INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATIONS 2
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CLASS | INDEX NUMBER

MATHEMATICS 9740/01

Paper 1 **10 September 2008**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)
Cover Page.

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, civics group and index number in the spaces at the top of this page.

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages and **1** blank page.

1 The curve $y = Ax + \frac{B}{\sqrt{x}} + Cx^2$ passes through the point $(9,-62)$ and the tangent to the curve at the point $(1,4)$ is $2y+3x=11$. Find the exact value of y when $x=5$. [5]

2 In triangle OAB, $\angle OAB = 90^\circ$ and the point C on AB is such that $AC = \frac{2}{3}CB$. With respect to the origin O, the position vectors of A and B are given as **a** and **b** respectively.

(i) Show that $\mathbf{a}.\mathbf{b} = |\mathbf{a}|^2$. [1]

(ii) Find **c**, the position vector of C in terms of **a** and **b**. [1]

(iii) Given that the lengths of OA and OB are 3 and 5 units respectively, find the length of projection of **c** onto **b**. [3]

3 Given that x is sufficiently small for x^3 and higher powers of x to be neglected, show that

$$\frac{(8+x)^{\frac{1}{3}}}{\cos 2x} \approx 2 + \frac{1}{12}x + kx^2,$$

where k is a fraction to be determined. [5]

4 The curve C has equation

$$4(x+1)^2 - (y-1)^2 = 1.$$

(i) Sketch the curve C, stating the equations of the asymptotes clearly. [4]

(ii) Find the greatest value of k, where k is a positive integer, for which the curve $y = \ln(x+k)$ cuts C at only one point. [2]

5 A curve is given parametrically by

$$x = \cos^{-1} 2t,$$

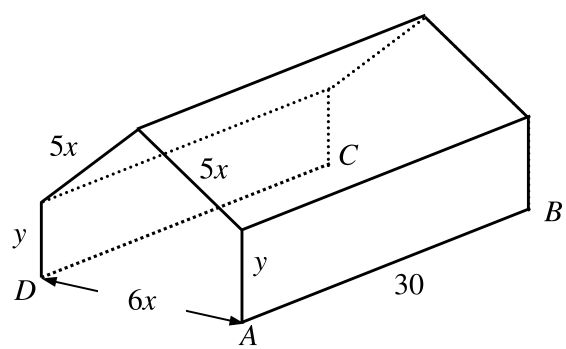
$$y = \frac{t}{\sqrt{1-4t^2}},$$

where $|t| < \frac{1}{2}$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [4]

(ii) Find the equation of the normal to the curve at the point where the curve cuts the x-axis, giving your answer in exact form. [3]

6 As part of an Art Project, a student designed a letterbox in the form of a prism. The cross-section forms a pentagon with two vertical sides of equal height, y cm, and two slant edges of equal length, $5x$ cm. The remaining faces are rectangles. A rectangular sheet of cardboard, $ABCD$, of area 1500 cm^2, is folded to make the surface $ABCD$ of the prism as shown in the diagram. The front, back and bottom surfaces are made of another material.



If the length of the letterbox is 30 cm and the width is $6x$ cm,

(i) show that the volume, V cm^3, enclosed by the letterbox is given by

$$V = 4500x - 540x^2,$$ [3]

