

# STATISTICAL COMPUTATIONAL METHODS

## Seminar Nr. 5, Counting Processes

1. On the average, 6 airplanes per minute land at a certain international airport. Assume the number of landings is modeled by a Binomial counting process.

a) What frame length should be used to guarantee that the probability of a landing does not exceed 0.1?

b) Using the chosen frames, compute the probability of no landings during the next half a minute;

c) Using the chosen frames, compute the probability of more than 170 landed airplanes during the next 30 minutes.

### Solution:

a) We have  $\lambda = 6/\text{min}$ . So, if we want  $p \leq 0.1$ , then

$$\Delta = \frac{p}{\lambda} \leq \frac{0.1}{6} \text{ min} = 1 \text{ sec.}$$

b) Let  $\Delta = 1 \text{ sec}$ . In  $t = 1/2 \text{ min} = 30 \text{ sec.}$ , there are  $n = \frac{t}{\Delta} = \frac{30}{1} = 30$  frames. The number of landings  $X(30)$  during 30 frames has Binomial distribution with parameters  $n = 30$  and  $p = 0.1$ . We want

$$P(X(30) = 0) = \text{binopdf}(0, 30, 0.1) = 0.0424.$$

c) Similarly, in 30 minutes = 1800 sec., there are  $n = \frac{1800}{1} = 1800$  frames. Thus, the number of landings  $X(1800)$  during the next half hour has Binomial distribution with parameters  $n = 1800$  and  $p = 0.1$ . We want to compute

$$P(X(1800) > 170) = 1 - P(X(1800) \leq 170) = 1 - \text{binocdf}(170, 1800, 0.1) = 0.7709.$$

2. Messages arrive at a communications center according to a Binomial counting process with 30 frames per minute. The average arrival rate is 40 messages per hour. How many messages can be expected to arrive between 10 a.m. and 10:30 a.m.? What is the standard deviation of that number of messages?

### Solution:

The arrival rate is  $\lambda = 40/\text{hr} = \frac{2}{3}/\text{min.}$  and the frame length is  $\Delta = \frac{1}{30}$  min. Then, the probability of a new message arriving during any given frame is

$$p = \lambda\Delta = \frac{2}{3} \cdot \frac{1}{30} = \frac{1}{45}.$$

Between 10 and 10:30 a.m., there are  $t = 30$  minutes, so there are  $n = \frac{t}{\Delta} = \frac{30}{1/30} = 900$  frames, hence, the number of new messages  $X(900)$  arriving during this time has Binomial distribution with  $n = 900$  and  $p = 1/45$ , so

$$\begin{aligned} E(X) &= np = 20 \text{ messages,} \\ \sigma(X) &= \sqrt{np(1-p)} = 4.4222 \text{ messages.} \end{aligned}$$

**3.** An internet service provider offers special discounts to every third connecting customer. Its customers connect to the internet according to a Poisson process with the rate of 5 customers per minute. Compute

- a) the probability that no offer is made during the first 2 minutes;
- b) the probability that no customers connect for 20 seconds;
- c) expectation and standard deviation of the time of first offer.

**Solution:**

a) We have  $\lambda = 5/\text{min.}$  In  $t = 2$  minutes, the number of connections  $X$  has Poisson distribution with parameter  $\lambda t = 5 \cdot 2 = 10$ . No offer is made if there are *fewer* than three connections during that time. So, we want

$$P(\text{no offer}) = P(X < 3) = P(X \leq 2) = \text{poisscdf}(2, 10) = 0.0028.$$

b) The time between connections (interarrival time)  $T$  has Exponential( $\lambda$ ) distribution. No customers connect for 20 sec. =  $1/3$  minutes, if the interarrival time exceeds that. So, we want

$$P(T > 1/3) = 1 - P(T \leq 1/3) = 1 - \text{expcdf}(1/3, 1/5) = 0.1889.$$

Or we can express  $\lambda$  in seconds,  $\lambda = 5/60 = 1/12/\text{sec.}$  Then we compute

$$P(T > 20) = 1 - P(T \leq 20) = 1 - \text{expcdf}(20, 12) = 0.1889.$$

Alternatively, we want 0 connections (arrivals) in  $t = 20$  seconds  $= 1/3$  minutes. Just like in part a), the number of connections  $X$  has Poisson distribution with parameter  $\lambda t = 5 \cdot 1/3 = 5/3$ . Then we compute

$$P(X = 0) = \text{poisspdf}(0, 5/3) = 0.1889.$$

**c)** The time  $T_3$  of the third connection (arrival) (and therefore, the first offer) is the sum of 3 independent  $\text{Exp}(5)$  times, so it has Gamma distribution with parameters  $\alpha = 3$  and  $\lambda = 1/5$ . Then

$$\begin{aligned} E(T_3) &= \alpha\lambda = 3/5 = 0.6 \text{ min}, \\ \sigma(T_3) &= \sqrt{V(T_3)} = \sqrt{\alpha\lambda^2} = 0.3464 \text{ min}. \end{aligned}$$

Note: We could have considered the time  $T_3$  of the first offer in part **a)**, as well. No offer is made during the first 2 minutes, if the time of the first offer *exceeds* 2 minutes. So, we want to compute

$$P(T_3 > 2) = 1 - P(T_3 \leq 2) = 1 - \text{gamcdf}(2, 3, 1/5) = 0.0028.$$