US GDP & Inflation

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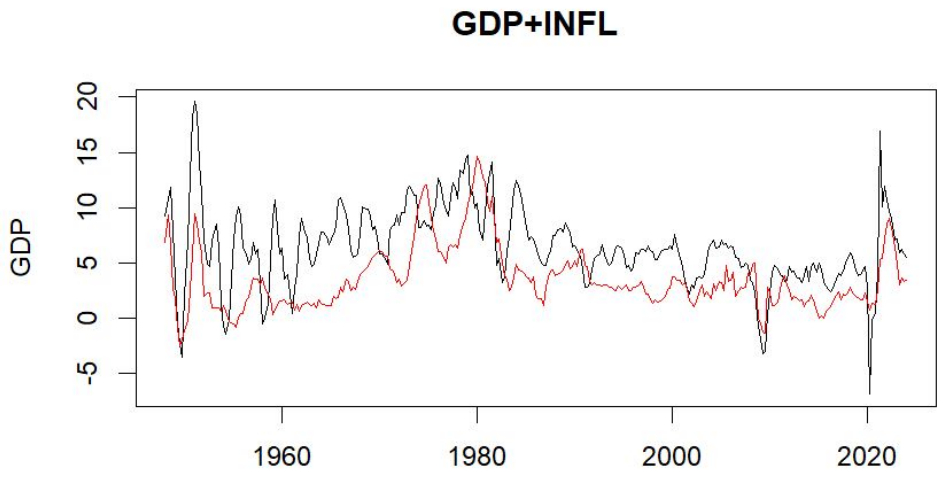
1. The dataset

Objectives of our analysis

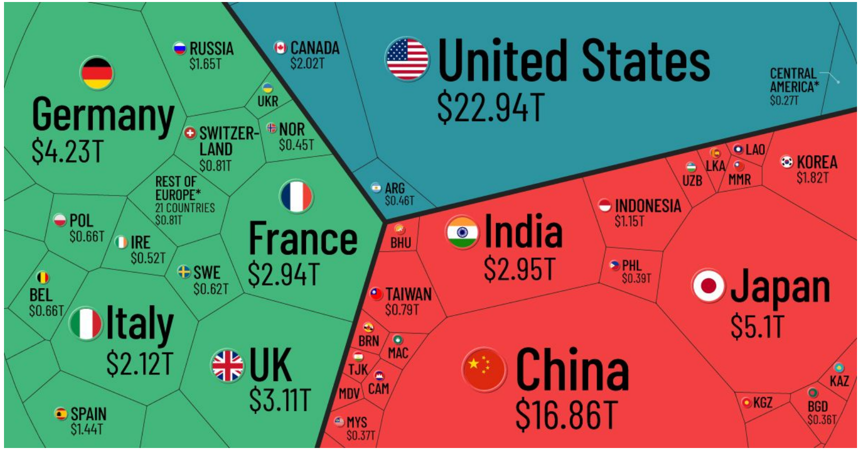
- Analyzing and modelling time series of these two key economic seen for the United States.

- Comparing the performance of these models using Information Criteria such as Bayesian Information Criterion (BIC), Deviance Information Criterion (DIC), and Watanabe-Akaike Information Criterion (WAIC).

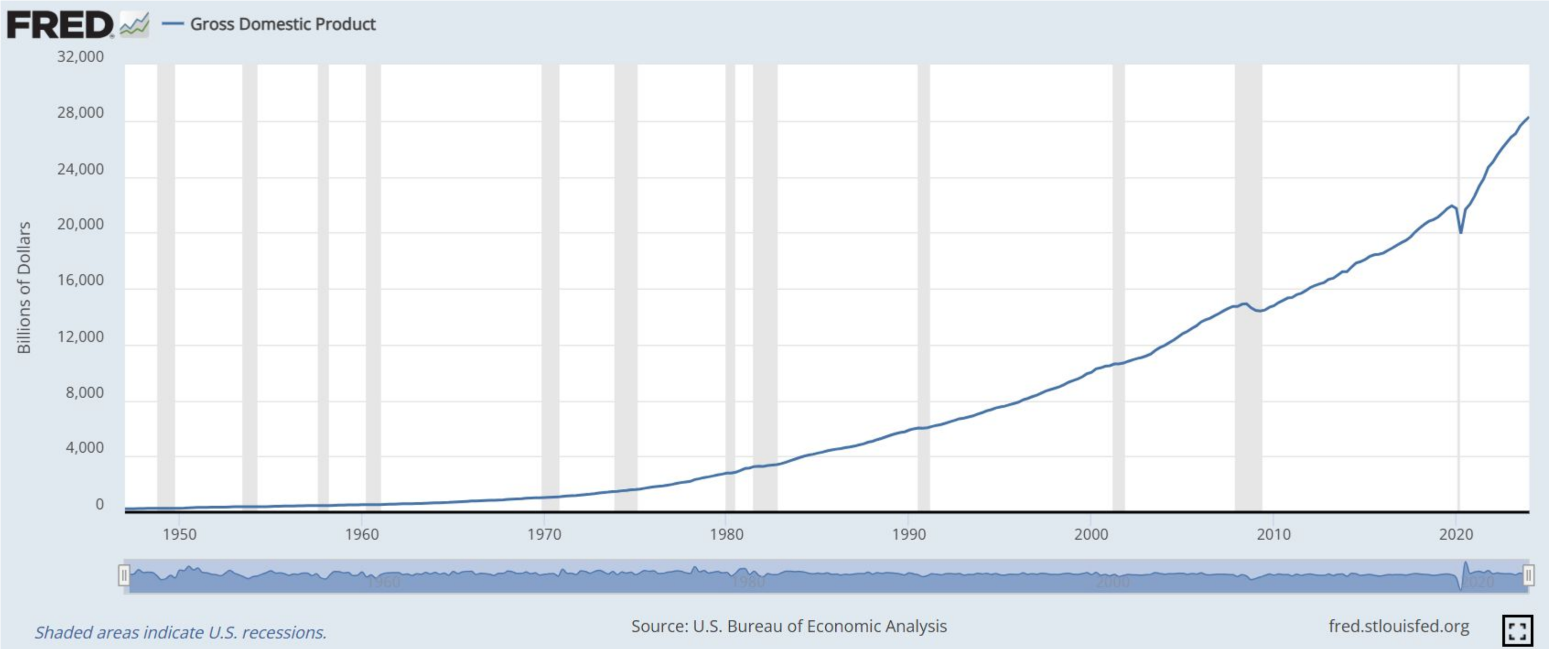


What is GDP?

- GDP is the market value of all goods and services produced by labor and property located in the United States.
- It is the principal measure of U.S. manufacturing and is used to evaluate the country's economic performance.
- GDP data is seasonally adjusted to eliminate the effects of seasonal variations



Original Gross Domestic Product Dataset



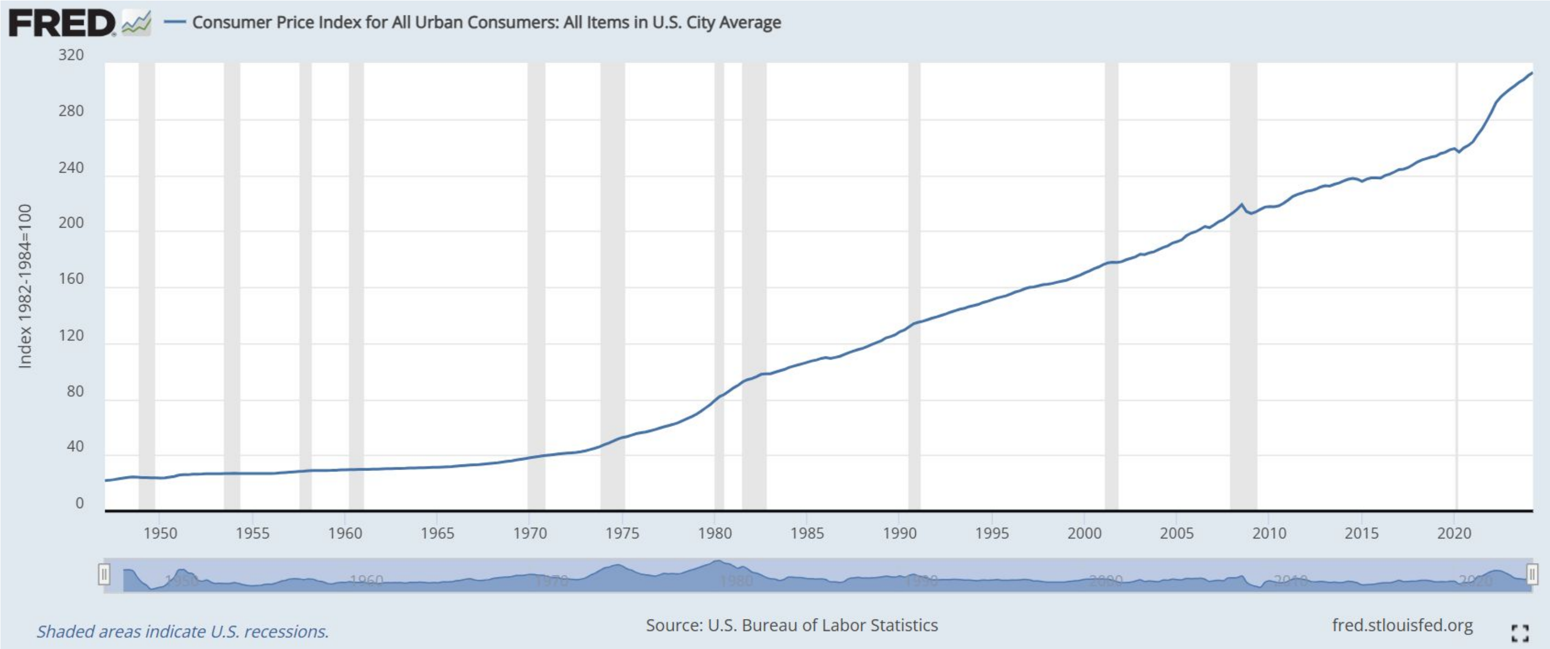
The data for this analysis is sourced from the Federal Reserve Economic Data (FRED) with series HSN1F for GDP. The dataset includes quarterly observations from 1948-01-01 to 2024- 01-01.

What is Original Consumer Price Index

- The CPI measures the average change over time in the prices paid by consumers for a market basket of goods and services.