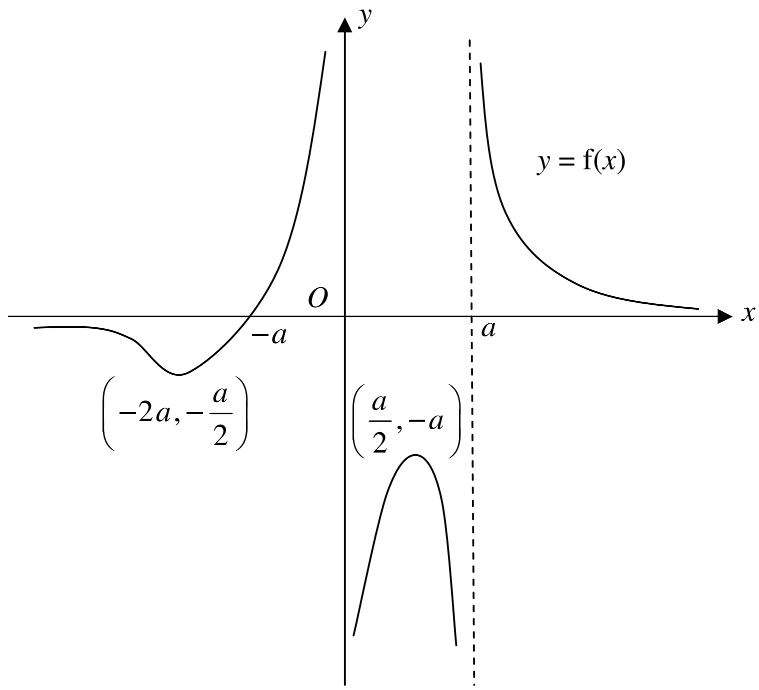
(ii) determine the maximum value of V. [4]

7 The diagram below shows the graph of $y = \mathrm{f}(x)$. There is a minimum at the point $\left(-2a, -\dfrac{a}{2}\right)$, a maximum at the point $\left(\dfrac{a}{2}, -a\right)$ and the curve cuts the x-axis at the point $(-a, 0)$. The curve has asymptotes $x = 0$, $x = a$ and $y = 0$. Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}(x+a)$, [2]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(iii) $y = \mathrm{f}'(x)$. [3]



8 The functions f and g are defined by

$$\mathrm{f} : x \mapsto 1 + \mathrm{e}^{-x}, \qquad x \in \mathbb{R},$$
$$\mathrm{g} : x \mapsto \lg(2x-3) - 2, \qquad x \geq 2.$$

(a) **(i)** State the range of f. [1]

(ii) Find an expression for $\mathrm{f}^{-1}(x)$. [2]

(iii) On the same diagram, sketch the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$. [5]

(b) Show that gf does not exist. [1]

The composite function gf exists if the domain of f is restricted to $x \leq k$, state the greatest value of *k*. [1]

9 **(a)** In an inter-house games competition, students from Green house decided to form a relay team. The first runner of the team will run 800 m, the second runner will run the next 808 m and each subsequent runner will increase the distance run by 8 m. Determine the minimum number of students needed to complete a total run of at least 80 km. [4]

(b) The sequence U_1, U_2, U_3, … is such that $U_1 = 1$, $U_2 = 3$ and $U_n = \frac{1}{2}U_{n-2} + 3$ for $n \geq 3$.

(i) Write down the values of U_4 and U_6. [1]

(ii) Show that $U_{2n} = 6\left(1 - \frac{1}{2^n}\right)$ for $n \geq 1$. [3]

(iii) Show that $\dfrac{U_{2(n+1)} - U_{2n}}{U_{2n} - U_{2(n-1)}}$ is a constant, and explain the significance of this result. [2]

10 **(a)** Given that

$$(4-\mathrm{i})^2+(8\lambda+\mathrm{i})(3\mu-\mathrm{i})+8\mathrm{i}=43,$$

find the exact values of the real numbers λ and μ. [4]

(b) Express the complex number $-\frac{1}{2}\left(1+\mathrm{i}\sqrt{3}\right)$ in exponential form. [2]

Solve the equation

$$\left(w+2\right)^4=-\frac{1}{2}\left(1+\mathrm{i}\sqrt{3}\right),$$

giving your answers in form $a+\mathrm{i}b$, where a and b are real values. [4]

Show that the points representing the roots of the given equation in an Argand diagram lie on a circle. Write down the centre and radius of this circle. [2]

11 **(i)** Express $\dfrac{2r^2-2r-1}{r\left(r-1\right)}$ in partial fractions. [2]

Hence find $\displaystyle\sum_{r=2}^{n}\frac{2r^2-2r-1}{r\left(r-1\right)}$, giving your answer in the form $k+\mathrm{f}\left(n\right)$, where k is a constant. [3]

