Part VI.
Models for Context-Free Languages

Context-Free Grammar (CFG)

Gist: A *grammar* is based on a finite set of grammatical rules, by which it generates strings of its language.

Illustration: **Start nonterminal** S

Grammar G:

Nonterminals: $A, B, \boxed{S}$

Terminals: a, b, c, d

Rules:

$S \to AB,$
$A \to aAb,$
$A \to ab,$
$B \to bBa,$
$B \to ba$

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$$S \Downarrow \overbrace{AB}$$

Rule: $S \rightarrow AB$

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S

$\Downarrow$ **Rule:** $S \to AB$

$\overbrace{AB}$

$\Downarrow$ **Rule:** $A \to ab$

$\overbrace{ab}B$

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$\overbrace{ab}B$

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$ab\overbrace{bBa}$

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Terminals:	a, b, c, d
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S

Rule: $S \rightarrow AB$

AB

Rule: $A \rightarrow ab$

abB

Rule: $B \rightarrow bBa$

$abbBa$

Rule: $B \rightarrow ba$

$abbbaa \in L(G)$

Context-Free Grammar: Definition

Definition: *A context-free grammar* (CFG) is a quadruple $G = (N, T, P, S)$, where

- N is an alphabet of *nonterminals*
- T is an alphabet of *terminals*, $N \cap T = \emptyset$
- P is a finite set of *rules* of the form $A \rightarrow x$, where $A \in N$, $x \in (N \cup T)^*$
- $S \in N$ is the *start nonterminal*

Mathematical Note on Rules:

- Strictly mathematically, P is a relation from N to $(N \cup T)^*$
- Instead of $(A, x) \in P$, we write $A \rightarrow x \in P$

- $A \rightarrow x$ means that A can be replaced with x
- $A \rightarrow \varepsilon$ is called ***ε-rule***

Convention

- $A, \ldots, F, S$: nonterminals
- S : the start nonterminal
- $a, \ldots, d$: terminals
- $U, \ldots, Z$: members of $(N \cup T)$
- $u, \ldots, z$: members of $(N \cup T)^*$
- π : sequence of productions

A subset of rules of the form:

$$A \rightarrow x_1, A \rightarrow x_2, \ldots, A \rightarrow x_n$$

can be simply written as:

$$A \rightarrow x_1 \mid x_2 \mid \ldots \mid x_n$$

Derivation Step

Gist: A change of a string by a rule.

Definition: Let $G = (N, T, P, S)$ be a CFG. Let $\boldsymbol{u}, \boldsymbol{v} \in (N \cup T)^*$ and $\boldsymbol{p} = \boldsymbol{A} \rightarrow \boldsymbol{x} \in P$. Then, $\boldsymbol{uAv}$ *directly derives* $\mathbf{uxv}$ *according to* $\boldsymbol{p}$ in G, written as $\boldsymbol{uAv} \Rightarrow \boldsymbol{uxv}$ $[p]$ or, simply, $\boldsymbol{uAv} \Rightarrow \boldsymbol{uxv}$.

Note: If $\boldsymbol{uAv} \Rightarrow \boldsymbol{uxv}$ in G, we also say that G makes a *derivation step* from $\boldsymbol{uAv}$ to $\boldsymbol{uxv}$.

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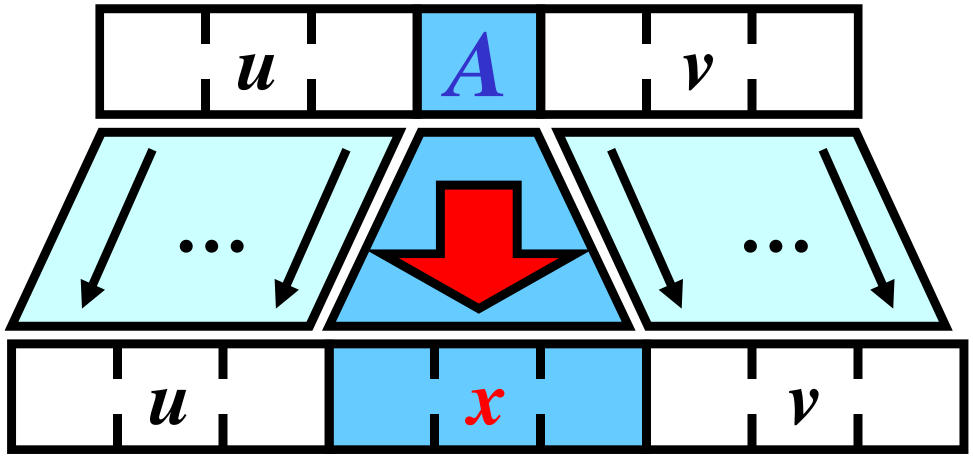
Rule: $A \rightarrow \boldsymbol{x}$



Sequence of Derivation Steps 1/2

Gist: Several consecutive derivation steps.

Definition: Let $u \in (N \cup T)^*$. G makes a *zero-step derivation* from u to u; in symbols,

$$u \Rightarrow^0 u\ [\varepsilon] \text{ or, simply, } u \Rightarrow^0 u$$

Definition: Let $u_0, \ldots, u_n \in (N \cup T)^*$, $n \geq 1$, and $u_{i-1} \Rightarrow u_i\ [p_i]$, $p_i \in P$, for all $i = 1, \ldots, n$; that is

$$u_0 \Rightarrow u_1\ [p_1] \Rightarrow u_2\ [p_2] \ldots \Rightarrow u_n\ [p_n]$$

Then, G makes *n derivation steps* from u_0 to u_n,

$$u_0 \Rightarrow^n u_n\ [p_1 \ldots p_n] \text{ or, simply, } u_0 \Rightarrow^n u_n$$

Sequence of Derivation Steps 2/2

If $u_0 \Rightarrow^n u_n$ [π] for some $n \geq 1$, then u_0 *properly derives* u_n in G, written as $u_0 \Rightarrow^+ u_n$ [π].

If $u_0 \Rightarrow^n u_n$ [π] for some $n \geq 0$, then u_0 *derives* u_n in G, written as $u_0 \Rightarrow^* u_n$ [π].

Example: Consider

$aAb \Rightarrow aaBbb$ [**1**: $A \rightarrow aBb$], and

$aaBbb \Rightarrow aacbb$ [**2**: $B \rightarrow c$].

Then, $aAb \Rightarrow^2 aacbb$ [**1 2**],

$aAb \Rightarrow^+ aacbb$ [**1 2**],

$aAb \Rightarrow^* aacbb$ [**1 2**]

Generated Language

Gist: ***G generates* a terminal string *w* by a sequence of derivation steps from *S* to *w***

Definition: Let $G = (N, T, P, S)$ be a CFG. The *language generated by* G, $L(G)$, is defined as

$$L(G) = \{w: w \in T^*, S \Rightarrow^* w\}$$

Illustration:

$G = (N, T, P, S)$, let $w = a_1a_2\ldots a_n$; $a_i \in T$ for $i = 1..n$

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Illustration:

$G = (N, T, P, S)$, let $w = a_1a_2\ldots a_n$; $a_i \in T$ for $i = 1..n$

if $S \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \underbrace{a_1a_2\ldots a_n}_{w}$ **then** $w \in L(G)$;

otherwise, $w \notin L(G)$

Context-Free Language (CFL)

Gist: A language generated by a CFG.

Definition: Let L be a language. L is a *context-free language* (CFL) if there exists a context-free grammar that generates L.

Example:

$G = (N, T, P, S)$, where $N = \{S\}$, $T = \{a, b\}$,

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$S \Rightarrow \varepsilon$ [2] $L(G)$

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$S \Rightarrow aSb$ [1] $\Rightarrow ab$ [2]

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$S \Rightarrow \varepsilon \quad [2]$

$S \Rightarrow aSb\ [1] \Rightarrow ab \quad [2]$

$S \Rightarrow aSb\ [1] \Rightarrow aaSbb\ [1] \Rightarrow aabb\ [2]$

$L(G)$

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⋮

$L(G) = \{a^n b^n : n \geq 0\}$

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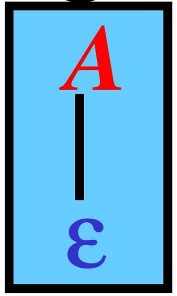
$\vdots$

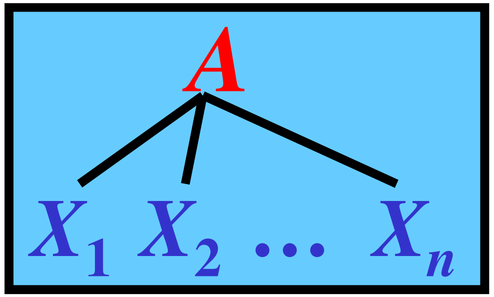
$L(G) = \{a^n b^n : n \geq 0\}$

$L = \{a^n b^n : n \geq 0\}$ is a CFL.

Rule Tree

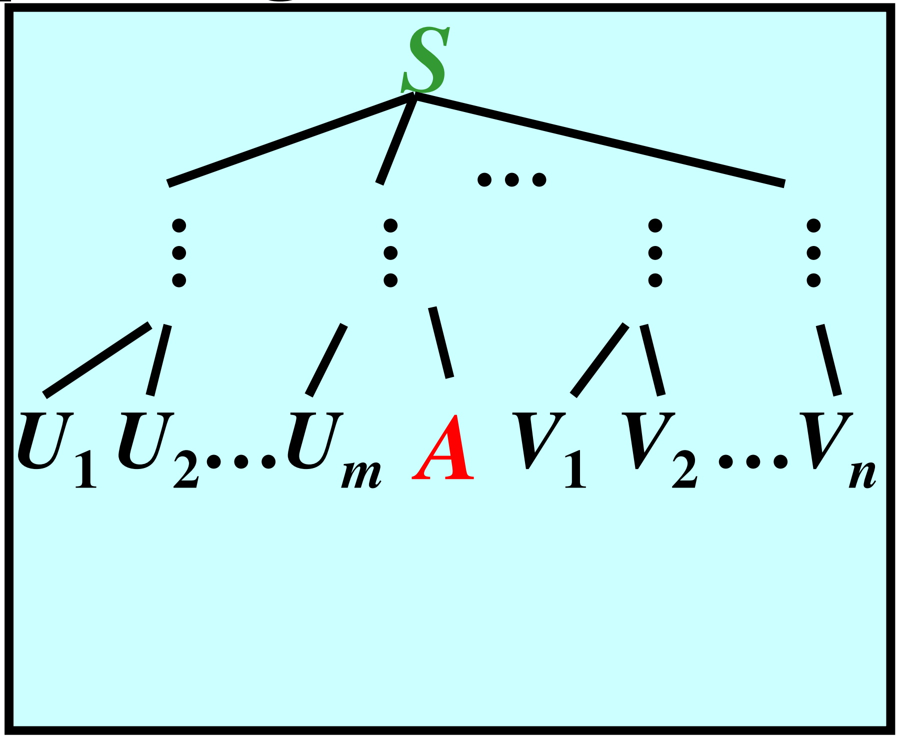
- **Rule tree graphically represents a rule**

1) $A \rightarrow \varepsilon$: 

2) $A \rightarrow X_1X_2\ldots X_n$: 

- **Derivation tree corresponding to a derivation**

$$S \Rightarrow \ldots$$
$$\vdots$$
$$\Rightarrow U_1U_2\ldots U_m A V_1V_2\ldots V_n$$



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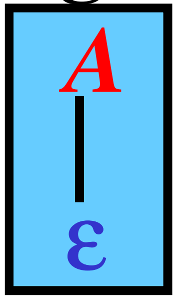

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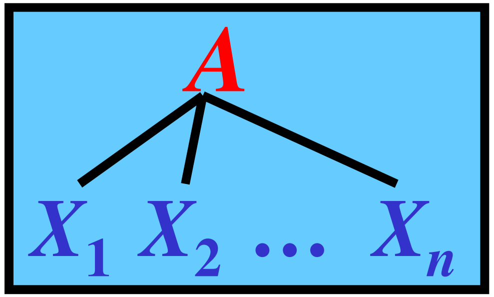

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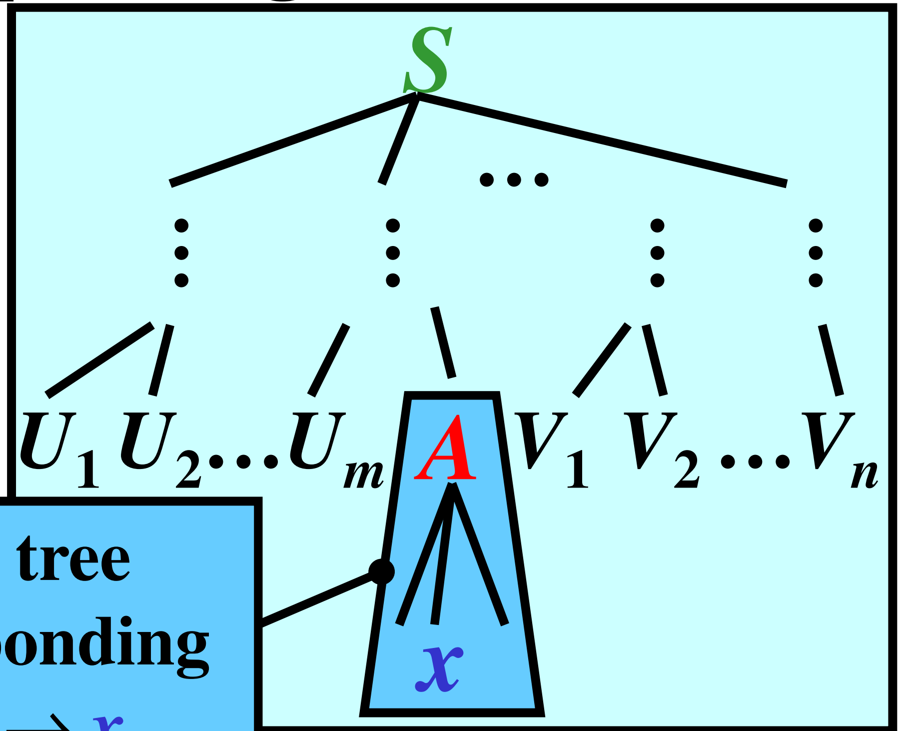
$S \Rightarrow \ldots$

$\vdots$

$\Rightarrow U_1U_2\ldots U_m A V_1V_2\ldots V_n$

$\Rightarrow U_1U_2\ldots U_m x\, V_1V_2\ldots V_n$



Derivation Tree: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Derivation:

Derivation tree:

Derivation Tree: Example

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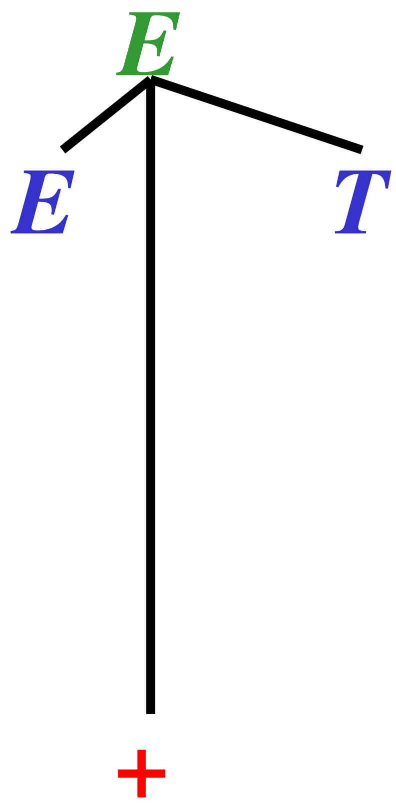
Derivation:

$\underline{E} \Rightarrow E + \underline{T}$ [1]

Derivation tree:



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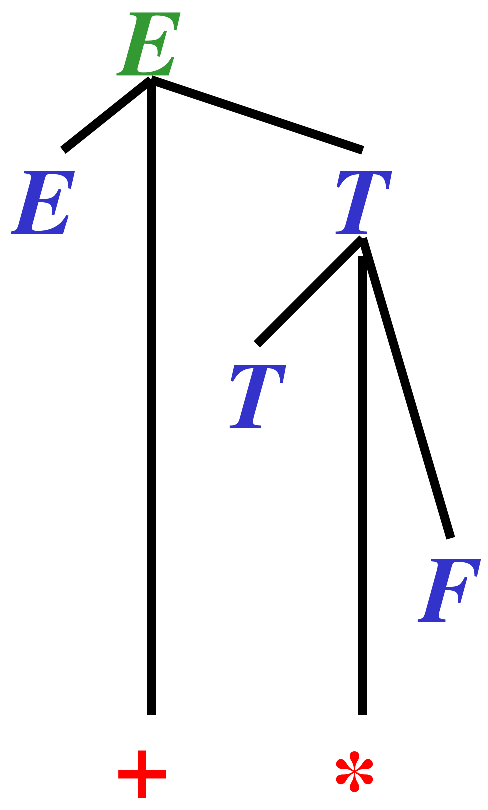
Derivation:

$\underline{E} \Rightarrow E + \underline{T}$ [1]

$\Rightarrow E + \underline{T} * F$ [3]

Derivation tree:



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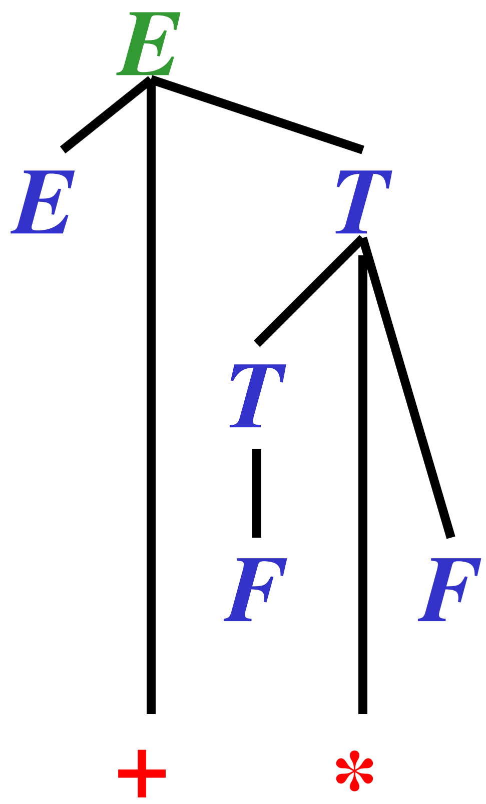
Derivation:

$$\underline{E} \Rightarrow E + \underline{T} \quad [1]$$
$$\Rightarrow E + \underline{T} * F \quad [3]$$
$$\Rightarrow E + \underline{F} * F \quad [4]$$

Derivation tree:



Derivation Tree: Example

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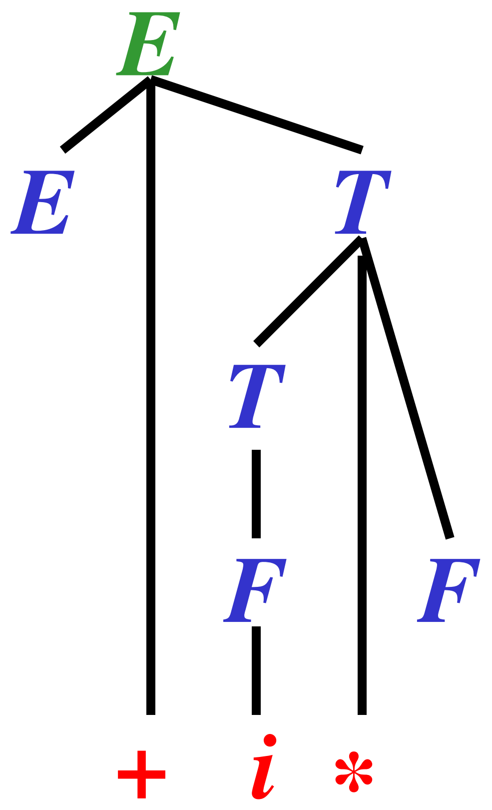
Derivation:

$$\underline{E} \Rightarrow E + \underline{T} \quad [1]$$
$$\Rightarrow E + \underline{T} * F \quad [3]$$
$$\Rightarrow E + \underline{F} * F \quad [4]$$
$$\Rightarrow \underline{E} + i * F \quad [6]$$

Derivation tree:



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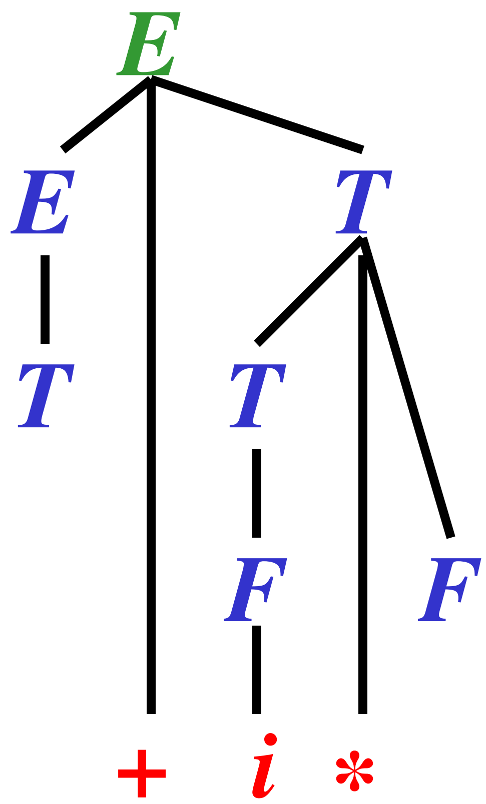
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Derivation:

$$
\begin{aligned}
\underline{E} &\Rightarrow E + \underline{T} && [1] \\
&\Rightarrow E + \underline{T} * F && [3] \\
&\Rightarrow E + \underline{F} * F && [4] \\
&\Rightarrow \underline{E} + i * F && [6] \\
&\Rightarrow T + i * \underline{F} && [2]
\end{aligned}
$$

Derivation tree:



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Derivation:

$$\underline{E} \Rightarrow E + \underline{T} \quad [1]$$
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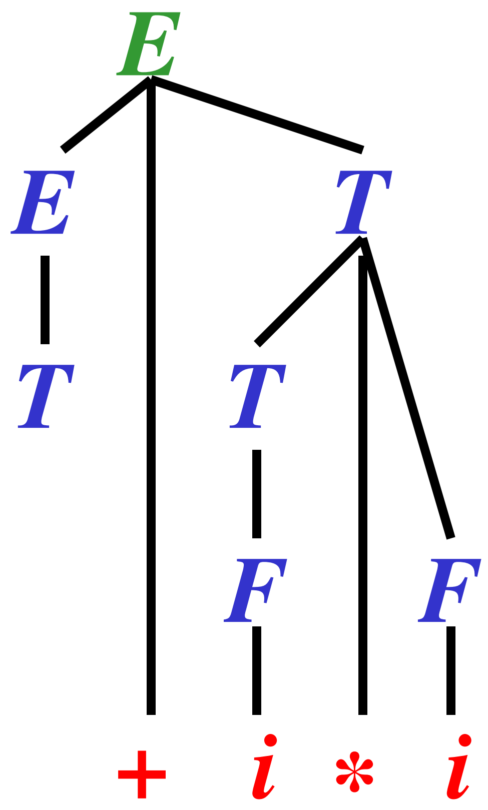
Derivation tree:



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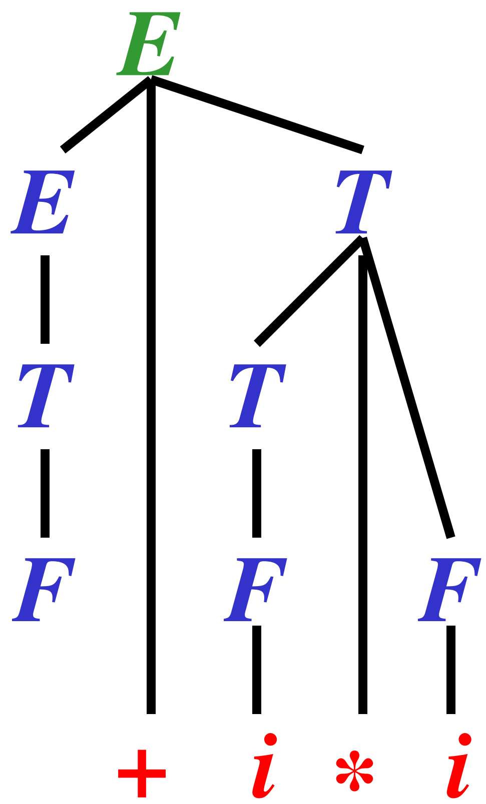

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Derivation:

$\underline{E} \Rightarrow E + \underline{T}$ [1]
$\Rightarrow E + \underline{T} * F$ [3]
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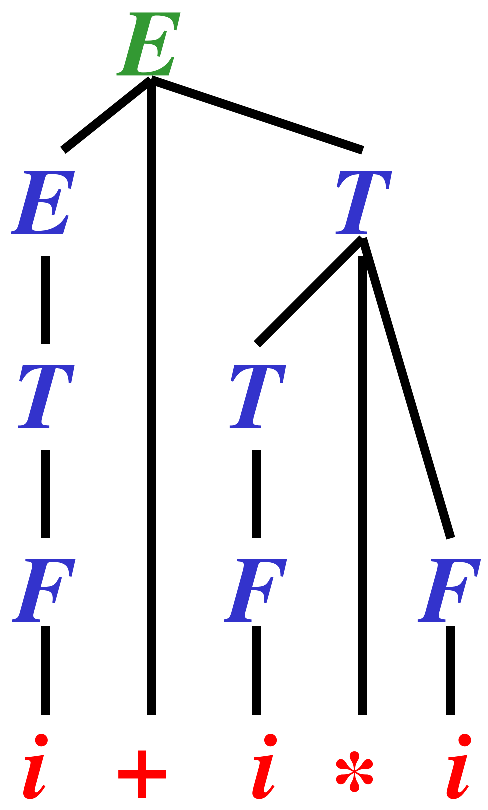
Derivation tree:



Leftmost Derivation

Gist: During a *leftmost derivation step*, the leftmost nonterminal is rewritten.

Definition: Let $G = (N, T, P, S)$ be a CFG, let $\boldsymbol{u} \in T^*$, $\boldsymbol{v} \in (N \cup T)^*$. Let $\boldsymbol{p} = \boldsymbol{A} \rightarrow \boldsymbol{x} \in P$ be a rule. Then, $\boldsymbol{uAv}$ directly derives $\boldsymbol{uxv}$ in the *leftmost way* according to $\boldsymbol{p}$ in G, written as

$$\boldsymbol{uAv} \Rightarrow_{lm} \boldsymbol{uxv}\ [\boldsymbol{p}]$$

Note: We define $\Rightarrow_{lm}^+$ and $\Rightarrow_{lm}^*$ by analogy with $\Rightarrow^+$ and $\Rightarrow^*$, respectively.

Leftmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$, 4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Leftmost derivation:

Derivation tree:

Leftmost Derivation: Example

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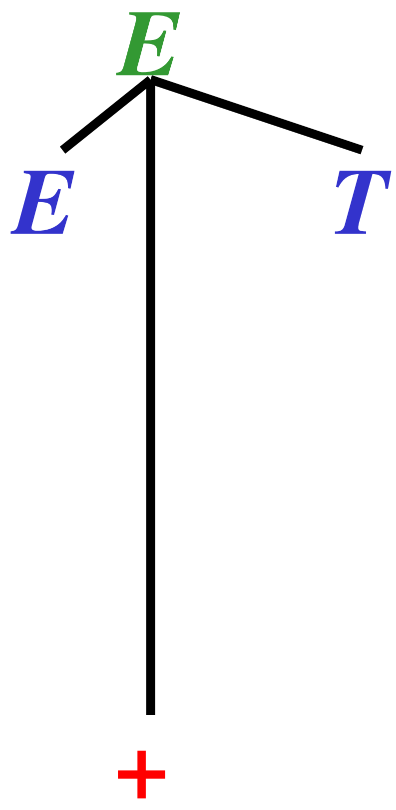
Leftmost derivation:

$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]

Derivation tree:



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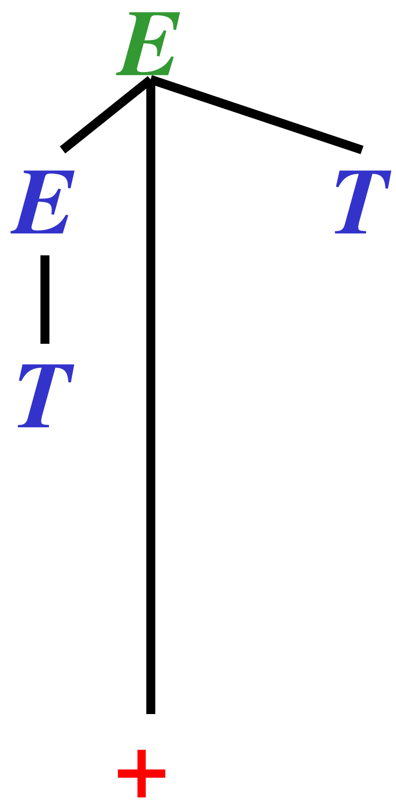
$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]

$\Rightarrow_{lm} \underline{T} + T$ [2]

Derivation tree:



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Leftmost derivation:

$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]

$\Rightarrow_{lm} \underline{T} + T$ [2]

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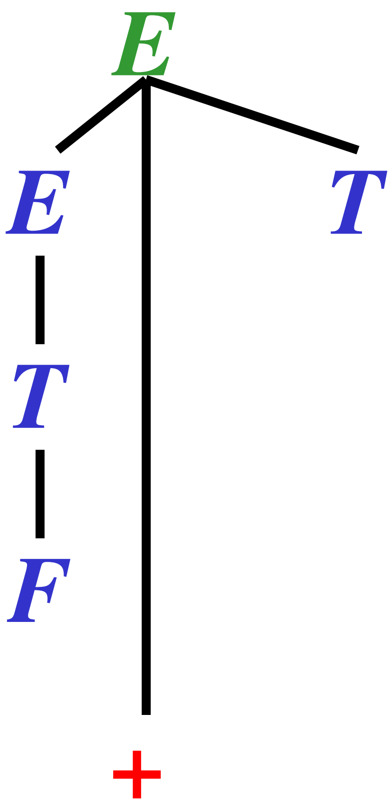
Derivation tree:



Leftmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Leftmost derivation:

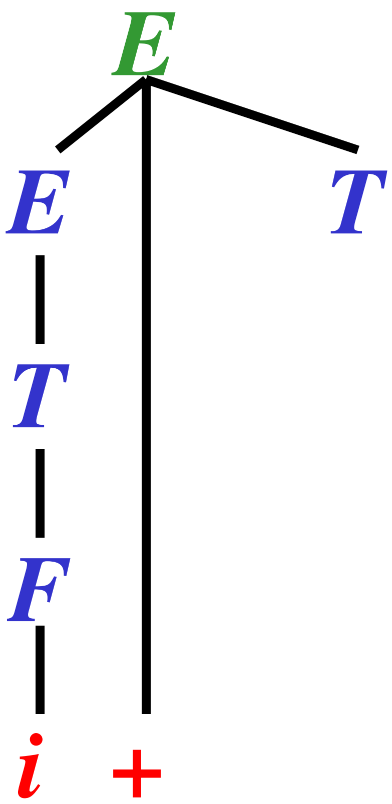
$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]

$\Rightarrow_{lm} \underline{T} + T$ [2]

$\Rightarrow_{lm} \underline{F} + T$ [4]

$\Rightarrow_{lm} i + \underline{T}$ [6]

Derivation tree:



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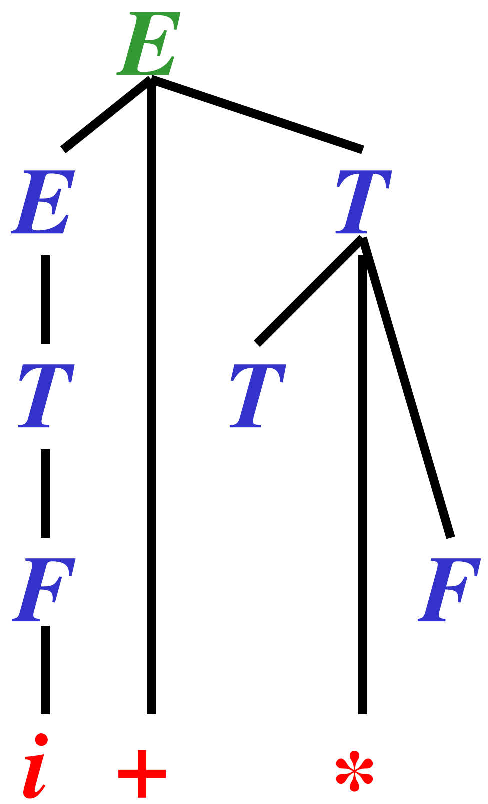
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Leftmost derivation:

$$\underline{E} \Rightarrow_{lm} \underline{E} + T \quad [1]$$
$$\Rightarrow_{lm} \underline{T} + T \quad [2]$$
$$\Rightarrow_{lm} \underline{F} + T \quad [4]$$
$$\Rightarrow_{lm} i + \underline{T} \quad [6]$$
$$\Rightarrow_{lm} i + \underline{T} * F \quad [3]$$

Derivation tree:



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$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

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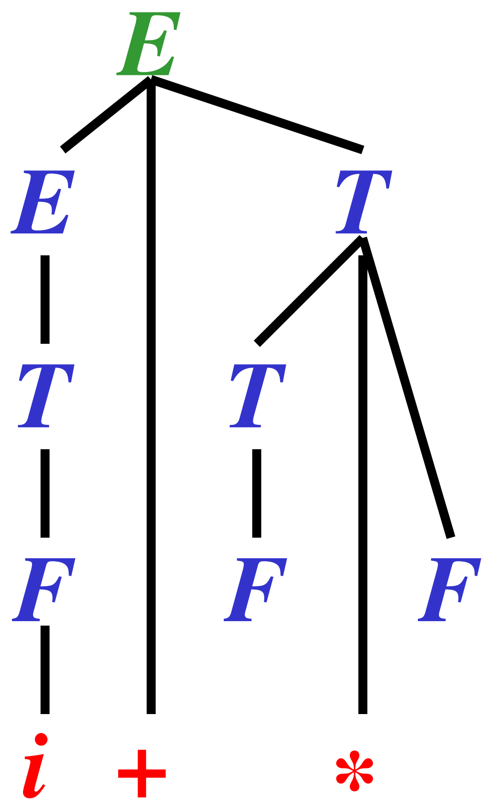
Leftmost derivation:

$\underline{E} \Rightarrow_{lm} \underline{E} + T$ [1]
$\Rightarrow_{lm} \underline{T} + T$ [2]
$\Rightarrow_{lm} \underline{F} + T$ [4]
$\Rightarrow_{lm} i + \underline{T}$ [6]
$\Rightarrow_{lm} i + \underline{T} * F$ [3]
$\Rightarrow_{lm} i + \underline{F} * F$ [4]

Derivation tree:



Leftmost Derivation: Example

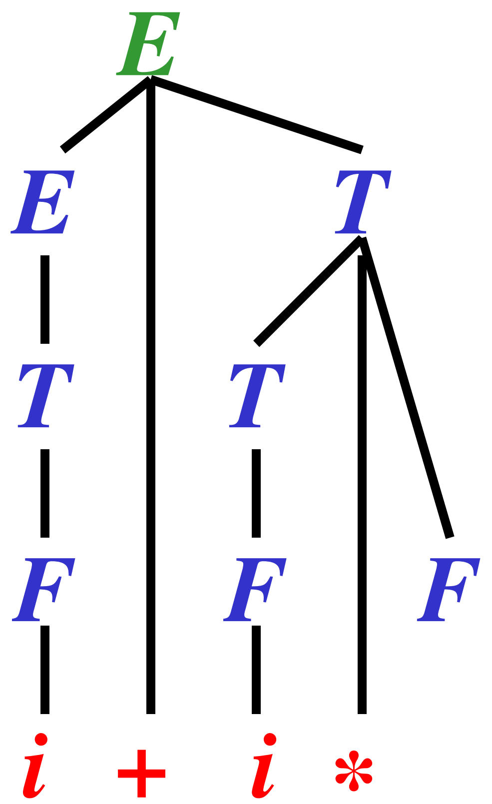
$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Leftmost derivation:

$$
\begin{aligned}
\underline{E} &\Rightarrow_{lm} \underline{E} + T && [1]\\
&\Rightarrow_{lm} \underline{T} + T && [2]\\
&\Rightarrow_{lm} \underline{F} + T && [4]\\
&\Rightarrow_{lm} i + \underline{T} && [6]\\
&\Rightarrow_{lm} i + \underline{T} * F && [3]\\
&\Rightarrow_{lm} i + \underline{F} * F && [4]\\
&\Rightarrow_{lm} i + i * \underline{F} && [6]
\end{aligned}
$$

Derivation tree:



Leftmost Derivation: Example

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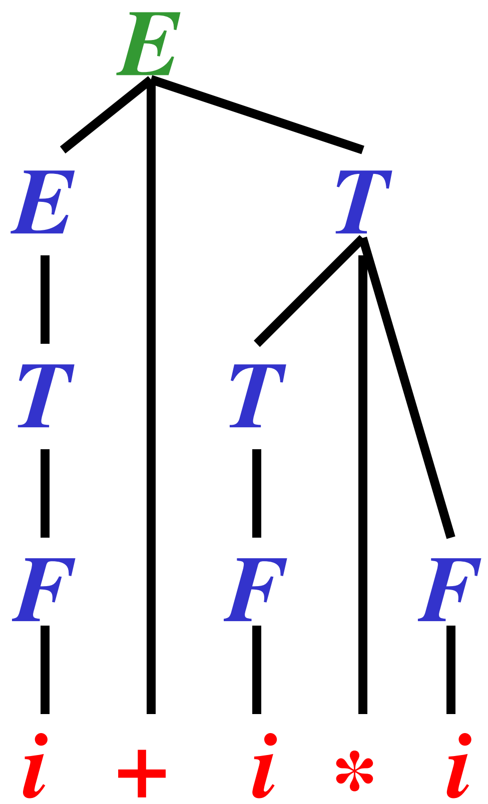
$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Leftmost derivation:

$$
\begin{aligned}
\underline{E} &\Rightarrow_{lm} \underline{E} + T && [1]\\
&\Rightarrow_{lm} \underline{T} + T && [2]\\
&\Rightarrow_{lm} \underline{F} + T && [4]\\
&\Rightarrow_{lm} i + \underline{T} && [6]\\
&\Rightarrow_{lm} i + \underline{T} * F && [3]\\
&\Rightarrow_{lm} i + \underline{F} * F && [4]\\
&\Rightarrow_{lm} i + i * \underline{F} && [6]\\
&\Rightarrow_{lm} i + i * i && [6]
\end{aligned}
$$

Derivation tree:



Rightmost Derivation

Gist: During a *rightmost derivation step*, the rightmost nonterminal is rewritten.

