INNOVA JUNIOR COLLEGE JC 2 PRELIMINARY EXAMINATIONS 2 in preparation for General Certificate of Education Advanced Level Higher 2						
CANDIDATE NAME						
CLASS		INDEX NUMBER				

MATHEMATICS

9740/02

Paper 2 17 September 2008

3 hours

Additional Materials: Answer Paper

List of Formulae (MF15)

Cover Page.

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, civics group and index number in the spaces at the top of this page.

Write your name and class on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [] at the end of each question or part question.

This document consists of 6 printed pages.

Section A: Pure Mathematics [40 marks]

1 Solve the inequality
$$\frac{2x+1}{x+1} \ge x+8$$
. [4]

- 2 On a single Argand diagram, sketch the following loci
 - (i) $|z-1-i| = \sqrt{2}$,

(ii)
$$\arg(z-1-i) = \frac{2}{3}\pi$$
. [3]

Hence, or otherwise, find the exact value of z satisfying both equations in parts (i) and (ii). [3]

- Given that $y = \sqrt{9 + \sin 2x}$, show that $y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y^2 = 18$. [3] By repeated differentiation of this result, or otherwise, find the series expansion of y in ascending powers of x up to and including the term in x^3 .[4]
- The line l has equation $\mathbf{r} = \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} 2\mathbf{j} + \mathbf{k}), \lambda \in \mathbb{R}$. The plane π_1 has equation x + 2y + 3z = 5. The point A on l is given by $\lambda = 2$ and the point B has position vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.
 - (i) Show that the line l lies in the plane π_1 . [2]
 - (ii) Find the acute angle between the line AB and the plane π_1 . [4]
 - (iii) The plane π_2 is perpendicular to the plane π_1 and parallel to the line l, and contains the point B. Find the equation of π_2 . [3]
 - (iv) The plane π_3 has equation 6x + y 4z = -3. Give a geometrical relationship between the 3 planes π_1, π_2 and π_3 . [1]

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5 (a) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sin^2 3x \,. \tag{4}$$

(b) In established forest fires, the proportion of the total area of the forest which has been destroyed is denoted by x, and the rate of change of x with respect to time, t hours, is called the destruction rate. Investigations show that the destruction rate is directly proportional to the product of x and (1-x). A particular fire is initially noticed when one half of the forest is destroyed, and it is found that the destruction rate at this time is such that, if it remained constant thereafter, the forest would be destroyed completely in a further 24 hours.

Show that
$$12\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x)$$
. [3]

Deduce that approximately 73% of the forest is destroyed 12 hours after it is first noticed. [6]

Section B: Statistics [60 marks]

- 6 (a) Find the number of different arrangements of the nine letters in the word CELEBRATE in which the two letters A and T are next to each other. [2]
 - (b) Find the number of 3-letter code-words that can be formed from the letters of the word CELEBRATE. [3]
- A certain game is played with two packs of cards and two unbiased dice. The first pack of cards contains the usual 52 cards. The second pack of cards contains only 36 cards, all the Aces, Kings, Queens and Jacks having been removed.

A card is drawn from the first pack. If the card drawn is an Ace, King, Queen or Jack, a card is drawn from the second pack. The number on the card drawn from the second pack is the player's score (i.e. two scores 2 points, three scores 3, and so on).

If any other card is drawn from the first pack, the two unbiased dice are thrown. The sum of the scores on the dice is the player's score.

Events *X* and *Y* are defined as follows:

X: the 2 unbiased dice are thrown,

Y: the player's score is less than 5.

Find the probabilities

(i)
$$P(Y)$$
, [3]

(ii)
$$P(X \cup Y')$$
. [3]

A manufacturer tests the quality of the jars of kaya he produces by taking a sample of 20 jars from a consignment consisting of 500 jars. Describe how he could use systematic random sampling. [2]

Each jar is labelled as containing m grams of kaya. A random sample of 20 jars is examined and the mass, x grams, of the contents of each jar is determined. It is found that

$$\sum (x-380) = 120$$
 and $\sum (x-380)^2 = 3100$.

