INNOVA JUNIOR COLLEGE

JC 2 PRELIMINARY EXAMINATION 2

in preparation for General Certificate of Education Advanced Level

Higher 1

CANDIDATE NAME	

Civics Group		INDEX NUMBER	

Mathematics **8863**

Paper 1

10 September 2008

3 hours

Additional materials:
Answer Papers, List of Formulae (MF15), Cover Page.

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, civics group and index number in the spaces at the top of this page.

Write in dark blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer all the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of 8 printed pages and NO blank page.

Section A: Pure Mathematics [40 marks]

1 Given that $\log_2 x - \log_2 (x + y) + 1 = 0$, show that it can be simplified to $y = x$. [1]
Hence solve the simultaneous equations

$$2e^{2x} = 5e^{y} + 3,$$
$$\log_2 x - \log_2 (x + y) + 1 = 0,$$

giving the answers in exact form. [4]

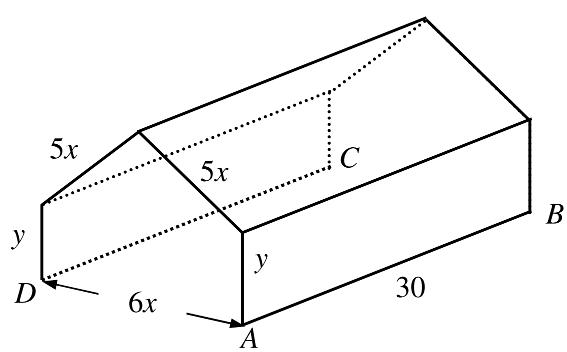
2 As part of an Art Project, a student designed a letterbox in the form of a prism. The cross-section forms a pentagon with two vertical sides of equal height, y cm, and two slant edges of equal length, $5x$ cm. The remaining faces are rectangles. A rectangular sheet of cardboard, $ABCD$, of area 1500 cm^2, is folded to make the surface $ABCD$ of the prism as shown in the diagram. The front, back and bottom surfaces are made of another material.

If the length of the letterbox is 30 cm and the width is $6x$ cm,

(i) show that the volume, V cm^3, enclosed by the letterbox is given by

$$V = 4500x - 540x^2,$$ [3]

(ii) determine the maximum value of V. [4]



3 **(i)** Show that the equation of the tangent to the curve $y = \dfrac{\left(e^x+1\right)^2}{e^x}$ at the point where $x = \ln 2$ is $y = \dfrac{3}{2}x - \dfrac{3}{2}\ln 2 + \dfrac{9}{2}$. Hence determine the coordinates of the point where this tangent cuts the y-axis. [5]

(ii) Find $\displaystyle\int \frac{\left(e^x+1\right)^2}{e^x}\,dx$. [1]

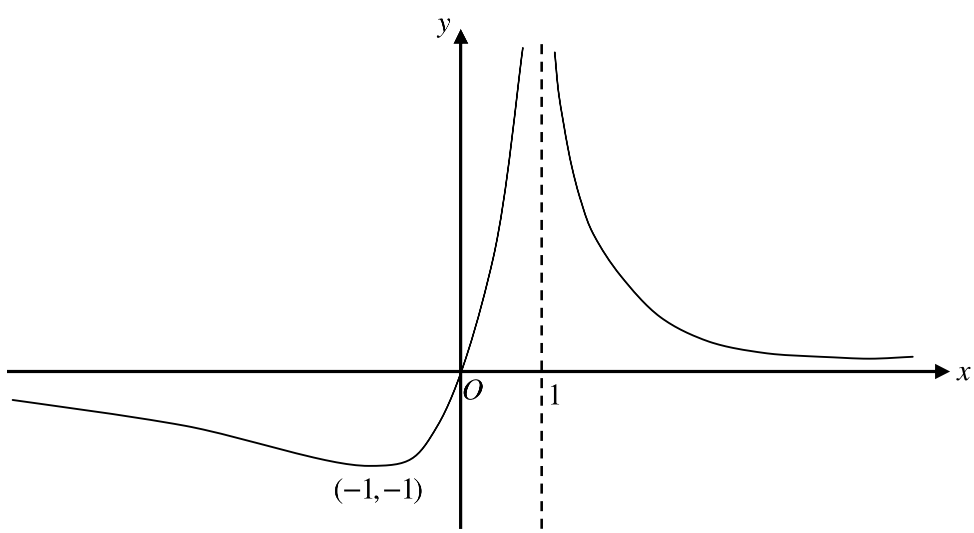
(iii) Find the exact area of the region bounded by the curve $y = \dfrac{\left(e^x+1\right)^2}{e^x}$, the tangent to the curve at the point where $x = \ln 2$ and the y-axis. [3]

