POLITECNICO MILANO 1863

Bayesian Learning And Montecarlo Simulation Project

GDP and Inflation

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1 Introduction

1.1 Objective

The objective of this project is to analyze and model the time series data of two key economic indicators for the United States: Gross Domestic Product (GDP) and the Consumer Price Index (CPI). In this project we aim to:

1. Fit various time series models to each of these economic indicators individually.

2. Compare the performance of these models using Information Criteria such as Bayesian Information Criterion (BIC), Deviance Information Criterion (DIC), and Watanabe-Akaike Information Criterion (WAIC).

3. Fit both GDP and CPI using a bivariate time series model: the Vector Autoregression (VAR) model.

1.2 Overview

In this project we use three time series models to analyze the GDP and CPI data:

1. The **Autoregressive (AR) Model:** The AR model uses only the past values of a series (lags) to describe the behaviour of a series in order to predict its future values.

2. **Autoregressive Moving Average (ARMA) Model:** This model combines the AR and MA models and is used to better capture the dynamics in the time series data.

3. **Vector Autoregression (VAR) Model:** For the bivariate analysis we use the VAR model which captures the linear interdependencies among multiple time series.

For each of these models we analyze the posterior distributions, make predictions and evaluate their performance using three Information Criteria: Bayesian Information Criteria (BIC), Deviance Information Criterion (DIC), Watanabe-Akaike Information Criterion (WAIC).

2 Description of the problem and the data

The data for this analysis is sourced from the Federal Reserve Economic Data (FRED) with series HSN1F for GDP and CPI-AUCSL for the Consumer Price Index. The dataset includes quarterly observations from 1948-01-01 to 2024-01-01. The data had already been preprocessed. The time series used for the analysis represented the percent change from the previous year, calculated using the following formula:

$$\left(\left(\frac{x_t}{x_{t-n\_obs\_per\_yr}}\right)-1\right)*100$$

1. Gross Domestic Product (GDP): Represents the market value of goods and services produced by labor and property in the US, measured in billions of dollars, seasonally adjusted.
   - Source: U.S. Bureau of Economic Analysis
   - Release: Gross Domestic Product
   - Units: Billions of Dollars, Seasonally Adjusted Annual Rate
   - Frequency: Quarterly
   - BEA Account Code: A191RC
   - Description: GDP is the market value of all goods and services produced by labor and property located in the United States. It is the principal measure of U.S. manufacturing and is used to evaluate the country's economic performance. GDP data is seasonally adjusted to eliminate the effects of seasonal variations.

1. Consumer Price Index for All Urban Consumers: All Items in U.S. City Average (CPIAUCSL): Measures the average change over time in the prices paid by urban consumers for a basket of goods and services, seasonally adjusted.
   - Source: U.S. Bureau of Labor Statistics
   - Release: Consumer Price Index
   - Units: Index 1982-1984=100, Seasonally Adjusted
   - Frequency: Monthly (aggregated to quarterly for this analysis)
   - Description: The CPI measures the average change over time in the prices paid by consumers for a market basket of goods and services. The CPI is also a measure for understanding inflation trends. The index is adjusted to remove the effects of seasonal changes.

3 Model Specification

3.1 AR(1)

3.1.1 Likelihood

The AR(1) model, a type of autoregressive model, uses only the previous observation of the time series to predict the next value. Although it directly uses just the immediate past value, it implicitly incorporates information from earlier observations as well. It is defines as:

$$y_{t+1} = \mu + \alpha y_t + \epsilon_t \qquad \epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2) \tag{1}$$

defined recursively from y_1 for $t \geq 1$, where:

- $\mathbf{y_t}$ is the GDP or CPI at time t.
- μ is the constant term of the series
- α is the autoregressive coefficient reflecting the impact of the lagged value on the current value.
- ϵ_t is the error term at time t.
- σ^2 is the variance of the white noise process ϵ_t. It measures the random fluctuations in data that are not explained by the model's autoregressive component.ϵ_t.

3.1.2 Priors

- $\alpha \sim \text{Uniform}(-1.5, 1.5)$: The prior for α was chosen as a uniform distribution over $[-1.5, 1.5]$ to create a non-informative prior and to include non-stationary models.
- $1/\sigma^2 \sim \text{Gamma}(0.1, 10)$ is the prior for the inverse of the error variance. The first hyperparameter α is the shape parameter, the second parameter, β is the rate parameter. These hyperparameters where chosen to have a small mean(α/β) and a small variance (α/β^2), corresponding to big values and variance for σ^2.
- $\mu \sim \mathcal{N}(0, 1.0\times10^4)$: Normal prior for the intercept with very high variance since we have no prior information on the mean.

3.2 ARMA(1,1)

3.2.1 Likelihood

The ARMA(1,1) model is a type of time series model that combines autoregressive (AR) and moving average (MA) components to express temporal dependencies and fluctuations in time series data.

