第五章 自由旋量场量子化

自然界中的粒子，按照它们在洛伦兹变换下性质的不同，分为标量粒子，如$\pi^{0,\pm}$，$K^{\pm}$，$\bar{K}$，$\bar{K}^{0}$，η，自旋为$S=0$；旋量粒子，如e，ν_e，δ，N，自旋为$\frac{1}{2}$，$\frac{3}{2}$，…；矢量粒子，如光子，胶子，中间玻色子，自旋为1。相应地，用来描述这些粒子的场分为标量场，旋量场，矢量场。

上一章介绍了自旋为零的标量场，在按照对易关系正则量子化后，标量场满足算符演化的海森堡方程，真空能量为正，体系粒子数表象下服从玻色－爱因斯坦统计，~~体系~~多个标量粒子能够同处同一个量子态。本章将介绍自旋为$\frac{1}{2}$的旋量场。若按照对易关系进行正则量子化，人们发现将出现真空能为负，体系粒子数表象下服从玻色－爱因斯坦统计，~~体系~~同一个量子态上可以存在多个费米子~~的情况~~。但是，这与泡利不相容原理，与费米子实际上服从费米－狄拉克统计显然矛盾。1928年，Jordan和Wigner提出费米子按照反对易关系进行正则量子化的方案，成功解决了上述矛盾。

一. 经典旋量场(狄拉克场)

参见上册 P_{32}, "§3.4 自由旋量场·自由Dirac方程一节", 定义 $\beta\equiv\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\alpha_i=\begin{pmatrix}0&\sigma_i\\\sigma_i&0\end{pmatrix}$

1. γ 矩阵

引入 $\gamma_0=\gamma^0=\beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\vec{\gamma}=\beta\vec{\alpha}=\begin{pmatrix}0&\vec{\sigma}\\-\vec{\sigma}&0\end{pmatrix}$

则 $\begin{cases}\gamma^\mu\equiv(\gamma^0,\vec{\gamma})\\ \gamma_\mu\equiv(\gamma_0,-\vec{\gamma})\\ \gamma^5=\gamma_5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}0&1\\1&0\end{pmatrix}\end{cases}$

其中 $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

伽玛矩阵性质

本征值±1　本征值±i　本征值±1

$\gamma_0^2=1$, $\gamma_i^2=-1$, $\gamma_5^2=1$

$\gamma_0^+=\gamma_0=\gamma_0^{-1}$, 幺正厄米

$\gamma_i^+=-\gamma_i=\gamma_i^{-1}$ 幺正反厄米.

$\gamma_5^+=\gamma_5=\gamma_5^{-1}$ 幺正厄米.

$\{\gamma^\mu,\gamma^\nu\}=\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$ 反对易

$\{\gamma^\mu,\gamma^5\}=0$

2. Dirac方程

$$E^2-p^2=m^2 \Longrightarrow (E+\sqrt{p^2+m})(E-\sqrt{p^2+m^2})=(\underbrace{E+\vec{\alpha}\cdot\vec{p}+\beta m}_{\text{Dirac}})(E-\vec{\alpha}\cdot\vec{p}-\beta m)$$

用算符表示 $H=\vec{\alpha}\cdot\vec{p}+\beta m=-i\vec{\alpha}\cdot\nabla+\beta m$, 作用于 $\psi(x)$.

$$i\frac{\partial}{\partial t}\psi(x)=(-i\vec{\alpha}\cdot\nabla+\beta m)\psi(x).\qquad\text{(Dirac方程)}$$

左乘 $\gamma^0=\beta$

$$(\gamma^0 i\partial_0+i\vec{\gamma}\cdot\nabla-m)\psi(x)=0.$$

即 $\boxed{(i\gamma^\mu\partial_\mu-m)\psi(x)=0 \quad 或\ (i\not{\partial}-m)\psi(x)=0}$