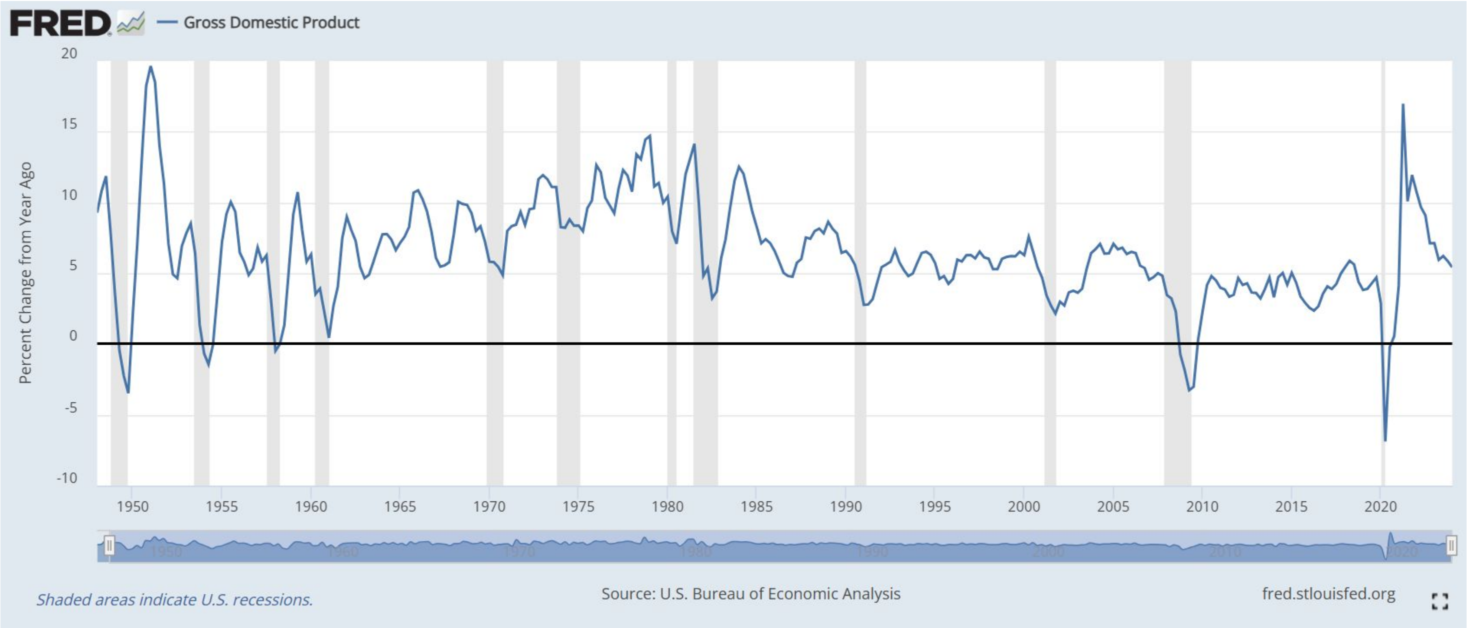
- The CPI is also a measure for understanding inflation trends. The index is adjusted to remove the effects of seasonal changes

Original Consumer Price Index Dataset



The data for this analysis is sourced from the Federal Reserve Economic Data (FRED) with series CPI-AUCSL for the Consumer Price Index. The dataset includes quarterly observations from 1948-01-01 to 2024- 01-01.

GDP Dataset with transformations



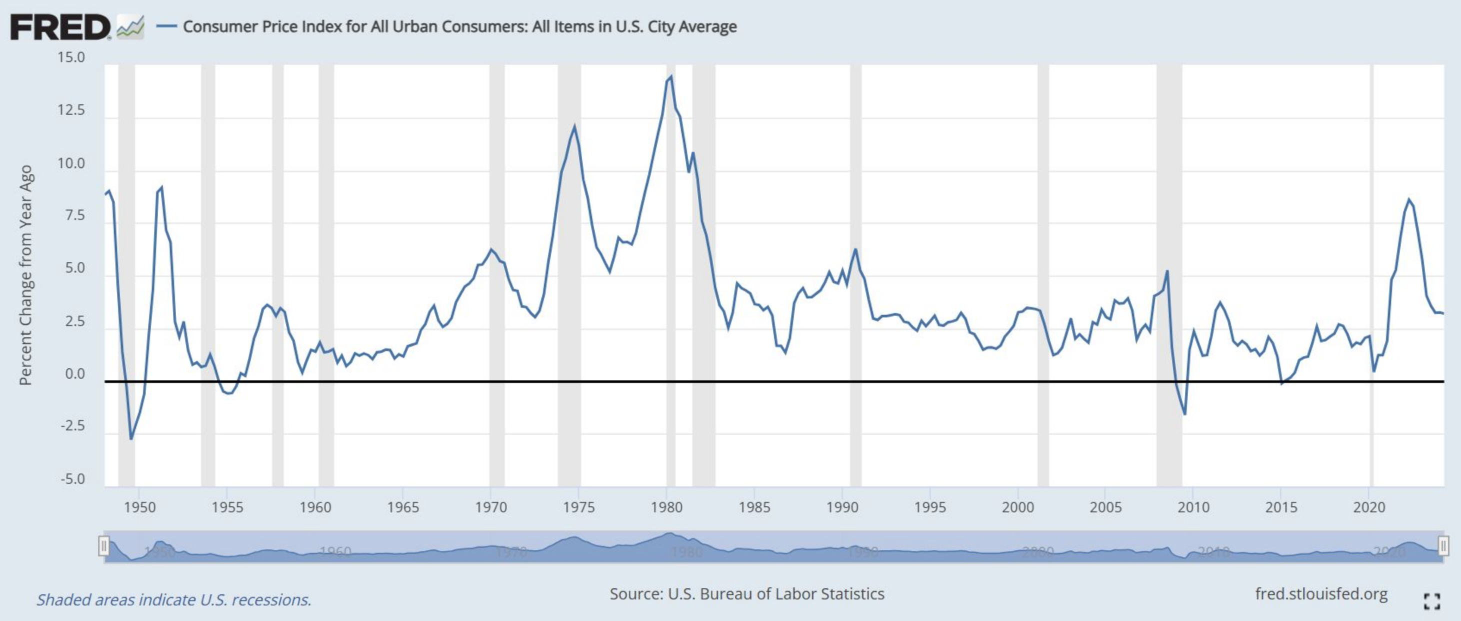
The version of the data we will be using is the percent change from the year immediately before.

Percent change from previous year

$$\left(\left(\frac{x_t}{x_{t-n\_obs\_per\_year}}\right)-1\right)*100$$

Because of this transformation over the dataset, no additional logarithmic transformations were applied

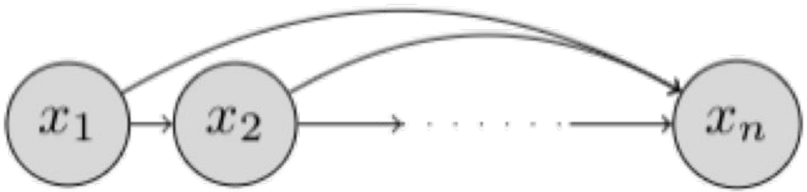
CPI Dataset with transformations



The version of the data we will be using is the percent change from the year immediately before.

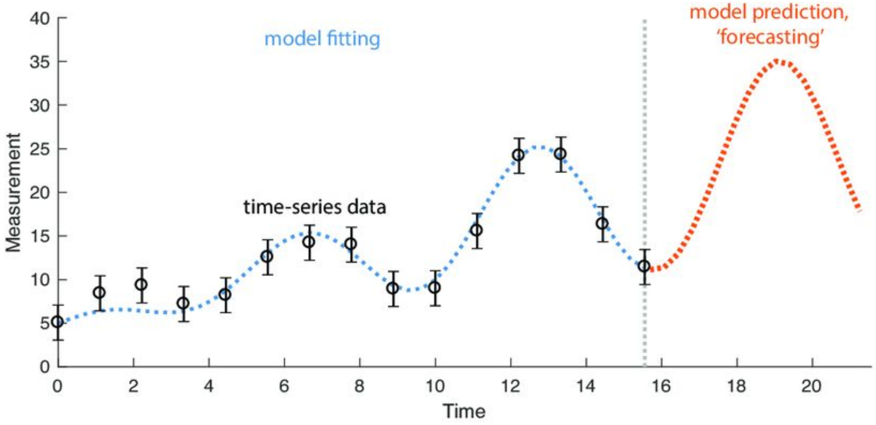
Proposed models

- Autoregressive (AR) Model with 1 and 2 time delays.



- Autoregressive Moving Average (ARMA) Model, with 1 time delay for both parts

- Vector Autoregressive (VAR) model



2. Models

AR(1)

The AR (Autoregressive) model uses only the past values of a series (lags) to describe the behaviour of a series in order to predict its future values.

$$x_t = c + \sum_{i=1}^{p} a_i x_{t-i} + \in_t$$

The AR(1) model uses only the previous observation of the time series to predict the next value. Although it directly uses just the immediate past value, it implicitly incorporates information from earlier observations as well. It is defined as:

$$y_{t+1} = \mu + \alpha y_t + \epsilon_t \qquad \epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$$

AR(1) specification

````
```{r}
modelAR.string <-"model {
  ## parameters: alpha,tau,m0
  #likeliohood
  mu[1]<-Y[1]
  Yp[1]=mu[1]
  LogLik[1] = 0
  for (i in 2:N) {
    Y[i] ~ dnorm(mu[i],tau)
    mu[i]<-m0+alpha*Y[i-1]
    Yp[i] ~ dnorm(mu[i],tau) # prediction in sample
    LogLik[i] <- log(dnorm( Y[i],mu[i],tau))
  }
  # prediction out of sample
  ypOut[1] ~dnorm(m0+alpha*Y[N],tau)
  for(k in 2:Npred){
  ypOut[k] ~dnorm(m0+alpha*ypOut[k-1],tau)
  }
  sigma2<-1/tau
  #prior
  alpha ~ dunif(-1.5,1.5)
  tau ~ dgamma(0.1, 10)
  m0 ~dnorm(0.0, 1.0E-4)
}"
```
````