The manufacturer assumed that the mass is normally distributed and used the above data to carry out a test at the 6% level of significance. The result led him to conclude that he had overstated the mean mass of kaya in a jar. Find the range of possible values of m. [6]

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- 9 The random variable *X* has the binomial distribution, B(n,p), where 0 .
 - (a) Given that $Var(X) = \frac{4}{5} E(X)$, find the least value of n such that $P(X \ge 1) > 0.92$.
 - (b) Given that n = 8 and $p = \frac{1}{3}$, the random variable S is the sum of 60 independent observations of X. Find the approximate value of P(S > 162).
- An athlete's best times for various distances are shown in the following set of

Distance (x metres)						
Best time (<i>t</i> seconds)	11.2	21.8	51.5	110.3	220.3	1775

It is suspected that x and t are related according to the formula $t = ax^b$, where a and b are constants.

- (i) Show that the relation between $\lg t$ and $\lg x$ is linear. By considering the line of regression of $\lg t$ on $\lg x$, find the estimated values of a and b. [4]
- (ii) Find the product moment correlation coefficient between $\lg t$ and $\lg x$, giving your answer to four decimal places. Comment on this value. [2]
- (iii) Estimate the athlete's time for a 42.2 km Marathon and comment on the reliability of your answer. [3]

- A garage has 2 vans and 3 cars, which can be hired out for a day at a time. Requests for the hire of a van follow a Poisson distribution with a mean of 1.5 requests per day and requests for the hire of a car follow an independent Poisson distribution with a mean of 4 requests per day.
 - (i) Find the probability that not all requests for the hire of a van can be met on any particular day. [2]
 - (ii) Find the least number of vans that the garage should have so that, on any particular day, the probability that a request for the hire of a van for that day has to be refused is less than 0.1. [2]
 - (iii) Find the probability that, on any particular day, there is at least one request for a van and at least two requests for a car, given that there are a total of 4 requests on that day. [3]
 - (iv) Using a suitable approximation, find the probability that the total number of requests for vans and cars in a randomly chosen five-day period exceeds 25. [4]
- A child is playing with a large set of wooden and plastic cubes. The random variable W denotes the length, in cm, of the edge of a wooden cube which is normally distributed with mean 7 and standard deviation σ . The length, in cm, of the edge of a plastic cube is an independent normal variable with mean 8 and standard deviation 0.1.

Given that 25P(W < 5) = P(W < 9), find the value of σ , giving your answer correct to 3 significant figures. [3]

- (i) The child picks two wooden cubes and one plastic cube at random and places them on top of each other in a box with a hinged lid. Find, correct to 2 decimal places, the smallest depth of the box for there to be a 95% probability that the lid will close fully. [4]
- (ii) Find the probability the sum of the length of the edges of three randomly chosen wooden cubes exceeds twice the length of the edge of one randomly chosen plastic cube by at least 6.2 cm. [3]
- (iii) 200 plastic cubes are chosen at random. Using a suitable approximation, find the probability that at most 10 of them are longer than 8.2 cm. [3]

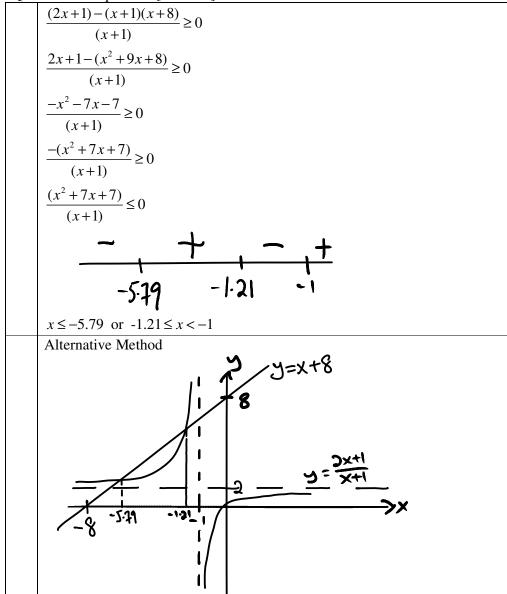
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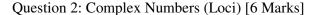
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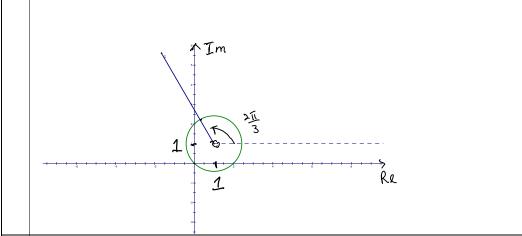
IJC 2008 Prelim 2 H2 Maths Paper 1 (Solutions)