Definition: Let $G = (N, T, P, S)$ be a CFG, let $\boldsymbol{u} \in (N \cup T)^*$, $\boldsymbol{v} \in T^*$. Let $\boldsymbol{p} = \boldsymbol{A} \rightarrow \boldsymbol{x} \in P$ be a rule. Then, $\boldsymbol{uAv}$ directly derives $\boldsymbol{uxv}$ in the *rightmost way* according to $\boldsymbol{p}$ in G, written as

$$\boldsymbol{uAv} \Rightarrow_{rm} \boldsymbol{uxv}\ [\boldsymbol{p}]$$

Note: We define $\Rightarrow_{rm}^{+}$ and $\Rightarrow_{rm}^{*}$ by analogy with $\Rightarrow^{+}$ and $\Rightarrow^{*}$, respectively.

Rightmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Rightmost derivation: **Derivation tree:**

Rightmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

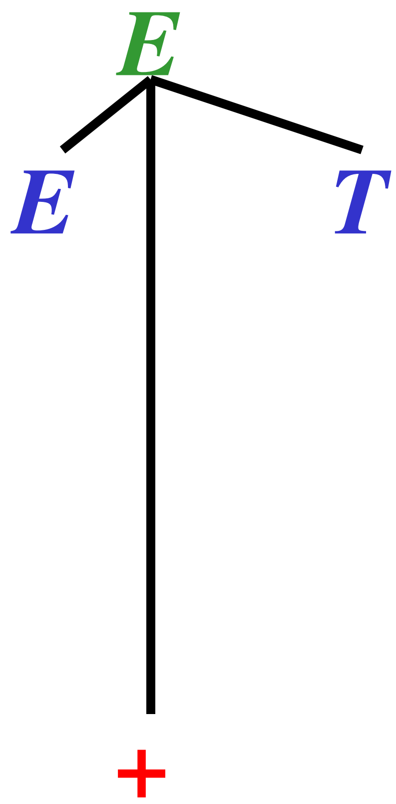
$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,

4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Rightmost derivation:

$\underline{E} \Rightarrow_{rm} E + \underline{T}$ [1]

Derivation tree:



Rightmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

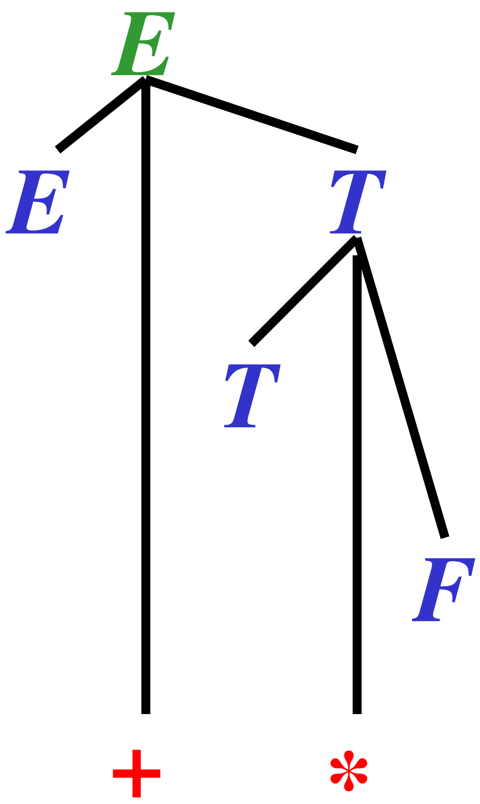
$P = \{$ 1: $E \to E+T$, 2: $E \to T$, 3: $T \to T*F$,
4: $T \to F$, 5: $F \to (E)$, 6: $F \to i$ $\}$

Rightmost derivation:

$\underline{E} \Rightarrow_{rm} E + \underline{T}$ [1]

$\Rightarrow_{rm} E + T * \underline{F}$ [3]

Derivation tree:



Rightmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

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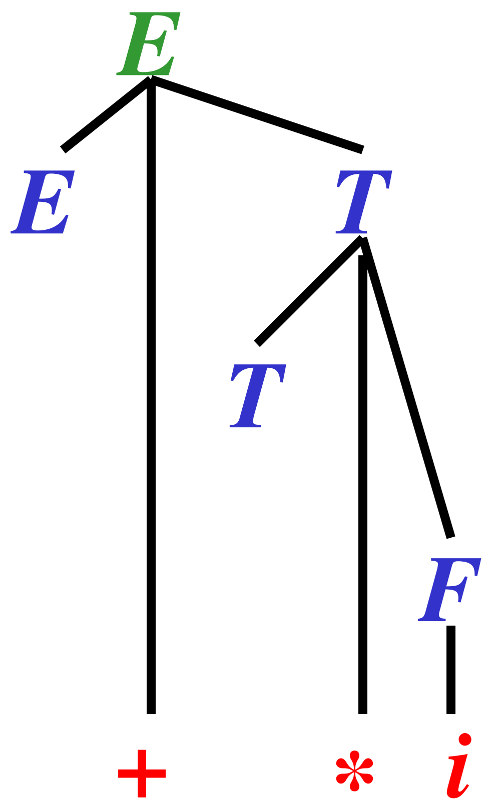
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Rightmost derivation:

$$\underline{E} \Rightarrow_{rm} E + \underline{T} \quad [1]$$
$$\Rightarrow_{rm} E + T * \underline{F} \quad [3]$$
$$\Rightarrow_{rm} E + \underline{T} * i \quad [6]$$

Derivation tree:



Rightmost Derivation: Example

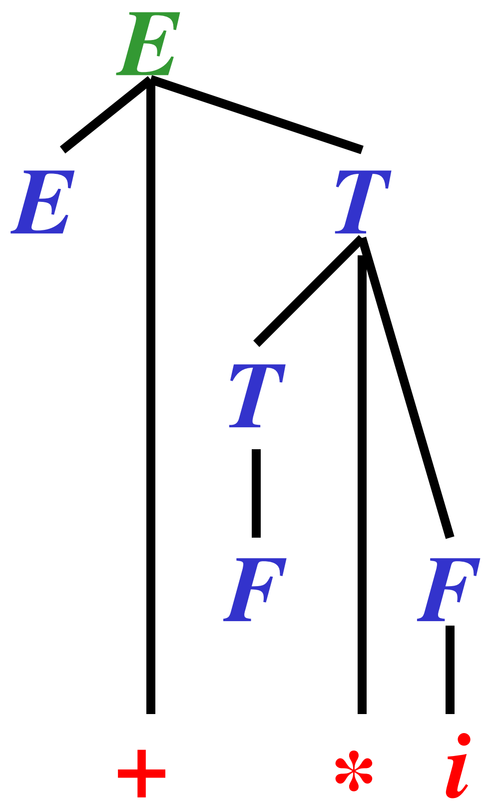
$G = (N, T, P, \boldsymbol{E})$, where $N = \{\boldsymbol{E}, \boldsymbol{F}, \boldsymbol{T}\}$, $T = \{\boldsymbol{i}, \boldsymbol{+}, \boldsymbol{*}, \boldsymbol{(}, \boldsymbol{)}\}$,

$P = \{$ 1: $E \rightarrow E+T$, 2: $E \rightarrow T$, 3: $T \rightarrow T*F$,
4: $T \rightarrow F$, 5: $F \rightarrow (E)$, 6: $F \rightarrow i$ $\}$

Rightmost derivation:

$\underline{E} \Rightarrow_{rm} E + \underline{T}$ [1]
$\Rightarrow_{rm} E + T * \underline{F}$ [3]
$\Rightarrow_{rm} E + \underline{T} * i$ [6]
$\Rightarrow_{rm} E + \underline{F} * i$ [4]

Derivation tree:



Rightmost Derivation: Example

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Rightmost derivation:

$$
\begin{aligned}
\underline{E} &\Rightarrow_{rm} E + \underline{T} && [1] \\
&\Rightarrow_{rm} E + T * \underline{F} && [3] \\
&\Rightarrow_{rm} E + \underline{T} * i && [6] \\
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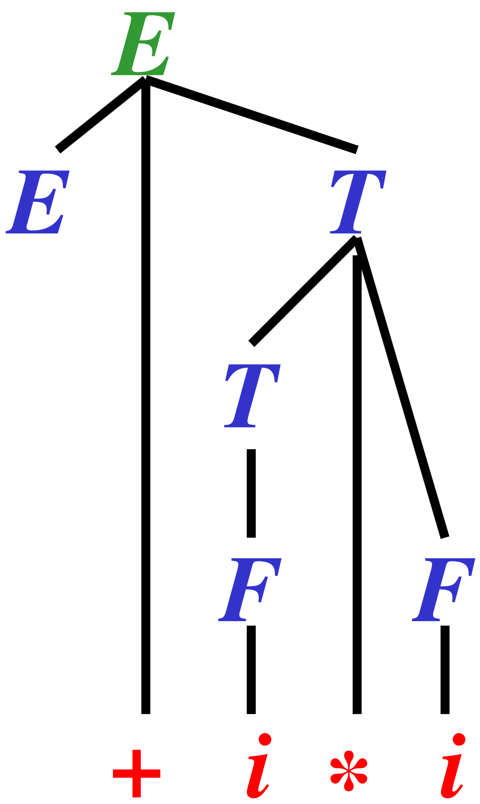
&\Rightarrow_{rm} \underline{E} + i * i && [6]
\end{aligned}
$$

Derivation tree:



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Rightmost derivation:

$$
\begin{aligned}
\underline{E} &\Rightarrow_{rm} E + \underline{T} && [1] \\
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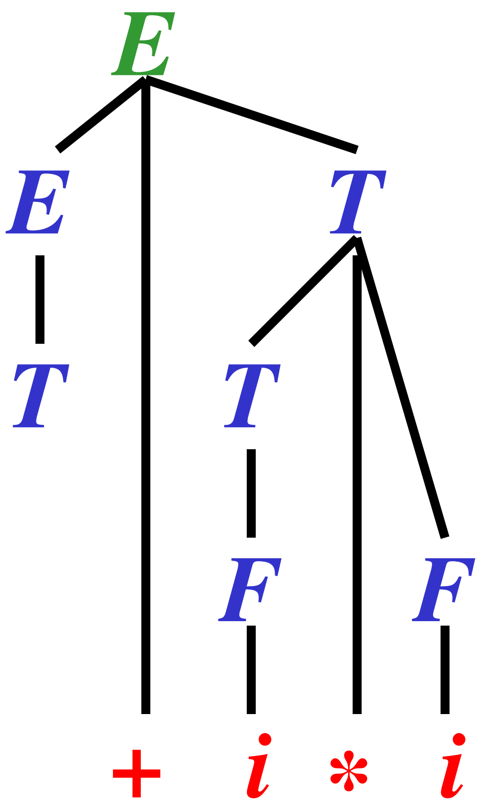
&\Rightarrow_{rm} E + \underline{T} * i && [6] \\
&\Rightarrow_{rm} E + \underline{F} * i && [4] \\
&\Rightarrow_{rm} \underline{E} + i * i && [6] \\
&\Rightarrow_{rm} \underline{T} + i * i && [2]
\end{aligned}
$$

Derivation tree:



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Rightmost derivation:

$$
\begin{aligned}
\underline{E} &\Rightarrow_{rm} E + \underline{T} && [1] \\
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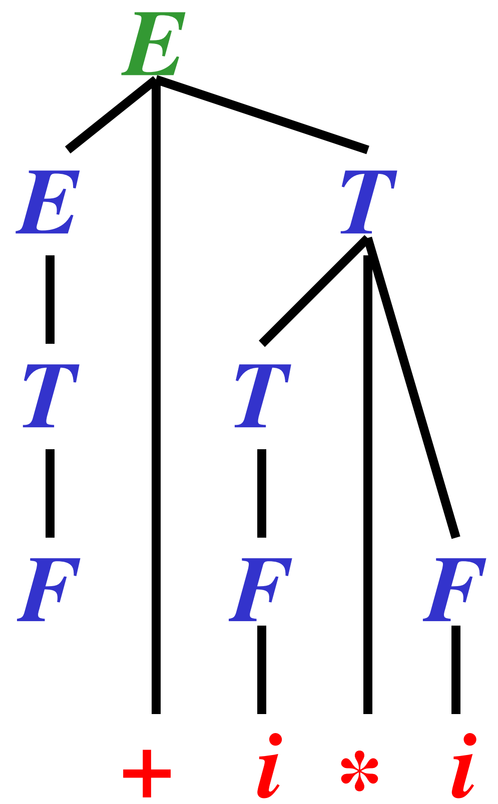
&\Rightarrow_{rm} E + \underline{T} * i && [6] \\
&\Rightarrow_{rm} E + \underline{F} * i && [4] \\
&\Rightarrow_{rm} \underline{E} + i * i && [6] \\
&\Rightarrow_{rm} \underline{T} + i * i && [2] \\
&\Rightarrow_{rm} \underline{F} + i * i && [4]
\end{aligned}
$$

Derivation tree:



Rightmost Derivation: Example

$G = (N, T, P, E)$, where $N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,

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Rightmost derivation:

$$
\begin{aligned}
\underline{E} &\Rightarrow_{rm} E + \underline{T} && [1] \\
&\Rightarrow_{rm} E + T * \underline{F} && [3] \\
&\Rightarrow_{rm} E + \underline{T} * i && [6] \\
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&\Rightarrow_{rm} \underline{E} + i * i && [6] \\
&\Rightarrow_{rm} \underline{T} + i * i && [2] \\
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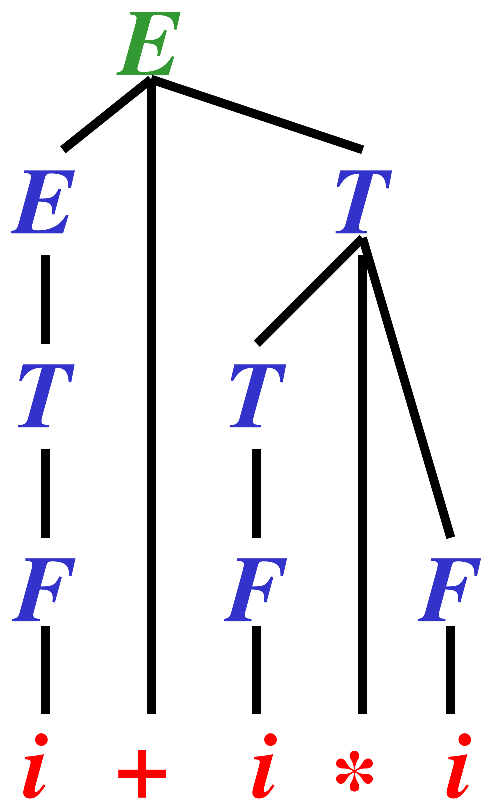
&\Rightarrow_{rm} i + i * i && [6]
\end{aligned}
$$

Derivation tree:



Derivations: Summary

- Let $A \rightarrow x \in P$ be a rule.

1) Derivation:

Let $u, v \in (N \cup T)^*$: $uAv \Rightarrow uxv$

Note: **<u>Any</u> nonterminal is rewritten**

2) Leftmost derivation:

Let $u \in T^*, v \in (N \cup T)^*$: $uAv \Rightarrow_{lm} uxv$

Note: **<u>Leftmost</u> nonterminal is rewritten**

3) Rightmost derivation:

Let $u \in (N \cup T)^*, v \in T^*$: $uAv \Rightarrow_{rm} uxv$

Note: **<u>Rightmost</u> nonterminal is rewritten**

Reduction of the Number of Derivations

Gist: Without any loss of generality, we can consider only leftmost or rightmost derivations.

Theorem: Let $G = (N, T, P, S)$ be a CFG. The next three languages coincide

(1) $\{w: w \in T^*, S \Rightarrow_{lm}^* w\}$

(2) $\{w: w \in T^*, S \Rightarrow_{rm}^* w\}$

(3) $\{w: w \in T^*, S \Rightarrow^* w\} = L(G)$

Introduction to Ambiguity

$G_{expr1} = (N, T, P, E)$, where
$N = \{E, F, T\}$, $T = \{i, +, *, (,)\}$,
$P = \{$ 1: $E \to E+T$, 2: $E \to T$,
3: $T \to T*F$, 4: $T \to F$,
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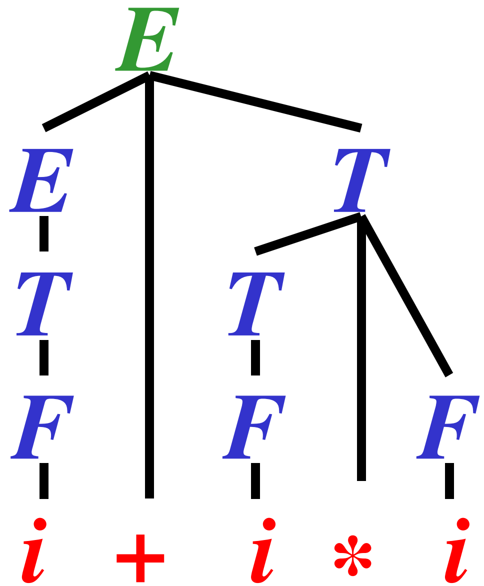
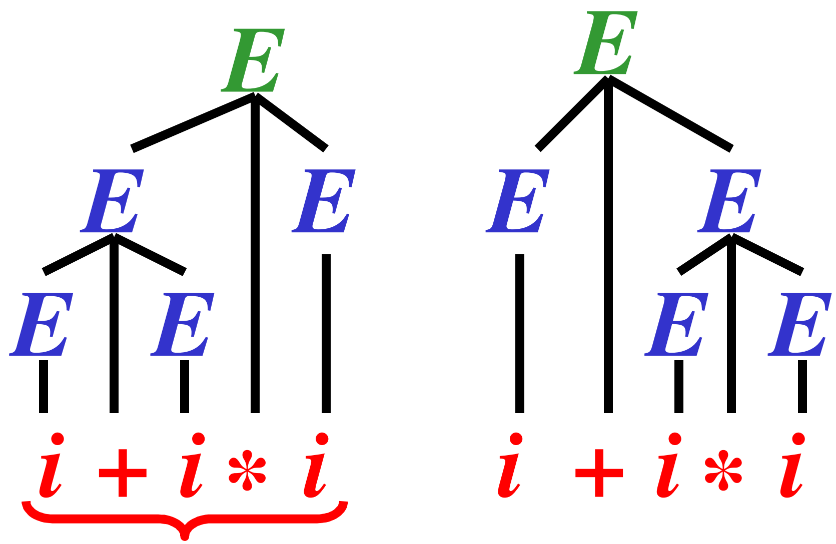


Theory: ☹ × Practice: ☺

$G_{expr2} = (N, T, P, E)$, where
$N = \{E\}$, $T = \{i, +, *, (,)\}$,
$P = \{$ 1: $E \to E+E$, 2: $E \to E*E$,
3: $E \to (E)$, 4: $E \to i$ $\}$

Theory: ☺ × Practice: ☹



Improper during compilation

Note: $L(G_{expr1}) = L(G_{expr2})$

Grammatical Ambiguity

Definition: Let $G = (N, T, P, S)$ be a CFG. If there exists $x \in L(G)$ with more than one derivation tree, then G is *ambiguous*; otherwise, G is *unambiguous*.

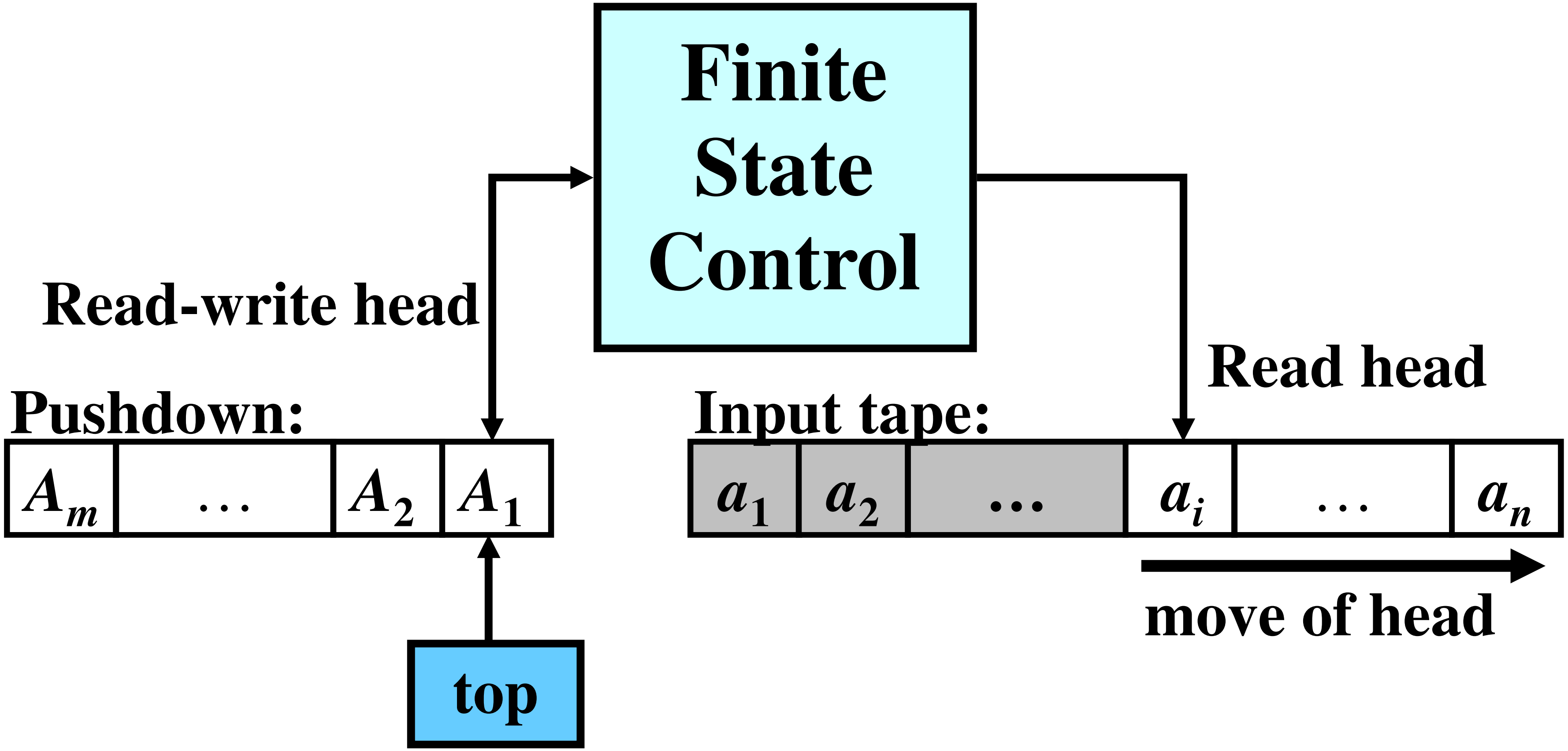
Definition: A CFL, L, is *inherently ambiguous* if L is generated by no unambiguous grammar.

Example:

- G_{expr1} is **unambiguous**, because for every $x \in L(G_{expr1})$ there exists **only one derivation tree**
- G_{expr2} is **ambiguous**, because for $i+i*i \in L(G_{expr2})$ there exist **two derivation trees**
- $L_{expr} = L(G_{expr1}) = L(G_{expr2})$ is **not inherently ambiguous** because G_{expr1} **is unambiguous**

Pushdown Automata (PDA)

Gist: An FA extended by a pushdown store.



Pushdown Automata: Definition

Definition: *A pushdown automaton* (PDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where

- Q is a *finite set of states*
- Σ is an *input alphabet*
- Γ is a *pushdown alphabet*
- R is a *finite set of rules* of the form: $Apa \rightarrow wq$ where $A \in \Gamma$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $w \in \Gamma^*$
- $s \in Q$ is the *start state*
- $S \in \Gamma$ is the *start pushdown symbol*
- $F \subseteq Q$ is a set of *final states*

Notes on PDA Rules

Mathematical note on rules:

- Strictly mathematically, R is a finite relation from $\Gamma \times Q \times (\Sigma \cup \{\varepsilon\})$ to $\Gamma^* \times Q$
- Instead of $(\boldsymbol{Apa}, \boldsymbol{wq}) \in R$, however, we write $\boldsymbol{Apa} \rightarrow \boldsymbol{wq} \in R$

Notes on PDA Rules

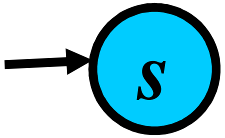
Mathematical note on rules:

- Strictly mathematically, R is a finite relation from $\Gamma \times Q \times (\Sigma \cup \{\varepsilon\})$ to $\Gamma^* \times Q$
- Instead of $(\boldsymbol{Apa}, \boldsymbol{wq}) \in R$, however, we write $\boldsymbol{Apa} \to \boldsymbol{wq} \in R$

- **Interpretation of** $\boldsymbol{Apa} \to \boldsymbol{wq}$: if the current state is $\boldsymbol{p}$, current input symbol is $\boldsymbol{a}$, and the topmost symbol on the pushdown is $\boldsymbol{A}$, then M can read $\boldsymbol{a}$, replace $\boldsymbol{A}$ with $\boldsymbol{w}$ and change state $\boldsymbol{p}$ to $\boldsymbol{q}$.
- **Note**: if $\boldsymbol{a} = \varepsilon$, no symbol is read

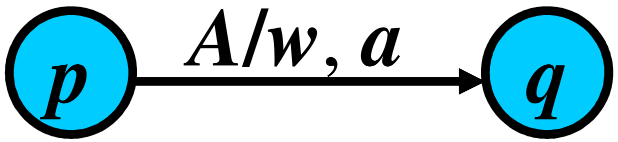
Graphical Representation

represents $q \in Q$



represents the initial state $s \in Q$

represents a final state $f \in F$



denotes $Apa \rightarrow wq \in R$

Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

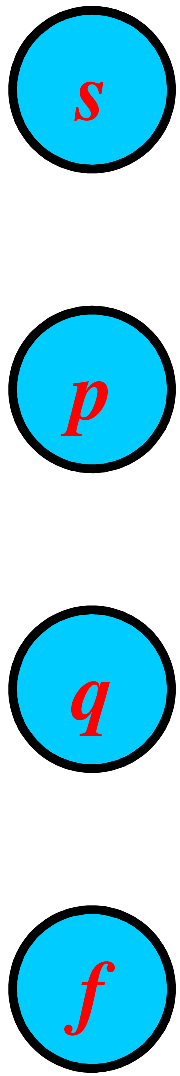
where:

Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{\boldsymbol{s}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f}\}$;

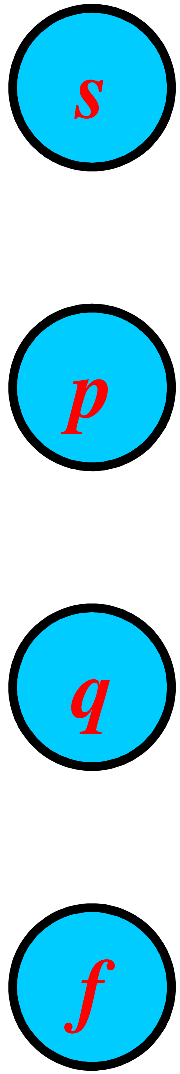


Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{s, p, q, f\}$;
- $\Sigma = \{a, b\}$;



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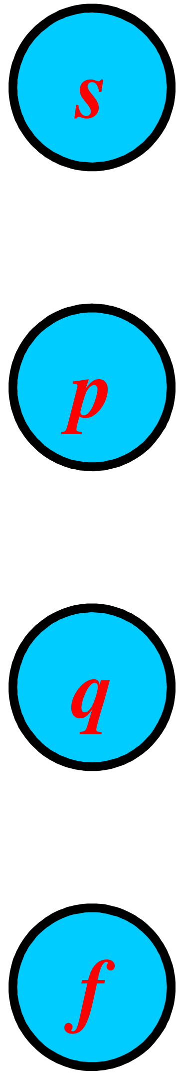
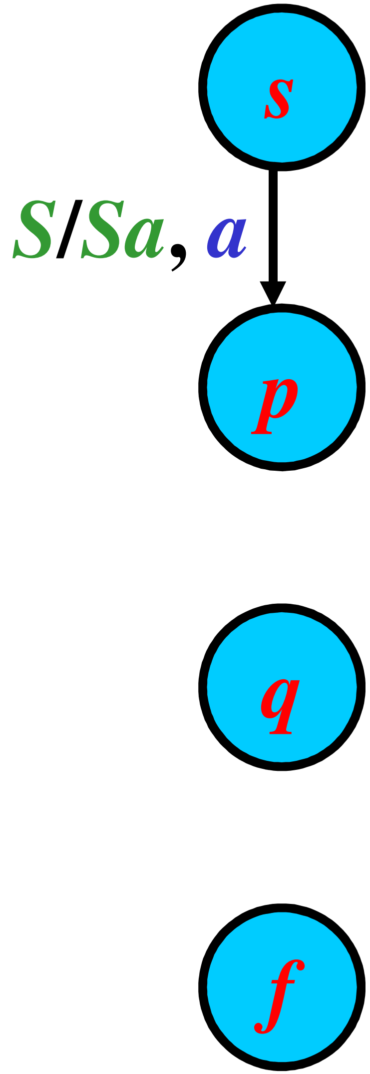


Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{s, p, q, f\}$;
- $\Sigma = \{a, b\}$;
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap,$



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where:

- $Q = \{s, p, q, f\}$;
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- $\Gamma = \{a, S\}$;
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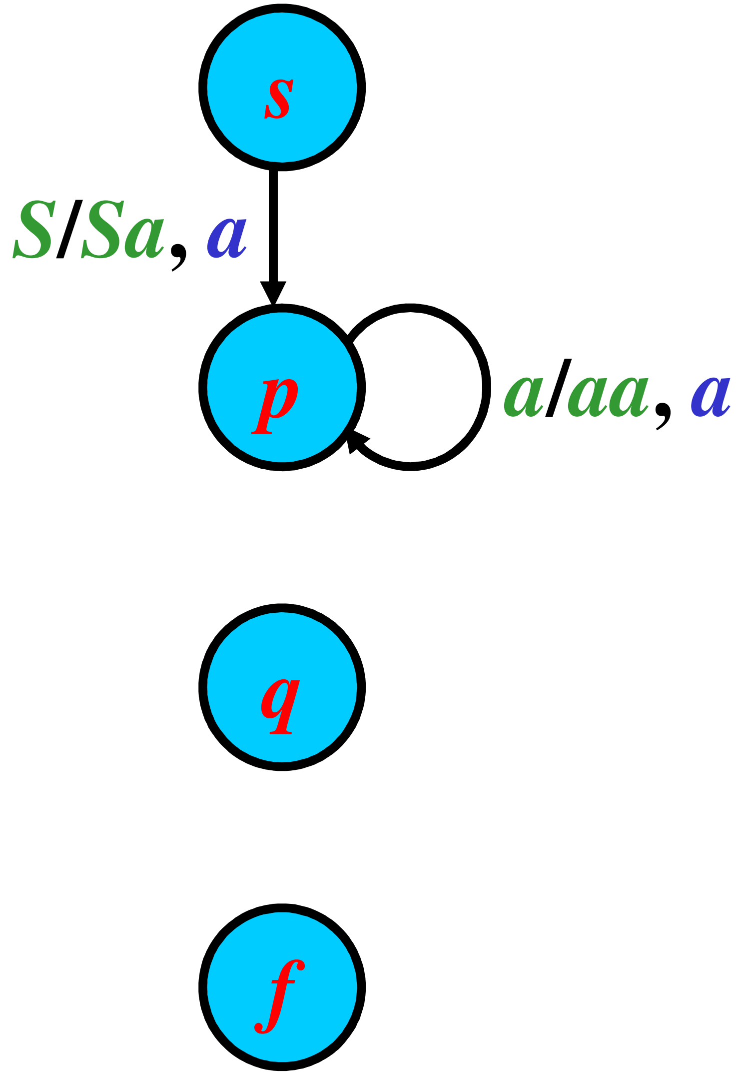

Graphical Representation: Example

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where:

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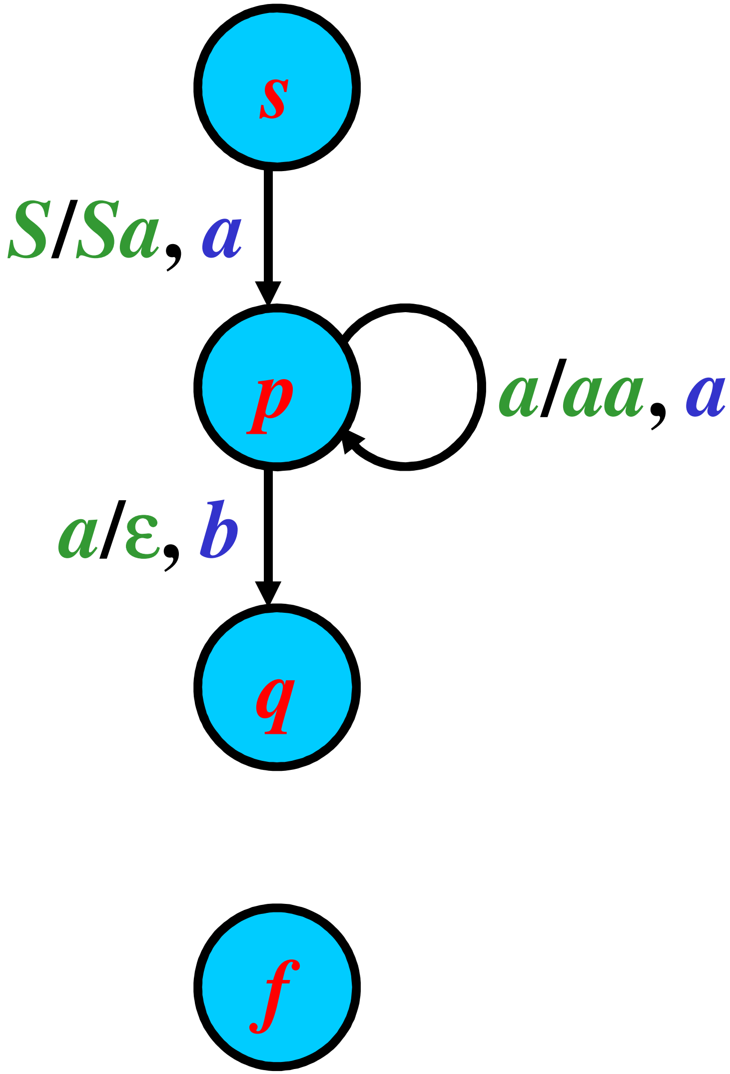
- $\Sigma = \{a, b\}$;
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap,$
  $apa \rightarrow aap,$
  $apb \rightarrow q,$



Graphical Representation: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{s, p, q, f\}$;
- $\Sigma = \{a, b\}$;
- $\Gamma = \{a, S\}$;
- $R = \{Ssa \rightarrow Sap,$
  $apa \rightarrow aap,$
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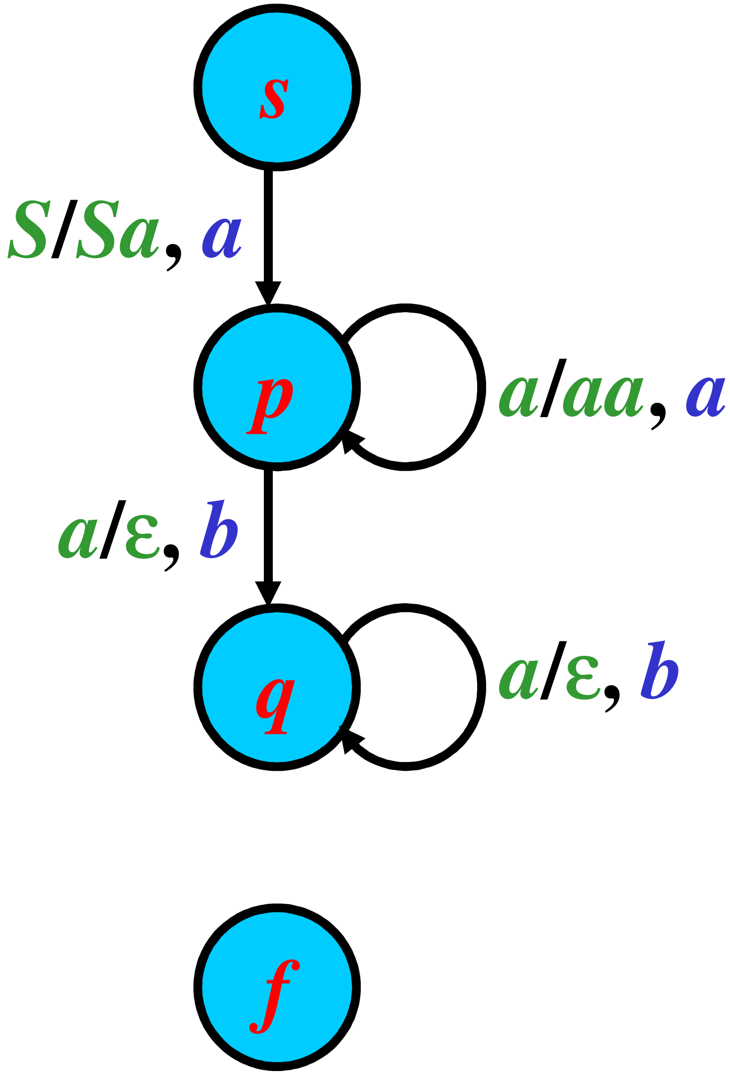
  $aqb \rightarrow q,$



Graphical Representation: Example

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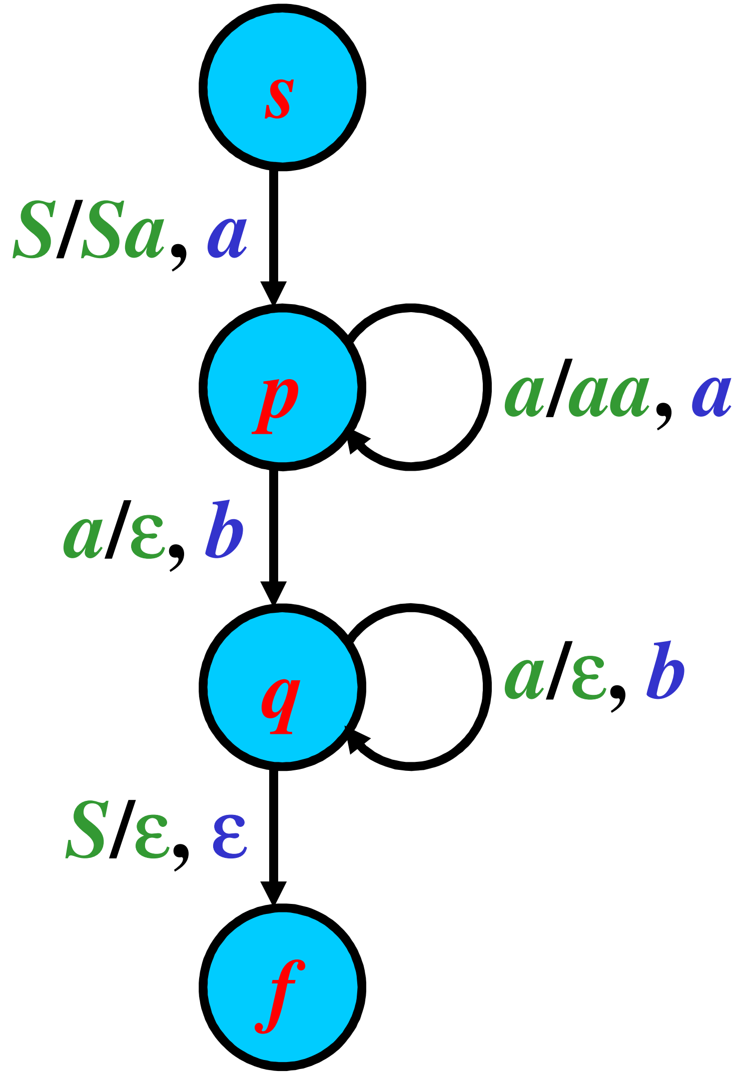

Graphical Representation: Example

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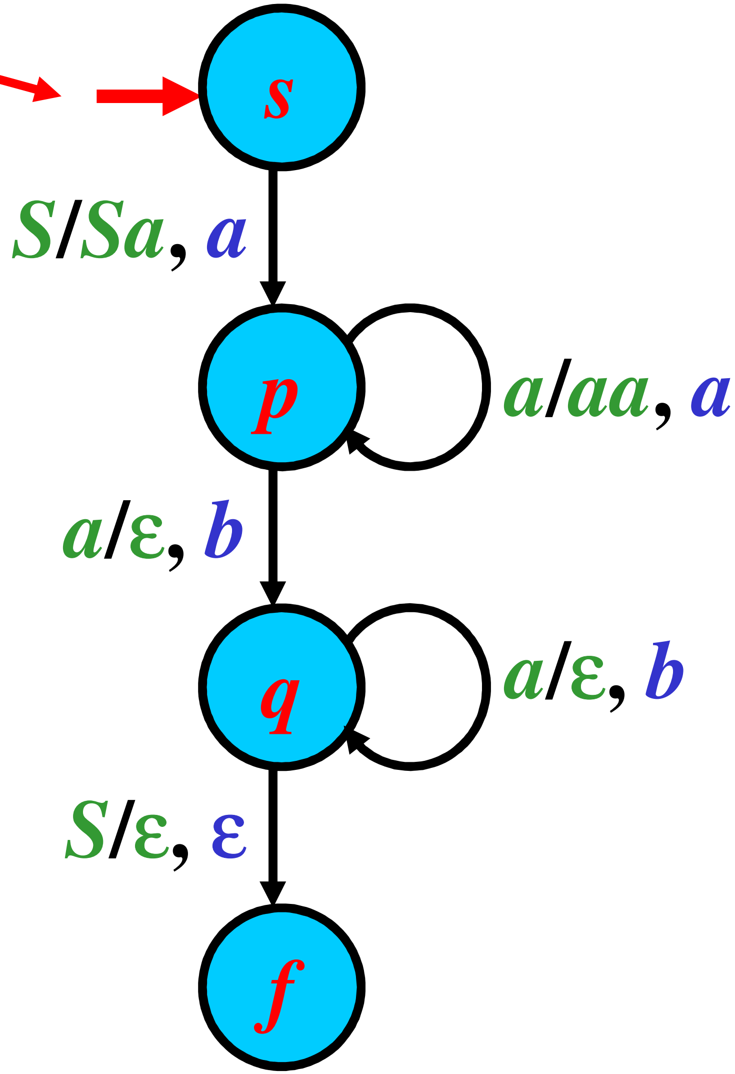

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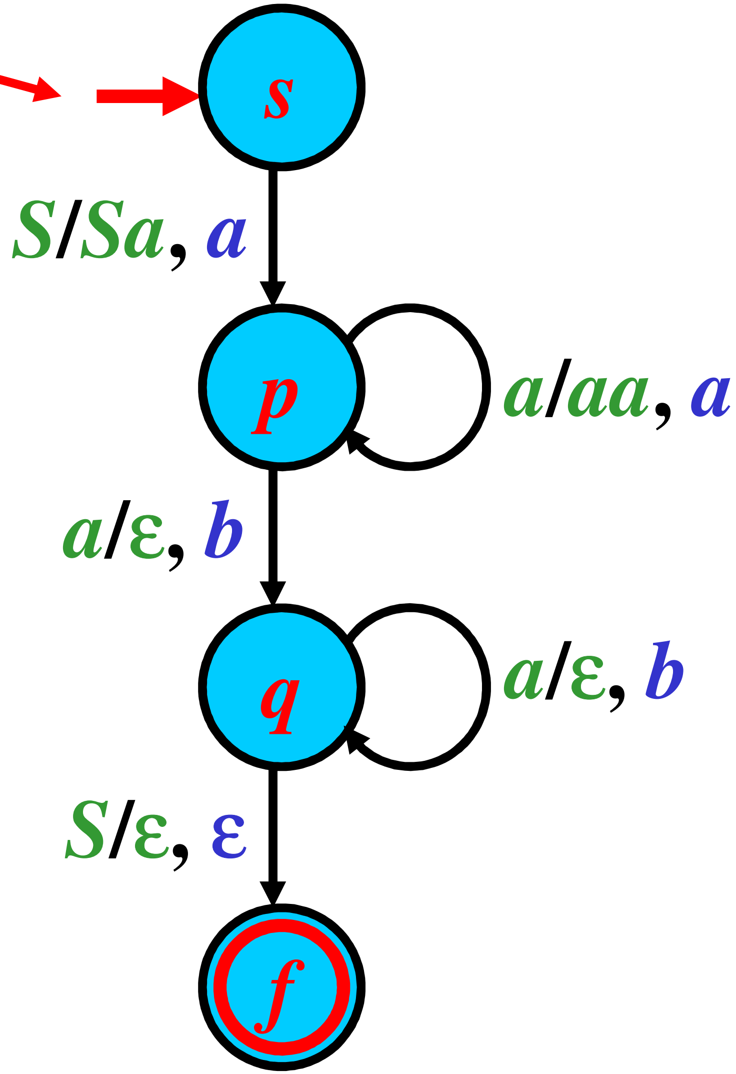
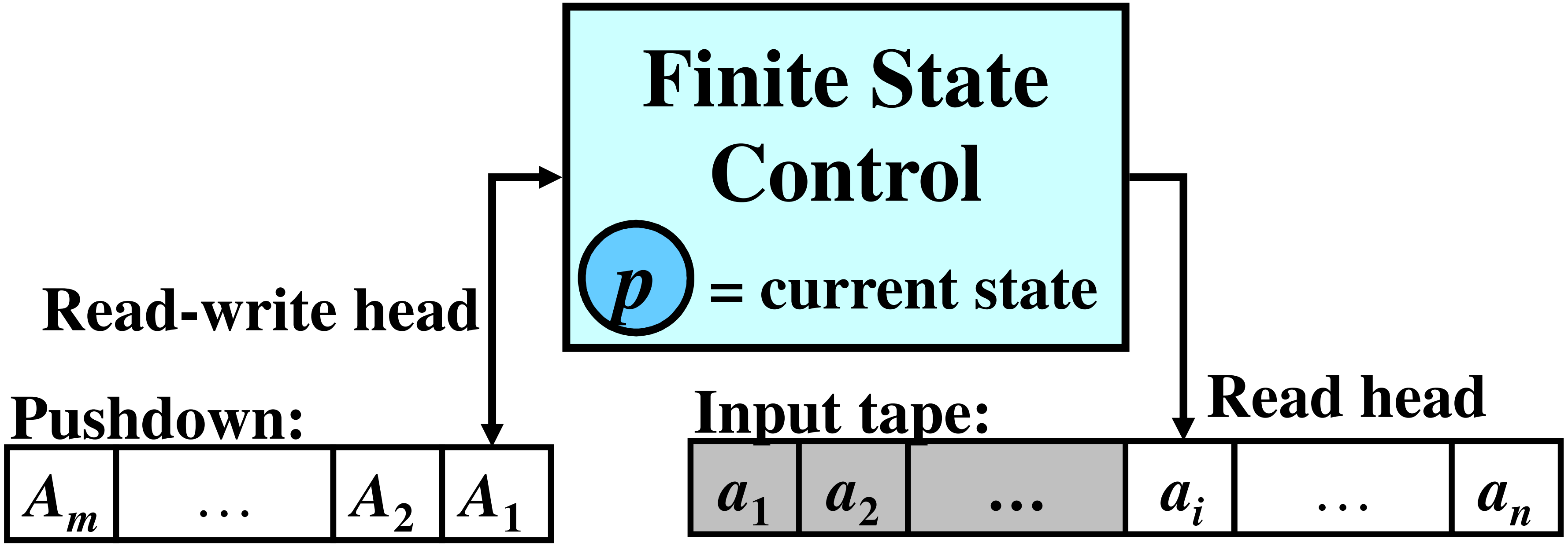


PDA Configuration

Gist: Instantaneous description of PDA

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA.
A configuration of M is a string $\chi \in \Gamma^* Q \Sigma^*$



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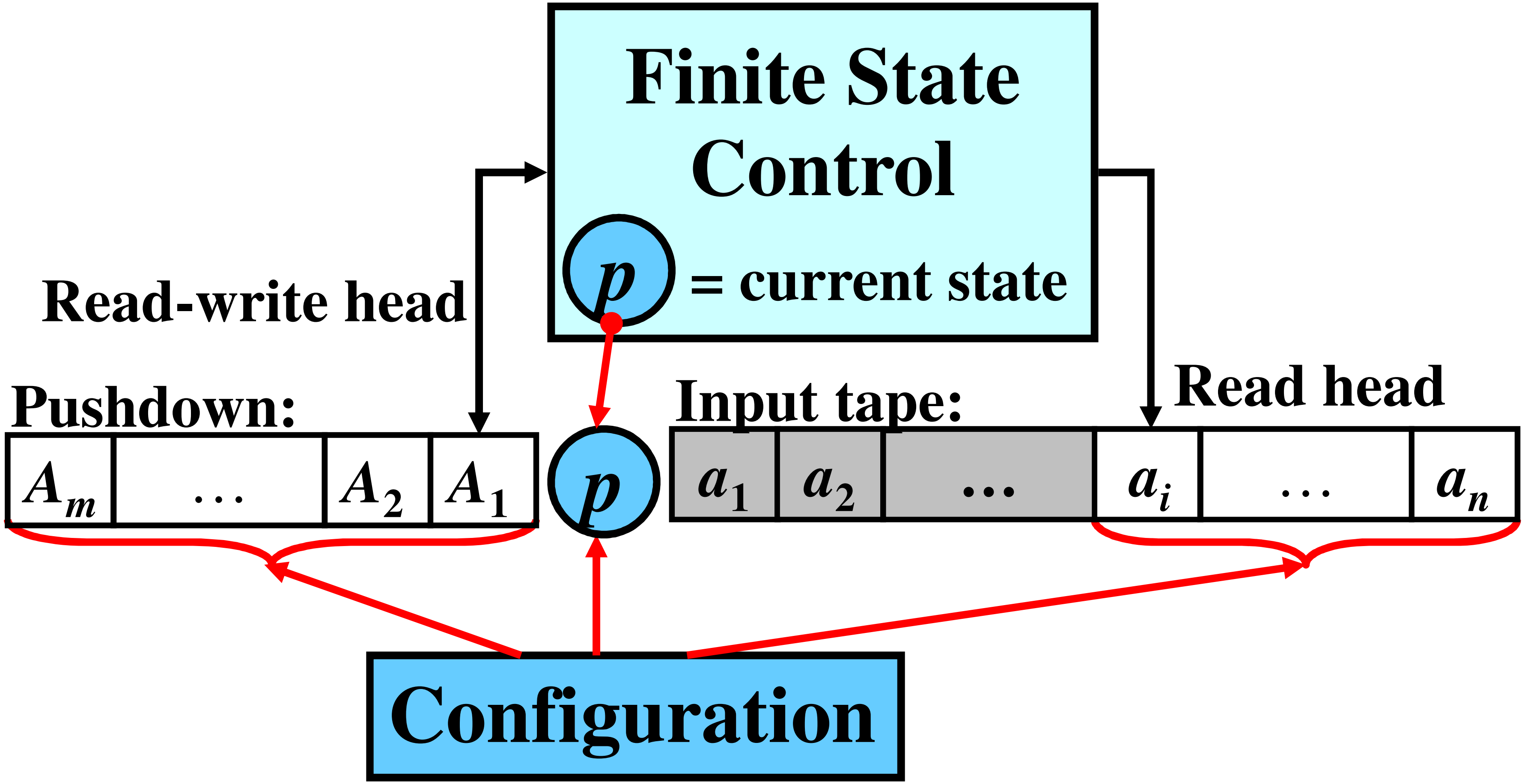

Move

Gist: A computational step made by a PDA

Definition: Let $xApay$ and $xwqy$ be two configurations of a PDA, M, where
$x, w \in \Gamma^*$, $A \in \Gamma$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and $y \in \Sigma^*$.
Let $r = Apa \rightarrow wq \in R$ be a rule. Then, M makes a *move* from $xApay$ to $xwqy$ according to r, written as $xApay \vdash xwqy$ [r] or, simply, $xApay \vdash xwqy$.

Note: if $a = \varepsilon$, no input symbol is read

Configuration:

Move

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Configuration:

Rule: $Apa \to wq$

Move

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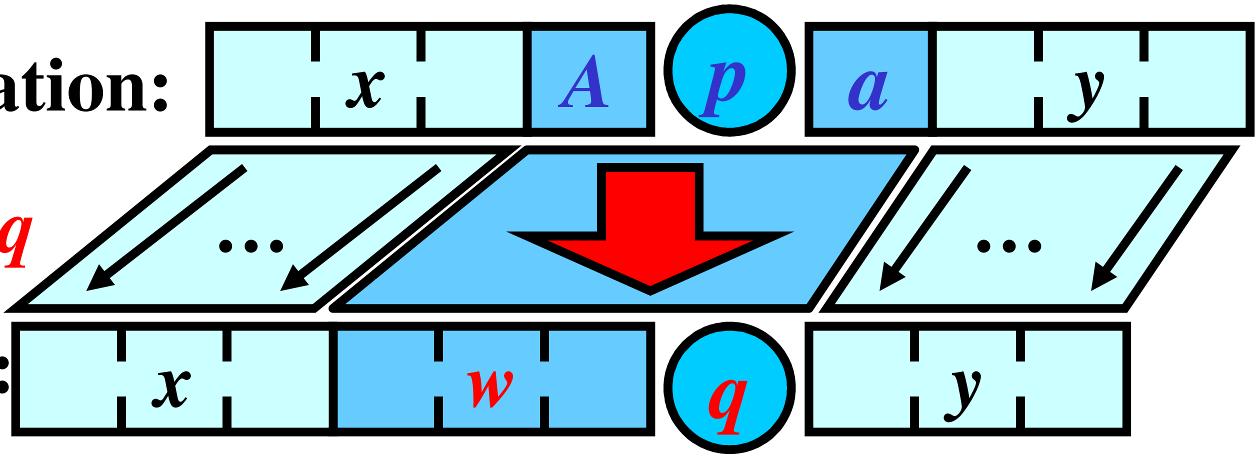
Configuration:

Rule: $Apa \rightarrow wq$

New configuration:



Sequence of Moves 1/2

Gist: Several consecutive computational steps

Definition: Let χ be a configuration. M makes *zero moves* from χ to χ; in symbols,

$$\chi \vdash^0 \chi\ [\varepsilon] \text{ or, simply, } \chi \vdash^0 \chi$$

Definition: Let $\chi_0, \chi_1, \ldots, \chi_n$ be a sequence of configurations, $n \geq 1$, and $\chi_{i-1} \vdash \chi_i\ [r_i]$, $r_i \in R$, for all $i = 1, \ldots, n$; that is,

$$\chi_0 \vdash \chi_1\ [r_1] \vdash \chi_2\ [r_2] \ \ldots \ \vdash \chi_n\ [r_n]$$

Then M makes *n moves* from χ_0 to χ_n,

$$\chi_0 \vdash^n \chi_n\ [r_1 \ldots r_n] \text{ or, simply, } \chi_0 \vdash^n \chi_n$$

Sequence of Moves 2/2

If $\chi_0 \vdash^n \chi_n$ [ρ] for some $n \geq 1$, then
$\chi_0 \vdash^+ \chi_n$ [ρ] or, simply, $\chi_0 \vdash^+ \chi_n$

If $\chi_0 \vdash^n \chi_n$ [ρ] for some $n \geq 0$, then
$\chi_0 \vdash^* \chi_n$ [ρ] or, simply, $\chi_0 \vdash^* \chi_n$

Example: Consider

$AApabc \vdash ABqbc$ [1: $Apa \rightarrow Bq$], and
$ABqbc \vdash ABCrc$ [2: $Bqb \rightarrow BCr$].

Then,
$AApabc \vdash^2 ABCrc$ [1 2],
$AApabc \vdash^+ ABCrc$ [1 2],
$AApabc \vdash^* ABCrc$ [1 2]

Accepted Language: Three Types

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA.

1) The *language that M accepts* ***by final state***, denoted by $\boldsymbol{L(M)_f}$, is defined as
$L(M)_f = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z \in \Gamma^*, f \in F\}$

2) The *language that M accepts* ***by empty pushdown***, denoted by $\boldsymbol{L(M)_\varepsilon}$, is defined as
$L(M)_\varepsilon = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z = \varepsilon, f \in Q\}$

3) The *language that M accepts* ***by final state and empty pushdown***, denoted by $\boldsymbol{L(M)_{f\varepsilon}}$, is defined as
$L(M)_{f\varepsilon} = \{w: w \in \Sigma^*, Ssw \vdash^* zf, z = \varepsilon, f \in F\}$

PDA: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

where:

- $Q = \{s, p, q, f\}$;
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  $apb \rightarrow q,$
  $aqb \rightarrow q,$
  $Sq \rightarrow f\}$
- $F = \{f\}$

Question: $aabb \in L(M)_{f\varepsilon}$**?**

S	s	a	a	b	b

$Ssaabb$

PDA: Example

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where:

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Question: $aabb \in L(M)_{f\varepsilon}$?

$\boxed{S}\ \boxed{s}\ \boxed{a}\boxed{a}\boxed{b}\boxed{b}$

Rule: $Ssa \rightarrow Sap$

$\boxed{S}\boxed{a}\ \boxed{p}\ \boxed{a}\boxed{b}\boxed{b}$

$Ssaabb \vdash Sapabb$

PDA: Example

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Question: $aabb \in L(M)_{f\varepsilon}$**?**

$\boxed{S}\ s\ \boxed{a}\boxed{a}\boxed{b}\boxed{b}$

Rule: $Ssa \rightarrow Sap$

$\boxed{S}\boxed{a}\ p\ \boxed{a}\boxed{b}\boxed{b}$

Rule: $apa \rightarrow aap$

$\boxed{S}\boxed{a}\boxed{a}\ p\ \boxed{b}\boxed{b}$

$Ssaabb \vdash Sapabb \vdash Saapbb$

PDA: Example

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$\boxed{S}\ s\ \boxed{a}\boxed{a}\boxed{b}\boxed{b}$

Rule: $Ssa \rightarrow Sap$

$\boxed{S}\boxed{a}\ p\ \boxed{a}\boxed{b}\boxed{b}$

Rule: $apa \rightarrow aap$

$\boxed{S}\boxed{a}\boxed{a}\ p\ \boxed{b}\boxed{b}$

Rule: $apb \rightarrow q$

$\boxed{S}\boxed{a}\ q\ \boxed{b}$

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb$

PDA: Example

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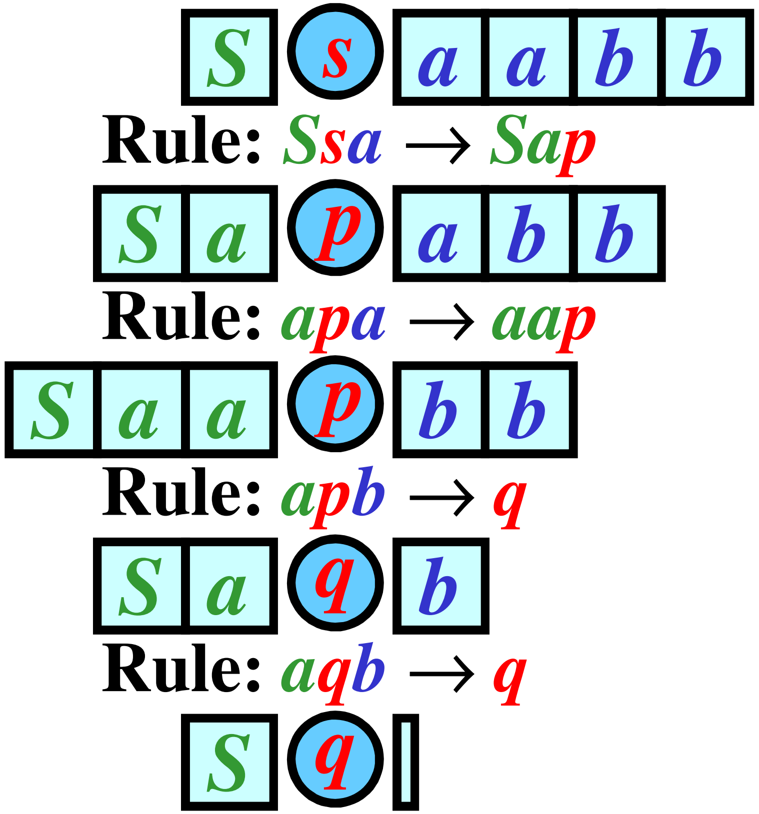

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb \vdash Sq$

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S s a a b b

Rule: $Ssa \rightarrow Sap$

S a p a b b

Rule: $apa \rightarrow aap$

S a a p b b

Rule: $apb \rightarrow q$

S a q b

Rule: $aqb \rightarrow q$

S q

Rule: $Sq \rightarrow f$

f

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb \vdash Sq \vdash f$

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Question: $aabb \in L(M)_{f\varepsilon}$?

S | s | $a\ a\ b\ b$

Rule: $Ssa \rightarrow Sap$

$S\ a$ | p | $a\ b\ b$

Rule: $apa \rightarrow aap$

$S\ a\ a$ | p | $b\ b$

Rule: $apb \rightarrow q$

$S\ a$ | q | b

Rule: $aqb \rightarrow q$

S | q |

Rule: $Sq \rightarrow f$

Empty pushdown | f |

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb \vdash Sq \vdash f$

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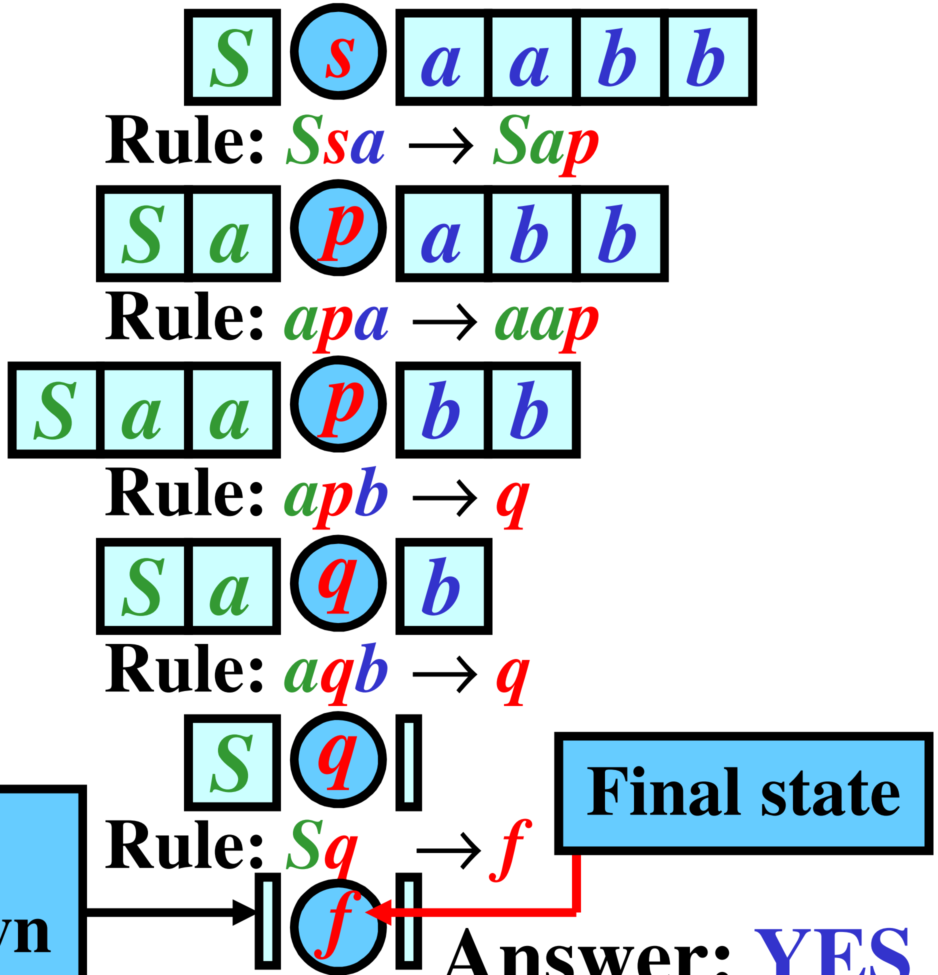

Answer: YES

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb \vdash Sq \vdash f$

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S s a a b b

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S a a p b b

Rule: $apb \rightarrow q$

S a q b

Rule: $aqb \rightarrow q$

S q

Rule: $Sq \rightarrow f$

f

Empty pushdown

Final state

Answer: YES

$Ssaabb \vdash Sapabb \vdash Saapbb \vdash Saqb \vdash Sq \vdash f$

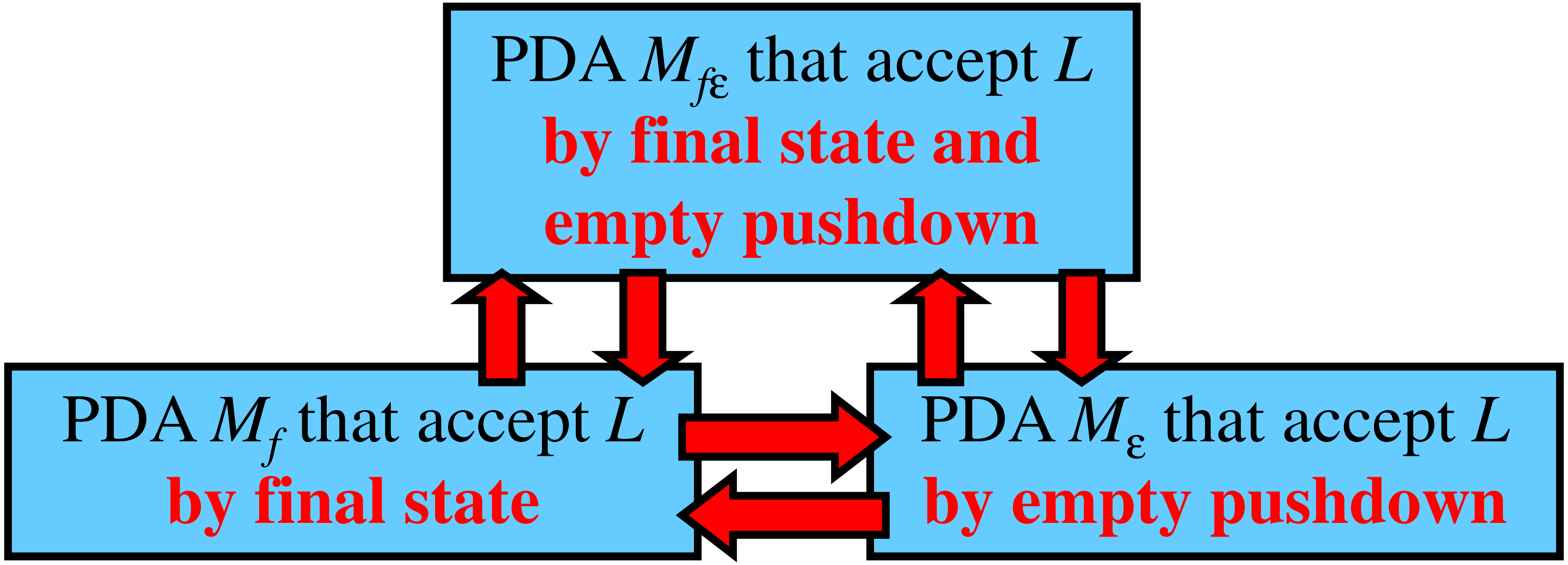
Note: $L(M)_f = L(M)_\varepsilon = L(M)_{f\varepsilon} = \{a^n b^n : n \geq 1\}$

Three Types of Acceptance: Equivalence

Theorem:

- $L = L(M_f)_f$ for a PDA $M_f \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for a PDA $M_{f\varepsilon}$
- $L = L(M_\varepsilon)_\varepsilon$ for a PDA $M_\varepsilon \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for a PDA $M_{f\varepsilon}$
- $L = L(M_f)_f$ for a PDA $M_f \Leftrightarrow L = L(M_\varepsilon)_\varepsilon$ for a PDA M_ε

Note: There exist these conversions:



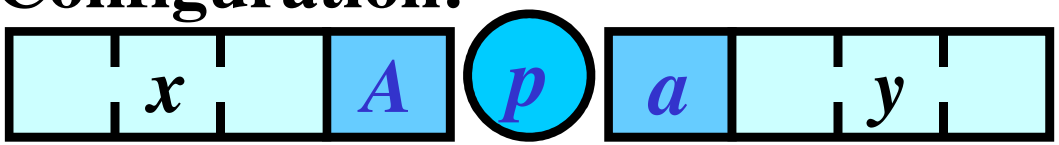
Deterministic PDA (DPDA)

Gist: Deterministic PDA makes no more than one move from any configuration.

