Part XII.
Normal Forms and Properties of CFLs

Chomsky Normal Form (CNF)

Definition: Let $G = (N, T, P, S)$ be a CFG. G is in *Chomsky normal form* if every rule in P has one of these forms

- $A \to BC$, where $A, B, C \in N$;
- $A \to a$, where $A \in N$, $a \in T$;

Example:

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b\}$,
$P = \{S \to CB, C \to AS, S \to AB, A \to a, B \to b\}$
is in Chomsky normal form.

Note: $L(G) = \{a^n b^n : n \geq 1\}$

Greibach Normal Form (GNF)

Definition: Let $G = (N, T, P, S)$ be a CFG.
G is in *Greibach normal form* if every rule in P is of this form

- $\boldsymbol{A} \rightarrow \boldsymbol{ax}$, where $A \in N$, $a \in T$, $x \in N^*$

Example:

$G = (N, T, P, S)$, where $N = \{\boldsymbol{B}, \boldsymbol{S}\}$, $T = \{\boldsymbol{a}, \boldsymbol{b}\}$,
$P = \{\boldsymbol{S} \rightarrow \boldsymbol{aSB}, \boldsymbol{S} \rightarrow \boldsymbol{aB}, \boldsymbol{B} \rightarrow \boldsymbol{b}\}$
is in Greibach normal form.

Note: $L(G) = \{a^n b^n: n \geq 1\}$

Generative Power of Normal Forms

Theorem: For every CFG G, there is an equivalent grammar G' in Chomsky normal form.

Proof: See page 348 in [Meduna: Automata and Languages]

Theorem: For every CFG G, there is an equivalent grammar G' in Greibach normal form.

Proof: See page 376 in [Meduna: Automata and Languages]

Note: Main properties of CNF and GNF:

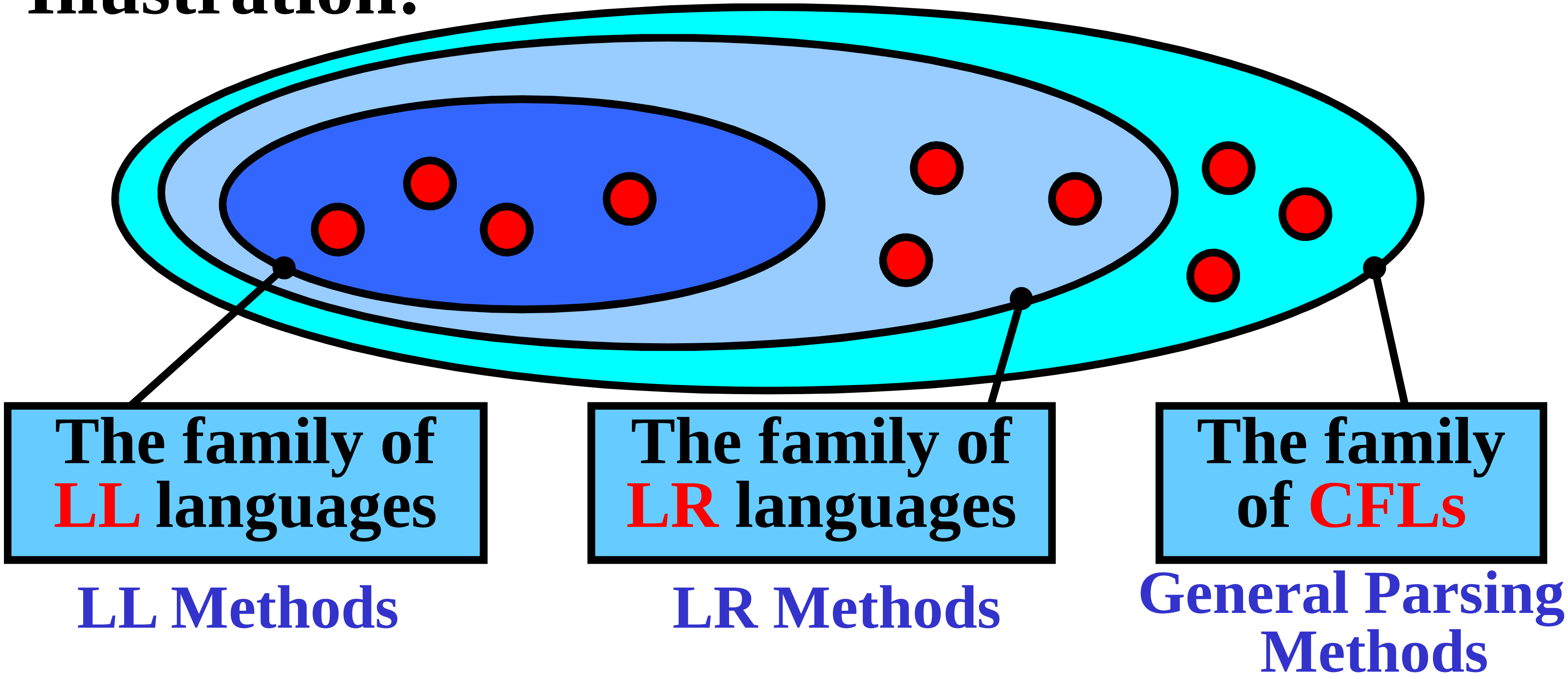
CNF: **if** $S \Rightarrow^n w$; $w \in T^*$ **then** $n = 2|w| - 1$

GNF: **if** $S \Rightarrow^n w$; $w \in T^*$ **then** $n = |w|$

General Parsing Methods

- ***G**eneral **P**arsing* **methods** (GP) are applicable to all context-free languages (CFLs)

Illustration:



- **Note:** The family of LR languages = the family of a **deterministic CFL**

GP Based on Chomsky Normal Form

$S[1, n]$

Idea:

$S[1, 3]$ … $S[n\text{-}2, n]$

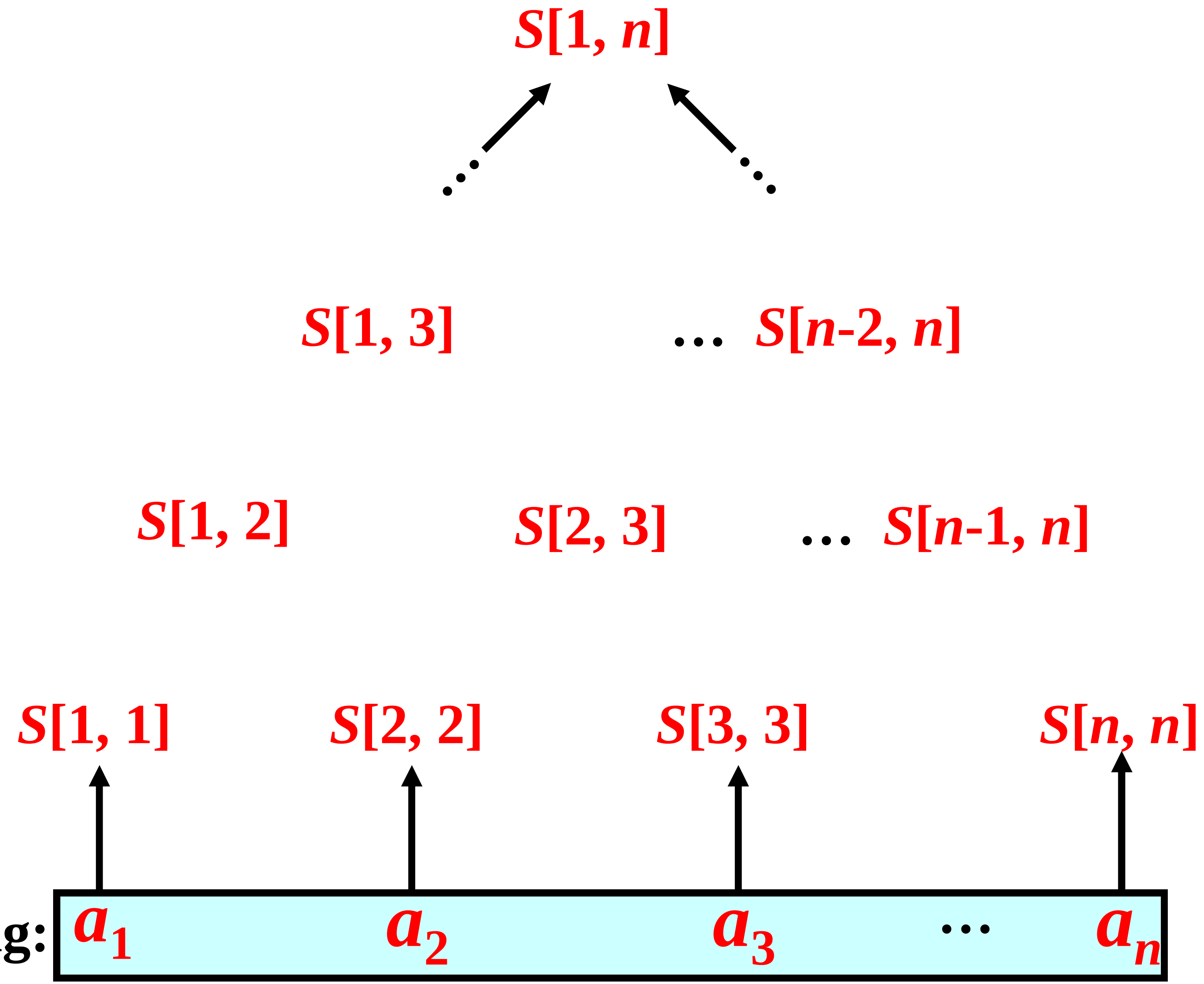
$S[1, 2]$ $S[2, 3]$ … $S[n\text{-}1, n]$

$S[1, 1]$ $S[2, 2]$ $S[3, 3]$ $S[n, n]$

- **Input string:** a_1 a_2 a_3 … a_n

GP Based on Chomsky Normal Form

Idea:



- **Input string:**

GP Based on Chomsky Normal Form

$S[1, n]$

Idea:

$S[1, 3]$ … $S[n\text{-}2, n]$

$S[1, 2]$ $S[2, 3]$ … $S[n\text{-}1, n]$

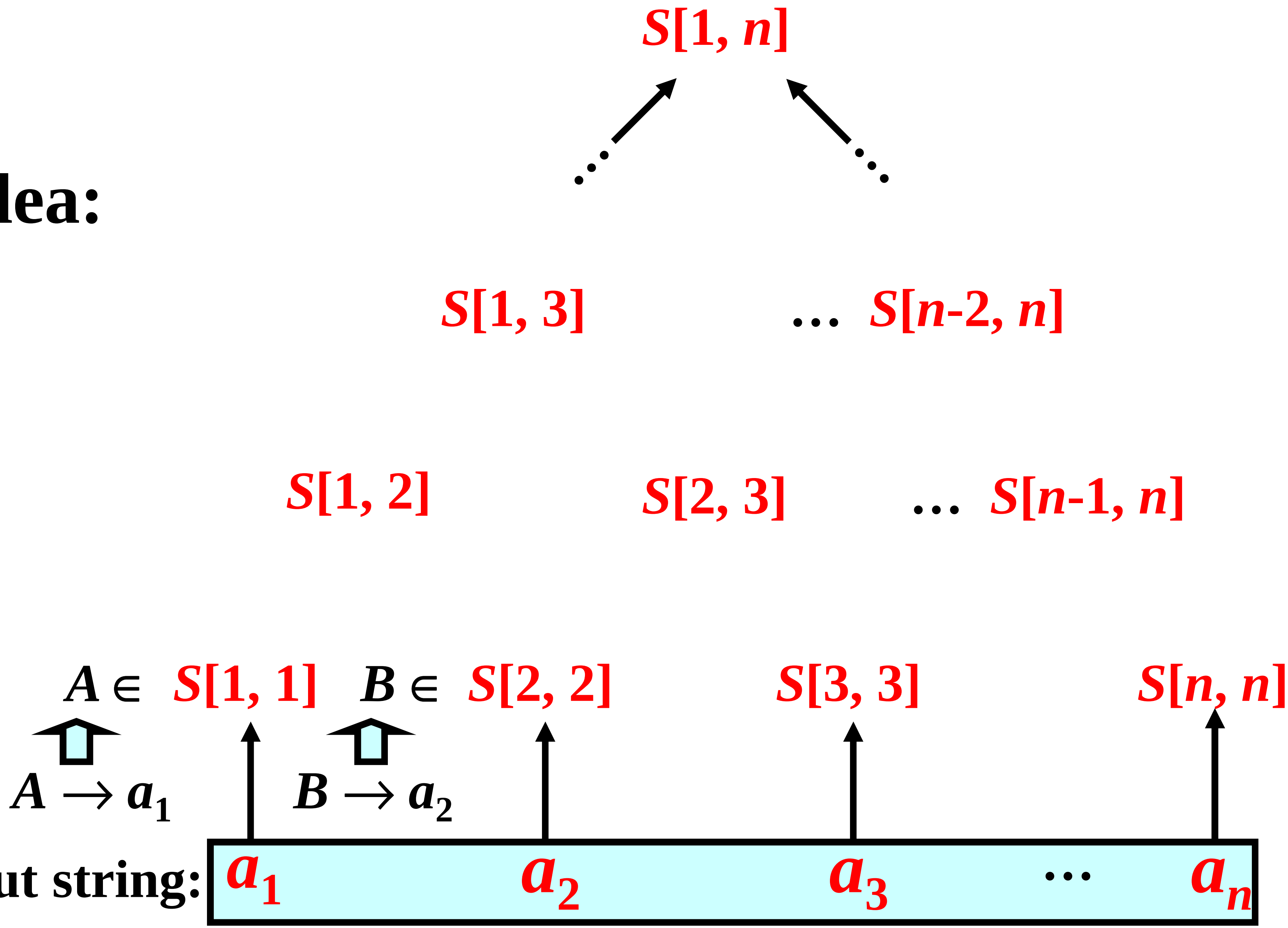
$A \in$ $S[1, 1]$ $S[2, 2]$ $S[3, 3]$ $S[n, n]$

$A \to a_1$

- **Input string:** a_1 a_2 a_3 $\cdots$ a_n

GP Based on Chomsky Normal Form

Idea:



