Classical Mechanics Exercises

Pilar Gil Pons, Enric Álvarez Lacalle, David Pino González, Maria Monguio

EETAC-UPC

February 24, 2021

Contents

Comments on the exercise collection

This is the icon code we will use for exercises:

- This is or should be easy: 
- You have seen this (or something very similar) in theory lessons. It is a good chance to review and fix what you learnt. 
- Heads up! This is important! Some of these exercises were part of exams! 
- This might be very long and or cumbersome. Take a look. Make sure you understand how it works, but do not get too obsessed. 

1

Fundamental principles in Mechanics. Solution to the equation of motion.

1.1 Questions and brief exercises

This section is aimed to help you make sure that you know the main concepts of the chapter and are ready for longer exercises. You must complete this section on your own 😊.

1. Fermi problem: how many piano tuners are there in Chicago?

2. What is the physical meaning of the equation of motion? What is the physical meaning of its solution?

3. Consider an object of mass m under the effect of a force $F = -mg$. The particle starts at rest at height h.
   a) Find $y(t)$ by writing the acceleration a as dv/dt.
   b) Find $y(t)$ by writing the acceleration a as vdv/dy.

4. Consider a mass m under the effect of an elastic force F=-kx.
   a) Find the equation of motion of m.
   b) Considering that the particle is at located at x_0 and at rest at t=0, solve the equation of motion.

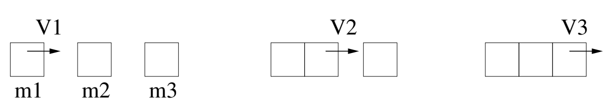
5. What is the terminal velocity? Give an example of motion in which the terminal velocity is reached.

1.2 Fundamentals of Mechanics.

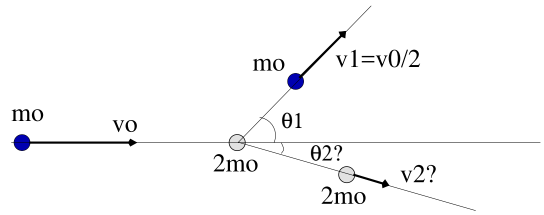
1. **A particle with mass m=2 kg moves on the top a table (X-Y plane) driven by an external force $\vec{F}(t)$ and the friction with the table $\vec{F}_{fr}$ (with dynamic friction coefficient $\mu = 0.3$). During the first 5 seconds of its movement the position of the mass as a function of time can be interpolated to be the function $\vec{r}(t) = (0.2 + 0.5e^{-t})\vec{i} + (0.4 + 0.5t - 0.2t^2)\vec{j}$ m.** 😊
   a) Compute the velocity $\vec{v}$ and the modulus of the velocity $|\vec{v}|$ at $t = 2$ s and $t = 5$ s.
   b) Compute the unitary vector tangent to the trajectory as a function of time. Is there any time when the trajectory is locally parallel to axis X?
   c) Compute the acceleration $\vec{a}$ as a function of time and split it up in the tangent a_t and normal a_n components. Check that a_t fulfills $a_t = \frac{d|\vec{v}|}{dt}$.
   d) Find the expression of the friction force $\vec{F}_{fr}$ at $t = 2$ s using that its direction is always tangent to the trajectory. Once you know $\vec{F}_{fr}$, find the external force $\vec{F}(t = 2)$. You may assume that at $t = 2$ s the entire tangential component of the acceleration is due to friction.
   e) Determine the particle's trajectory.

2. **A mass $m = 1$ kg is subject to two forces whose value depends on the position of the particle $\vec{F}_1 = -4z^2\vec{k}$ N and $\vec{F}_2 = \frac{1}{(x+1)^2}\vec{i}$ N. The particle moves in 3d along a rail which only generates forces normal to the trajectory. Friction is negligible.** 😊
   a) Setting the total potential energy $U = 0$ at the origin, compute an expression for the total potential energy $U(x, y, z)$
   b) Compute the change in potential energy when the particle moves from $\vec{r}_A = 0.2\vec{j} + 0.4\vec{k}$ to $\vec{r}_B = 0.3\vec{i}$
   c) What is the change in mechanical energy when the particle moves from $\vec{r}_A$ to $\vec{r}_B$? Is it possible for the particle to have no velocity at $\vec{r}_A$ and reach $\vec{r}_B$? If so, what will be the velocity at $\vec{r}_B$ in this case?

3. **Consider three boxes on an ice table of masses m1, m2 and m3. At the beginning only m1 is moving and it has velocity v1. When m1 collides against m2 the two boxes stick together and start moving with v2. Next, the system m1+m2 collides against m3 and the three boxes stick together and start moving with v3. Knowing that 10% of the initial energy is lost during the first collision and that another 10% of the remaining energy is lost during the second collision, determine m3 as a function of m1 and m2.** 😊



4. **One particle of mass $m_1 = m_0$ moves with velocity v_0 and collides with another particle $m_2 = 2\,m_0$. After the collision, m_1 is deviated $\theta_1 = 45^o$ with respect to its initial direction and moves with velocity $v_1 = v_0/2$.** 😊
   a) Obtain the final velocity of m_2 and the angle that it forms with the initial direction of m_1.
   b) Is the energy conserved in this collision?



1.3 Problems of variable forces

1.3.1 Forces depending on time

1. **A particle with mass m and electric charge q is placed in the presence of an oscillating electric field given by the expression: $E = E_0 cos(wt + \theta)$, where E_0, w and θ are constants.**
   a) Determine the particle's acceleration.
   b) Determine the equation of movement for the particle.
   c) Obtain the velocity and the position of the particle as functions of time. Consider that at time $t = 0$ the particle is at rest and its position is $x(0) = x_0$.

1.3.2 Forces depending on velocity

2. **An aircraft of mass $m = 2000kg$ lands at a speed of $100m/s$ and then deploys a parachute in order to stop. The resistance to the onward motion given by the parachute is given by $D = \frac{1}{2} C_D \rho A v^2$, where C_D is the drag coefficient, ρ is the density of air, and A is the area of the front section of the parachute. Take $C_D = 1.2$ and $\rho = 1.2475kg/m^3$ to determine:**
   a) the area of the front section of the parachute if the deceleration achieved is $a = -5g$, where $g = 9.81m/s^2$.
   b) the interval of time that the aircraft needs to decrease its speed from $100m/s$ to $60m/s$.
   c) the distance covered in that interval of time.

3. **A ball is thrown vertically, from negligible height, with initial velocity v_0. Assuming that the force of friction is proportional to the velocity of the ball, determine:**
   a) the equation of motion of the ball.
   b) its velocity as a function of time and the time at which the ball reaches its maximum height. Is there a limit speed? If so, determine it.
   c) its position as a function of time and the time at which the ball will reach the ground.
   d) Show that if the effect of friction is very small, the motion of the ball corresponds to a case of uniform acceleration.

1.3.3 Forces depending on position

4. **A particle with mass m is repelled by the origin of coordinates by a force inversely proportional to the cube of the distance between m and the origin. Assume that at time t=0 its velocity is zero and it is located at a distance x_0 from the origin.**

a) Determine the equation of motion of the particle.
b) Obtain the position and the velocity of the particle versus time.
c) Represent qualitatively the position and velocity of the particle.

5. **The potential energy corresponding to the force between two atoms in a molecule is given approximately by the expression:**
$V(x) = -\frac{a}{x^6} + \frac{b}{x^{12}}$
where x is the distance between the two atoms and a and b are positive constants.
a) Determine the expression of the force between the atoms.
b) Assuming that the mass one of the atoms (M) is much larger than that of the other atom (m), so that M remains at rest, while m moves along a straight line, describe qualitatively the possible trajectories of the second atom.
c) Determine the distance of equilibrium and the period of small oscillations around the point of equilibrium.

6. **Para intentar encontrar la forma funcional de la fuerza de fricción del aire con objetos pequeños, un alumno realiza el siguiente experimento. Se sube a la Sagrada Familia (altura de $H_s = 130$ m) y deja caer una moneda de masa m en una dia sin viento y a temperatura-presion del aire conocida. Un amigo mide el tiempo de caida t_S. A continuación, toma una moneda idéntica y se va a la calle 33W de Nueva York para subirse al Empire State Building (altura $H_E = 390$), donde en las mismas condiciones de falta de viento y temperatura-presión, deja caer la moneda midiendo de nuevo el tiempo de caida t_E.**
a) Calcula cómo evoluciona la altura en función del tiempo durante la caida de la moneda si la fuerza de fricción fuese lineal con la velocidad $|F| = b|v|$.
b) Halla la diferencia $t_E - t_S$ en el límite en que los tiempos de caída son muy grandes porque b/m es muy grande.
c) Calcula cuánto vale la diferencia $t_E - t_S$ si la fuerza de fricción fuese cuadrada con la velocidad $|F| = bv^2$ en el mismo limite del apartado anterior.
d) Si suponemos que el estudiante no conoce el coeficiente de rozamiento b pero sí la masa m ¿Tiene alguna manera de determinar cuál de las dos dependencias es la correcta?

1.3.4 Motion in two dimensions

7. **A robot of mass $m = 1$ kg moves over an $x - y$ horizontal plane (the weight is balanced by the normal force), experiencing a friction $\vec{F}_f = -b\vec{v}$, with $b = 0.1$ kg/s and $\vec{v}$ the speed of the robot. Moreover, the robot has an engine that produces a force $\vec{F}_m = F_0\hat{\text{ß}}$, with $F_0 = 10$ N. Assuming the initial conditions are: $x(0) = y(0) = 0$ and $v_x(0) = 0$, $v_y(0) = 1$ m/s, calculate:**
a) the two components of the speed of the robot as a function of time.
b) the robot's position as a function of time $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$.
c) Justify that for long time, the robot moves at constant speed only along the x axis. What is the value of this constant speed?