(ii) Prove by induction that

$$\sum_{r=1}^{n}\frac{r+1}{2^r}=3-\left(\frac{1}{2}\right)^n\left(n+3\right).$$ [5]

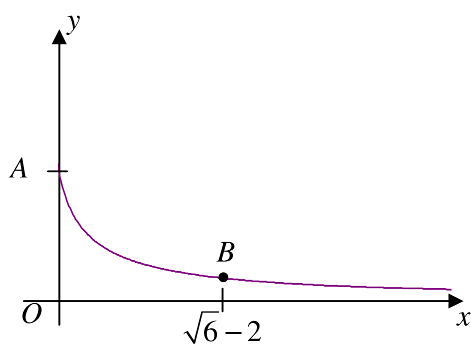
(iii) Using the above results, evaluate

$$\lim_{n\to\infty}\sum_{r=2}^{n}\left(\frac{2r^2-2r-1}{nr\left(r-1\right)}-\frac{r+1}{2^r}\right).$$ [3]

12 **(i)** Use the substitution $y = \dfrac{1}{x+2}$ to show that

$$\int_0^{\sqrt{6}-2} \frac{1}{(x+2)\sqrt{x^2+4x+1}}\,\mathrm{d}x = \frac{\pi}{12\sqrt{3}}. \qquad [6]$$

(ii) The graph of $y = \dfrac{1}{(x+2)\sqrt{x^2+4x+1}}$, for $0 \le x \le \sqrt{6}-2$, is shown in the diagram. A and B are points on the curve $y = \dfrac{1}{(x+2)\sqrt{x^2+4x+1}}$ where $x = 0$ and $x = \sqrt{6}-2$ respectively.



(a) The region R is enclosed by the chord AB and the arc AB of the curve. Using the result in part **(i)**, find the exact area of R. [3]

(b) The region S is enclosed by the arc AB, the y-axis and the line $y = \dfrac{1}{3\sqrt{2}}$. Find the volume of the solid of revolution formed when S is rotated through $360°$ about the x-axis, giving your answer correct to 3 significant figures. [3]

End of Paper

	IJC 2008 Prelim 2 H2 Maths Paper 1 (Solutions)	
1	At $(9,-62)$, $27A+B+243C=-186 \quad \cdots\cdots(1)$ At $(1,4)$, $A+B+C=4 \quad \cdots\cdots(2)$ $2y+3x=11$ $y=-\frac{3}{2}x+\frac{11}{2}$ At $(1,4)$, gradient = $-\frac{3}{2}$ $\frac{dy}{dx}=A-\frac{B}{2}x^{-\frac{3}{2}}+2Cx$ $2A-B+4C=-3 \quad \cdots\cdots(3)$ Using GC, Solving eq (1), (2) & (3) simultaneously $A=2,\ B=3,\ C=-1$ $y=2x+\frac{3}{\sqrt{x}}-x^2$ When $x=5$, $y=-15+\frac{3\sqrt{5}}{5}$	
2i	$\overline{AB}=\overline{OB}-\overline{OA}$ $=\mathbf{b}-\mathbf{a}$ Since $\overline{OA}$ is perpendicular to $\overline{AB}$, $\overline{OA}\bullet\overline{AB}=0$ $\mathbf{a}\bullet(\mathbf{b}-\mathbf{a})=0$ $\mathbf{a}\bullet\mathbf{b}-\lvert\mathbf{a}\rvert^2=0$ $\therefore \mathbf{a}\bullet\mathbf{b}=\lvert\mathbf{a}\rvert^2$	
ii	$\overline{AC}=\frac{2}{3}\overline{CB}$ or: $\overline{OC}-\overline{OA}=\frac{2}{3}\left(\overline{OB}-\overline{OC}\right)$ $\mathbf{c}=\frac{3}{5}\mathbf{a}+\frac{2}{5}\mathbf{b}$	Using ratio theorem, $\mathbf{c}=\frac{2\mathbf{b}+3\mathbf{a}}{5}=\frac{3}{5}\mathbf{a}+\frac{2}{5}\mathbf{b}$

iii Length of projection of $\mathbf{c}$ onto $\mathbf{b}$

$$= \left| \frac{\overline{OC} \cdot \overline{OB}}{\left|\overline{OB}\right|} \right| = \left| \frac{\left(\frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b} \right) \cdot \mathbf{b}}{|\mathbf{b}|} \right|$$