4 **(a)** Sketch the graph of $y = \dfrac{1-2x}{1-x}$, giving the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

Hence, by sketching a suitable graph on the same diagram, solve the inequality

$$\frac{1-2x}{1-x} > 2^x.$$ [3]

(b) The diagram below shows the graph of $y = f(x)$. The curve passes through the origin O and has a minimum point $(-1,-1)$. The asymptotes of the graph are $x = 1$ and $y = 0$. Sketch the graph of $y = f'(x)$. [3]



[Turn over

5 The functions f and g are defined by

$$\mathrm{f}(x) = (x+1)^2, \qquad x \in \mathbb{R},$$
$$\mathrm{g}(x) = |\sin x|, \qquad 0 \le x \le 2\pi.$$

(i) The function f has an inverse if its domain is restricted to $x \ge k$. State the least value of k. Find an expression for $\mathrm{f}^{-1}(x)$ corresponding to this domain. [3]

(ii) Sketch the graph of y = g(x). Show that the composite function fg exists and find an expression for fg(x). State the range of fg. [5]

(iii) Solve the equation ff(x) = 4. [2]

Section B: Statistics [60 marks]

6 In a factory of 1500 employees, there are 150 management staff, 300 administrative staff and 1050 production workers. A survey is to be taken to check the factory's working conditions. A sample of 30 employees is needed for the survey. The factory uses a list with all the employees' names arranged in alphabetical order.

(i) One method of obtaining this sample is to select every fiftieth name on the name list. Identify the sampling method used here. Give a reason why the sample obtained may not be a good representation of the employees. [2]

(ii) Describe an alternative sampling method which would be better in this case. [2]

7 The life of fluorescent tubes made by a particular manufacturer has mean 758 hours and standard deviation 12 hours.

(i) Find the probability that the mean life of a random sample of 100 fluorescent tubes is greater than 756 hours. State, giving a reason, whether it is necessary to assume that the life of the fluorescent tubes has a normal distribution. [3]

(ii) A large random sample, n, of fluorescent tubes is randomly selected. Find the least value of n such that the probability the mean life of the fluorescent tubes exceeds 762 hours is at most 0.01. [4]

8 The volume of perfume in the bottles of perfume made by a particular manufacturer has mean μ ml. A random sample of 50 bottles is taken and the volume of the perfume in each bottle, x ml, is measured. The data are summarized by

$$\sum x = 9500 \text{ and } \sum x^2 = 1817544.$$

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) The manufacturer claims that $\mu = 195$. Test, at the 4% significance level, whether the manufacturer's claim overstates the value of μ. [3]

(iii) The null hypothesis $\mu = \mu_o$ is being tested. Find the range of possible values of μ_o for which $\mu = \mu_o$ is rejected in favour of $\mu \neq \mu_o$ at the 6% level of significance. [3]

[Turn over

9 A manufacturer of soft drinks launched a new drink. The table shows the weekly advertising expenditure (x) and weekly sales (y) during the launch period.