We defined the ARMA(1,1) model that we used:

$$y_t = \mu + \alpha y_{t-1} + \beta\epsilon_{t-1} + \epsilon_t \qquad \epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2) \tag{2}$$

where: y_t is the GDP at time t, μ is a constant term, α is the autoregressive coefficient reflecting the impact of the lagged value on the current value. β is the moving average coefficient reflecting the impact of the lagged error term on the current value. ϵ_t is the error term at time t. σ^2 is the variance of the error term ϵ_t

3.2.2 Priors

- $\alpha \sim \text{Uniform}(-1.5, 1.5)$ is the uniform prior for the autoregressive coefficient. The autoregressive coefficient determines how much the current value is influenced by its past values. A higher α indicates stronger persistence in GDP fluctuations over time. The prior for α was chosen as a uniform distribution over $[-1.5, 1.5]$ to create a non-informative prior and to include non-stationary models.

- $\beta \sim \text{Uniform}(-1.5, 1.5)$ is the uniform prior for the lagged error coefficient. The moving average coefficient reflects how much the current data depends on past forecast errors. A higher β suggests a stronger short term impact of past errors.

- $1/\sigma^2 \sim \text{Gamma}(0.1, 10)$ is the prior for the inverse of the error variance. The first hyperparameter α is the shape parameter, the second parameter, β is the rate parameter. These hyperparameters where chosen to have a small mean(α/β) and a small variance (α/β^2), corresponding to big values and variance for σ^2.

- $\mu \sim \mathcal{N}(0, 1.0 \times 10^4)$: Normal prior for the intercept with very high variance since we have no prior information on the mean.

3.3 AR(2)

3.3.1 Likelihood

The AR(2) model takes the two previous observations to calculate the new possible value of the time series. This can be seen in the following formulation of the process:

$$y_{t+1} = \mu + \alpha y_t + \beta y_{t-1} + \epsilon_t \qquad \epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2) \tag{3}$$

defined recursively from y_2 for $t \geq 2$. The first 2 observations are taken from the dataset and then it is possible to start making predictions. This model shares the same parameters as the AR(1), except we now include the β parameter for the coefficient of the second delay.

3.3.2 Priors

- $\alpha \sim \text{Uniform}(-1.5, 1.5)$ is the uniform prior for the autoregressive coefficient. The autoregressive coefficient determines how much the current value is influenced by its past values. A higher α indicates stronger persistence in GDP fluctuations over time.

- $\beta \sim \text{Uniform}(-1.5, 1.5)$ is the uniform prior for the second autoregressive coefficient.

- $1/\sigma^2 \sim \text{Gamma}(0.1, 10)$ is the prior for the inverse of the error variance. The first hyperparameter α is the shape parameter, the second parameter, β is the rate parameter. These hyperparameters where chosen to have a small mean(α/β) and a small variance (α/β^2), corresponding to big values and variance for σ^2.

- $\mu \sim \mathcal{N}(0, 1.0 \times 10^4)$: ormal prior for the intercept with very high variance since we have no prior information on the mean.

3.4 VAR(1,1)

3.4.1 Likelihood

The VAR (Vector Autoregressive) model considers both time series for predictions, assuming a correlation between the GDP and inflation time series. The VAR(1) model extends the AR(1) model to multiple variables. In this model, dependencies between the two time series are encoded by parameters a_{12} and a_{21} : y_{1t} depends linearly on y_{2t-1} with coefficient a_{12} and y_{2t} depends linearly on y_{1t-1} with coefficient a_{21}.
Additionally, the VAR(1) model incorporates stochastic dependencies through the error terms, where correlations between these errors $Cov(\varepsilon_{1t}, \varepsilon_{2t})$, can exist. This feature allows the model to capture how disturbances in one time series affect the others over time.

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \tag{4}$$

$$\boldsymbol{\epsilon} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, S \right) \tag{5}$$

3.4.2 Priors

- $a_{11}, a_{12}, a_{21}, a_{22} \overset{iid}{\sim} \text{Uniform}(-1.5, 1.5)$. Uniform distributions were chosen so to have a non informative prior.

- $S^{-1} \sim \text{Wishart}(R, 3)$ with $R = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$ These values were selected based on the property that when R (the scale matrix) is diagonal and k (the degrees of freedom) equals the dimension of the matrix plus one, the correlation factors exhibit a uniform distribution over [-1, 1]. This prior was chosen because there is no prior assumption that the errors are correlated, ensuring that the degree of correlation follows a uniform distribution.

- $\mu_1, \mu_2 \overset{iid}{\sim} \mathcal{N}(0, 1.0 \times 10^4)$: normal prior for the intercept with very high variance since we have no prior information on the mean.

4 Methods

All models were fitted using R with JAGS. The models were written in JAGS, and posterior distributions were built from samples collected over multiple iterations. The parameters used for JAGS were: n.adapt = 1000, n.iter = 10000, n.chains = 1, n.burnin = 2000. In JAGS, the normal distributions take as input the precision (inverse of variance) rather than the variance. Therefore, we specified a prior distribution on the precision and subsequently calculated the covariance.