$$Y_i \sim N(\mu_i, \tau)$$

$$\mu_i = \mu_0 + \alpha Y_{i-1}$$

$$\alpha \sim U(-1.5, 1.5)$$

$$\tau \sim \mathcal{G}(0.1, 10)$$

$$\mu_0 \sim \mathcal{N}(0, 10^4)$$

Parameter selection

$$\alpha \sim U(-1.5, 1.5)$$

Create a non-informative prior and include non-stationary models

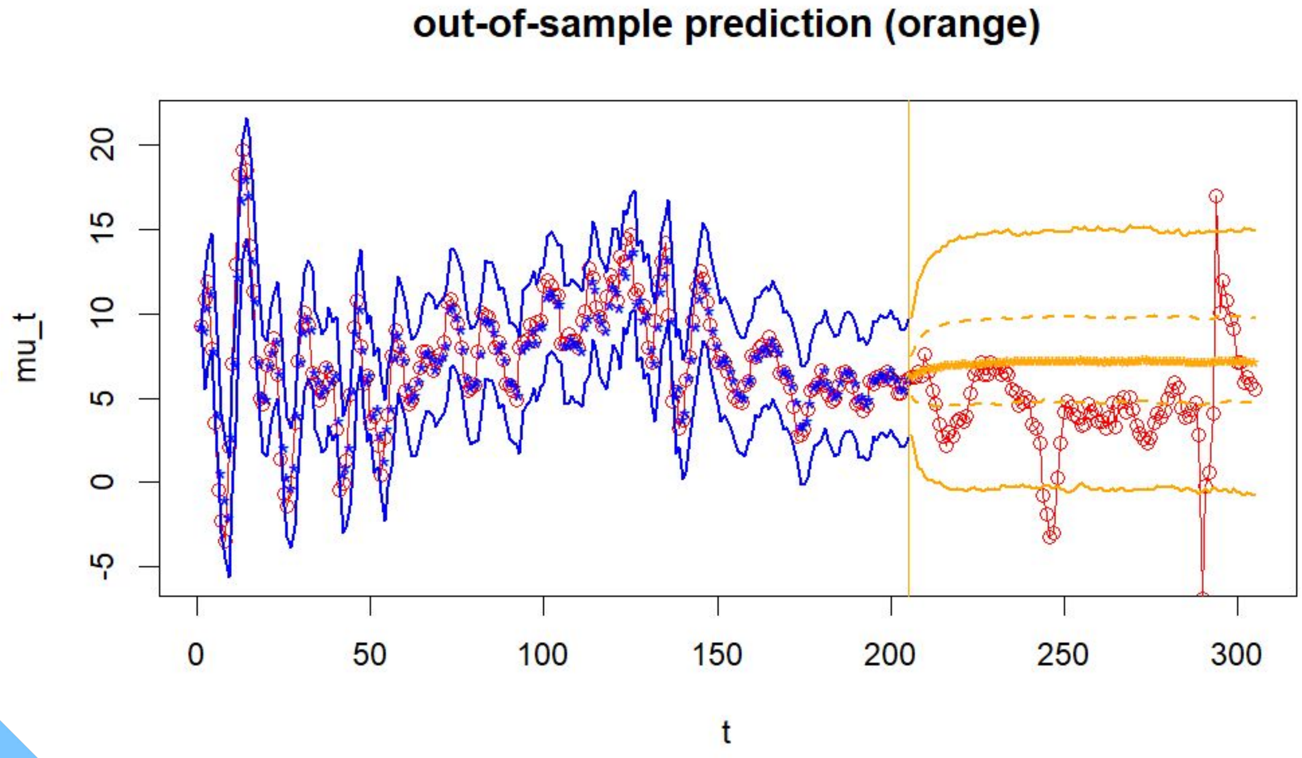
$$\mu_0 \sim \mathcal{N}(0, 10^4)$$

Normal prior for the intercept with very high variance since we have no prior information on the mean.

Simulating the model

```{r}

# prepare the data
gdpData = as.numeric(gdpData)
Ntot=length(gdpData)
Npred=100 # horizon for out-of-sample prediction
N=Ntot-Npred
data_subsample=gdpData[1:N]

line_data <- list("Y" =data_subsample,"N" = length(data_subsample),"Npred"=Npred)

outputmcmcAR_GDP <- jags(model.file=textConnection(modelAR.string),
                     data=line_data,
                     parameters.to.save=
c('alpha',"sigma2","m0","Yp","ypOut","LogLik"),
                     n.adapt=1000, n.iter=10000,n.chains = 1,n.burnin = 2000)

```

GDP Simulation results

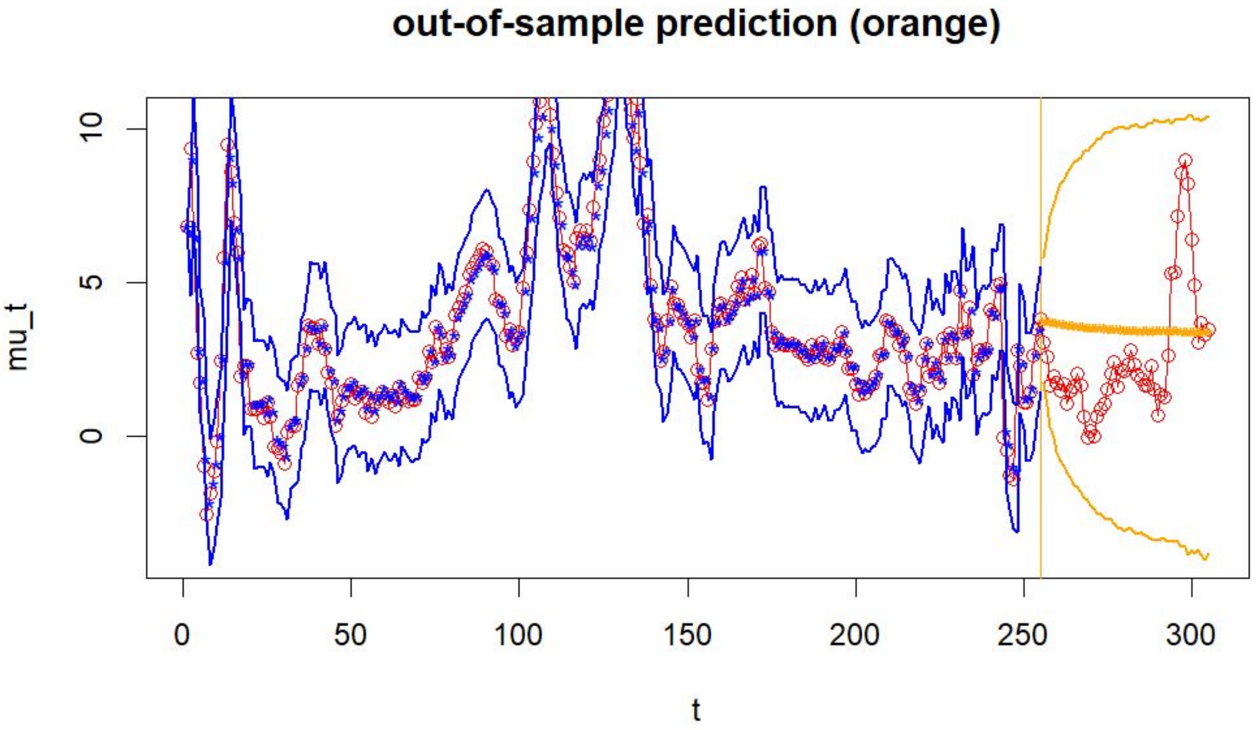


Red: Real data

Blue: In sample prediction

Orange: Out of sample prediction

CPI Simulation results



AR(2)

The AR(2) model takes the two previous observations to calculate the new possible value of the time series. This can be seen in the following formulation of the process:

$$Y_t = \alpha + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \cdots + \beta_p Y_{t-p} + \epsilon_t$$

defined recursively from y2 for t ≥ 2. The first 2 observations are taken from the dataset and then it is possible to start making predictions.

$$y_{t+1} = \mu + \alpha y_t + \beta y_{t-1} + \epsilon_t \qquad \epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$$

AR(2) specification

```
modelAR2.string <-"model {
  ## parameters: alpha,tau,m0
  #likeliohood
  mu[1]<-Y[1]
  Yp[1]=mu[1]
  mu[2]<-Y[2]
  Yp[2]=mu[2]
  LogLik[1] = 0
  LogLik[2] = 0
  for (i in 3:N) {
    Y[i] ~ dnorm(mu[i],tau)
    mu[i]<-m0+alpha*Y[i-1]+beta*Y[i-2]
    Yp[i] ~ dnorm(mu[i],tau) # prediction in sample
    LogLik[i] <- log(dnorm( Y[i],mu[i],tau))
  }
  # prediction out of sample
  ypOut[1] ~dnorm(m0+alpha*Y[N]+beta*Y[N-1],tau)
  ypOut[2] ~dnorm(m0+alpha*ypOut[1]+beta*Y[N],tau)
  for(k in 3:Npred){
    ypOut[k] ~dnorm(m0+alpha*ypOut[k-1]+beta*ypOut[k-2],tau)
  }
  sigma2<-1/tau
  #prior
  alpha ~ dunif(-1.5,1.5)
  beta ~ dunif(-1.5,1.5)
  tau ~ dgamma(0.1, 10)
  m0 ~dnorm(0.0, 1.0E-4)
```