取厄米共轭,右乘 γ_0, 记 $\bar{\psi}=\psi^+\gamma_0$

$$\psi^+(x)(-i\gamma^0\overleftarrow{\partial}_0+i\gamma^i\overleftarrow{\partial}_i-m)\gamma_0=0$$

$$\bar{\psi}(x)(-i\gamma^0\overleftarrow{\partial}_0-\gamma^i\overleftarrow{\partial}_i-m)=0$$

即 $\boxed{\bar{\psi}(x)(i\gamma^\mu\overleftarrow{\partial}_\mu+m)\psi(x)=0 \quad 或\ \bar{\psi}(x)(i\overleftarrow{\not{\partial}}+m)=0}$

* Dirac方程描述 $s=\frac{1}{2}$ 费米子. 泡利不相容. $\psi_{4\times1}$
* 真空态: 负能态被填满, 正能态空着, 故真空具有负能量。不可观测.
* 正能态: 负能态填满, 正能态有粒子。可观测。
* 空穴态: 负能态有空着, 正能态都空着。可观测.　负能態解釋!

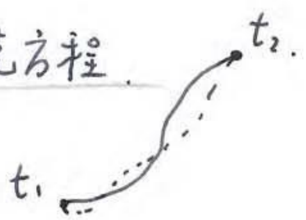
$$E^2=\vec{p}^2+m^2 \longrightarrow E=\pm\sqrt{\vec{p}^2+m^2}$$

二. 经典旋量场的量子化 ($s=(2k+1)\frac{\hbar}{2}$)

1. 经典旋量场

$$\mathcal{L}(x)=\bar\psi(x)(i\gamma^\mu\partial_\mu-m)\psi(x)\quad,\quad 独立量\ \psi,\bar\psi\ (\bar\psi,\bar\psi)$$

利用作用量 $S=\int_{t_1}^{t_2}d^4x\,\mathcal{L}(x)$ 取极值原理 $\delta S=0$，可得狄拉克方程.



证明：

对场量取轨道变分

$$\psi(x)\longrightarrow\psi(x)+\delta\psi(x),\qquad \delta\psi(x)\big|_{t_1}=0,\quad \delta\psi(x)\big|_{t_2}=0.$$

$$\bar\psi(x)\longrightarrow\bar\psi(x)+\delta\bar\psi(x),\qquad \delta\bar\psi(x)\big|_{t_1}=0,\quad \delta\bar\psi(x)\big|_{t_2}=0.$$

$S=\int_{t_1}^{t_2}d^4x\,\mathcal{L}(x)$ 变为

$$S\to S+\delta S=\int_{t_1}^{t_2}d^4x\,(\bar\psi(x)+\delta\bar\psi(x))(i\partial\!\!\!/-m)(\psi(x)+\delta\psi(x))$$

一阶变分

$$=\underbrace{\int_{t_1}^{t_2}d^4x\,\bar\psi(x)(i\partial\!\!\!/-m)\psi(x)}_{S}+\int_{t_1}^{t_2}d^4x\left[\delta\bar\psi(x)(i\partial\!\!\!/-m)\psi(x)+\bar\psi(x)(i\partial\!\!\!/-m)\delta\psi(x)\right]$$

$$\therefore\ \delta S=+\int_{t_1}^{t_2}d^4x\,\delta\bar\psi(x)(i\partial\!\!\!/-m)\psi(x)+\int_{t_1}^{t_2}d^4x\,\bar\psi(x)(-i\overleftarrow{\partial\!\!\!/}-m)\delta\psi(x)+\underbrace{\int_{t_1}^{t_2}d^4x\,\partial_\mu(i\bar\psi\gamma^\mu\delta\psi(x))}_{\bigstar}$$

