Dynamic behaviour of chaotic cellular automata

A comparative entropy analysis of regular lattices and small-world structures

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***Abstract*—This work investigates elementary cellular automata comparing their dynamic evolution on two topologies: the usual regular lattice and a small-world network constructed as a directed graph. The number of iterations for any initial perturbation to propagate over all cells of the automaton corresponds to how quickly an initial configuration reaches a high entropy state, and the temporal rate of propagation of the perturbation can be evaluated from a chaotic set of elementary cellular automata rules using an entropy measure. Results indicate that the average entropy can be nearly tripled in the small-world network topology, suggesting much faster applications (*e.g.* in Cryptography) from alterations in the topological arrangement of usual cellular automata regular lattices.**

Keywords-cellular automata; dynamic behaviour; small-world network.

I. INTRODUCTION

Cellular automata (CA) are mathematical models of complex systems with a large number of identical components and local interactions [1]. CA has applicability in various fields of knowledge, from Sociology to Simulation of Physical Systems [2].

The standard connection structure of cellular automata is a regular lattice, which is convenient for the definition of updating rules and analysis of regularities, but causes the propagation of a disturbance between its cells to become limited and conditioned on the number of iterations. Some studies have been conducted in order to explore other connection structures [3-9], as a matter of fact, recent studies indicate that depending on the structure of links and rules, generalized network automata (GNA) are more efficient than the usual lattice structures for solving classical CA problems, such as the problems of density classification and synchronization task [6-8]. Unlike CA, which have dynamic behavior determined solely by the transition rules, the dynamics of GNA also depends on the graph, making the study of GNA particularly challenging. The evolution of networks with small-world connections using rules known as "majority rules" or a variation of this type of rule were studied in [4-7], however a limitation found in those works is the difficulty in exploring the rules space such as they are exploited by traditional CA. This limitation is related to the varied and loosely controlled pattern of connections for each network element in such structures. We propose here the use of a set of small-world networks that use the same rule space of standard CA. The primary objective was to compare the dynamic behavior of elementary transition rules in the evolution of CA with a regular connection structure and small-world topologies. We performed a modification in the construction of the network proposed by [3], so that all elements have the same degree, allowing its evolution using standard CA transition rules. Aiming to evaluate the CA dynamic behavior, experiments were performed calculating the lattice entropy in order to measure how much a disturbance affects the states of the cellular space: networks capable of propagating a disturbance within a few steps of evolution reach higher values of entropy. The main motivation for the present work is therefore analyzing the possibility of reducing the number of evolution steps needed by CA applications only by changing the connection structure, enabling applications that require perturbation propagation (e.g., Criptography) to become faster.

The paper is structured as follows. Section II describes basic concepts of CA. Small-world networks are presented in Section III. Section IV explains the process of building small-world networks suitable to our purposes. Section V introduces concepts of elementary CA transition rules. Experiments, analysis of the results and discussions are in Section VI, and Section VII presents the concluding remarks.

II. CELLULAR AUTOMATA

CA are defined by their cellular space and transition rules. The cellular space is formed by a grid of N cells in a d-

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dimensional lattice. Each cell in the lattice takes one of the possible states at any given time. The transition rule is a mapping using neighboring cells to determine the new state of the central cell of the neighborhood. One time step or iteration corresponds to the application of the transition rule to all the cells of the lattice, thus resulting a new lattice configuration. Typically, a CA lattice is subject to the transition rule for several time steps, producing a temporal evolution of the CA.

We can distinguish various types of cellular automata by the following characteristics:

- Timing: if all cells evolve at the same time step, the CA is called synchronous (asynchronous otherwise).
- Homogeneity: if all cells evolve using the same transition rule, the CA is called homogeneous (heterogeneous otherwise).
- State Set Cardinality: if cells only assume the states ON or OFF (1 or 0), it is called a binary CA.
- Boundary condition on edge cells: among the possible boundary conditions, we can have (a) periodical, if edge cells are neighbors (torus-like structure); (b) fixed, if the values of the edge cells are repeated to complete a neighborhood; (c) reflective, if the edge states behave like a mirror, (d) null, if we consider edge cells neighbors to some of the possible valid states of the CA (state 0 being the most common use for a null binary CA).