• **Input string:**

GP Based on Chomsky Normal Form

$S[1, n]$

Idea:

$S[1, 3]$ … $S[n\text{-}2, n]$

$S[1, 2]$ $S[2, 3]$ … $S[n\text{-}1, n]$

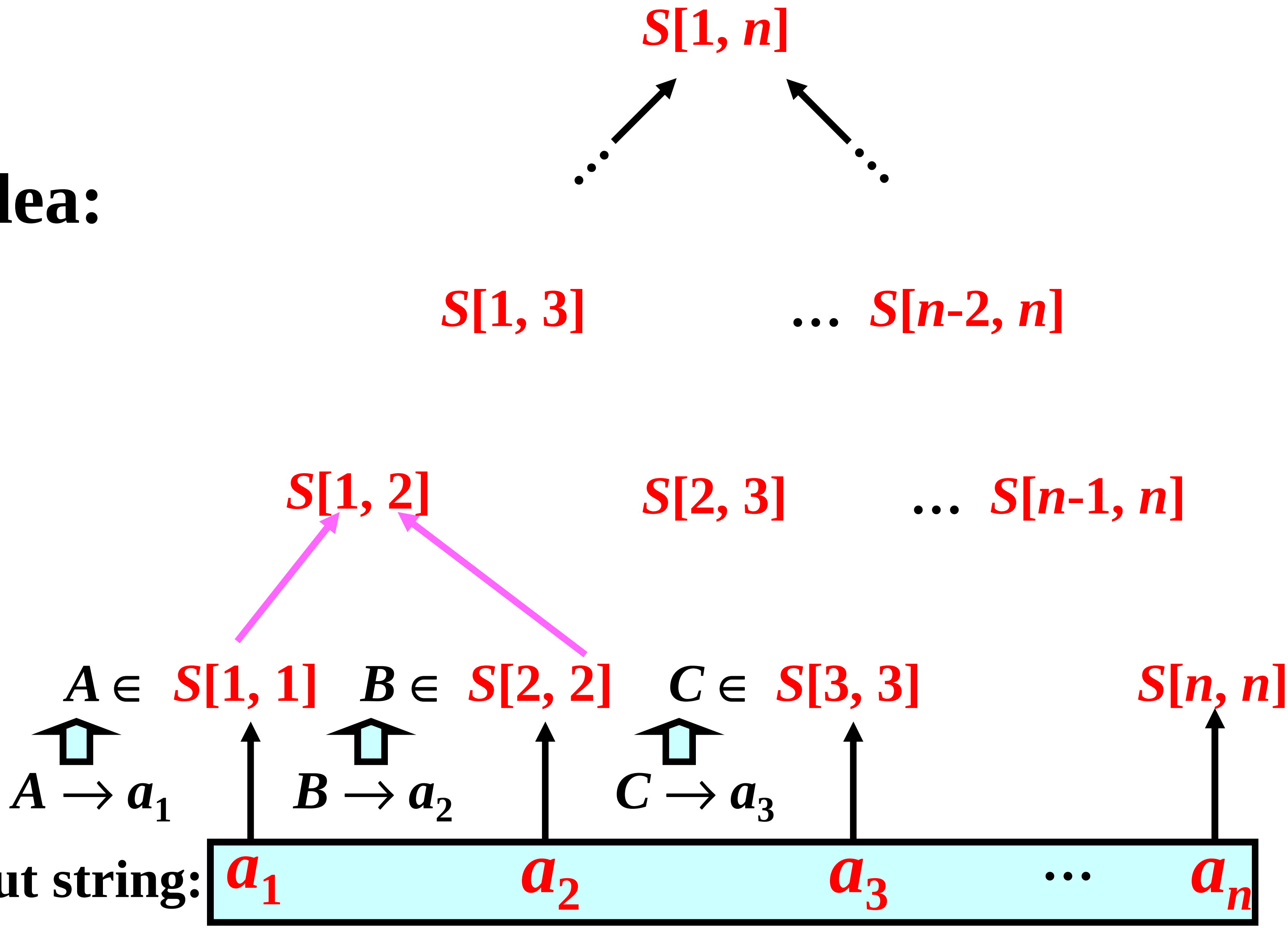
$A \in S[1, 1]$ $B \in S[2, 2]$ $C \in S[3, 3]$ $S[n, n]$

$A \rightarrow a_1$ $B \rightarrow a_2$ $C \rightarrow a_3$

- **Input string:** a_1 a_2 a_3 … a_n

GP Based on Chomsky Normal Form

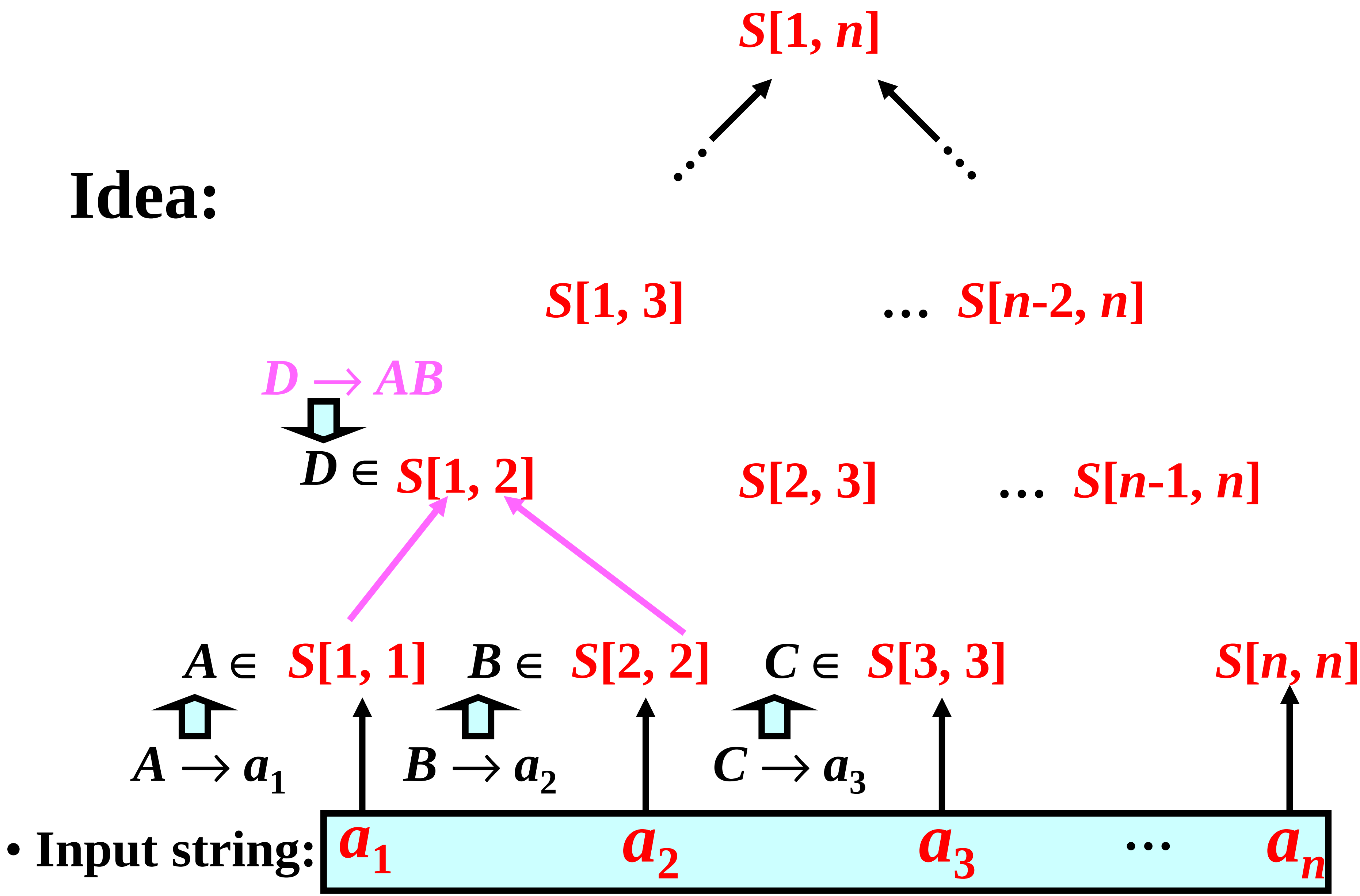
Idea:



• **Input string:**

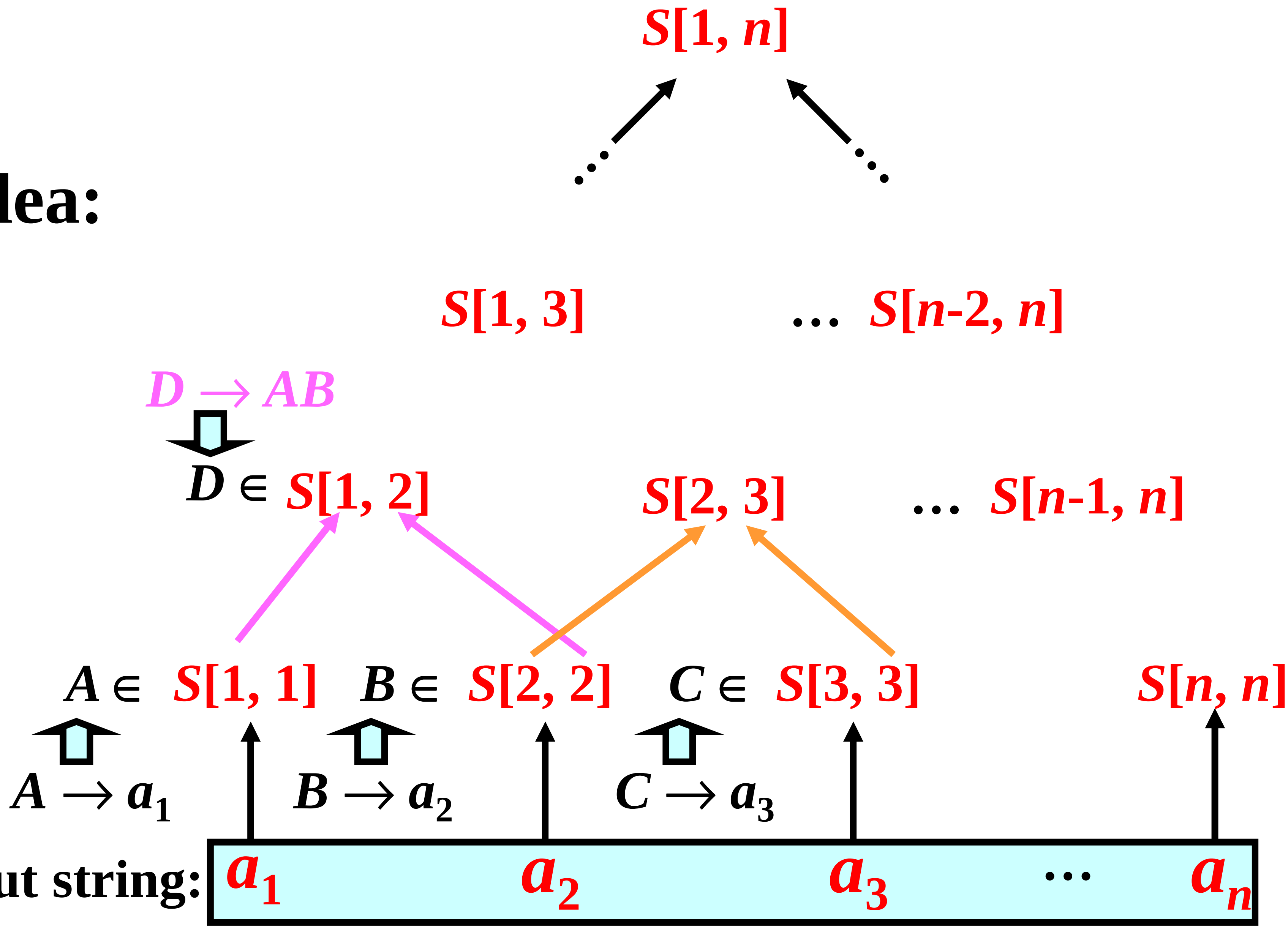
GP Based on Chomsky Normal Form

Idea:



GP Based on Chomsky Normal Form

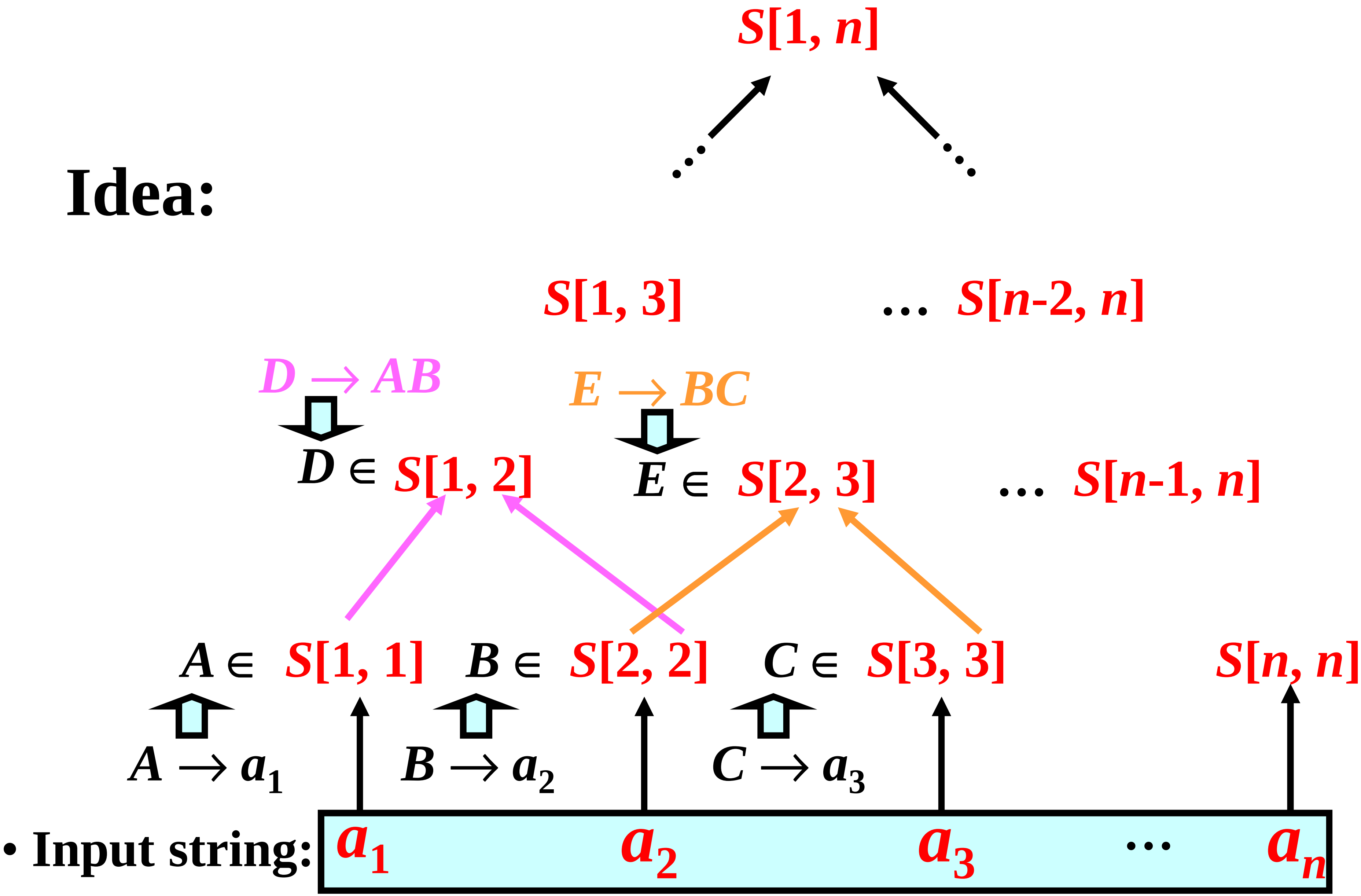
Idea:



• **Input string:**

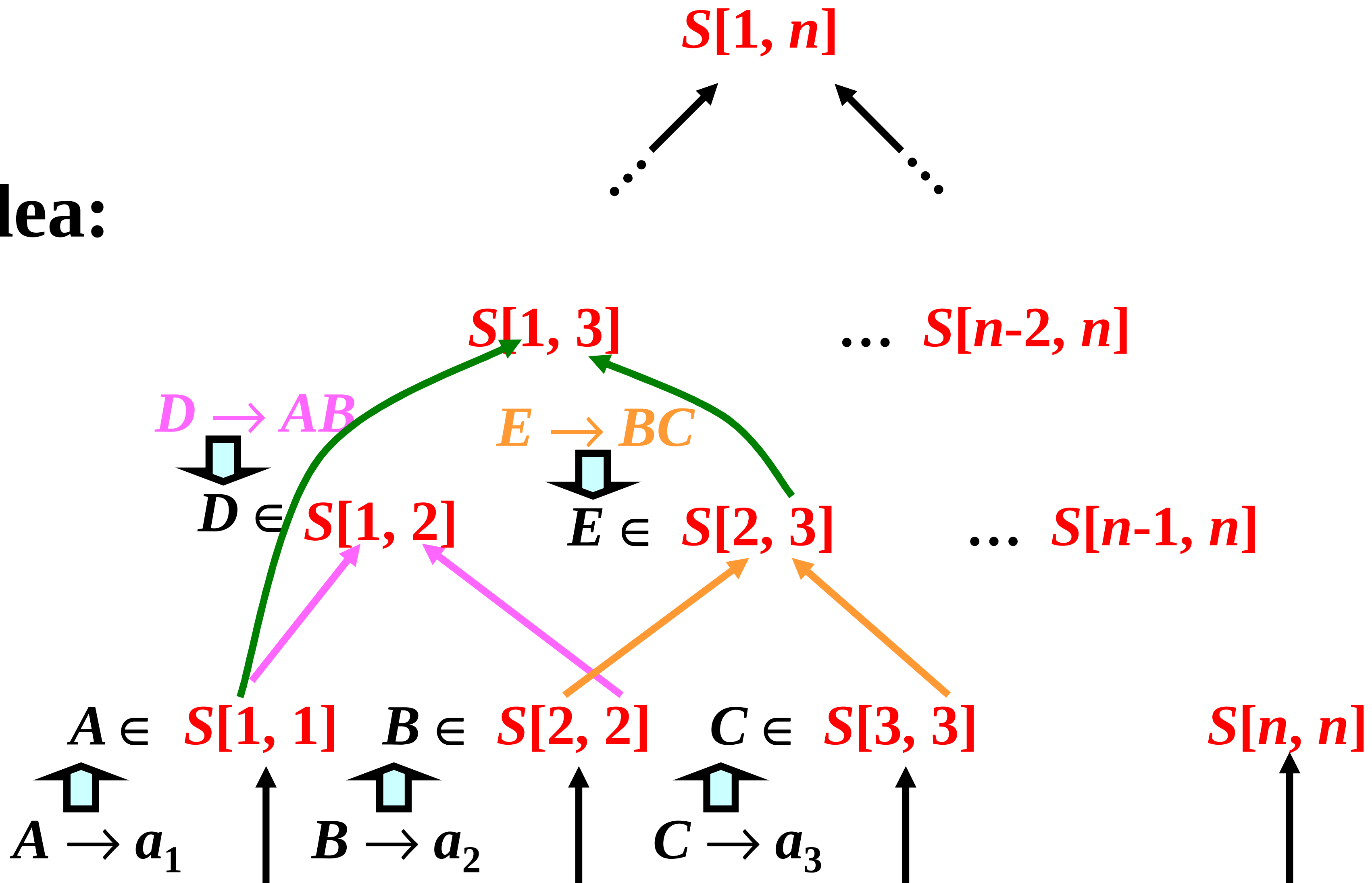
GP Based on Chomsky Normal Form

Idea:



GP Based on Chomsky Normal Form

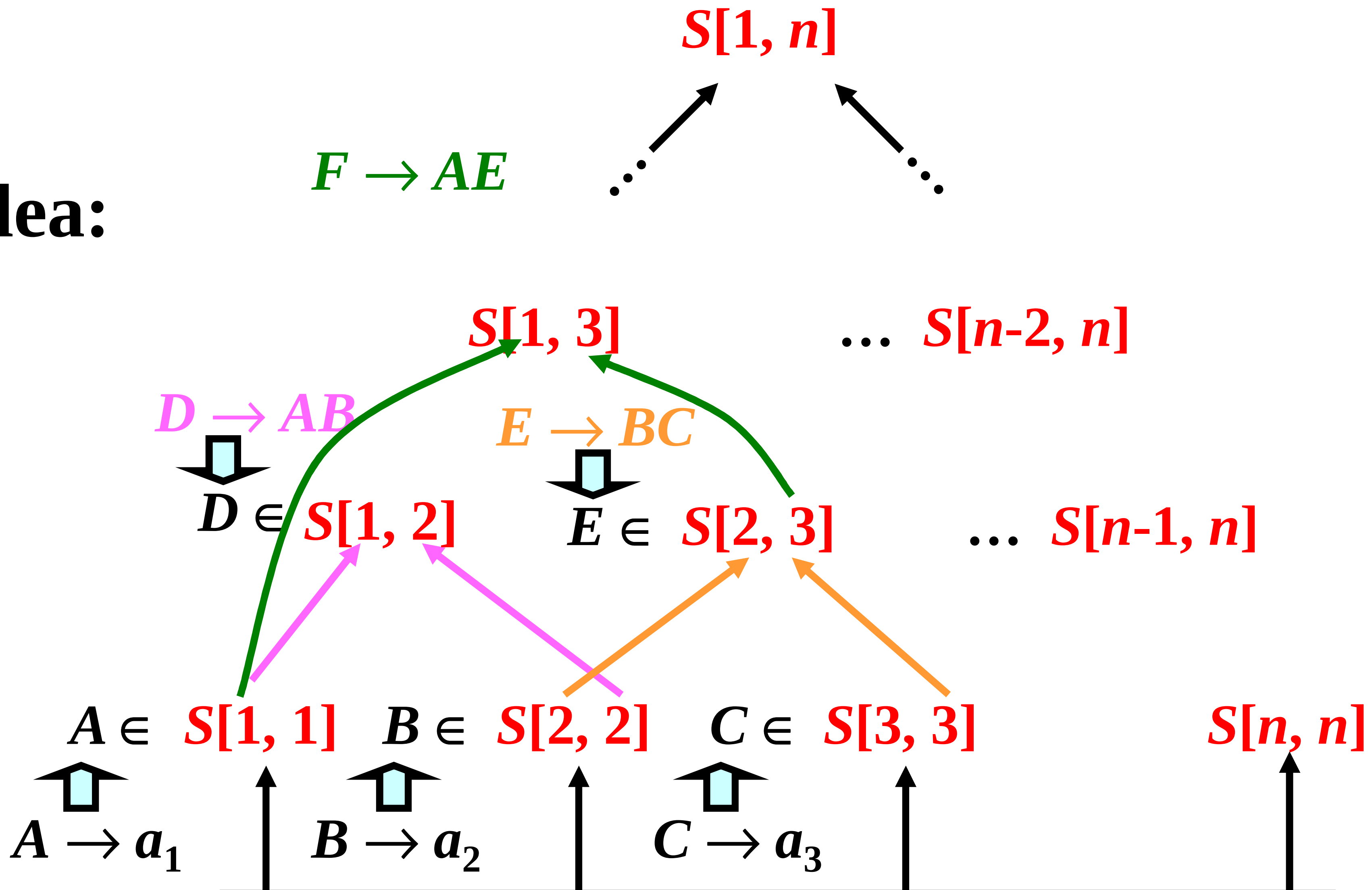
Idea:



- **Input string:** a_1 a_2 a_3 ⋯ a_n

GP Based on Chomsky Normal Form

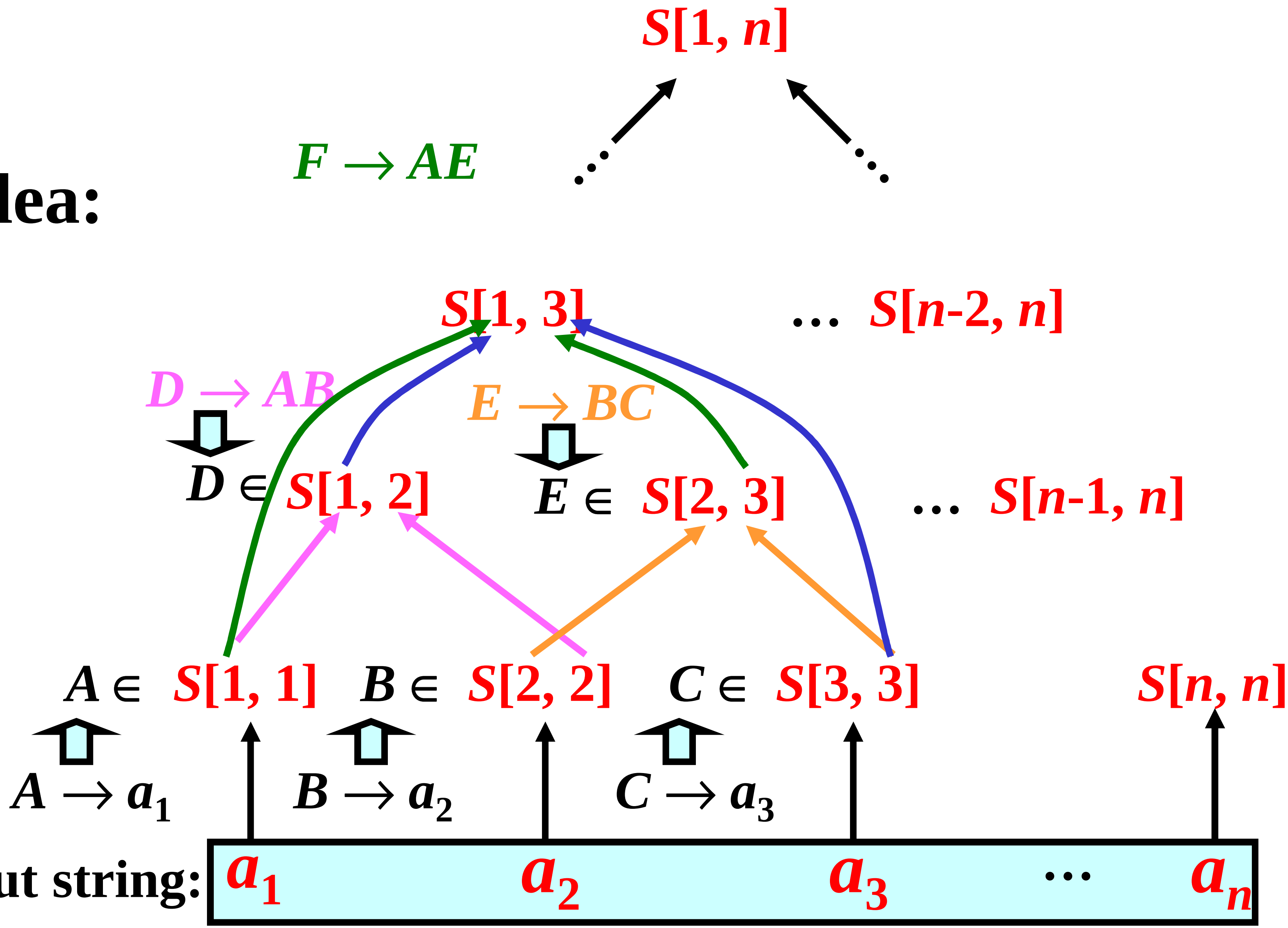
Idea:



- **Input string:** a_1 a_2 a_3 … a_n

GP Based on Chomsky Normal Form

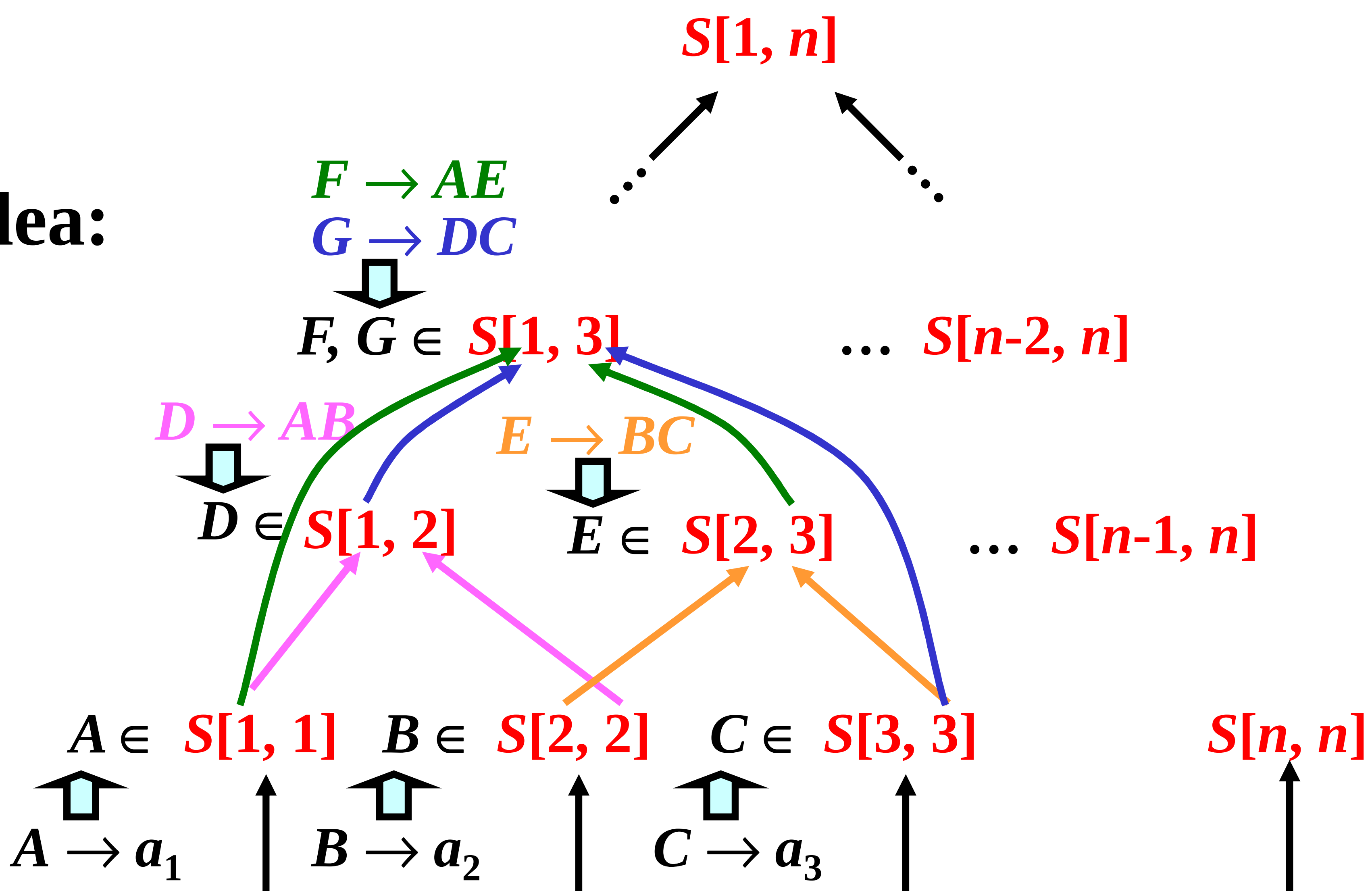
Idea:



• **Input string:**

GP Based on Chomsky Normal Form

Idea:

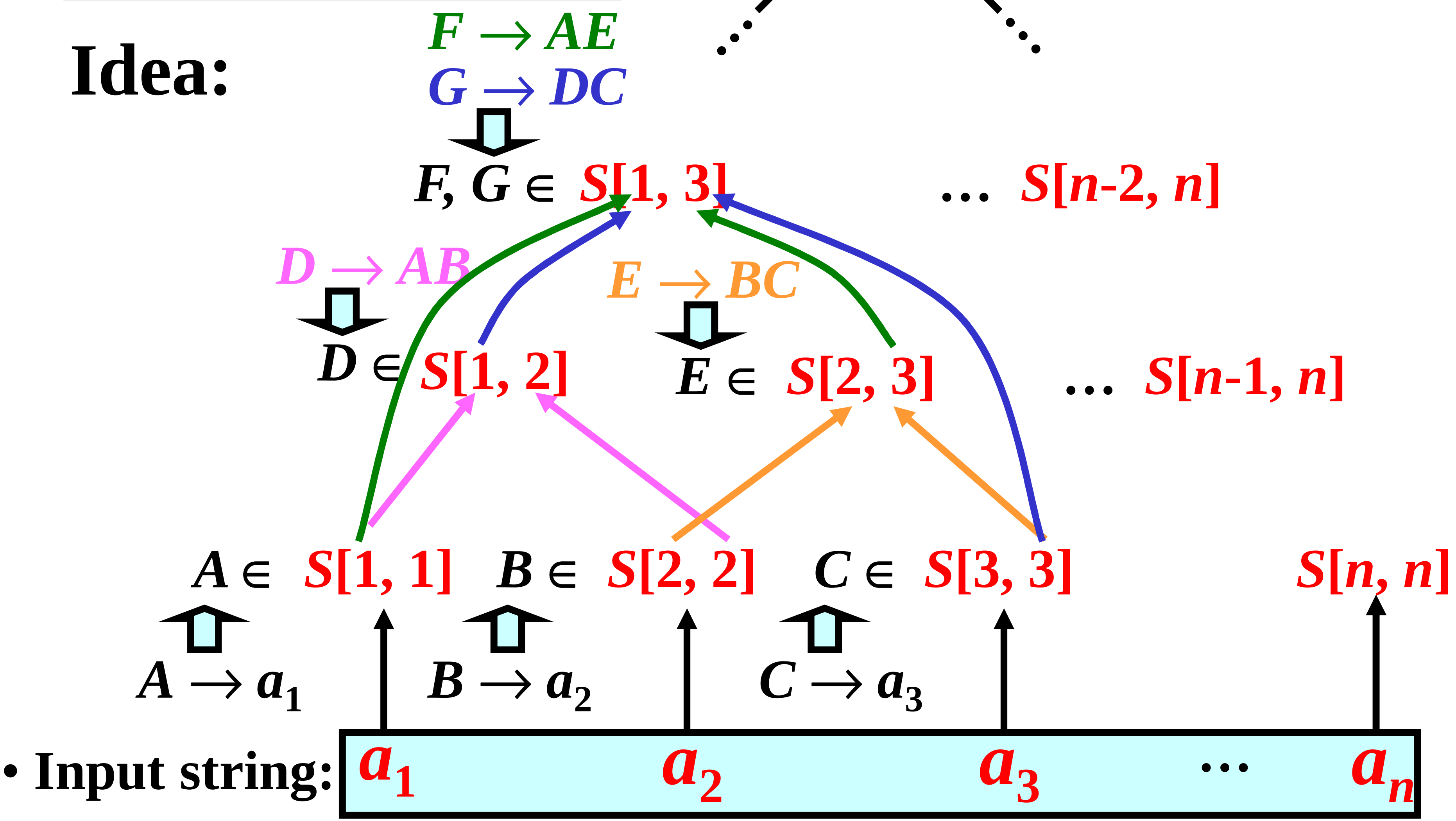


• **Input string:** a_1 a_2 a_3 … a_n

GP Based on Chomsky Normal Form

if $S \in$ $S[1, n]$ then
$S \Rightarrow^* a_1 \ldots a_n$

Idea:



Algorithm: GP Based on CNF

- **Input:** $G = (N, T, P, S)$ in CNF, $\boldsymbol{w} = \boldsymbol{a_1 \ldots a_n}$
- **Output:** **YES** if $w \in L(G)$

  NO if $w \notin L(G)$

- **Method:**
- **for each** $\boldsymbol{a_i}$, $\boldsymbol{i} = 1, \ldots, \boldsymbol{n}$ **do**

  $\boldsymbol{S}[\boldsymbol{i}, \boldsymbol{i}] := \{\boldsymbol{A} : \boldsymbol{A} \rightarrow \boldsymbol{a_i} \in \boldsymbol{P}\}$
- **Apply the following rule until no $\boldsymbol{S}[\boldsymbol{i}, \boldsymbol{k}]$ can be changed:**

  if $\boldsymbol{A} \rightarrow \boldsymbol{BC} \in \boldsymbol{P}$, $\boldsymbol{B} \in \boldsymbol{S}\,[\boldsymbol{i}, \boldsymbol{j}]$, $\boldsymbol{C} \in \boldsymbol{S}\,[\boldsymbol{j}+1, \boldsymbol{k}]$,

  where $1 \leq \boldsymbol{i} \leq \boldsymbol{j} < \boldsymbol{k} \leq \boldsymbol{n}$ **then** add $\boldsymbol{A}$ to $\boldsymbol{S}[\boldsymbol{i}, \boldsymbol{k}]$
- **if** $\boldsymbol{S} \in \boldsymbol{S}[1, \boldsymbol{n}]$ **then** write ('**YES**')

  else write ('**NO**')

GP Based on CNF: Example 1/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$

Question: $aacbb \in L(G)$**?**

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

$A \rightarrow a$ $A \rightarrow a$ $S \rightarrow c$ $B \rightarrow b$ $B \rightarrow b$

a a c b b

GP Based on CNF: Example 2/5

$G = (N, T, P, S)$, where $N = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{S}\}$, $T = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$,
$P = \{\boldsymbol{S} \rightarrow \boldsymbol{AC}, \boldsymbol{C} \rightarrow \boldsymbol{SB}, \boldsymbol{A} \rightarrow \boldsymbol{a}, \boldsymbol{B} \rightarrow \boldsymbol{b}, \boldsymbol{S} \rightarrow \boldsymbol{c}\}$
Question: ***aacbb*** $\in L(G)$**?**

***S*[1, 2]** ***S*[2, 3]** ***S*[3, 4]** ***S*[4, 5]**

***S*[1, 1]={*A*}** ***S*[2, 2]={*A*}** ***S*[3, 3]={*S*}** ***S*[4, 4]={*B*}** ***S*[5, 5]={*B*}**

a	***a***	***c***	***b***	***b***

GP Based on CNF: Example 2/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$
Question: $aacbb \in L(G)$?

$? \rightarrow AA$

$S[1, 2]$ $S[2, 3]$ $S[3, 4]$ $S[4, 5]$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 2/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \to AC, C \to SB, A \to a, B \to b, S \to c\}$
Question: $aacbb \in L(G)$?

$? \cancel{\to} AA$

$S[1, 2]=\emptyset$ $S[2, 3]$ $S[3, 4]$ $S[4, 5]$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	*a*	*c*	*b*	*b*

GP Based on CNF: Example 2/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$
Question: $aacbb \in L(G)$**?**

$? \cancel{\rightarrow} AA$ $\quad$ $? \rightarrow AS$

$S[1, 2]=\emptyset$ $\quad$ $S[2, 3]$ $\quad$ $S[3, 4]$ $\quad$ $S[4, 5]$

$S[1, 1]=\{A\}$ $\quad$ $S[2, 2]=\{A\}$ $\quad$ $S[3, 3]=\{S\}$ $\quad$ $S[4, 4]=\{B\}$ $\quad$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 2/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$
Question: $aacbb \in L(G)$?

$? \cancel{\rightarrow} AA$ $\quad$ $? \cancel{\rightarrow} AS$

$S[1, 2]=\emptyset$ $\quad$ $S[2, 3]=\emptyset$ $\quad$ $S[3, 4]$ $\quad$ $S[4, 5]$

$S[1, 1]=\{A\}$ $\quad$ $S[2, 2]=\{A\}$ $\quad$ $S[3, 3]=\{S\}$ $\quad$ $S[4, 4]=\{B\}$ $\quad$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 2/5

$G = (N, T, P, S)$, where $N = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{S}\}$, $T = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$,
$P = \{\mathbf{S} \rightarrow \mathbf{AC}, \mathbf{C} \rightarrow \mathbf{SB}, \mathbf{A} \rightarrow \mathbf{a}, \mathbf{B} \rightarrow \mathbf{b}, \mathbf{S} \rightarrow \mathbf{c}\}$
Question: $\mathbf{aacbb} \in L(G)$**?**

$? \cancel{\rightarrow} AA$ $\quad$ $? \cancel{\rightarrow} AS$ $\quad$ $C \rightarrow SB$

$S[1, 2] = \emptyset$ $\quad$ $S[2, 3] = \emptyset$ $\quad$ $S[3, 4]$ $\quad$ $S[4, 5]$

$S[1, 1] = \{A\}$ $\quad$ $S[2, 2] = \{A\}$ $\quad$ $S[3, 3] = \{S\}$ $\quad$ $S[4, 4] = \{B\}$ $\quad$ $S[5, 5] = \{B\}$

a	a	c	b	b

GP Based on CNF: Example 2/5

$G = (N, T, P, S)$, where $N = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{S}\}$, $T = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$,
$P = \{\mathbf{S} \to \mathbf{AC}, \mathbf{C} \to \mathbf{SB}, \mathbf{A} \to \mathbf{a}, \mathbf{B} \to \mathbf{b}, \mathbf{S} \to \mathbf{c}\}$
Question: $\mathbf{aacbb} \in L(G)$**?**

$? \cancel{\to} \mathbf{AA}$ $\quad ? \cancel{\to} \mathbf{AS}$ $\quad \boxed{\mathbf{C} \to \mathbf{SB}}$

$\mathbf{S[1, 2]}=\emptyset$ $\quad \mathbf{S[2, 3]}=\emptyset$ $\quad \mathbf{S[3, 4]}=\{\mathbf{C}\}$ $\quad \mathbf{S[4, 5]}$

$\mathbf{S[1, 1]}=\{\mathbf{A}\}$ $\quad \mathbf{S[2, 2]}=\{\mathbf{A}\}$ $\quad \mathbf{S[3, 3]}=\{\mathbf{S}\}$ $\quad \mathbf{S[4, 4]}=\{\mathbf{B}\}$ $\quad \mathbf{S[5, 5]}=\{\mathbf{B}\}$

a	a	c	b	b

GP Based on CNF: Example 2/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \to AC, C \to SB, A \to a, B \to b, S \to c\}$
Question: $aacbb \in L(G)$?