8. **A football player shoots a free kick at a distance of 25 m from the goal. She kicks the ball ($m = 0.5$ kg) towards the goal with an initial velocity v_0=20 m/s and an angle**

of 30° with respect to the horizontal. The frictional force with the air is modeled as $F_f = -bv$, with $b = 0.08$ kg/s. Use $g = \mathbf{10\,m/s^2}$.

a) Write the equations of motion for the ball.
b) Write the expressions for the ball's velocity as a function of time. When will it reach its maximum height?
c) Write the expressions for the ball's position as a function of time. How long will it take to reach the goal? Knowing that the goal's height is 2.4m, that the the goalkeeper's height is 1.75 m, and that she doesn't move, will the attacker score? And if the wall is located at 9m from the ball, and the player located there are 1.7m tall?

9. **Two people jump out of a plane flying at an altitude of 1500 m and a speed of 270 km/h. After $t_1 = 3$ s the first person ($m_1 = 110$ kg) opens the parachute. The second one ($m_2 = 60$ kg) waits 7 s longer to open it, i.e., $t_2 = 10$ s after jumping. Assuming that before opening the parachute they have a free fall movement ($g = 9.8$ m/s ^{2}) and with the parachute open there is a friction with the air proportional to the speed ($\vec{F}_f = -b\vec{v}$), answer the following questions:**
   a) What is the expression for the terminal speed, $\vec{v}_l$? What are the values for each person if $b = 20$ kg/s?
   b) Integrate the equation of motion to obtain the expression for the variation of the speed vector as a function of time, starting from the time of the jump, $\vec{v}(t)$, and without substituting any constant. If we don't take into account that the masses and times to open the parachute are different, is this equation the same for the two people? Justify your answer.
   c) Schematically plot in a single graph the variation of $v_x(t)$ for each person. In another graph plot $v_y(t)$ for each person. Clearly mark on each graph which line represents each person's speed and their terminal speed.
   d) Assume that they open the parachute at the same time. Who goes the furthest from the position in which they jumped from the plane? Which person is flying the longest? Justify your answers. You don't need to do any calculation.

1.4 Solving problems with a computer

1. **Consider an engine able to develop a maximum thrust F_0. This engine drives an aircraft that moves under the effect of a force of friction proportional to the square of its speed ($F_r = -bv^2$).If the initial velocity of the aircraft is zero and it accelerates with the maximum thrust given by the engine, determine:**
   a) its velocity as a function of time.
   b) its terminal velocity.

2. **The Typhoon Eurofighter, with mass $M = 1.6 \times 10^4 kg$, maximum thrust F_0 and maximum speed of 2 Mach (in altitude) and 1.2 Mach (at sea level). Using the expression obtained in a), plot its velocity versus time for different values of b (b=0.1, b=1, b=100, b=1000). What can you tell about the approximate value of b in order to fit the given values for the maximum speed? Solve the problem both analytically and numerically (by Euler's method).**

2

Oscillations

2.1 Questions and brief exercises

1. Give an expression for the average potential energy (V) of a simple harmonic oscillator along one period. Do the same for its kinetic energy (K). Is the total energy E=V+K constant? Justify your answer.

2. Write the equation of motion of a damped oscillator and establish the conditions of underdamping, critical damping and overdamping. What is the general solution of a damped oscillator? Can you write alternative expressions for the under- and critically damped oscillators?

3. Find an expression for the energy of the extremely underdamped oscillator. Is this energy constant? Justify your answer. Describe the concepts of relaxation time τ and quality factor Q. Say for which type of oscillators these concepts make sense. How can you tell the number of pulsations an underdamped oscillator experiences before it stops?

4. Describe briefly the characteristics of a driven oscillators and say how we can find their general solutions.

5. Describe the phenomenon of resonance?

6. Write down and sketch the function $P(\omega)$, that represents the power of an oscillator driven with an harmonic force, and give expressions for the power at the frequency of resonance and its bandwidth.

7. An oscillator of mass m and elastic constant k is placed in a fluid environment in which the force of friction is proportional to the velocity of m, $F_r = -bv$. If we apply on the oscillator a periodic external force, $F_e = F_0 \cos(\omega t)$:
   a) Write down the equation of motion and find a valid solution as a function of time. Express the result as a function of the problem's data (m, k, b, F_0, ω). Sketch an approximate solution $x(t)$ for long time values ($t \gg$), in the cases in which $\omega = 0$, $\omega = \sqrt{k/m}$ y $\omega = 2\sqrt{k/m}$.
   b) What can you tell about the amplitude, the frequency, and the phase of the motion for ($t \gg$)?
   c) Which expression must the external frequency ω take (as function of m, k and b), so that the amplitude of the motion is maximum?

2.2 Oscillations: SHO, damped oscillators and driven oscillators.

1. **A mass m oscillates on a spring with spring constant k. The amplitude of the motion is d. At the moment (let it be t=0) when the mass is at position $x = d/2$ and moving to the right, it collides and sticks to another mass m. From momentum conservation, the speed of the resulting mass $2m$ right after the collision is half the speed of m right before the collision. Determine:**
   a) the velocity just before and just after the collision as functions of m, k and d.
   b) the final amplitude of the oscillation and x(t) after the collision.

2. **A mass $m = 0.1$ kg oscillates around the minimum of a potential energy in 1D defined by $V(x) = 100(2x^2 - x^6)$ Determine the period of its motion T.**

3. **A mass of 100 g, driven by a force $F = -kx - bv$, is at rest at the equilibrium position $x(0) = 0$. Suddenly an initial speed v_0 is given to the mass in the positive direction of the x axis and we observe that after 0.5 s the mass reaches a maximum displacement of 15 cm and it reaches again the equilibrium position every 1.3 s.**
   a) Which type of movement will the mass follow? Sketch the displacement as a function of time. Indicate the pseudo-frequency and write down $x(t)$ without determining here the constants.
   b) Use the conditions indicated above (initial position and maximum displacement) to determine k and b. Write down $x(t)$ including the values of all constants,
   c) Determine the initial speed, the time when the absolute value of the friction force bv is maximum, and its value.

4. **An oscillator is designed such that its amplitude after 10 pseudo-periods diminishes a 10% with respect to its initial position x_0. In addition the time interval between two consecutive zeros in elongation is 0.05 seconds. Assume that the oscillator starts from rest at a position $x_0 = 0.1$ m.**
   a) Determine the type of oscillator designed, and make a sketch, indicating the information of the data in the exercise. Give the values of the pseudo-period T_1, the corresponding pseudo-frequency, and check that the damping constant γ is iqual to 0.11 s^{-1}. Knowing that the mass of the oscillator is $m = 0.01$ kg, determine its elastic constant its natural frequency.
   b) Determine the solution to the equation of motion, $x(t)$, including all the constants as function of known data.
   c) Assume that the oscillator is extremely under-damped ($\omega_0 >> \gamma$). Give an approximate expression and sketch its energy as a function of time and of the given data. Determine the relaxation time and the quality factor of the oscillator.
   d) Because we intend the oscillator to operate at constant amplitude, we apply an external force $F(t) = F_0 \cos(\omega_0 t)$. Determine the solution to the equation of motion in the permanent (stationary) regime, $x_p(t)$. Determine the value of F_0 for the amplitude of x_p to be equal to the initial position of the system, x_0.
   e) Sketch the power curve of the oscillator in d) indicating the frequency of the resonance, the maximum power developed, and the bandwidth $\Delta\omega$ (or FWHM).
5. **A 300 kg sac, initially at rest, falls from a height $H = 2.5$ m above a horizontal platform. The platform is located on top of a vertical spring fixed to the ground by its lower end. The spring hosts a damper which slows down the system through a force proportional to its velocity ($F_b = -bv$). Assume that the masses of the platform, the spring and the damper are negligible compared to the mass of the sac. When the sac reaches the platform, the equilibrium position of the platform is**

displaced $d = 15$ cm below its original equilibrium position (when the sac was not on top of it). We know that the spring operates as a critically damped oscillator.

a) Which are the values of the spring's elastic constant, k, and of the damper's constant b?
b) Determine the interval between the time of impact between the sac and the platform, and the time in which the platform reaches its new equilibrium position. Which velocity will the platform have in that position?
c) Which is the maximum distance below the new equilibrium position which the platform will reach?
d) Determine the work done by each of the intervening forces, that is, the work due to gravity, to the spring and to the damper, between impact time and maximum compression.

6. **It is desired to design a bathroom scale with a platform deflection of 2 cm under a 60 kg man.**
   a) If the motion is to be critically damped find the required spring constant K, the natural frequency ω_0 and the damping constants b and γ.
   b) If a lighter person (a 50 kg man) uses this scale (same K and b), determine the new natural frequency ω_0 and constant γ of the oscillator. Is the new oscillator critically, under or overdamped?
   c) What type of oscillator would we have if a heavier person used the bathroom scale?
   d) For the case in which the oscillator is underdamped, determine the decay constant τ and the quality factor Q of the scale as a function of the mass of the user.