Regarding the connection structure (topology) of CAs, it is possible to consider different types of regular geometry (linear, square, triangular, hexagonal, among others), the most common being a linear structure for one-dimensional CAs, square for two-dimensional CAs, cubic for three dimensional CAs, and so forth. Thus, the concept of neighborhood depends on the structure to be employed and on the radius r, a parameter which defines the access to neighboring cells. A more detailed survey on CAs can be found in [1].

A. *Elementary Cellular Automata*

The simplest CA is the elementary cellular automaton, characterized as being synchronous, homogeneous and binary. Topologically, it is a linear lattice with radius $r = 1$ and dimension $d = 1$. From this point on, we consider this kind of automaton as the canonical CA structure. The neighborhood of an elementary CA is formed by only 3 cells (left, center and right cell), so there are only 8 (2^3) different neighborhoods and 256 (2^{2^3}) possible transition rules.

A cellular automaton A is defined [6] by the 4-tuple

$$A = (\Sigma, U, d, f), \tag{1}$$

where Σ corresponds to the finite set of possible states, U is the cell neighborhood, $d \in Z^+$ is the size of A, and f is the local transition function responsible for the transition of states in the evolution of the CA, also called the transition rule.

For elementary CA, the set of possible states is $\Sigma = \{0, 1\}$. The neighborhood of a cell with state c^i is:

$$U^i = (c^{i-1}, c^i, c^{i+1}), \tag{2}$$

where i corresponds to the cell position in the lattice. The transition function f maps the state of a cell c^i at i to another state in Σ ($f : \Sigma^n \rightarrow \Sigma$), according to the neighborhood U^i. As the elementary CA is homogeneous, the same transition rule is applied for all cells. Considering a cellular space of size N, the CA configuration at a given instant of time t is defined by

$$C_t = (c_t^0, c_t^1, \cdots, c_t^{N-1}), \tag{3}$$

where $c_t^i \in \Sigma$ corresponds to the state of cell i at time t. The dynamic evolution of the CA is performed by the global operator Φ, responsible for mapping a configuration C_t to the next time step (Φ: $C_t \rightarrow C_{t+1}$), through the simultaneous application of the local transition function f by every cell in the lattice, *i.e.* $C_{t+1}^i = \Phi(U_t^i)$.

Fig. 1 shows an example of elementary transition rule. It presents the mapping for all possible neighborhoods and their respective output bits (the next state of the central cell). Usually, CA transition rule are represented by the binary string correspondent to its output bits; in this example, <01111000>. Alternatively, it can be represented by the decimal number corresponding to the conversion of the output bits when read from the right (neighborhood 111) to the left (neighborhood 000). In this example, $(00011110)_2 = (30)_{10}$, the transition rule in Fig. 1 is called rule 30.

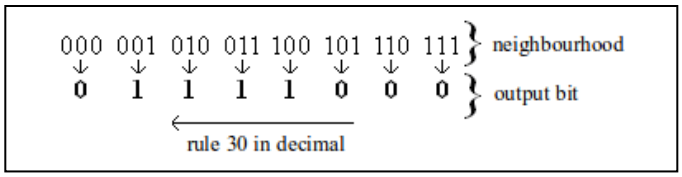

Figure 1. Transition rule <01111000> or rule 30.

B. *Generalized Automata Networks*

The definitions of the CA connection structure given above is generalized to arbitrary graph topologies in the so called Generalized Automata Networks (GAN) [6], Network-based Cellular Automata (NBCA) [8] or Networked Cellular Automata (NCA) [9].

A GAN is defined as a 4-tuple

$$R = (G, \Sigma, U, \{fi \mid i \in V\}), \tag{4}$$

where $G = (V, E)$ is a graph with a set V of vertices or nodes and a set E of (possibly directed) edges or links that define binary relations on V.

Generalizing a CA to a GAN causes the neighborhood size, which is always constant for a standard CA, to have different

sizes, since nodes can have different degrees. Thus, the typical rules used in CAs usually cannot express GANs relations, and as a consequence studies on GANs tend to more strongly consider topological aspects involving concepts of graph theory and complex networks. Initial work was on random GANs (that is, random graphs), and currently focus has been on the so called small-world ([3-7]) and scale-free [6] topologies.