Question 1:Inequalities [4 Marks]

 $x \le -5.79$ or $-1.21 \le x < -1$







$$\sin \frac{\pi}{3} = \frac{y}{\sqrt{2}}$$

$$y = \sqrt{\frac{3}{2}}$$

$$\cos \frac{\pi}{3} = \frac{x}{\sqrt{2}}$$

$$x = \frac{1}{\sqrt{2}}$$

$$z = \left(1 - \frac{1}{\sqrt{2}}\right) + i\left(1 + \sqrt{\frac{3}{2}}\right)$$

Question 3: Maclaurin's Expansion [7 Marks]

$$y = \sqrt{9 + \sin 2x} \qquad \text{or } \frac{dy}{dx} = \frac{\cos 2x}{\sqrt{9 + \sin 2x}}$$

$$y^2 = 9 + \sin 2x$$

$$2y \frac{dy}{dx} = 2\cos 2x$$

$$y \frac{dy}{dx} = \cos 2x$$

$$\left(\frac{dy}{dx}\right)^2 + y \frac{d^2y}{dx^2} = -2\sin 2x$$

$$\left(\frac{dy}{dx}\right)^2 + y \frac{d^2y}{dx^2} = -2\left(y^2 - 9\right)$$

$$y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y^2 = 18$$

$$y\frac{d^{3}y}{dx^{3}} + \frac{dy}{dx}\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx}\frac{d^{2}y}{dx^{2}} + 4y\frac{dy}{dx} = 0$$
$$y\frac{d^{3}y}{dx^{3}} + 3\frac{dy}{dx}\frac{d^{2}y}{dx^{2}} + 4y\frac{dy}{dx} = 0$$

When
$$x = 0$$
,

$$y = 3$$
, $\frac{dy}{dx} = \frac{1}{3}$, $\frac{d^2y}{dx^2} = -\frac{1}{27}$, $\frac{d^3y}{dx^3} = -\frac{107}{81}$

$$y \approx 3 + \frac{1}{3}x - \frac{1}{54}x^2 - \frac{107}{486}x^3$$

Question 4: Vectors [10 Marks]

$$= (2+3) + \lambda(1-4+3)$$

$$=5$$

Therefore the line l lie on π_1 ,

(ii)
$$\overline{AB} = \overline{OB} - \overline{OA}$$

$$= \begin{pmatrix} 2\\1\\4 \end{pmatrix} - \begin{pmatrix} 2\\-3\\3 \end{pmatrix} = \begin{pmatrix} 0\\4\\1 \end{pmatrix}$$

$$\cos \beta = \frac{\mathbf{d}_{AB} \bullet \mathbf{n}_1}{|\mathbf{d}_{AB}||\mathbf{n}_1|}$$

$$= \frac{\begin{pmatrix} 0\\4\\1 \end{pmatrix} \bullet \begin{pmatrix} 1\\2\\3 \end{pmatrix}}{\sqrt{0^2 + 4^2 + 1^2} \sqrt{1^2 + 2^2 + 3^2}}$$
$$= \frac{11}{\sqrt{238}}$$