$? \cancel{\to} AA$ $\quad$ $? \cancel{\to} AS$ $\quad$ $C \to SB$ $\quad$ $? \to BB$

$S[1, 2] = \emptyset$ $\quad$ $S[2, 3] = \emptyset$ $\quad$ $S[3, 4] = \{C\}$ $\quad$ $S[4, 5]$

$S[1, 1] = \{A\}$ $\quad$ $S[2, 2] = \{A\}$ $\quad$ $S[3, 3] = \{S\}$ $\quad$ $S[4, 4] = \{B\}$ $\quad$ $S[5, 5] = \{B\}$

a	*a*	*c*	*b*	*b*

GP Based on CNF: Example 2/5

$G = (N, T, P, S)$, where $N = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{S}\}$, $T = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$,
$P = \{\boldsymbol{S} \rightarrow \boldsymbol{AC}, \boldsymbol{C} \rightarrow \boldsymbol{SB}, \boldsymbol{A} \rightarrow \boldsymbol{a}, \boldsymbol{B} \rightarrow \boldsymbol{b}, \boldsymbol{S} \rightarrow \boldsymbol{c}\}$
Question: $\boldsymbol{aacbb} \in L(G)$**?**

$? \cancel{\rightarrow} AA$ $\quad ? \cancel{\rightarrow} AS$ $\quad C \rightarrow SB$ $\quad ? \cancel{\rightarrow} BB$

$\boldsymbol{S}[1, 2] = \emptyset$ $\quad \boldsymbol{S}[2, 3] = \emptyset$ $\quad \boldsymbol{S}[3, 4] = \{C\}$ $\quad \boldsymbol{S}[4, 5] = \emptyset$

$\boldsymbol{S}[1, 1] = \{A\}$ $\quad \boldsymbol{S}[2, 2] = \{A\}$ $\quad \boldsymbol{S}[3, 3] = \{S\}$ $\quad \boldsymbol{S}[4, 4] = \{B\}$ $\quad \boldsymbol{S}[5, 5] = \{B\}$

a	***a***	***c***	***b***	***b***

GP Based on CNF: Example 3/5

$G = (N, T, P, S)$, where $N = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{S}\}$, $T = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$,
$P = \{\boldsymbol{S} \rightarrow \boldsymbol{AC}, \boldsymbol{C} \rightarrow \boldsymbol{SB}, \boldsymbol{A} \rightarrow \boldsymbol{a}, \boldsymbol{B} \rightarrow \boldsymbol{b}, \boldsymbol{S} \rightarrow \boldsymbol{c}\}$
Question: $\boldsymbol{aacbb} \in L(G)$**?**

S[1, 3] **S[2, 4]** **S[3, 5]**

S[1, 2]$=\emptyset$ **S[2, 3]**$=\emptyset$ **S[3, 4]**$=\{\boldsymbol{C}\}$ **S[4, 5]**$=\emptyset$

S[1, 1]$=\{\boldsymbol{A}\}$ **S[2, 2]**$=\{\boldsymbol{A}\}$ **S[3, 3]**$=\{\boldsymbol{S}\}$ **S[4, 4]**$=\{\boldsymbol{B}\}$ **S[5, 5]**$=\{\boldsymbol{B}\}$

a	***a***	***c***	***b***	***b***

GP Based on CNF: Example 3/5

$G = (N, T, P, S)$, where $N = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{S}\}$, $T = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$,
$P = \{\boldsymbol{S} \to \boldsymbol{AC}, \boldsymbol{C} \to \boldsymbol{SB}, \boldsymbol{A} \to \boldsymbol{a}, \boldsymbol{B} \to \boldsymbol{b}, \boldsymbol{S} \to \boldsymbol{c}\}$
Question: $\boldsymbol{aacbb} \in L(G)$**?**

S[1, 3] **S[2, 4]** **S[3, 5]**

S[1, 2]$=\emptyset$ **S[2, 3]**$=\emptyset$ **S[3, 4]**$=\{C\}$ **S[4, 5]**$=\emptyset$

S[1, 1]$=\{A\}$ **S[2, 2]**$=\{A\}$ **S[3, 3]**$=\{S\}$ **S[4, 4]**$=\{B\}$ **S[5, 5]**$=\{B\}$

a	*a*	*c*	*b*	*b*

GP Based on CNF: Example 3/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \to AC, C \to SB, A \to a, B \to b, S \to c\}$
Question: $aacbb \in L(G)$?

$S[1, 3]=\emptyset$ $S[2, 4]$ $S[3, 5]$

$S[1, 2]=\emptyset$ $S[2, 3]=\emptyset$ $S[3, 4]=\{C\}$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	*a*	*c*	*b*	*b*

GP Based on CNF: Example 3/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$

Question: $aacbb \in L(G)$?

$S \rightarrow AC$

$S[1, 3]=\varnothing$ $S[2, 4]$ $S[3, 5]$

$S[1, 2]=\varnothing$ $S[2, 3]=\varnothing$ $S[3, 4]=\{C\}$ $S[4, 5]=\varnothing$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	*a*	*c*	*b*	*b*

GP Based on CNF: Example 3/5

$G = (N, T, P, S)$, where $N = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{S}\}$, $T = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$,
$P = \{\boldsymbol{S} \rightarrow \boldsymbol{AC}, \boldsymbol{C} \rightarrow \boldsymbol{SB}, \boldsymbol{A} \rightarrow \boldsymbol{a}, \boldsymbol{B} \rightarrow \boldsymbol{b}, \boldsymbol{S} \rightarrow \boldsymbol{c}\}$
Question: $\boldsymbol{aacbb} \in L(G)$?

$\boldsymbol{S} \rightarrow \boldsymbol{AC}$

$S[1, 3]=\emptyset$ $S[2, 4]$ $S[3, 5]$

$S[1, 2]=\emptyset$ $S[2, 3]=\emptyset$ $S[3, 4]=\{C\}$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	***a***	***c***	***b***	***b***

GP Based on CNF: Example 3/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \to AC,\ C \to SB,\ A \to a,\ B \to b,\ S \to c\}$
Question: $aacbb \in L(G)$?

$S \to AC$

$S[1, 3]=\emptyset$ $S[2, 4]=\{S\}$ $S[3, 5]$

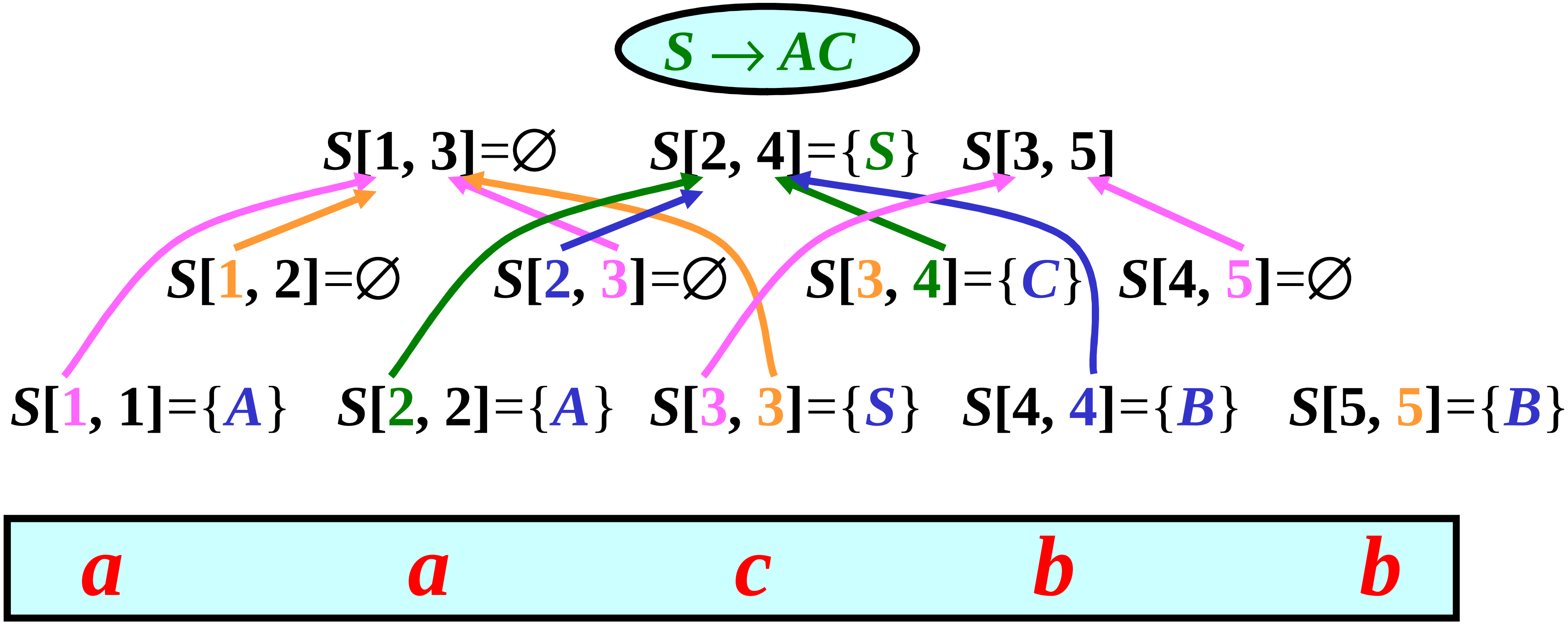
$S[1, 2]=\emptyset$ $S[2, 3]=\emptyset$ $S[3, 4]=\{C\}$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 3/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$
Question: $aacbb \in L(G)$?



GP Based on CNF: Example 3/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \to AC, C \to SB, A \to a, B \to b, S \to c\}$
Question: $aacbb \in L(G)$?

$S \to AC$

$S[1, 3]=\emptyset$ $S[2, 4]=\{S\}$ $S[3, 5]= \emptyset$

$S[1, 2]=\emptyset$ $S[2, 3]=\emptyset$ $S[3, 4]=\{C\}$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	*a*	*c*	*b*	*b*

GP Based on CNF: Example 4/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$

Question: $aacbb \in L(G)$?

$S[1, 4]$ $S[2, 5]$

$S[1, 3]=\emptyset$ $S[2, 4]=\{S\}$ $S[3, 5]=\emptyset$

$S[1, 2]=\emptyset$ $S[2, 3]=\emptyset$ $S[3, 4]=\{C\}$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 4/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$
Question: $aacbb \in L(G)$?

$S[1, 4]$ $S[2, 5]$

$S[1, 3]=\emptyset$ $S[2, 4]=\{S\}$ $S[3, 5]=\emptyset$

$S[1, 2]=\emptyset$ $S[2, 3]=\emptyset$ $S[3, 4]=\{C\}$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 4/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \to AC, C \to SB, A \to a, B \to b, S \to c\}$

Question: $aacbb \in L(G)$?

$S[1, 4]$ $S[2, 5]$

$S[1, 3]=\varnothing$ $S[2, 4]=\{S\}$ $S[3, 5]=\varnothing$

$S[1, 2]=\varnothing$ $S[2, 3]=\varnothing$ $S[3, 4]=\{C\}$ $S[4, 5]=\varnothing$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 4/5

$G = (N, T, P, S)$, where $N = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{S}\}$, $T = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$,
$P = \{\boldsymbol{S} \rightarrow \boldsymbol{AC}, \boldsymbol{C} \rightarrow \boldsymbol{SB}, \boldsymbol{A} \rightarrow \boldsymbol{a}, \boldsymbol{B} \rightarrow \boldsymbol{b}, \boldsymbol{S} \rightarrow \boldsymbol{c}\}$

Question: $\boldsymbol{aacbb} \in L(G)$?

$\boldsymbol{S}[1, 4]=\emptyset$ $\quad$ $\boldsymbol{S}[2, 5]$

$\boldsymbol{S}[1, 3]=\emptyset$ $\quad$ $\boldsymbol{S}[2, 4]=\{\boldsymbol{S}\}$ $\quad$ $\boldsymbol{S}[3, 5]=\emptyset$

$\boldsymbol{S}[1, 2]=\emptyset$ $\quad$ $\boldsymbol{S}[2, 3]=\emptyset$ $\quad$ $\boldsymbol{S}[3, 4]=\{\boldsymbol{C}\}$ $\quad$ $\boldsymbol{S}[4, 5]=\emptyset$

$\boldsymbol{S}[1, 1]=\{\boldsymbol{A}\}$ $\quad$ $\boldsymbol{S}[2, 2]=\{\boldsymbol{A}\}$ $\quad$ $\boldsymbol{S}[3, 3]=\{\boldsymbol{S}\}$ $\quad$ $\boldsymbol{S}[4, 4]=\{\boldsymbol{B}\}$ $\quad$ $\boldsymbol{S}[5, 5]=\{\boldsymbol{B}\}$

a	***a***	***c***	***b***	***b***

GP Based on CNF: Example 4/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,

$P = \{S \to AC, C \to SB, A \to a, B \to b, S \to c\}$

Question: $aacbb \in L(G)$?

$S[1, 4]=\emptyset$ $S[2, 5]$

$S[1, 3]=\emptyset$ $S[2, 4]=\{S\}$ $S[3, 5]= \emptyset$

$S[1, 2]=\emptyset$ $S[2, 3]=\emptyset$ $S[3, 4]=\{C\}$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 4/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
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Question: $aacbb \in L(G)$**?**

$S[1, 4]=\emptyset$ $\quad$ $S[2, 5]$

$S[1, 3]=\emptyset$ $\quad$ $S[2, 4]=\{S\}$ $\quad$ $S[3, 5]= \emptyset$

$S[1, 2]=\emptyset$ $\quad$ $S[2, 3]=\emptyset$ $\quad$ $S[3, 4]=\{C\}$ $\quad$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $\quad$ $S[2, 2]=\{A\}$ $\quad$ $S[3, 3]=\{S\}$ $\quad$ $S[4, 4]=\{B\}$ $\quad$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 4/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \to AC, C \to SB, A \to a, B \to b, S \to c\}$
Question: $aacbb \in L(G)$?