Advertising, \$$x$ (in thousands)	1.60	3.05	0.45	2.00	0.75	2.65
Weekly sales, \$$y$ (in thousands)	240	450	130	324	200	365

(a) **(i)** Find the equation of the regression line of y on x in the form $y = a + bx$, giving the values of a and b correct to 1 decimal place. [1]

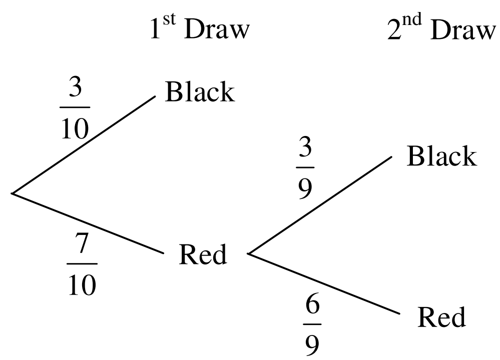
(ii) Interpret the values of a and b in terms of the amount spent on advertising and the weekly sales. [2]

(b) Find the equation of the regression line of x on y. [1]

(c) Find the product moment correlation coefficient of the data, and say what it leads you to expect about the scatter diagram for the data. [2]

(d) Some time later, the manufacturer launched another new drink. The manufacturer wants a weekly sale of $\$500000$ for this new drink. Using the regression line of y on x, estimate the amount of money the manufacturer has to spend on advertising. Comment on the reliability of the estimate. [2]

10 A bag contains 3 black balls and 7 red balls. A player draws balls at random from the bag, one by one without replacement, continuing until he gets a black ball. The tree diagram below illustrates the possibilities for the first 2 draws.



(i) Draw the tree diagram until the fifth draw, showing the probability for each branch. [2]

(ii) Show that the probability the player gets his first black ball on the fourth draw is $\frac{1}{8}$. [1]

(iii) Find the probability that the player needs at least 4 draws to get his first black ball. [4]

(iv) Given that the player needs at least 4 draws to get his first black ball, find the probability that he needs fewer than 6 draws to get his first black ball. [3]

[Turn over

11 The random variable X has a normal distribution with mean 15 and standard deviation 3. The independent random variable Y has a normal distribution with mean μ and standard deviation $\frac{\sqrt{7}}{2}$.

(i) Find the value of the constant a such that $\mathrm{P}(15-a<X<15+a)=0.90$. [3]

(ii) Given $\mathrm{P}(2Y-X>54.27)=0.165$, show that the value of μ is 32.7, correct to three significant figures. [5]

(iii) Two independent observations of X are denoted by X_1 and X_2. Find

$$\mathrm{P}(X_1+X_2>Y+3).$$

[3]

12 **(a)** A bag contains red and green balls. An experiment is conducted where n balls are drawn one by one with replacement. The colour of the ball drawn is noted before it is placed back in the bag. The random variable X denotes the number of red balls drawn. When the experiment was repeated a large number of times, it was found that the mean number of red balls drawn is 10.8 and the variance is 4.32. Find $\mathrm{P}(X\geq 16)$. [5]

(b) Market research has shown that 4 out of 7 clients of a bank make use of its internet banking services.

(i) A random sample of N clients is selected. Find the smallest value of N required so that the probability the sample contains at least one client who makes use of the bank's internet banking services is greater than 99.9 %. [3]

