# Chapter 1. Summary of Mechanics.

# Solving the equation of motion for variable force cases

## 1 A very summary of principles of mechanics

### 1.1 Newton's Principles

1) **Objects remain either at rest or at constant velocity unless a force acts upon them.** This principle works for inertial frames of reference (**SRI**), that is, for frames of reference which are either at rest or moving with constant velocity. Note tat velocity must be understood as a vector. Therefore, constant velocity means constant magnitude and constant direction, that is, rectilinear motion.

2) **In SRI**, the relation between the net force $\vec{F}_{net}$ (the sum of all forces) acting upon an object and its acceleration $\vec{a}$ is given by the expression:

$$\vec{F}_{net} = m\vec{a}, \tag{1}$$

where $m$ is the inertial mass of the object. The more massive the object, the lower the acceleration it will undergo for a given $\vec{F}_{net}$.

3) If an object $A$ exerts a force upon object $B$, $\vec{F}_{A\rightarrow B}$, then , object $B$ will exert a force upon object $A$, $\vec{F}_{B\rightarrow A}$, such that:

$$\vec{F}_{B\rightarrow A} = -\vec{F}_{A\rightarrow B}. \tag{2}$$

That is, the reaction force $\vec{F}_{B\rightarrow A}$ will have the same magnitude, same direction and opposite sign to the force $\vec{F}_{A\rightarrow B}$).

### 1.2 Concept of linear momentum and linear momentum conservation principle

Given a system of n particles of masses $\{m_i\}$, with i=1,..., n, we can define the linear momentum of the system as:

$$\vec{P} = \sum_{i=1}^{n} m_i \vec{v}_i \tag{3}$$

If we consider the time variation of $\vec{P}$, assuming that the masses $m_i$ remain constant, using Newton's second principle [1], and realising that the sum of all internal forces will cancel out *(example system Earth-Moon and external force due to the Sun)*:

$$\frac{d\vec{P}}{dt} = \sum_{i=1}^{n} m_i \frac{d\vec{v}_i}{dt} = m_i \vec{a}_i = \vec{F}_{net}^{ext}$$

$$\Rightarrow \frac{d\vec{P}}{dt} = \vec{F}_{net}^{ext} \tag{4}$$

That is, the time variation of linear momentum equals to the net force acting upon the system. Therefore, if the net force acting upon the system is zero, the linear momentum will be conserved. This is the **Principle of linear momentum conservation**:

$$\vec{F}_{net} = 0 \Rightarrow \vec{P} = constant \tag{5}$$

In many instances it is important to consider the centre of mass of a system of particles. Its position ($\vec{R}$), velocity ($\vec{V}$) and acceleration ($\vec{A}$) are given by the following expressions:

$$\vec{R} = \frac{1}{M_{tot}} \sum_{i=1}^{n} m_i \vec{r}_i$$

where $\vec{r}_i$ is the position vector of the particle of mass $m_i$, and $M_{tot} = \sum_{i=1}^{n} m_i$ is the total mass of the system.

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{1}{M_{tot}} \sum_{i=1}^{n} m_i \vec{v}_i$$

$$\vec{A} = \frac{d\vec{V}}{dt} = \frac{1}{M_{tot}} \sum_{i=1}^{n} m_i \vec{a}_i$$

Note that $\sum_{i=1}^{n} m_i \vec{v}_i = \vec{P}_i$, and therefore $\vec{V} = \frac{\vec{P}}{M_{tot}}$. If we consider the latter expression together with the principle of linear momentum conservation, it is easy to see that the velocity of centre of mass $\vec{V}$ will remain constant for systems upon which the net force is zero.

## 1.3 Concept of angular momentum and angular momentum conservation principle

The angular momentum of a particle of mass $m_i$, velocity $\vec{v}_i$, and position $\vec{r}$ with respect to the origin can be written as:

$$\vec{L} = \sum_{i=1}^{n} m_i \vec{r}_i \times \vec{v}_i \tag{6}$$

[1] We will consider variable mass systems in the system later in this course. Be patient by now.

It may be interpreted as the analog of linear momentum, $\vec{P}$, in the case of rotating systems. The time variation of angular momentum is given by:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\sum_{i=1}^{n} m_i\vec{r}_i \times \vec{v}_i) = \sum_{i=1}^{n}(m_i\frac{d\vec{r}_i}{dt} \times \vec{v}_i + m_i\vec{r}_i \times \frac{d\vec{v}_i}{dt}) =$$
$$= \vec{0} + \sum_{i=1}^{n}\vec{r}_i \times m_i\vec{a}_i = \sum_{i=1}^{n}\vec{r}_i \times \vec{F}_i = \sum_{i=1}^{n}\vec{\tau}_i^{ext} = \vec{\tau}_{net}^{ext}$$
$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{\tau}_{net}^{ext} \qquad (7)$$

where $\tau$ is the torque applied on the system. You should note that the sum of internal torques also cancels out, think of it ;).

