Apèndix A: Taula de primitives

Per a la comoditat del lector, recollim en aquest apèndix algunes de les primitives més usuals que apareixen al llarg del text. Una taula molt més completa es troba al llibre *Fórmulas y tablas de matemática aplicada*, citat a la bibliografia.

1. $\displaystyle\int (uv') = uv - \int (vu')$.
2. $\displaystyle\int (a^u u') = \frac{a^u}{\ln a} + C\,, \quad a \neq 1\,, \quad a > 0.$
3. $\displaystyle\int (u' \cos u) = \sin u + C.$
4. $\displaystyle\int (u' \sin u) = -\cos u + C.$
5. $\displaystyle\int (ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)} + C\,, \quad n \neq -1.$
6. $\displaystyle\int (ax+b)^{-1} = \frac{1}{a} \ln \lvert ax+b \rvert + C.$
7. $\displaystyle\int x(ax+b)^{-1} = \frac{x}{a} - \frac{b}{a^2} \ln \lvert ax+b \rvert + C.$
8. $\displaystyle\int x(ax+b)^{-2} = \frac{1}{a^2} \left[\ln \lvert ax+b \rvert + \frac{b}{ax+b} \right] + C.$
9. $\displaystyle\int \frac{1}{x(ax+b)} = \frac{1}{b} \ln \left\lvert \frac{x}{ax+b} \right\rvert + C.$
10. $\displaystyle\int (\sqrt{ax+b})^n = \frac{2}{a} \frac{(\sqrt{ax+b})^{n+2}}{n+2} + C\,, \quad n \neq -2.$

11. $\displaystyle\int \frac{\sqrt{ax+b}}{x} = 2\sqrt{ax+b} + b\int \frac{1}{x\sqrt{ax+b}}.$
12. a) $\displaystyle\int \frac{1}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\,\mathrm{arctg}\,\sqrt{\frac{ax+b}{-b}} + C\,, \quad \text{si} \quad b<0.$ b) $\displaystyle\int \frac{1}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left\|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right\| + C\,, \quad \text{si} \quad b>0.$
13. $\displaystyle\int \frac{\sqrt{ax+b}}{x^2} = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{1}{x\sqrt{ax+b}} + C.$
14. $\displaystyle\int \frac{1}{a^2+x^2} = \frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a} + C.$
15. $\displaystyle\int \frac{1}{a^2-x^2} = \frac{1}{2a}\ln\left\|\frac{x+a}{x-a}\right\| + C.$
16. $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}} = \mathrm{Arg\ sinh}\,\frac{x}{a} + C = \ln\left\|x+\sqrt{a^2+x^2}\right\| + C.$
17. $\displaystyle\int \sqrt{a^2+x^2} = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\,\mathrm{Arg\ sinh}\,\frac{x}{a} + C.$
18. $\displaystyle\int \frac{1}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\ln\left\|\frac{a+\sqrt{a^2+x^2}}{x}\right\| + C.$
19. $\displaystyle\int \frac{1}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x} + C.$
20. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C.$
21. $\displaystyle\int \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C.$
22. $\displaystyle\int x^2\sqrt{a^2-x^2} = \frac{a^4}{8}\arcsin\frac{x}{a} - \frac{1}{8}x\sqrt{a^2-x^2}(a^2-2x^2) + C.$
23. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x} = \sqrt{a^2-x^2} - a\ln\left\|\frac{a+\sqrt{a^2-x^2}}{x}\right\| + C.$
24. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x^2} = -\arcsin\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{x} + C.$