Definition: Let $M = (Q, \Sigma, \Gamma, R, s, S, F)$ be a PDA. M is a *deterministic PDA* if for each rule $\boldsymbol{Apa} \rightarrow \boldsymbol{wq} \in R$, it holds that $R - \{\boldsymbol{Apa} \rightarrow \boldsymbol{wq}\}$ contains no rule with the left-hand side equal to $\boldsymbol{Apa}$ or $\boldsymbol{Ap}$.

Illustration:

Configuration:



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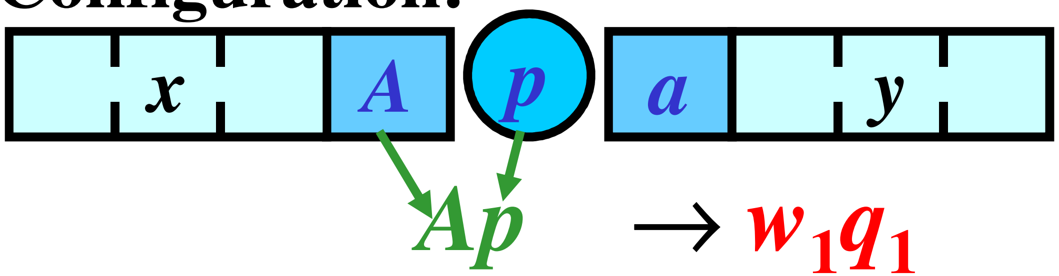

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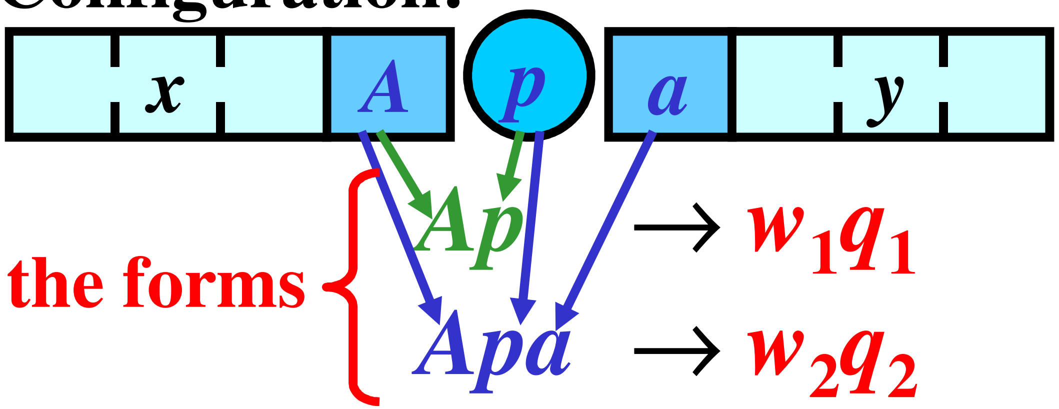

No more that one rule of the forms

PDAs are Stronger than DPDAs

Theorem: There exists no DPDA $M_{f\varepsilon}$ that accepts
$$L = \{xy: x, y \in \Sigma^*, y = reversal(x)\}$$

Proof: See page 431 in [Meduna: Automata and Languages]

Illustration:

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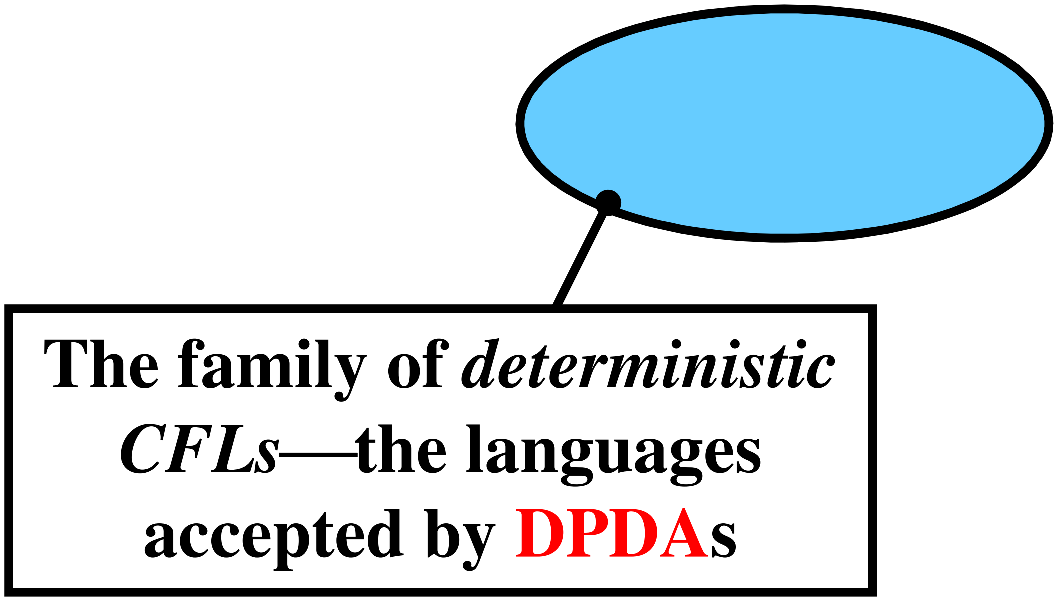

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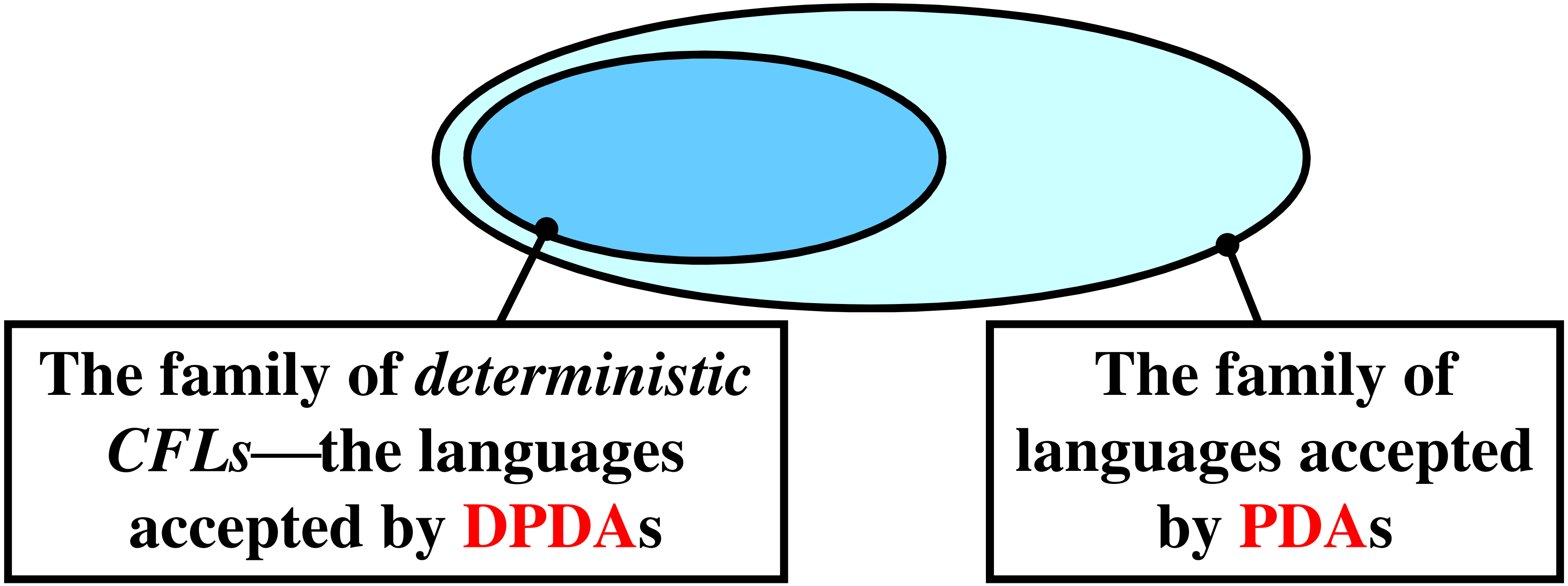

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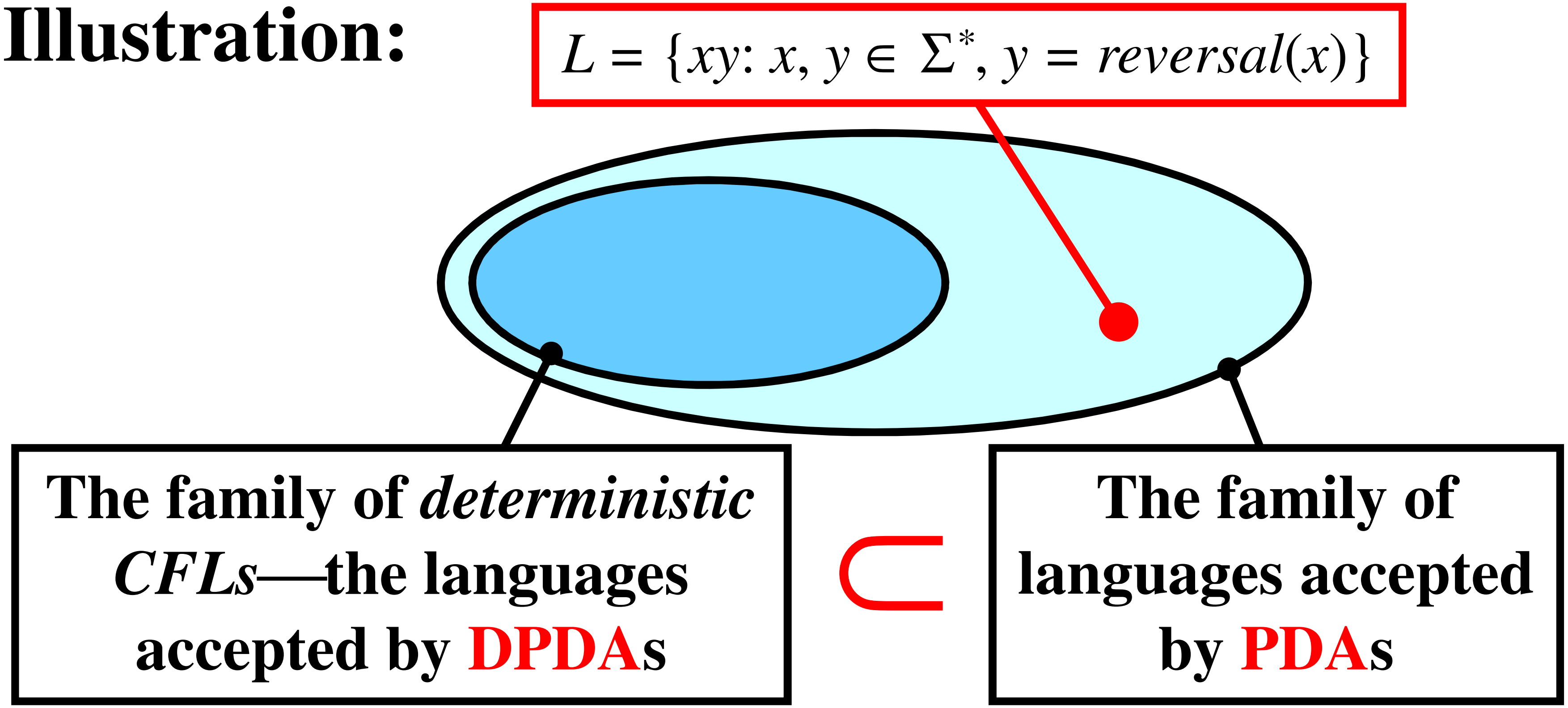


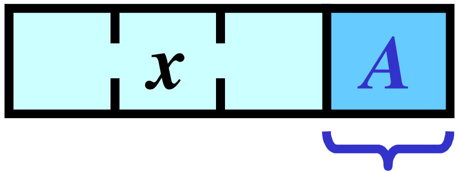
Extended PDA (EPDA)

Gist: The pushdown top of an EPDA represents a string rather than a single symbol.

Definition: *An Extended Pushdown automaton* (EPDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where $Q, \Sigma, \Gamma, s, S, F$ are defined as in an PDA and R is a *finite set of rules* of the form: $vpa \rightarrow wq$, where $v, w \in \Gamma^*$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$

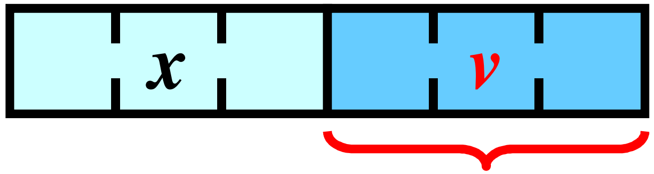
Illustration:

Pushdown of PDA:



PDA has a single symbols as the pushdown top

Pushdown of EPDA:

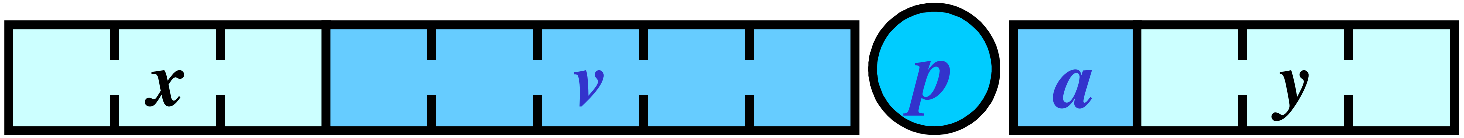


EPDA has a string as the pushdown top

Move in EPDA

Definition: Let $\boldsymbol{xvpay}$ and $\boldsymbol{xwqy}$ be two configurations of an EPDA, M, where $\boldsymbol{x}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \Gamma^*$, $\boldsymbol{p}$, $\boldsymbol{q} \in Q$, $\boldsymbol{a} \in \Sigma \cup \{\varepsilon\}$, and $\boldsymbol{y} \in \Sigma^*$. Let $\boldsymbol{r} = \boldsymbol{vpa} \rightarrow \boldsymbol{wq} \in R$ be a rule. Then, M makes a *move* from $\boldsymbol{xvpay}$ to $\boldsymbol{xwqy}$ according to $\boldsymbol{r}$, written as $\boldsymbol{xvpay} \vdash \boldsymbol{xwqy}$ $[\boldsymbol{r}]$ or $\boldsymbol{xvpay} \vdash \boldsymbol{xwqy}$.

Configuration:



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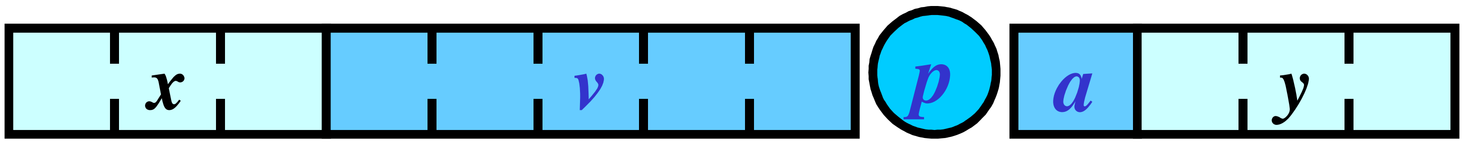
Configuration:



Rule: $vpa \rightarrow wq$

Move in EPDA

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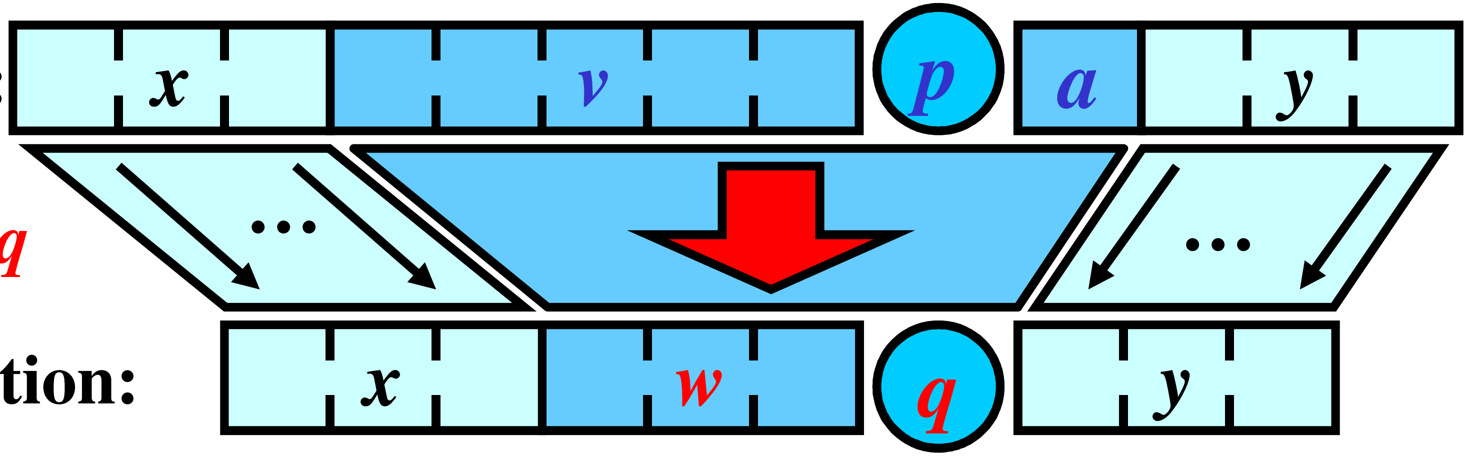
Configuration:

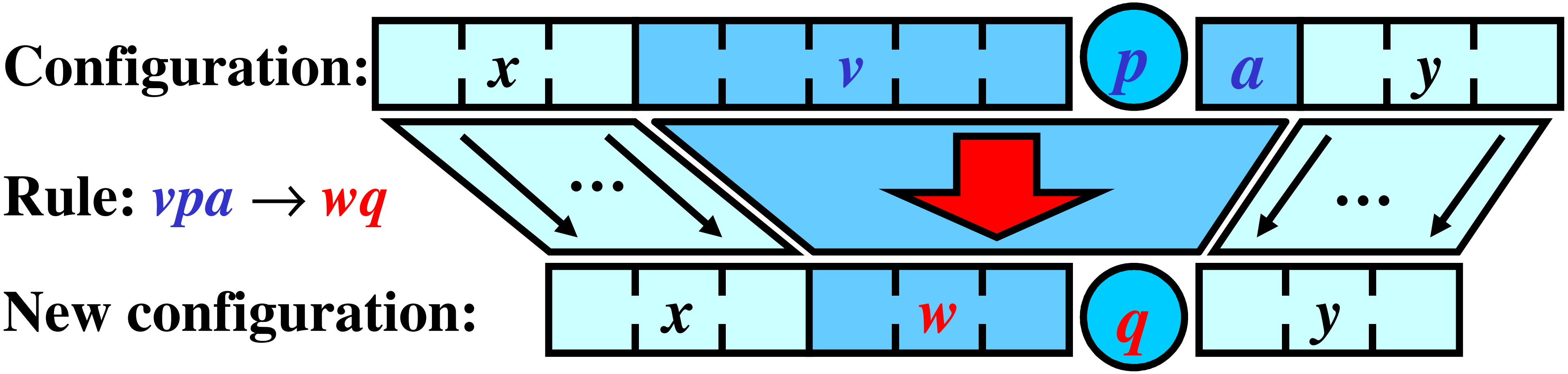
Rule: $vpa \rightarrow wq$

New configuration:



Move in EPDA

Definition: Let $xvpay$ and $xwqy$ be two configurations of an EPDA, M, where $x, v, w \in \Gamma^*$, $p, q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and $y \in \Sigma^*$. Let $r = vpa \to wq \in R$ be a rule. Then, M makes a *move* from $xvpay$ to $xwqy$ according to r, written as $xvpay \vdash xwqy$ [r] or $xvpay \vdash xwqy$.



Note: $\vdash^n$, $\vdash^+$, $\vdash^*$, $L(M)_f$, $L(M)_\varepsilon$, and $L(M)_{f\varepsilon}$ are defined analogically to the corresponding definitions for PDA.

EPDA: Example

$M = (Q, \Sigma, \Gamma, R, s, S, F)$

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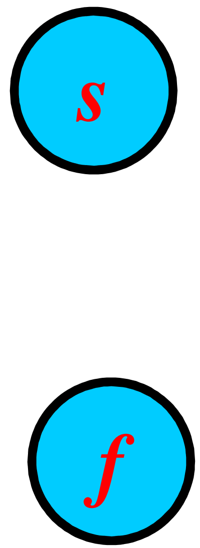

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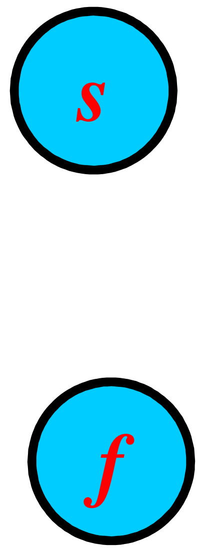

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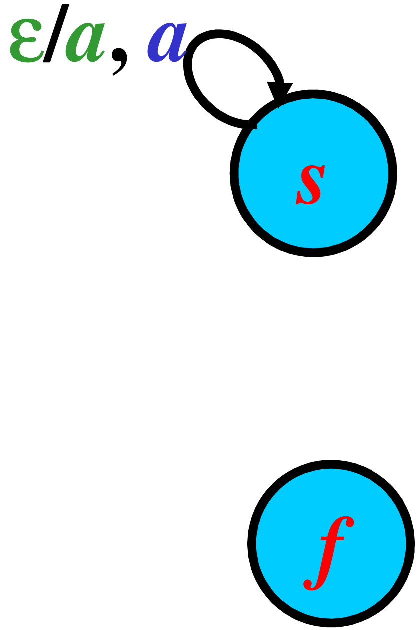

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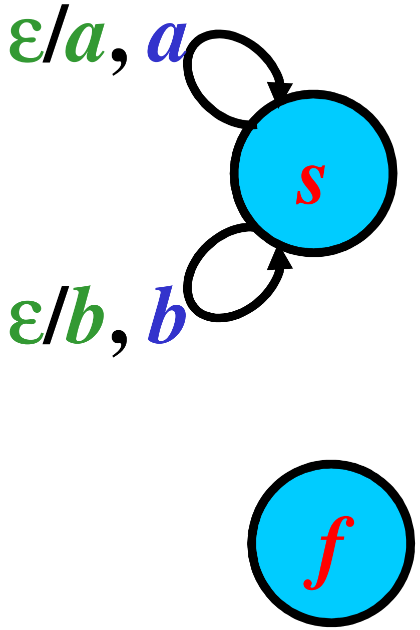

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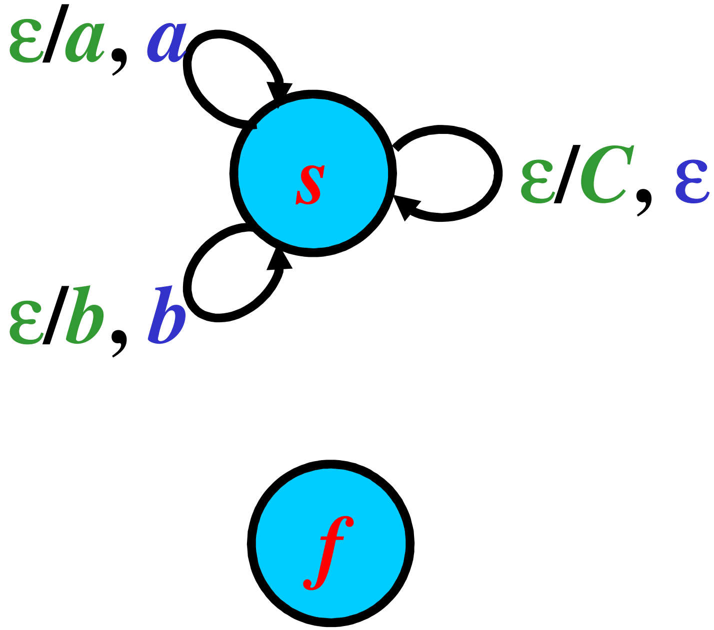

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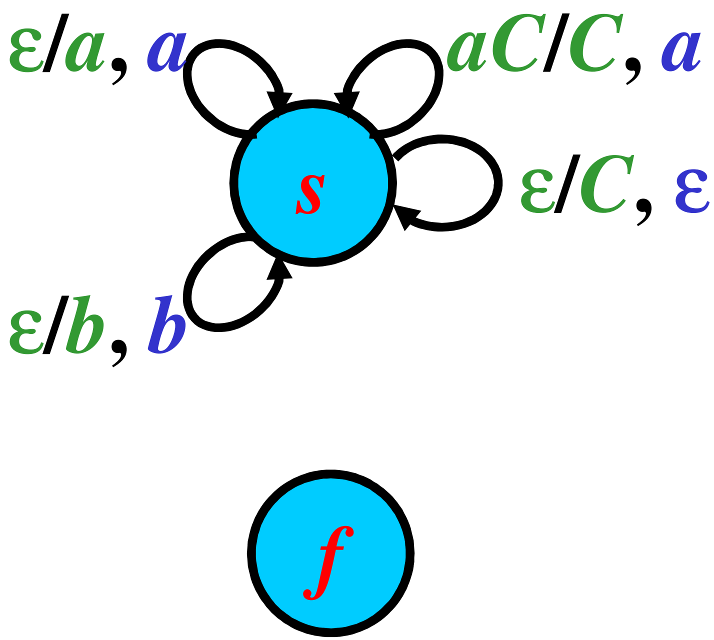

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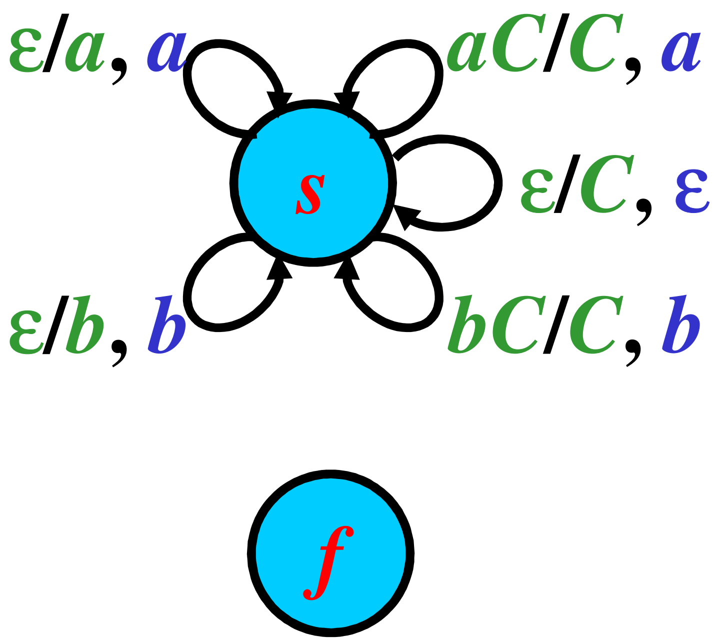

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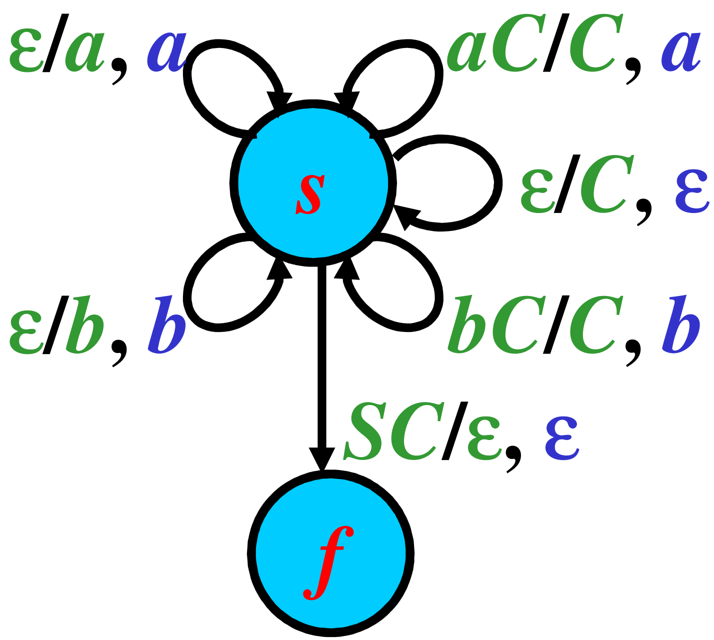

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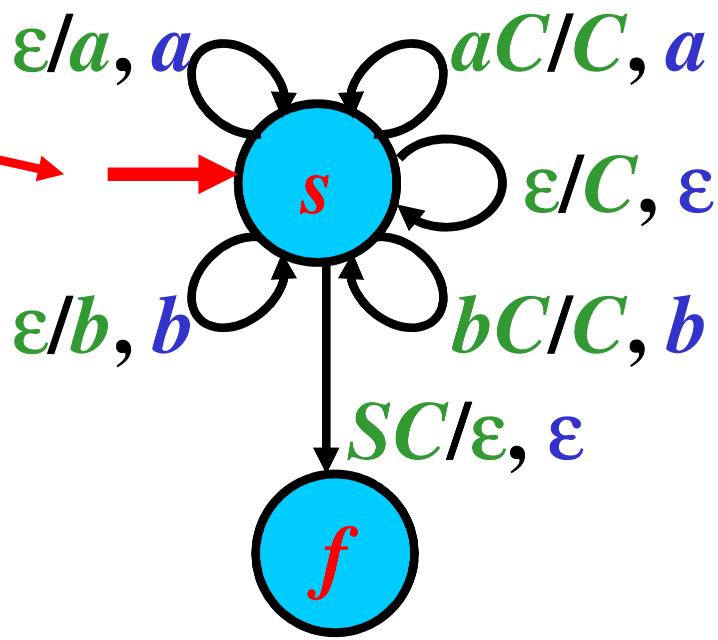

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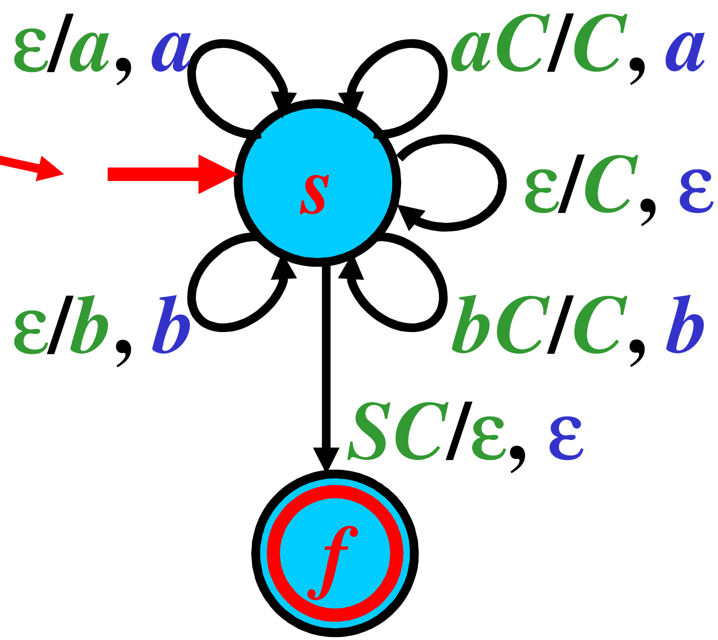

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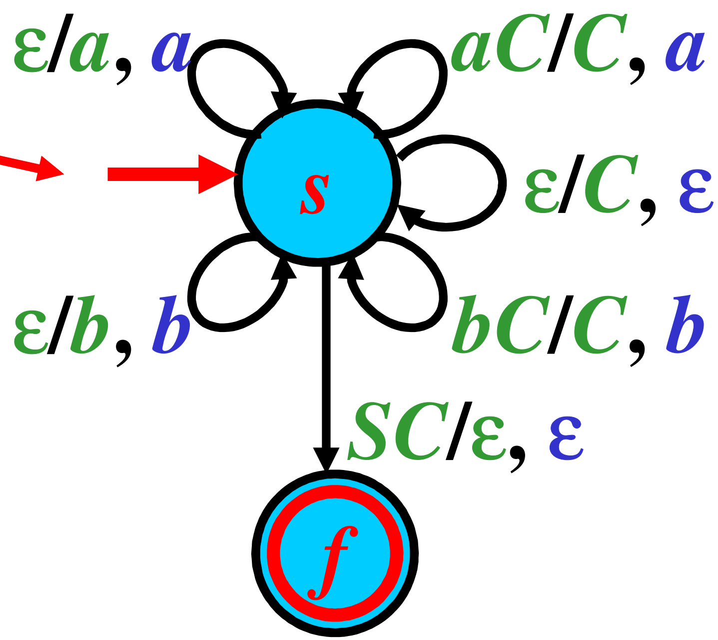

Question: $abba \in L_{f\varepsilon}(M)$?