$$\beta = \cos^{-1} \frac{11}{\sqrt{238}} = 44.5185^{\circ}$$

Hence angle between line AB and plane π_1

$$=90^{\circ} - 44.5185^{\circ}$$

$$=45.4815^{\circ} \approx 45.5^{\circ}$$

Alternative method using sin

Let θ be the acute angle between the line and the plane

$$\sin \theta = \frac{\mathbf{d}_{AB} \cdot \mathbf{n}_1}{\left| \mathbf{d}_{AB} \right| \left| \mathbf{n}_1 \right|}$$

$$= \frac{\begin{pmatrix} 0\\4\\1 \end{pmatrix} \cdot \begin{pmatrix} 1\\2\\3 \end{pmatrix}}{\sqrt{0^2 + 4^2 + 1^2} \sqrt{1^2 + 2^2 + 3^2}}$$
$$= \frac{11}{\sqrt{238}}$$

$$\therefore \theta = \sin^{-1} \frac{11}{\sqrt{238}} = 45.5^{\circ}$$

(iii) Normal of plane π_2 ,

$$\mathbf{n}_{2} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

Equation of plane π_2

 $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$$\mathbf{r} \cdot \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}$$

$$\mathbf{r} \cdot \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix} = 2 \text{ or } \mathbf{r} \cdot \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = 1$$

(Accept parametric form)

(iv) The 3 planes do not have a common point of intersection OR common line of intersection.

Question 5: Differential Equation [13 Marks]

(a)
$$\frac{d^2 y}{dx^2} = \sin^2 3x$$
$$\frac{dy}{dx} = \int (\sin^2 3x) dx$$
$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 6x\right) dx$$
$$= \frac{1}{2}x - \frac{1}{12}\sin 6x + C$$

$$y = \int \left(\frac{1}{2}x - \frac{1}{12}\sin 6x + C\right) dx$$
$$= \frac{x^2}{4} + \left(\frac{1}{12}\right) \frac{\cos 6x}{6} + Cx + D$$
$$= \frac{x^2}{4} + \frac{\cos 6x}{72} + Cx + D$$

(b)
$$\frac{dx}{dt} = kx(1-x)$$

When $x = \frac{1}{2}$; $\frac{dx}{dt} = \frac{1/2}{24} = \frac{1}{48}$

$$\frac{1}{48} = k \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)$$

$$k = \frac{1}{12}$$

$$k = \frac{1}{12}$$
$$\frac{dx}{dt} = \frac{1}{12}x(1-x)$$

$$12\frac{dx}{dt} = x(1-x) \qquad \text{(Shown)}$$

Using separable variables,

$$\int \frac{1}{x(1-x)} dx = \int \frac{1}{12} dt$$

$$\int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \int \frac{1}{12} dt$$
$$\ln|x| - \ln|1 - x| = \frac{t}{12} + c$$

$$\ln\left|\frac{x}{1-x}\right| = \frac{t}{12} + c$$

When
$$t = 0$$
, $x = \frac{1}{2}$

$$\ln\left|\frac{1}{2}\right| - \ln\left|1 - \frac{1}{2}\right| = c$$

$$c = 0$$

$$\therefore \ln \left| \frac{x}{1-x} \right| = \frac{t}{12}$$

When t = 12,

$$\ln\left|\frac{x}{1-x}\right| = 1$$

$$\frac{x}{1-x} = e$$

$$x = e - ex$$

$$x = \frac{e}{1+e} = 0.731 \approx 73\%$$
 (Shown)

Therefore approximately 73% destroyed.

Question 6: P&C [5 Marks]

(a) AT 7 others 8 items

Number of ways = $\frac{8!}{3!} \times 2!$ = 13 440

(b) Case 1: EEE

Number of ways = 1

Case 2: EE 1 others

Number of ways = ${}^{6}C_{1} \times \frac{3!}{2!} = 18$

Case 3: E 2 others

Number of ways = ${}^6C_2 \times 3! = 90$

Case 4: 3 others

Number of ways = ${}^6P_3 = 120$

Alternatively, combining Case 3 and Case 4:

All letters are different: ${}^{7}P_{3} = 210$

Total number of ways = 1 + 18 + 90 + 120 = 229

Question 7: Probability [6 Marks]