其中 $\bigstar=i\int_{t_1}^{t_2}d^4x\,\nabla\cdot(\bar\psi\vec\gamma\,\delta\psi(x))+i\int_{t_1}^{t_2}d^4x\,\frac{\partial}{\partial t}(\bar\psi(x)\gamma^0\delta\psi(x))$

$$=i\int_{t_1}^{t_2}d^4x\,\partial^i(\bar\psi(x)\gamma_i\delta\psi(x))+i\int_{t_1}^{t_2}d^4x\,\frac{\partial}{\partial t}(\bar\psi(x)\gamma^0\delta\psi(x))$$

利用周期边界条件.

$$\int d^3x\,\nabla\cdot\vec A=\iiint dx\,dy\,dz\left(\frac{\partial A_1}{\partial x}(x,y,z,t)+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)$$

$$=\iint dy\,dz\,\underbrace{A_1(x,y,z,t)\Big|_{x=-\frac{L}{2}}^{x=\frac{L}{2}}}_{=0}+\iint dz\,dx\,\underbrace{A_2(x,y,z,t)\Big|_{y=-\frac{L}{2}}^{y=\frac{L}{2}}}_{=0}+\iint dx\,dy\,\underbrace{A_3(x,y,z,t)\Big|_{z=-\frac{L}{2}}^{z=\frac{L}{2}}}_{=0}=0$$

而固定端点的变分得

$$\int_{t_1}^{t_2}d^4x\,\frac{\partial}{\partial t}(\bar\psi(x)\gamma^0\delta\psi(x))=\int d^3x\,\underbrace{\bar\psi(x)\gamma^0\delta\psi(x)\Big|_{t=t_1}^{t=t_2}}_{=0}=0.$$

$\therefore\ \bigstar=0.$

根据在物理轨道上，作用量取极值原理 $\delta S\big|_{物理轨道}=0$，则

$$0=\delta S=\int_{t_1}^{t_2}d^4x\,\delta\bar\psi(x)(i\partial\!\!\!/-m)\psi(x)+\int_{t_1}^{t_2}d^4x\,\bar\psi(x)(-i\overleftarrow{\partial\!\!\!/}-m)\delta\psi(x)=0.$$

又轨道变分时，$\delta\psi$ 和 $\delta\bar\psi$ 可任意独立选取．故

$$(i\partial\!\!\!/-m)\psi(x)=0\qquad \bar\psi(x)(i\overleftarrow{\partial\!\!\!/}+m)=0.$$

2. Jordon-Wigner 量子化

$$\mathcal{L}(x) = \bar\psi(x)(i\partial\!\!\!/ - m)\psi(x) = \bar\psi(x)(i\gamma^0\partial_0 + i\vec\gamma\cdot\nabla - m)\psi(x)$$

引入正则动量

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\psi} = i\psi^\dagger(x)$$

(由于$\mathcal{L}$不含$\bar\psi(x)$的时间导数，故$\bar\psi(x)$不是正则坐标)

1) 哈密顿量

$$H \equiv \int(\pi\dot\psi - \mathcal{L})d^3x = \int\left[\cancel{i\psi^\dagger\dot\psi} - \bar\psi(x)(\cancel{i\gamma^0\partial_0} + i\vec\gamma\cdot\nabla - m)\psi(x)\right]d^3x = \int\psi^\dagger(-i\vec\alpha\cdot\nabla + \beta m)d^3x$$

$$\mathcal{H} = \psi^\dagger(-i\vec\alpha\cdot\nabla + \beta m)\psi$$

假定ψ，π为厄米算符，满足正则反对易关系

$$\{\psi_\alpha(\vec x,t),\ \psi^\dagger_\beta(\vec x',t)\} = \delta_{\alpha\beta}\delta^3(\vec x - \vec x').$$

$$\{\psi_\alpha(\vec x,t),\ \psi_\beta(\vec x',t)\} = 0,\qquad \{\psi^\dagger_\alpha(\vec x,t),\ \psi^\dagger_\beta(\vec x',t)\} = 0$$