III. Small-World Networks

Small-world networks, originally proposed by [3], correspond to an alternative and intermediate model to random and regular networks. The term small-world appeared in analogy to the famous conjecture [10] on the six degrees of separation, which suggests that the average distance between any two persons in a society is approximately of six links (a link identifying an acquaintance relationship between two individuals in the social graph).

The process of constructing a small-world network presented by [3] initiates from a ring-like one-dimensional regular lattice. Initially, each node i has r connections with neighboring elements to the left and to the right, which together add up to the degree of the vertex $z^i = 2 * r$. Then, with probability p, a reconnection is created for each connection in the network, under two single restrictions: (i) self-connections are not allowed and (ii) multiple connections between the same nodes are not allowed. Experiments in [3] showed that it is possible to obtain networks with small-world characteristics even with a low probability of reconnection ($p \approx 0.01$).

Small-world networks constructed as such have a low average length of the shortest internodes path (Average Path Length – *APL*) and a high clustering coefficient (Clustering Coefficient - *CC*) between network elements. The clustering coefficient of a network is defined as

$$CC = \frac{1}{n}\sum_{i=1}^{n}\tau^i \,, \tag{5}$$

where n is the number of nodes and τ^i is the fraction of neighbors of i that are neighbors themselves:

$$\tau^i \equiv \frac{\left|\{jk \in E : ij \in E \wedge ik \in E\}\right|}{\frac{z^i\left(z^i-1\right)}{2}}, \tag{6}$$

If $z^i < 2$ we define ($\tau^i = 0$). The average path length is defined as

$$APL = \frac{1}{n(n-1)}\sum_{i,j} d_{ij} \,, \tag{7}$$

where d_{ij} is the minimum distance between nodes i and j.

On average, random networks produce low values for *APL* and *CC*, and regular lattices reach high values for *CC* and *APL*. Small-world networks give rise to high values for *CC* and low values for *APL*.

IV. Building Small-World Networks for Neighborhood Structure from Elementary CA

The process of constructing small-world networks proposed by [3] produces networks with varying degrees of connectivity between elements, difficulting the investigation of the dynamic behavior of standard CA rules in a GAN based on such topologies. On the other hand, we would like to explore the low *APL* characteristics of small-world topologies for CA applications such as Cryptography [13-14].

Aiming to enable the use of CA rules, we propose a change in the process of building the network based on the use of directed node links. The idea is to consider a reconnection scheme that does respect the constraints of the reconnection probability p, self-connections and multiple connections, therefore not interfering with the resulting node output degrees. The neighborhood of an element is constructed from its outgoing arcs, however, an element i can belong to the neighborhood of an element j while the other way around is not true. The main difference from the construction process proposed by [3] is therefore the use of a directed graph. The whole process is defined as follows:

- Build a directed regular ring lattice with n nodes and k neighbors per node, $k / 2$ on each side of any node. That is, considering the node set $v_0, v_1, \ldots, v_{n-1}$, there is an arc (v_i, v_j) if and only if $0 < |i - j| \ mod \ n <= k / 2$. It is important to notice that $(v_i, v_j) \neq (v_j, v_i)$.
- For each arc in the graph repeat: keep the origin of the arc unchanged and, with probability p, change the destination to a vertex chosen uniformly at random over the entire ring, obeying the same restrictions proposed by [3] relative to self-connections and multiple connections. That is, rewiring is done by replacing (v_i, v_j) with (v_i, v_k) where k is chosen with uniform probability from all possible nodes that avoid self-loops ($k \neq i$) and arc duplication.

One of the major motivations to investigate the dynamic behavior of the target networks, as proposed, is that they enable the use of the same set of rules of traditional CAs since it keeps a regular neighborhood size. This characteristic allows easy adaptation to various applications of CAs already developed. In the remainder of the paper we will refer to theses networks as Directed Small World Cellular Automata (DSW-CA).