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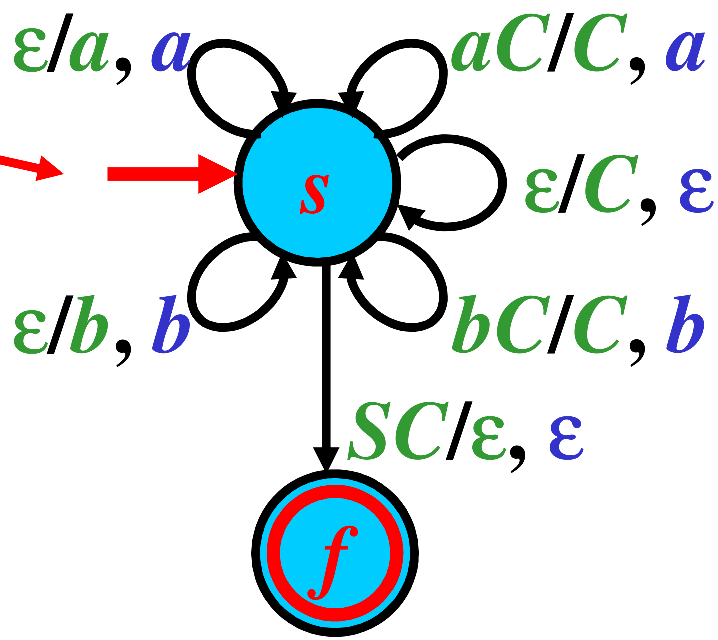

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$S\underline{sa}bba$

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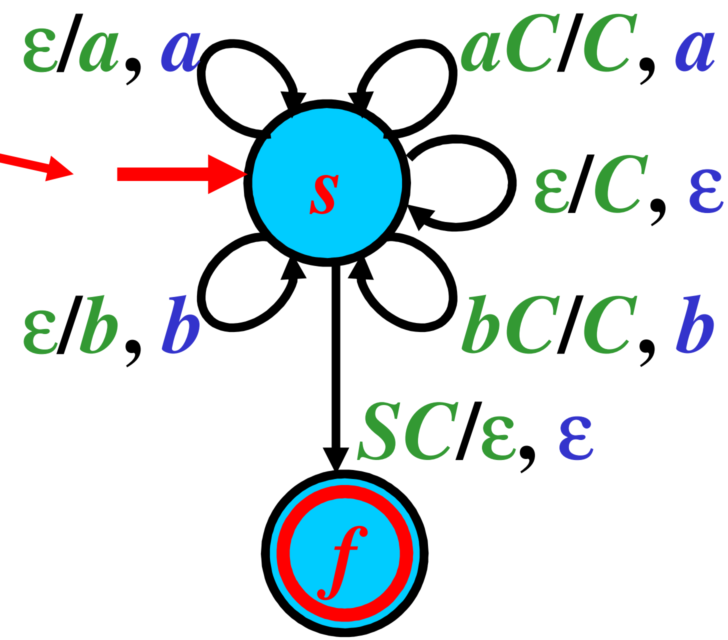

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$Ssabba \vdash Sasbba$

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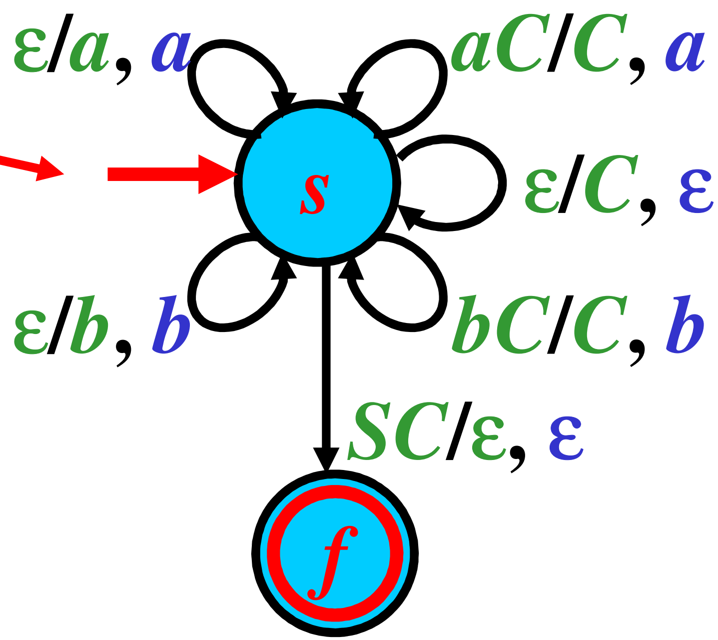

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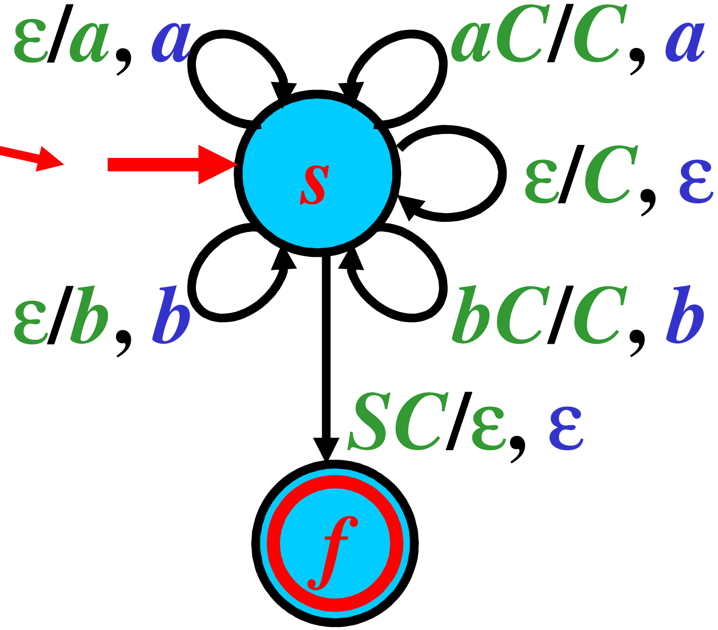

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$\vdash SabCsba$

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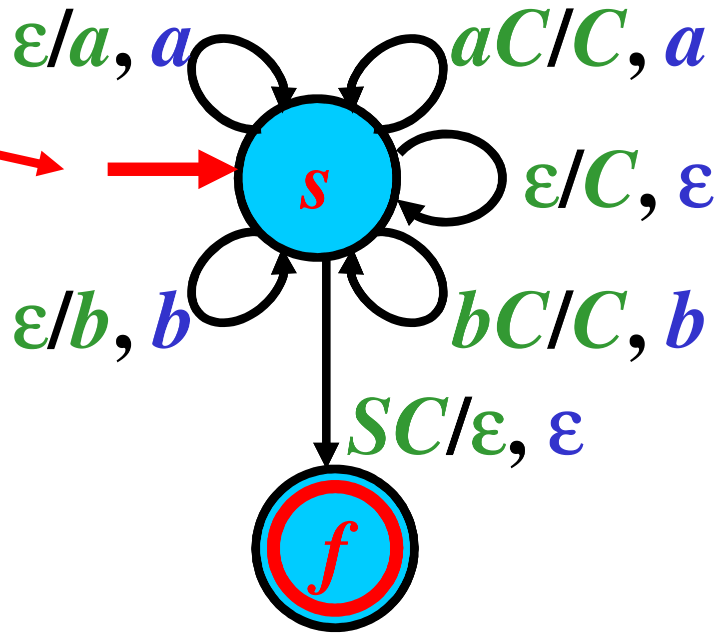

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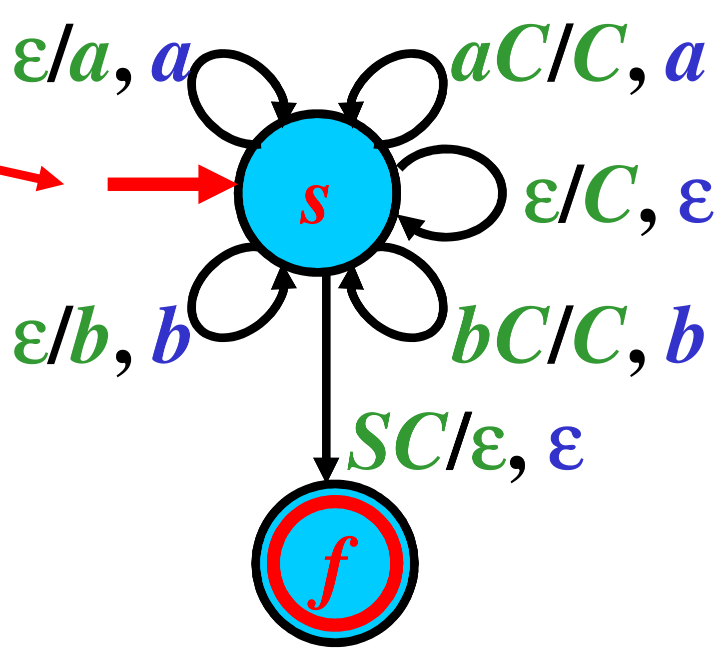

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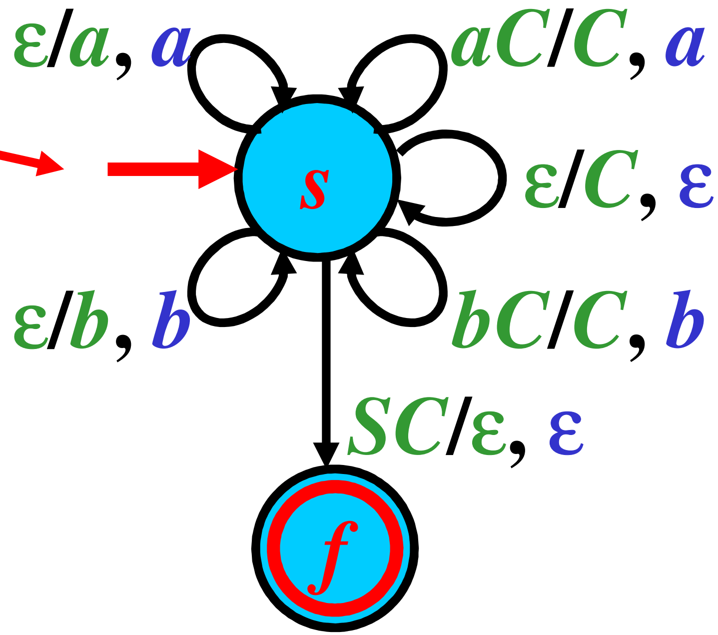

Question: $abba \in L_{f\varepsilon}(M)$?

$\underline{S}\underline{s}\underline{a}bba \vdash Sa\underline{s}\underline{b}ba \vdash Sab\underline{s}ba$
$\vdash Sab\underline{C}\underline{s}\underline{b}a \vdash Sa\underline{C}\underline{s}\underline{a}$
$\vdash \underline{SC}\underline{s} \vdash f$

Answer: YES

EPDA: Example

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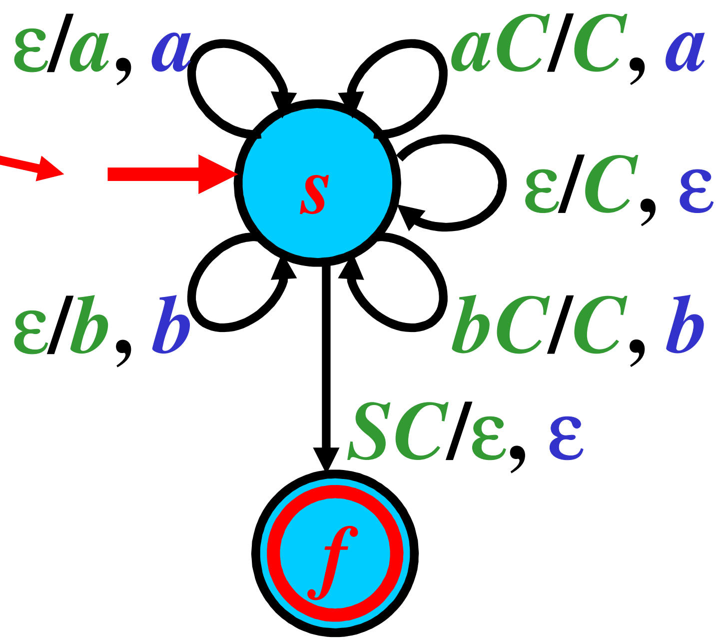

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Answer: YES

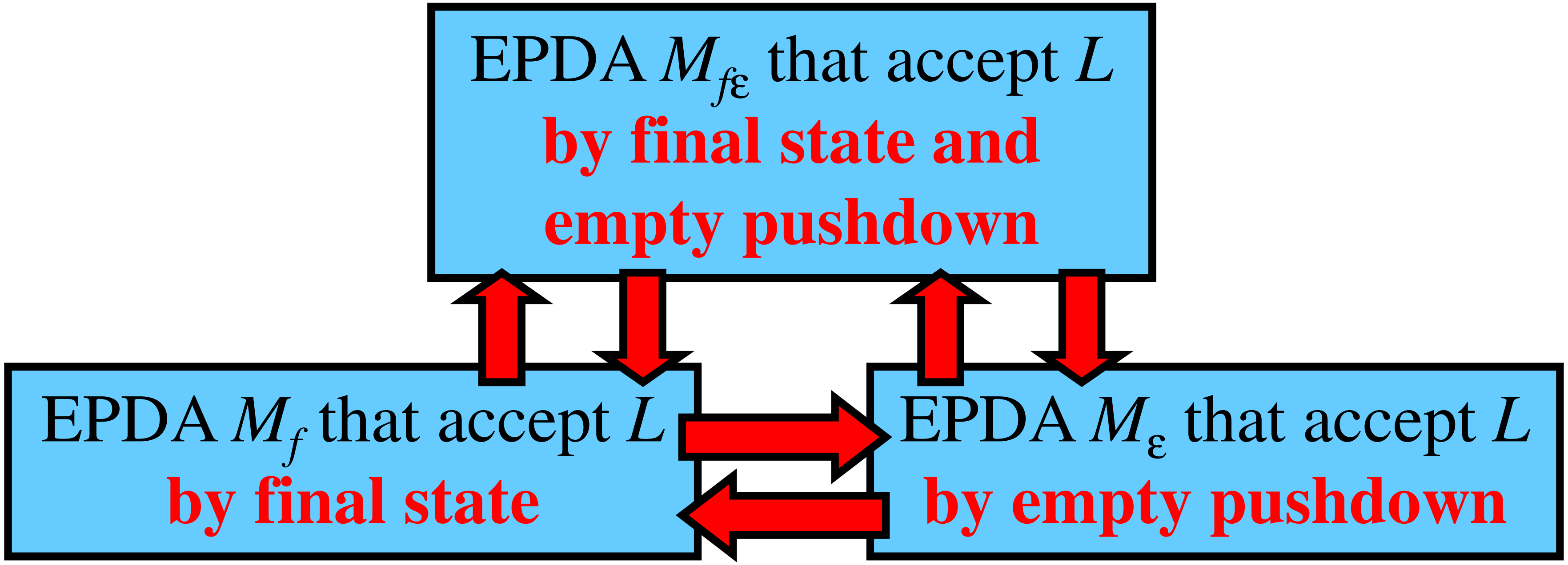
Note: $L(M)_f = L(M)_\varepsilon = L(M)_{f\varepsilon} = \{xy: x, y \in \Sigma^*, y = \text{reversal}(x)\}$

Three Types of Acceptance: Equivalence

Theorem:

- $L = L(M_f)_f$ for an EPDA $M_f \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for an EPDA $M_{f\varepsilon}$
- $L = L(M_\varepsilon)_\varepsilon$ for an EPDA $M_\varepsilon \Leftrightarrow L = L(M_{f\varepsilon})_{f\varepsilon}$ for an EPDA $M_{f\varepsilon}$
- $L = L(M_f)_f$ for an EPDA $M_f \Leftrightarrow L = L(M_\varepsilon)_\varepsilon$ for an EPDA M_ε

Note: There exist these conversion:



EPDAs and PDAs are Equivalent

Theorem: For every EPDA M, there is a PDA M', and $L(M)_f = L(M')_f$.

Proof: See page 419 in [Meduna: Automata and Languages]

Illustration:

EPDAs and PDAs are Equivalent

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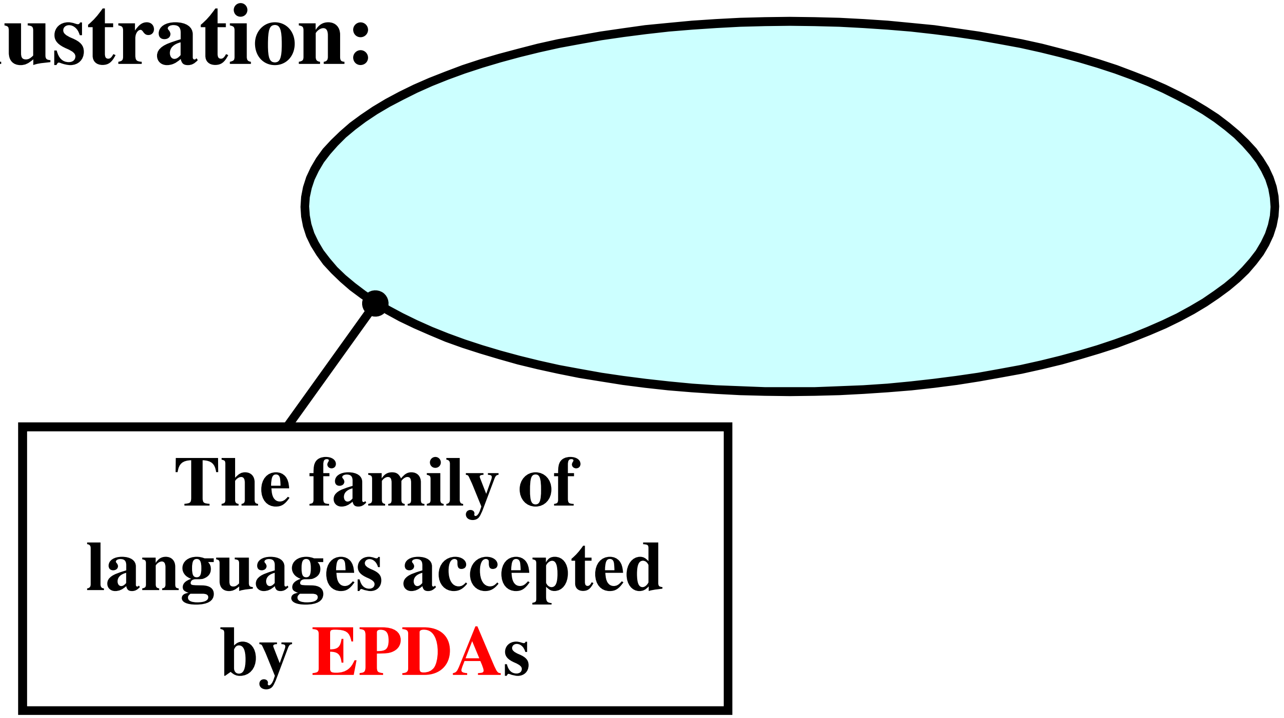
Illustration:



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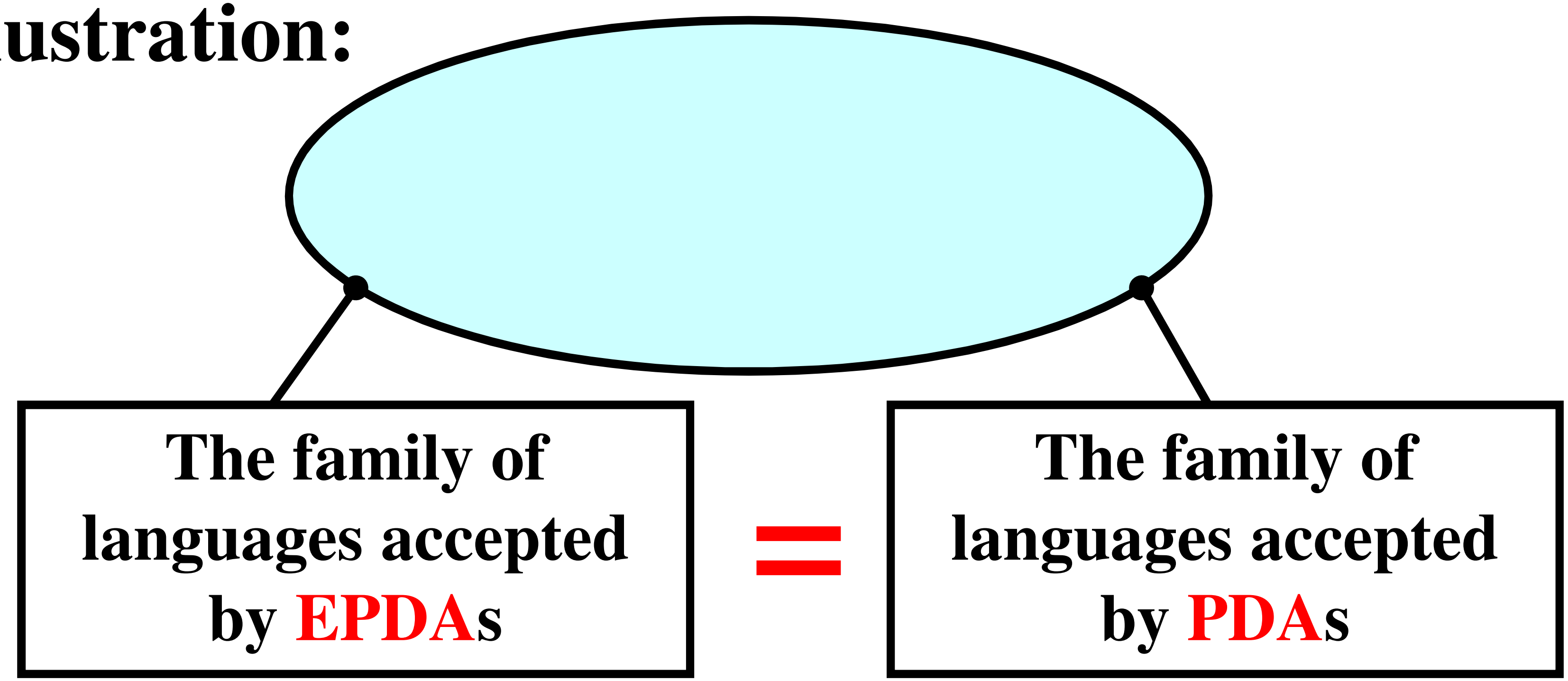
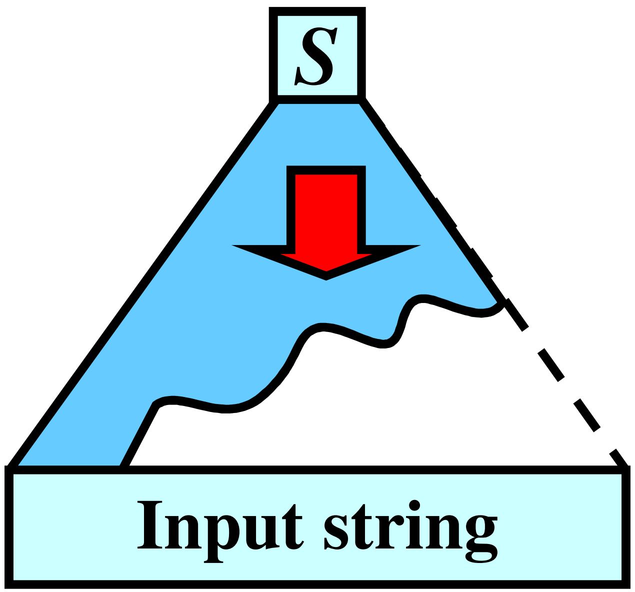


EPDAs and PDAs as Parsing Models for CFGs

Gist: An EPDA or a PDA can simulate the construction of a derivation tree for a CFG

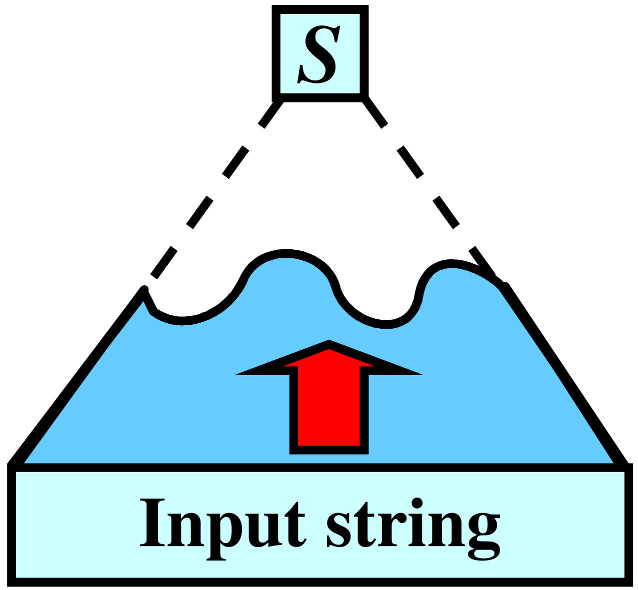
- **Two basic approaches:**

1) Top-Down Parsing



From *S* towards the input string

2) Bottom-Up Parsing

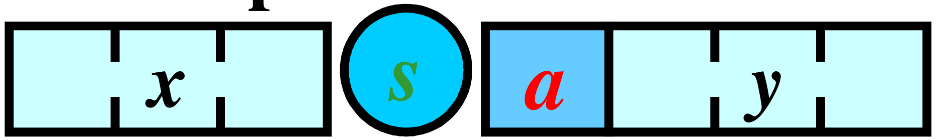


From the input string towards *S*

EPDAs as Models of Bottom-Up Parsers 1/2

Gist: An EPDA *M* underlies a bottom-up parser

1) *M* contains *shift* rules that copy the input symbols onto the pushdown:



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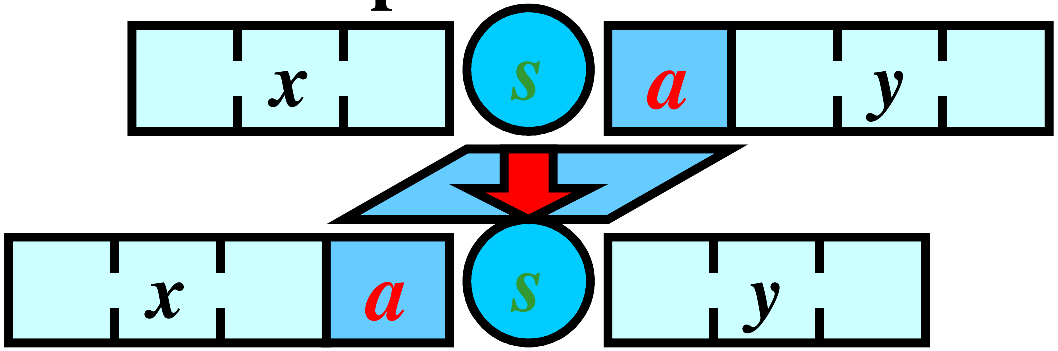

for every $a \in \Sigma$:

add $sa \rightarrow as$ to R;

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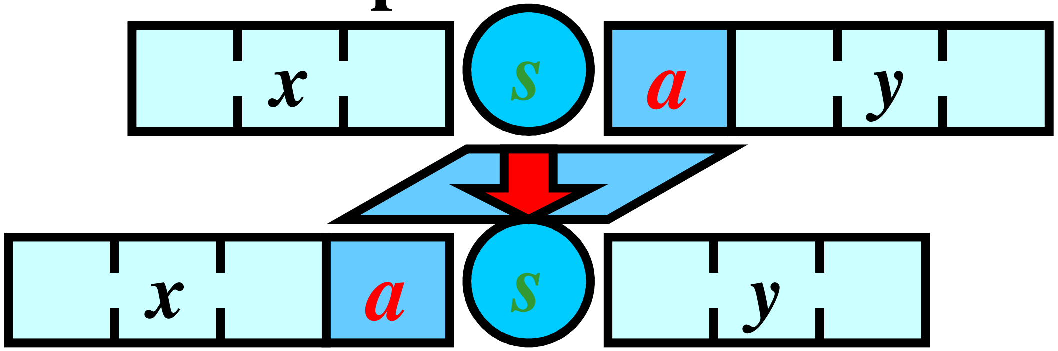

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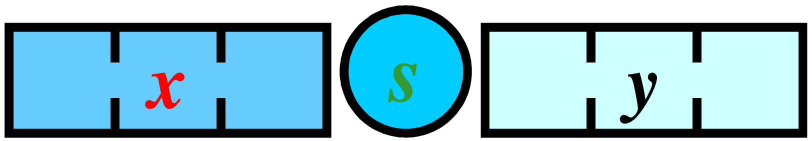
2) *M* contains *reduction* rules that simulate the application of a grammatical rule in reverse:



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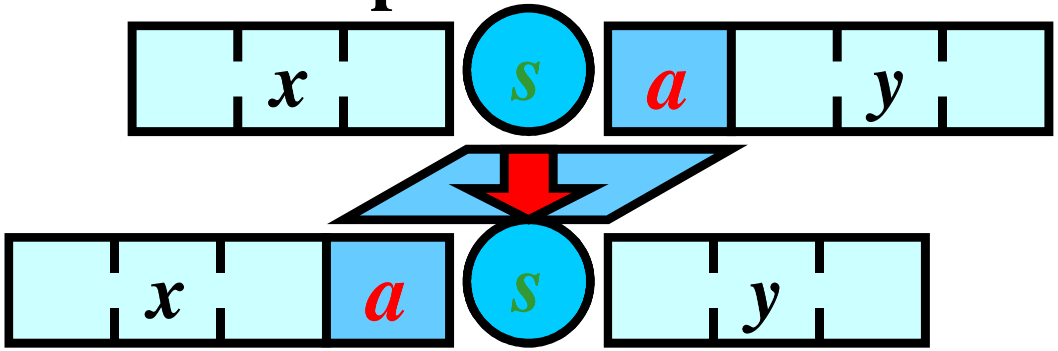

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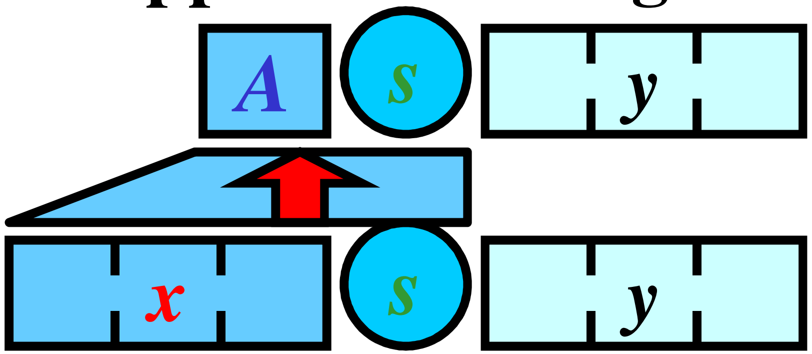

for every $A \rightarrow x \in P$ in G:
add $xs \rightarrow As$ to R;

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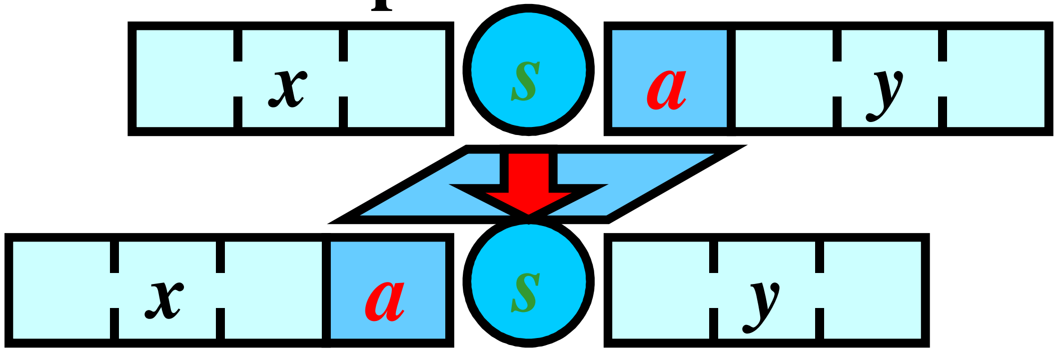

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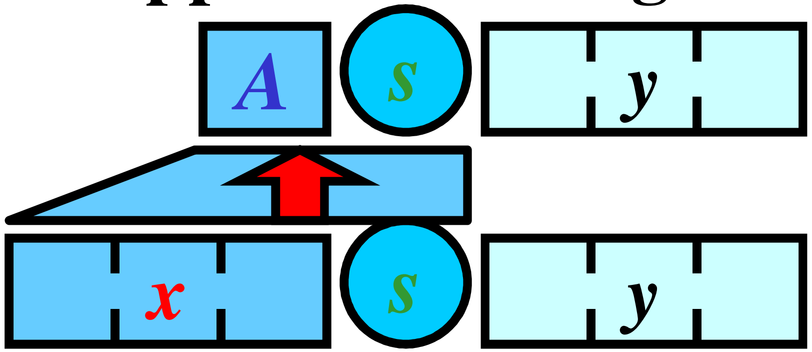
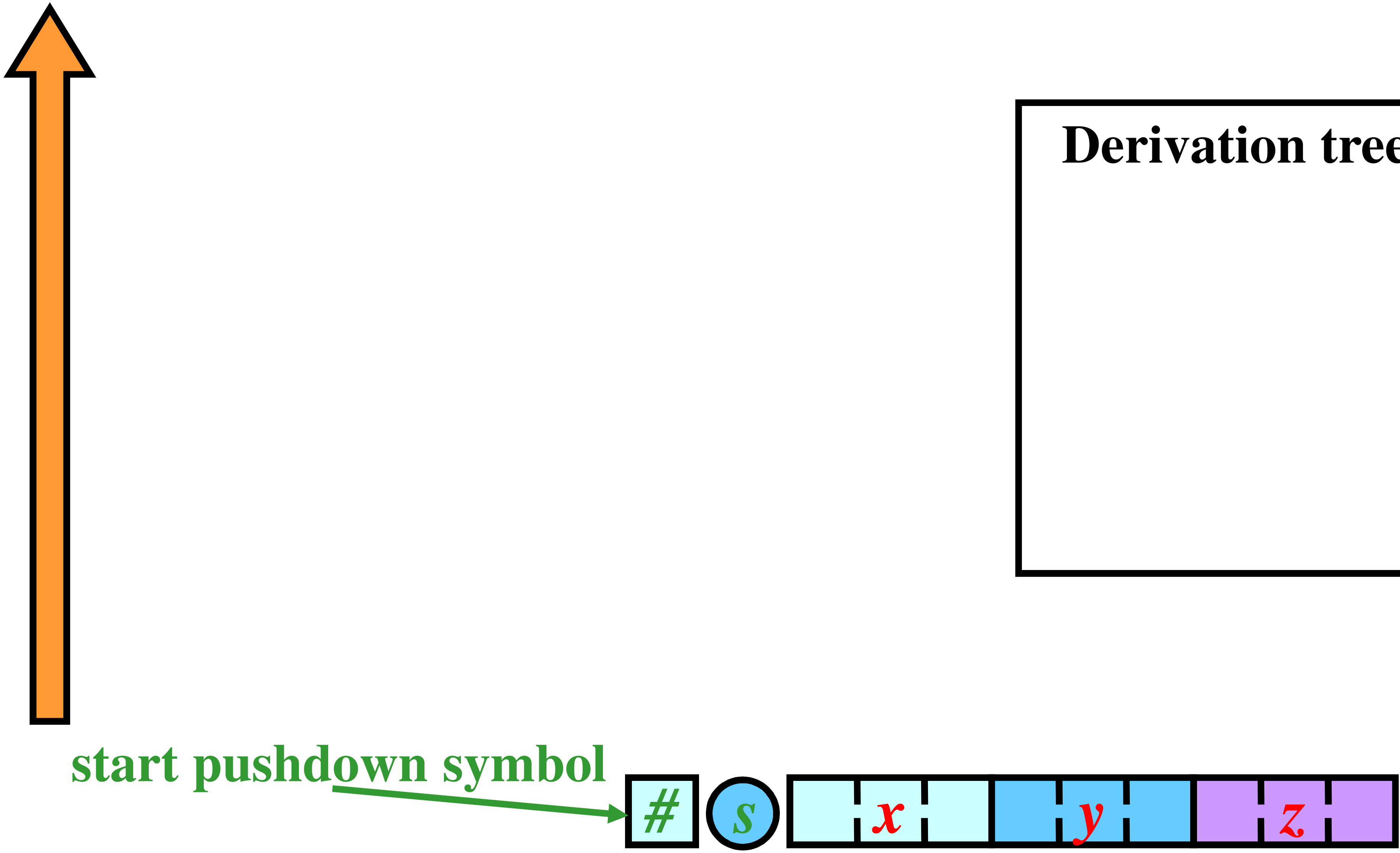


for every $A \rightarrow x \in P$ in G:
add $xs \rightarrow As$ to R;

3) *M* also contains the rule $\#Ss \rightarrow f$ that takes *M* to a final state f

EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



EPDAs as Models of Bottom-Up Parsers 2/2

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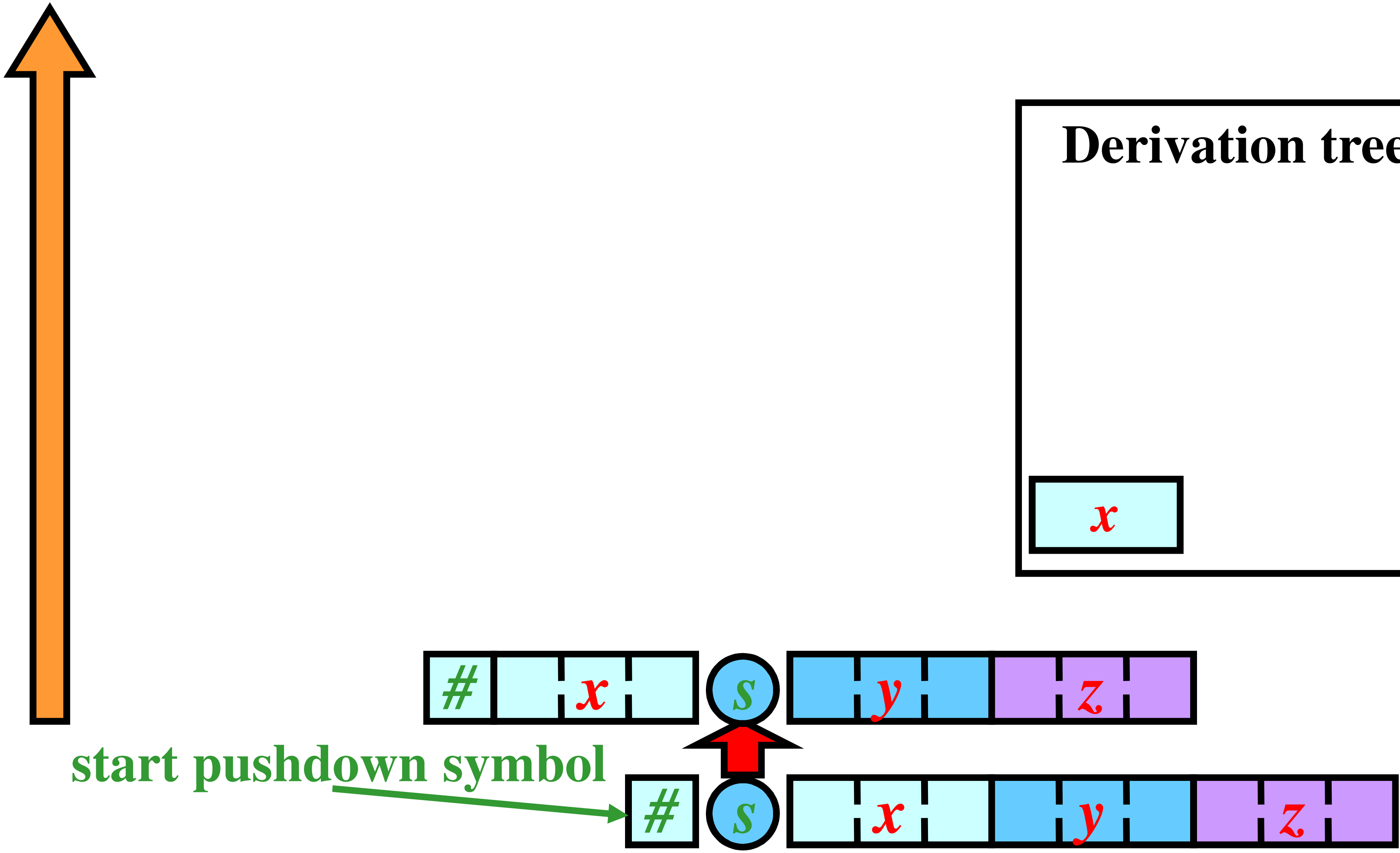

EPDAs as Models of Bottom-Up Parsers 2/2

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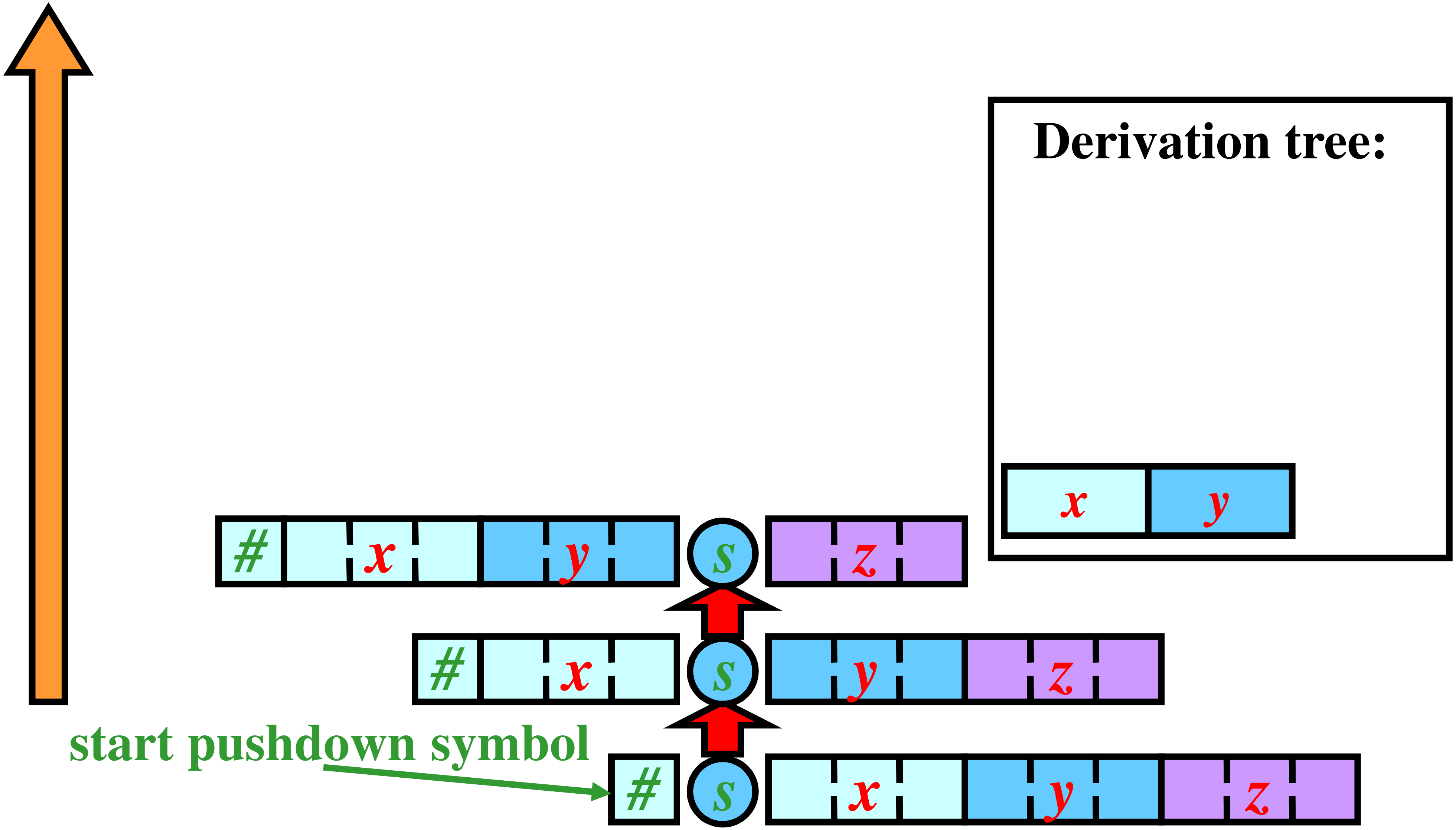

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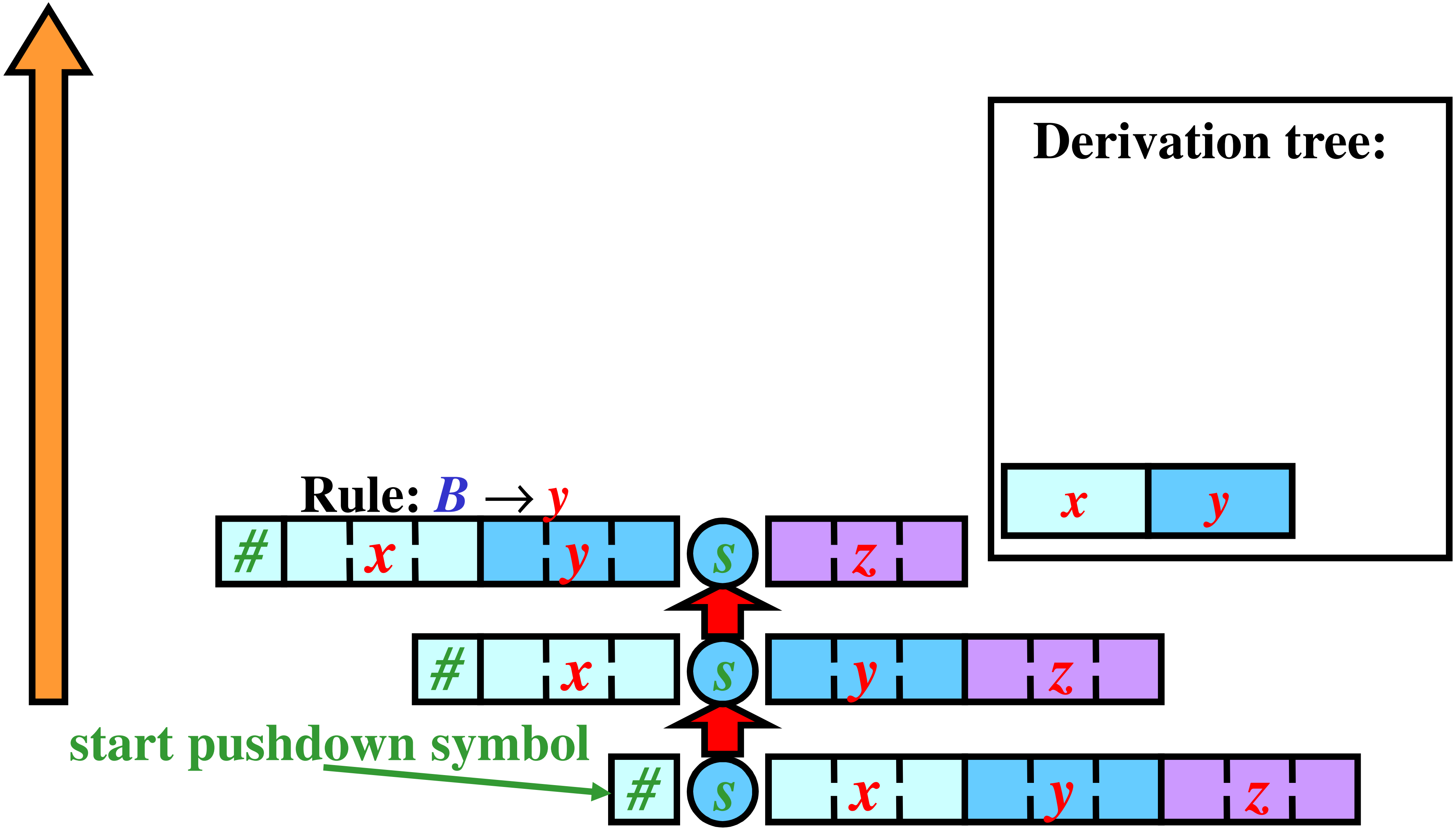

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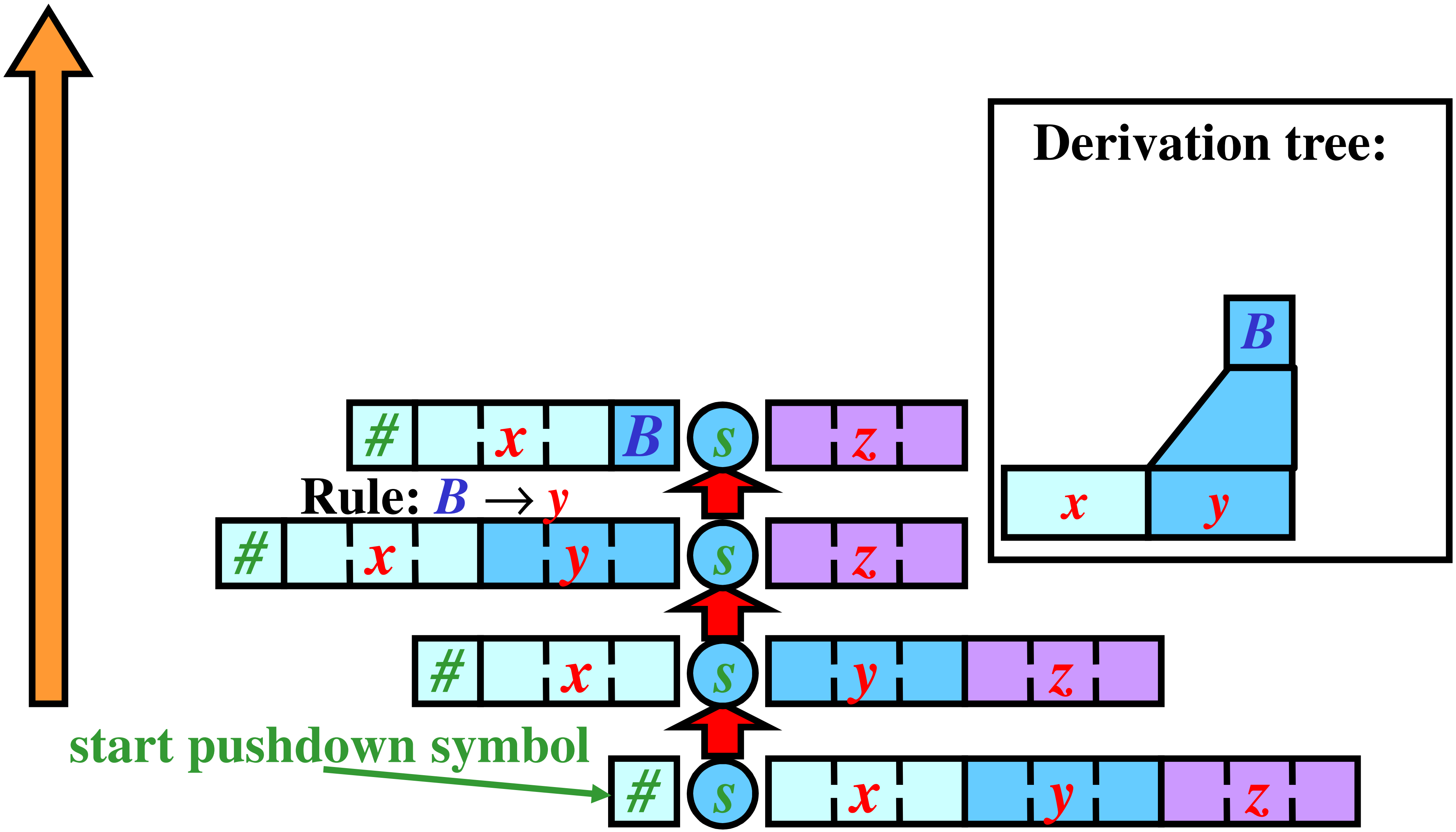
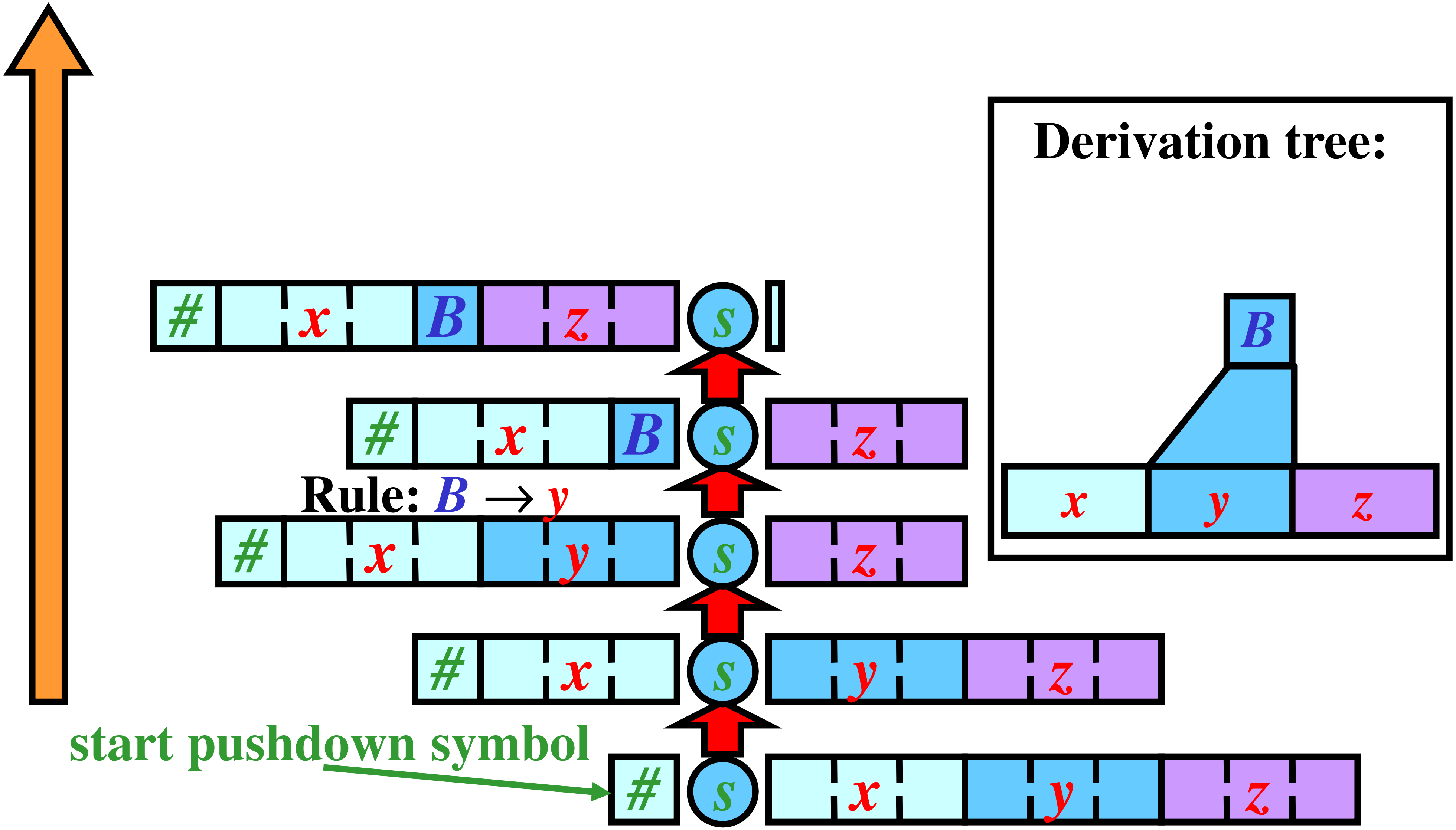
Bottom-up construction of a derivation tree:



EPDAs as Models of Bottom-Up Parsers 2/2

Bottom-up construction of a derivation tree:



EPDAs as Models of Bottom-Up Parsers 2/2

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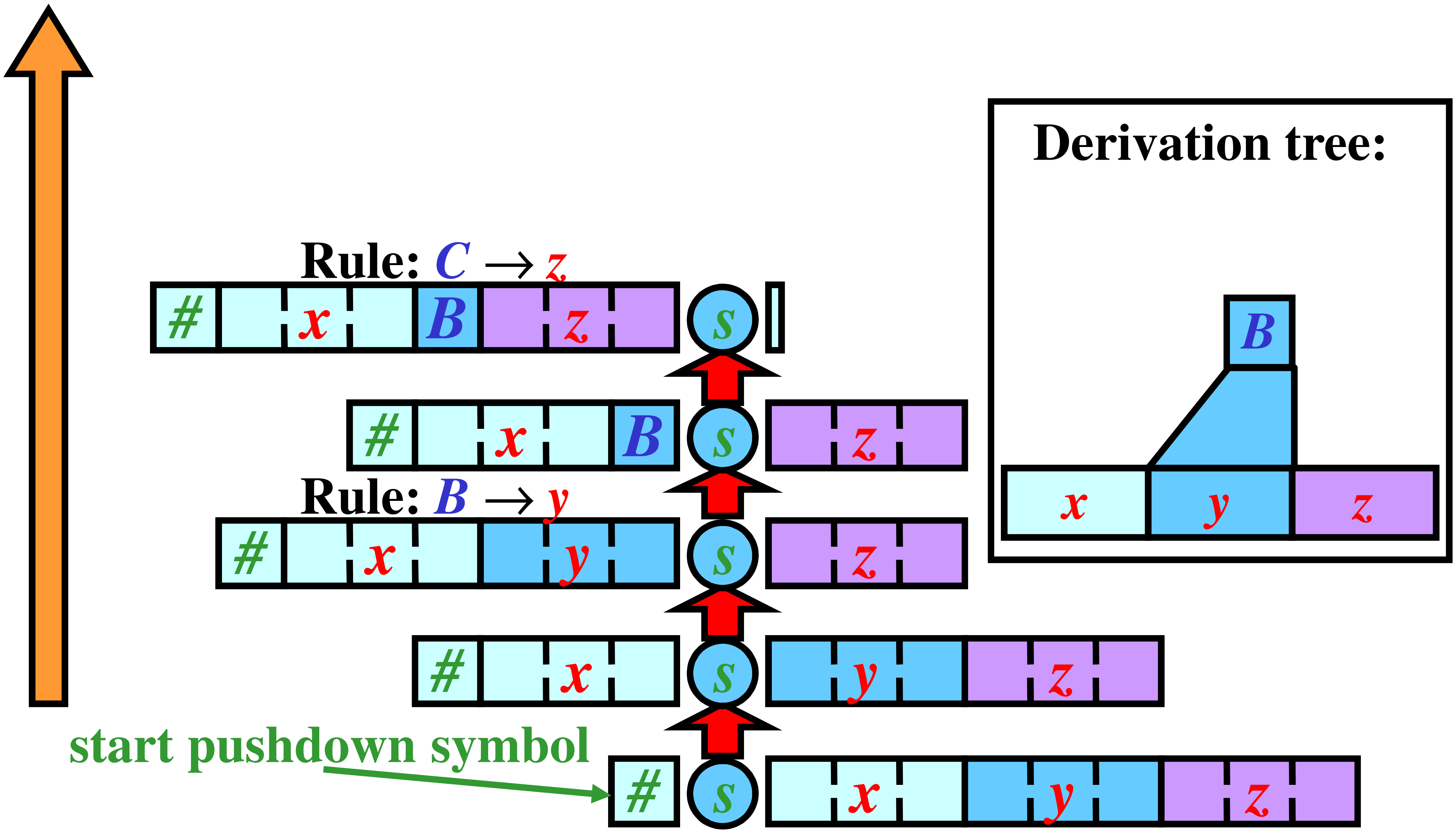

EPDAs as Models of Bottom-Up Parsers 2/2

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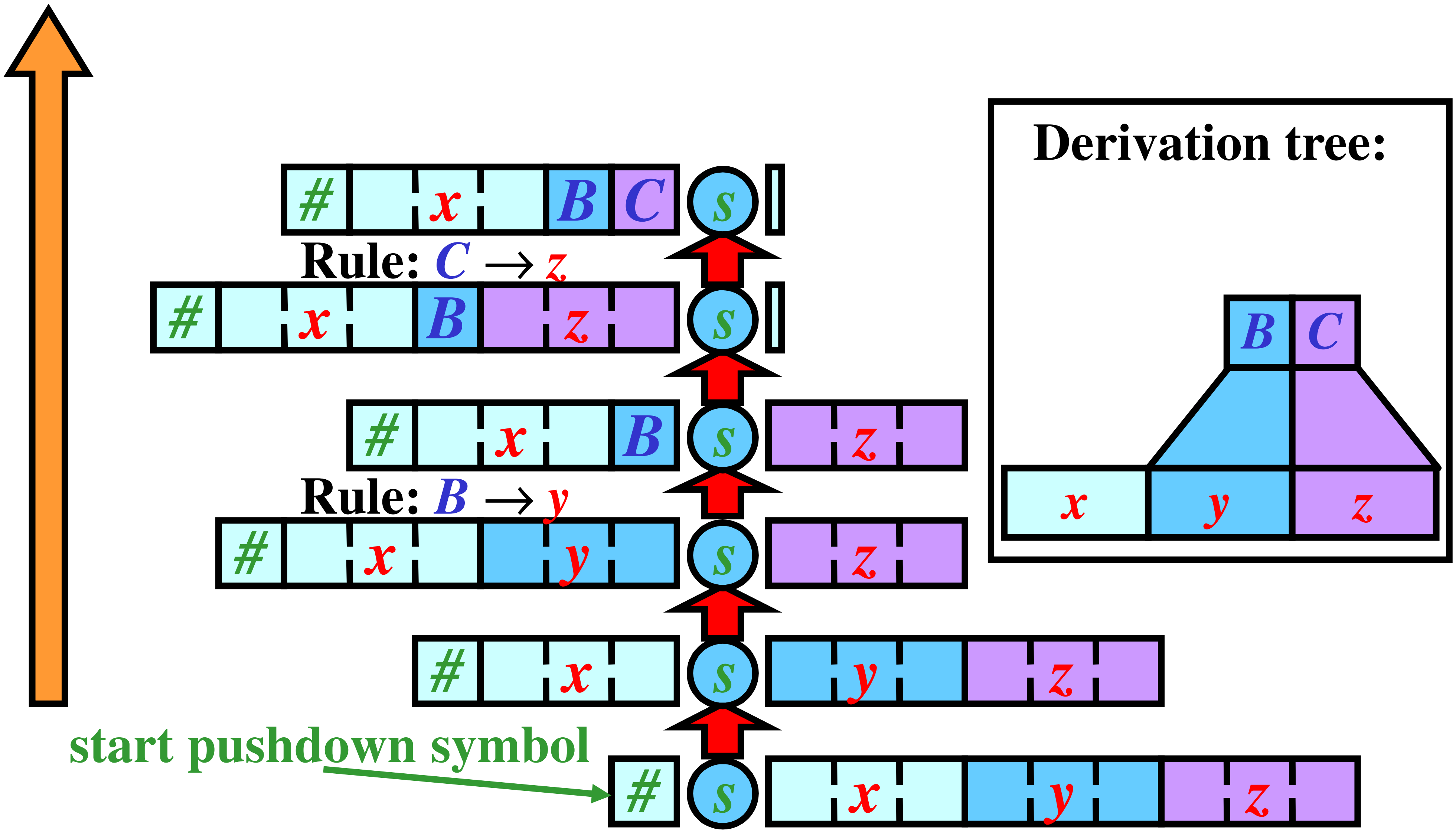

EPDAs as Models of Bottom-Up Parsers 2/2

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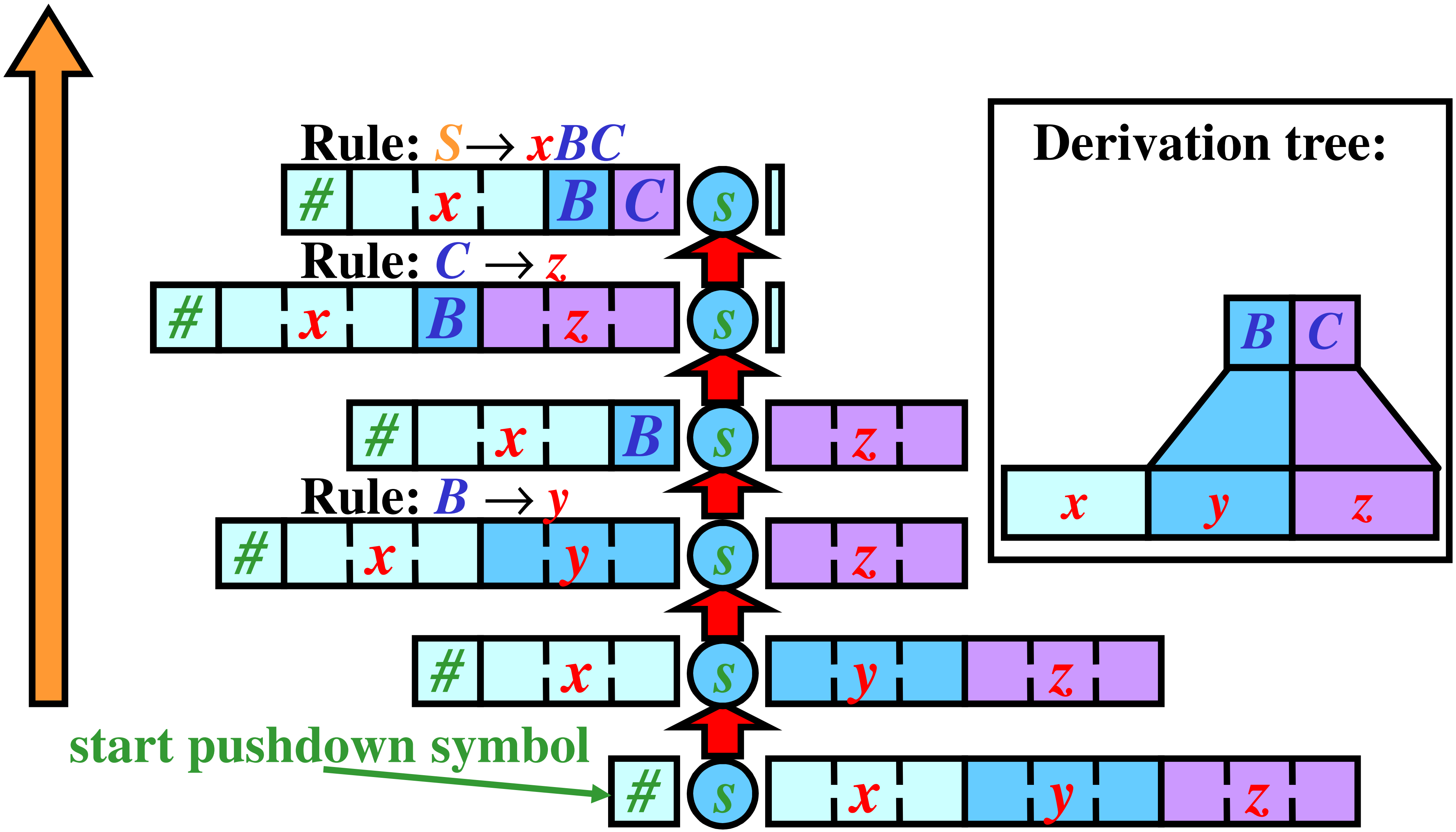

EPDAs as Models of Bottom-Up Parsers 2/2

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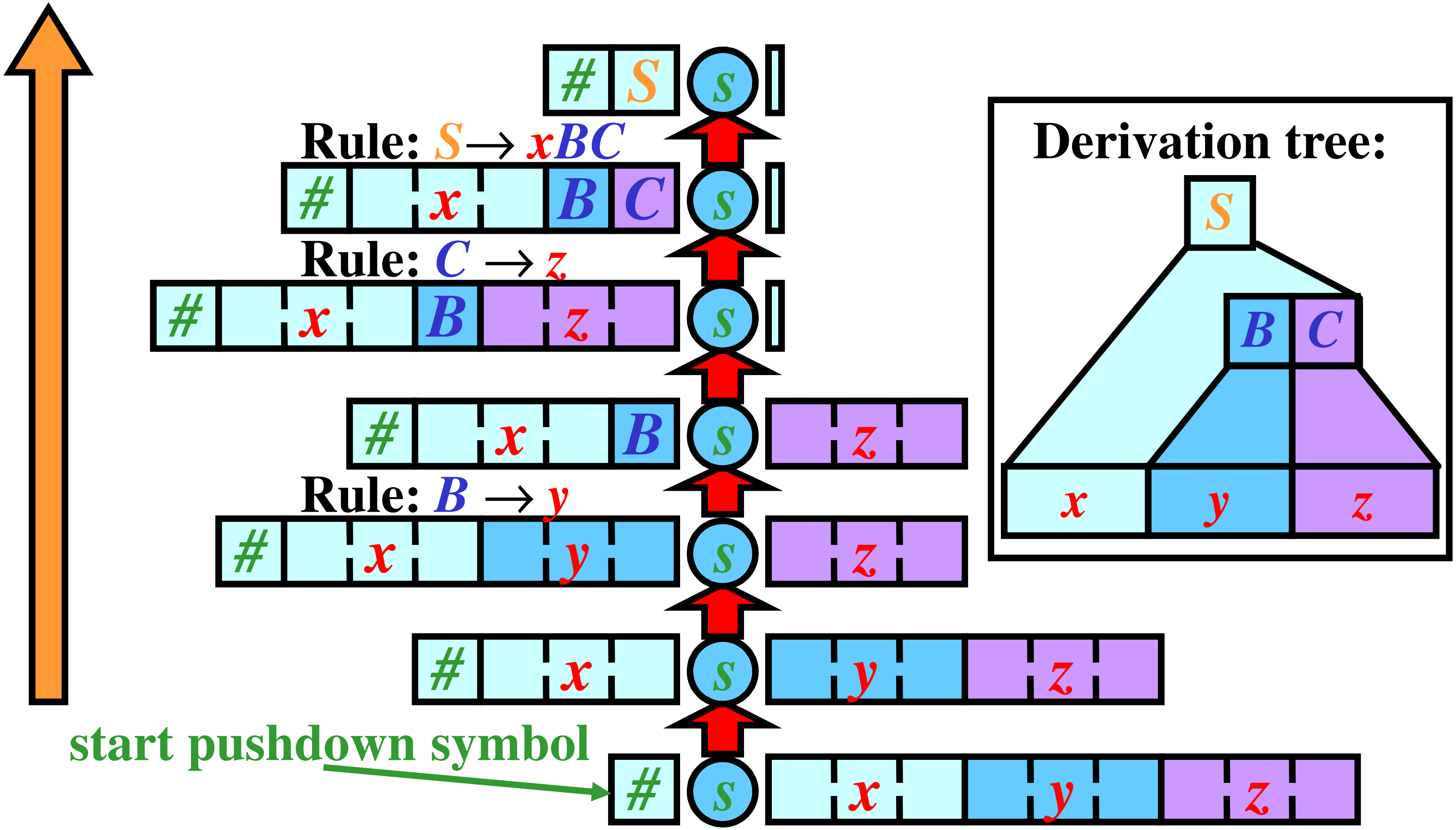

EPDAs as Models of Bottom-Up Parsers 2/2

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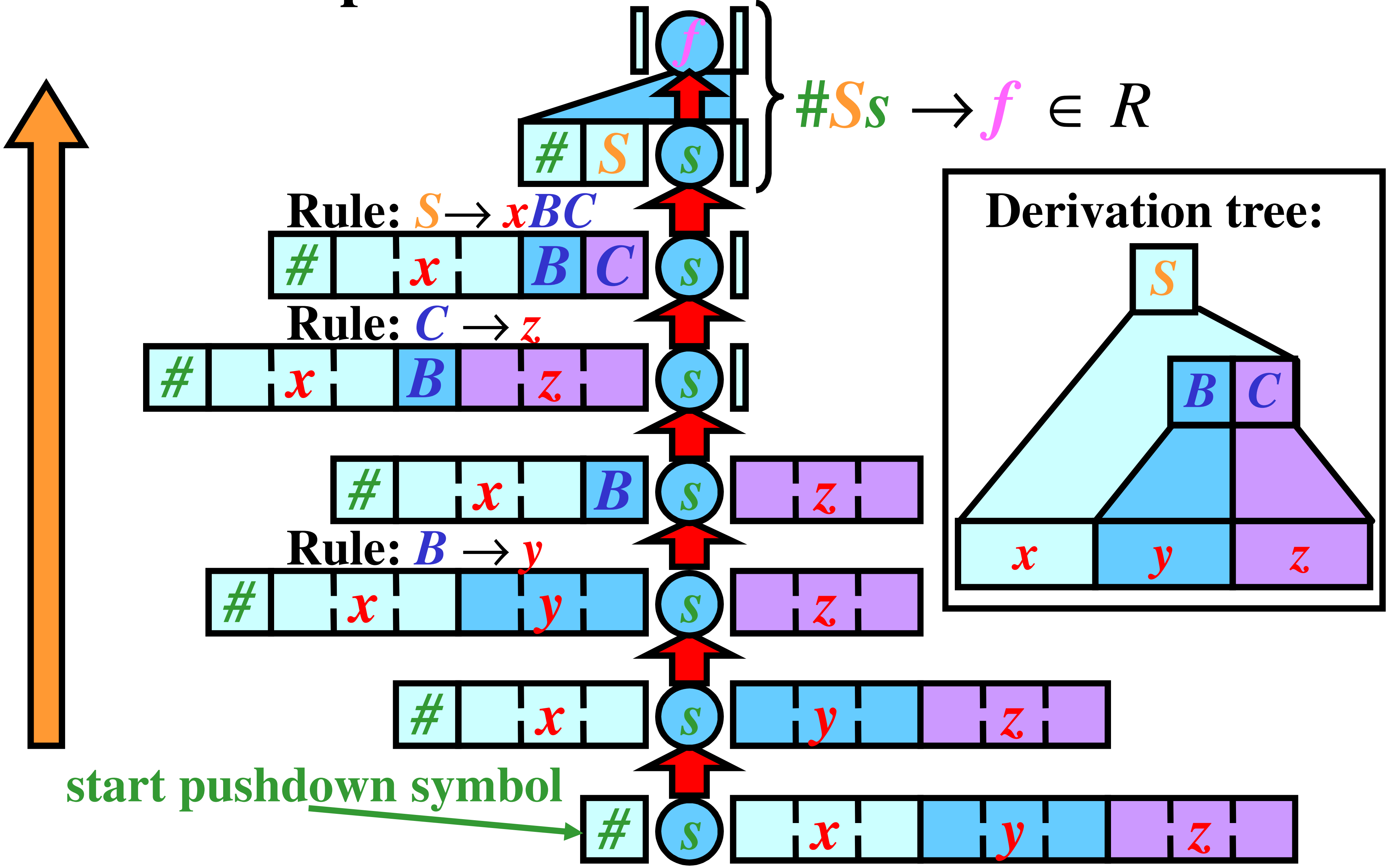

Algorithm: From CFG to EPDA

- **Input:** CFG $G = (N, T, P, S)$
- **Output:** EPDA $M = (Q, \Sigma, \Gamma, R, s, \#, F)$; $L(G) = L(M)_f$

- **Method:**
- $Q := \{s, f\}$;
- $\Sigma := T$;
- $\Gamma := N \cup T \cup \{\#\}$;
- Construction of R:
    - for every $a \in \Sigma$, add $sa \rightarrow as$ to R;
    - for every $A \rightarrow x \in P$, add $xs \rightarrow As$ to R;
    - add $\#Ss \rightarrow f$ to R;
- $F := \{f\}$;

From CFG to EPDA: Example 1/2

- $G = (N, T, P, S)$, where:

  $N = \{S\}$, $T = \{(,)\}$, $P = \{S \to (S), S \to ()\}$

  Objective: An EPDA M such that $L(G) = L(M)_f$

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$ where:

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$R = \{s(\rightarrow (s,\quad s) \rightarrow)s,\quad (S)s \rightarrow Ss,$

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$$R = \{\underbrace{s(\rightarrow (s,\ \ s) \rightarrow)s,}_{\textbf{shift rules}}\ \underbrace{(S)s \rightarrow Ss,\ \ ()s \rightarrow Ss,}_{\textbf{reduction rules}}\ \#Ss \rightarrow f\}$$

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$F = \{f\}$

From CFG to EPDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, \#, F)$, where:

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Question: $(())\in L(M)_f$?