Therefore, if net external torques are zero, the angular momentum will be conserved. This is the **Angular momentum conservation principle**:

$$\vec{\tau}_{net} = \vec{0} \Rightarrow \vec{L} = constant \qquad (8)$$

Note that, in the case of extended systems $\vec{L} = I\vec{\omega}$, where $I$ is the moment of inertia of the system with respect to a certain axis, and $\omega$ is the rotational velocity along that axis. In this case the angular momentum conservation principle also holds.

### 1.4 Energy balance and energy conservation

The kinetic energy for a system of n particles is:

$$K = \sum_{i=1}^{n}\frac{1}{2}m_i\vec{v}_i^2 \qquad (9)$$

If we write Newton's second law as $\sum_{i=1}^{n}\vec{F}_i = \sum_{i=1}^{n} m_i\vec{a}_i = \sum_{i=1}^{n} m_i\frac{d\vec{v}_i}{dt}$ and multiply the left and right side of the equation by $\vec{v}_i$ (inside the sum) we get:

$$\sum_{i=1}^{n}\vec{v}_i \cdot m_i\frac{d\vec{v}_i}{dt} = \sum_{i=1}^{n}\vec{v}_i \cdot \vec{F}_i = \mathcal{P},$$

where $\mathcal{P}$ represents the power. It is easy to see that the left hand side in the above equation corresponds to the time variation of the kinetic energy of the system, and that the power can be divided into power due to conservative forces, $\mathcal{P}_{cons}$, and the power due to non conservative forces, $\mathcal{P}_{noc}$.

$$\Rightarrow \frac{d}{dt}(\sum_{i=1}^{n}\frac{1}{2}m_i\vec{v}_i^2) = \mathcal{P}_{cons} + \mathcal{P}_{noc}$$
$$\Rightarrow \frac{dK}{dt} = \mathcal{P} = \mathcal{P}_{cons} + \mathcal{P}_{noc} \qquad (10)$$

You might be more familiar with the integral version of this theorem, which is also named the work-(kinetic) energy theorem (by rather obvious reasons):

$$W = \Delta K = K_{fin} - K_{ini} \tag{11}$$

Let us recall that the power is the time variation of work and, in particular $\mathcal{P}_{cons} = \frac{dW_{cons}}{dt}$. Besides, the work due to conservative forces is:

$$W_{cons} = \int \vec{F}_{cons} \cdot d\vec{r}, \tag{12}$$

and the potential energy is :

$$V = -\int \vec{F}_{cons} \cdot d\vec{r} = -W_{cons} \quad (\vec{F}_{cons} = -\vec{\nabla} V) \tag{13}$$

*You may recall here the example of the variation of potential energy of a rock of mass m falling a certain height h: V=-mgh, and the work done by the weight W=(-mg)(-h)=mgh $\Rightarrow$ V=-W.*

Therefore:

$$\mathcal{P}_{cons} = \frac{dW_{cons}}{dt} = -\frac{dV}{dt}$$

Therefore:

$$\frac{dK}{dt} = -\frac{dV}{dt} + \mathcal{P}_{noc} \Rightarrow \frac{d(K+V)}{dt} = \frac{dE_{mec}}{dt} = P_{noc},$$

And its perhaps more familiar integral form:

$$\Delta E_{mec} = W_{noc}, \tag{14}$$

where $E_{mec}$ is the total mechanical energy (K+V). Note that if only conservative forces act upon the system, its mechanical energy will be conserved. This is the **Energy conservation principle**.

$$\mathcal{P}_{noc} = 0 \Rightarrow E_{mec} = constant \tag{15}$$

## 1.5 Some considerations on acceleration

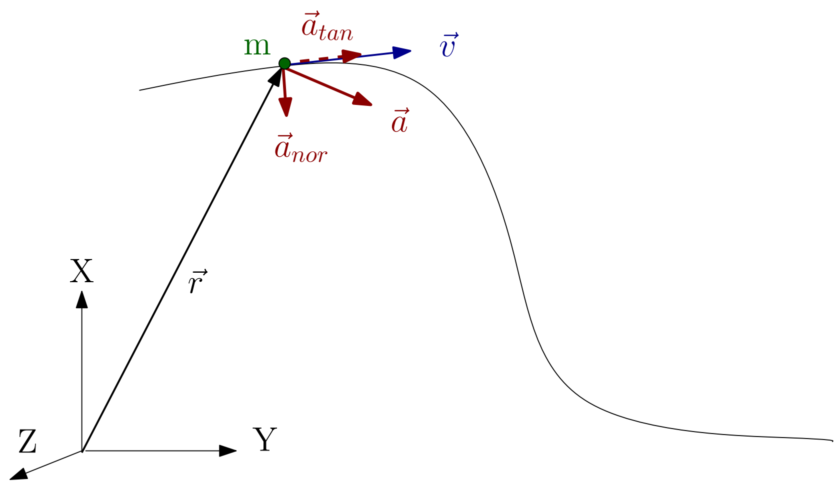


Figure 1: Normal and tangent components of the acceleration vector.