则可推出海森堡方程　　利用$[AB,C] = A\{B,C\} - \{A,C\}B$

$$[H,\psi(\vec x,t)] = \int d^3x'\left[\psi^\dagger(\vec x',t)(-i\vec\alpha\cdot\nabla' + \beta m)\psi(\vec x',t),\ \psi(\vec x,t)\right] = \cancel{\int d^3x'[\psi^\dagger(\vec x',t),\psi(\vec x,t)](-i\vec\alpha\cdot\nabla' + \beta m)}\ \psi(\vec x',t)$$

$$= -\int d^3x'\{\psi^\dagger(\vec x',t),\psi(\vec x,t)\}(-i\vec\alpha\cdot\nabla' + \beta m)\psi(\vec x',t) = -(-i\vec\alpha\cdot\nabla + \beta m)\psi(\vec x,t)$$

$$= -i\partial_0\psi(\vec x,t) = -i\dot\psi(\vec x,t).$$

$$\therefore \dot\psi(\vec x,t) = i[H,\psi(\vec x,t)]$$

$$[H,\psi^\dagger(\vec x,t)] = \int d^3x'\left[\psi^\dagger(\vec x',t)(-i\vec\alpha\cdot\nabla + \beta m)\psi(\vec x',t),\ \psi^\dagger(\vec x,t)\right] = \int d^3x'\psi^\dagger(\vec x',t)\{(-i\vec\alpha\cdot\nabla' + \beta m)\psi(\vec x',t),\psi^\dagger(\vec x,t)\}$$

$$= \int d^3x'\psi^\dagger(\vec x',t)\left(\nabla'\cdot\{-i\vec\alpha\,\psi(\vec x',t),\psi^\dagger(\vec x,t)\} + \beta m\{\psi(\vec x',t),\psi^\dagger(\vec x,t)\}\right)$$

$$= \int d^3x'\psi^\dagger(\vec x',t)\left(\nabla'\cdot(-i\vec\alpha\,\delta(\vec x - \vec x')) + \beta m\,\delta(\vec x - \vec x')\right)$$

$$= \nabla\cdot\int d^3x'\,\cancel{\psi^\dagger(\vec x,t)}\,\psi^\dagger(\vec x',t)\,i\vec\alpha\,\delta(\vec x - \vec x') + \psi^\dagger(\vec x,t)\beta m$$

$$= \psi^\dagger(\vec x,t)(i\vec\alpha\cdot\overleftarrow\nabla + \beta m) = -\psi^\dagger(\vec x,t)\,i\overleftarrow{\partial_0}$$

$$\therefore \dot\psi^\dagger(\vec x,t) = i[H,\psi^\dagger(\vec x,t)]$$

综上，海森堡方程

$$\boxed{\begin{cases}\dot\psi(\vec x,t) = i[H,\psi(\vec x,t)]\\ \dot\psi^\dagger(\vec x,t) = i[H,\psi^\dagger(\vec x,t)]\end{cases}}$$

2). 动量算符.

$$T_{\mu\nu} = i\bar{\psi}\gamma_\mu \partial_\nu \psi. \qquad P_\nu = \int d^3x\, T_{0\nu} = i\int d^3x\, \psi^+ \partial_\nu \psi$$

取. $\vec{p} = i\int d^3x\, \psi^+ \partial^i \psi = -i\int d^3x\, \psi^+ \nabla\psi$

则 $[P, \psi(x)] = \int d^3x' [-i\psi^+(\vec{x}',t)\nabla'\psi(\vec{x}',t), \psi(\vec{x},t)] = \int d^3x' \{i\psi^+(\vec{x}',t), \psi(\vec{x},t)\}\nabla'\psi(\vec{x}',t)$

$$= \int d^3x'\, i\delta(\vec{x}-\vec{x}')\nabla'\psi(\vec{x}',t) = i\nabla\psi(\vec{x},t)$$