7. **Consider a damped oscillator that at t=0 is at the origin and has an initial velocity $v_0 \neq 0$. If its natural frequency ω_0 and its damping constant γ are known (Note that you do not know which type of damped oscillator we are dealing with and therefore you should use the generic solution):**
   a) Determine its position as a function of time. Give the constants as a function of the initial conditions.
   b) Determine the time a which the velocity of the oscillator is maximum.

8. **Consider a critically damped oscillator of mass $m = 0.1kg$, and elastic constant $k = 10^{-3}\ N/m$.**
   a) Write the equation of motion and a generic expression for its solution (without calculating the constants).
   b) Determine the solution for the equation of motion, using that the oscillator starts from rest at a separation x_0 from the equilibrium position.
   c) Can the oscillator reach the equilibrium position?

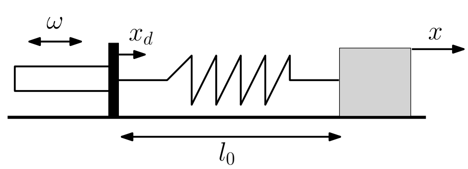
9. **One harmonic oscillator of mass m and natural frequency $\omega = \sqrt{\frac{k}{m}}$ is placed inside a fluid. Due to the presence of this fluid the oscillator experiences a force of friction $F_r = -bv$. The initial conditions are the following: $x(t = 0) = 0$ and $v(t = 0) = v_0 \neq 0$. The constants are such that $\frac{b^2}{4m^2} < \omega_0^2$.**
   a) Say if the oscillator is under-, critically or overdamped. Give an expression for the equation of motion.
   b) Solve the equation of motion and give $x(t)$ and $v(t)$ as functions of the initial conditions.
   c) If we apply a driving force $F = F_0 cos(\omega_0 t)$ (the frequency of the force is equal to the natural frequency of the oscillator), give the stationary solution of the movement.
   d) If we apply a harmonic force of arbitrary frequency ω, it can be shown that the average dissipated power in each period is: $P_{average} = \frac{F_0^2}{m} \frac{\gamma\omega^2}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$.

Use this expression to determine the frequency of resonance and briefly explain its meaning.

10. When the Earth vibrates it period of resonance is of 54 minutes and its quality factor is Q=400. After a strong earthquake the vibration lasts for about two months.

a) Determine the percentage of energy dissipated per cycle.
b) Show that after n periods the energy of the vibration can be expressed as $E_n = E_0(0.984)^n$.
c) If the initial energy is E_0, determine the energy after two days.

11. To solve the problem, please use the sinus function for the expression of position versus time. As shown in the figure, a body of mass $m = 1$ kg which can move without friction with the ground is attached to a spring of negligible mass. The other end of the spring is attached to a platform which can vibrate horizontally with displacement $x_d(t)$.



The force that the spring exerts on the body (taking the positive direction to the right) can be modelled as $F = -k\Delta l - bv$, where $\Delta l = l - l_0$ is the length variation of the spring, $k = 4$ kg s^{-2} is the spring constant, v is the speed of the body and $b = 2$ kg s^{-1} is the friction constant.

a) Considering there is no vibration of the platform and the body initially moves to the right with $v_0 = 0.1$ m s^{-1}.
   a1) What is the displacement function $x(t)$ of the body?
   a2) What is the maximum distance reached by the body and how long will it need to reach that maximum distance?
   a3) How long will the body take to return to its equilibrium position for the first time?
b) Considering the body at rest, the platform starts to move with angular frequency ω, and $x_d = A_d sin(\omega t)$.
   b1) Show that the force acting on the body is $F = -kx - bv + kA_d sin(\omega t)$, and consequently the body will follow a driven harmonic oscillation.
   b2) If $A_d = 0.03$ m and $\omega = 4$ rad s^{-1}, determine $x(t)$ once the stationary regime has been reached.
   b3) Which is the angular frequency which provides the highest speed? Which is the maximum speed in this case?

12. Consider a mass m attached to a spring of constant k. m is released from rest at a position x_0 and then oscillates in the void. The experiment is repeated under the same initial conditions, but in a medium such that the movement is overdamped with a certain γ. Note: express results as functions of k, γ and the initial conditions.

a) For the case in which the experiment is done in the void, give the position and velocity of m as a function of time. Determine the time at which m reaches its maximum velocity and the expression of this velocity v_a.
b) For the case in which the experiment is done in the medium such that the damping constant is γ, give the position and the velocity of m as a function of time.

c) Find the time at which the maximum velocity is reached for the case with damping.
d) Find the maximum velocity in the case with damping: v_b.
e) Use that the quotient $v_a/v_b = (\frac{\gamma+\sqrt{\gamma^2-\omega_0^2}}{\gamma-\sqrt{\gamma^2-\omega_0^2}})^{\frac{\gamma}{2\sqrt{\gamma^2-\omega^2}}}$ to approximate v_a/v_b in the case $\gamma >> \omega_0$, where ω_0 is the natural frequency of the oscilator. Use Taylor's expansion $\sqrt{1-x^2} \approx 1 - \frac{1}{2}x^2$, valid for small x.

2.3 Additional problems

1. **Q is designing the dampers for James Bond's last supercar. First, Q decides to use a damper for which the position versus time follows the expression:**
$x_1(t) = e^{-\gamma t}(Ae^{-\sqrt{\gamma^2-\omega_0^2}t} + Be^{\sqrt{\gamma^2-\omega_0^2}t})$.
Then he finds an alternative system such that:
$x_2(t) = (C + Dt)e^{-\gamma t}$,
where A, B, C and D are constants to be determined, $\omega_0 = 10 rad/s$ **and** $\gamma = 31.6 rad/s$ **in** $x_1(t)$ **and** $\gamma = 31.6\ rad/s$ **in** $x_2(t)$**. The initial conditions are** $x(t = 0) = 0$ **and** $v(t = 0) = v_0 \neq 0$**.**
a) Write down the equation of motion that corresponds to both systems.
b) Which type of oscillators is Q considering?
c) Determine the constants A and B, and express $x_1(t)$ as a function of the given data.
d) Determine the constants C and D, and express $x_2(t)$ as a function of the given data.
e) Express the velocity of both of Q's systems as functions of the given data.
f) If the initial velocity is $v_0 = 1\ m/s$, calculate the velocity of both dampers at time $t = 0.5\ s$ and explain which system is the best to achieve a velocity close to zero.
NB: You can use that $cosh(x) = \frac{e^x+e^{-x}}{2}$ and $sinh(x) = \frac{e^x-e^{-x}}{2}$.

2. **En el tren de control de Airbus se estudia el comportamiento de un ala de un avión ante un forzamiento oscilatorio externo para evitar resonancias graves. Estudiando la curva de potencia se ve que la máxima absorción se produce cuando el forzamiento exterior tiene un periodo de** $T = 0.1$ **s y que el Full Width Half Maximum es** $\Delta\omega = 10$ **s**$^{-1}$
a) Cuánto vale la frecuencia natural del sistema ω_o?
b) Cuánto vale la constante de fricción γ (donde $\gamma = b/2m$, con fuerza de fricción $F_f = -bv$)?
c) Si la masa del ala de una airbus es tipicamente $m = 5000$ kg y la fuerza máxima al forzar el ala son $F_o = 20000$ N. Cuánto vale la amplitud de las oscilaciones del ala si realizamos un forzamiento con la frecuencia de máxima resonancia?

2.4 Solving problems with a computer

1. **We can measure the viscosity of different types of oil by measuring the decay time of an oscillator immersed in the oil. Because the velocity of the oscillators is relatively low we can assume there is no turbulence and the drag force of the fluid on a sphere of radius a that moves at velocity v is** $F_d = 6\pi a\eta v$**, where** η **is the viscosity of the fluid. Assuming that the oscillator is composed of a small spring of constant** $k = 350N/cm$ **and a** $6cm$ **sphere made of gold, determine:**

a) the viscosity of the fluid and the quality factor Q if the decay time $\tau = 2.8s$.
b) the viscosity of the fluid and the quality factor Q if the decay time $\tau = 5.6s$.
c) Assuming a reasonable value for the amplitude of the oscillation, plot the position versus time for the cases in a) and b).
d) Plot the energy versus time for the cases a) and b). Mark the corresponding values of τ and check that the energies associated to each τ correspond to the expected values (E_0/e).
e) Determine in each case the number of oscillations before $t = \tau$.

2. An object of mass $m = 1.5kg$ is attached to a spring of constant $K = 600N/m$. We know that m loses 3% of its energy in each cycle. The system is driven by an external harmonic force of amplitude $F_0 = 0.5N$.
a) Determine the Q factor of the system.
b) Determine the frequence and the width of resonance.
c) Determine the amplitude at the frequence of resonance and the amplitude when the frequence is $\omega = 19rad/s$.
d) Plot the amplitude of the oscillation in the stationary regime versus the frequence of the driving force and identify the amplitudes obtained in c).
e) Plot aplitude versus frequency of the driving force if the Q of the system is decreased to one fifth of the original value and if it is increased to five times the original value of Q.

3

Central forces: gravity

3.1 Questions and brief exercises

1. What is a central force? Are all central forces conservative? Which magnitudes are constant when motion occurs only under the action of central forces? Show that all central forces are conservative.

2. Enunciate Kepler's Laws.

3. Would any of Kepler's laws hold if the Sun and consequently, its gravitational pull on the planets, suddendly disappeared? To answer this question you can assume that the planets are so far apart from each other that their mutual interaction is negligible.