It is frequently convenient to separate the acceleration vector into two components: one component tangent to the object's trajectory, and another component perpendicular to the trajectory:

$$\vec{a} = \vec{a}_{tan} + \vec{a}_{nor} \tag{16}$$

The unitary vector tangent to the trajectory in a certain instant $t$ is given by:

$$\vec{u} = \frac{\vec{v}(t)}{|\vec{v}(t)|},$$

and therefore, the acceleration component along the direction given by $\vec{u}$ will be:

$$a_{tan} = \vec{a} \cdot \vec{u} = \frac{d\vec{v}}{dt} \cdot \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

The above expression for $a_{tan}$ must of course coincide with the well known $a_{tan} = \frac{d|\vec{v}|}{dt}$. Expressed as a vector:

$$\vec{a}_{tan} = (\vec{a} \cdot \vec{u}) \cdot \vec{u} = \left(\frac{d\vec{v}}{dt} \cdot \frac{\vec{v}(t)}{|\vec{v}(t)|}\right) \cdot \frac{\vec{v}(t)}{|\vec{v}(t)|} \tag{17}$$

The normal component $\vec{a}_{nor}$ will be simply given by:

$$\vec{a}_{norm} = \vec{a} - \vec{a}_{tan}$$

It is easy to show that $a_{norm} = \frac{v^2}{\rho}$, where $\rho$ is the curvature radius at the given position. In the case of circular motion $\rho$ is the radius of the circle associated to the trajectory. In the case of rectilinear motion, $\rho \to \infty$.

# 2 Solving the equation of motion for variable forces

We are very familiar with the motion of objects under no force (and thus with no acceleration and constant velocity $\vec{v}_0$). Position versus time in these cases varies linearly and can be expressed as:

$$\vec{r}(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{v}(t)dt = \vec{r}_0 + \vec{v}_0(t - t_0)$$

where $\vec{r}_0$ represents the initial position.

We also know well the motion of objects under constant forces (which of course imply constant accelerations $\vec{a}_0$). In this case the velocity varies linearly with time:

$$\vec{v}(t) = \vec{v}_0 + \vec{a}_0(t - t_0)$$

where $\vec{v}_0$ in this case is the initial velocity. Position versus then time varies as a quadratic function:

$$\vec{r}(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{v}(t)dt = \vec{r}_0 + \vec{v}_0(t - t_0) + \frac{1}{2}\vec{a}_0(t - t_0)^2$$

However, most interesting cases in mechanics are not so simple. Forces may depend on time, on velocity (as in the case of viscous friction), or on position (as in the case of gravity or the elastic force). In this section we will consider the former cases. We will see that $\vec{r}(t)$ will be represented by functions which, in general, will not be simply linear or quadratic.

Given a certain equation of motion $F = ma$, we will now show how to integrate this equation to obtain its solution, $\vec{r}(t)$, for different cases of variable forces. We will start working in 1 dimension, but our results can be trivially generalised to problems in 3 dimensions in the cases we will consider in this chapter. Note however that, again, reality tends to be more complex. The equations of motion in the different dimension (X, Y and Z) can be coupled. For instance, the acceleration along the X axis may depend on the velocity along the Y and Z components, as in the case of friction forces which depend on $\left|\vec{v}^2\right|$, or when the Coriolis effect is taken into account.

Finally, note also that this is all about solving second order differential equations, and in each case we will need two constants (two initial conditions $x(t = 0) = x_0$ and $v(t = 0) = v_0$) in order to get their solution.

## 2.1 Forces depending on time

If forces depend on time the equation of motion will be easy to integrate if we simple express the acceleration a as the time derivative of velocity:

$$\sum F = F(t) = m\,a = m\,\frac{d\vec{v}}{dt} \Rightarrow dv = \frac{F(t)}{m}dt \Rightarrow \int_{t_0}^{t} v(t) = \int_{t_0}^{t} \frac{F(t)}{m}dt$$

$$\Rightarrow v(t) = v_0 + \int_{t_0}^{t} \frac{F(t)}{m}dt = v_0 + \int_{t_0}^{t} a(t)dt \tag{18}$$

After calculating the former integral we get an expression for the velocity as a function of time. After a second integration with time we get $r(t)$.

$$x(t) = x_0 + \int_{t_0}^{t} v(t)dt \tag{19}$$

The concept of terminal velocity.