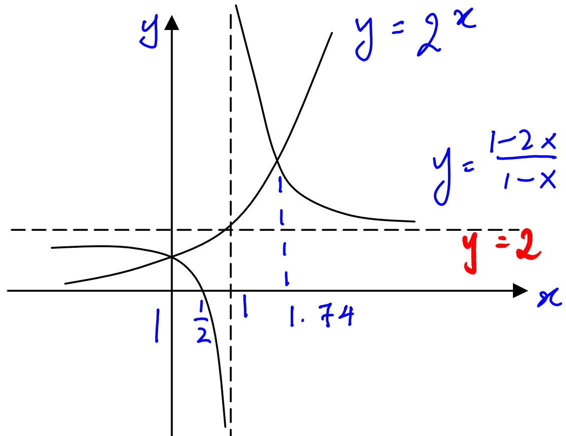
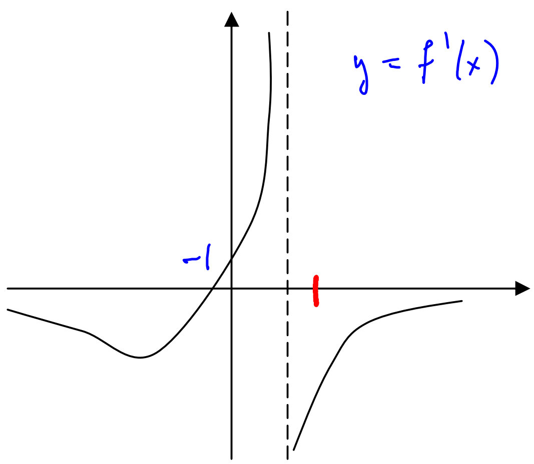
(ii) A random sample of 210 clients is chosen. Using a suitable approximation, find the probability that more than 100 clients use the bank's internet banking services. [4]

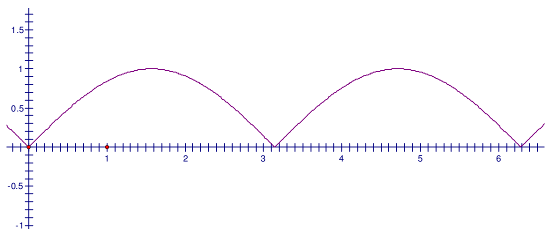
End of Paper

Innova Junior College **2008 Prelim 2 Exam H1 Maths**

Qn	Solutions
1	$\log_2 x-\log_2(x+y)+1=0 \Rightarrow \log_2\left(\frac{x}{x+y}\right)=-1$ $\frac{x}{x+y}=2^{-1} \Rightarrow 2x=x+y \Rightarrow x=y$ $2e^{2x}=5e^{y}+3$ $2e^{2x}-5e^{x}-3=0$ Let w be e^x, so $2w^2-5w-3=0$ Solving, $w=3$ or $w=-0.5$ (rejected) $\Rightarrow$ $x=\ln 3$ and $y=\ln 3$
2	$(2y+10x)\times 30=1500$ $2y+10x=50$ $y=25-5x$ $V=\left[6xy+\frac{1}{2}(6x)(4x)\right]\times 30=\left(6xy+12x^2\right)\times 30$ $=180xy+360x^2=180x(25-5x)+360x^2$ $=4500x-540x^2$ (Shown) $\frac{dV}{dx}=4500-1080x$ For maximum volume, $\frac{dV}{dx}=0$ $4500-1080x=0$ $x=\frac{25}{6}$ *or* 4.167 Need to show that Volume is maximum when $x=\frac{25}{6}$, $\frac{d^2V}{dx^2}=-1080<0$ Since $\frac{d^2V}{dx^2}<0$, V is maximum when $x=\frac{25}{6}$. $\text{Maximum Volume}=4500\left(\frac{25}{6}\right)-540\left(\frac{25}{6}\right)^2$ $=9375\,\text{cm}^3$