5 Models Comparison

The best model was assessed using three criteria: BIC, DIC, and WAIC, each addressing model fit and complexity differently.

- **BIC (Bayesian Information Criterion)** penalizes model complexity based on the likelihood and number of parameters. The AR(1) model had the lowest BIC value, suggesting it as the preferred model according to BIC for GDP while AR(2) for the CPI.
- **DIC (Deviance Information Criterion)** combines model fit (deviance) with complexity. The AR(1) model had the lowest DIC value, suggesting it as the preferred model according to DIC.
- **WAIC (Watanabe-Akaike Information Criterion)** considers predictive accuracy and model complexity. The AR(1) model had the lowest WAIC value , indicating it as the best model by WAIC for GDP while AR(2) for CPI.

In summary, while BIC favored the AR(1) and AR(2) model, both DIC and WAIC indicated that the AR(1) model is the best choice based on their respective criteria.

Model	**BIC**	**DIC**	**WAIC**
AR(1)	820.9816	807.6827	808.6
AR(2)	880.1	862.2862	863.2
ARMA	968.782	1385.48	970.3

Table 1: Results on GDP data

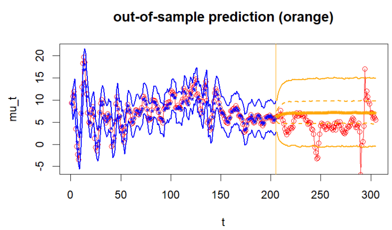
Model	BIC	DIC	WAIC
AR(1)	742.4853	730.1602	731.2
AR(2)	720.0796	862.2862	705.1
ARMA	742.0565	796.5156	730.5

Table 2: Results on INF data

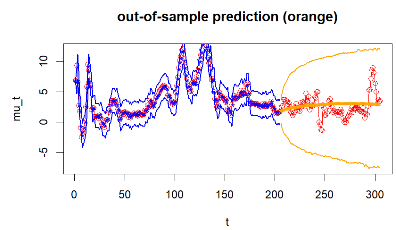
Model	BIC	DIC	WAIC
VAR(1)	1790.418	1750.123	1761.5

Table 3: Results of VAR on GDP and INF data

6 Posterior Analysis



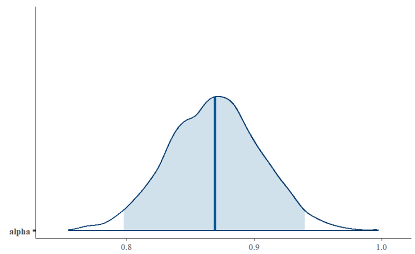
(a) AR(1) GDP



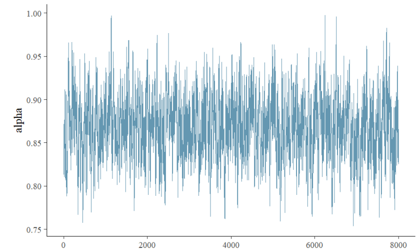
(b) AR(1) INF

Figure 1: AR(1) GDP and INF

The posterior analysis of the AR(1) models for GDP and inflation shows the predictive distributions obtained through MCMC sampling. The graphs highlight time series point estimates compared to aggregate forecasts and the confidence intervals. The posterior distributions reflect variability in historical data and uncertainty in future predictions, providing a detailed assessment of AR(1) model performance for both time series

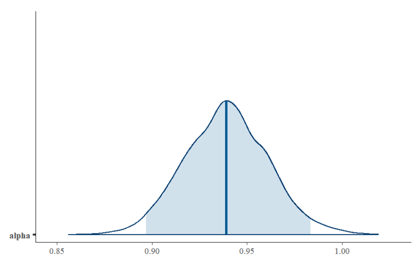


(a) GDP Area of Alpha with 0,95 confidence interval

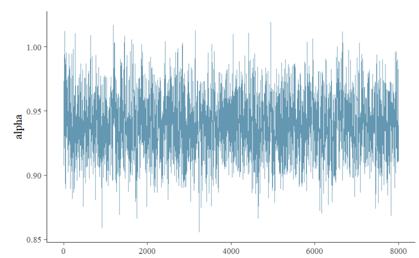


(b) GDP alpha trace plot

Figure 2: AR(1) GDP Alpha graphics



(a) Inflation Area of Alpha with 0,95 confidence interval



(b) AR(1) INF

Figure 3: AR(1) INF alpha graphics

7 Final comments and conclusions

In conclusion, this project analyzed GDP and inflation time series data using various Bayesian models: AR, ARMA, AR(2), and VAR. Based on BIC, DIC and WAIC, the AR(1) model emerged as the preferred choice for both GDP and inflation. These findings highlight its better balance between model fit and complexity. Future research could explore additional factors or more complex models to further refine predictive accuracy.

8 References

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