*Example: Consider a particle of mass m, initial position $x(t=0)=x_0$, and initial velocity $v(t=0)=v_0$ known. The particle moves under the effect of a force $F(t)=A+Be^{\gamma t}$.*

1. *Express the equation of motion for m.*

$$F = ma \Rightarrow A + Be^{\gamma t} = m\frac{dv}{dt}$$

2. *Give an expression for the velocity of m v=v(t). Check that for time t=0 you recover the corresponding initial condition.*

$$a = \frac{dv}{dt} = \frac{A}{m} + \frac{B}{m}e^{\gamma t}$$

$$\Rightarrow v(t) = v_0 + \frac{1}{m}\int_0^t A + Be^{\gamma t}dt$$

$$\Rightarrow v(t) = v_0 + \frac{A}{m}t + \frac{B}{\gamma m}(e^{\gamma t} - 1)$$

*Simply substituting t=0 in the above expression we can see that $v(t=0) = v_0$.*

3. *Give an expression for the position of m x=x(t). Check that for time t=0 you recover the corresponding initial condition.*

$$x(t) = x_0 + \int_0^t v(t)dt = x_0 + \int_0^t v_0 + \frac{A}{m}t + \frac{B}{\gamma m}(e^{\gamma t} - 1)dt$$

$$\Rightarrow x(t) = x_0 + (v_0 - \frac{B}{\gamma m})t + \frac{A}{2m}t^2 + \frac{B}{\gamma^2 m}(e^{\gamma t} - 1)$$

*Substituting t=0 in the above expression we can see that $x(t=0) = x_0$.*

4. *Which must be the dimensions of A, B and $\gamma$?*

   *By consistency, A and B must have dimensions of force: $[A] = [B] = MLT^{-2}$. $\gamma$ must have dimensions $[\gamma] = T^{-1}$.*

## 2.2 Forces depending on velocity

In this case we will proceed as before, that is, we will also express the acceleration a as the time derivative of velocity and integrate:

$$\sum F = F(v) = m\,a = m\,\frac{d\vec{v}}{dt} \Rightarrow \frac{dv}{F(v)} = \frac{1}{m}dt$$

$$\Rightarrow \int_{v_0}^{v} \frac{dv}{F(v)} = \frac{1}{m}\int_{t_0}^{t} dt = \frac{1}{m}t \tag{20}$$

Again, after calculating the former integral we get an expression for the velocity as a function of time, $v(t)$. The position versus time $r(t)$ is also obtained as before.

$$\vec{r}(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{v}(t)dt$$

*Example: Consider a particle of mass $m$, initial position $x(t=0) = x_0$, and initial velocity $v(t=0) = v_0$ known. The particle moves under the effect of a constant force $F = F_0$, and friction which varies linearly with its velocity: $F_{fric} = -bv$, where $b$ is a known constant with dimensions $[b] = MT^{-1}$.*

1. *Express the equation of motion for m.*

$$F = ma \Rightarrow F_0 - bv = m\frac{dv}{dt}$$

2. *Give an expression for the velocity of m v=v(t). Check that for time t=0 you recover the corresponding initial condition.*

$$\int_{v_0}^{v} \frac{dv}{F(v)}dv = \frac{1}{m}t$$

$$\Rightarrow \int_{v_0}^{v} \frac{dv}{F_0 - bv}dv = \frac{1}{m}t \Rightarrow \frac{-1}{b}\log|F_0 - bv|_{v_0}^{v} = \frac{1}{m}t$$

$$\Rightarrow v(t) = \frac{F_0}{b} + (v_0 - \frac{F_0}{b})e^{-\frac{b}{m}t}$$

*The term $\frac{b}{m}$ in the above exponential has dimensions of $T^{-1}$, and one can define the* characteristic time $\tau = \frac{m}{b}$. *Then we can write $v(t)$:*

$$v(t) = \frac{F_0 \tau}{m} + (v_0 - \frac{F_0 \tau}{m})e^{-\frac{t}{\tau}}$$

*Simply substituting t=0 in the above expression we can see that $v(t = 0) = v_0$.*

3. *Give an expression for the position of m x=x(t). Check that for time t=0 you recover the corresponding initial condition.*

$$x(t) = x_0 + \int_0^t v(t)dt = x_0 + \int_0^t \frac{F_0}{b} + (v_0 - \frac{F_0}{b})e^{-\frac{b}{m}t}$$

$$\Rightarrow x(t) = x_0 + \frac{F_0}{b}t + \frac{m}{b}(v_0 - \frac{F_0}{b})(1 - e^{-\frac{b}{m}t})$$

*Substituting t=0 in the above expression we can see that $x(t = 0) = x_0$. We can also write x(t) in terms of the characteristic time $\tau$:*

$$x(t) = x_0 + \frac{F_0 \tau}{m}t + \tau(v_0 - \frac{F_0 \tau}{m})(1 - e^{-\frac{t}{\tau}})$$