4. Which type of orbits describe objects under the effect of central forces?
   a) Give the trajectory equation of an elliptical orbit. How must be the energy of the orbit in this case? Which values can have the eccentricity in this case?
   b) Give the trajectory equation of an parabolic orbit. How must be the energy of the orbit in this case? Which values can have the eccentricity in this case?
   c) Give the trajectory equation of an hyperbolic orbit. How must be the energy of the orbit in this case? Which values can have the eccentricity in this case?
   d) How do energy and angular momentum relate to the orbital parameters?

5. What is the escape velocity? Derive its expression.

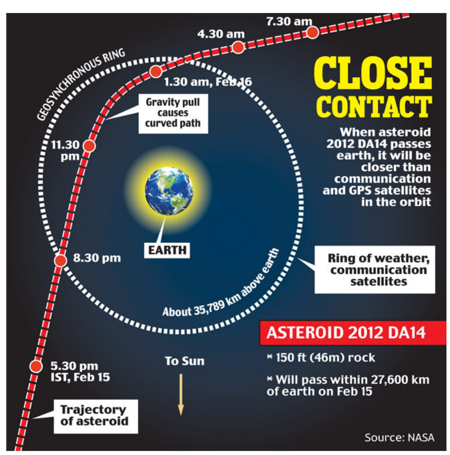
3.2 Potential energy

1. **Obtain the potential energy associated to the following central forces:**
   a) $F = -kx$
   b) $F = k/r^2$
   Use the expressions for the potential energy in a) and b) to compute the cartesian components of the force in each case.

3.3 Orbits under central forces

1. **Sputnik I had a perigee (point of closest approach to Earth) 215 km above Earth's surface, at which point its speed was $8.18 \times 10^3 m/s$. Using that $G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$, the radius of the Earth $R_E = 6370 km$ and the mass of the Earth $M_E = 5.98 \times 10^{24} kg$, find:**
   a) the angular momentum per unit mass and the energy per unit mass of Sputnik in its orbit.
   b) the eccentricity and semimajor axis of Sputnik's orbit.
   c) its apogee an its corresponding velocity.

2. **Asteroid 2012 DA14 reached its closest point to the Earth on February 15 2013. At that moment the distance between the Asteroid and the Earth's surface was 27700 km. Take $M_{Sun} = 1.989 \times 10^{30} kg$**
   a) Compare this value to that of an artificial satellite in geostationary orbit.
   b) Knowing that the period of the asteroid in its orbit around the Sun was 366 days before its approach to the Earth and that this period decreased to a value of 317 days afterwards, determine the major semiaxis of the asteroid's orbit before and after approaching the Earth.
   c) If the eccentricity of 2012 DA14 is $\epsilon = 0.089$ and assuming that it did not change during the approach to Earth, determine the perihelion and the aphelion of the asteroid before and after the approach. Give the values in Astronomical Units ($1 A.U. = 150 \times 10^6 km$).
   d) Give the equation of the orbit of 2012 DA14 before and after approaching the Earth. What can be the cause of the variation in the orbital parameters of the asteroid?
   e) Give an estimate of the asteroid's energy and angular momentum per unit mass before and after its approach to Earth.



3. **For a body of mass m, the potential energy along a straight line between the Earth and the Moon is given by the expression:**

   $V(r) = -\frac{GM_t m}{r} - \frac{GM_L m}{D-r}$
   a) Find the point of maximum potential energy.
   b) If a spacecraft were to reach the Moon, starting from the Earth's surface and following a straight line, determine the minimum required initial velocity.

   DATA: distance Earth-Moon $D = 3.84 \times 10^8 m$; mass of the Earth: $M_E = 5.84 \times 10^{24} kg$; mass if the Moon $M_M = 7.36 \times 10^{22} kg$, $G = 6.67 \times 10^{-11} m^3/kg/s^2$.

3.4 Transfer orbits

1. **The aircraft Orion is at a parking orbit at a height of 500 km above the Earth's surface. The total mass of the aircraft is 29157 kg.**
   a) Determine the total energy of the aircraft, its orbital velocity and its period at the parking orbit.
   b) Assuming that the aircraft's orbit is in the same plane as the Moon's orbit, determine the required variations in the velocity in order to transfer the aircraft from the parking orbit to the Moon's orbit (use a Hohmann orbit for the transfer and assume that the Moon's eccentricity is zero. Take $R_{Moon} = 383000$ km)
   c) Specify if the angular momentum has increased, decreased or remained constant along the different steps of the process. Proof your answer analytically.
   d) Once the aircraft is in the Moon's orbit, the aircraft is by mistake injected into an orbit of total energy zero with respect to Earth. Give the orbital parameters for this orbit and comment what will happen to the aircraft.

2. **A rocket is launched with initial velocity v_0 and forming an angle α with the vertical, in such way that, without additional energy the satellite's velocity only has a horizontal component at height h_1. At that time the second phase of the rocket ignites and increases its velocity a certain Δv_1. Considering that the final orbit must be an ellipse of perigee $r_1 = R_T + h_1$ and apogee $r_2 = R_T + h_2$, determine the required v_0 and Δv_1 as functions of R_T, α, h_1, h_2 and g.**

4

Systems of particles: the rocket's equation

4.1 Questions and brief exercises

1. What is the linear momentum, $\vec{p}$, of a system? Express the temporal variation of $\vec{p}$. Under which conditions is $\vec{p}$ conserved?

2. Calculate the centre of mass of a homogeneous cone of mass M, radius of base R and height H.

3. Calculate the total mass of a sphere of radius R such that its density varies with its radial coordinate as $\rho(r) = ar$, where a is a positive constant.

4. Express the velocity of a particle that belongs to a discrete system with respect to its centre of mass. Show that the linear momentum of the system with respect to its centre of mass is zero.

5. Describe briefly the concepts of elastic and inelastic collisions. Describe briefly the concept of exothermic and endothermic collision.

4.2 Centre of mass

1. Compute the centre of mass of a semisphere of homogeneous density ρ and radius R.

2. Compute the centre of mass of a uniform density cone of mass M, radius R and height H.

4.3 Momentum conservation

1. **A helicopter of mass M has a main helix with radius R. The air above the circular zone described by the motion of the helix is pushed vertically downwards with velocity v_0. The density of air is ρ.**
   a) If the helicopter keeps a constant height, show that the velocity of the air v_0 is given by: $v_0 = \frac{1}{R}\sqrt{\frac{Mg}{\pi\rho}}$.
   b) If the helicopter has a mass of 10 tones and a radius R = 10 m, determine the minimum value of v_0 that allows the helicopter take off. You can take $\rho = 1.3 kg/m^3$.

4.4 Rockets

1. **A rocket of initial mass M_0 ejects gas at a constant rate $|\frac{dM}{dt}| = \mu$ and velocity u with respect to the rocket.**
   a) Determine the thrust and initial acceleration of the rocket, assuming that the launch is vertical.
   b) If u = 2000m/s, which must be the value of μ to allow a 10 tone rocket rise with an initial acceleration of 0.5g?. (Approximate g = 10m/s^2).

2. **The rocket Saturn V, used in the Apollo mission, had an initial mass $M_0 = 2.85\times10^6 kg$ and its payload mass represented the 2.7% of the initial mass. It ejected gas at a constant rate $|\frac{dM}{dt}| = \mu$ and velocity u with respect to the rocket. Given that the gas combustion rate was $\mu = 1.38 \times 10^4\ kg/s$ and that the thrust was $T = 3.4 \times 10^7\ N$, determine:**
   a) the velocity of ejection of gases,
   b) the duration of the gas combustion process,
   c) the initial acceleration of the rocket.
   d) the final acceleration of the rocket. Is this value realistic in terms of the safety of the crew or the payload? Which is the important missing physics which we are not considering in this exercise?

3. **Considerad un cohete de 2 fases, tal que su carga útil es m_0, su masa inicial es $n_1 m_0$, y su masa al principio de la segunda fase es $n_2 m_0$. Lanzamos el cohete desde la superficie de la Tierra, partiendo del Ecuador y hacia el Este, con velocidad inicial vertical respecto del sistema de referencia Tierra (en rotación). Suponed que la ignición de fuel es instantánea, y que la fricción atmosférica es despreciable. Utilizad las siguientes constantes: masa de la Tierra: $M_E = 5.98 \times 10^{24}\ kg$, $GM_E = 3.99 \times 10^{14}\ m^3/s^2$ (G es la constante de gravitación universal), Radio de la Tierra $R_E = 6.378 \times 10^6\ m$, $GM_E R_E = 2.54 \times 10^{21}\ m^3/s^2$, $GM_E/R_E = 6.26 \times 10^7\ m^2/s^2$ y velocidad angular de rotación de la Tierra $\omega_E = 7.27 \times 10^{-5}\ rad/s$.**
   a) Teniendo en cuenta que el lanzamiento es vertical respecto de la Tierra, determina el momento angular específico del cohete (respecto de un sistema de referencia inercial en reposo) tras la ignición de la primera fase. Deseamos que el cohete entre en una órbita elíptica muy excéntrica (excentricidad ϵ=0.9983) y semieje mayor a=1.02 R_E. Determinad la energía que ha de tener el cohete al final de la primera fase, el radio del apogeo de su órbita, y la altura máxima sobre la superficie terrestre que alcanzará.
   b) Determinad las componentes horizontal y vertical de la velocidad tras la ignición de la primera fase, v_{hor} and v_{ver}. Si la velocidad de eyección de los gases es u=3000m/s, y $n_2 = 3$, obtened el valor de n_1 necesario para obtener la velocidad vertical inicial que permite que el cohete describa la órbita del apartado a).
   c) Determina la velocidad del cohete cuando está en su apogeo. Haz un pequeño esquema mostrando la órbita del cohete, la posición del centro de la Tierra, el radio del apogeo y la velocidad en el apogeo. Si deseamos transferir el cohete a una órbita circular de radio igual al radio del apogeo de la órbita anterior, determinad la variación de velocidad necesaria y el valor de la velocidad de eyección de gas en la segunda fase.
4. **This problem shows the advantages of 2-phase rockets. Imagine a payload, of mass m, is mounted on a 2-phase rocket as shown in the figure. The total mass of the rocket when it is loaded with fuel (payload plus 2 fuel capsules plus fuel) is Nm. The mass of the second phase plus the payload, when the capsule of fuel of the first phase has been ejected is nm. In each phase the quotient between the final mass**

of the fuel capsule and the mass of the fuel capsule plus the fuel is r. The escape velocity is u.