25. $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}} = \frac{a^2}{2}\arcsin\frac{x}{a} - \frac{1}{2}x\sqrt{a^2-x^2} + C.$
26. $\displaystyle\int \frac{1}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C.$
27. $\displaystyle\int \frac{1}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C.$
28. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}} = \text{ Arg } \cosh\frac{x}{a} + C = \ln\left|x+\sqrt{x^2-a^2}\right| + C.$
29. $\displaystyle\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\text{ Arg } \cosh\frac{x}{a} + C.$
30. $\displaystyle\int x^2\sqrt{x^2-a^2} = \frac{x}{8}\left(2x^2-a^2\right)\sqrt{x^2-a^2} - \frac{a^4}{8}\text{ Arg } \cosh\frac{x}{a} + C.$
31. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x} = \sqrt{x^2-a^2} - a\,\text{arcsec }\left|\frac{x}{a}\right| + C.$
32. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^2} = \text{ Arg } \cosh\frac{x}{a} - \frac{\sqrt{x^2-a^2}}{x} + C.$
33. $\displaystyle\int \frac{x^2}{\sqrt{x^2-a^2}} = \frac{a^2}{2}\text{ Arg } \cosh\frac{x}{a} + \frac{x}{2}\sqrt{x^2-a^2} + C.$
34. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}} = \frac{1}{a}\text{arcsec }\left|\frac{x}{a}\right| + C = \frac{1}{a}\arccos\left|\frac{a}{x}\right| + C.$
35. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C.$
36. $\displaystyle\int \frac{1}{\sqrt{2ax-x^2}} = \arcsin\left(\frac{x-a}{a}\right) + C.$
37. $\displaystyle\int \sqrt{2ax-x^2} = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x-a}{a}\right) + C.$
38. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x} = \sqrt{2ax-x^2} + a\arcsin\frac{x-a}{a} + C.$
39. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x^2} = -2\sqrt{\frac{2a-x}{x}} - \arcsin\left(\frac{x-a}{a}\right) + C.$

40. $\displaystyle\int \frac{x}{\sqrt{2ax-x^2}} = a\arcsin\frac{x-a}{a} - \sqrt{2ax-x^2} + C.$
41. $\displaystyle\int \frac{1}{x\sqrt{2ax-x^2}} = -\frac{1}{a}\frac{\sqrt{2a-x}}{x} + C.$
42. $\displaystyle\int \sin ax = -\frac{1}{a}\cos ax + C.$
43. $\displaystyle\int \cos ax = \frac{1}{a}\sin ax + C.$
44. $\displaystyle\int \sin^2 ax = \frac{x}{2} - \frac{\sin 2ax}{4a} + C.$
45. $\displaystyle\int \cos^2 ax = \frac{x}{2} + \frac{\sin 2ax}{4a} + C.$
46. $\displaystyle\int \sin^n ax = \frac{-\sin^{n-1} ax\cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax.$
47. $\displaystyle\int \cos^n ax = \frac{\cos^{n-1} ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax.$
48. a) $\displaystyle\int \sin ax\cos bx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} + C\,, \quad a^2 \neq b^2.$ b) $\displaystyle\int \sin ax\sin bx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}\,, \quad a^2 \neq b^2.$ c) $\displaystyle\int \cos ax\cos bx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}\,, \quad a^2 \neq b^2.$
49. $\displaystyle\int \sin ax\cos ax = -\frac{\cos 2ax}{4a} + C.$
50. $\displaystyle\int \sin^n ax\cos ax = \frac{\sin^{n+1} ax}{(n+1)a} + C\,, \quad n \neq -1.$
51. $\displaystyle\int \frac{\cos ax}{\sin ax} = \frac{1}{a}\ln\vert\sin ax\vert + C.$
52. $\displaystyle\int \cos^n ax\sin ax = -\frac{\cos^{n+1} ax}{(n+1)a} + C\,, \quad n \neq -1.$
53. $\displaystyle\int \frac{\sin ax}{\cos ax} = -\frac{1}{a}\ln\vert\cos ax\vert + C.$

54. $\displaystyle\int \sin^n ax \cos^m ax = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(m+n)} + \frac{n-1}{m+n}\int \sin^{n-2} ax \cos^m ax$, $n \neq -m$
55. $\displaystyle\int \sin^n ax \cos^m ax = \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(m+n)} + \frac{m-1}{m+n}\int \sin^{n} ax \cos^{m-2} ax$, $m \neq -n$
56. $\displaystyle\int x \sin ax = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C.$
57. $\displaystyle\int x \cos ax = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C.$
58. $\displaystyle\int \operatorname{tg} ax = -\frac{1}{a}\ln\lvert\cos ax\rvert + C.$
59. $\displaystyle\int \operatorname{cotg} ax = \frac{1}{a}\ln\lvert\sin ax\rvert + C.$
60. $\displaystyle\int \operatorname{tg}^2 ax = \frac{1}{a}\operatorname{tg} ax - x + C.$
61. $\displaystyle\int \operatorname{cotg}^2 ax = -\frac{1}{a}\operatorname{cotg} ax - x + C.$
62. $\displaystyle\int \sec ax = \frac{1}{a}\ln\lvert\sec ax + \operatorname{tg} ax\rvert + C.$
63. $\displaystyle\int \operatorname{cosec} ax = -\frac{1}{a}\ln\lvert\operatorname{cosec} ax + \operatorname{cotg} ax\rvert + C.$
64. $\displaystyle\int \sec^2 ax = \frac{1}{a}\operatorname{tg} ax + C.$
65. $\displaystyle\int \operatorname{cosec}^2 ax = -\frac{1}{a}\operatorname{cotg} ax + C.$
66. $\displaystyle\int \arcsin ax = x \arcsin ax + \frac{1}{a}\sqrt{1-a^2x^2} + C.$
67. $\displaystyle\int \arccos ax = x \arccos ax - \frac{1}{a}\sqrt{1-a^2x^2} + C.$