*Both the expressions for velocity and position versus time have terms proportional to the decaying exponential: $e^{-\frac{b}{m}t}$. We can define the characteristic time $\tau = \frac{1}{b/m} = \frac{m}{b}$, and thus express $e^{-\frac{b}{m}t} = e^{-\frac{t}{\tau}}$. We can then weite $v(t)$ and $x(t)$ as:*

Note that, in the former example, $F_0$ always remains constant, whereas the force of friction $F_f = -bv$ varies with time. Eventually, it may occur that $F_0 - bv = 0$. At this point, when forces balance each other, the acceleration becomes zero and, of course, velocity will be constant. This constant velocity is the so-called **terminal velocity**. It is easy to see that, in the above example $v_{lim} = \frac{F_0}{b}$.

Terminal velocity may be calculated from $\sum F(v) = 0$, or as $v_{lim} = Lim_{t\to\infty} v(t)$.

In some cases the quotient $\frac{b}{m}$ is sufficiently small so that the Taylor expansion for the exponential function reasonably holds:

$$e^{-\frac{b}{m}t} \sim 1 - \frac{b}{m}t + \frac{1}{2}\frac{b^2}{m^2}t^2$$

In these cases v(t) and x(t) can be easily approximated by substituting the former expansion.

*For instance, if we substitute in the expression for $v(t$ in the above example we get:*

$$v(t) = \frac{F_0}{b} + (v_0 - \frac{F_0}{b})e^{-\frac{b}{m}t} \sim \frac{F_0}{b} + (v_0 - \frac{F_0}{b})(1 - \frac{b}{m}t + \frac{1}{2}\frac{b^2}{m^2}t^2)$$

$$\Rightarrow v(t) = v_0 + (\frac{F_0}{m} - \frac{bv_0}{m})t + \frac{1}{2}\frac{b^2}{m^2}(v_0 - \frac{F_0}{b})t^2$$

$$\Rightarrow v(t) = v_0 + (\frac{F_0}{m} - \frac{v_0}{\tau})t + \frac{1}{2\tau^2}(v_0 - \frac{F_0\tau}{m})t^2$$

*Note that if the term $(b/m)^2$ is sufficiently small (or the characteristic time is very long) to be neglected:*

$$v(t) = v_0 + (\frac{F_0}{m} - \frac{bv_0}{m})t = v_0 + (\frac{F_0}{m} - \frac{v_0}{\tau})t$$

*And thus, the position versus time would correspond to a constant acceleration case:*

$$x(t) = x_0 + v_0 t + \frac{1}{2}(\frac{F_0}{m} - \frac{bv_0}{m})t^2 = x_0 + v_0 t + \frac{1}{2}(\frac{F_0}{m} - \frac{v_0}{\tau})t^2$$

*If $b/m$ is practically zero (or the characteristic time tends to infinity), we would simply get:*

$$x(t) = x_0 + v_0 t + \frac{1}{2}\frac{F_0}{m}t^2$$

*Obviously $\frac{F_0}{m} = a$ (acceleration), and thus you recover the expression for rectilinear motion at constant acceleration. Besides, you can easily check that if $F_0 = -mg$ you recover the expressions for v(t) and x(t) in the case of a free-falling object.*

## 2.3 Forces depending on position

In this case, because F=F(r), expressing the acceleration $a = \frac{dv}{dt}$ leaves a differential equation with 3 variables, and thus we must find an alternative. We can actually approach this problem in 3 different ways.

### 2.3.1 Rewriting acceleration:

Using the chain's rule $a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ In this case the equation of motion is:

$$F(x) = m\,a = mv\frac{dv}{dx} \Rightarrow mvdv = F(x)dx \Rightarrow m\int_{v_0}^{v} vdv = \int_{x_0}^{x} F(x)dx$$

Note that m$\int_{v_0}^{v} vdv = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$, that is, the variation of kinetic energy. Thus:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x} F(x)dx = W \qquad (21)$$

So we just recovered the K-W (kinetic energy-work) theorem. Besides, if F(x) is conservative $\int_{x_0}^{x} F(x)dx = W = -(V - V_0)$, that is, minus the variation of potential energy. Therefore $\Delta K = -\Delta V$, that is, we recover the principle of energy conservation.

