TEMASEK JUNIOR COLLEGE

2022 JC 1 H2 Mathematics (9758)
VECTORS
Tutorial 1: Basic Concepts of 3-D Vectors

[Level 0]

1 Vectors **a**, **b**, **c** and **d** are given by $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{c} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and $\mathbf{d} = -\mathbf{j} + \mathbf{k}$. Prove that the vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{d}-\mathbf{c}$ are parallel and find the ratio of their magnitudes.

[Solution]

$$\mathbf{a} = \begin{pmatrix}1\\1\\1\end{pmatrix},\quad \mathbf{b} = \begin{pmatrix}2\\3\\0\end{pmatrix},\quad \mathbf{c} = \begin{pmatrix}3\\5\\-2\end{pmatrix},\quad \mathbf{d} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

$$\mathbf{b}-\mathbf{a} = \begin{pmatrix}1\\2\\-1\end{pmatrix};\quad \mathbf{d}-\mathbf{c} = = \begin{pmatrix}-3\\-6\\3\end{pmatrix} = -3\begin{pmatrix}1\\2\\-1\end{pmatrix} = -3(\mathbf{b}-\mathbf{a})$$

$\therefore$ $\mathbf{b}-\mathbf{a}$ and $\mathbf{d}-\mathbf{c}$ are parallel and $|\mathbf{b}-\mathbf{a}| : |\mathbf{d}-\mathbf{c}| = 1:3$

Remarks:

To find the ratio of magnitudes, division of vectors eg, $\dfrac{\mathbf{b}-\mathbf{a}}{\mathbf{d}-\mathbf{c}}$ is meaningless.

2 Given that $\mathbf{p} = 4\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}$, $\mathbf{q} = -2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$, and $\mathbf{r} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. If $\mathbf{s} = \mathbf{p} + a\mathbf{q} + b\mathbf{r}$ and $\mathbf{t} = -2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, find the values of a and b so that **s** is in the opposite direction to **t**.

[Solution]

$$\mathbf{p} = \begin{pmatrix}4\\-4\\8\end{pmatrix},\quad \mathbf{q} = \begin{pmatrix}-2\\-3\\-4\end{pmatrix},\quad \mathbf{r} = \begin{pmatrix}1\\1\\-1\end{pmatrix},\quad \mathbf{t} = \begin{pmatrix}-2\\-1\\-3\end{pmatrix}$$

Since $\mathbf{s} \mathbin{/\!/} \mathbf{t}$, Let $\mathbf{s} = -\lambda\mathbf{t}$ for some $\lambda > 0$ [A common mistake is to let $s = -t$]

Negative sign since **s** is in the opp direction to **t**

$$\begin{pmatrix}4\\-5\\8\end{pmatrix} + a\begin{pmatrix}-2\\-3\\-4\end{pmatrix} + b\begin{pmatrix}1\\1\\-1\end{pmatrix} = -\lambda\begin{pmatrix}-2\\-1\\-3\end{pmatrix}$$

$$\begin{cases}-2a + b - 2\lambda = -4\\ -3a + b - \lambda = 5\\ -4a - b - 3\lambda = -8\end{cases}$$

Using GC, $a = -3,\ b = 2,\ \lambda = 6$

3 Vector **a** and **b** are given by $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \lambda \mathbf{a}$ where $\lambda \in \mathbb{R}$.

(i) Find the magnitude of **a** and the unit vector in the direction of **a**.

(ii) Given that $|\mathbf{a} + 2\mathbf{b}| = 6$, find the possible values of λ .

[Solution]

(i) $|\underset{\sim}{a}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

$$\hat{\underset{\sim}{a}} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

(ii) $|\underset{\sim}{a} + 2\lambda \underset{\sim}{a}| = 6$ (since $\mathbf{b} = \lambda \mathbf{a}$)

$|1 + 2\lambda||\underset{\sim}{a}| = 6$

$|1 + 2\lambda| = 2$

$1 + 2\lambda = \pm 2$

$\lambda = \frac{1}{2}$ or $-\frac{3}{2}$

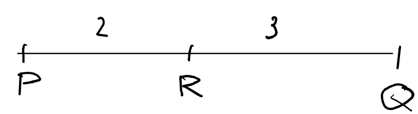
4 Point P and Q have position vectors $\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ respectively. Using ratio theorem in each of the following cases, find the position vector of R given that

(i) R divides PQ in the ratio $2:3$.

(ii) Given that point R lies on QP produced such that PR : $RQ = 3 : 4$.