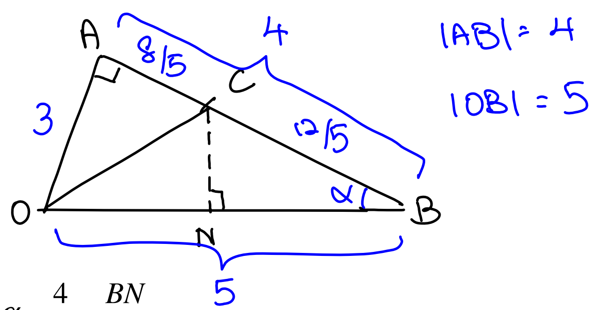
$$= \left| \frac{\frac{3}{5}\mathbf{a} \bullet \mathbf{b} + \frac{2}{5}|\mathbf{b}|^2}{|\mathbf{b}|} \right|$$

$$= \left| \frac{\frac{3}{5}|\mathbf{a}|^2 + \frac{2}{5}|\mathbf{b}|^2}{|\mathbf{b}|} \right|$$

$$= \left| \frac{\frac{3}{5}(3)^2 + \frac{2}{5}(5)^2}{5} \right|$$

$$= 3\frac{2}{25} = 3.08$$

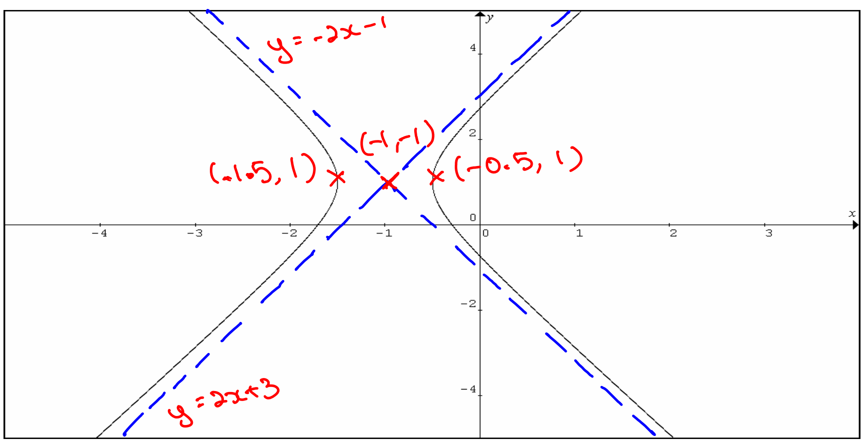
Alternative:



$$\cos\alpha = \frac{4}{5} = \frac{BN}{12/5}$$

$$BN = \frac{48}{25}$$

$$ON = 5 - \frac{48}{25} = 3.08$$

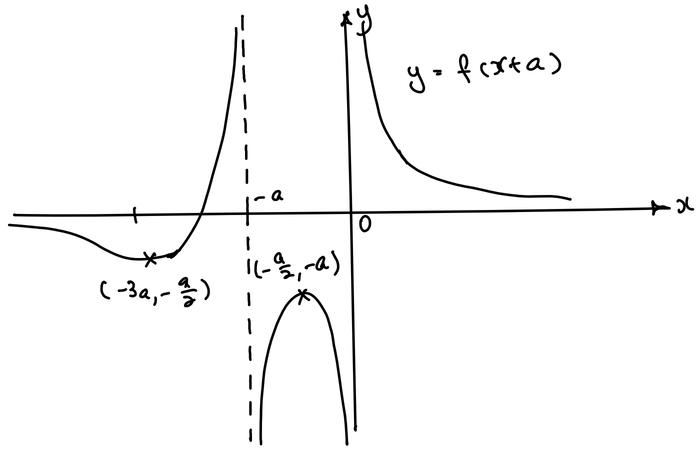
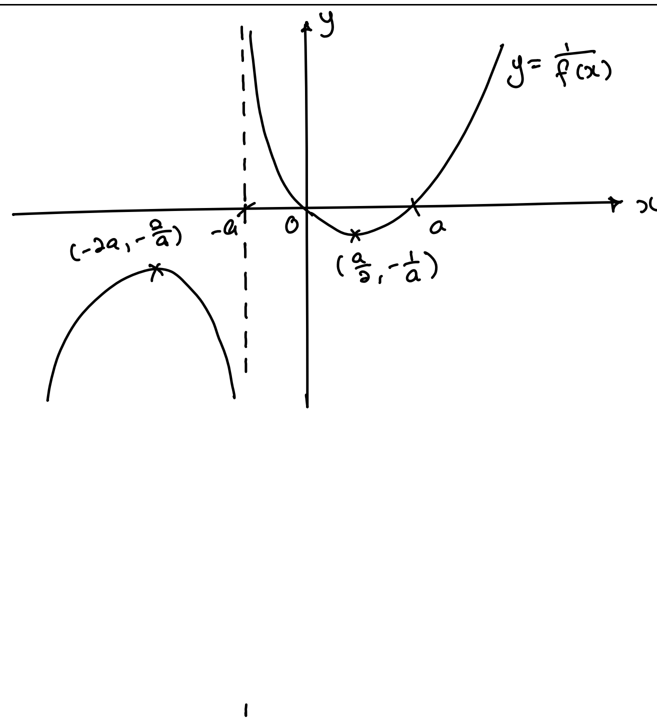
3	$(8+x)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(1+\frac{x}{8}\right)^{\frac{1}{3}}$ $= 2\left(1+\frac{x}{24}-\frac{x^2}{576}+\cdots\right)$ $= 2+\frac{x}{12}-\frac{x^2}{288}+\cdots$ Since x is small, $\frac{1}{\cos 2x} \approx \frac{1}{(1-2x^2)}$ $= (1-2x^2)^{-1}$ $= 1+2x^2+\cdots$ $\frac{(8+x)^{\frac{1}{3}}}{\cos 2x} = \left(2+\frac{x}{12}-\frac{x^2}{288}+\cdots\right)(1+2x^2+\cdots)$ $\approx 2+\frac{1}{12}x+\frac{1151}{288}x^2$ $k = \frac{1151}{288}$
4i	
ii	$k = 2$

5i	$x=\cos^{-1}2t$ $\qquad$ $y=\dfrac{t}{\sqrt{1-4t^2}}$ $\dfrac{dx}{dt}=-\dfrac{2}{\sqrt{1-4t^2}}$ $\qquad$ $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\sqrt{1-4t^2}-\frac{1}{2}t\left(\frac{-8t}{\sqrt{1-4t^2}}\right)}{1-4t^2}$ $\dfrac{dy}{dt}=\dfrac{1}{\left(1-4t^2\right)^{\frac{3}{2}}}$ $\dfrac{dy}{dx}=-\dfrac{1}{2\left(1-4t^2\right)}$
ii	When y = 0, t = 0 and $x=\dfrac{\pi}{2}$ $\dfrac{dy}{dx}=-\dfrac{1}{2}$ $y-0=2\left(x-\dfrac{\pi}{2}\right)$ $y=2x-\pi$
6i	$(2y+10x)\times 30=1500$ $2y+10x=50$ $y=25-5x$ $V=\left[6xy+\frac{1}{2}(6x)(4x)\right]\times 30$ $=(6xy+12x^2)\times 30$ $=180xy+360x^2$ $=180x(25-5x)+360x^2$ $=4500x-540x^2$ (Shown)
ii	$\dfrac{dV}{dx}=4500-1080x$ For maximum volume, $\dfrac{dV}{dx}=0$ $4500-1080x=0$ $x=\dfrac{25}{6}$ *or* 4.167

<table>
<tr><td></td><td>Need to show that Volume is maximum when $x=\dfrac{25}{6}$,

$\dfrac{d^2V}{dx^2}=-1080<0$

Since $\dfrac{d^2V}{dx^2}<0$, V is maximum when $x=\dfrac{25}{6}$.