In any case, we can always isolate $v = v(x)$ from the K-W theorem and express the velocity as $v = \frac{dx}{dt}$.

$$v(x) = \frac{dx}{dt} \Rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{\int_{x_0}^{x} F(x)dx + \frac{1}{2}mv_0^2}} = \int_{t_0}^{t} \sqrt{\frac{2}{m}}dt \tag{22}$$

After performing the former integral we will get x(t).
Note that if F(x) is conservative:

$$\int_{x_0}^{x} \frac{dx}{\sqrt{V_0 - V(x) + \frac{1}{2}mv_0^2}} = \int_{x_0}^{x} \frac{dx}{\sqrt{E - V(x)}} = \int_{t_0}^{t} \sqrt{\frac{2}{m}}dt \tag{23}$$

#### 2.3.2 Using directly E=K+V(x)

This method is actually equivalent to the former. If E is constant (F(x) conservative), we can obtain E using the initial conditions $x_0$ and $v_0$, and that the potential energy $V(x) = -\int_{x_0}^{x} F(x)dx$. Then:

$$E = K + V(x) = \frac{1}{2}mv^2 + V(x) = \frac{1}{2}m(\frac{dx}{dt})^2 + V(x) \Rightarrow \frac{dx}{\sqrt{E - V(x)}} = \sqrt{\frac{2}{m}}dt$$

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{E - V(x)}} = \int_{t_0}^{t} \sqrt{\frac{2}{m}}dt$$

This expression is equivalent to Equation 23. Obviously, as a result from the former integral we can get the solution $x(t)$.

#### 2.3.3 Using what we know about Ordinary Differential Equations

We will talk about this in Chapter 2 :).

*Example: Consider a homogeneous density non-extensible chain of mass $M$ and total length $L$. The chain lays at rest, on top of a frictionless table, in such a way that only a tiny bit of its length, $x_0$, hangs from the surface of the table (see figure).*

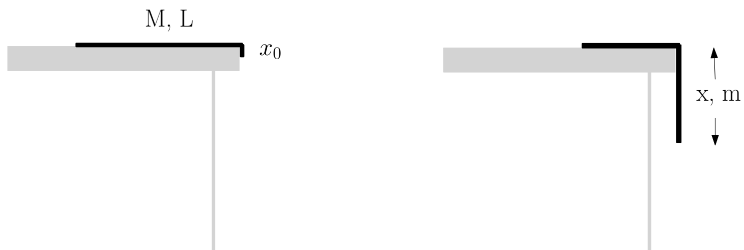


Figure 2: Example, the falling chain.

1. *Express the mass hanging from the table, $m$, when the hanging length has an arbitrary value $x$.*

   *Knowing that the chain is homogeneous:*

$$\frac{M}{L} = \frac{m}{x} = \frac{dm}{dx} \Rightarrow m = \frac{M}{L}x$$

2. *Express the net force acting upon the chain when an arbitrary length $x$ hangs from the table. Assuming that its total energy at time t=0 is $E_0 = 0$. Determine an expression for its potential energy.*

   *As the table is frictionless, the only net force is the weight on the hanging part of the chain. Note that the weight on the part of the chain which lays on top of the table is balanced by a contact (normal) force. Therefore:*

$$F = mg = \frac{M}{L}gx$$

   *The force on the chain depends on its hanging length, we expect an increasing acceleration (and thus an increasing velocity as a longer part of the chain hangs from the table.). Its potential energy will be:*

$$V(x) = -\int_{x_0}^{x} F(x)dx = -\int_{x_0}^{x} \frac{M}{L} g\, x\, dx = \frac{Mg}{2L}(x_0^2 - x^2)$$

   *Indeed, if $x = x_0$ (at the starting point of our problem), $E_0 = V(x_0) = 0$, as initially required. Note that we are also taking into account that the chain is initially at rest, and thus its initial kinetic energy $K_0 = 0$.*

3. *Express the equation of motion for m.*

$$F = ma \Rightarrow \frac{M}{L} gx = M \frac{d^2x}{dt^2}$$

4. *Solve the equation of motion of the chain (that is, find x=x(t)). Check that for time t=0 you recover the corresponding initial condition.*

   *We can use equation 23*:

$$\int_{x_0}^{x} \frac{dx}{\sqrt{E - V(x)}} = \int_{t_0}^{t} \sqrt{\frac{2}{M}} dt \Rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{Mg}{2L}(x^2 - x_0^2)}} = \int_{t_0}^{t} \sqrt{\frac{2}{M}} dt$$

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{(x^2 - x_0^2)}} = \sqrt{\frac{g}{L}} t \Rightarrow Log \left| \sqrt{x^2 - x_0^2} + x \right|_{x_0}^{x} = \sqrt{\frac{g}{L}} t$$

$$\Rightarrow x + \sqrt{x^2 - x_0^2} = x_0 e^{\sqrt{\frac{g}{L}} t}$$

*At this point, simply by squaring the former equation and isolating x, we get:*

$$x = \frac{x_0}{2} (e^{\sqrt{\frac{g}{L}} t} + e^{-\sqrt{\frac{g}{L}} t}) = x_0 \cosh \sqrt{\frac{g}{L}} t$$