$C \to SB$

$S[1, 4]=\emptyset$ $S[2, 5]$

$S[1, 3]=\emptyset$ $S[2, 4]=\{S\}$ $S[3, 5]=\emptyset$

$S[1, 2]=\emptyset$ $S[2, 3]=\emptyset$ $S[3, 4]=\{C\}$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 4/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$
Question: $aacbb \in L(G)$?

$C \rightarrow SB$

$S[1, 4]=\emptyset$ $S[2, 5]=\{C\}$

$S[1, 3]=\emptyset$ $S[2, 4]=\{S\}$ $S[3, 5]=\emptyset$

$S[1, 2]=\emptyset$ $S[2, 3]=\emptyset$ $S[3, 4]=\{C\}$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 5/5

$G = (N, T, P, S)$, where $N = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{S}\}$, $T = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$,
$P = \{\boldsymbol{S} \rightarrow \boldsymbol{AC}, \boldsymbol{C} \rightarrow \boldsymbol{SB}, \boldsymbol{A} \rightarrow \boldsymbol{a}, \boldsymbol{B} \rightarrow \boldsymbol{b}, \boldsymbol{S} \rightarrow \boldsymbol{c}\}$
Question: $\boldsymbol{aacbb} \in L(G)$**?**

$\boldsymbol{S}$ **[1, 5]**

$\boldsymbol{S}$**[1, 4]**$=\emptyset$ $\boldsymbol{S}$**[2, 5]**$=\{\boldsymbol{C}\}$

$\boldsymbol{S}$**[1, 3]**$=\emptyset$ $\boldsymbol{S}$**[2, 4]**$=\{\boldsymbol{S}\}$ $\boldsymbol{S}$**[3, 5]**$=\emptyset$

$\boldsymbol{S}$**[1, 2]**$=\emptyset$ $\boldsymbol{S}$**[2, 3]**$=\emptyset$ $\boldsymbol{S}$**[3, 4]**$=\{\boldsymbol{C}\}$ $\boldsymbol{S}$**[4, 5]**$=\emptyset$

$\boldsymbol{S}$**[1, 1]**$=\{\boldsymbol{A}\}$ $\boldsymbol{S}$**[2, 2]**$=\{\boldsymbol{A}\}$ $\boldsymbol{S}$**[3, 3]**$=\{\boldsymbol{S}\}$ $\boldsymbol{S}$**[4, 4]**$=\{\boldsymbol{B}\}$ $\boldsymbol{S}$**[5, 5]**$=\{\boldsymbol{B}\}$

a	***a***	***c***	***b***	***b***

GP Based on CNF: Example 5/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$
Question: $aacbb \in L(G)$?

$S \rightarrow AC$

S [1, 5]

$S[1, 4]=\emptyset$ $S[2, 5]=\{C\}$

$S[1, 3]=\emptyset$ $S[2, 4]=\{S\}$ $S[3, 5]=\emptyset$

$S[1, 2]=\emptyset$ $S[2, 3]=\emptyset$ $S[3, 4]=\{C\}$ $S[4, 5]=\emptyset$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 5/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$
Question: *aacbb* $\in L(G)$?

$S \rightarrow AC$

S [1, 5]

S[1, 4]=$\emptyset$ **S[2, 5]=$\{C\}$**

S[1, 3]=$\emptyset$ **S[2, 4]=$\{S\}$** **S[3, 5]= $\emptyset$**

S[1, 2]=$\emptyset$ **S[2, 3]=$\emptyset$** **S[3, 4]=$\{C\}$** **S[4, 5]=$\emptyset$**

S[1, 1]=$\{A\}$ **S[2, 2]=$\{A\}$** **S[3, 3]=$\{S\}$** **S[4, 4]=$\{B\}$** **S[5, 5]=$\{B\}$**

a	*a*	*c*	*b*	*b*

GP Based on CNF: Example 5/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \to AC, C \to SB, A \to a, B \to b, S \to c\}$
Question: ***aacbb*** $\in L(G)$**?**

$S \to AC$

$S\,[1, 5]$

$S[1, 4]=\varnothing$ $S[2, 5]=\{C\}$

$S[1, 3]=\varnothing$ $S[2, 4]=\{S\}$ $S[3, 5]=\varnothing$

$S[1, 2]=\varnothing$ $S[2, 3]=\varnothing$ $S[3, 4]=\{C\}$ $S[4, 5]=\varnothing$

$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	a	c	b	b

GP Based on CNF: Example 5/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
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Question: $aacbb \in L(G)$?

$S \rightarrow AC$

$S[1, 5]$

$S[1, 4]=\varnothing$ $S[2, 5]=\{C\}$

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$S[1, 1]=\{A\}$ $S[2, 2]=\{A\}$ $S[3, 3]=\{S\}$ $S[4, 4]=\{B\}$ $S[5, 5]=\{B\}$

a	*a*	*c*	*b*	*b*

GP Based on CNF: Example 5/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \to AC, C \to SB, A \to a, B \to b, S \to c\}$

Question: $aacbb \in L(G)$?

$S \to AC$

$S[1, 5] = \{S\}$

$S[1, 4] = \emptyset$ $S[2, 5] = \{C\}$

$S[1, 3] = \emptyset$ $S[2, 4] = \{S\}$ $S[3, 5] = \emptyset$

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a	***a***	***c***	***b***	***b***

GP Based on CNF: Example 5/5

$G = (N, T, P, S)$, where $N = \{A, B, C, S\}$, $T = \{a, b, c\}$,
$P = \{S \rightarrow AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b, S \rightarrow c\}$
Question: $aacbb \in L(G)$?

$S \rightarrow AC$

$S[1, 5]=\{S\}$ $\quad S \in S[1, 5] \Rightarrow$ **YES**

$S[1, 4]=\emptyset$ $\quad S[2, 5]=\{C\}$

$S[1, 3]=\emptyset$ $\quad S[2, 4]=\{S\}$ $\quad S[3, 5]=\emptyset$

$S[1, 2]=\emptyset$ $\quad S[2, 3]=\emptyset$ $\quad S[3, 4]=\{C\}$ $\quad S[4, 5]=\emptyset$

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a	a	c	b	b

Pumping Lemma for CFL

• Let L be CFL. Then, there exists $k \geq 1$ such that:
if $z \in L$ and $|z| \geq k$ **then** there exist u, v, w, x, y so
$z = uvwxy$ and
1) $vx \neq \varepsilon$ **2)** $|vwx| \leq k$ **3)** for each $m \geq 0$, $uv^m wx^m y \in L$

Example:

$G = (\{S, A\}, \{a, b, c\}, \{S \rightarrow aAa, A \rightarrow bAb, A \rightarrow c\}, S)$
generate $L(G) = \{ab^n cb^n a : n \geq 0\}$, so $L(G)$ is CFL.

There is $k = 5$ such that **1)**, **2)** and **3)** holds:

• for $z = abcba$: $z \in L(G)$ and $|z| \geq 5$:

$u\,v\,w\,x\,y$

$vx = bb \neq \varepsilon$

$|vwx| = 3$: $1 \leq 3 \leq 5$

$uv^0wx^0y = ab^0cb^0a = aca \in L(G)$

$uv^1wx^1y = ab^1cb^1a = abcba \in L(G)$

$uv^2wx^2y = ab^2cb^2a = abbcbba \in L(G)$

⋮

• for $z = abbcbba$: $z \in L(G)$ and $|z| \geq 5$:

⋮

Pumping Lemma: Illustration

- L = any context-free language:

Pumping Lemma: Illustration

- L = any context-free language:

$\overbrace{\boxed{\quad z \quad}}^{k} \in L$ $\Rightarrow$ ***nothing interesting***

Pumping Lemma: Illustration

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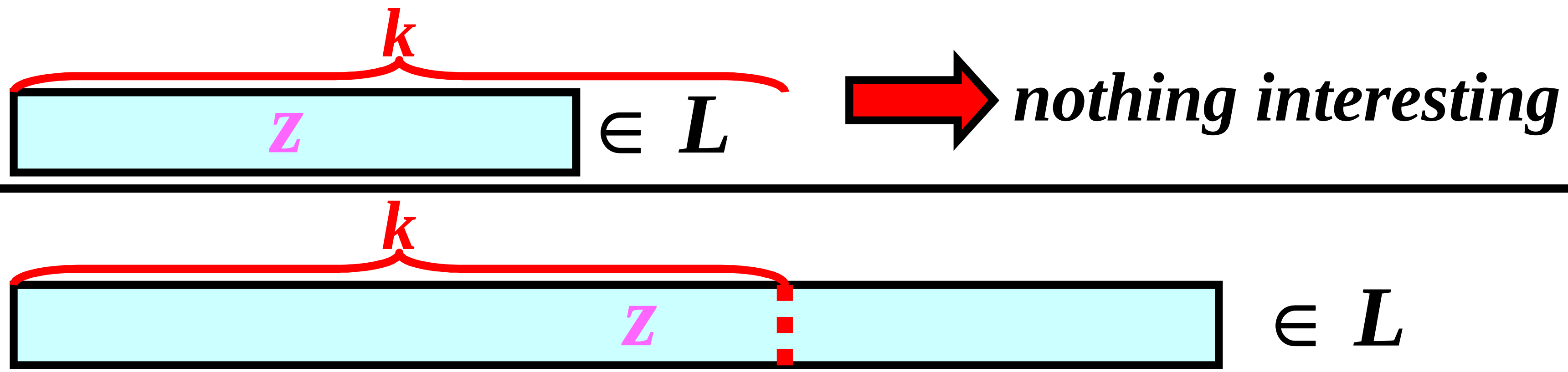
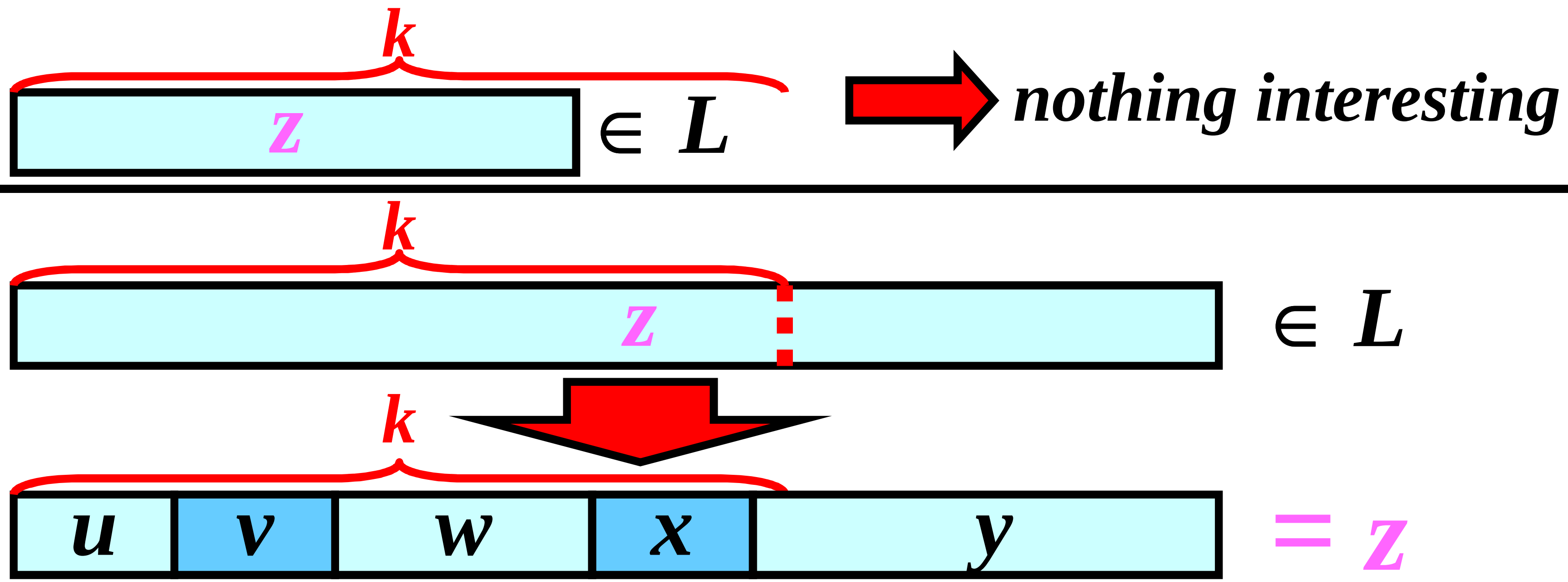


Pumping Lemma: Illustration

• L = any context-free language:



Pumping Lemma: Illustration

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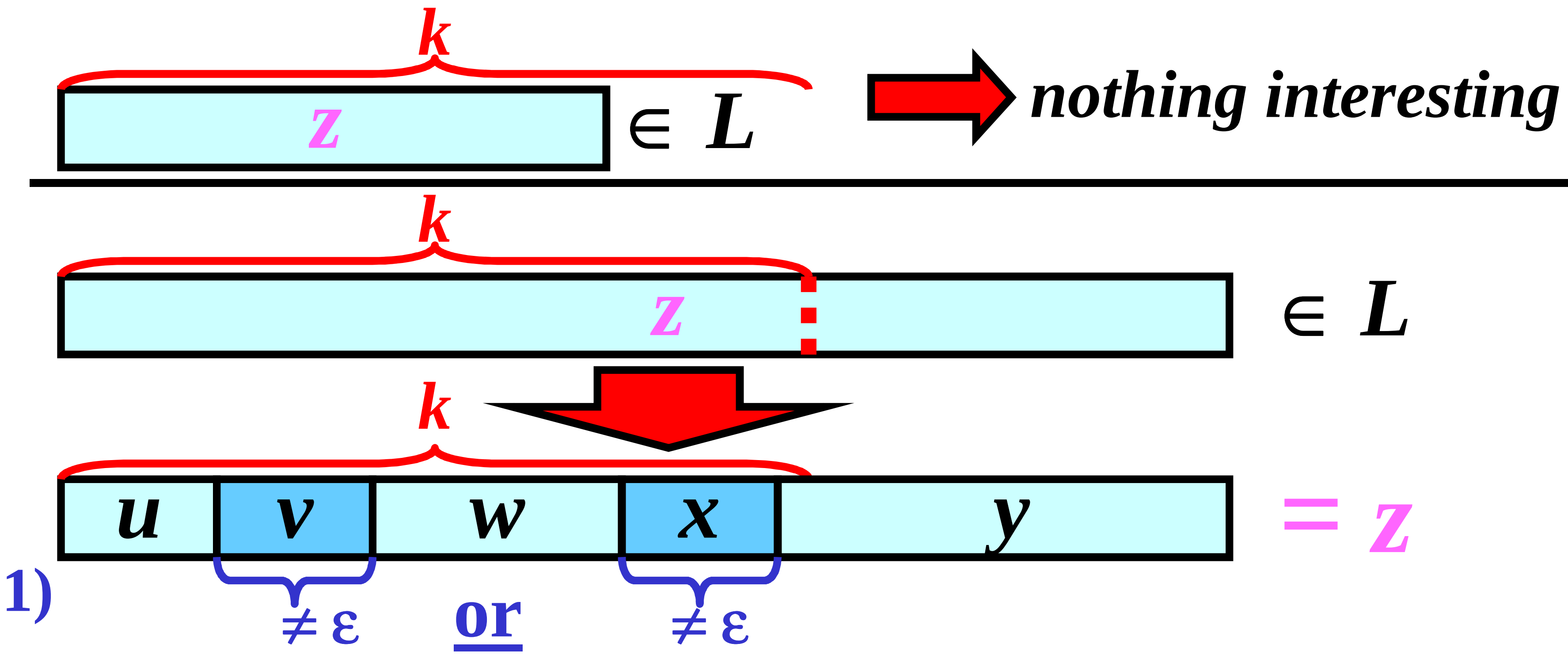
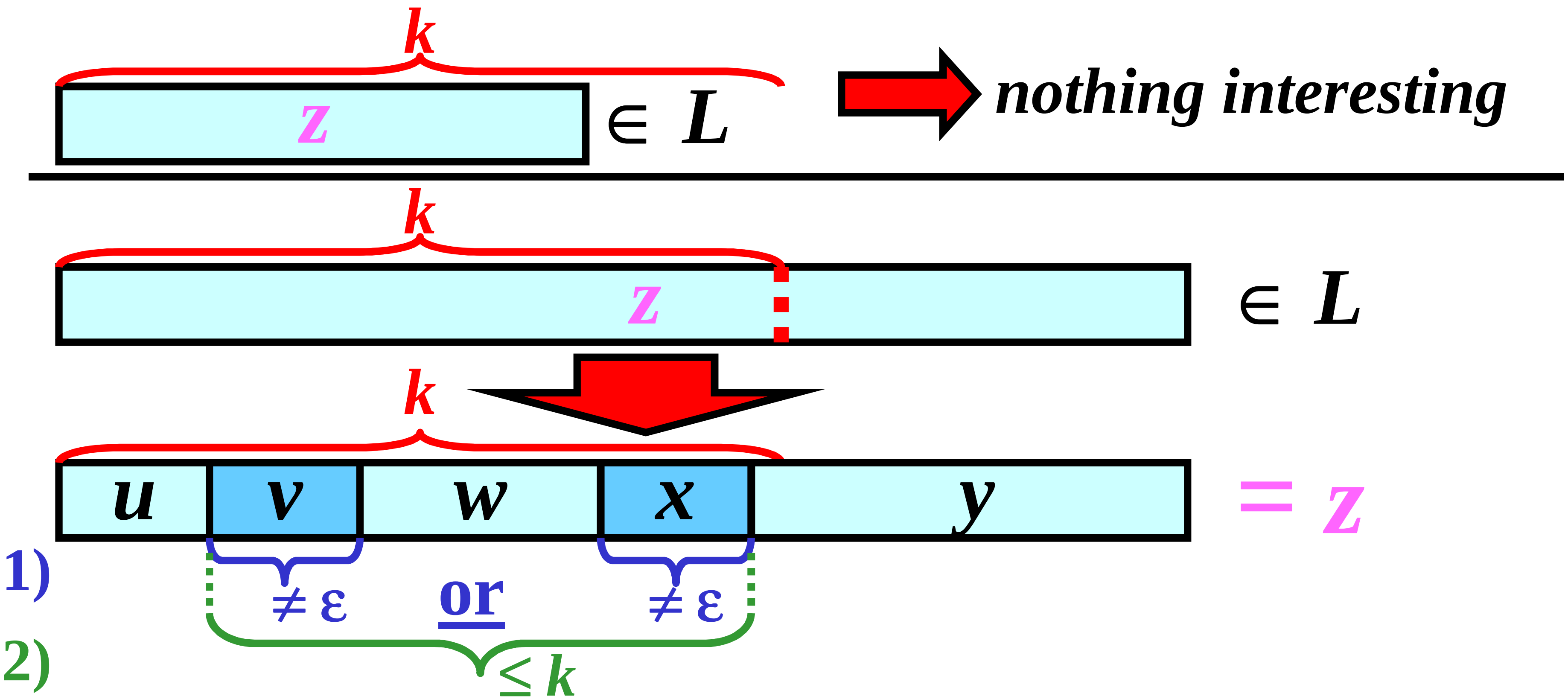


Pumping Lemma: Illustration

- *L* = any context-free language:



Pumping Lemma: Illustration

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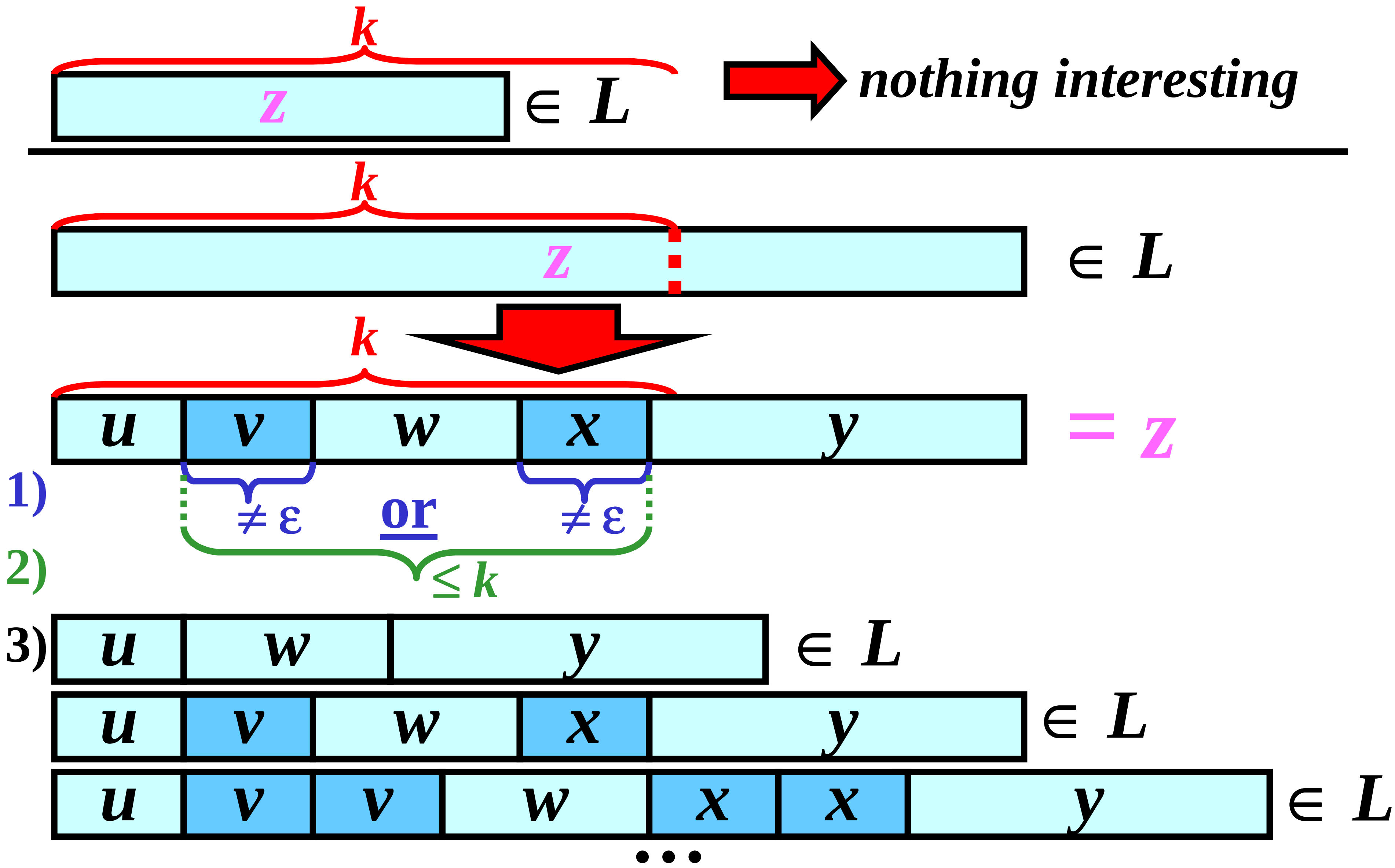

Pumping Lemma: Application

• Based on the pumping lemma for CFL, we often make a proof by contradiction to demonstrate that a language is **not** a CFL.

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Assume that L is a CFL.

↓

Consider the PL constant $\boldsymbol{k}$ and select $\boldsymbol{z} \in L$, whose length depends on $\boldsymbol{k}$ so $|\boldsymbol{z}| \geq \boldsymbol{k}$ is surely true.

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↓

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↓

For all decompositions of $\boldsymbol{z}$ into $\boldsymbol{uvwxy}$: $vx \neq \varepsilon$, $|vwx| \leq k$, show that

$$\left.\begin{array}{l}\text{there exists } m \geq 0 \text{ such that } \boldsymbol{uv^m wx^m y} \notin \boldsymbol{L}; \\ \text{from the pumping lemma, } \boldsymbol{uv^m wx^m y} \in \boldsymbol{L}\end{array}\right\} \textbf{contradiction}$$

Pumping Lemma: Application

- Based on the pumping lemma for CFL, we often make a proof by contradiction to demonstrate that a language is **not** a CFL.

Assume that L is a CFL.

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false assumption

Pumping Lemma: Application

• Based on the pumping lemma for CFL, we often make a proof by contradiction to demonstrate that a language is **not** a CFL.

Assume that L is a CFL.

Consider the PL constant $\boldsymbol{k}$ and select $\boldsymbol{z} \in L$, whose length depends on $\boldsymbol{k}$ so $|\boldsymbol{z}| \geq \boldsymbol{k}$ is surely true.

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false assumption

Therefore,
L is not a CFL

Pumping Lemma: Example 1/2

Prove that $L = \{a^n b^n c^n : n \geq 1\}$ is not CFL.

1) Assume that L is a CFL. Let $k \geq 1$ be the pumping lemma constant for L.

2) Let $\boldsymbol{z} = a^k b^k c^k$: $a^k b^k c^k \in L$, $|\boldsymbol{z}| = |a^k b^k c^k| = 3k \geq k$

3) All decompositions of $\boldsymbol{z}$ into $\boldsymbol{uvwxy}$; $vx \neq \varepsilon$, $|vwx| \leq k$:

$$\overbrace{aaaaa\ldots aa}^{k}\overbrace{bb\ldots bb\ldots bb}^{k}\overbrace{cc\ldots ccccc}^{k}$$

a) $vwx \in \{a\}^*\{b\}^*$, $vx \neq \varepsilon$

b) $vwx \in \{b\}^*\{c\}^*$, $vx \neq \varepsilon$

Pumping Lemma: Example 2/2

a) $vwx \in \{a\}^*\{b\}^*$:

- Pumping lemma: $uv^0wx^0y \in L$

$$\underbrace{a\ldots a}_{u}\ \overbrace{\underbrace{a\ldots aabb\ldots b}_{vwx}}^{\leq k}\ \underbrace{b\ldots bcc\ldots cc}_{y}$$

- $uv^0wx^0y = uwy =$ $\underbrace{a\ldots a}_{u}\ \underbrace{\ldots aabb\ldots}_{w}\ \underbrace{b\ldots bcc\ldots cc}_{y} \notin L$

Note: uwy contains k cs, but fewer than k as or bs.

---

b) $vwx \in \{b\}^*\{c\}^*$:

- Pumping lemma: $uv^0wx^0y \in L$

$$\underbrace{aa\ldots ab\ldots b}_{u}\ \overbrace{\underbrace{b\ldots bbcc\ldots c}_{vwx}}^{\leq k}\ \underbrace{c\ldots c}_{y}$$

- $uv^0wx^0y = uwy =$ $\underbrace{aa\ldots ab\ldots b}_{u}\ \underbrace{\ldots bbcc\ldots}_{w}\ \underbrace{c\ldots c}_{y} \notin L$

Note: uwy contains k as, but fewer than k bs or cs.

All these decompositions lead to a contradiction!

Pumping Lemma: Example 2/2

a) $vwx \in \{a\}^*\{b\}^*$:

- Pumping lemma: $uv^0wx^0y \in L$

$$\underbrace{a\ldots a}_{u}\overbrace{\underbrace{a\ldots aabb\ldots b}_{vwx}}^{\leq k}\underbrace{b\ldots bcc\ldots cc}_{y}$$

- $uv^0wx^0y = uwy = \underbrace{a\ldots a}_{u}\underbrace{\ldots aabb\ldots}_{w}\underbrace{b\ldots bcc\ldots cc}_{y} \notin L$

Note: uwy contains k cs, but fewer than k as or bs.

---

b) $vwx \in \{b\}^*\{c\}^*$:

- Pumping lemma: $uv^0wx^0y \in L$

$$\underbrace{aa\ldots ab\ldots b}_{u}\overbrace{\underbrace{b\ldots bbcc\ldots c}_{vwx}}^{\leq k}\underbrace{c\ldots c}_{y}$$

- $uv^0wx^0y = uwy = \underbrace{aa\ldots ab\ldots b}_{u}\underbrace{\ldots bbcc\ldots}_{w}\underbrace{c\ldots c}_{y} \notin L$

Note: uwy contains k as, but fewer than k bs or cs.

All these decompositions lead to a contradiction!

---

4) Therefore, L is not a CFL.

Closure properties of CFL

Definition: The family of CFLs is closed under an operation ***o*** if the language resulting from the application of ***o*** to **any** CFLs is a CFL as well.