(i) P(Y) $= P(\text{Ace, King, Queen, Jack drawn from 1st Pack}) \times P(\text{score from 2nd Pack} < 5)$ $+ P(\text{others drawn from 1st Pack}) \times P(\text{total score on both dice} < 5)$ $= \frac{16}{52} \times \frac{12}{36} + \frac{36}{52} \times \frac{6}{36}$ $= \frac{17}{78}$ (ii) $P(X \cup Y') = P(X) + P(Y') - P(X \cap Y')$ $= \frac{9}{13} + (1 - \frac{17}{78}) - \frac{36}{52} \times \frac{30}{36}$ $= \frac{35}{12}$

Q8: Sampling + Hypothesis Testing [8 Marks]

The jars of kaya are labeled from 1 to 500.

1 jar is randomly chosen from the first 25 jars, thereafter every 25th jar is chosen.

Alternatively

Write down the explicit sequence of labeled jars eg 3^{rd} , 8^{th} , 13^{th} , ... to fulfill the requirement for 2^{nd} B1.

Let X be the random variable "mass of kaya in each jar."

Unbiased estimate for population mean

$$= \bar{x} = \frac{120}{20} + 380 = 386$$

Unbiased estimate for population variance

$$= \frac{1}{20 - 1} \left[3100 - \frac{120^2}{20} \right] = 125.2631579 \approx 125$$

To test $H_0: \mu = m$

against $H_0: \mu < m$

Perform a 1-tail test at 6 % level of significance.

Under Ho, $T \sim t(19)$

Since H_0 is rejected, t < -1.6279723

$$\frac{386 - m}{\sqrt{\frac{125.2631}{20}}} < -1.6279723$$
$$m > 390$$

Q9: Binomial Distribution [8 Marks]

(a)
$$X \sim B(n, p)$$

 $npq = \frac{4}{5}np$
 $q = \frac{4}{5}$ and $p = \frac{1}{5}$
 $P(X \ge 1) > 0.92$
 $1 - P(X = 0) > 0.92$
 $P(X = 0) < 0.08$

$$\left(\frac{4}{5}\right)^n < 0.08 \quad OR$$

$$\left(0.8\right)^{11} = 0.0859 > 0.08$$

$$\left(0.8\right)^{12} = 0.0687 < 0.08$$

$$n > 11.3$$
Least value of n is 12

(b) $X \sim B(8, \frac{1}{3})$

$$E(X) = \frac{8}{3} \text{ and } Var(X) = \frac{16}{9}$$
By CLT, $S \sim N(\frac{8}{3} \times 60, \frac{16}{9} \times 60)$

$$S \sim N(160, \frac{320}{3})$$

$$P(S > 162) = 0.423 \quad (3 \text{ s.f})$$

Question 10 Correlation & Regression [9 Marks]

 $t = ax^b \implies \lg t = \lg ax^b$

(i)

$$\Rightarrow \lg t = \lg a + b \lg x$$
Hence, the relation between $\lg x$ and $\lg t$ is linear
Using G.C., regression line of $\lg t$ on $\lg x$ is:
$$\lg t = 1.10879 \lg x - 1.18259$$

$$\therefore b = 1.10879 = 1.11 \quad (3 \text{ s.f.})$$

$$\lg a = -1.1826 \Rightarrow a = 10^{-1.1787} = 0.0657 \quad (3 \text{ s.f.})$$

(ii) Using G.C., the product moment correlation coefficient between $\lg t$ and $\lg x$ is 0.9998 (to 4 d.p.)

There is a high positive linear correlation between $\lg t$ and $\lg x$ which means that $t = ax^b$ is an appropriate linear model to provide a good fit to the data points.

(iii) When
$$x = 42\ 200$$
,
 $\lg t = 1.10879 \lg(42200) - 1.18259$
 $t = 8828.97 \text{ secs.}$
 $\approx 8830 \text{ secs}$

The answer is not reliable as the value $x = 42\ 200$ is outside the data range for x where the linear relation between $\lg x$ and $\lg t$ may no

longer hold.

