STATISTICAL COMPUTATIONAL METHODS

Seminar Nr. 6, Queuing Systems

1. A metered parking lot with two parking spaces is modeled by a Bernoulli two-server queuing system with capacity limited by two cars and 30-second frames. Cars arrive at the rate of one car every 4 minutes and each car is parked for 5 minutes, on the average.

a) find the transition probability matrix for the number of parked cars;

b) find the steady-state distribution for the number of parked cars;

c) what fraction of the time are both parking spaces vacant?

d) what fraction of arriving cars will not be able to park?

e) every 2 minutes of parking costs 25 cents; assuming all drivers use all the parking time they pay for, how much money is the parking lot going to raise every 24 hours?

Solution:

This is a B2SQS with

$$C = 2, k = 2, \Delta = 1/2 \text{ min}, \lambda_A = 1/4 \text{ min}^{-1} \text{ and } \lambda_S = 1/5 \text{ min}^{-1}.$$

a) $p_A = \lambda_A \Delta = 1/8$ and $p_S = \lambda_S \Delta = 1/10$. There are 3 states $\{0, 1, 2\}$ and the transition probabilities are

$$\begin{aligned}
p_{00} &= 1 - p_A = 7/8, \\
p_{01} &= p_A = 1/8, \\
p_{02} &= 0; \\
p_{10} &= p_S(1 - p_A) = 7/80, \\
p_{11} &= (1 - p_A)(1 - p_S) + p_A p_S = 4/5, \\
p_{12} &= p_A(1 - p_S) = 9/80; \\
p_{20} &= p_S^2(1 - p_A) = 7/800, \\
p_{21} &= 2p_S(1 - p_A)(1 - p_S) + p_S^2 p_A = 127/800, \\
p_{22} &= (1 - p_A)(1 - p_S)^2 + 2p_A p_S(1 - p_S) + p_A(1 - p_S)^2 = 333/400.
\end{aligned}$$

So, the transition probability matrix is

$$P = \begin{bmatrix} 7/8 & 1/8 & 0 \\ 7/80 & 4/5 & 9/80 \\ 7/800 & 127/800 & 333/400 \end{bmatrix}.$$

b) The steady-state distribution is

$$\pi = [\pi_0 \;\; \pi_1 \;\; \pi_2] = [0.3089 \quad 0.4135 \quad 0.2777].$$

c) That would be

$$P(X = 0) \;=\; \pi_0 \;=\; 0.3089,$$

or 30.89% of the time.

d) A car cannot park if both spaces are taken, so

$$P(X = 2) \;=\; \pi_2 \;=\; 0.2777,$$

or 27.77% of cars.

e) The expected number of parked cars is

$$E(X) \;=\; \sum_{0}^{2} x\pi_x \;=\; 0 \cdot 0.3089 + 1 \cdot 0.4135 + 2 \cdot 0.2777 \;=\; 0.9689.$$

Then the total revenue in 24 hours is

$$E(X) \cdot 24 \cdot 60 \cdot 0.25/2 \;=\; 174.4020 \text{ dollars.}$$

2. Trucks arrive at a weigh station according to a Poisson process with average rate of 1 truck every 10 minutes. Inspection times are Exponential with the average of 3 minutes. When a truck is on the scale, the other arrived trucks stay in line waiting for their turn. Compute
a) the expected number of trucks at the weigh station at any time;
b) the proportion of time when the weigh station is empty;
c) the expected time each truck spends at the station, from arrival to departure;
d) the fraction of time there are fewer than 2 trucks in the weigh station.

Solution:

A Poisson process of arrivals implies Exponential interarrival times, so the described system is M/M/1 with $\mu_A = 10$ min and $\mu_S = 3$ min. Hence, $\lambda_A = 1/10\,\mathrm{min}^{-1}$, $\lambda_S = 1/3\,\mathrm{min}^{-1}$ and $r = \lambda_A/\lambda_S = 0.3 < 1$.

a) The expected number of trucks at the weigh station is

$$E(X) \;=\; \frac{r}{1-r} \;=\; 3/7 \;=\; 0.4286.$$

b) The proportion of time when the weigh station is empty is

$$P(X=0) \;=\; 1-r \;=\; 0.7,$$

or 70% of time.

c) This is the expected response time

$$E(R) \;=\; \frac{\mu_S}{1-r} \;=\; \frac{3}{0.7} \;=\; 30/7,$$

or 4.2857 minutes.

d) This is the probability

$$\begin{aligned} P(X<2) \;&=\; P(\{X=0\}\cup\{X=1\}) \;=\; P(0)+P(1) \\ &=\; \pi_0+\pi_1 \;=\; (1-r)+r(1-r) \\ &=\; 1-r^2 \;=1-0.09 \;=\; 0.91, \end{aligned}$$

or 91% of the time.