$[P, \psi^+(x)] = \int d^3x' [-i\psi^+(\vec{x}',t)\nabla'\psi(\vec{x}',t), \psi^+(\vec{x},t)] = \int d^3x' (-i\psi^+(\vec{x}',t))\{\nabla'\psi(\vec{x}',t), \psi^+(\vec{x},t)\}$

$$= -i\int d^3x'\, \psi^+(\vec{x}',t)\nabla'\{\psi(\vec{x}',t), \psi^+(\vec{x},t)\} = -i\int d^3x'\, \psi^+(\vec{x}',t)\nabla'\delta(\vec{x}-\vec{x}')$$

$$= i\nabla\int d^3x'\, \psi^+(\vec{x}',t)\delta(\vec{x}-\vec{x}') = i\nabla\psi^+(\vec{x},t)$$

综上 $\boxed{\begin{cases} \nabla\psi(\vec{x},t) = i[-\vec{p}, \psi(\vec{x},t)] \\ \nabla\psi^+(\vec{x},t) = i[-\vec{p}, \psi^+(\vec{x},t)] \end{cases}}$

考虑到 $P_\mu = i\frac{\partial}{\partial x^\mu} = i\partial_\mu = i(\frac{\partial}{\partial t}, \nabla)$, 且 $P_\mu = (H, -\vec{p})$, 故

$$\begin{cases} \frac{\partial}{\partial x^\mu}\psi(\vec{x},t) = i[P_\mu, \psi(\vec{x},t)] \\ \frac{\partial}{\partial x^\mu}\psi^+(\vec{x},t) = i[P_\mu, \psi^+(\vec{x},t)] \end{cases}$$

旋量场在时空平移下的变化性质

解为 $\psi(x+b) = e^{iPb}\psi(x)e^{-iPb}$

3). 守恒荷 (U(1) 变换)

旋量场 满足内禀空间的 U(1) 规范变换不变性。

$$\psi(x) \longrightarrow \psi'(x) = e^{i\alpha}\psi(x)$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = \bar{\psi}(x)e^{-i\alpha}$$

利用作用量变分原理 $\delta S = 0$, 得 $\partial_\mu j^\mu = 0$, 从而引入流密度.

守恒流 $\quad j^\mu = \bar{\psi}\gamma^\mu\psi$

守恒荷 $\quad Q \equiv \int d^3x\, j^0 = \int d^3x\, \bar{\psi}\gamma^0\psi = \int d^3x\, \psi^+\psi$

注: 乘 e, 得 ej^μ 为电流 电荷-密度, 而 eQ 为电荷.

上式与 ψ 场的费米数守恒律相关.

三. 傅立叶展开 (李政道 P_{32})

利用空间 V 内的系统, $\psi(\vec{x},t)$ 满足周期性边界条件, 狄拉克方程, 求得 ψ 傅展解.

1. 在周期性边界条件下, 得满足正交归一、完备性的本征函数.

$$\left\{ e^{i\vec{p}_n\cdot\vec{x}} \;\middle|\; \vec{p}_n = \vec{n}\frac{2\pi}{L},\ \vec{n}=0,\pm1,\cdots \right\}$$

2. 旋量场满足周期边条, 可傅展如下.

1). $\psi(\vec{x},t) = \sum_{\vec{p}} S_{\vec{p}}(t)\dfrac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{V}}$, $S_{\vec{p}}(t)$ 为 4×1 矩阵, 矩阵元为算符 (希尔伯特空间), 空间为旋量空间 (4×4)