68. $\displaystyle\int \operatorname{arctg}\, ax = x \operatorname{arctg}\, ax - \frac{1}{2a}\ln(1+a^2x^2) + C.$
69. $\displaystyle\int e^{ax} = \frac{1}{a}e^{ax} + C.$
70. $\displaystyle\int b^{ax} = \frac{1}{a}\frac{b^{ax}}{\ln b} + C\,, \quad b > 0\,, \quad b \neq 1.$
71. $\displaystyle\int xe^{ax} = \frac{e^{ax}}{a^2}(ax-1) + C.$
72. $\displaystyle\int x^n e^{ax} = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}.$
73. $\displaystyle\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$
74. $\displaystyle\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C.$
75. $\displaystyle\int \ln ax = x\ln ax - x + C.$
76. $\displaystyle\int x^n \ln ax = \frac{x^{n+1}}{n+1}\ln ax - \frac{x^{n+1}}{(n+1)^2} + C\,, \quad n \neq -1.$
77. $\displaystyle\int x^{-1}\ln ax = \frac{1}{2}(\ln ax)^2 + C.$
78. $\displaystyle\int \frac{1}{x\ln ax} = \ln\|\ln ax\| + C.$
79. $\displaystyle\int \sinh ax = \frac{1}{a}\cosh ax + C.$
80. $\displaystyle\int \cosh ax = \frac{1}{a}\sinh ax + C.$
81. $\displaystyle\int x\sinh ax = \frac{x}{a}\cosh ax - \frac{1}{a^2}\sinh ax + C.$
82. $\displaystyle\int x\cosh ax = \frac{x}{a}\sinh ax - \frac{1}{a^2}\cosh ax + C.$

83. $\displaystyle\int \tanh ax = \frac{1}{a}\ln(\cosh ax) + C.$
84. $\displaystyle\int \text{cotgh } ax = \frac{1}{a}\ln\lvert\sinh ax\rvert + C.$
85. $\displaystyle\int \text{cosech } ax = \frac{1}{a}\ln\left\lvert\tanh\frac{ax}{2}\right\rvert + C.$

Apèndix B: Les funcions d'Euler

Al llarg del text hem suposat conegudes, en diferents punts, com ara quan hem establert la fórmula de Dirichlet, les funcions Γ i B d'Euler. En aquest breu apèndix recordem la definició d'aquestes funcions, així com algunes de les fórmules associades.

La funció gamma: Γ

Si $x > 0$, es defineix la funció Γ com la integral impròpia

$$\boxed{\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.}$$

La funció Γ està ben definida (en el sentit que és una integral convergent) per a tot $x > 0$, i és una funció C^∞ per a aquests valors. La propietat fonamental de Γ és la relació funcional que estableix el resultat següent.

Proposició. $\Gamma(x+1) = x\Gamma(x)$.

En efecte, el resultat se segueix d'integrar per parts:

$$\int_0^\infty e^{-t} t^x dt = -t^x e^{-t}\Big|_0^\infty + x \int_0^\infty e^{-t} t^{x-1} dt = x\Gamma(x).$$

■

Com que $\Gamma(1) = 1$, de la proposició anterior se segueix que

$$\Gamma(n+1) = n!\,.$$

A més, al capítol 2 hem provat que

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,,$$

d'on se segueix que

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}\,.$$

La funció beta: B

Per a $x, y > 0$ es defineix la funció beta d'Euler per

$$\boxed{B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.}$$

De les propietats de la funció B destaquem:

Proposició.

1. $B(x,y) = B(y,x)$.
2. $B(x,y) = 2\displaystyle\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta$.
3. $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.
4. $B(x,y) = \displaystyle\int_0^{\infty} \frac{v^{x-1}}{(1+v)^{x+y}}\,dv$.

