form $\Psi \sim e^{-i\omega(t \pm r_*)}$. Hence, we must require that our solutions behave as:

$$\Psi \sim e^{-i\omega(t+r_*)}\,\text{when}\,r \to -r_+ \quad \text{and} \quad \Psi \sim e^{-i\omega(t-r_*)}\,\text{when}\,r \to \infty$$

The waves can run away either into the BH or to ∞, which is the reason why BHs are dissipative system for QNMs and why QNMs frequencies have imaginary parts.

Obtaining the eigenmodes and frequencies with this boundary conditions analytically are not possible in general and we have to do it numerically [27]. However we can check that for the even case the potential independently of l has a peak at $R = 3M$, the position of the so called light-ring [28] and for $l \gg 1$ (the *WKB limit*) one can obtain exact solutions relating the QNMs with waves orbiting around these unstable photon orbit and slowly running away [25]. With regards to the odd part, Chandrashekar realised that the equations (25) presented a duality symmetry, explaining the **isospectrality** of the even and odd sectors [9].

Instead of displaying random frequency values for different l and n, we redirect to [29] for tabulated tables of QNMs frequencies for the most relevant events in GW astronomy. However, since we are interested in detecting these feautures in the ringdown, it would be useful to calculate the least damped modes. In most cases, this corresponds to the fundamental mode with $l = 2$. Numerically one can shown that [25] this corresponds to an oscillation frequency and damping time of:

$$f = \omega_R/2\pi = 1.207\left(\frac{10\ M_\odot}{M}\right)\text{kHz}\,, \tau = 1/|\omega_I| = 0.5537\left(\frac{M}{10\ M_\odot}\right)\text{ms} \tag{27}$$

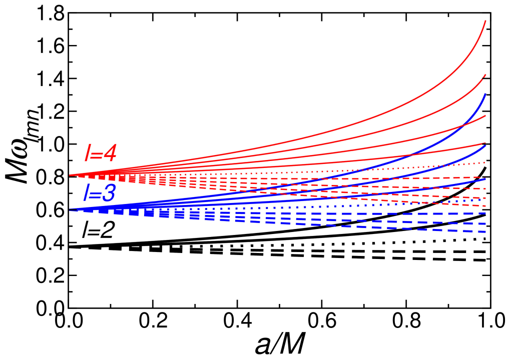

Figure 2: Extracted from [25]. Imaginary parts of QNMs as a function of the spin.

$10M_\odot$ means 10 solar masses, a typical value for a stellar-mass BH. The most sensitive band of LIGO is around $f_{GW} \simeq 150Hz$ so an equilibrium for the mass is needed since for bigger BHs the sensitivity is better but the damping increases too.

3.2.2 QUASI-NORMAL MODES IN KERR

How a pertubed BH settles down to a Kerr BH is a complex problem we cannot treat it with the conventional formalism used for static uncharged BHs. As with LNs, the Teukolsky's equation (14) simplifies the dynamics of the linear gravitational pertubations in Kerr BHs dramatically, up to the point of reducing the problem to a one-dimensional one, just like before.

To make a comparison with the Schwarzschild case, one can take the radial and angular equations of (14) and transform them by a redefinition of the variables [23], recasting them into Schrödinger equations. We do not show the form of these equations since the potentials are too long but one can check them in [30]. In spite of having the same form as in Schwarzschild, the separation constant $A_{l,m}$ depends on the solution of the angular part, which are spin-weighted spheroidal harmonics since $a\omega \neq 0$, and no closed form for $A_{l,m}$ is known. Aditionally, opposed to LNs, the radial equation is now the Confluent Heunn equation, which are less amenable to solve. Moreover, recall that the spin breaks the spherical symmetry, so we have to label each mode with l, m.

Nonetheless, the angular part is a hypergeometric function, and some algorithms allow us to numerically calculate $A_{l,m}$ and with them calculate the free modes of vibration. The most famous one is the leaver method [27]. The effects of the rotation in these QNMs resemble the *Zeeman effect* in Atomic and Molecular physics, where the presence of a magnetic field splits the energy levels of the atom. In this case, the QNMs will unfold as the spin increases, labelling with the magnetic quantum number m, as we can see in Fig. 2.

To obtain the eigenmodes of $R(r)$ subject to the proper boundary conditions, as in Sec. 3.2.1, we can do it by inspecting the form of the potentials in those limits. Dropping the subdominant terms in our equations we will have to impose that:

$$R(r) \sim \frac{e^{-iKr_*}}{\Delta^s} \,\text{when}\, r \to r_+ \quad \text{and} \quad R(r) \sim \frac{e^{i\omega r_*}}{r^{2s+1}} \,\text{when}\, r \to \infty$$

With an eye toward the detection of GWs signals in the ringdown, we show in Fig. 2 the numerical results for the imaginary part of the fundamental mode frequencies varying l, as a function of the spin. As in Schwarzschild, $l = 2$ is the least damped mode. When the spin increases, the lines unfold into different values of m ($m = l$ to $m = -l$ from top to bottom). For a more in-depth look check [25].

4 CONCLUSIONS

In this work we have provided a thorough and detailed introduction to the theory of gravitational perturbations on Kerr's spacetime.

We begun by reviewing the mathematical framework that is required to introduce Petrov's classification in a convenient form. Taking advantage of the Petrov's classification, we derived in detail Teukolsky's equations and specialised them to Kerr's solution. Remarkably, these can be reduced to a pair of decoupled ODE's.

Finally, we have discussed two of the main applications of Teukolsky's equations. One consists in the tidal deformability of BHs, (a topic that has generated a lot of interest recently [13],[14],[19],[22]) encoded in the so-called tidal Love numbers, which are significant and (in principle) observable in the late stages of the inspiral phase of a merger. We have shown why they vanish identically for $D = 4$ static BHs by requiring regularity at the horizon. For Kerr's BHs, we have solved the radial Teukolsky equation in order to neatly obtain their non-vanishing response coefficient. It is purely imaginary and accounts for the dissipative effect caused by the friction of the rotating black hole with the static tidal field.

The other application focuses on the extraction of the black hole's QNMs when they settle down to an equlibrium state at the last phase of the coalescence. The impositions of suitable boundary conditions fixes the discrete set of frequencies. They dominate the ring-down signal, specially the fundamental mode, and they encode information about the resulting BH.

We are witnessing a revolutionary epoch where the study of GW signals is going to be a leading research topic [1]. BHs are excellent candidates to test GR predictions as well as natural places to search for discrepancies between those predicitons and others from modified theories of gravity. The upcoming Earth-based interferometers and even space-based interferometers such as LISA [31] will provide a unique opportunity to test with un unprecedented accuracy our theoretical predictions with regards to BHs and other compact objects. LNs and QNMs of BHs could be measured by these next generation interferometers, verifying the no-hair theorems and ellucidating their misterious nature. Moreover, the properties and structure of compact objects such as NSs could also be obtained by their TLNs, which could help us understand the properties of matter in extreme conditions of density and temperature [19],[32].

The observation of BHs through their GW emission will be crucial in the next decades in order to get further insights into the nature of gravity.

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