$\text{Maximum Volume}=4500\left(\dfrac{25}{6}\right)-540\left(\dfrac{25}{6}\right)^2$
$=9375\,\text{cm}^3$</td></tr>
<tr><td>7i</td><td>
</td></tr>
<tr><td>ii</td><td>
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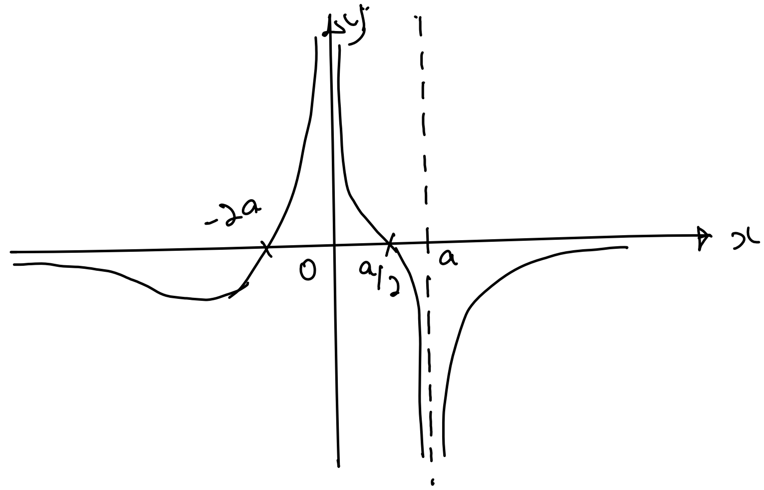
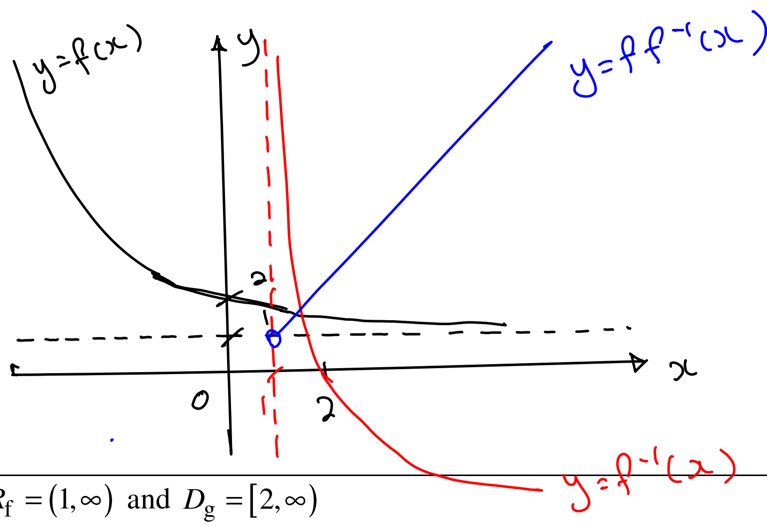
iii	
8ai	$R_{\mathrm{f}} = (1, \infty)$
ii	$y = 1 + \mathrm{e}^{-x}$ $x = -\ln(y-1)$ $\mathrm{f}^{-1}(x) = -\ln(x-1),\ x > 1$
iii	
b	$R_{\mathrm{f}} = (1, \infty)$ and $D_{\mathrm{g}} = [2, \infty)$ $R_{\mathrm{f}} \not\subset D_{\mathrm{g}}$ gf does not exist. $1 + \mathrm{e}^{-x} = 2$ $\mathrm{e}^{-x} = 1$ $x = 0$ Greatest $k = 0$