$$Y_i \sim N(\mu_i, \tau)$$

$$\mu_i = \mu_0 + \alpha Y_{i-1} + \beta Y_{i-2}$$

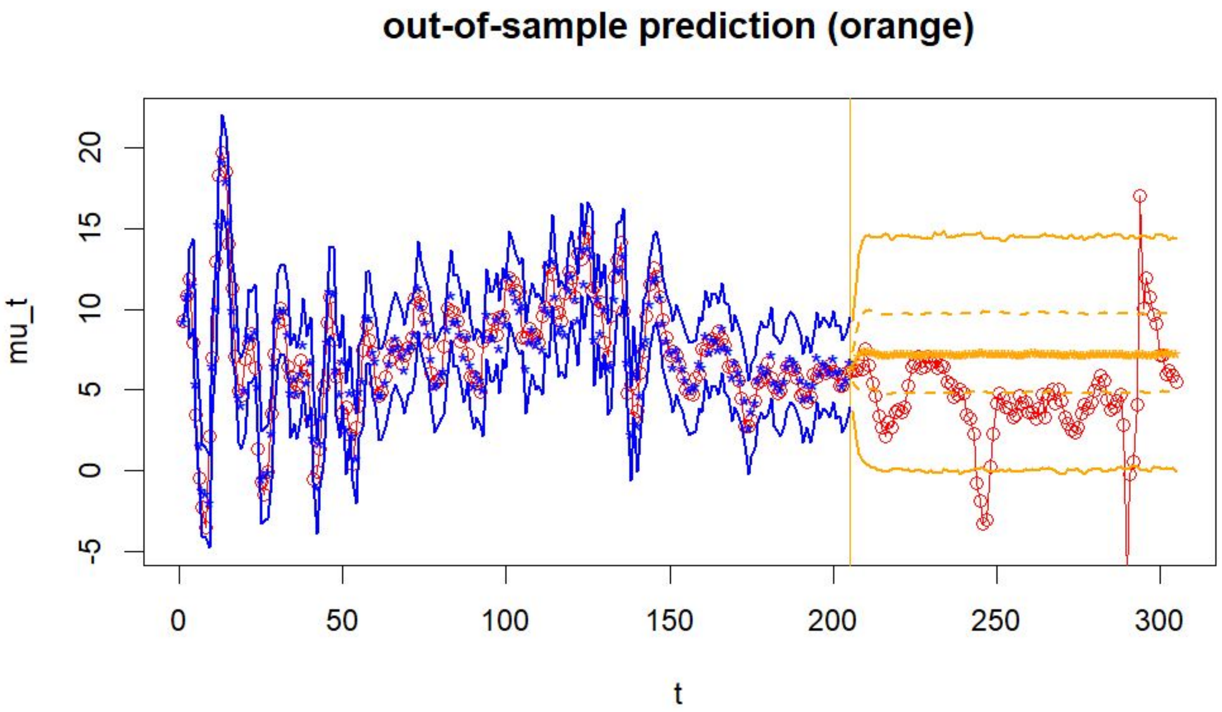
$$\beta \sim U(-1.5, 1.5)$$

$$\alpha \sim U(-1.5, 1.5)$$

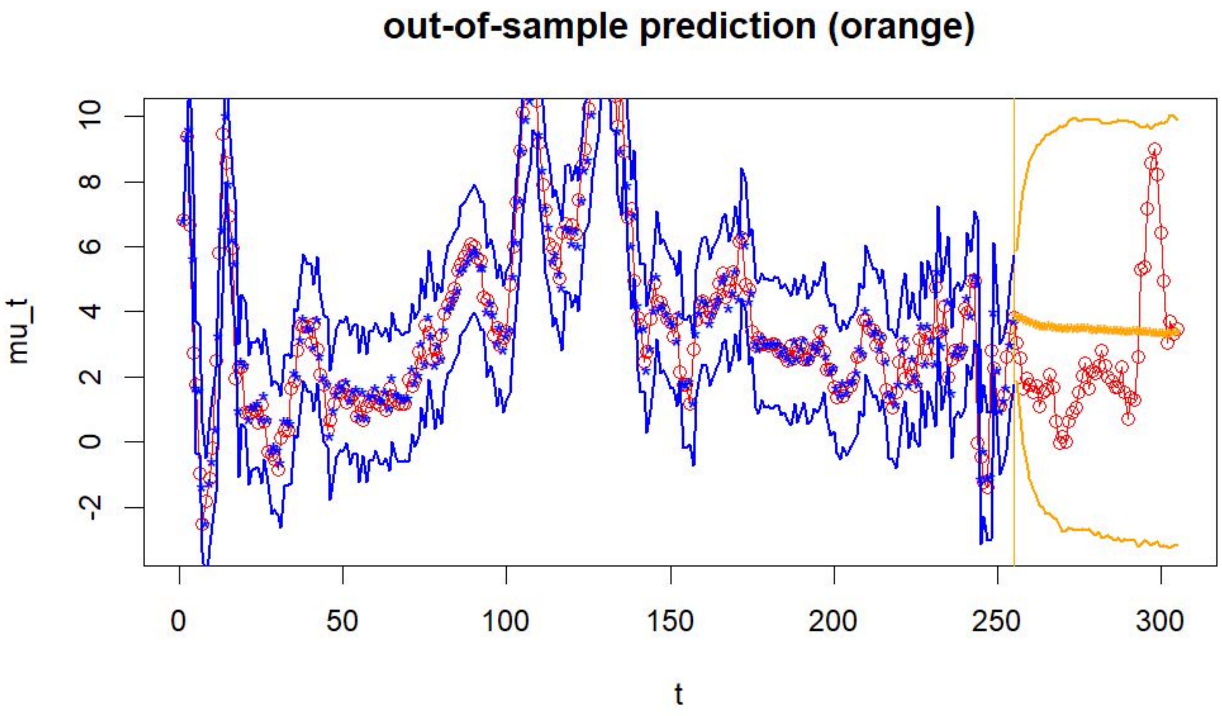
$$\tau \sim \mathcal{G}(0.1, 10)$$

$$\mu_0 \sim \mathcal{N}(0, 10^4)$$

GDP Simulation results



CPI Simulation results



VAR

The Vector AutoRegression (VAR) model is a statistical model used to capture the temporal dynamics and interactions among multiple time series. In a VAR model, each variable depends not only on its own past values but also on the past values of all other variables in the system.

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \ldots + A_p Y_{t-p} + \epsilon_t$$

A VAR(1) model is a Vector AutoRegression model of order 1. This means that each variable in the model depends on the previous time period's values of all the variables in the system. Specifically, for a VAR(1) model, each variable at time ttt is a linear function of all variables at time t−1t-1t−1.

$$Y_t = c + A_1 Y_{t-1} + \epsilon_t$$

VAR(1,1) Specification

```
modelVAR.string <- "model {
   ## parameters:  a11, a12, a21, a22, [m01 m02]

   #likelihood
   mu[1:2,1]<- Y[1:2,1]
   Yp[1:2,1]<- mu[1:2,1]
   LogLik[1]<-logdensity.mnorm(Y[1:2,1], mu[1:2,1], omega)
   for (i in 2:N) {
      mu[1,i]<- m0[1] + (a11*Y[1,i-1]) + (a12*Y[2, i-1])
      mu[2,i]<- m0[2] + (a21*Y[1, i-1]) +(a22*Y[2, i-1])
      Y[1:2,i] ~ dmnorm(mu[1:2,i], omega[1:2,1:2])
      Yp[1:2,i] ~ dmnorm(mu[1:2,i], omega)
      LogLik[i]<-logdensity.mnorm(Y[1:2,i], mu[1:2, i], omega)
   }
   #prediction out of sample
   mp[1,1]<- m0[1] + (a11* Y[1,N]) + (a12*Y[2,N])
   mp[2,1]<- m0[2] + (a21*Y[2,N]) + (a22*Y[2,N])
   ypOut[1:2,1] ~ dmnorm(mp[1:2,1], omega)
   for ( k in 2:Npred){
      mp[1,k]<- m0[1] +(a11*ypOut[1,k-1]) + (a12 * ypOut[2,k-1])
      mp[2,k]<- m0[2] +(a21*ypOut[1,k-1]) + (a22*ypOut[2,k-1])
      ypOut[1:2,k] ~ dmnorm(mp[1:2,k], omega)
   }
   # priors
   a11 ~ dunif(-1.5, 1.5)
   a12 ~ dunif(-1.5, 1.5)
   a21 ~ dunif(-1.5, 1.5)
   a22 ~ dunif(-1.5, 1.5)
   m<- 3
   omega ~ dwish(R,m)
   m0 ~ dmnorm(vec, S)
```

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

$$\epsilon \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, S\right)$$

$$a_{ij} \sim U(-1.5, 1.5) \quad i,j = \{1,2\}$$

$$\omega \sim \mathcal{W}(R, m)$$

$$\mu_0 \sim \mathcal{N}(0, 10^4)$$

Parameter selection

$$S^{-1} \sim \mathcal{W}(R, 3) \quad R = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

R is the scale matrix.

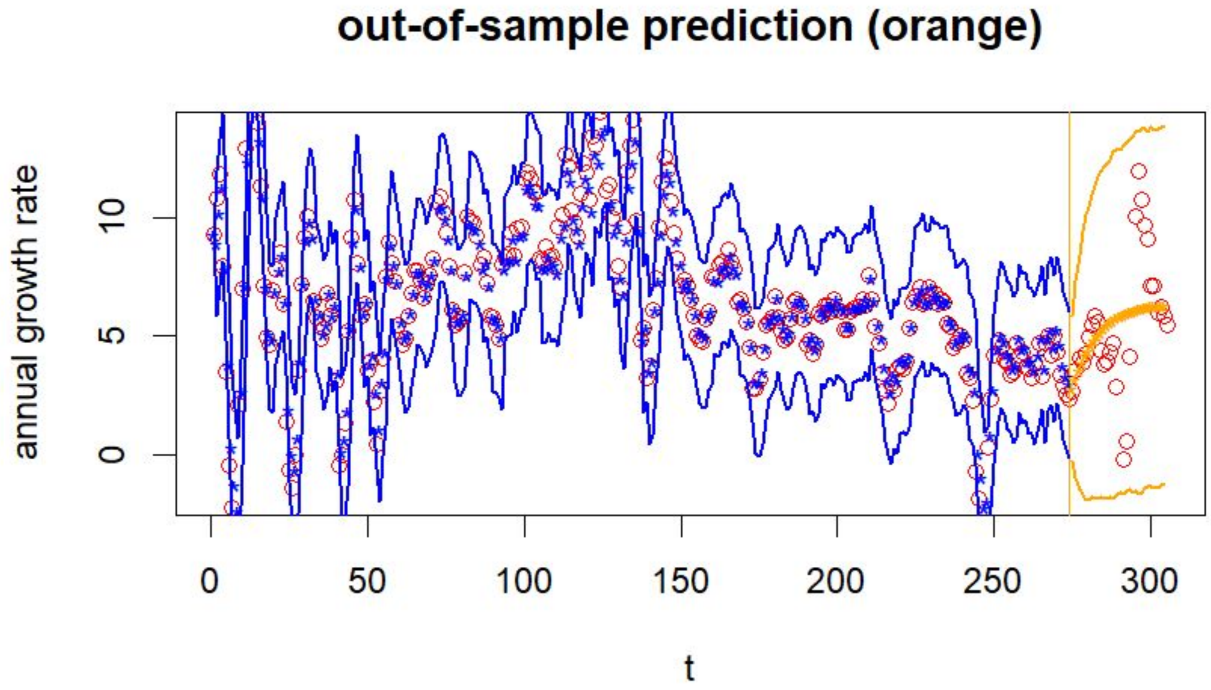
k = dimension of matrix + 1

Given this, the correlation factors exhibit a uniform distribution over [-1, 1].

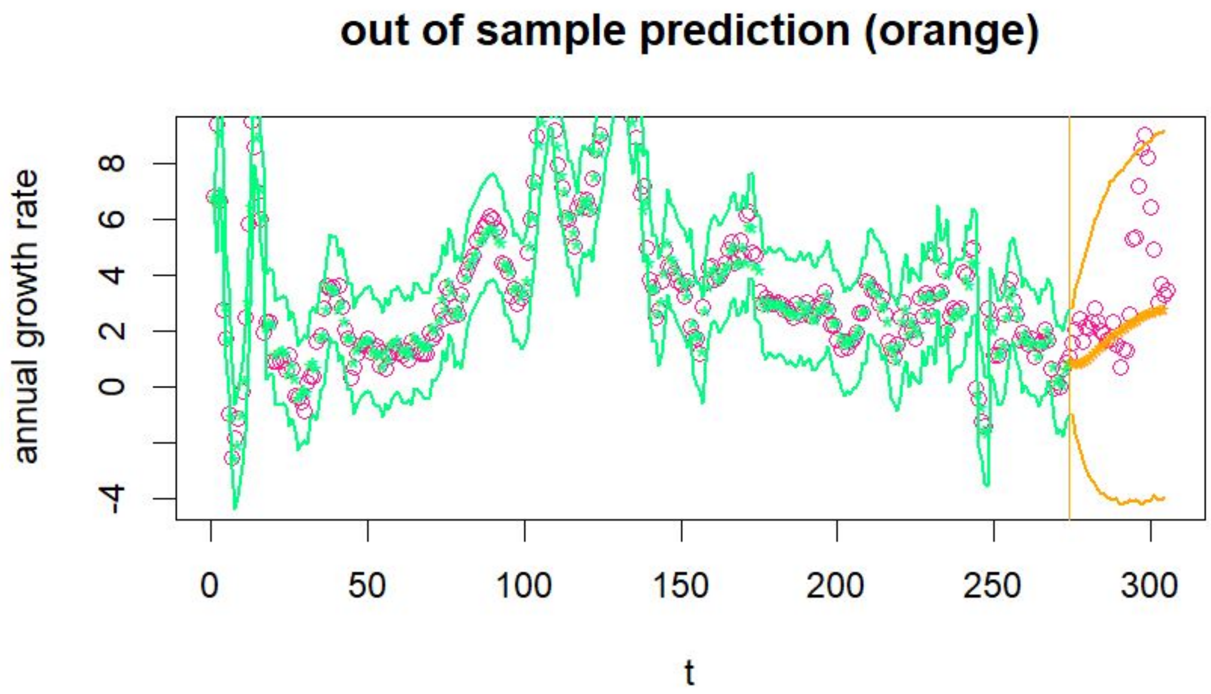
There is no prior assumption that the errors are correlated, ensuring that the degree of correlation follows a uniform distribution.