3	y = $\dfrac{e^{2x}+2e^{x}+1}{e^{x}} = e^{x}+2+e^{-x}$ $\dfrac{dy}{dx} = e^{x}-e^{-x}$ $x=\ln 2,\ y=\dfrac{9}{2}$ and $\dfrac{dy}{dx}=\dfrac{3}{2}$ $y-\dfrac{9}{2}=\dfrac{3}{2}\left(x-\ln 2\right)$ $y=\dfrac{3}{2}x-\dfrac{3}{2}\ln 2+\dfrac{9}{2}$ The coordinates are $\left(0,\dfrac{9}{2}-\dfrac{3}{2}\ln 2\right)$
3ii	$\displaystyle\int \frac{(e^{x}+1)^{2}}{e^{x}}\,dx$ $=\displaystyle\int \frac{e^{2x}+2e^{x}+1}{e^{x}}\,dx$ (*) $=\displaystyle\int e^{x}+2+e^{-x}dx$ $=e^{x}+2x-e^{-x}+c$
3iii	$\left(e^{\ln 2}+2\ln 2-e^{-\ln 2}\right)-\left(1+0-1\right)-\dfrac{1}{2}\ln 2\left(\dfrac{9}{2}+\dfrac{9}{2}-\dfrac{3}{2}\ln 2\right)$ $=\left(\dfrac{3}{2}+2\ln 2\right)-\dfrac{1}{2}\ln 2\left(9-\dfrac{3}{2}\ln 2\right)$ $=\dfrac{3}{2}-\dfrac{5}{2}\ln 2+\dfrac{3}{4}\left(\ln 2\right)^{2}$

4	 Asymptotes: $x=1$, $y=2$ Coordinates of points of intersection with axes: $(0.5,0)$ and $(0,1)$ To satisfy the inequality, $x<0$ or $1<x<1.74$
	

5i	$k=-1$ $y=(x+1)^2$ $x=-1\pm\sqrt{y}$ Since $x\geq -1$, $x=-1+\sqrt{y}$ $f^{-1}(x)=-1+\sqrt{x}$, $x\geq 0$
ii	
ii	$R_g=[0,1]\subseteq D_f=\mathbb{R}$, therefore fg exists. $\mathrm{fg}(x)=\mathrm{f}(\lvert\sin x\rvert)$ $=(\lvert\sin x\rvert+1)^2$, $0\leq x\leq 2\pi$ Range of $\mathrm{fg}(x)=[0,4]$
iii	$\mathrm{f}(x)=(x+1)^2$ $\mathrm{ff}(x)=4$ $\mathrm{f}\left[(x+1)^2\right]=4$ $\left((x+1)^2+1\right)^2=4$ $(x+1)^2+1=\pm 2$ $(x+1)^2=1$ or $(x+1)^2=-3$ (rejected) $x+1=1$ or $x+1=-1$ $x=0$ $\quad$ $x=-2$

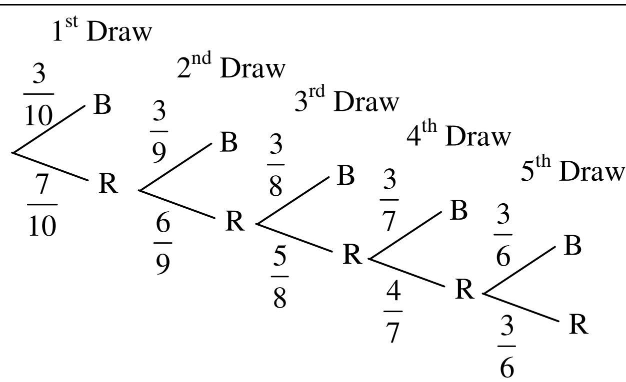
<table>
<tr><td>6(i)</td><td>Systematic random sampling
Biased as may not select enough management staff.</td></tr>
<tr><td>(ii)</td><td>Stratified Random Sampling
<table><tr><th>Strata</th><th>No. of employees</th></tr><tr><td>Management Staff</td><td>(150/1500) x 30 = 3</td></tr><tr><td>Admin Staff</td><td>(300/1500) x 30 = 6</td></tr><tr><td>Workers</td><td>(1050/1500) x 30 = 21</td></tr></table>The employees selected from each strata are to be chosen randomly.</td></tr>
<tr><td>7(i)</td><td>Let X be the random variable denoting the life of a fluorescent tube (in hours).
Since $n = 100$ is large, by Central Limit Theorem
$\overline{X} \sim N(758, \frac{12^2}{100})$ approx
$P(\overline{X} > 756) = 0.95221 = 0.952$ (3 s.f.)

