

Lab Nr. 2, Numerical Calculus

Taylor Polynomials and Taylor Series

• Short theory review

Theorem 1. [Taylor's Theorem] Let $f : [a, b] \rightarrow \mathbb{R}$ be a function with $n + 1$ continuous derivatives on $[a, b]$, for some $n \geq 0$ and let $x, x_0 \in [a, b]$. Then

$$\begin{aligned} f(x) &= T_n f(x) + R_{n+1}(x), \text{ with} \\ T_n f(x) &= f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \dots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) \text{ and} \\ R_{n+1}(x) &= \frac{1}{n!} \int_{x_0}^x (x - t)^n f^{(n+1)}(t) dt = \frac{(x - x_0)^{n+1}}{(n + 1)!} f^{(n+1)}(\xi), \xi \text{ between } x \text{ and } x_0. \end{aligned} \quad (1)$$

Using Taylor's theorem with $x_0 = 0$, we obtain MacLaurin's formula:

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n + 1)!} f^{(n+1)}(\xi), \xi \text{ between } x \text{ and } x_0.$$

MacLaurin's formula for some common functions:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n + 1)!} e^{\xi_x}, \quad (2)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1} \frac{x^{2n+2}}{(2n + 2)!} \cos \xi_x, \quad (3)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n - 1)!} + (-1)^n \frac{x^{2n+1}}{(2n + 1)!} \sin \xi_x, \quad (4)$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + (-1)^n \frac{x^{n+1}}{n + 1} \frac{1}{(1 + \xi_x)^{n+1}}, \quad (5)$$

$$(1 + x)^a = 1 + \binom{a}{1} x + \binom{a}{2} x^2 + \dots + \binom{a}{n} x^n + \binom{a}{n + 1} \frac{x^{n+1}}{(1 + \xi_x)^{n+1-a}}, \quad (6)$$

$$\binom{a}{k} = \frac{a(a - 1) \cdots (a - k + 1)}{k!}, \quad k = 1, 2, 3, \dots, a \in \mathbb{R}.$$

As $n \rightarrow \infty$, the Taylor series on the right-hand-sides of (2) – (4) converge for $x \in \mathbb{R}$, and those in (5) – (6), for $|x| < 1$.

• **Matlab function** *taylor*

Applications

1. Use formula (2) to do the following:

- a) Graph on the same set of axes the function e^x and its Taylor polynomials of degrees 1, 2, 3 and 4, for $x \in [-3, 3]$;
- b) Approximate e with 6 correct decimals.

2. Using formula (4), do the following:
- a) Graph on the same set of axes the function $\sin x$ and its Taylor polynomials of degrees 3 and 5, for $x \in [-\pi, \pi]$;
 - b) Approximate $\sin\left(\frac{\pi}{5}\right)$ with 5 correct decimals;
 - c) What happens for $x = \frac{10\pi}{3}$? Explain the phenomenon and find a solution.
- 3.
- a) Using formula (5), graph on the same set of axes the function $\ln(1+x)$ and its Taylor polynomials of degrees 2 and 5, for $x \in [-0.9, 1]$;
 - b) How many terms would be necessary in (5) to approximate $\ln 2$ with 5 correct decimals?
 - c) Use formula (5) to find a Taylor series expansion for the function $\ln(1-x)$.
 - d) Use formula (5) and the one found in part c) to derive a Taylor series expansion for the function $\ln \frac{1+x}{1-x}$.
 - e) Use the formula found in part d) to approximate $\ln 2$ with 5 correct decimals. How many terms are necessary?

Optional

4. Use formula (6) to approximate $\sqrt[3]{999}$ with 10 correct decimals (**Caution!!** The series in (6) converges *only* for $|x| < 1$).