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}$$

GDP Simulation results



CPI Simulation results



ARMA

An ARMA (Autoregressive Moving Average) model combines autoregressive (AR) and moving average (MA) components. An ARMA(p,q) model uses p autoregressive terms and q moving average terms.

$$Y_t = \mu + \sum_{i=1}^{p} \alpha_i Y_{t-i} + \sum_{j=1}^{q} \beta_j \epsilon_{t-j} + \epsilon_t$$

An ARMA(1,1) model, which is a specific implementation of the general ARMA(p,q) model, combines one autoregressive term (AR(1)) and one moving average term (MA(1))

$$Y_t = \mu + \alpha Y_{t-1} + \beta \epsilon_{t-1} + \epsilon_t$$

ARMA(1,1) Specification

$$Y_t = \mu + \alpha Y_{t-1} + \beta \epsilon_{t-1} + \epsilon_t$$

$$\epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_e^2)$$

```{r}
modelARMA.string <-"
model {
  ## parameters: alpha, beta, tau,tau_eps, m0
  # likelihood
  mu[1] <- Y[1]
  Yp[1] <- mu[1]
  eps[1] <- 0
  LogLik[1] <- log(dnorm(Y[1], mu[1], tau))

  for (i in 2:N) {
    eps[i] ~ dnorm(0, tau_eps)
    Y[i] ~ dnorm(mu[i], tau)
    mu[i] <- m0 + alpha * Y[i-1] + beta * eps[i-1]
    Yp[i] ~ dnorm(mu[i], tau) # prediction in sample

    # Calcolo della log-verosimiglianza per ogni osservazione
    LogLik[i] <- log(dnorm(Y[i], mu[i], tau))


  }


 # prediction out of sample
  ypOut[1] ~ dnorm(m0 + alpha * Y[N] + beta * eps[N], tau)
  for (k in 2:Npred) {
    eps[N + k - 1] ~ dnorm(0, tau_eps)
    ypOut[k] ~ dnorm(m0 + alpha * ypOut[k-1] + beta * eps[N + k - 1], tau)
  }
  sigma2 <- 1 / tau
  sigma_eps2 <- 1 / tau_eps
  # priors
  alpha ~ dunif(-1.5, 1.5)
  beta ~ dunif(-1.5, 1.5)
  tau ~ dgamma(0.1, 10)
  tau_eps ~ dgamma(0.1, 10)
  m0 ~ dnorm(0.0, 1.0E-4)
}"
```

Parameter selection

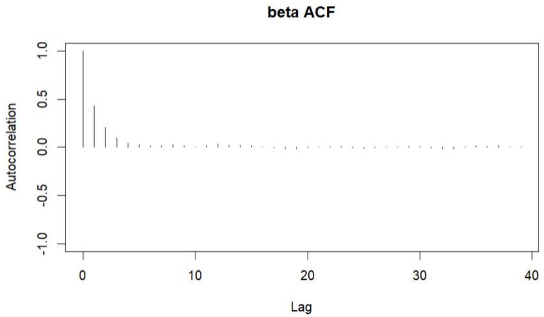
$$\beta \sim U(-1.5, 1.5)$$

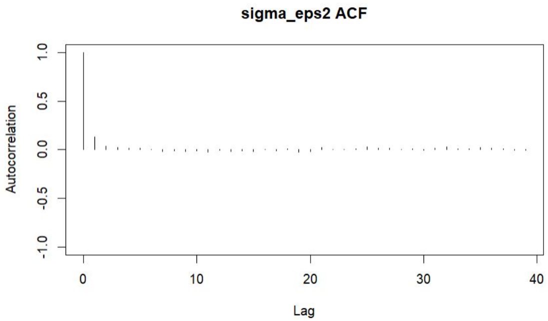
The moving average coefficient reflects how much the current data depends on past forecast errors. A higher β suggests a stronger short term impact of past errors.

$$1/\sigma_e^2 \sim \mathcal{G}(0.1, 10)$$

The first hyperparameter α is the shape parameter, the second parameter, β is the rate parameter. These hyperparameters where chosen to have a small mean(α/β) and a small variance (α/β^2), corresponding to big values and variance for σ ^2 .

$$\beta \sim \mathcal{B}e(20, 20)$$

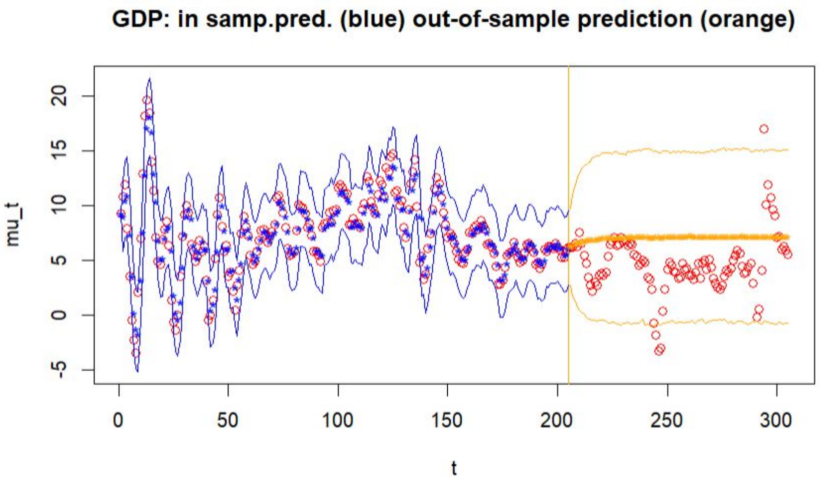


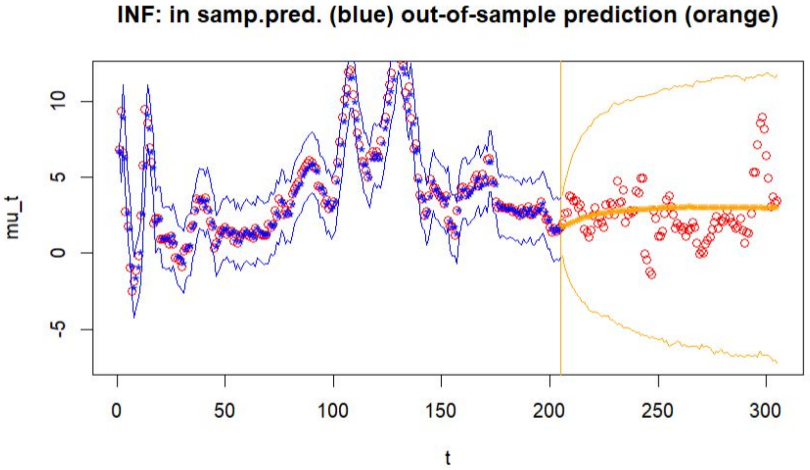



```{r}
modelARMA.string <-"
model {
  ## parameters: alpha, beta, tau,tau_eps, m0
  # likelihood
  mu[1] <- Y[1]
  Yp[1] <- mu[1]
  eps[1] <- 0
  LogLik[1] <- log(dnorm(Y[1], mu[1], tau))

  for (i in 2:N) {
    eps[i] ~ dnorm(0, tau_eps)
    Y[i] ~ dnorm(mu[i], tau)
    mu[i] <- m0 + alpha * Y[i-1] + beta * eps[i-1]
    Yp[i] ~ dnorm(mu[i], tau) # prediction in sample

    # Calcolo della log-verosimiglianza per ogni osservazione
    LogLik[i] <- log(dnorm(Y[i], mu[i], tau))


  }


 # prediction out of sample
  ypOut[1] ~ dnorm(m0 + alpha * Y[N] + beta * eps[N], tau)
  for (k in 2:Npred) {
    eps[N + k - 1] ~ dnorm(0, tau_eps)
    ypOut[k] ~ dnorm(m0 + alpha * ypOut[k-1] + beta * eps[N + k - 1], tau)
  }
  sigma2 <- 1 / tau
  sigma_eps2 <- 1 / tau_eps
  # priors
  alpha ~ dunif(-1.5, 1.5)
  beta ~ dbeta(20,20)
  tau ~ dgamma(0.1, 10)
  tau_eps ~ dgamma(100, 100)
  m0 ~ dnorm(0.0, 1.0E-4)
}"
```

Simulation Results

````
```{r}
t = seq(1, Ntot)
tt = seq(1, N)

# Previsioni in-sample
yp = outputmcmcARMA_GDP$mean$Yp
q1 = outputmcmcARMA_GDP$q2.5$Yp
q2 = outputmcmcARMA_GDP$q97.5$Yp

# Previsioni out-of-sample
yp_pred = outputmcmcARMA_GDP$mean$ypOut
q1_pred = outputmcmcARMA_GDP$q2.5$ypOut
q2_pred = outputmcmcARMA_GDP$q97.5$ypOut

plot(t, gdpData, col="red", ylab="mu_t", ylim=c(min(q1), max(q2)),
     main="GDP: in samp.pred. (blue) out-of-sample prediction (orange)")
abline(v=N, col="orange")
lines(tt, yp, type="p", pch="*", col="blue")
lines(tt, q1, type="l", col="blue", lwd=1.5)
lines(tt, q2, type="l", col="blue", lwd=1.5)
points(seq((N+1), Ntot, 1), yp_pred, pch="*", col="orange")
lines(seq((N+1), Ntot, 1), q1_pred, col="orange", lwd=1.5)
lines(seq((N+1), Ntot, 1), q2_pred, col="orange", lwd=1.5)
```
````





3. Comparison

BIC, DIC and WAIC

BIC (Bayesian Information Criterion) penalizes model complexity based on the likelihood and number of parameters. The AR(1) model had the lowest BIC value, suggesting it as the preferred model according to BIC for GDP while AR(2) for the CPI.

Model	BIC	DIC	WAIC
AR(1)	820.9816	807.6827	808.6
AR(2)	880.1	862.2862	863.2
ARMA	968.782	1385.48	970.3

Table 1: Results on GDP data

$$BIC = -2 * \log(L) + k * \log(n)$$

Model	BIC	DIC	WAIC
AR(1)	742.4853	730.1602	731.2
AR(2)	720.0796	862.2862	705.1
ARMA	742.0565	796.5156	730.5

Table 2: Results on INF data

Model	BIC	DIC	WAIC
VAR(1)	1790.418	1750.123	1761.5

Table 3: Results of VAR on GDP and INF data

BIC, DIC and WAIC

DIC (Deviance Information Criterion)} combines model fit (deviance) with complexity. The AR(1) model had the lowest DIC value, suggesting it as the preferred model according to DIC.

Model	BIC	DIC	WAIC
AR(1)	820.9816	807.6827	808.6
AR(2)	880.1	862.2862	863.2
ARMA	968.782	1385.48	970.3

Table 1: Results on GDP data

Model	BIC	DIC	WAIC
AR(1)	742.4853	730.1602	731.2
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Table 2: Results on INF data

Model	BIC	DIC	WAIC
VAR(1)	1790.418	1750.123	1761.5

Table 3: Results of VAR on GDP and INF data

BIC, DIC and WAIC

WAIC (Watanabe-Akaike Information Criterion)} considers predictive accuracy and model complexity. The AR(1) model had the lowest WAIC value , indicating it as the best model by WAIC for GDP while AR(2) for CPI.