La primera propietat és clara. Quant a la segona, considerem el canvi de variable

$$t = \sin^2\theta\,, \qquad 0 < \theta < \pi/2\,.$$

Aleshores, es té que

$$\begin{aligned}\int_0^1 t^{x-1}(1-t)^{y-1}dt &= \int_0^{\pi/2} \sin^{2x-2}\theta\cos^{2y-2}\cdot 2\sin\theta\cos\theta d\theta \\ &= 2\int_0^{\pi/2}\sin^{2x}\theta\cos^{2y-1}\theta d\theta\,.\end{aligned}$$

Provem ara 3: en el producte $\Gamma(x)\Gamma(y)$,

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-t}t^{x-1}dt\int_0^\infty e^{-t}t^{y-1}dt\,,$$

fem el canvi de variable $t = u^2$, $t = v^2$ i calculem la integral resultant utilitzant coordenades polars:

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty e^{-u^2}u^{2x-2}\cdot 2udu\cdot\int_0^\infty e^{-v^2}v^{2y-2}2vdv \\ &= 4\int_0^\infty\int_0^\infty e^{-u^2-v^2}u^{2x-1}v^{2y-1}dudv \\ &= 4\int_0^{2\pi}d\theta\int_0^\infty e^{-r^2}r^{2x-1}\cos^{2x-1}\theta\, r^{2y-1}\sin^{2y-1}rdr \\ &= 2\int_0^{2\pi}\cos^{2x-1}\theta\sin^{2y-1}\theta d\theta\cdot 2\int_0^\infty e^{-r^2}r^{2x+2y-1}dr \\ &= B(x,y)\cdot\Gamma(x+y)\,.\end{aligned}$$

Deixem la quarta propietat com a exercici. ■

La propietat 3, juntament amb el càlcul de Γ en els enters i els semienters, permet calcular la funció B en punts de coordenades enteres i semienteres. Per exemple,

$$B\left(\frac{3}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)}=\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}=\frac{\pi}{2}\,.$$

Conjugant les propietats 2 i 3 trobem que

$$2\int_0^{\pi/2}\sin^{2x-1}\theta\cos^{2y-1}d\theta=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,.$$

En particular,

$$2\int_0^{\pi/2}\sin^n\theta\cos^m\theta d\theta=\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+m+2}{2}\right)}\,.$$

Així, per exemple,

$$\int_0^{\pi/2}\sin^4\theta\cos^2\theta d\theta=\frac{1}{2}\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)}=\frac{1}{2}\frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{3\cdot 2}=\frac{\pi}{96}\,.$$

Apèndix C: Sistemes de coordenades curvilínies

Recollim aquí les fórmules dels operadors clàssics en sistemes ortogonals que hem provat al llarg del text, així com diversos exemples de sistemes ortogonals de coordenades.

Sistema ortogonal de coordenades

Si (u, v, w) és un sistema ortogonal de coordenades, els coeficients de dilatació es defineixen per

$$h_1 = \parallel \varphi_u \parallel, \quad h_2 = \parallel \varphi_v \parallel, \quad h_3 = \parallel \varphi_w \parallel,$$

i la base ortonormal associada és

$$e_u = \frac{1}{h_1}\varphi_u, \quad e_v = \frac{1}{h_2}\varphi_v, \quad e_w = \frac{1}{h_3}\varphi_w.$$

Sigui f una funció escalar i $F = F_u e_u + F_v e_v + F_w e_w$ un camp vectorial.

1. Gradient: $\nabla f = \dfrac{1}{h_1}\dfrac{\partial f}{\partial u} e_u + \dfrac{1}{h_2}\dfrac{\partial f}{\partial v} e_v + \dfrac{1}{h_3}\dfrac{\partial f}{\partial w} e_w.$
2. Rotacional:
$$\frac{1}{h_2h_3}\left(\frac{\partial(h_3F_w)}{\partial v} - \frac{\partial(h_2F_v)}{\partial w}\right)e_u + \frac{1}{h_1h_3}\left(\frac{\partial(h_1F_u)}{\partial w} - \frac{\partial(h_3F_w)}{\partial u}\right)e_v + \frac{1}{h_1h_2}\left(\frac{\partial(h_2F_v)}{\partial u} - \frac{\partial(h_1F_u)}{\partial v}\right).$$
3. Divergència: $\operatorname{div} F = \dfrac{1}{h_1h_2h_3}\left(\dfrac{\partial(h_2h_3F_u)}{\partial u} + \dfrac{\partial(h_1h_3F_v)}{\partial v} + \dfrac{\partial(h_1h_2F_w)}{\partial w}\right).$
4. Laplaciana: $\Delta f = \dfrac{1}{h_1h_2h_3}\left(\dfrac{\partial}{\partial u}\left(\dfrac{h_2h_3}{h_1}\dfrac{\partial f}{\partial u}\right) + \dfrac{\partial}{\partial v}\left(\dfrac{h_1h_3}{h_2}\dfrac{\partial f}{\partial v}\right) + \dfrac{\partial}{\partial w}\left(\dfrac{h_1h_2}{h_3}\dfrac{\partial f}{\partial w}\right)\right).$