a) Show that the velocity v_1 reached after the combustion of the first phase, when the initial velocity is zero and ignoring gravity, is:
$v_1 = u\, Ln\frac{N}{rN+n(1-r)}$

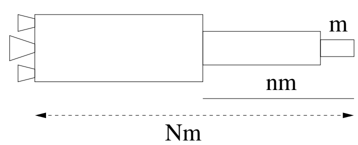
b) Show that the corresponding expression for the extra velocity attained after the first phase capsule is ejected and the second phase has burnt is:
$v_2 = u\, Ln\frac{n}{rn+(1-r)}$

c) When the two phases ignite consecutively the final velocity of the rocket as a function of u, r, n and N is obtained adding v_1 and v_2. Assuming u, r and N constant show that the rocket's maximum velocity is achieved for $n = \sqrt{N}$.

d) Show that the condition for maximum velocity obtained in c) implies that the velocity increments obtained in each phase are equal. Obtain the maximum value of the velocity and determine and explain its meaning for the limit cases r=0 and r=1.

e) Show that the final velocity for a one-phase rocket with the same values of u, r and N is:
$v_1 = u\, Ln\frac{N}{rN+(1-r)}$

f) Imagine that you wish to obtain a final velocity of v = 10 km/s, using rockets such that u=2.5 km/s and r = 0.1. Show that this can only be done with a 2-phase rocket.



5. **A two-phase rocket is launched from the Earth's surface. Its payload (final mass) is m_L, its mass after the first phase is nm_L and its initial mass is Nm_L. Assume that the mass of the structures containing the fuel is practically zero, and that friction with air and fuel burning times are negligible. The height reached by the rocket is such that the acceleration of gravity canot be assumed constant.**

a) Find expressions for the increments of velocity after each phase, Δv_1 y Δv_2, as function of u, n, N and m_L.

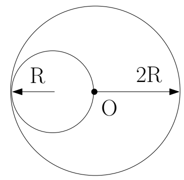
b) Assuming that the second phase burns when the rocket has reached its maximum height after the first phase, find the final height reached by the rocket as a function of the exercise's data.

c) Assuming that the second phase burns just after the first phase, find the final height reached by the rocket as a function of the exercise's data. Determine which procedure (b) or c)) is better to reach a higher altitude.

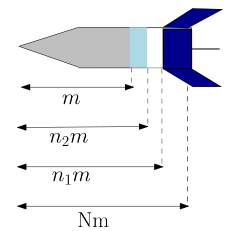
d) Proof the rocket's equation in the vertical case (with gravity).

4.5 Additional problems

1. **A solid spherical metal ball of radius 2R has had a smaller sphere of radius R removed from it. The metal in the sphere is of uniform density. Find the centre of mass of the larger sphere after the smaller sphere has been scooped out.**

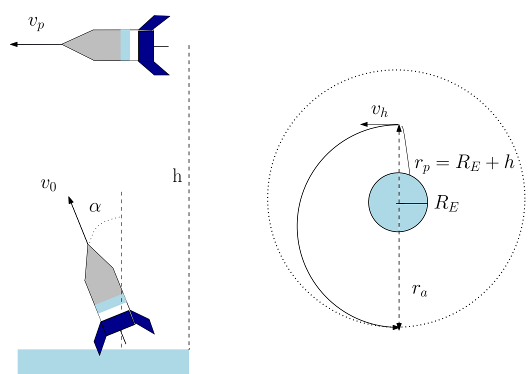


2. A 3-phase rocket has a payload mass m. Its initial total mass is Nm, the mass after the first phase is ejected is $n_1\,m$, and the mass after the second phase is $n_2\,m$. Gas scape velocity is u along the entire exercise. Data: $G = 6.67{\times}10^{-11}N/m^2/kg^2$, mass of the Earth $M_E = 5.98{\times}10^{24}kg$, radius of the Earth $R_E = 6370km$, $N/n_1 = 4$, $u = 2500\,m/s$. In sections b) to f) express your result algebraically as functions of given data, and then obtain the corresponding numerical values.



a) Determine the algebraic expressions of the velocity variations after the ejection of each phase: Δv_1, Δv_2, and Δv_3, assuming that the burning and ejection processes are instantaneous and u, N, n_1 and n_2 are known.

b) If the first phase is used to launch the rocket, from rest, at the Earth's surface (R_E), forming an angle $\alpha = 10°$ with the vertical, determine the maximum height that the rocket will reach, h (use $N/n_1 = 4$). Assume that there is no friction.



c) Obtain the value of the orbital parameter a and show that the rocket will crash against the Earth if no additional phases are ignited. (0.5 pt)

d) Assume that if the rocket did not ignite additional fuel under the conditions given at b), it would crash against the Earth's surface. Instead, the rocket's second phase is ignited when it reaches its maximum height, and then it is transferred to an elliptical orbit of perigee $r_p = h + R_E$ and apogee $r_a = 1.5\,r_p$. Obtain the rocket's velocity at h, before (v_p) and after the ignition of the second phase (v'_p). Determine the variation of velocity, $\Delta v_p = v'_p - v_p$. item [e)] When the rocket reaches the apogee of the elliptical Hohmann orbit, the third phase is ignited to take it from the elliptical to a circular orbit of radius r_a. Determine the Δv_a required.

f) Determine n_2, n_1 and N required to provide Δv_3, Δv_2 and $\Delta v_1 = v_0$.

g) PROOF: Use energy and angular momentum conservation to obtain the expressions of the velocities at the apogee (r_a) and the perigee (r_p) of an elliptical orbit. Express your results as functions of r_a, r_p, G and M_E.

h) Describe briefly the concept of effective potential V_{eff}. Give its expression and make a qualitatively plot of V_{eff} in the case of the present exercise, including the energy values in each part of the problem.

3. Consider the launch of a 2-stage rocket. The gas exhaust velocity is u=2000m/s in both stages, and the mass quotient in the first stage is $\mathbf{m_{10}/m_{1f}=2}$. For the sake of simplicity, we will assume that air friction is negligible and that combustion and launch time are zero.

a) In order to get maximum profit from Earth's rotation the rocket is launched from the Equator. Besides, we will assume that the launch is practically perpendicular to the ground. Determine the horizontal and vertical components of the velocity right after the launch. Determine the rocket's angular momentum ($l = R_T\, v_{horizontal}$) and energy per unit mass after the launch.

b) Calculate a and ϵ parameters of the rocket's orbit, and its radius at the apogee. Compare this result with the maximum height which the rocket will reach, h_{max}, assuming a vertical launch and using energy conservation. Determine the horizontal component of its velocity at the apogee.

c) Consider now a launch from the North Pole. If we desire the X component of the rocket's velocity be the same as in a), calculate the angle with respect to the vertical with which the rocket should be launched. Determine the rocket's energy, its angular momentum per unit mas, and the radius at the apogee in this case.

d) When we are at the maxium height (apogee), we desire to transfer the rocket to an eliptical orbit of perigee $R_T + h_{max} = r_{apogee,before}$ and apogee $R_T + 300km$. Determine the velocity variations required both in case b) and in case c). Determine the mass quotients required in the rocket's second stage in both cases.

5

Non-inertial systems of reference

5.1 Questions and brief exercises

1. Give the general expression that relate position, velocity and acceleration in an inertial and a non-inertial system of reference.

2. Write an expression for the Earth's rotational velocity at a point of Earth placed at latitude λ. Does this expression vary when we are at the Northern or the Southern Hemisphere?

5.2 Problems

1. **Consider the movement in the X-Y plane given by $\vec{r} = 5t\vec{\text{ß}} + (40 - 5t^2)\vec{\text{æ}}$ in an inertial system of reference which we call A-reference-system. Now take two other system of references (B and C) which have the same unitary vectors as A $\vec{\text{ß}}, \vec{\text{æ}}, \vec{k}$ but whose origin of coordinates does not coincide with the one in A. For the B-system the origin of coordinates with respect to the origin of A reads $\vec{R_{oB}} = 5t\vec{\text{ß}} - 3t\vec{\text{æ}}$, and for the C system $\vec{R_{oc}} = 5t^2\vec{\text{ß}} - 3t\vec{\text{æ}}$**
   a) Compute vector $\vec{r}$ at $t = 1$ s in system A, B and C.
   b) Compute vector $\vec{r}$ at any time in system A, B and C.
   c) Compute the velocity in system A, B, and C at $t = 1$s. Are the velocities the same in all systems?
   c) Compute the velocity in system A, B, and C at any time.
   e) Compute the acceleration in system A, B, and C at $t = 1$s. Are the accelerations the same in all systems? Which reference-system are inertial? Write down the fictional force you have to add in the non-inertial system to make it look like the Newton's second law holds.