Model	BIC	DIC	WAIC
AR(1)	820.9816	807.6827	808.6
AR(2)	880.1	862.2862	863.2
ARMA	968.782	1385.48	970.3

Table 1: Results on GDP data

Model	BIC	DIC	WAIC
AR(1)	742.4853	730.1602	731.2
AR(2)	720.0796	862.2862	705.1
ARMA	742.0565	796.5156	730.5

Table 2: Results on INF data

Model	BIC	DIC	WAIC
VAR(1)	1790.418	1750.123	1761.5

Table 3: Results of VAR on GDP and INF data

BIC, DIC and WAIC

In summary, while BIC favored the AR(1) and AR(2) model, both DIC and WAIC indicated that the AR(1) model is the best choice based on their respective criteria.

Model	BIC	DIC	WAIC
AR(1)	820.9816	807.6827	808.6
AR(2)	880.1	862.2862	863.2
ARMA	968.782	1385.48	970.3

Table 1: Results on GDP data

Model	BIC	DIC	WAIC
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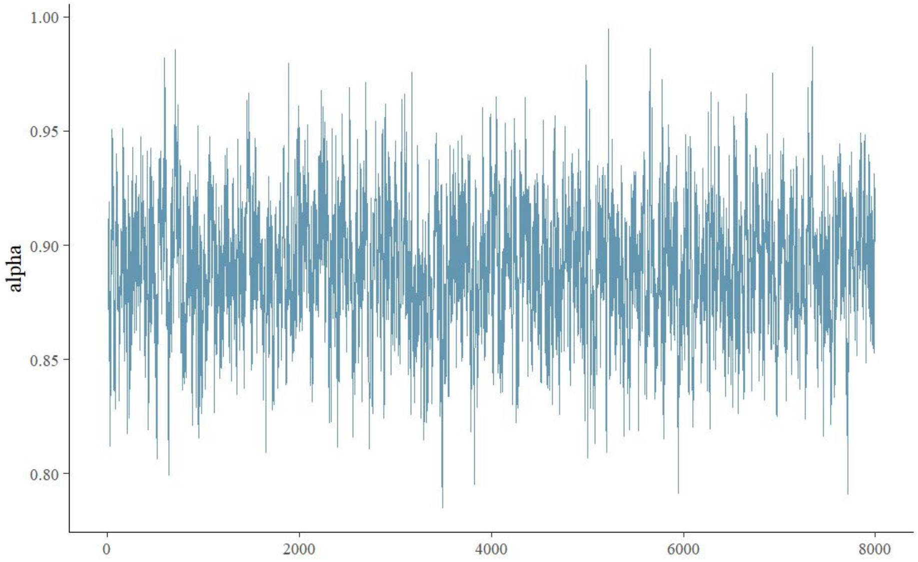
Table 2: Results on INF data

Model	BIC	DIC	WAIC
VAR(1)	1790.418	1750.123	1761.5

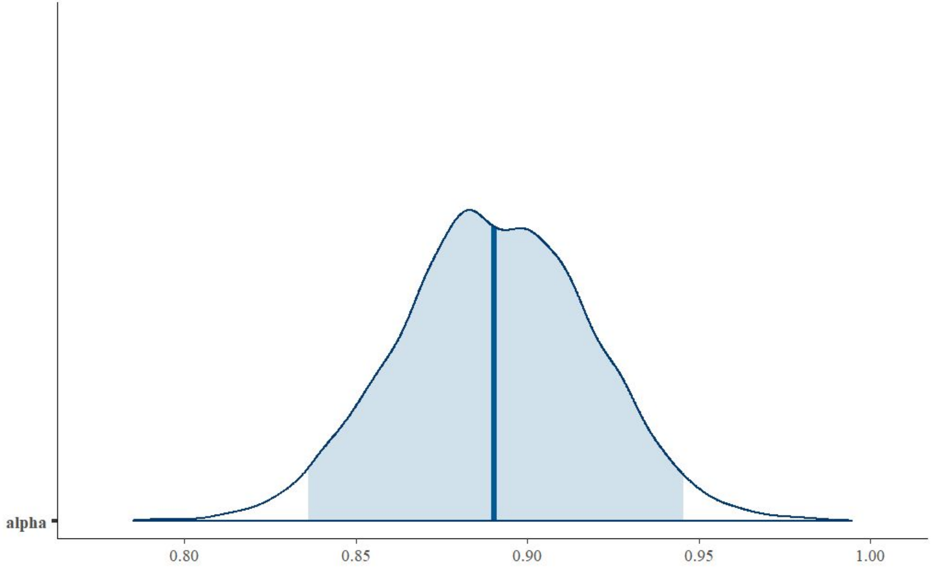
Table 3: Results of VAR on GDP and INF data