No. Since $n = 100$ is large, Central Limit Theorem applies.</td></tr>
<tr><td>(ii)</td><td>Let $\overline{Y}$ be the random variable denoting the mean life of n fluorescent tubes.
$\overline{Y} \sim N(758, \frac{12^2}{n})$ approx by CLT since n is large
$P(\overline{Y} > 762) \le 0.01$
$P\left(Z > \frac{762-758}{\frac{12}{\sqrt{n}}}\right) \le 0.01$
when $n = 48$ $P(\overline{Y} > 762) = 0.01046 > 0.01$
when $n = 49$ $P(\overline{Y} > 762) = 0.00981 < 0.01$

Hence the least value of n is 49 fluorescent tubes.

Alternatively: (Algebraic Method)
Let $\overline{Y}$ be the random variable denoting the mean life of n fluorescent tubes.
$\overline{Y} \sim N(758, \frac{12^2}{n})$ approx by CLT since n is large
$P(\overline{Y} > 762) \le 0.01$
$1 - P(\overline{Y} \le 762) \le 0.01$
$P(\overline{Y} \le 762) \ge 0.9$
$P(Z \le \frac{762-758}{(12/\sqrt{n})}) \ge 0.99 \Rightarrow P(Z \le \frac{\sqrt{n}}{3}) \ge 0.99$
$\Rightarrow \frac{\sqrt{n}}{3} \ge 2.326347877 \Rightarrow n \ge 48.7$
Least value of $n = 49$.</td></tr>
</table>

8i	Let X be the random variable "volume of contents in each bottle." Unbiased estimate for population mean $=\bar{x}=\dfrac{9500}{900}=190$ Unbiased estimate for population variance $=\dfrac{1}{50-1}\left[1817544-\dfrac{9500^2}{50}\right]=256$
ii	To test $H_0: \mu=195$ against $H_0: \mu<195$ Perform a 1-tail test at 4 % level of significance. Under Ho, since $n=50$ is large, by CLT, $\overline{X} \sim N(195, \frac{256}{50})$ <u>Using GC</u> $\mu_0=195$ $\sigma=\sqrt{256}$ $\bar{x}=190$ $n=50$ Then p-value $=0.0135$ Since p-value $=0.0135<0.04$ Reject H_0. We conclude that there is sufficient evidence at the 4 % level of significance that $\mu<195$. Alternatively Critical Region: Reject H_0 if $z<-1.750686$ Test statistic: $z=\dfrac{190-195}{\sqrt{\frac{256}{50}}}$ Since $z=-2.209708<-1.750686$, reject H_0.
iii	To test $H_0: \mu=\mu_o$ against $H_0: \mu\neq\mu_o$ Perform a 2-tail test at 6 % level of significance. H_0 is rejected $\Rightarrow \dfrac{190-\mu_o}{\sqrt{\frac{256}{50}}}<-1.88079$ or $\dfrac{190-\mu_o}{\sqrt{\frac{256}{50}}}>1.88079$ $\Rightarrow \mu_o>194.255$ or $\mu_o<185.744$ $\Rightarrow \mu_o<186$ or $\mu_o>194$