2. **Consider the movement in the X-Y plane given by$\vec{r} = 5t\vec{\text{ß}} + (40 - 5t^2)\vec{\text{æ}}$ in a inertial ssystem of reference (SRI). To get the idea that equal vectors are expressed different in different coordiante systems, but they might be intrinsically different when differentiation is involved, let's see how this trajectory would look like in a reference systen whose origin is the same as the inertial one but it rotates as $\vec{\omega} = \frac{\pi}{4}\vec{k}$ rad/s (take that at $t = 0$ the inertial system (SRI) and the system which rotates (SRG) are the same).**
   a) At $t = 2$, give $\vec{r}$ using the unitary vectors of the inertial system and the unitary vectors of the rotating one $\vec{I}, \vec{J}, \vec{K}$ (Note that $\vec{k} = \vec{K}$).

b) Find now how $\vec{r}$ is written in the rotating unitary vectors for any time..
c) Differentiate $\vec{r}$ in the inertial frame ($\frac{d\vec{r}}{dt}|_{SRI}$) and do the same with $\vec{r}$ in the rotating frame ($\frac{d\vec{r}}{dt}|_{SRG}$) . The first is expressed in unitary vectors $\vec{ß}, \vec{æ}$, and the second in $\vec{I}, \vec{J}$.
d) Write now the vector $\frac{d\vec{r}}{dt}|_{SRI}$ at $t = 2$ using the unitary vectors of the axis that rotate $\vec{I}, \vec{J}$
e) If you have compute properly $\frac{d\vec{r}}{dt}|_{SRI}$ using the unitary vectors of the rotating system you will notice that it is not at all the same as $\frac{d\vec{r}}{dt}|_{SRG}$. Check that on the other hand it is indeed equal to $\frac{d\vec{r}}{dt}|_{SRG} + \vec{\omega} \times \vec{r}$.

3. **At some moment in history the Space Shuttle was located at a point A, of altitude h = 240 km, and exactly above the Equator, traveling from West to East. At the same moment we are on holidays and watch the Shuttle from a point B, also in the Equator, and right in the vertical of the Shuttle. Determine:**
   a) The velocity of the Shuttle ($\mathbf{v_A}$), our velocity ($\mathbf{v_B}$) and the relative velocity ($\mathbf{v_A} - \mathbf{v_B}$) with respect to an inertial system of reference whose origin is at the centre of the Earth.
   b) The the relative velocity ($\mathbf{v_A^*} - \mathbf{v_B^*}$) that we measure from Earth (non-inertial system of reference).
   c) The acceleration of the Shuttle ($\mathbf{a_A}$), our acceleration ($\mathbf{v_B}$) and the relative acceleration ($\mathbf{a_A} - \mathbf{a_B}$) with respect to an inertial system of reference whose origin is at the centre of the Earth.
   d) The the relative acceleration ($\mathbf{a_A^*} - \mathbf{a_B^*}$) that we measure from Earth (non-inertial system of reference).

   Help: You can relate the time variation of a magnitude in an inertial and a non inertial system of reference using the expression: $\frac{dA}{dt} = \frac{dA^*}{dt} - \omega \wedge A^*$ (OJO PILAR!!! Diria que esto es un + tal como has definido aqui asterisco $\frac{dA}{dt}|_{SRI} = \frac{dA}{dt}|_{SRNI} + \vec{\omega} \times A$).

4. **A plumb line is a line (as of cord) that has at one end a weight and it points in the direction of the local effective gravity $\mathbf{g_{eff}}$. The local effective gravity is the result of adding the gravity that the object would experience if the Earth were not rotating ($\mathbf{g_0} = 9.814 m/s^2$) and the local centrifugal acceleration. Therefore the effective gravity is the value that we can actally measure on Earth.**
   a) Make a sketch of $\mathbf{g_0}$, $\mathbf{g_{eff}}$ and the centrifugal acceleration at a certain latitude λ.
   b) Determine an expression for $\mathbf{g_{eff}}$.
   c) Determine the angle of deviation between $\mathbf{g_{eff}}$ and $\mathbf{g_0}$.
   d) Do the results in b) and c) depend on whether we are at the Northern or Southern hemisphere?
   e) Give expressions for $\mathbf{g_{eff}}$ at the Poles and at the Equator and comment the results.

5. **Determine the angle of deviation due to Coriolis acceleration, for a free falling body at latitude λ, when it is dropped from height h.**

6. **A particle located at $\lambda = 45^o$ North moves along a meridian towards the South.**
   a) Determine its centrifugal acceleration.
   b) Determine its Coriolis acceleration.
   c) Repeat the exercise for the a latitude $\lambda = 45^o$ South.

7. **Consider a conic pendulum of length L, mass m and such that it forms an angle θ with the local vertical.**
   a) Determine its frequency of rotation if the Earth were an inertial system of reference.

b) Show that, due to Earth's rotation, the actual frequency on Earth depends on the direction of the pendulum's rotation and follows the expressions: $w_{-} = -\sqrt{\frac{g}{l} + (\Omega\, sin\lambda)^2} - \Omega\, sin\lambda$ and $w+ = -\sqrt{\frac{g}{l} + (\Omega\, sin\lambda)^2} + \Omega\, sin\lambda$

5.3 Solving problems with a computer

An object of mass m = 10 kg is launched along a meridian towards the South. Its initial velocity is $v_0 = 200m/s$ and it forms an angle $\theta = 30^o$ with the local horizontal.

a) In the absence of air friction, determine the components its acceleration, its velocity and its position as functions of time.
b) If friction is modeled as $F_{r,x} = -bv_x$, $F_{r,y} = -bv_y$ and $F_{r,z} = -bv_z$ in the axis X, Y and Z respectively, where $b = 0.1kg/s$ use Euler's method to determine the acceleration, the velocity and the position of the object as a function of time.
c) Make plots of the magnitudes obtained in a) and b) and comment the results..

6

Rotation about a fix axis

6.1 Questions and brief exercises

1. Give expressions for the moments of inertia, I, in a discrete and in a continuous system of particles. Do these values depend on the distribution of matter along the axis of rotation? Comment the analogies and differences between the concepts of mass and moment of inertia.

2. Give expressions for the angular moments, L, in a discrete and in a continuous system of particles. Give an expression for the time variation of L and state under which conditions L is conserved.

3. Consider a ladder leaned against a smooth (without friction) vertical wall. The ladder has mass m and length L. The friction coefficient between the ground and the ladder is μ. Determine the maximum angle that the ladder and the wall can form without the ladder falling down.

4. Consider a homogeneous semisphere of density ρ and radius R such that its equatorial plane lays in the plane XY and the axis Z goes through its centre. Calculate the centre of mass and the moment of inertia of the semisphere with respect to the Z axis.

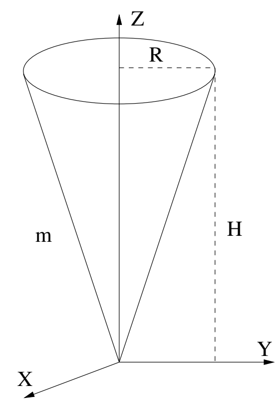
6.2 Moments of inertia

1. **Consider a disc of radius R, such that one of its halves has density σ and the other half has density 2σ. The disc is placed on the plane XY.**
   a) Determine its centre of mass;
   b) Determine its moment of inertia with respect to an axis perpendicular to the disc through its geometric centre;
   c) Determine its moment of inertia with respect to an axis in the plane of the disc, along the diameter that separates the zones of different densities;
   d) Determine its moment of inertia with respect to an axis perpendicular to the axes in b) and c);
   e) Determine its moment of inertia with respect to an axis perpendicular to the disc through its centre of mass.

2. **The Earth has no uniform density. This magnitude varies as a function of r, the distance to the Earth's centre, according to the following expression: $\rho = C(1.22 - r/R)$, where R is the radius of the Earth and C is a constant.**
   a) Determine C as a function of the mass of the Earth, M, and its radius, R.
   b) Determine the moment of inertia of Earth with respect to an axis crossing its centre of mass.

3. **Show that the moment of inertia of the cone in the Figure with respect to the axis X, Y and Z are:**
   - $I_x = I_y = \frac{3}{20}mR^2 + \frac{3}{5}mH^2$,
   - $I_z = \frac{3}{10}mR^2$.

   Compute the location of the centre of mass of the cone and compute its moment of inertia with respect to a system of reference parallel to the former and centered at the centre of mass.



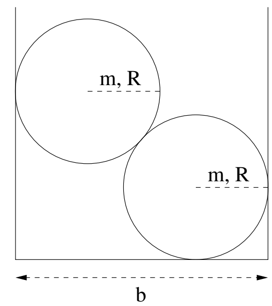
6.3 Dynamics of rotation

1. **A stick of length L and mass M hangs from the ceiling through a pivot located at one of its ends and about which the stick can rotate freely. A bullet of mass m reaches the stick with velocity v and gets stuck in the stick at a distance a from the pivot.**
   a) Find the angular momentum of the stick and the bullet with respect to the pivot before the collision.
   b) Find the angular momentum of the stick and the bullet with respect to the pivot after the collision.
   c) Find the angular velocity ω just after the collision and the maximum angle with respect to the vertical that the stick will form after the collision, θ_{Max}.
   d) Consider the same problem but take into account that the linear density of the stick varies as $\lambda = \lambda_0 + bx$, where λ_0 and b are constants and x=0 at the end of the stick located at the pivot. Compute the new momentum of inertia of the stick and get the new expressions for ω and θ_{Max}.