4. Additional analysis

AR(1) GDP Posterior Parameter Insights

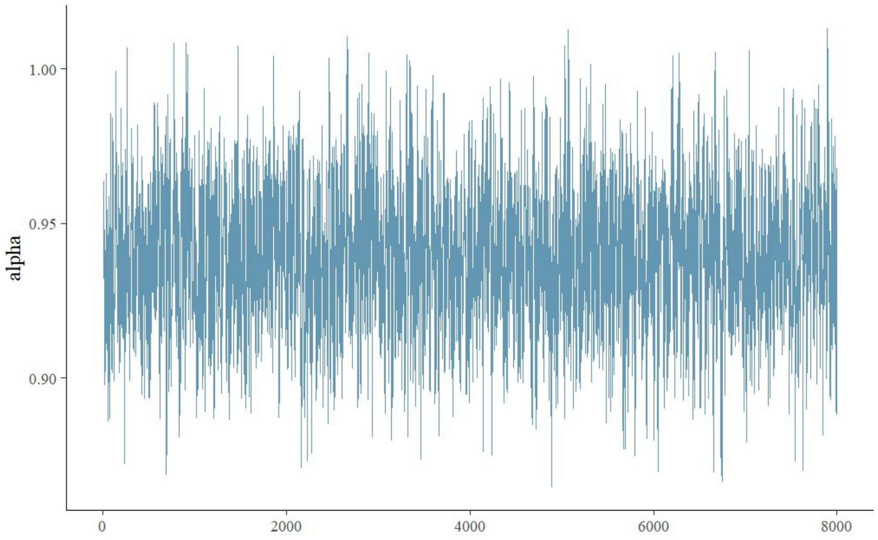


Alpha Traceplot

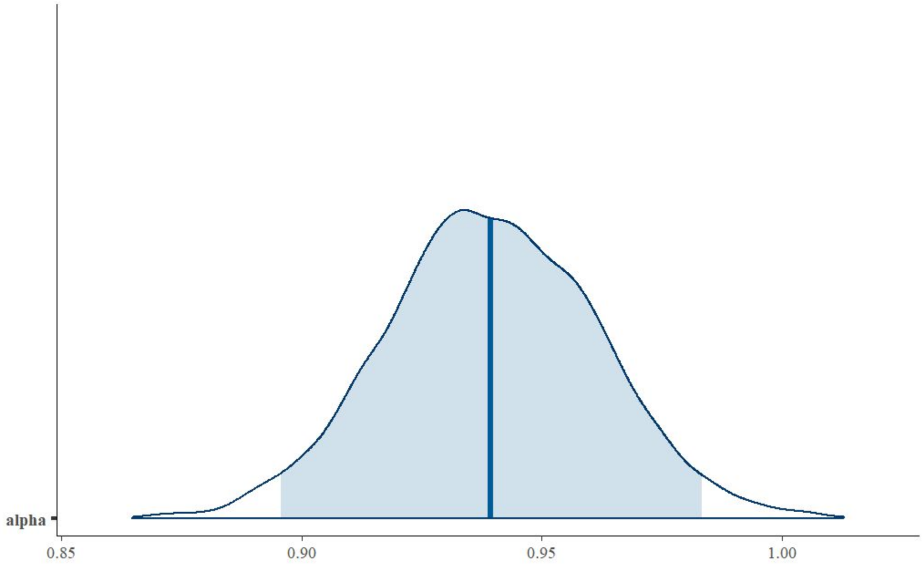


Area of Alpha with 95% credible interval

AR(1) CPI Posterior Parameter Insights

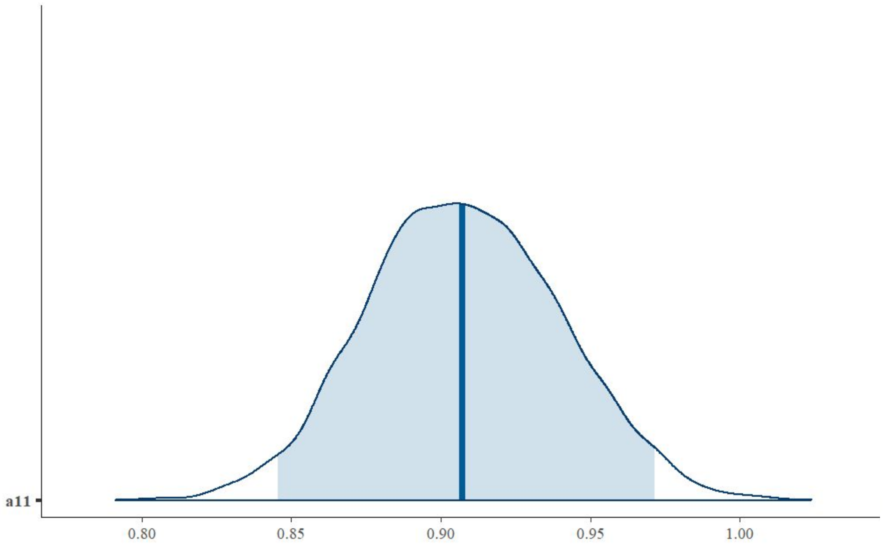


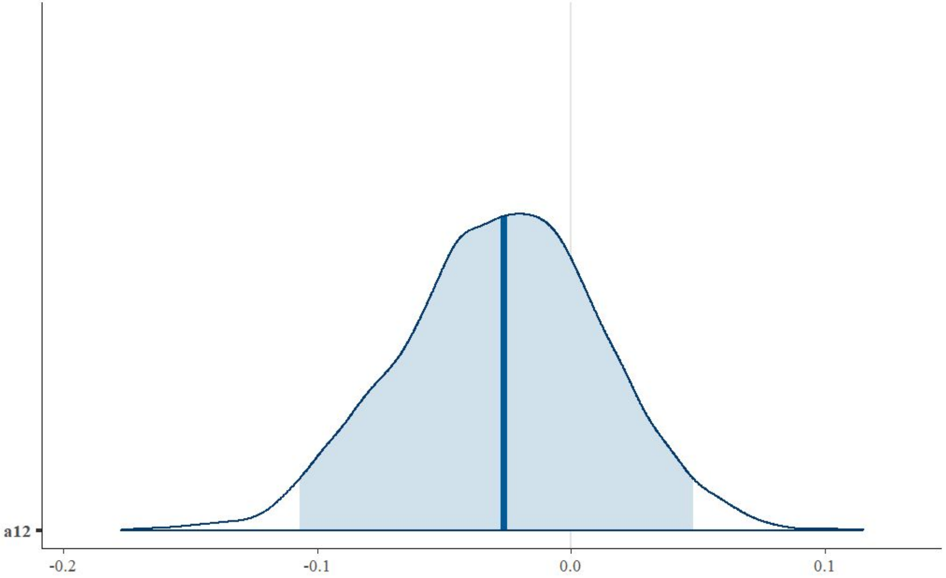
Alpha Traceplot



Area of Alpha with 95% credible interval

VAR(1,1) Posterior Parameter Insights

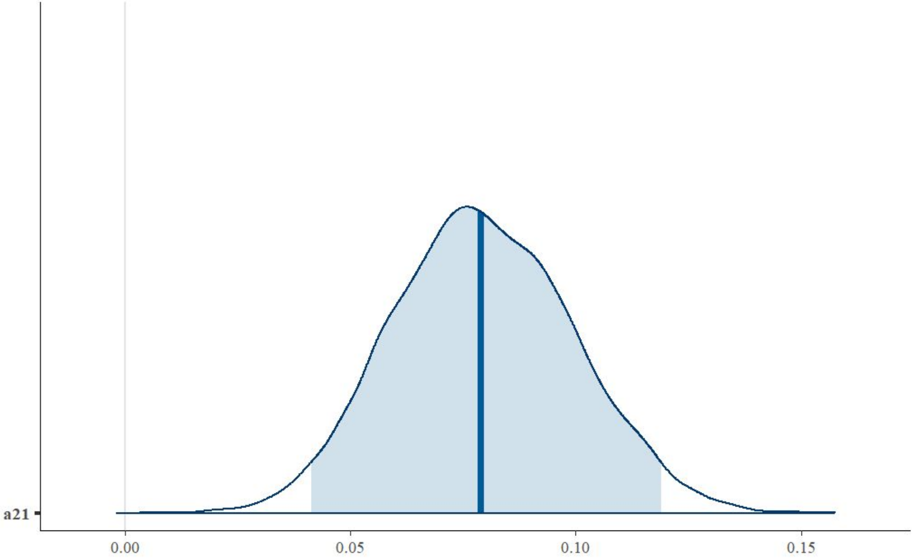


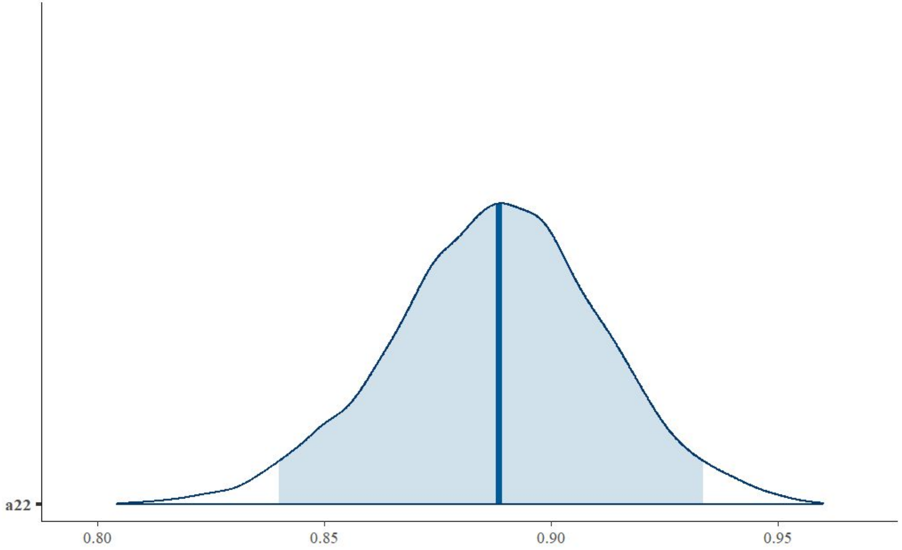


a11 Area with 95% credible interval

a12 Area with 95% credible interval

VAR(1,1) Posterior Parameter Insights

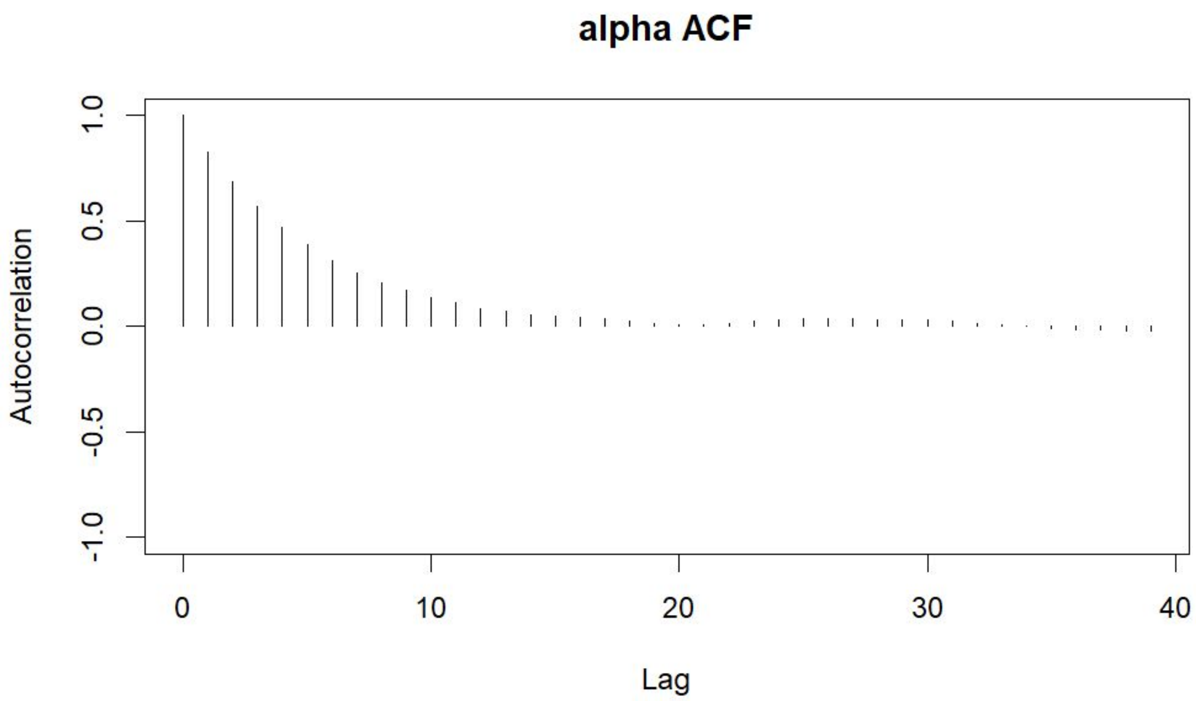




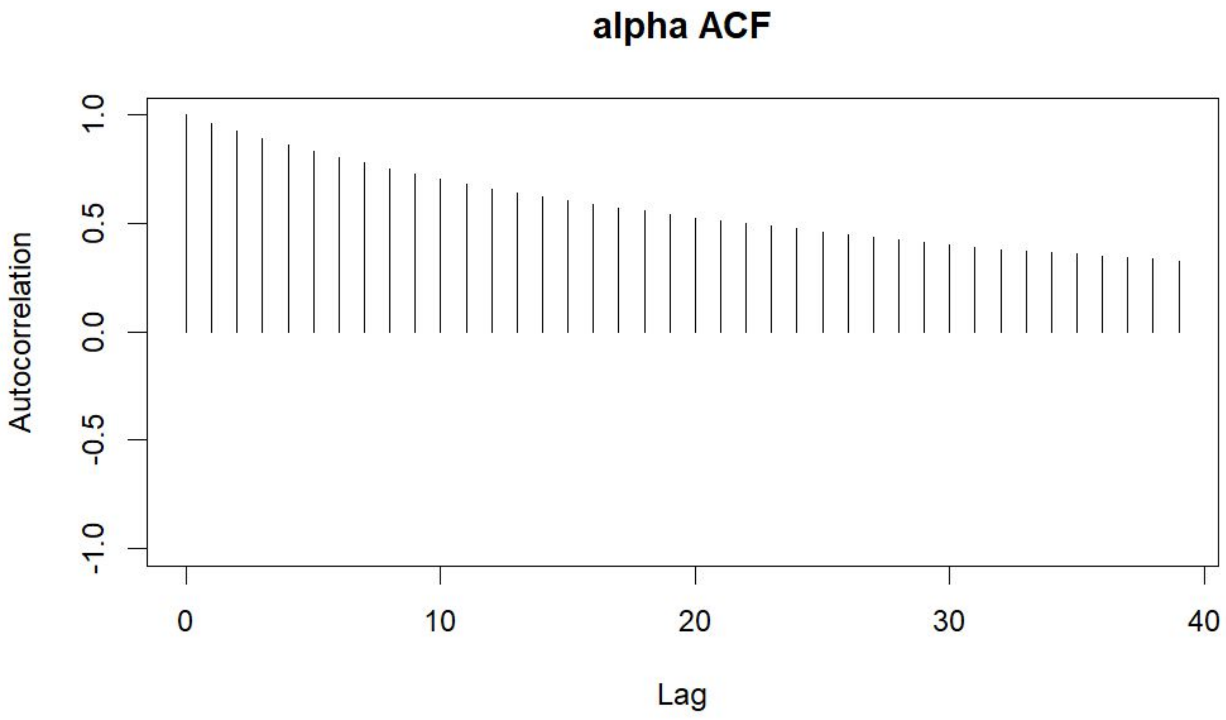
a21 Area with 95% credible interval

a22 Area with 95% credible interval

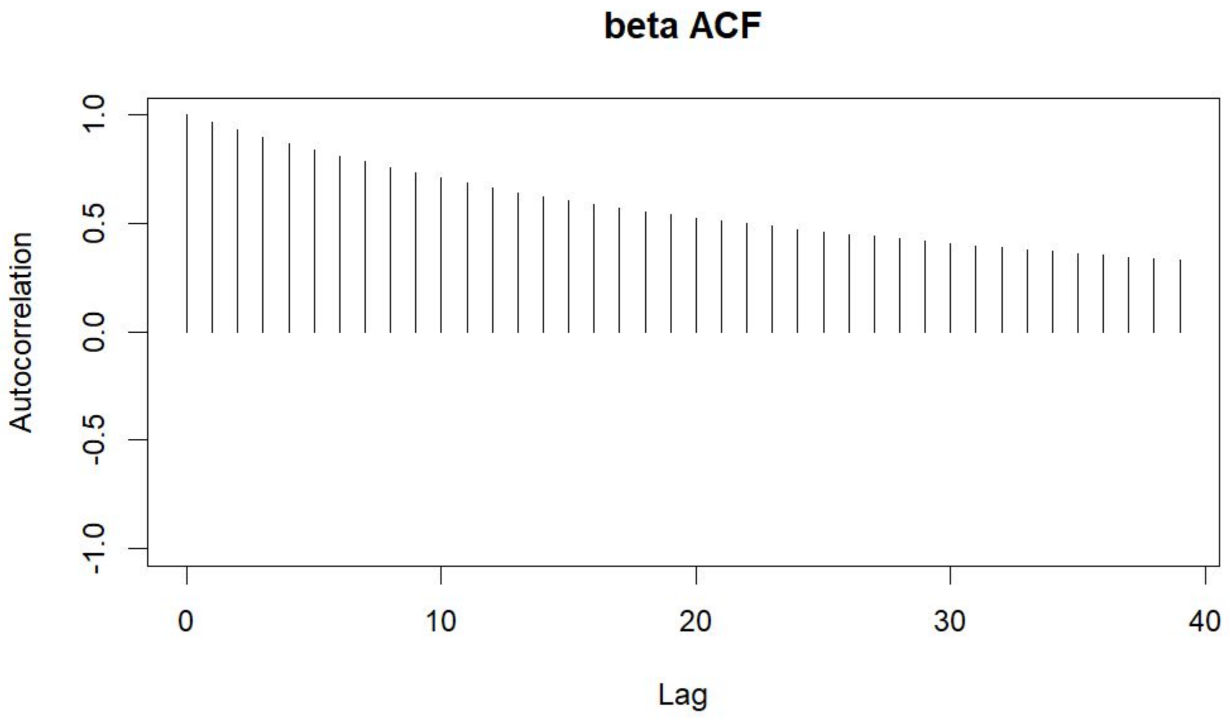
AR(1) GDP Alpha correlation



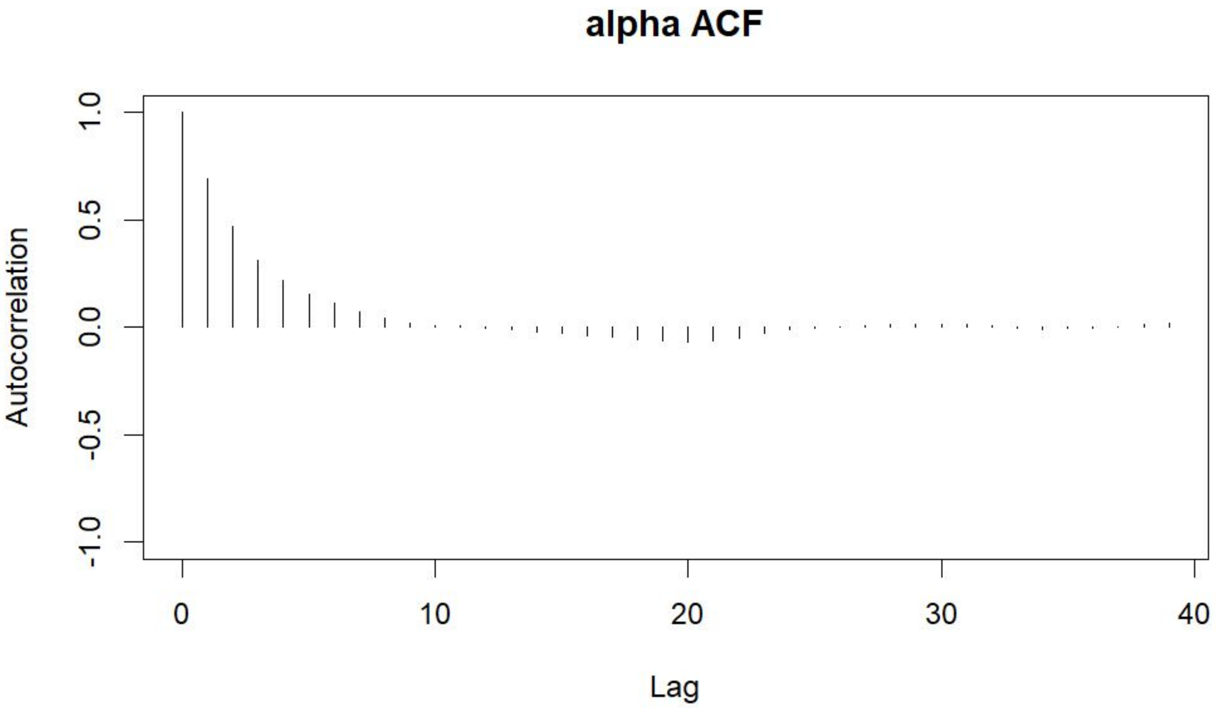
AR(2) GDP Alpha autocorrelation



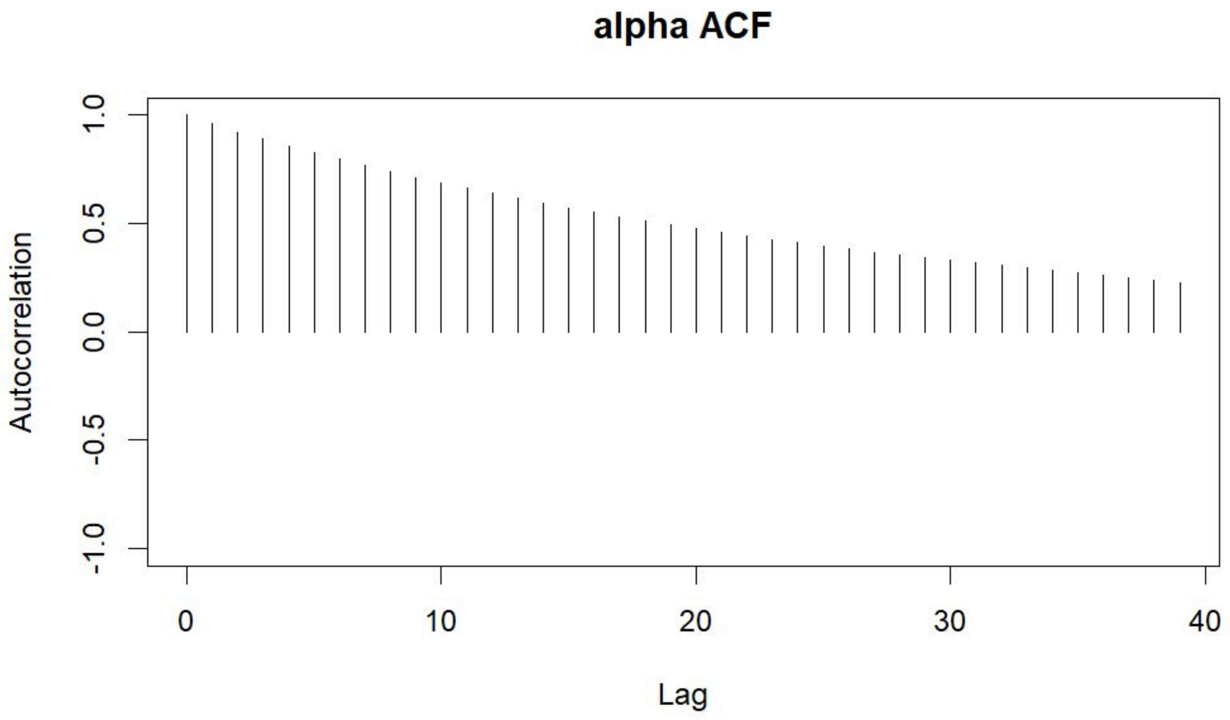
AR(2) GDP Beta autocorrelation