V. Elementary Rules

Due to their neighborhood with 3 cells, the rule space of elementary CA is formed by 256 different transition rules. Wolfram [1] proposed a classification for the dynamic behavior of those 256 elementary rules into four classes: fixed point (Class 1); cyclic or periodic (Class 2); chaotic (Class 3) and complex (Class 4). While not all researchers agree on the proposed classification, it is currently the most used in the literature.

The dynamical behavior of Class 3 is of special interest here since such rules are very sensitive to their initial lattice configuration. Given an initial configuration C_0 and its modified version C_0' - being that C_0 and C_0' differ in a single bit – it is expected that their temporal evolution generate substantially different lattice configurations after few steps. Due to this characteristic, Class 3 rules are widely employed in CA-based cryptographic systems [12-16].

Some elementary rules can be represented by additive logical operations. For example, the transition rule <01011010>, or rule 90, can be alternatively represented by the logical operation ($c_{t+1}^{i} = c_t^{i-1} \oplus c_t^{i+1}$). If a transition rule can be represented using the logical operator XOR, it is called a linear rule. Similarly, it is a complementary rule if it can be represented using the logical operator NXOR. Non-additive rules involve AND or OR logical operators on their descriptions [15].

A particular characteristic in some CA transition rules is related to the sensitivity to a specific cell of the neighborhood [12]. In such rules, any modification in this sensitive cell necessarily provokes a change in the state of the center cell in the next time step. We considered with particular attention those rules sensitive to the external bordering bits of a neighborhood, the so called left-toggle (or right-toggle) rules [12]. Fig. 2 shows right-toggle (a) and left-toggle (b) elementary rules and highlights the sensitive bit of the neighborhood. In general, such rules exhibit a chaotic behavior.

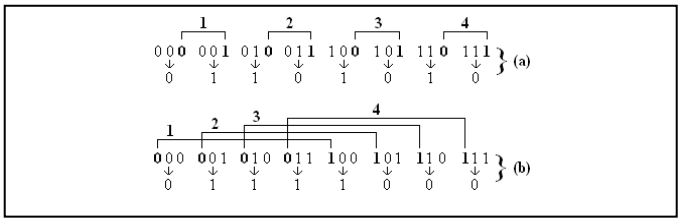

Figure 2. Toggle rules: (a) sensitivity to the right and (b) to the left.

Toggle rules are commonly employed in cryptographic methods based on CA pre-image calculus [12-14], while linear and complementary rules are employed in cryptographic methods based on algebraic properties of additive CA [14-16].

VI. EXPERIMENTS

Experiments were carried out in order to measure how the small-world-based network used to define the neighborhood can influence the dynamical behavior of cellular automata. The metric used to calculate the entropy is the classical one proposed by Shannon [11], and the experiments consist of measuring the entropy propagated from an arbitrarily small perturbation applied in the initial configuration. Given a configuration C_0 and its corresponding perturbed version C_0', we obtain the difference between the changes using an exclusive-or (XOR) operation. Finally, we perform the calculation of the entropy S over the resulting configuration. This entire process is formally represented by

$$S_t = \lambda\left(C_t \oplus C_t^{'}\right), \tag{8}$$

where λ is the operator responsible for calculating the entropy, $\oplus$ is the XOR operator and t represents the iteration step.

Fig. 3 shows an example of the comparative evolutions performed in our experiments. In the first temporal evolution (Fig. 3a) the initial configuration (arbitraly chosen) is C_0 = <11000001110000000000011110110101>. Rule 30 was applied over the lattice for 16 time steps using the standard CA neighborhood structure (p = 0) until the last configuration was obtained. In the second temporal evolution (Fig. 3b) the initial configuration was complemented in only the 15th bit (c^{14}) generating C_0' = <11000001110000**1**00000011110110101>. Rule 30 was also applied starting from this initial configuration for 16 time steps using the standard CA neighborhood. When a CA rule has a good perturbation spread, as rule 30 in Fig. 3, the configurations become more different as time goes by.