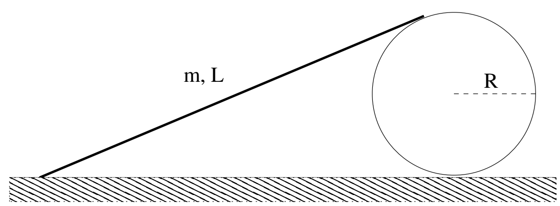
2. **The helix of a plane has a moment of inertia I. It rotates under the effect of a torque $\tau = \tau_0(1+\alpha cos(\omega_0 t))$ given by an engine, and by an additional torque $\tau_f = -b\dot{\theta}$, caused by friction, where α, ω_0 and b are constants. Find the equation for its stationary motion.**

6.4 Statics

1. **Consider two identical spheres of mass m and radius R placed inside a box of base length b (see Figure). Determine the expressions of the contact (normal) forces acting between the two spheres and between the spheres and the box.**



2. **Consider the system of the Figure, consisting of a sphere of radius R and mass M and a stick of mass m and length L, with one end placed on the floor and the other end on the sphere. Assume there is friction between all objects and determine:**
   a) The expresions for the contact (normal) forces acting in the system.
   b) The expresions for the friction forces acting on the system.



7

Rotation about variable axes

7.1 Questions and brief exercises.

1. Show that both the angular momentum and the rotational kinetic energy of a rigid solid is related to the instantaneous angular velocity vector $\vec{\omega}$ through a tensor. Which tensor is it?

2. Consider the 9 components of the tensor of inertia of a uniform sphere. The tensor is given with respect to a system of axes centered at the centre of the sphere. Determine which components of the tensor are different or equal to zero. How many different non-zero components are there? Obtain the same information in the case of a uniform cylinder.

3. Consider now a non-homogeneous sphere such that its density depends on its latitude. It can be assumed equal to zero at the south pole, and linearly growing to reach a maximum value at the north pole. Perform the same analysis as in question 2.

4. What is the meaning of principal axis of inertia? In terms of the properties of rotation, which is the main difference between a principal and a non-principal axis of inertia?

5. Briefly describe Euler's angles, and how a general rotation can be expressed in terms of these angles.

6. Describe the relationship between the time derivatives of Euler angles (Euler angle rates), and the instantaneous angular velocity vector. Can you define a vector using the Euler angle rates?

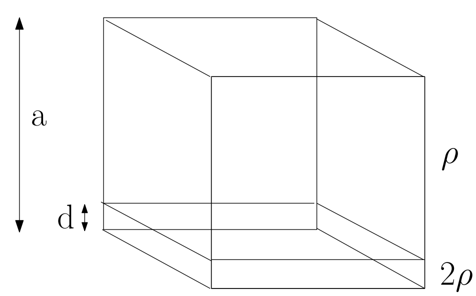
7. Describe each term of Euler equations for rotation. Tell in which system of reference Euler equations are expressed. Why do we need to know the transformation relation for a given vector between an inertial and a non-inertial system of reference, in order to express Euler equations?

7.2 Tensor of Inertia

1. **Consider a rigid solid composed of 4 point masses m, placed at the corners of a square of side a.**
   a) Obtain the tensor of inertia with respect to a system of reference centred at the centre of the square, with XYZ axes perpendicular to each side of the square.
   b) Obtain the tensor of inertia with respect to a system of reference parallel to the system in a), but centred at one of the corners of the square.
   c) Obtain the principal axes of inertia for case b).

2. **Consider a homogeneous cone, of mass M, height H, and radius at the base R. Assume a system of reference such that its Z axis coincides with the symmetry axis of the cone, and that the origin is located at its vertex.**
   a) Obtain the principal momenta of inertia with respect to the given system of reference.
   b) Find the centre of mass of the cone, and its tensor of inertia, for a system of reference parallel to the one in a), but with origin is located at its centre of mass.
   c) Find the principal momenta of inertia in the latter case.

3. **Consider a loaded dice, that is, one such that its mass is not uniformly distributed in order to favour certain results. Assume that the dice is a cube of side a and constant density ρ in the entire volume, except for a thin square slab of density 2ρ and dimensions (a,a,d) , located at one of the sides of the cube. Consider the geometric centre of the dice as the origin of the system of reference, and its axes perpendicular to the sides of the cube.**
   a) Obtain the mass of the dice as functions of ρ, a and d.
   b) Locate the position of the centre of mass of the dice.
   c) Obtain the tensor of inertia of the dice for the system of reference described above.
   d) Assume a=4cm, d=8mm, M=100g (for the entire dice). Obtain the angular momentum in three cases: when an initial angular velocity is applied along the X axis, along the Y axis, and along the Z axis. Will the dice keep constant angular velocity in any of the three cases, in the absence of external torques? Is any of the axes a principal axis of inertia?
   e) Describe a simple experiment to be performed in the absence of gravity, in order to find out whether a dice is loaded.



4. **Consider a homogeneous slab of unknown shape surface density σ. Assume that the slab is located in the plane XY, and consider a system of reference with origin at its centre of mass, and Z axis perpendicular to the slab.**
   a) Show that 4 elements of the tensor of inertia will always be zero, regardless the shape of the slab.

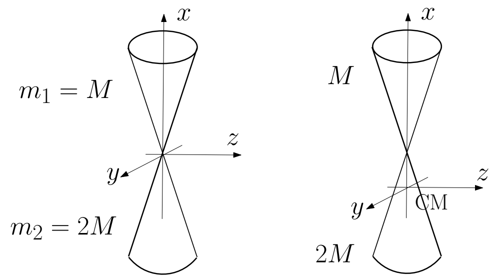
b) Show that, in order to find the non-zero elements of the tensor of inertia, it is enough to calculate 3 integrals. Write the tensor of inertia in terms of these integrals (say their results are A, B and C).
c) Determine the new tensor of inertia with respect to a system of axes rotated an angle θ abot the original Z axis.

5. An airplane takes off by gradually lifting its nose, as the height of its centre of mass increases.
a) Describe the lift of the nose in terms of Euler angles.
b) A 100 m long airplane lifts its nose 250 m, and its centre of mass 200 m above the ground, during the first 3 seconds. Write the transformation matrix which allows to obtain the location of an arbitrary point of the plane (x',y',z') in terms of its position at time t=0 (x,y,z).

6. You throw a dice and get a six.
a) Say which rotation (in terms of Euler angles) You would need to implement on the dice, in order to get a one (recall that six and one are in opposite sides of the dice). Is the solution to the former question unique?
b) Calculate the general rotation matrix to change from 6 to one.

7.3 Euler Equations and Euler Angles. Aeronautics notation.

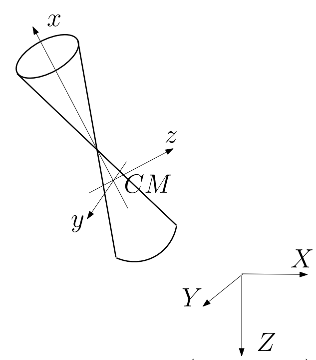
3. We build a device by fixing together two solid cones of radius of the base R and height H. One cone has mass M, and the other has mass $2M$. You may assume that the experiments described in this exercise are performed in space, and thus **you should neglect gravity**. (4p) *HELP: The distance between the vertex of a homogeneous cone of height H and its centre of mass is* $d = \frac{3}{4}H$.



(**A**) (**B**)

   a) The inertia tensor of a single cone of mass m, radius of the base R and height H, with respect to a frame of reference centred at its centre of mass, and with the X-axis located along its revolution axis is: $I_{xx} = \frac{3}{10}mR^2$ and $I_{yy} = I_{zz} = \frac{3}{20}mR^2 + \frac{3}{80}mH^2$. Determine the inertia tensor of the entire system with respect to a frame of reference centred in its centre of mass, as in Figure (B). (0.75p)

   b) From now on, use the following notation for the moments of inertia of the system: $I_x = I_0$, $I_y = I_z = I$. Consider the frame of reference of axes $\{x, y, z\}$ fixed to the body (SRB), with x axis along the revolution axes of the cones. Consider also the frame of reference $\{XYZ\}$, a horizontal frame of reference (SRH) at rest, and with Z axis pointing along the direction of the weight. Justify briefly whether the system can rotate with constant angular velocity about Z in the absence of external torques. (0.75p)



   c) Consider a point charge q located in vertex of the cones. The system rotates with $(\omega_1, \omega_2, \omega_3)$ about its centre of mass, which is a fixed point. If an external electric field $\vec{E} = E_0\vec{I}$ (with E_0 constant) acts upon q, determine the electric force $\vec{F}_E = q\vec{E}$ and the torque due this force $\vec{\tau}_E$ in SRB. (0.75p)