$\#\ s\ (\ (\ )\ )$

From CFG to EPDA: Example 2/2

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$\# \; s \; (\; (\;) \;)$

Rule: $s(\to (s$

$\# \; (\; s \; (\;) \;)$ $\quad ($

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Rule: $s(\rightarrow (s$

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$\# \; (\; (\; s \;) \;)$ $\qquad (\; ($

Rule: $s) \rightarrow)s$

$\# \; (\; (\;) \; s \;)$ $\qquad (\; (\;)$

Rule: $(\,)s \rightarrow S$

$\# \; (\; S \; s \;)$ $\qquad (\; \underbrace{(\;)}_{} \to S$

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$\#\ s\ (())$

Rule: $s(\rightarrow (s$

$\#(\ s\ ())$

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$\#(()\ s\ )$

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Rule: $()s \rightarrow S$

$\#(S\ s\ )$

S

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| # | s | (| (|) |) |

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(

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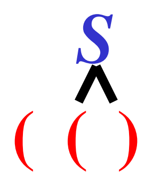
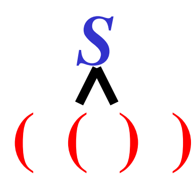


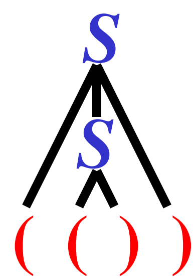
Rule: $s) \rightarrow)s$

| # | (| S |) | s |



Rule: $(S) \rightarrow S$

| # | S | s |



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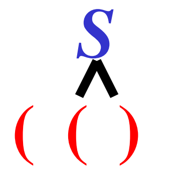

Rule: $s) \to)s$

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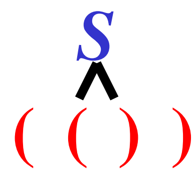
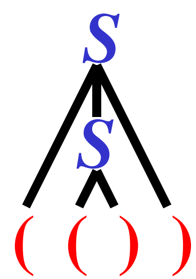


Rule: $(S) \to S$

$\#S\ s$



Rule: $\#Ss \to f$

f

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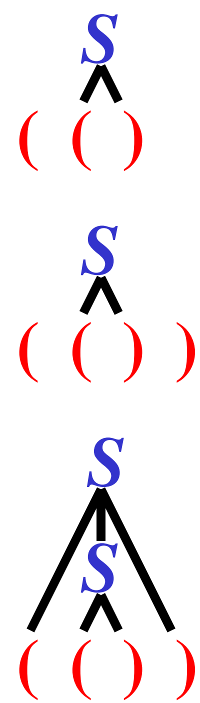
$\#(S)\ s$

Rule: $(S)s \rightarrow S$

$\#S\ s$

Rule: $\#Ss \rightarrow f$

Final state f

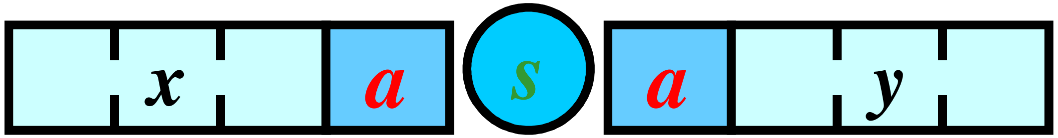


Answer: YES

PDAs as Models of Top-Down Parsers 1/2

Gist: An PDA *M* underlies a top-down parser

1) *M* contains *popping* rules that pops the top symbol from the pushdown and reads the input symbol if both coincide:



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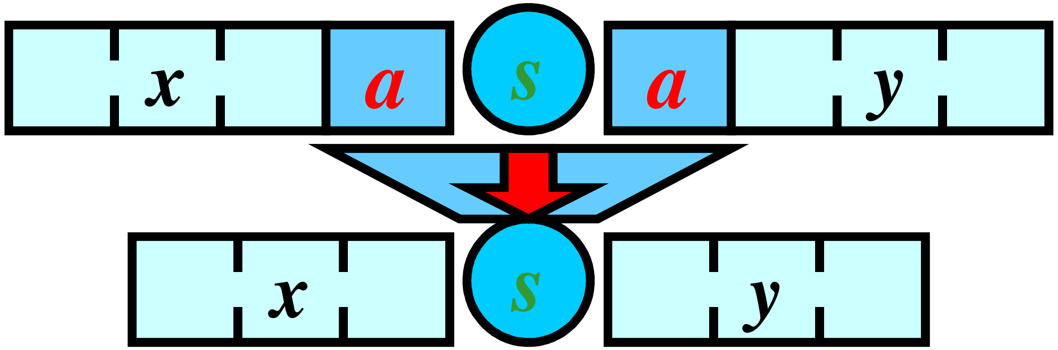

for every $a \in \Sigma$:

add $asa \rightarrow s$ to R;

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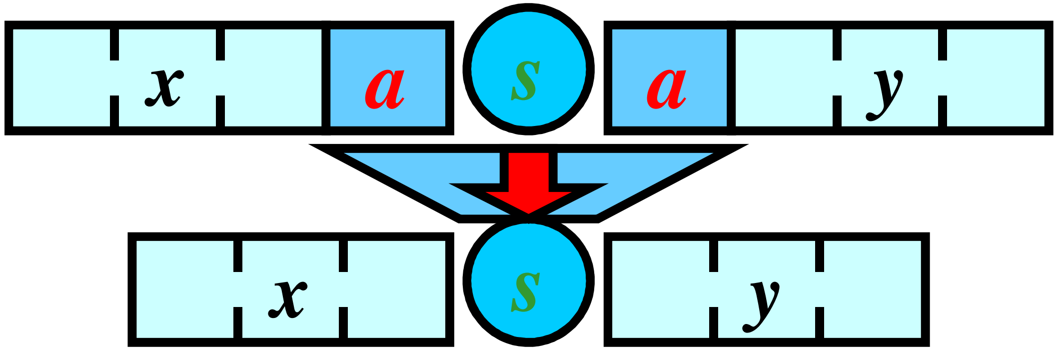

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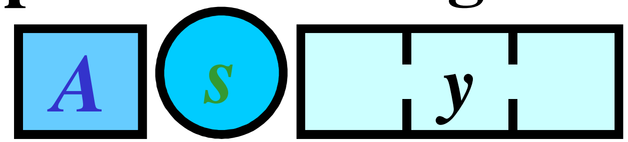
2) *M* contains *expansion* rules that simulate the application of a grammatical rule:



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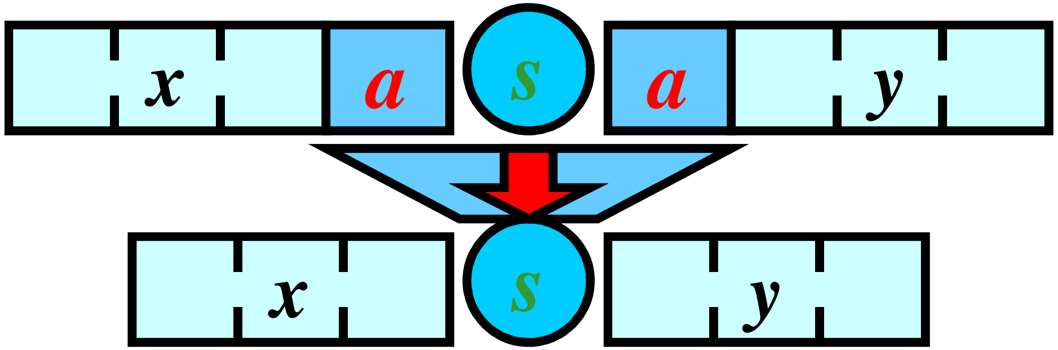

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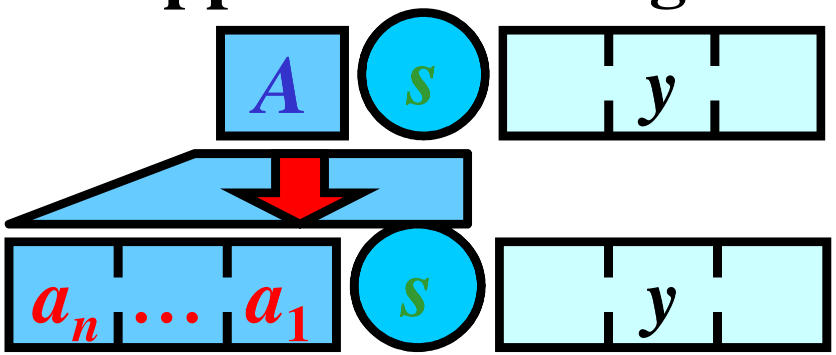

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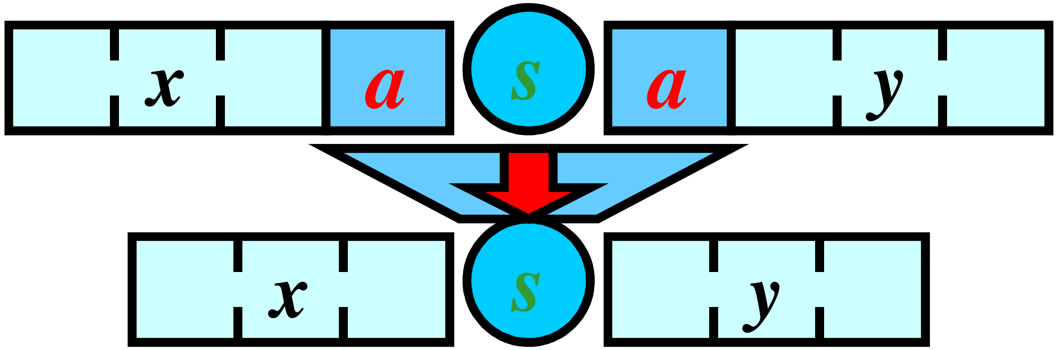

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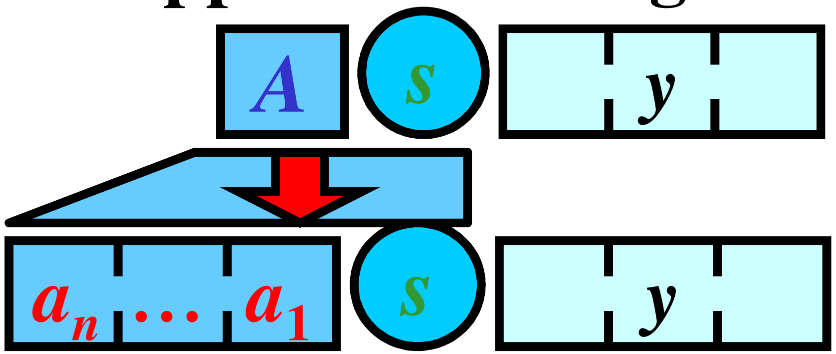
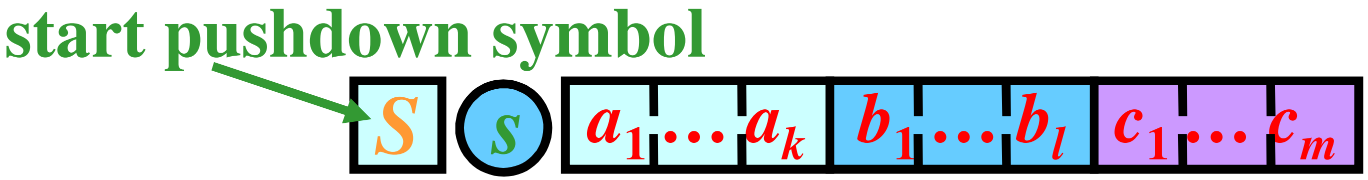


for every $A \rightarrow a_1 \dots a_n \in P$ in G,
add $As \rightarrow \underbrace{a_n \dots a_1}_{= \textbf{reversal}(a_1 \dots a_n)}s$ to R;

PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:

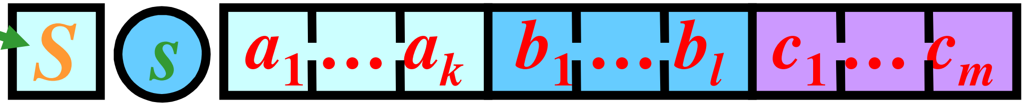


Derivation tree:

PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:

start pushdown symbol



$S \rightarrow a_1..a_kBC$

Derivation tree:

PDAs as Models of Top-Down Parsers 2/2

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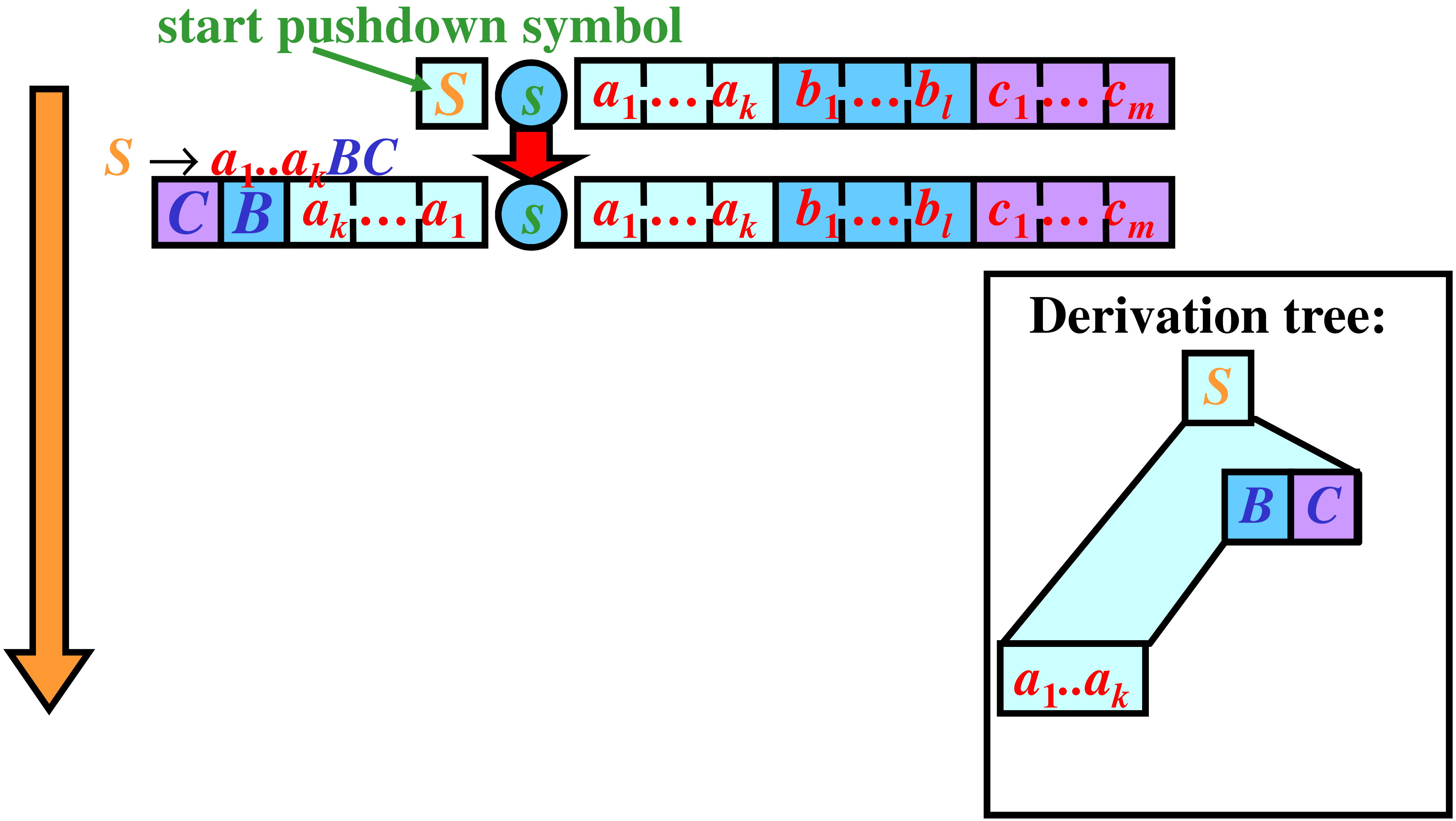

PDAs as Models of Top-Down Parsers 2/2

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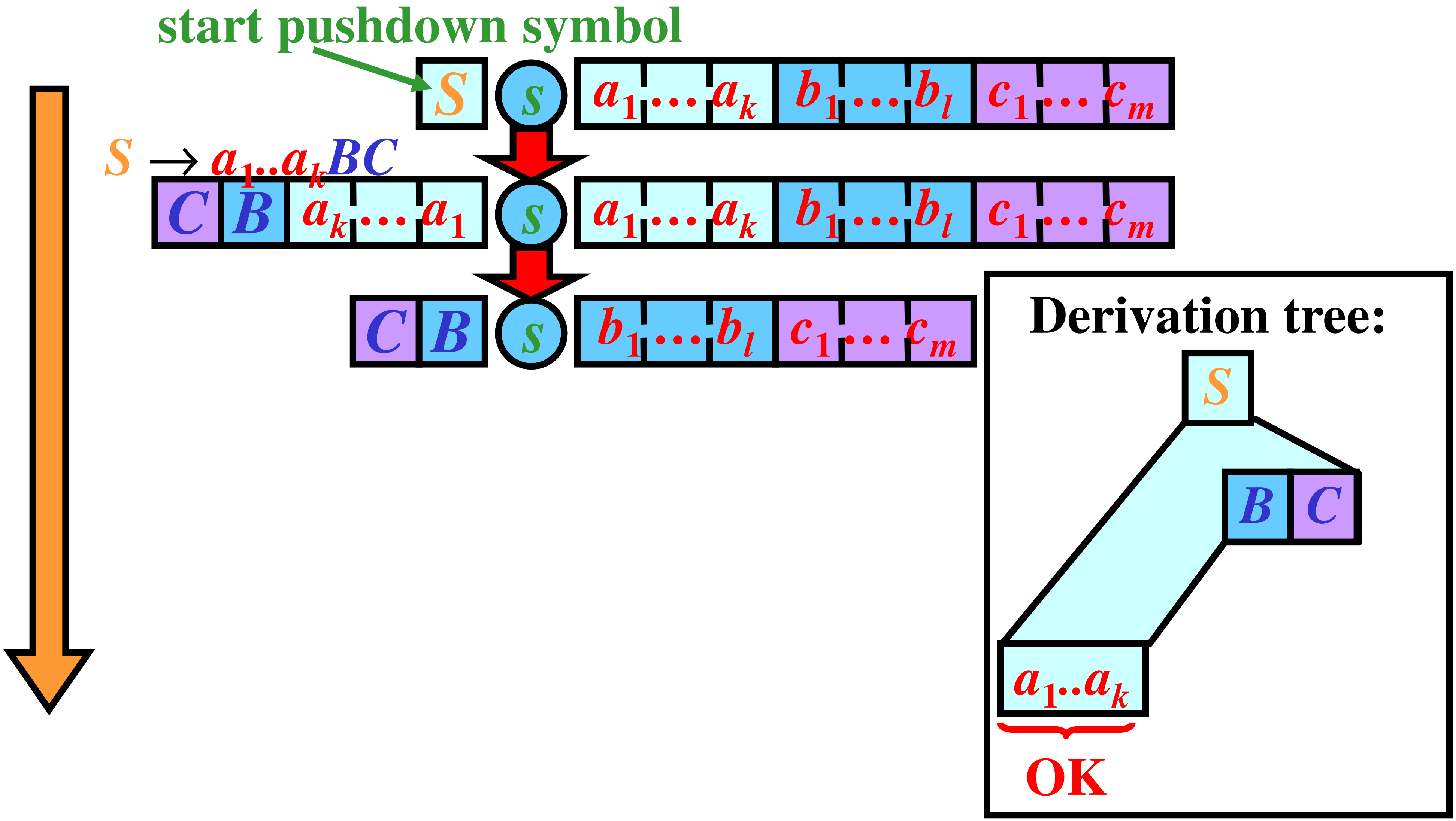

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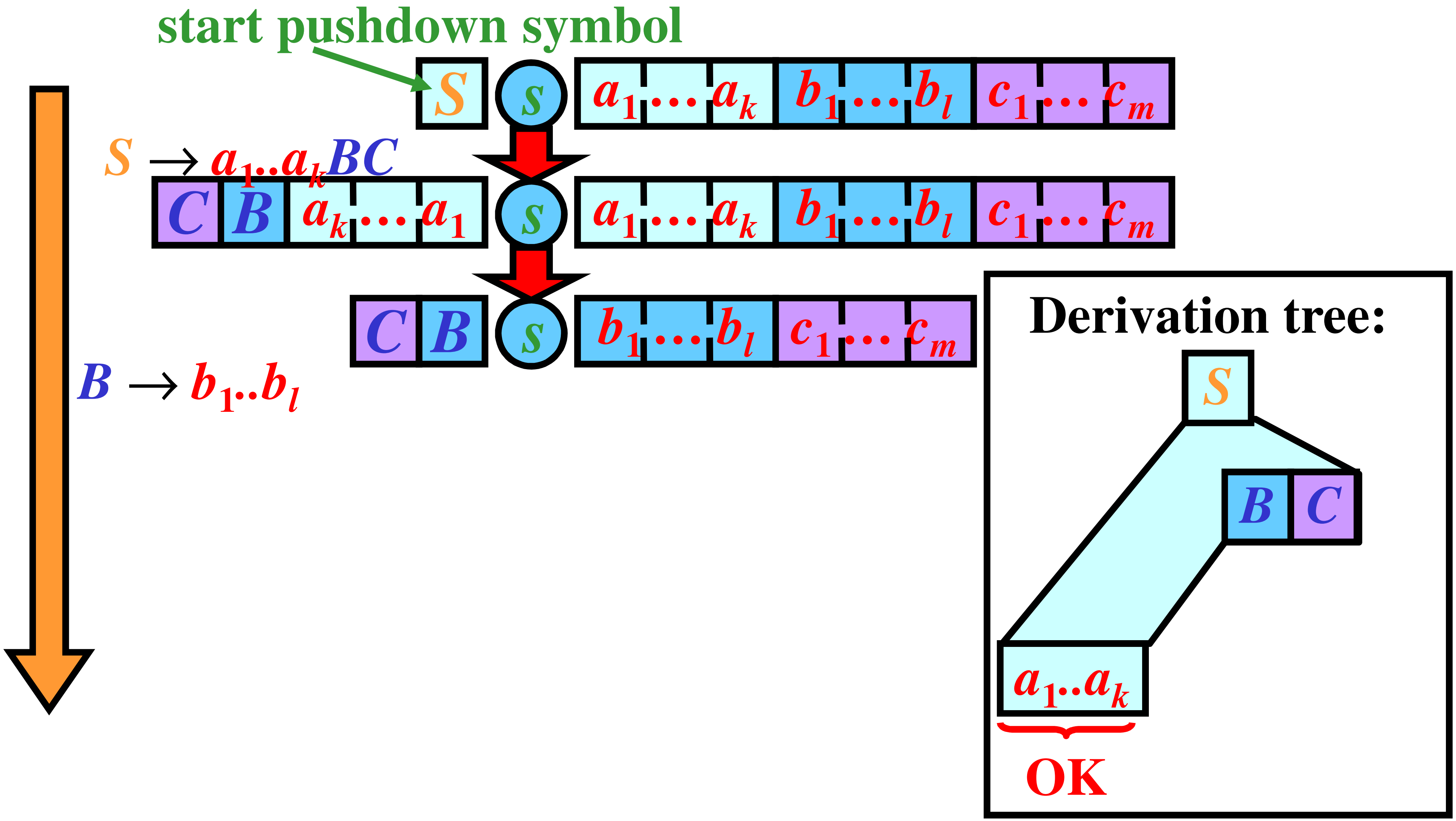

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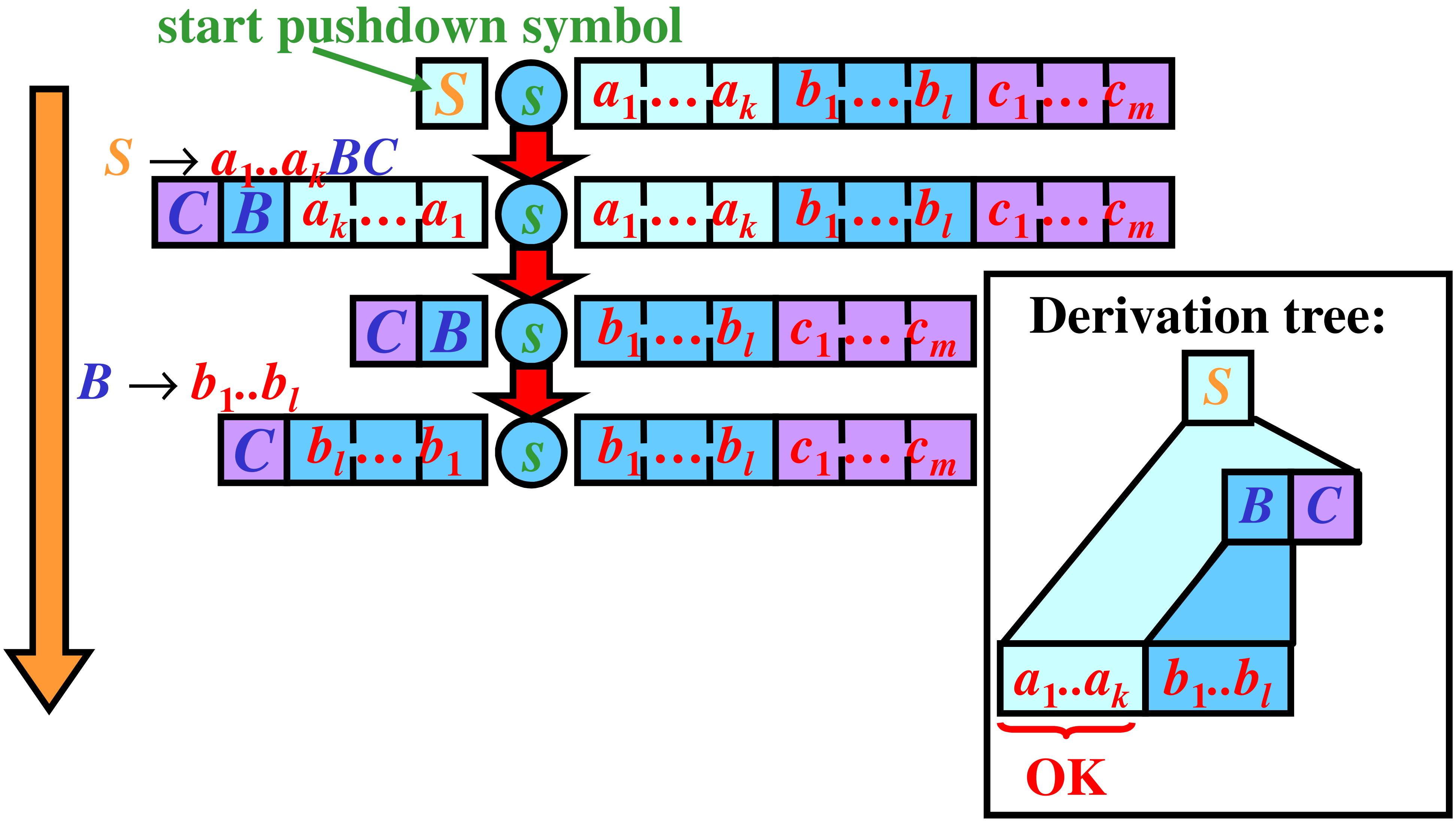

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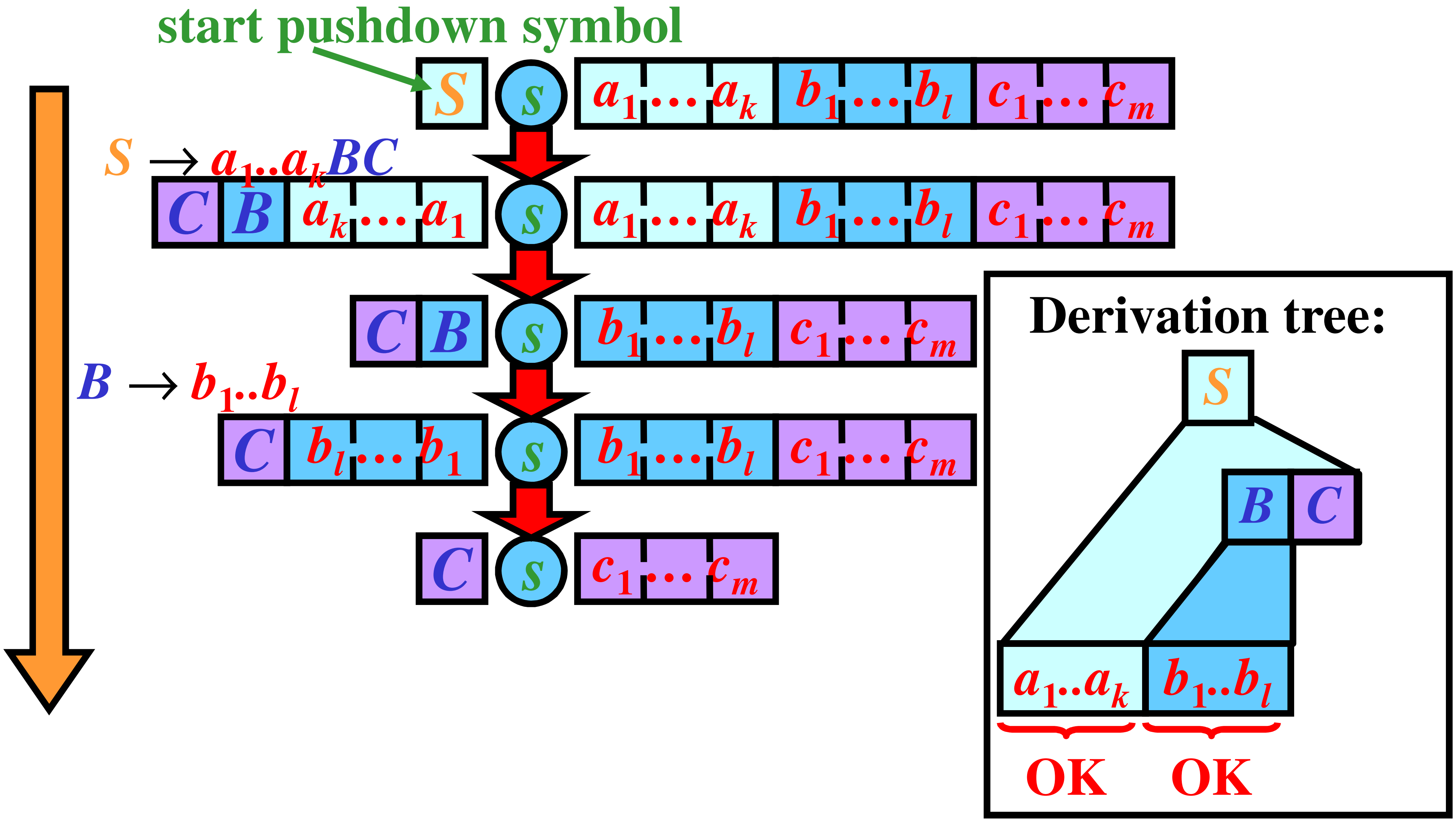
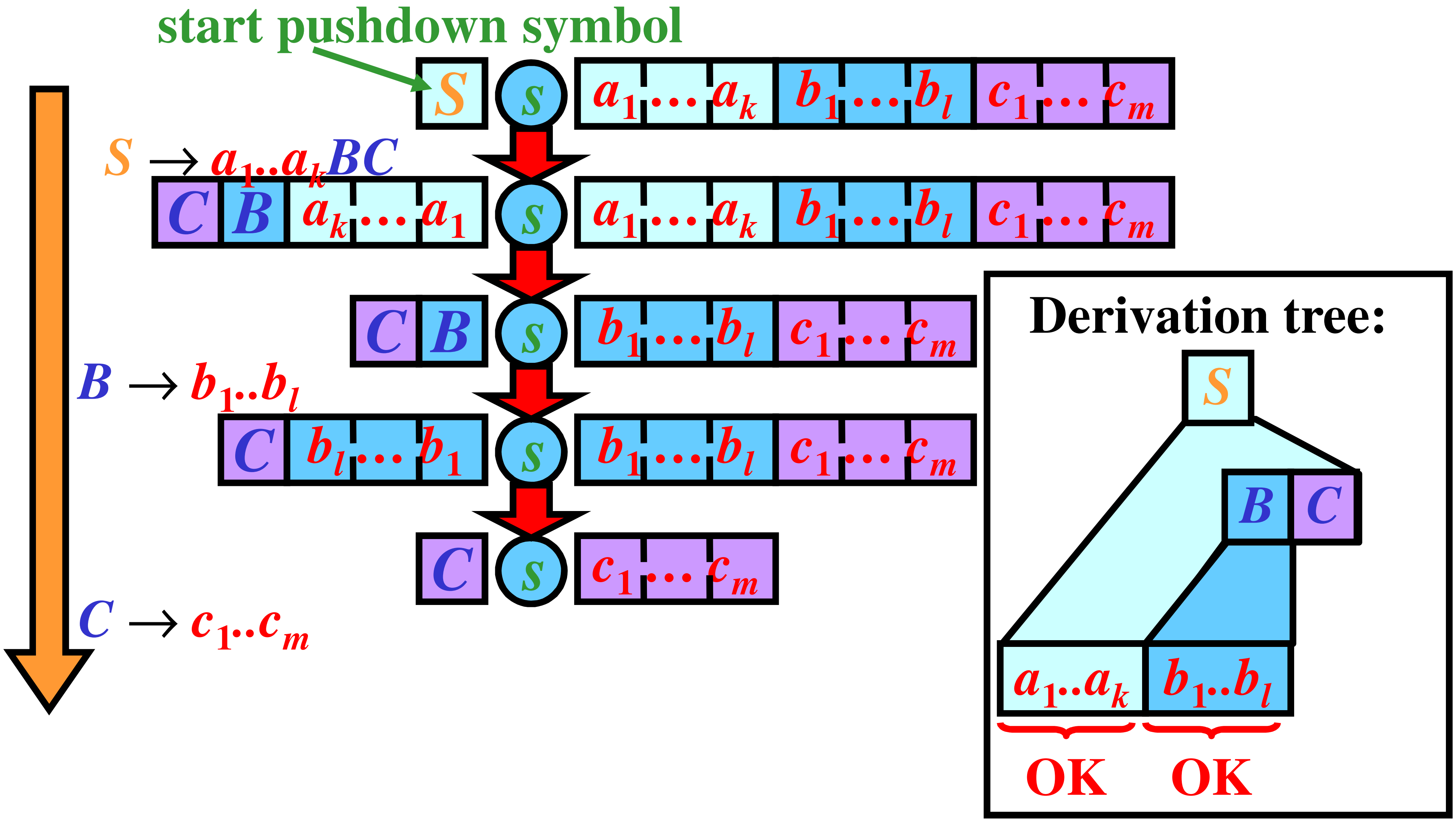


PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



PDAs as Models of Top-Down Parsers 2/2

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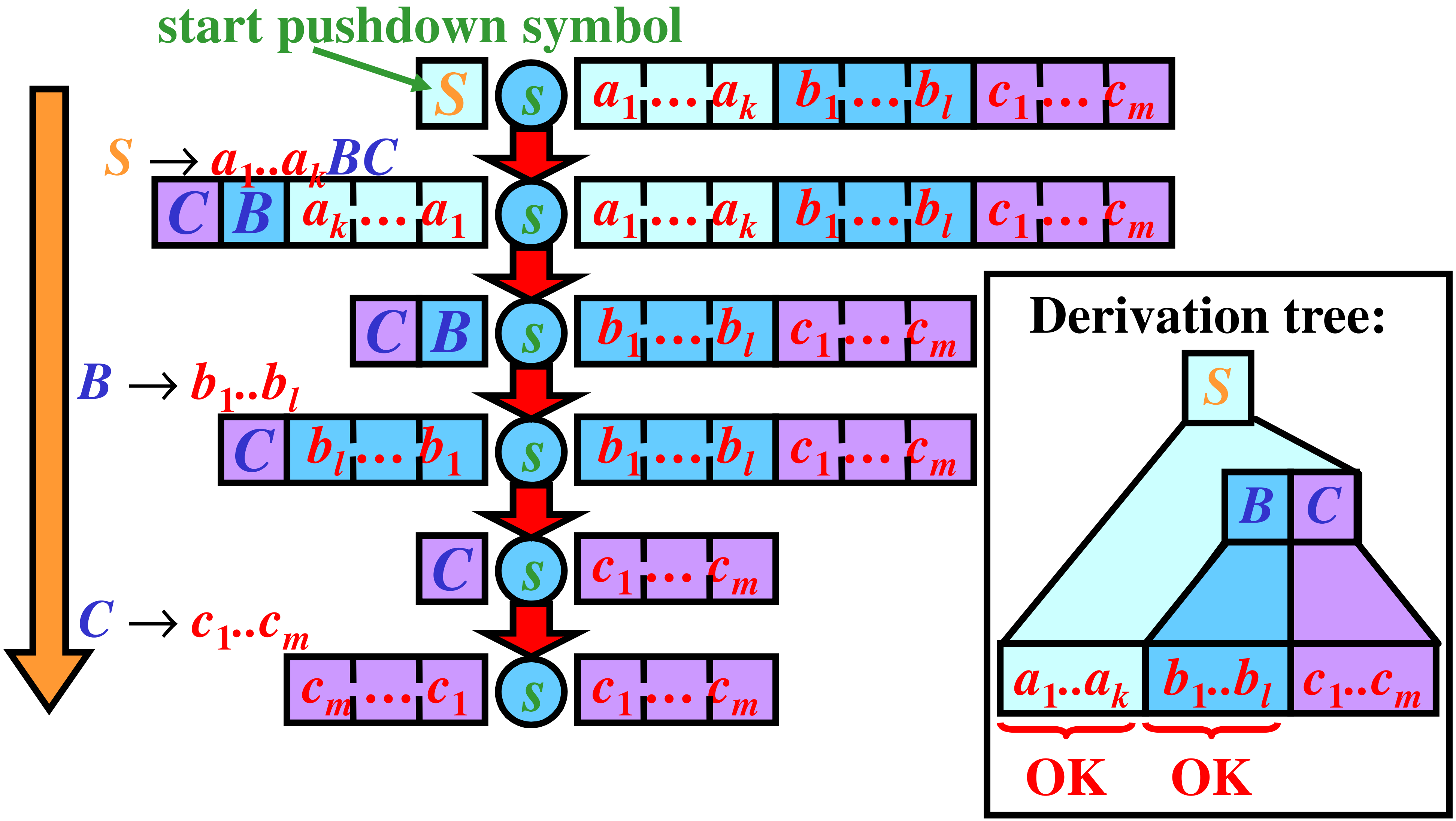
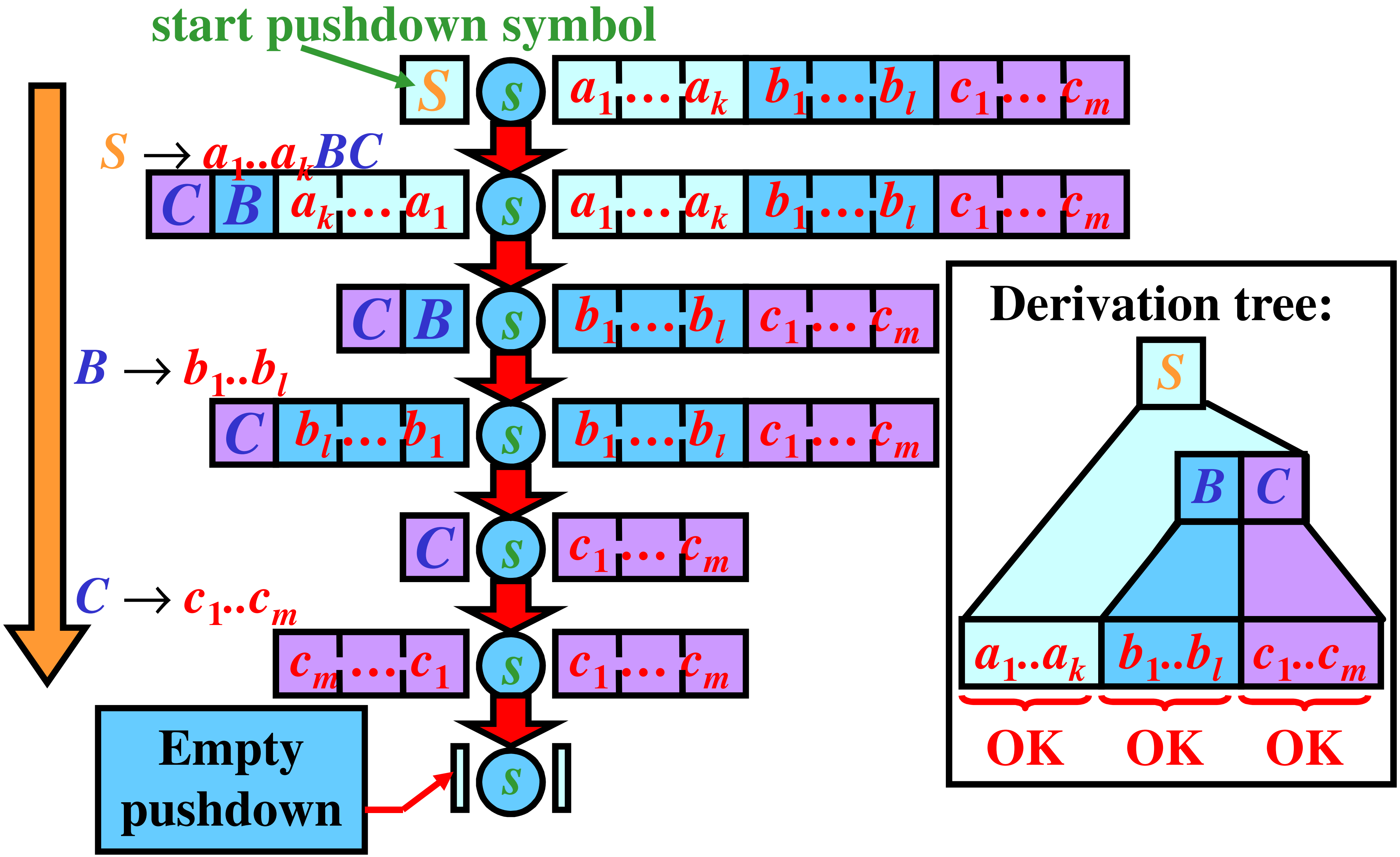


PDAs as Models of Top-Down Parsers 2/2

Top-down construction of a derivation tree:



Algorithm: From CFG to PDA

- **Input:** CFG $G = (N, T, P, S)$
- **Output:** PDA $M = (Q, \Sigma, \Gamma, R, s, S, F)$; $L(G) = L(M)_\varepsilon$

- **Method:**
- $Q := \{s\}$;
- $\Sigma := T$;
- $\Gamma := N \cup T$;
- Construction of R:
  - for every $a \in \Sigma$, add $asa \rightarrow s$ to R;
  - for every $A \rightarrow x \in P$, add $As \rightarrow ys$ to R, where y = reversal(x);
- $F := \emptyset$;

From CFG to PDA: Example 1/2

- $G = (N, T, P, S)$, where:

$N = \{S\}$, $T = \{(,)\}$, $P = \{S \rightarrow (S), S \rightarrow ()\}$

Objective: An PDA M such that $L(G) = L(M)_{\varepsilon}$

$M = (Q, \Sigma, \Gamma, R, s, S, F)$ where:

From CFG to PDA: Example 1/2

- $G = (N, T, P, S)$, where:

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"(" $\in T$ ")" $\in T$

$\Downarrow$ $\Downarrow$

$R = \{(s(\rightarrow s,\ \ )s) \rightarrow s,$

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"(" $\in$ ***T*** ")" $\in$ ***T*** $\underbrace{S} \rightarrow \underbrace{(S)}_{\textbf{rev}} \in$ ***P***

$R = \{(s(\rightarrow s,\ \ )s) \rightarrow s,\ \ Ss \rightarrow)S(s,$

From CFG to PDA: Example 1/2

- $G = (N, T, P, S)$, where:

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rev **rev**

$R = \{(s(\to s,\ \ )s) \to s,\ \ Ss \to)S(s,\ \ Ss \to)(s\ \}$

From CFG to PDA: Example 1/2

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$Q = \{s\}$; $\quad \Sigma = T = \{(,)\}$; $\quad \Gamma = N \cup T = \{S, (,)\}$

$\text{“(”} \in T \qquad \text{“)”} \in T \qquad S \to (S) \in P \qquad S \to (\,) \in P$

rev **rev**

$$R = \{\underbrace{s(\to s,\quad)s) \to s,}_{\textbf{popping rules}}\quad \underbrace{Ss \to)S(s,\quad Ss \to)(s}_{\textbf{expansion rules}}\}$$

From CFG to PDA: Example 1/2

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$\boxed{\textbf{rev}}$ $\boxed{\textbf{rev}}$

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$F = \emptyset$

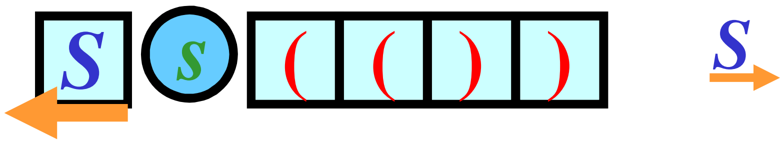
From CFG to PDA: Example 2/2

$M = (Q, \Sigma, \Gamma, R, s, S, F)$, where:

$Q = \{s\}$, $\Sigma = T = \{(,)\}$, $\Gamma = \{(,), S\}$, $F = \emptyset$

$P = \{(s(\rightarrow s, \quad)s) \rightarrow s, \quad Ss \rightarrow)S(s, \quad Ss \rightarrow)(s \}$

Question: $(())\in L(M)_\varepsilon$?



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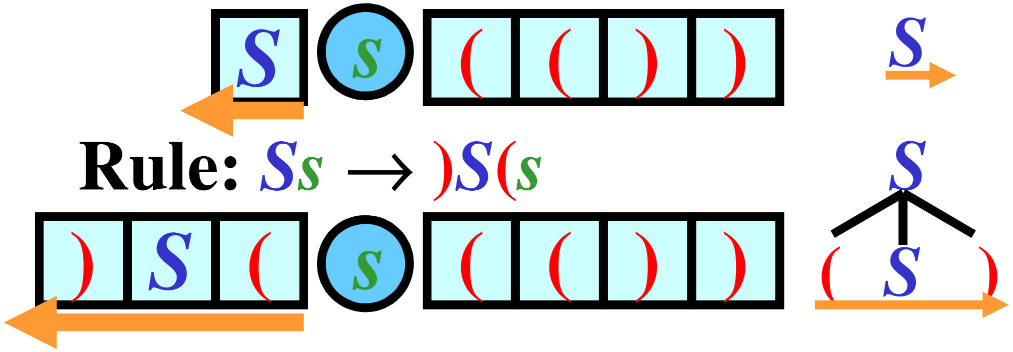

Rule: $Ss \to)S(s$

From CFG to PDA: Example 2/2

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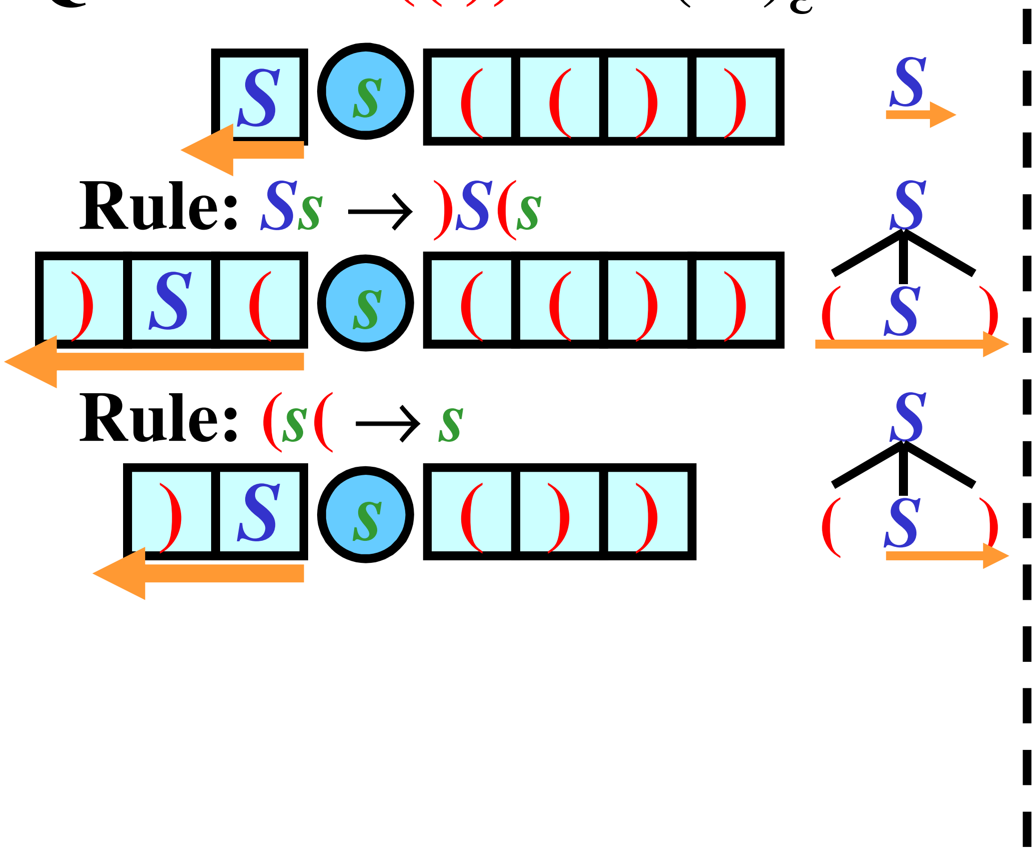

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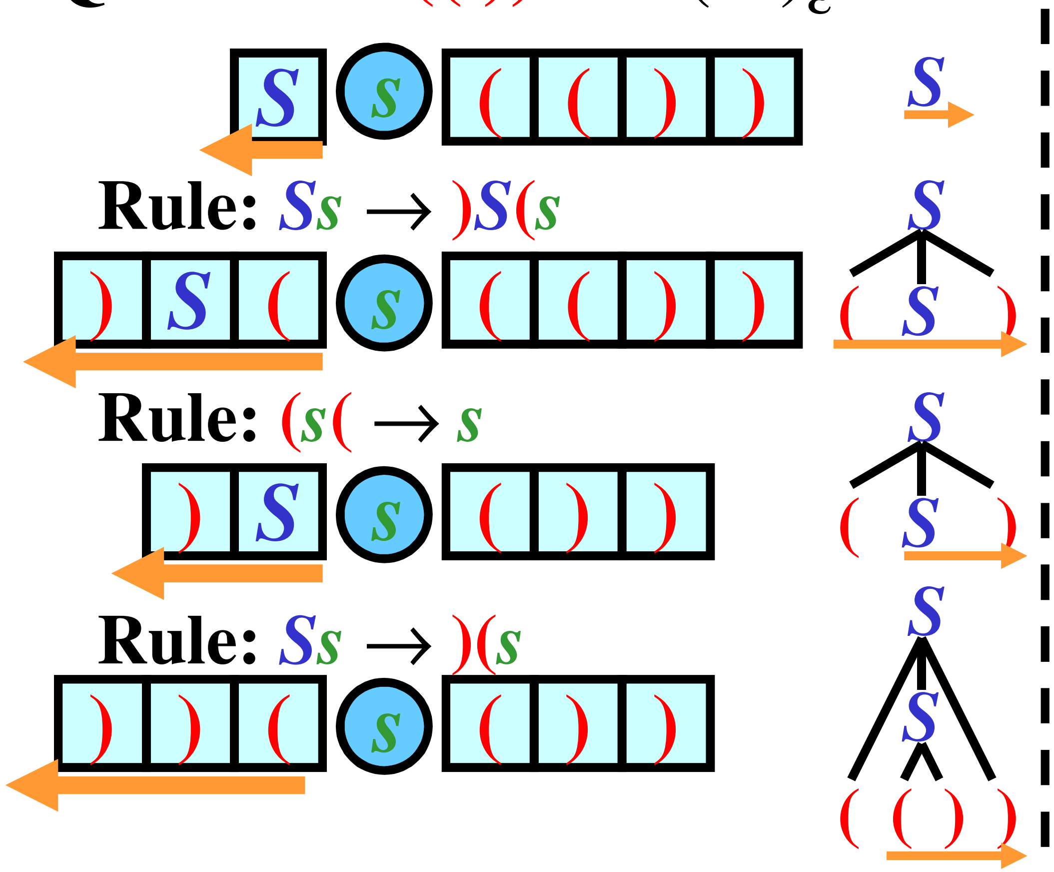

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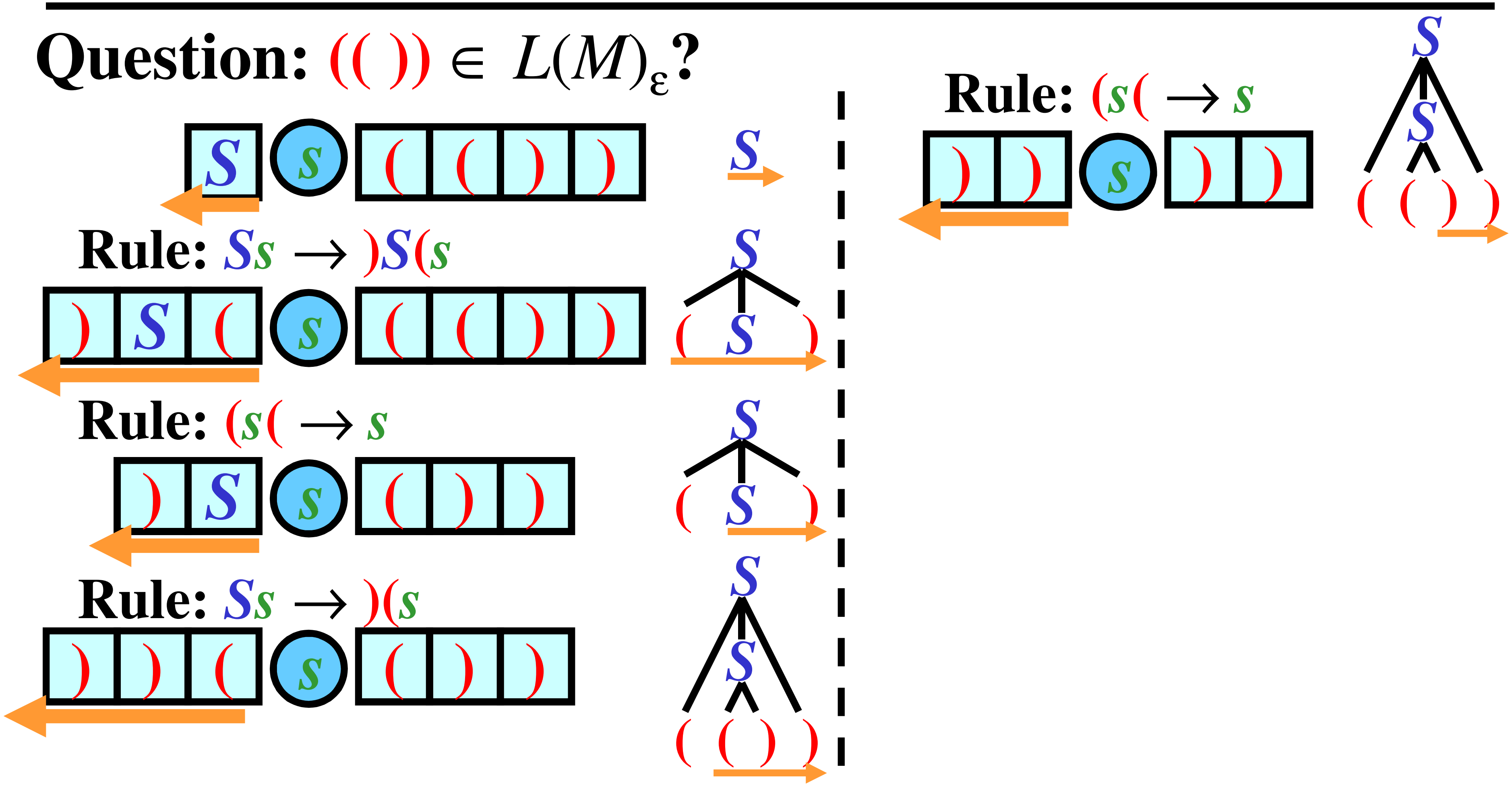

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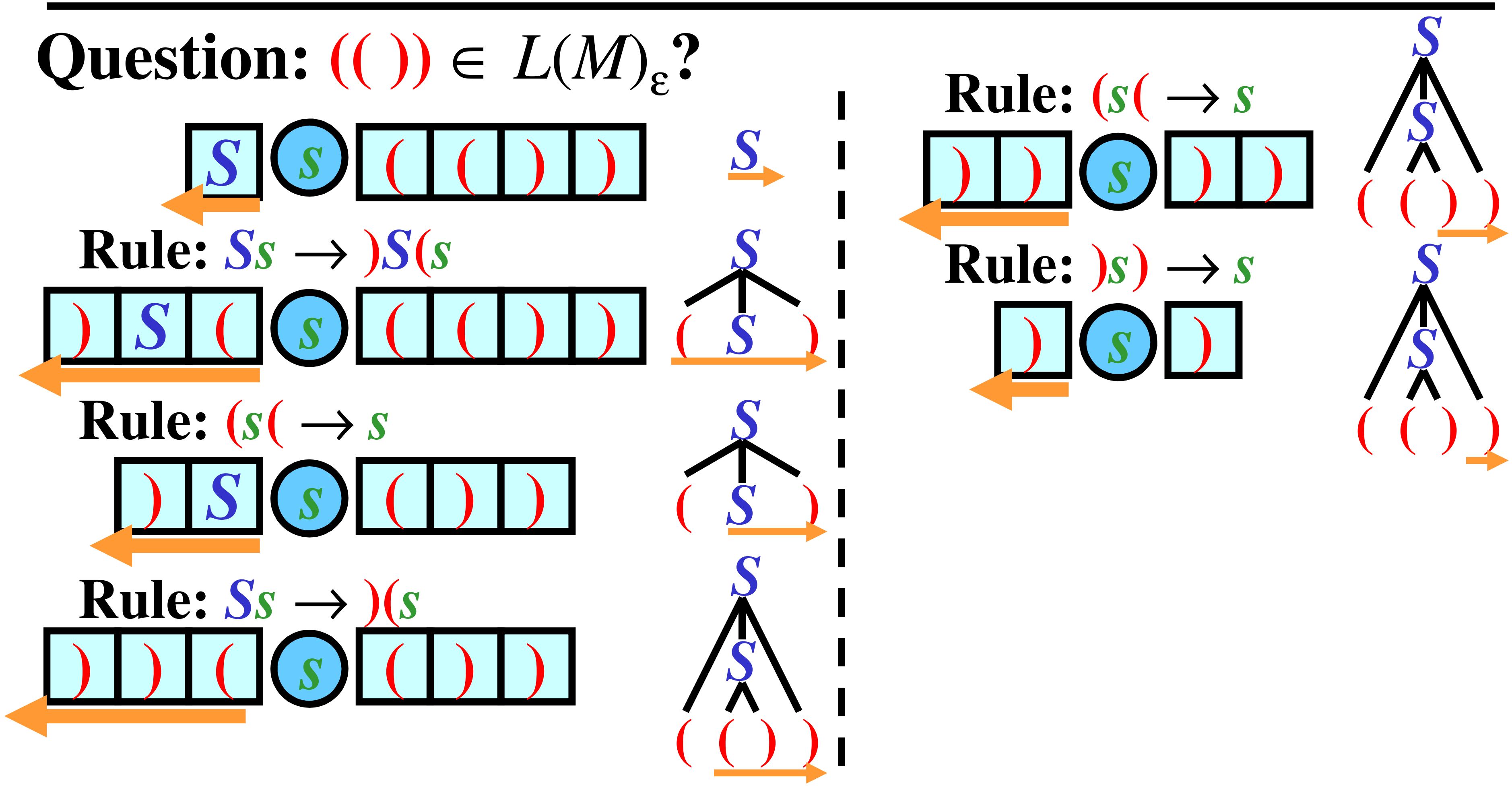

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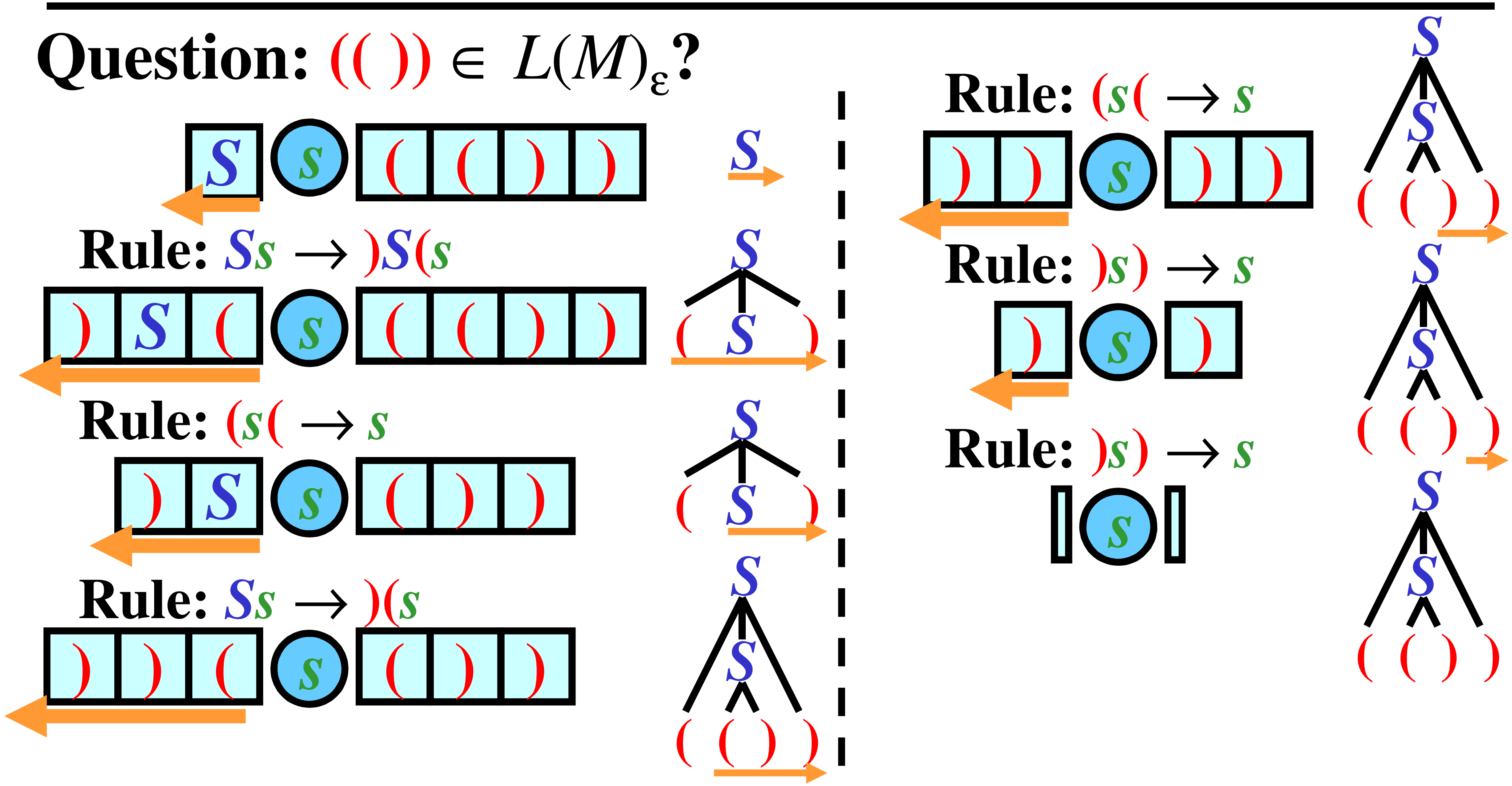

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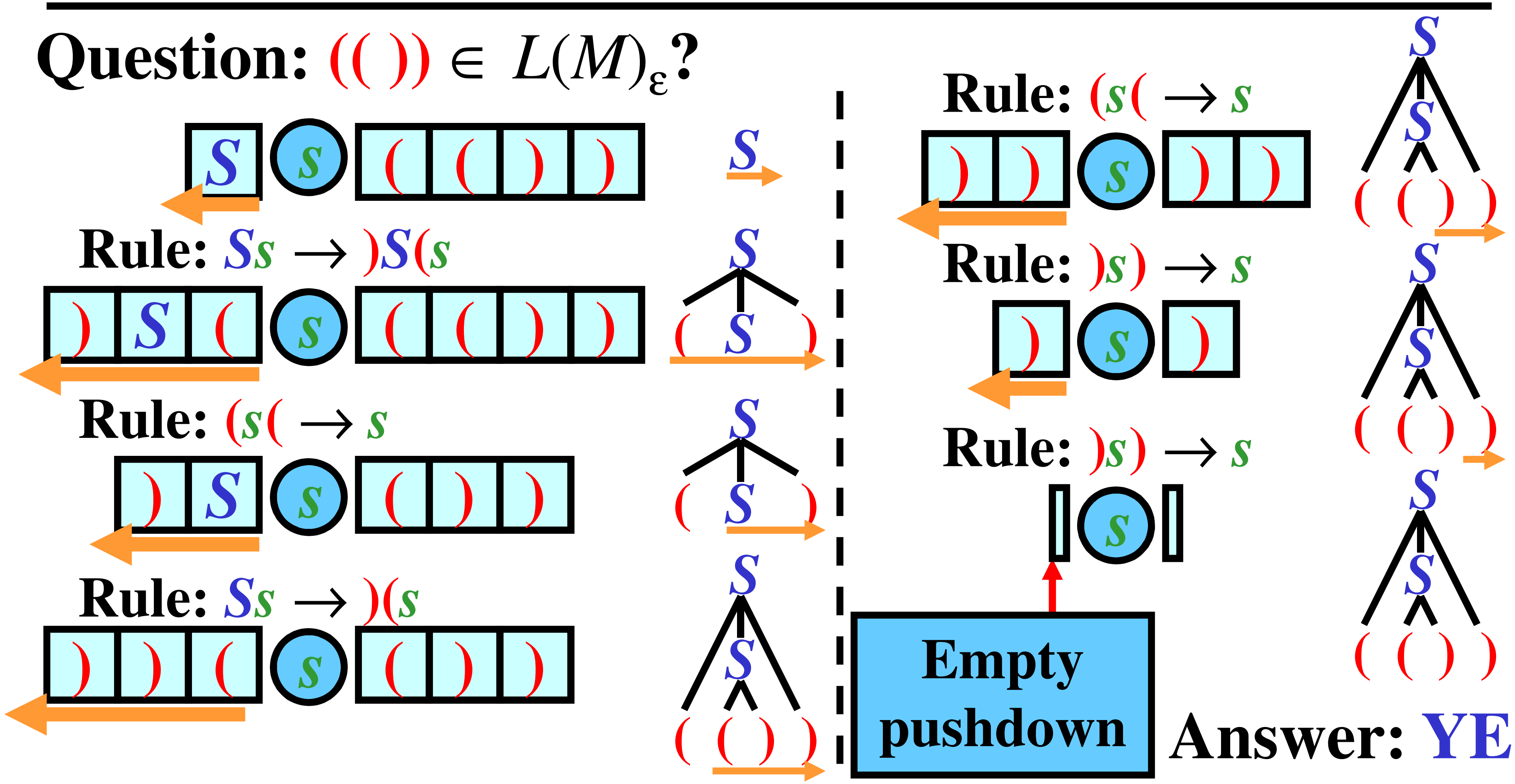

Answer: YES

Models for Context-free Languages

Theorem: For every CFG G, there is an PDA M such that $L(G) = L(M)_\varepsilon$.

Proof: See the previous algorithm.

Theorem: For every PDA M, there is a CFG G such that $L(M)_\varepsilon = L(G)$.

Proof: See page 486 in [Meduna: Automata and Languages]

Conclusion: The fundamental models for context-free languages are

1) **Context-free grammars** 2) **Pushdown automata**