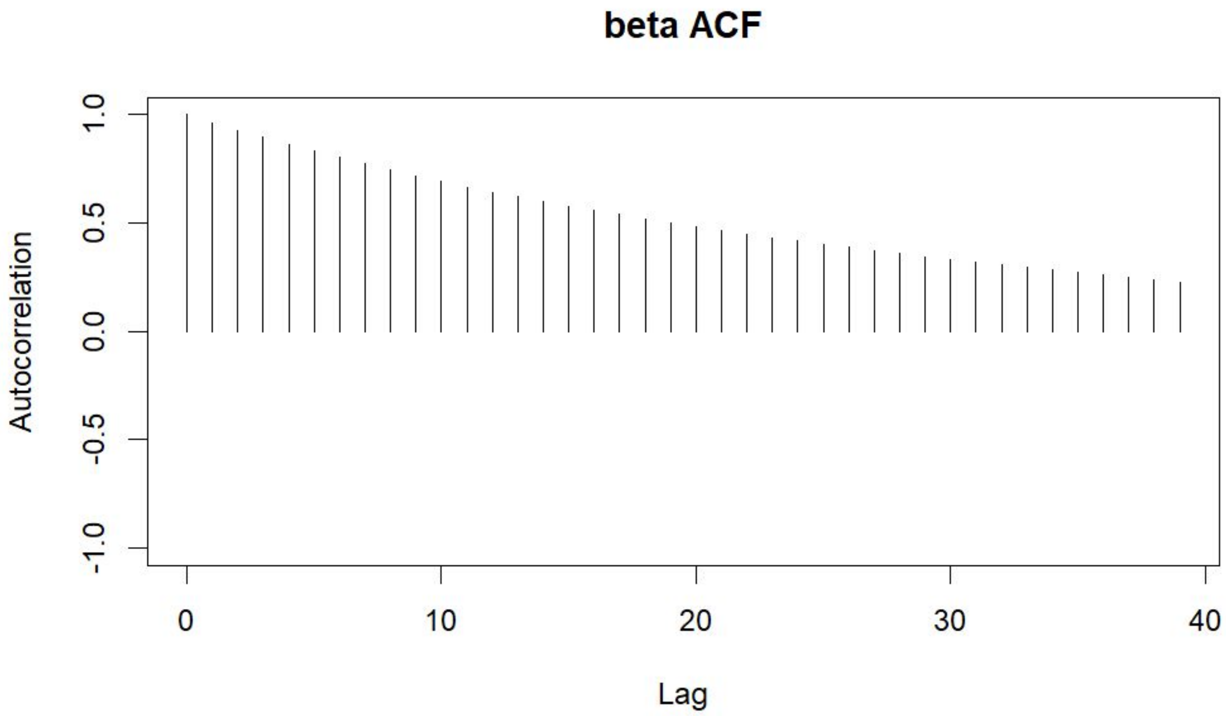
AR(1) CPI Alpha autocorrelation



AR(2) CPI Alpha autocorrelation



AR(2) CPI Beta autocorrelation



5. Comments and conclusions

Selected Model

Based on BIC, DIC and WAIC, the AR(1) model emerged as the preferred choice for GDP and the AR(2) was slightly preferred over the AR(1) to model the behaviour of the inflation.

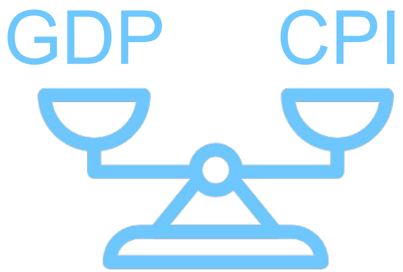
- This probably indicates that the events happening each year are the most relevant for the next year's GDP and CPI.
- Also goes to show that oftentimes the simpler models can be the best ones.

Keep it simple.

Correlation between GDP and CPI?

The VAR(1,1) model showed us that there's little correlation between these two critical economic indicators.

- Even though at first glance they would seem to be related, our investigation showed that it is probably not the case.
- Further investigation could try to find out the most influential factors for the change in these metrics other than its value the previous years.



Thanks!