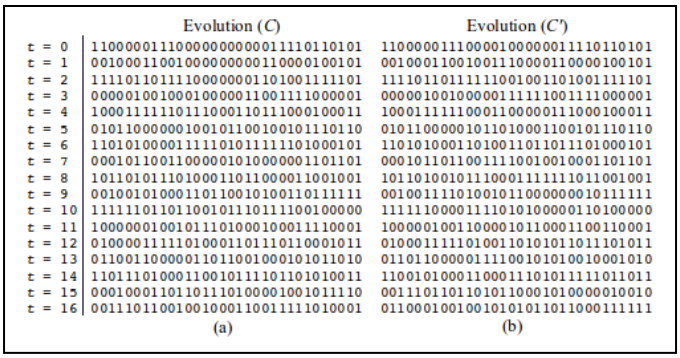

Figure 3. Temporal evolutions of Rule 30 starting from two similar initial lattices: (a) C_0. (b) C_0' (identical to C_0, except for the 15th cell).

This behaviour is highlighted in Fig. 4, which presents the lattice resultant from the XOR operation $C_t \oplus C_t'$, in which bit 1 represents a cell with different values comparing C_t and C_t' and bit 0 represents a cell with the same value in both evolutions. It is possible to observe that modifications on the cell states spreads from the center of the lattice (15th cell) growing to the borders as the transition rule is applied over time. Fig. 4 also presents the respective lattice entropy calculated for each time step. It starts from a very low entropy (S = 0.197613) correpondent to the configuration with only one bit 1 to high values of entropy (above 0.8), indicating that the perturbation spreads well over the lattice. A simple experiment was performed to verify the entropy level associated to random binary strings. A sample of 10,000 32-bits strings was randomly generated and their entropies were calculated obtaining an average of 0.841191 with a standard deviation of 0.002262. Thus, we concluded that for the comparative evolution showed in Fig. 4 a entropy level above 0.8 can be considered a good pertubation spread. It was obtained in step 12 (S = 0.815564). A similar experiment was performed using a different structure for the neighborhood, now based on a small-world network DSW-CA constructed with a low probability of reconnection (p = 0.1). Rule 30 was applied again for the temporal evolution and the same initial configurations C_0 and C_0' were used, obtaining similar evolutions to Fig. 3. Fig. 5 is analogous to Fig. 4 and presents the resulting configurations from the XOR operation $C_t \oplus C_t'$, and the respective entropies. As in Fig. 4, it is possible to observe the perturbation spreading over the lattice. However,

the small-world-based neighborhood ($p = 0.1$) provokes a faster spreading of the perturbation as it needs only 6 time steps to reach a entropy level above 0.8. Therefore, DSW-CA can be more efficient than the standard CA w.r.t. spreading the perturbation.

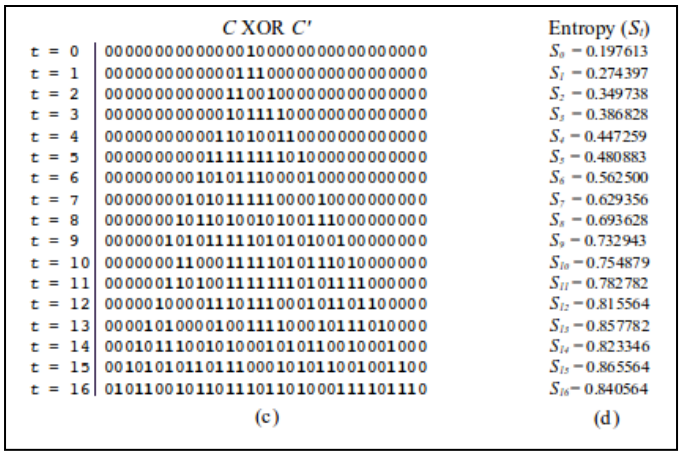

Figure 4. Perturbation spreading by comparing two temporal evolutions C and C' with the respective entropies (traditional CA neighborhood $p = 0$).