*Recall that the hyperbolic cosine,* $\cosh z$*, is a monotonically increasing function, which grows more and more quickly as z increases. In terms of interpreting our result, it means that the chain will fall increasingly faster with increasing time, just as we expected :).*

*Final comment on this example:*
*Let us consider the equation of motion we obtained in section 3):* $M \frac{d^2x}{dt^2} = \frac{M}{L} gx$*, which can be rewritten as* $\frac{d^2x}{dt^2} - \frac{g}{L} x = 0$*, with* $\frac{g}{L} > 0$*. This equation of motion is similar to that of the pendulum, which is a particular example of the simple harmonic oscillator:* $\frac{d^2x}{dt^2} + \frac{g}{L} x = 0$.
*Note that the 'only' difference between both equations of motions is the sign (- for the falling chain, + for the simple pendulum). Yet, the behaviour of the systems is completely different. The chain will fall with increasingly higher rates* ($x(t) \propto \cosh \sqrt{\frac{g}{l}} t$)*, whereas the simple pendulum oscillates forever around an equilibrium position* ($x(t) \propto \cos \sqrt{\frac{g}{l}} t$)*. One has to love Mechanics!*

## 2.4 An example of motion in two dimensions: projectile motion with friction $\vec{F}_f = -b\vec{v}$

Consider an projectile of mass m, moving under the effect of gravity ($\vec{g} = -g\vec{j}$), and friction which may be modelled as $\vec{F}_f = -b\vec{v}$. The object is launched from the origin, with initial velocity $\vec{v}_0 = v_{0x}\vec{i} + v_{0y}\vec{j}$. Determine the position versus time of the projectile, $\vec{r}(t)$ and its trajectory equation $y = y(x)$. Determine the maximum height of the projectile for any time t, and the range of the projectile when $b/m$ is small. Check that you recover parabolic motion when $b/m$ is negligible.

- First, we write the equation of motion along X and Y:

$$X)\ F_x = ma_x \Rightarrow -bv_x = m\frac{dv_x}{dt}$$
$$Y)\ F_y = ma_y \Rightarrow -bv_y - mg = m\frac{dv_y}{dt}$$

- X) It is easy to separate variables and integrate once to get $v_x(t)$:

$$\frac{dv}{v} = -\frac{b}{m}dt \Rightarrow Log\ v|_{v_{0x}}^{v_x} = -\frac{b}{m}t \Rightarrow v_x(t) = v_{0x}(t)e^{-\frac{b}{m}t}$$

  Or, in terms of the characteristic time $\tau = m/b$:

$$v_x(t) = v_{0x}(t)e^{-\frac{t}{\tau}}$$

  And a second integral will allow us to get x(t):

$$x(t) = \frac{mv_{0x}}{b}(1 - e^{-\frac{b}{m}t}) = v_{0x}\tau(1 - e^{-\frac{t}{\tau}})$$

- Y) From the example in Section 2.2, with $F_0 = -mg$, $v(t) = v_y(t)$ and $v_0 = v_{0y}$, we get:

$$v_y(t) = \frac{F_0}{b} + (v_{0y} - \frac{F_0}{b})e^{-\frac{b}{m}t} \Rightarrow v_y(t) = -\frac{mg}{b} + (v_{0y} + \frac{mg}{b})e^{-\frac{b}{m}t}$$

  In terms of the characteristic time $\tau$:

$$v_y(t) = -\tau g + (v_{0y} + \tau g)e^{-\frac{t}{\tau}}$$

  And a second integral will allow us to get y(t):

$$y(t) = -\frac{mg}{b}t + \frac{m}{b}(v_{0y} + \frac{mg}{b})(1 - e^{-\frac{b}{m}t}) = -g\tau t + \tau(v_{0y} + \tau g)(1 - e^{-t/\tau})$$

  Noteu que treballant amb temps característics és molt fàcil veure que les nostres equacions són correctees des del punt de vista dimensional.

- The trajectory equation can be obtained isolating t in the solution x(t) ($t = -\frac{m}{b}Log(1-\frac{bx}{bv_{0x}})$) and substituting in y(t):

$$y = \frac{m^2 g}{b^2}Log(1-\frac{bx}{mv_{0x}}) + \frac{m}{b}\frac{bv_{0y}+mg}{bx-mv_{0x}}$$

$$y = \tau^2 gLog(1-\frac{x}{\tau v_{x0}}) + \tau\frac{v_{0y}+\tau g}{x-\tau v_{0x}}$$