3. A toll area on a highway has three toll booths and works as an M/M/3 queuing system. On the average, cars arrive at the rate of one car every 5 seconds, and it takes 12 seconds to pay the toll, not including the waiting time. Compute the fraction of time when there are ten or more cars waiting in the line.

<u>Solution:</u>

We have

$$\lambda_A \;=\; 1/5\,\mathrm{sec}^{-1},\; \lambda_S \;=\; 1/12\,\mathrm{sec}^{-1},\; k \;=\; 3 \text{ and } r \;=\; \lambda_A/\lambda_S \;=\; 12/5 \;=\; 2.4 \;< 3.$$

We want to compute

$$P(X_w \geq 10).$$

Since there are 3 toll booths (all busy), this is the same as

$$P(X \geq 13) \;=\; \sum_{x=13}^{\infty} \pi_x.$$

Now, for the steady-state distribution, we have

$$\begin{aligned}
\pi_0 &= \frac{1}{1 + r + \dfrac{r^2}{2} + \dfrac{r^3}{6(1 - r/3)}} \;=\; 0.0562, \\
\pi_1 &= \ldots, \\
\pi_2 &= \ldots, \\
\pi_x &= \frac{r^3}{3!}\pi_0 \left(\frac{r}{3}\right)^{x-3}, \text{ for any } x \geq 3.
\end{aligned}$$

So,

$$\begin{aligned}
P(X_w \geq 10) &= \frac{r^3\pi_0}{3!}\sum_{x=13}^{\infty}\left(\frac{r}{3}\right)^{x-3} \;=\; \frac{r^3\pi_0}{3!}\left(\frac{r}{3}\right)^{10}\left[1 + \left(\frac{r}{3}\right) + \left(\frac{r}{3}\right)^2 + \ldots\right] \\
&= \frac{r^3\pi_0}{6} \cdot \frac{(r/3)^{10}}{1 - r/3} \;=\; 0.0695,
\end{aligned}$$

or 6.95% of the time. Note that we used the formula for the sum of a Geometric series with ratio $q < 1$,

$$\sum_{k=a}^{\infty} q^k \;=\; q^a \sum_{k=a}^{\infty} q^{k-a} \;=\; q^a \sum_{i=0}^{\infty} q^i \;=\; \frac{q^a}{1-q}.$$

4. Sports fans tune to a local sports radio station according to a Poisson process with the rate of three fans every two minutes and listen to it for an Exponential amount of time with the average of 20 minutes.

a) what queuing system is the most appropriate for this situation?

b) compute the expected number of concurrent listeners at any time;

c) find the fraction of time when 40 or more fans are tuned to this station.

<u>Solution:</u>

We have

$$\lambda_A \;=\; 3/2\,\mathrm{min}^{-1},\; \lambda_S \;=\; 1/20\,\mathrm{min}^{-1},\; k \;\approx\; \infty \;\text{ and }\; r \;=\; \lambda_A/\lambda_S \;=\; 60/2 \;=\; 30.$$

a) We have a Poisson process of arrivals (so Exponential interarrival times), Exponential service times and "infinitely" many servers (because any number of people can listen to a radio station simultaneously), therefore, an $\mathrm{M}/\mathrm{M}/\infty$ queuing system is the most appropriate to model this situation.

b) The expected number of concurrent listeners is

$$E(X) \;=\; r \;=\; 30 \;\text{ listeners.}$$

c) The number of concurrent listeners, X, has Poisson distribution with parameter $r = 30$. So,

$$P(X \geq 40) \;=\; 1 - P(X < 40) \;=\; 1 - P(X \leq 39) \;=\; 1 - \mathit{poisscdf}(39, 30) \;=\; 0.0463,$$

or 4.63% of the time.