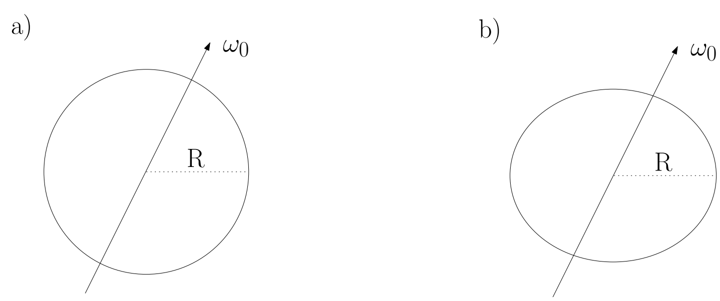
   d) Write Euler equations in terms of $(\omega_1, \omega_2, \omega_3)$, and briefly describe the behaviour of ω_1. Consider the particular case in which the roll angle is $\phi = 0$ constant. Express Euler equations in terms of the Euler angles (ϕ, θ, and ψ) and their temporal variations ($\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$). (0.75p)

   e) If in addition to $\phi = 0$ constant we know that $\ddot{\psi} = 0$ constant, use the result from the former section to rewrite Euler equations in terms of Euler angles and their rates. Describe the possible types of motion of the body. (1p)

$$\mathcal{R} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & cos\phi\cos\theta \end{bmatrix}$$

$$\mathcal{H} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

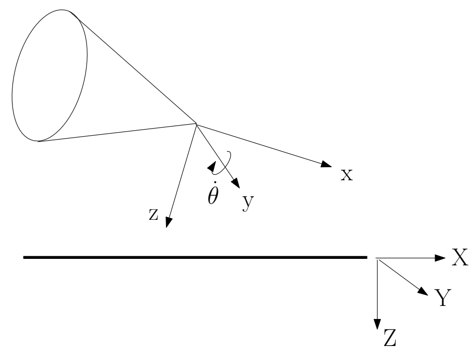
1. **We get to travel to the ISS and of course take with us our favourite ball, which can be approximated by a solid sphere of radius R = 10 cm and mass M = 0.1 kg.**
   a) Determine the tensor of inertia of the ball (under the approximation that it is a perfect solid sphere), for a system of orthogonal axes centered in the centre of the sphere. HELP: $\int sin^3 x dx = \frac{1}{12}cos(3x) - \frac{3}{4}cos(x)$. (1p)
   b) We give an initial angular velocity to the ball $\omega_0 = 1\, rad/s$, so that it starts rotating about its centre (Figure a). Determine (with a numerical value) the angular momentum of the ball. What type of motion of the ball would you expect if it were a perfect sphere? Justify your answer.
   c) After closer inspection we suspect that the ball is not a perfect sphere. Instead it appears to be oblate, as shown in Figure b). What type of motion do we actually observe if $\vec{\omega}_0$ has an arbitrary direction? (
   d) Make a sketch of the location of the principal axes of inertia of the non-spherical ball, including the initial angular velocity. Assuming that the moments of inertia with respect to the principal axes are $I_1 = I_2 = I$, $I_3 = I_0$, write the Euler equations for the non-spherical ball (Figure b)).
   e) Solve Euler's equations to sketch and give expressions for the angular velocity of the ball in terms of its components about the principal axes of inertia. Jusify that your solution agrees with the type of motion you expected in item c).
   f) We measure $w_{30} = 1\, rad/s$ and observe that the sphere wobbles about its principal axis of inertia X_3, with a period $T_\Omega = 44\, s$. If we know that $I_0 = 4 \times 10^{-4}\, kg\, m$, determine the moments of inertia $I_1 = I_2 = I$.



2. **The payload of a rocket is a small conic-shaped space capsule falling towards Mars under the effect of gravity. The capsule has mass M and homogeneous density, base radius R and height H. Its tensor of inertia with respect to a system of reference centred in its centre of mass (SRCM), with X axis along the axis of the cone, is such that:**
   $\mathbf{I_{xx} = 3/10MR^2}$, $\mathbf{I_{yy} = I_{zz} = 3/80MH^2 + 3/20MR^2}$,
   $\mathbf{I_{xy} = I_{xz} = I_{yz} = 0}$.
   Consider also a system of reference attached to the ground, with Z axis aimed towards the centre of the Earth (SRI).
   a) Determine the tensor of inertia of the capsule w.r.t. a SR parallel to SRCM and centred in its vertex. The new system of reference is SRBody. Help: $V_{cono} = \frac{\pi}{3}R^2 H$.
   b) While the capsule falls, something forces it to rotate with constant angular velocity around its vertex (the vertex is actually the fixed point of rotation), so that $\dot{\theta} = \dot{\theta}_0$=constant$\neq 0$, and $\dot{\psi} = \dot{\phi} = 0$. Write the rotation matrix associated to the motion of the capsule after a certain time t. Express the weight of the capsule in SRBody.

c) Given the motion described in b), write Euler equations of rotation in terms of the angular velocities about the principal axes of inertia associated to SRBody $(\omega_1, \omega_2, \omega_3)$. Does the weight exert any torque on the capsule? If so, give an expression for the associated torque in SRBody. Justify whether the motion of the cone described above is possible without additional torques.

d) Express Euler equations for rotation in SRBody in terms of the Euler angle rates ($\dot{\psi}$, $\dot{\theta}$ and $\dot{\phi}$) in the case in which the cone describes a rotation in 'roll' and 'pitch', both at constant rates $\dot{\phi}_0$ y $\dot{\theta}_0$ respectively. Help: You may use the H matrix below to express $\omega_1, \omega_2, \omega_3$ in terms of $\dot{\phi}, \dot{\theta}, \dot{\psi}$.



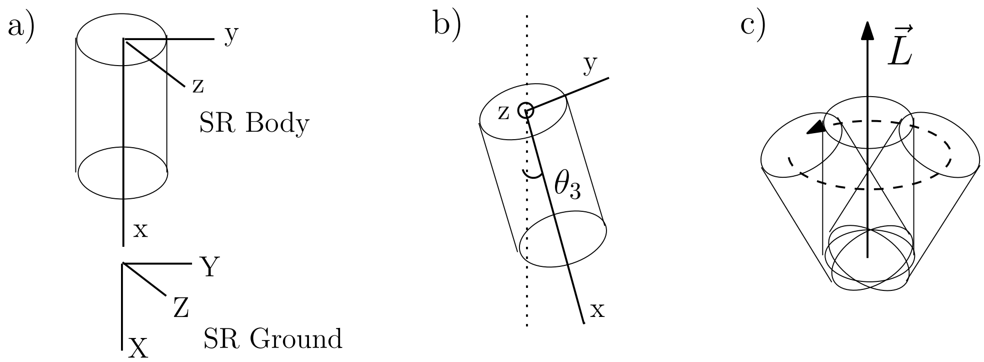
3. Consider an homogeneous cylinder of mass M, radius R and height H. Take an ortogonal system of reference, attached to the body (SRBody), centered at the centre of the base of the cylinder, with its X axis along the cylinder axis, and axes Y and Z placed at the plane of the base. Consider also an inertial system of reference fixed at the ground, with its X axis towards the centre of the Earth (SRGround).

a) Proof (by integration an symmetry arguments) that its tensor of inertia with respect to the SRBody is such that: $\mathrm{I_{xx}} = \frac{1}{2}\mathrm{MR}^2$, $\mathrm{I_{yy}} = \mathrm{I_{zz}} = \frac{1}{3}\mathrm{MH}^2 + \frac{1}{4}\mathrm{MR}^2$, $\mathrm{I_{xy}} = \mathrm{I_{xz}} = \mathrm{I_{yz}} = 0$.

b) Assume that the cylinder is fixed at the origin of the SRBody. We displace the cylinder a certain angle θ_3 about the z axis, and then release it from rest. Write the weight of the cylinder and its associated torque in SRBody. Write Euler equations for rotation, solve them and describe the motion of the cylinder when θ_3 is very small.

c) Assume that that cylinder rotates about the origin, following the motion of Figure c), with constant ω_1 and and variable ω_2 and ω_3. Write Euler equations for rotation in terms of ω_1, ω_2 and ω_3 (in SRBody) and check that the τ_1 component of the net torque must be zero. If $\omega_2 = A cos(\Omega\, t)$ and $\omega_3 = A sin(\Omega\, t)$, with constant A and Ω, determine the expression of Ω for the cylinder to move as in Figure c), when the net external torque is $\vec{\tau}{=}\vec{0}$.

d) Consider a different type of motion: the cylinder rotates with constant zero pitch and roll angles ($\theta{=}\phi{=}0$), and constant rate of yaw $\dot{\psi}{=}\dot{\psi}_0$. Write Euler equation in terms of Euler angles (aeronautical notation) and their derivatives. Check that the cylinder may follow the motion described above with $\vec{\tau}{=}\vec{0}$. If $\theta{=}0$ constant, but ϕ and ψ vary at constant rate ($\dot{\phi}_0$ and $\dot{\psi}_0$ respectivelly, with $\phi_0{=}0$), determine the new Euler equations for rotación and determine $\vec{\tau}$ in terms of Euler angles and their variation rates and other constants.



HELP:

$$R = \begin{bmatrix} cos\theta \cos\psi & cos\theta sin\psi & -sin\theta \\ -cos\phi sin\psi + sin\phi sin\theta cos\psi & cos\phi cos\psi + sin\phi sin\theta sin\psi & sin\phi cos\theta \\ sin\phi sin\psi + cos\phi sin\theta cos\psi & -sin\phi cos\psi + cos\phi sin\theta sin\psi & cos\phi cos\theta \end{bmatrix}$$

$$, H = \begin{bmatrix} 1 & 0 & -sin\theta \\ 0 & cos\phi & sin\theta sin\phi \\ 0 & -sin\phi & cos\theta cos\phi \end{bmatrix}$$