Coordenades cilíndriques

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z.$$

$$h_1 = 1,\ h_2 = r,\ h_3 = 1.$$

1. Gradient: $\nabla f = \dfrac{\partial f}{\partial r} e_r + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta} e_\theta + \dfrac{\partial f}{\partial z} e_z.$

2. Rotacional:

$$\frac{1}{r}\left(\frac{\partial F_z}{\partial \theta} - \frac{\partial (rF_\theta)}{\partial z}\right) e_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right) e_\theta + \frac{1}{r}\left(\frac{\partial (rF_z)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right) e_z \,.$$

3. Divergència: $\operatorname{div} F = \dfrac{1}{r}\dfrac{\partial (rF_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial F_\theta}{\partial \theta} + \dfrac{\partial F_z}{\partial z}.$

4. Laplaciana: $\Delta f = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial \theta^2} + \dfrac{\partial^2 f}{\partial z^2}.$

Coordenades esfèriques

$$x = r\cos\phi\cos\theta,\ \ y = r\cos\phi\sin\theta,\ \ z = r\sin\phi.$$

$$h_1 = 1,\ \ h_2 = r\cos\phi,\ \ h_3 = r.$$

1. Gradient: $\nabla f = \dfrac{\partial f}{\partial r} e_r + \dfrac{1}{r\cos\phi}\dfrac{\partial f}{\partial \theta} e_\theta + \dfrac{1}{r}\dfrac{\partial f}{\partial \phi} e_\phi.$

2. Rotacional:

$$\frac{1}{r\cos\phi}\left(\frac{\partial F_\phi}{\partial \theta} - \frac{\partial (F_\theta \cos\phi)}{\partial \phi}\right) e_r + \frac{1}{r}\left(\frac{\partial F_r}{\partial \phi} - \frac{\partial (rF_\phi)}{\partial r}\right) e_\theta + \frac{1}{r\cos\phi}\left(\frac{\partial (r\cos\phi F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right) e_\phi \,.$$

3. Divergència: $\operatorname{div} F = \dfrac{1}{r^2}\dfrac{\partial (r^2 F_r)}{\partial r} + \dfrac{1}{r\cos\phi}\dfrac{\partial F_\theta}{\partial \theta} + \dfrac{1}{r\cos\phi}\dfrac{\partial (\cos\phi F_\phi)}{\partial \phi}.$

4. Laplaciana: $\Delta f = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2\cos^2\phi}\dfrac{\partial^2 f}{\partial \theta^2} + \dfrac{1}{r^2\cos\phi}\dfrac{\partial}{\partial \phi}\left(\cos\phi\dfrac{\partial f}{\partial \phi}\right).$

Coordenades cilíndriques parabòliques

$$x = \frac{1}{2}(u^2 - v^2)\,, \quad y = uv, \quad z = z,$$

$$-\infty < u < \infty, \quad v > 0, \quad -\infty < z < \infty,$$

$$h_1 = h_2 = \sqrt{u^2 + v^2}\,, \quad h_3 = 1.$$

Les corbes coordenades són paràboles homofocals amb un eix comú.

Coordenades cilíndriques el·líptiques

$$x = a \cosh u \cos v \,, \quad y = a \sinh u \sin v, \quad z = z,$$
$$n \geq 0, \quad 0 < v < 2\pi, \quad -\infty < z < \infty,$$
$$h_1 = h_2 = a\sqrt{\sinh^2 u + \sin^2 v}\,, \quad h_3 = 1.$$

Les corbes coordenades són el·lipses i hipèrboles homofocals.

Coordenades esferoïdals allargades

$$x = a \sinh \xi \sin \eta \cos \varphi \,, \quad y = a \sinh \xi \sin \eta \sin \varphi, \quad z = a \cosh \xi \cos \eta,$$
$$\xi \geq 0, \quad 0 \leq \eta \leq \pi, \quad 0 \leq \varphi \leq 2\pi,$$
$$h_1 = h_2 = a\sqrt{\sinh^2 \xi + \sin^2 v}\,, \quad h_3 = a \sinh \xi \sin \eta.$$