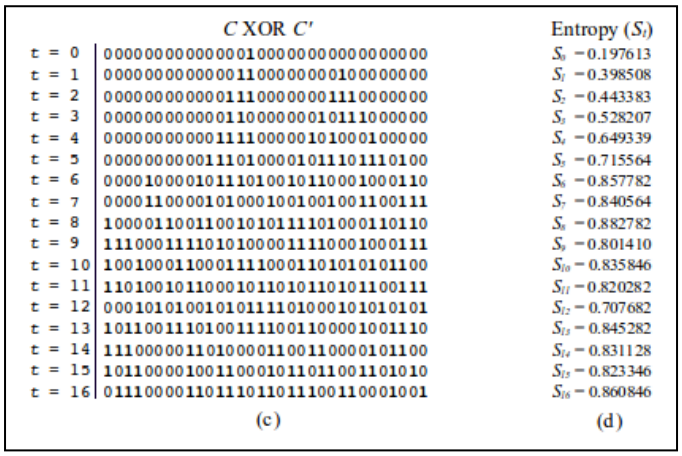

Figure 5. Perturbation spreading by comparing two temporal evolutions C and C' with the respective entropies (DSW_CA neighborhood $p = 0.1$).

A second series of experiments was performed to confirm DSW-CA faster perturbation spreading using a more robust scenario. A set of 500 128-bit initial lattices was randomly generated and a set of 12 rules was selected from the elementary rule space - Class 3 [3], non-linear, elementary toggle rules: 30, 45, 75, 86, 89, 101, 106, 120, 135, 149, 169 and 225. A set of 20 network-based neighborhoods for DSW-CA was generated for each one of 51 different probabilities of reconnection (p = {0, 0.01, 0.02, ..., 0.5}), totaling 1,020 networks. Each network was evolved using each one of the 12 selected rules over the 500 128-bit initial configurations. Table I summarizes the parameter settings. Calculation of the entropy was carried out in specific stages of DSW-CA evolution (t = {7, 14, 28, 64, 96, 128}). Given a network i, a time t, the set of rules j (= 1, 2, …, 12) and the initial configurations k (= 1, 2, … , 500), expression (9) was used for calculating the entropy:

$$R_{it} = \frac{1}{12}\sum_{j=1}^{12}\left(\frac{1}{500}\sum_{k=1}^{500} S_t\right). \quad (9)$$

Finally, we averaged R_{it} over 20 networks for each reconnection probability p = {0, 0.01, 0.02, …, 0.5}, thus generating an *average entropy index* $\langle R_t \rangle$ for a given reconnection probability of a network over the set of all possible rules and initial configurations.

TABLE I. PARAMETERS SETTINGS

	Exp. 1	**Exp. 2**	**Exp. 3**
Network	1 ($p = 0$)	1 ($p = 0.1$)	1,020 (p = {0 - 0.5})
Rule	1 (rule 30)	1 (rule 30)	12
Initial Lattice	1	1	500
Lattice Size	32-bit	32-bit	128-bit

An efficiency metric [17] was also considered for this set of experiments. It measures the global communication efficiency (E_{glob}) considering that the communication efficiency between any two vertices is inversely proportional to the length of the shortest path between them:

$$E_{glob}(G) = \frac{1}{n(n-1)}\sum_{v_i \neq v_j \in G}\frac{1}{d_{ij}}. \quad (10)$$

where G is a n nodes graph and d is the minimal distance between nodes v_i and v_j.

Experimental results can be seen in Fig. 6, where the abscissa represents the probability p of reconnection and the axis of ordinates represents the average entropy index $\langle R_t \rangle$ and the overall efficiency (E_{glob}) of the network. Results in Fig. 6 indicates that the standard CA ($p = 0$) needs more than 64 steps to reach an average entropy above 0.8. With $t = 96$ the average entropy is 0.808104 and with t = 128 it is 0.862469 (for reference, an additional experiment was performed where 10,000 128-bits strings were randomly generated and their entropies were calculated, obtaining an average of 0.883425 with a standard deviation of 0.000311). However, the average entropy 0.639255 at t = 64 indicates an intermediate level of entropy/spreading. And for a few time steps (t = 14 or t = 28) the standard CA was not able to propagate the perturbation since the entropy was below 0.4. In fact, for a small number of steps the effect of the small-world-based neighborhood is remarkable. Using t = 14, the entropy average jumps from 0.206006 ($p = 0$, standard CA neighborhood) to 0.615409 (p = 0.5, reconstructed network with 50% of rewiring probability). Using $t = 28$, it goes from 0.333543 ($p = 0$) to 0.604793 using a 10% rewiring probability ($p = 0.10$). Such results indicate that reconnections in cell neighborhoods using a small-world topology can indeed induce faster spreading of perturbations using few time steps of evolution. Table II highlights these results. Notice that a CA with 50% rewiring probability (p = 0.5 and t = 14) can produce results near to standard CA ($p = 0$ and $t = 64$), saving 50 steps of evolution.