[Solution]

(i) $PR : RQ = 2 : 3$

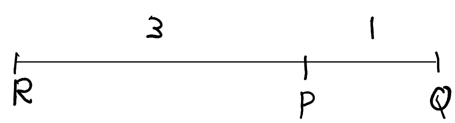


By Ratio Theorem,

$$\overrightarrow{OR} = \frac{1}{5}\left[2\overrightarrow{OQ} + 3\overrightarrow{OP}\right]$$

$$= \frac{1}{5}\left[2\begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}\right] = \frac{1}{5}\begin{pmatrix} 1 \\ 12 \\ -1 \end{pmatrix}.$$

(ii) By Ratio Theorem,



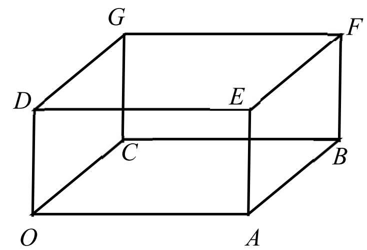
$$\overrightarrow{OP} = \frac{1}{4}\left[3\overrightarrow{OQ} + \overrightarrow{OR}\right]$$

$$\overrightarrow{OR} = 4\overrightarrow{OP} - 3\overrightarrow{OQ}$$

$$= 4\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} -10 \\ 16 \\ 27 \end{pmatrix}$$

5 The diagram shows a rectangular cuboid $OABCDEFG$ such that $\overrightarrow{OA}=\begin{pmatrix}1\\2\\4\end{pmatrix}$, $\overrightarrow{OC}=\begin{pmatrix}0\\2\\-1\end{pmatrix}$ and $\overrightarrow{OD}=\begin{pmatrix}-10\\1\\2\end{pmatrix}$.



(i) Find vector $\overrightarrow{CG}$ and the position vector of F.

(ii) Find the position vector of point M, the midpoint of GF.

[Solution]

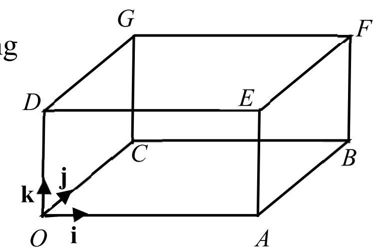
(i) $\overrightarrow{CG}=\overrightarrow{OD}=\begin{pmatrix}-10\\1\\2\end{pmatrix}$

$$\overrightarrow{OF}=\overrightarrow{OC}+\overrightarrow{CG}+\overrightarrow{GF}$$
$$=\begin{pmatrix}0\\2\\-1\end{pmatrix}+\begin{pmatrix}-10\\1\\2\end{pmatrix}+\begin{pmatrix}1\\2\\4\end{pmatrix}$$
$$=\begin{pmatrix}-9\\5\\5\end{pmatrix}$$

(ii) By ratio theorem, $\overrightarrow{OM}=\frac{1}{2}\left(\overrightarrow{OG}+\overrightarrow{OF}\right)$

$$=\frac{1}{2}\left(\overrightarrow{OC}+\overrightarrow{CG}+\overrightarrow{OF}\right)$$
$$=\frac{1}{2}\left[\begin{pmatrix}0\\2\\-1\end{pmatrix}+\begin{pmatrix}-10\\1\\2\end{pmatrix}+\begin{pmatrix}-9\\5\\5\end{pmatrix}\right]$$
$$=\frac{1}{2}\begin{pmatrix}-19\\8\\6\end{pmatrix}$$

6 The diagram shows a rectangular cuboid *OABCDEFG*
The point *O* is the origin and unit vectors **i**, **j**, **k** are taken along the direction of *OA*, *OC* and *OD* respectively. It is given that
$OA = 4$ units, $OC = 2$ units and $OD = \frac{3}{2}$ units



(i) Find vector $\overrightarrow{CG}$ and the position vector of *F*.
(ii) Find the position vector of point *N* which divides *BF* in the ratio 2 : 1

[Solution]

(i) $\overrightarrow{CG} = \overrightarrow{OD} = \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix}$

$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BF}$

$= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2} \end{pmatrix} + = \begin{pmatrix} 4 \\ 2 \\ \frac{3}{2} \end{pmatrix}$

(ii) $BN : NF = 2 : 1$

By ratio theorem, $\overrightarrow{ON} = \frac{1}{3}\left(2\overrightarrow{OF} + \overrightarrow{OB}\right)$

$= \frac{1}{3}\left[2\begin{pmatrix} 4 \\ 2 \\ \frac{3}{2} \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}\right]$

$= \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$