9ai	$y = 89.2646 + 111.754x$ $y = 89.3 + 111.8x$
aii	The weekly sale of the soft drink is \$89300 when no money is spent on advertising. There is an increase of \$111.8 in weekly sales for every dollar invested in advertising.
b	$x = -0.704066 + 0.00861579y$ $x = -0.704 + 0.00862y$
c	$r = 0.981$ Since $r = 0.981$ is close to 1, the points representing the data on a scatter diagram would be close to a line with a positive gradient.
d	Sub $y = 500$ into $y = 89.2646 + 111.754x$ Then $x = 3.675353$. The manufacturer has to spend \$3680 on advertising. Since $y = 500$ is outside the range [130,450], the estimate is an extrapolation, it is not reliable.
10(i)	
(ii)	P(obtain 1st black ball in 4th draw) = $\frac{7}{10}.\frac{6}{9}.\frac{5}{8}.\frac{3}{7} = \frac{1}{8}$
(iii)	P(needs at least 4 draws) = 1 – [P(1 draw) + P(2 draws) + P(3 draws)] $= 1-[\frac{3}{10}+\frac{7}{10}.\frac{3}{9}+\frac{7}{10}.\frac{6}{9}.\frac{3}{8}]$ $= 1-\frac{17}{24} = \frac{7}{24}$
(iv)	P(needs fewer than 6 draws | needs at least 4 draws) $=\frac{P(\text{needs fewer than 6 draws} \cap \text{needs at least 4 draws})}{P(\text{needs at least 4 draws})}$ $=\frac{P(\text{needs 4 draws})+P(\text{needs 5 draws})}{P(\text{needs at least 4 draws})}$ $=\frac{\frac{7}{10}.\frac{6}{9}.\frac{5}{8}.\frac{3}{7}+\frac{7}{10}.\frac{6}{9}.\frac{5}{8}.\frac{4}{7}.\frac{3}{6}}{\left(\frac{7}{24}\right)} = \frac{5}{7}$

11(i)	Given: $X \sim N(15, 3^2)$ and $Y \sim N(\mu, \frac{7}{4})$ $P(15-a < X < 15+a) = 0.90$ $P(X < 15+a) = 0.95$ $15+a = 19.935$ $a = 4.935 = 4.93$ (3 s.f.)
(ii)	Consider $2Y - X$: $E(2Y-X) = 2E(Y) - E(X) = 2\mu - 15$ $Var(2Y-X) = 2^2 Var(Y) + Var(X) = 4(\frac{7}{4}) + 9 = 16$ $2Y - X \sim N(2\mu - 15,\ 16)$ $P(2Y - X > 54.27) = 0.165$ $P(2Y - X \le 54.27) = 0.835$ $P(Z \le \frac{54.27-(2\mu-15)}{\sqrt{16}}) = 0.835 \Rightarrow \frac{69.27-2\mu}{4} = 0.974114$ $\Rightarrow 69.27 - 2\mu = 3.896456 \Rightarrow \mu = 32.686772 = 32.7$ (3 s.f.) (shown)
(iii)	Consider $X_1 + X_2 - Y$: $E(X_1 + X_2 - Y) = 2E(X_1) - E(Y) = 2(15) - (32.7) = -2.7$ $Var(X_1 + X_2 - Y) = 2Var(X_1) + Var(Y) = 2(9) + \left(\frac{7}{4}\right) = 19.75$ $X_1 + X_2 - Y \sim N(-2.7,\ 19.75)$ $P(X_1 + X_2 > Y + 3) = P(X_1 + X_2 - Y > 3) = 0.09981615 = 0.0998$ (3 s.f.)
12a	Let X be the r.v. “number of red balls”. $X \sim B(n,\ p)$. $np = 10.8$ and $npq = 4.32$ $\Rightarrow q = \frac{4.32}{10.8} = 0.4$, $p = 0.6$, $n = \frac{10.8}{0.6} = 18$ $P(X \ge 16)\ =\ 1 - P(X \le 15) =\ 0.00823$
bi	$1 - \left(\frac{3}{7}\right)^n > 0.999 \Rightarrow \left(\frac{3}{7}\right)^n < 0.001$ $n > \ln(0.001)/\ln\left(\frac{3}{7}\right) \Rightarrow n > 8.15$ A sample size of at least 9 clients is needed.
ii	Let Y be the r.v. “number of clients using internet banking out of 210 clients.” $Y \sim B(210, \frac{4}{7})$ Since $n = 210$ is large, $np = 210(\frac{4}{7}) = 120 > 5$ and $nq = 210(\frac{3}{7}) = 90 > 5$, $Y \sim N(120, \frac{360}{7})$ approximately $P(Y > 100) \xrightarrow{cc} P(Y > 100.5) =\ 0.996727 =\ 0.997$ (3 s.f.)