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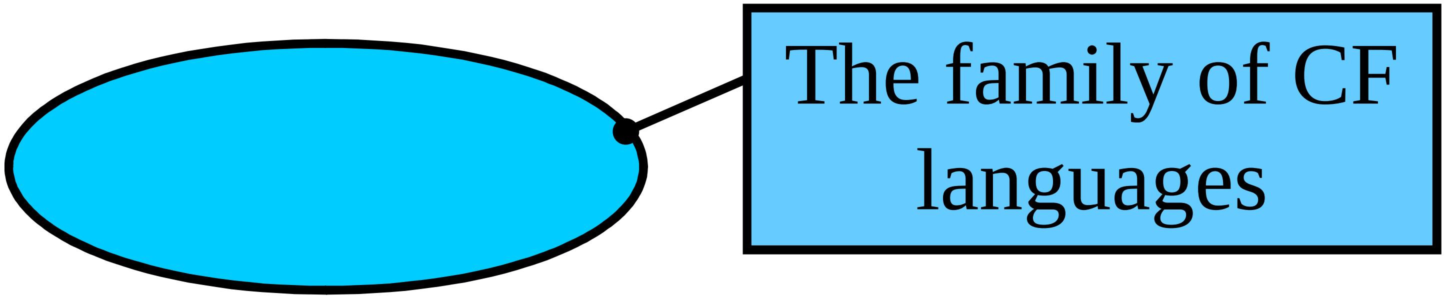
Illustration:

- The family of CF languages is closed under ***union***.
  It means:



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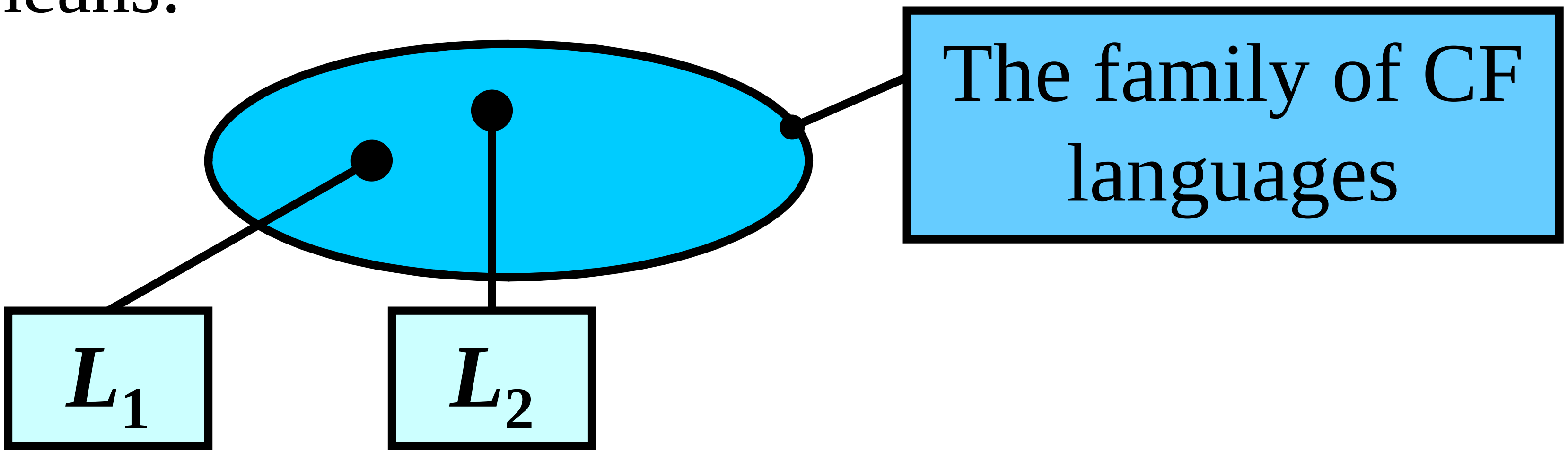

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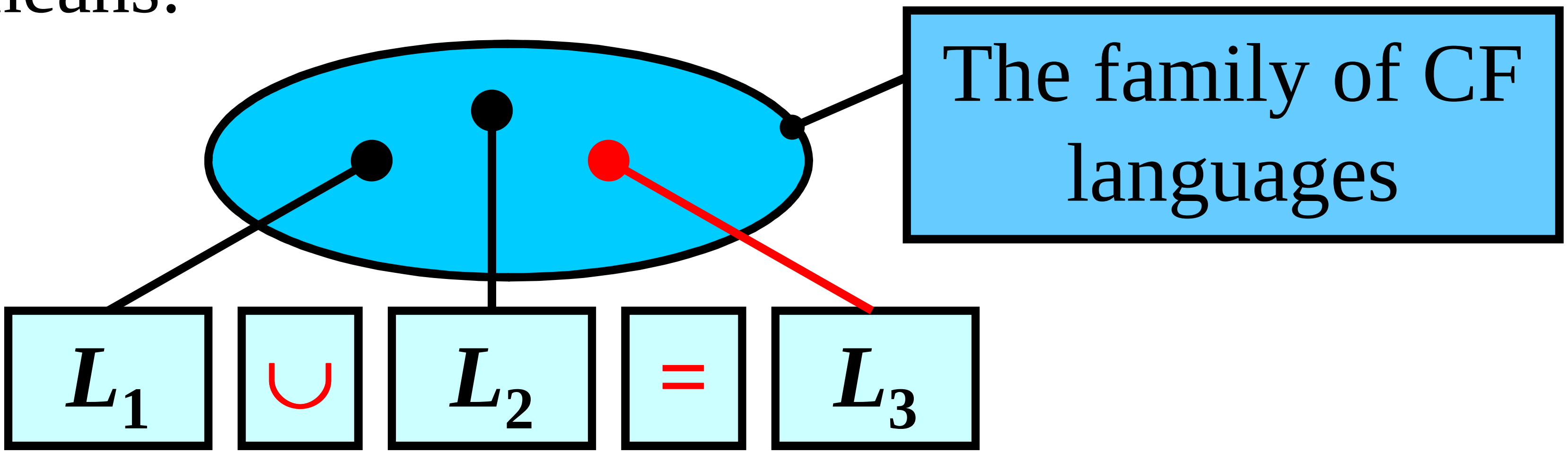

Algorithm: CFG for Union

- **Input:** Grammars $G_1 = (N_1, T, P_1, S_1)$ and $G_2 = (N_2, T, P_2, S_2)$;
- **Output:** Grammar $G_u = (N, T, P, S)$ such that $L(G_u) = L(G_1) \cup L(G_2)$

- **Method:**
- let $S \notin N_1 \cup N_2$, let $N_1 \cap N_2 = \emptyset$:
  - $N := \{S\} \cup N_1 \cup N_2$;
  - $P := \{S \rightarrow S_1, S \rightarrow S_2\} \cup P_1 \cup P_2$;

Algorithm: CFG for Concatenation

- **Input:** $G_1 = (N_1, T, P_1, S_1)$ and
  $G_2 = (N_2, T, P_2, S_2)$;
- **Output:** $G_c = (N, T, P, S)$ such that
  $L(G_c) = L(G_1) \,.\, L(G_2)$

- **Method:**
- let $S \notin N_1 \cup N_2$, let $N_1 \cap N_2 = \varnothing$:
  - $N := \{S\} \cup N_1 \cup N_2$;
  - $P := \{S \rightarrow S_1S_2\} \cup P_1 \cup P_2$;

Algorithm: CFG for Iteration

- **Input:** $G_1 = (N_1, T, P_1, S_1)$
- **Output:** $G_i = (N, T, P, S)$ such that $L(G_i) = L(G_1)^*$

- **Method:**
- let $S \notin N_1$:
  - $N := \{S\} \cup N_1$;
  - $P := \{S \to S_1 S, S \to \varepsilon\} \cup P_1$;

Closure properties

Theorem: The family of CFLs is closed under **union**, **concatenation**, **iteration**.

Proof:

- Let $\boldsymbol{L_1}$, $\boldsymbol{L_2}$ be two **CFLs.**
- Then, there exist two CFGs G_1, G_2 such that $L(G_1) = \boldsymbol{L_1}$, $L(G_2) = \boldsymbol{L_2}$;
- Construct grammars
  - G_u such that $L(G_u) = L(G_1) \cup L(G_2)$
  - G_c such that $L(G_c) = L(G_1) \,.\, L(G_2)$
  - G_i such that $L(G_i) = L(G_1)^*$

  by using the previous three algorithms
- Every CFG denotes CFL, so
- $\boldsymbol{L_1 L_2}$, $\boldsymbol{L_1 \cup L_2}$, $\boldsymbol{L_1^*}$ are **CFLs.**

Intersection: Not Closed

Theorem: The family of CFLs is **not** closed under **intersection**.

Proof:

- The intersection of some CFLs is not a CFL:

- $L_1 = \{a^m b^n c^n : m, n \geq 1\}$ is a CFL
- $L_2 = \{a^n b^n c^m : m, n \geq 1\}$ is a CFL
- $L_1 \cap L_2 = \{a^n b^n c^n : n \geq 1\}$ is not a CFL (proof based on the pumping lemma) *QED*

Complement: Not Closed

Theorem: The family of CFLs is not closed under **complement**.

Proof by contradiction:

- Assume that family of CFLs is closed under complement.
- $L_1 = \{a^m b^n c^n : m, n \geq 1\}$ is a **CFL**
- $L_2 = \{a^n b^n c^m : m, n \geq 1\}$ is a **CFL**
- $\overline{L_1}$, $\overline{L_2}$ are **CFLs**
- $\overline{L_1} \cup \overline{L_2}$ is a **CFL** (the family of CFLs is closed under union)
- $\overline{\overline{L_1} \cup \overline{L_2}}$ is a **CFL** (assumption)
- DeMorgan's law implies $L_1 \cap L_2 = \{a^n b^n c^n : n \geq 1\}$ is a **CFL**
- $\{a^n b^n c^n : n \geq 1\}$ is not a **CFL** $\Rightarrow$ **Contradiction**

Main Decidable Problems

1. Membership problem:

- **Instance:** CFG G, $w \in T^*$;**Question:** $w \in L(G)$**?**

2. Emptiness problem:

- **Instance:** CFG G; **Question:** $L(G) = \emptyset$**?**

3. Finiteness problem:

- **Instance:** CFG G; **Question:** Is $L(G)$ finite**?**

Algorithm: Membership

- **Input:** CFG $G = (N, T, P, S)$ in Chomsky normal form; $w \in T^+$
- **Output:** **YES** if $w \in L(G)$
  NO if $w \notin L(G)$

- **Method I:**
- **if** $S \Rightarrow^{\boldsymbol{n}} w$, where $1 \leq \boldsymbol{n} \leq 2|w| - 1$, **then** write ('**YES**')
  else write ('**NO**')
- **Method II:**
- See: **The general parsing method based on CNF**

Summary:

The membership problem for CFLs is decidable

Accessible Symbols

Gist: Symbol X is *accessible* if $S \Rightarrow^* \ldots X \ldots$, where S is the start nonterminal.

Definition: Let $G = (N, T, P, S)$ be a CFG. A symbol $X \in N \cup T$ is *accessible* if there exists $u, v \in (N \cup T)^*$ such that $S \Rightarrow^* uXv$; otherwise, X is *inaccessible*.

Note: Each inaccessible symbol can be removed from CFG

Example:

$G = (\{S, A, B\}, \{a, b\}, \{S \rightarrow SB, S \rightarrow a, A \rightarrow ab, B \rightarrow aB\}, S)$

S - accessible: for $u = \varepsilon$, $v = \varepsilon$: $S \Rightarrow^0 S$

A - **inaccessible**: there is no $u, v \in \Sigma^*$ such that $S \Rightarrow^* uAv$

B - accessible: for $u = S$, $v = \varepsilon$: $S \Rightarrow^1 SB$

a - accessible: for $u = \varepsilon$, $v = \varepsilon$: $S \Rightarrow^1 a$

b - **inaccessible**: there is no $u, v \in \Sigma^*$ such that $S \Rightarrow^* ubv$

Terminating Symbols

Gist: Symbol *X* is *terminating* if *X* derives a terminal string.

Definition: Let $G = (N, T, P, S)$ be a CFG. A symbol $X \in N \cup T$ is *terminating* if there exists $w \in T^*$ such that $X \Rightarrow^* w$; otherwise, X is *nonterminating*

Note: Each nonterminating symbol can be removed from any CFG.

Example:

$G = (\{S, A, B\}, \{a, b\}, \{S \rightarrow SB, S \rightarrow a, A \rightarrow ab, B \rightarrow aB\}, S)$

Symbol S - terminating: for $w = a$: $S \Rightarrow^1 a$

Symbol A - terminating: for $w = ab$: $A \Rightarrow^1 ab$

Symbol B - **nonterminating**: there is no $w \in T^*$ such that $B \Rightarrow^* w$

Symbol a - terminating: for $w = a$: $a \Rightarrow^0 a$

Symbol b - terminating: for $w = b$: $b \Rightarrow^0 b$

Algorithm: Emptiness

- **Input:** CFG $G = (N, T, P, S)$;
- **Output:** **YES** if $L(G) = \varnothing$

  NO if $L(G) \neq \varnothing$

- **Method:**
- **if** S is nonterminating **then** write ('**YES**')

  else write ('**NO**')

Summary:

The emptiness problem for CFLs is decidable

Algorithm: Finiteness

- **Input:** CFG $G = (N, T, P, S)$ in CNF;
- **Output: YES** if $L(G)$ is finite
  NO if $L(G)$ is infinite

- **Method:**
- Let $k = 2^{\mathrm{card}(N)}$
- **if** there exist $z \in L(G)$, $k \leq |z| < 2k$ **then** write ('**NO**')
  else write ('**YES**')

Summary:

The finiteness problem for CFLs is decidable

Main Undecidable Problems

1. Equivalence problem:

- **Instance:** CFGs G_1, G_2;**Question:** $L(G_1) = L(G_2)$**?**

2. Ambiguity problem:

- **Instance:** G; **Question:** Is G ambiguous**?**

Note:

It is mathematically proved that there exists no algorithm, which solve these problems in finite time.