Question 11: Poisson Distribution [11 Marks]

 $V_5 + C_5 \sim N(27.5, 27.5)$ approx

 $P(V_5 + C_5 > 25) \xrightarrow{c.c} P(V_5 + C_5 > 25.5)$

= 0.6485411= 0.649 (3 s.f.)

Let *V* be the r.v. "number of vans for hire in 1 day." $V \sim Po(1.5)$ Let C be the r.v. "number of cars for hire in 1 day." $C \sim Po(4)$ $P(V > 2) = 1 - P(V \le 2)$ =0.19115= 0.191 (3 s.f.)(ii) P(V > n) < 0.1 $1 - P(V \le n) < 0.1$ $P(V \le n) > 0.9$ Using GC: $P(V \le 2) = 0.80885 < 0.9$ $P(V \le 3) = 0.93436 > 0.9$ Hence the least value of n = 3[GC: Y_1 = poissoncdf (1.5, X)] (iii) $V + C \sim Po(5.5)$ $P(V \ge 1 \cap C \ge 2 \mid V + C = 4)$ $= \frac{P(V \ge 1 \cap C \ge 2 \cap V + C = 4)}{}$ P(V+C=4) $= \frac{P(V=1)P(C=3) + P(V=2)P(C=2)}{P(C=3) + P(V=3)P(C=3)}$ P(V+C=4)=0.656 (3 s.f.) Let V_5 be the r.v. "number of vans for hire in 5 days." (iv) $V_5 \sim Po(7.5)$ Let C₅ be the r.v. "number of cars for hire in 5 days." $C_5 \sim Po(20)$ $V_5 + C_5 \sim Po(27.5)$ Since $\lambda = 27.5 > 10$

Question 12:Normal Distribution [13 Marks]

	Let W be the r.v. denoting "the length of a wooden cube."				
	$W \sim N(7, \sigma^2)$				
	25P(W < 5) = P(W < 9)				
	$25P(Z < \frac{5-7}{\sigma}) = P(Z < \frac{9-7}{\sigma})$				
	$25P(Z < -\frac{2}{\sigma}) = P(Z < \frac{2}{\sigma})$				
	$25P(Z > \frac{2}{\sigma}) = P(Z < \frac{2}{\sigma})$				
	$25[1 - P(Z < \frac{2}{\sigma})] = P(Z < \frac{2}{\sigma})$				
	$P(Z < \frac{2}{\sigma}) = \frac{25}{26}$				
	$\frac{2}{\sigma} = 1.768825$				
	$\sigma = 1.13$				
(i)	Let <i>P</i> be the r.v. denoting "length of the edge of a plastic cube."				
	$P \sim N(8, 0.1^2)$				
	Consider $T = W_1 + W_2 + P$				
	$E(T) = E(W_1 + W_2 + P) = 2(7) + 8 = 22$				
	$Var(T) = Var(W_1 + W_2 + P) = 2(1.13^2) + 0.1^2 = 2.5638$				
	$T \sim N(22, 2.5638)$				
	$P(T \le d) \ge 0.95$ OR $P(T \le d) = 0.95$				
	$d \ge 24.63372$				
9ii0	Smallest depth is 24.63 (2 d.p.)				
9110	$P \sim N(8, 0.1^2)$ and $W \sim N(7, 1.1306941^2)$				
	Let $S = W_1 + W_2 + W_3 - 2P \sim N(3 \times 7 - 2 \times 8, 3 \times 1.1306941^2 + 4 \times 0.1^2)$				
	$S \sim N(5, 3.87541)$				
	P(S > 6.2) = 0.271 (3s.f)				
(iii)	P(P > 8.2) = 0.02275				
	Let <i>X</i> be r.v. denoting "number of plastic cubes, out of 200, that are longer than 8.20 cm."				
	$X \sim Bin(200, 0.02275)$				
	Since $n = 200 > 50$				
	np = 200(0.02275) = 4.55001 < 5				
	$X \sim Po(4.55001)$ approx				

 $P(X \le 10) = 0.99279 = 0.993$ (3 s.f.)