TABLE II. EVOLUTION TIME VERSUS AVERAGE ENTROPY

Time (*t*)	Average Entropy		
	p = 0	***p = 0.1***	***p = 0.5***
t = 14	0.206006	0.413596	0.615409
t = 28	0.333543	0.604793	0.634088
t = 64	0.639255	0.655398	0.628847

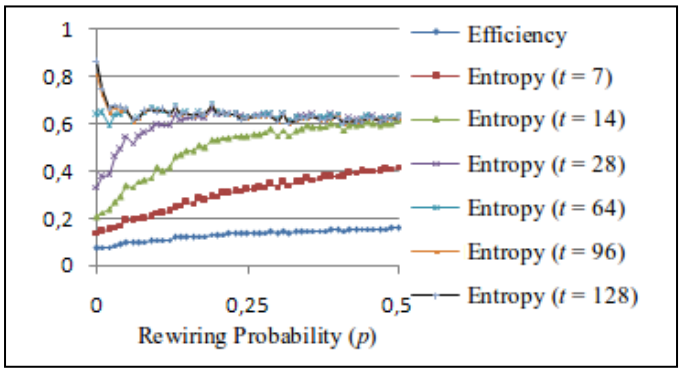

Figure 6. Average Entropy Index < R_t > calculated for t = {7, 14, 28, 64, 96, 128} and Global Communication Efficiency (Eglob) varying the rewiring probability in the neighborhood network from 0 to 0.5.

It was also possible to observe a tendency to obtain entropy averages next to 0.6 using small-world-networks (DSW-CA). A good level of spreading in 128-bit strings should return entropies above 0.8 as it was possible to observe using a standard CA and t = 96. Observing the individual values of entropy used to compose the average, it was possible to detect that although several evolutions have indeed reached an entropy level above 0.8, there are some DSW-CA evolutions which returned entropy next to 0, indicating that the initial perturbation was not propagated. These particular evolutions decrease the entropy average to around 0.6. Preliminary results indicate that this behavior is explained by two phenomena that are under scrutiny in our current work: a) the resulting network after rewiring can get disconnected, producing isolated components (islands) and a perturbation in one cell of an island will not propagate to cells of other components; b) some cells of the lattice after rewiring may not maintain any exit edge (only enter edges), and an eventual modification in its state will not affect any other cell (such cells works as barriers to perturbation propagation). Finally, it was possible to note a relation between E_{glob} and the entropy values returned by DSW-CA, indicating that the higher the communication efficiency, the faster the propagation of a single perturbation over the lattice, enabling us to apply fewer time steps.

VII. CONCLUSIONS

Inspired by [4-6, 8], we presented a variation in the construction process of small-world networks that generates structures that evolve from CA rules. We focused here in a particular set of elementary (toggle) rules, which have chaotic dynamic behavior and are applied in cryptographic systems [12-14]. Experiments showed that, in a short period of time, the propagation of a perturbation is more effective in small-world structures than in regular lattices. In fact, due to their shortcutting connections, small-world networks produced a faster propagation of a disturbance compared to regular structures, suggesting that some applications of CAs could have a reduced number of iterations – and an economy of computing resources – by replacing its lattice structure by a small-world topology. For the case of directed models such as those presented here, examples of application are the cryptographic systems proposed in [12-14] or those that use the dynamical behavior of CA [6-8]. However, it is important to point out that rewiring must be made in a way that the structure for running the application is not disturbed. For the case of these cryptographic systems, the local topological arrangement for calculating pre-images must be kept intact. For future work we will design a cryptographic system based on CA rules on a small-world network whose topology is globally effective w.r.t. perturbation propagation, and locally controlled in such a way that pre-images can be calculated. Promising initial results have been obtained and will be presented in another article.

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