9a	$\frac{n}{2}\left[2(800)+(n-1)8\right]\geq 80000$ $n^2+199n-20000\geq 0$ $n\geq 73.42$ or $n\leq -272.42$ The minimum number of students needed is 74.
bi	$U_4=4.5$ $U_6=5.25$
ii	$U_{2n}=\frac{1}{2}U_{2n-2}+3$ $=\left(\frac{1}{2}\right)^2 U_{2n-4}+\left(\frac{1}{2}\right)3+3$ $=\left(\frac{1}{2}\right)^{\frac{2n-2}{2}} U_{2n-(2n-2)}+3\left[\left(\frac{1}{2}\right)^{\frac{2n-2}{2}-1}+\left(\frac{1}{2}\right)^{\frac{2n-2}{2}-2}+.....+3\right]$ $=\left(\frac{1}{2}\right)^{n-1}3+3\left[\frac{1-\left(\frac{1}{2}\right)^{n-1}}{\frac{1}{2}}\right]=3\left(\frac{1}{2}\right)^{n-1}+6\left[1-\left(\frac{1}{2}\right)^{n-1}\right]$ $=6-3\left(\frac{1}{2}\right)^{n-1}=6\left(1-\frac{1}{2^n}\right)$
iii	$\frac{U_{2(n+1)}-U_{2n}}{U_{2n}-U_{2(n-1)}}=\frac{\frac{6}{2^{n+1}}}{\frac{6}{2^n}}=\frac{1}{2}$ $(U_{2(i+1)}-U_{2i})$ forms geometric sequence with common ratio as $\frac{1}{2}$.

10a	$(4-\mathrm{i})^2+(8\lambda+\mathrm{i})(3\mu-\mathrm{i})=43-8\mathrm{i}$, $(16-8\mathrm{i}-1)+(24\lambda\mu-8\lambda\mathrm{i}+3\mu\mathrm{i}+1)=43-8\mathrm{i}$ $(16+24\lambda\mu)+\mathrm{i}(3\mu-8\lambda-8)=43-8\mathrm{i}$ Equating real and imaginary parts: $16+24\lambda\mu=43 ----(1)$ $3\mu-8\lambda-8=-8----(2)$ From (2) : $3\mu=8\lambda----(3)$ Sub (3) into (2): $16+3\mu(3\mu)=43$ $\mu^2=3 \Rightarrow \mu=\pm\sqrt{3}$ $\lambda=\pm\frac{3\sqrt{3}}{8}$
b	$\left\|-\frac{1}{2}\left(1+\mathrm{i}\sqrt{3}\right)\right\|=1,\ \arg\left[-\frac{1}{2}\left(1+\mathrm{i}\sqrt{3}\right)\right]=-\pi+\tan^{-1}\sqrt{3}=-\frac{2\pi}{3}$ $-\frac{1}{2}\left(1+\mathrm{i}\sqrt{3}\right)=\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$. $(w+2)^4=-\frac{1}{2}\left(1+\mathrm{i}\sqrt{3}\right)$ $\Rightarrow (w+2)^4=\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}$ $\Rightarrow w+2=\mathrm{e}^{\mathrm{i}\left(\frac{-\frac{2\pi}{3}+2k\pi}{4}\right)}$, where k = 0, 1, 2, 3. $\Rightarrow w=\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}+\frac{k\pi}{2}\right)}-2$, where k = 0, 1, 2, 3. $\Rightarrow w=\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}-2, \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}\right)}-2, \mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{6}\right)}-2, \mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}-2$ $=\left(\frac{\sqrt{3}}{2}-2\right)-\frac{1}{2}\mathrm{i}, -\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{i}, \left(-\frac{\sqrt{3}}{2}-2\right)+\frac{1}{2}\mathrm{i}, -\frac{5}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$. $\|w+2\|^4=\left\|-\frac{1}{2}\left(1+\mathrm{i}\sqrt{3}\right)\right\|=1$ $\Rightarrow \|w+2\|=1$ Hence, the points representing the roots of the given equation in an Argand diagram lie on a circle with centre (-2,0) and radius 1 unit.