- The time at maximum height, $t_H$ is given by $v_y(t_H) = 0$. Therefore:

$$t_H = -\frac{m}{b}Log\frac{mg/b}{mg/b+v_{0y}} = -\frac{m}{b}Log\frac{1}{1+\frac{bv_{0y}}{mg}} = \frac{m}{b}Log(1+\frac{bv_{0y}}{mg})$$

$$\Rightarrow t_H = \tau Log(1+\frac{v_{0y}}{g\tau})$$

And the corresponding height:

$$y(t_H) = -\frac{mg}{b}\frac{m}{b}Log(1+\frac{bv_{0y}}{mg}) + \frac{m}{b}(v_{0y}+\frac{mg}{b})\frac{bv_{0y}}{mg}$$

$$\Rightarrow y(t_H) = -\frac{m^2 g}{b^2}Log(1+\frac{bv_{0y}}{mg}) + \frac{v_{0y}}{g}(v_{0y}+\frac{mg}{b})$$

$$\Rightarrow y(t_H) = -\tau^2 gLog(1+\frac{v_{0y}}{g\tau}) + \frac{v_{0y}}{g}(v_{0y}+g\tau)$$

- The maximum range $x_R$ can be obtained by imposing $y_R = 0$. Therefore we should solve for time t:

$$0 = y_R(t) = -\frac{mg}{b}t + \frac{m}{b}(v_{0y}+\frac{mg}{b})(1-e^{-\frac{b}{m}t})$$

We cannot solve the former expression analytically, but if $b/m$ is sufficiently small, we can approximate the exponential function by its Taylor expansion: $e^{-\frac{b}{m}t} \sim 1-\frac{b}{m}t+\frac{1}{2}\frac{b^2}{m^2}t^2$, and express y(t) as follows:

$$y(t) = -\frac{mg}{b}(1-\frac{b}{m}t+\frac{1}{2}\frac{b^2}{m^2}t^2) + \frac{m}{b}(v_{0y}+\frac{mg}{b})(\frac{b}{m}t-\frac{1}{2}\frac{b^2}{m^2}t^2)$$

$$\Rightarrow y(t) = -\frac{mg}{b}t + \frac{m}{b}(v_{0y}+\frac{mg}{b})(\frac{b}{m}t-\frac{1}{2}\frac{b^2}{m^2}t^2)$$

$$\Rightarrow y(t) = v_{0y}t - \frac{1}{2}(g+\frac{bv_{0y}}{m})t^2$$

$$\Rightarrow y(t) = v_{0y}t - \frac{1}{2}(g+\frac{v_{0y}}{\tau})t^2$$

Now it is easy to obtain $t_R$ for which $y(t_R) = 0$: t=0 is the trivial result (we knew the projectile started its motion at the origin), and $t_R = \frac{2v_{0y}}{g+\frac{bv_{0y}}{m}}$ is the time we needed. Now substituting for $x(t)$ in the small $b/m$ approximation:

$$\Rightarrow x(t) = \frac{mv_{0x}}{b}(1 - e^{-\frac{b}{m}t}) \sim \frac{mv_{0x}}{b}(\frac{b}{m}t - \frac{1}{2}\frac{b^2}{m^2}t^2) = v_{0x}t - \frac{bv_{0x}}{2m}t^2$$

$$\Rightarrow x_R = x(t) = \frac{mv_{0x}}{b}(1-e^{-\frac{b}{m}t}) \sim \frac{mv_{0x}}{b}(\frac{b}{m}t-\frac{1}{2}\frac{b^2}{m^2}t^2) = v_{0x}t-\frac{bv_{0x}}{2m}t^2$$

$$\Rightarrow x_R = \frac{2v_{0x}v_{0y}}{g + \frac{bv_{0y}}{m}} \frac{1}{1 + \frac{bv_{0y}}{mg}} = \frac{2v_{0x}v_{0y}}{g + \frac{v_{0y}}{\tau}} \frac{1}{1 + \frac{v_{0y}}{g\tau}}$$

Note that if $b/m$ is negligible ($\sim 0$), or equivalently $\tau$ is very large, then:

$$x(t) = v_{0x}t$$
$$y(t) = v_{0y}t - \frac{1}{2}t^2$$

which are the well known expressions of parabolic motion.
[2]

[2]Note that the former method of solution would not be possible if friction were modelled as $F_f = -bv^2$. In such case $F_{fx} = -b\,|\vec{v}|\,v_x = -b(v_x^2+v_y^2)^{1/2}v_x$, $F_{fy} = -b\,|\vec{v}|\,v_= -b(v_x^2+v_y^2)^{1/2}v_y$ and thus, the equations of motion would be coupled. We will learn how to solve these type of problems using numerical methods.

$$X)\ \ F_x = ma_x \Rightarrow -b(v_x^2 + v_y^2)^{1/2}v_x = m\frac{dv_x}{dt}$$
$$Y)\ \ F_y = ma_y \Rightarrow -b(v_x^2 + v_y^2)^{1/2}v_y - mg = m\frac{dv_y}{dt}$$