11i	$\frac{2r^2-2r-1}{r(r-1)}=2-\frac{1}{r(r-1)}$ $=2+\frac{1}{r}-\frac{1}{r-1}$ $\sum_{r=2}^{n}\frac{2r^2-2r-1}{r(r-1)}=\sum_{r=2}^{n}\left(2+\frac{1}{r}-\frac{1}{r-1}\right)$ $=\sum_{r=2}^{n}2+\sum_{r=2}^{n}\left(\frac{1}{r}-\frac{1}{r-1}\right)$ $=2(n-1)+$ $\frac{1}{2}-\frac{1}{1}$ $+\frac{1}{3}-\frac{1}{2}$ $+\frac{1}{4}-\frac{1}{3}$ $\vdots$ $+\frac{1}{n-1}-\frac{1}{n-2}$ $+\frac{1}{n}-\frac{1}{n-1}$ $=2(n-1)-1+\frac{1}{n}$ $=-3+\left(2n+\frac{1}{n}\right)$
ii	Let P_n be the statement $\sum_{r=1}^{n}\frac{r+1}{2^r}=3-\left(\frac{1}{2}\right)^n(n+3)$ for $n\in\mathbb{Z}^+$. Show P_1 is true. $LHS=\frac{1+1}{2}=1$ $RHS=3-\frac{1}{2}(1+3)=1=LHS$ $\therefore P_1$ is true. Assume P_k is true for some $k\in\mathbb{Z}^+$, i.e $\sum_{r=1}^{k}\frac{r+1}{2^r}=3-\left(\frac{1}{2}\right)^k(k+3)$ Show that P_{k+1}, i.e. $\sum_{r=1}^{k+1}\frac{r+1}{2^r}=3-\left(\frac{1}{2}\right)^{k+1}(k+4)$

	$\sum_{r=1}^{k+1}\frac{r+1}{2^r}=3-\left(\frac{1}{2}\right)^k(k+3)+\frac{k+2}{2^{k+1}}$ $=3-\left(\frac{1}{2}\right)^{k+1}\left[2(k+3)-(k+2)\right]$ $=3-\left(\frac{1}{2}\right)^{k+1}(k+4)$ Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is also true. By mathematical induction, P_n is true for $n\in\mathbb{Z}^+$.
iii	$\lim_{n\to\infty}\sum_{r=2}^{n}\left(\frac{2r^2-2r-1}{nr(r-1)}-\frac{r+1}{2^r}\right)$ $=\lim_{n\to\infty}\left[\left(-\frac{3}{n}+2+\frac{1}{n^2}\right)-\left(3-\left(\frac{1}{2}\right)^n(n+3)-1\right)\right]$ $=0$
12(i)	$\int_0^{\sqrt{6}-2}\frac{1}{(x+2)\sqrt{x^2+4x+1}}\,\mathrm{d}x$ $=-\int_{\frac{1}{2}}^{\frac{1}{\sqrt{6}}}\frac{y}{\sqrt{\left(\frac{1}{y}\right)^2-3}}\frac{\mathrm{d}y}{y^2}$ $=\int_{1/\sqrt{6}}^{1/2}\frac{1}{\sqrt{1-3y^2}}\,\mathrm{d}y$ $=\frac{1}{\sqrt{3}}\left[\sin^{-1}\frac{y}{1/\sqrt{3}}\right]_{\frac{1}{\sqrt{6}}}^{\frac{1}{2}}=\frac{\pi}{12\sqrt{3}}$
(ii) (a)	When $x=0$, $y=\frac{1}{2}$ When $x=\sqrt{6}-2$, $y=\frac{1}{3\sqrt{2}}$ Required Area = $\frac{1}{2}\left(\frac{1}{2}+\frac{\sqrt{2}}{6}\right)\left(\sqrt{6}-2\right)-\frac{\pi}{12\sqrt{3}}$
(b)	Required Volume $=\pi\int_0^{\sqrt{6}-2}\left(\frac{1}{(x+2)\sqrt{x^2+4x+1}}\right)^2\mathrm{d}x-\pi\left(\frac{1}{3\sqrt{2}}\right)^2\left(\sqrt{6}-2\right)$ $=0.0888$