2) 对给定 $\vec{p}$, 在旋量空间中引进一组 C数基矢 $u_{\vec{p},s}$, $v_{-\vec{p},s}$, 满足

是 $(\alpha\cdot\vec{p}+\beta m)$ 和 $\vec{\tau}\cdot\hat{p}$ 的本征态, 即

$$\begin{cases}(\vec{\alpha}\cdot\vec{p}+\beta m)\,u_{\vec{p},s} = E_p u_{\vec{p},s}\\ (\alpha\cdot\vec{p}+\beta m)\,v_{-\vec{p},s} = -E_p v_{-\vec{p},s}\end{cases},\quad \begin{cases}\vec{\tau}\cdot\hat{p}\,u_{\vec{p},s} = 2s\,u_{\vec{p},s}\\ \vec{\tau}\cdot\hat{p}\,v_{-\vec{p},s} = 2s\,v_{-\vec{p},s}\end{cases}$$

其中 $\begin{cases}E_p = \sqrt{\vec{p}^2+m^2}>0\\ \vec{\tau} = \begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}\end{cases}$, 而 $\hat{p} = \dfrac{\vec{p}}{|\vec{p}|}$, $s=\pm\frac{1}{2}$ 称为螺旋度.

* 由于 $u_{\vec{p},s}$, $v_{-\vec{p},s}$ 相应的本征值 (厄米算符) 不同, 故二者正交

* $u_{\vec{p},s}$, $v_{-\vec{p},s}$ 归一化后, 满足完备性

$$u^{+}_{\vec{p},s}u_{\vec{p},s}=1,\quad v^{+}_{-\vec{p},s}v_{-\vec{p},s}=1$$

$$\begin{cases}\bar{u}_{\vec{p},s}u_{\vec{p},s'} = \dfrac{E}{m}\delta_{ss'}\\ u^{+}_{\vec{p},s}v_{-\vec{p},s'} = v^{+}_{\vec{p},s}u_{-\vec{p},s'} = 0\end{cases}$$

3). 由于 $u_{\vec{p},s}$, $v_{-\vec{p},s}$ 构成正交归一完备基矢, 故可将 $S_{\vec{p}}(t)$ 按这组基矢展开.

$$S_{\vec{p}}(t) = \sum_{s=\pm\frac{1}{2}}\left(a_{\vec{p},s}(t)\,u_{\vec{p},s} + b^{+}_{-\vec{p},s}v_{-\vec{p},s}\right),$$

$a_{\vec{p},s}(t)$, $b^{+}_{-\vec{p},s}$ 为希伯空间算符, 不是旋量

故 $$\begin{cases}\psi(\vec{x},t) = \dfrac{1}{\sqrt{V}}\sum_{\vec{p},s}\left(a_{\vec{p},s}(t)u_{\vec{p},s}e^{i\vec{p}\cdot\vec{x}} + b^{+}_{\vec{p},s}(t)v_{\vec{p},s}e^{-i\vec{p}\cdot\vec{x}}\right)\\ \psi^{+}(\vec{x},t) = \dfrac{1}{\sqrt{V}}\sum_{\vec{p},s}\left(a^{+}_{\vec{p},s}(t)u^{+}_{\vec{p},s}e^{-i\vec{p}\cdot\vec{x}} + b_{\vec{p},s}(t)v^{+}_{\vec{p},s}e^{i\vec{p}\cdot\vec{x}}\right)\end{cases}$$

3. $\psi(x)$ 满足狄拉克方程的解:

$(i\gamma^\mu\partial_\mu - m)\psi(x)=0$, 即 $(\gamma^0 i\partial_0 + i\vec{\gamma}\cdot\nabla - m)\sum_{\vec{p}} S_p(t)\,e^{i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{V}} = 0$

$$\Rightarrow \sum_{\vec{p}}\left[\gamma^0 i\,\dot{S}_p(t)e^{i\vec{p}\cdot\vec{x}} + (i\vec{\gamma}\cdot i\vec{p}-m)S_{\vec{p}}(t)\,e^{i\vec{p}\cdot\vec{x}}\right]=0$$

左乘 γ^0，移项

$$\sum_{\vec p} i\dot S_{\vec p}(t) = \sum_{\vec p}(\vec\alpha\cdot\vec p+\beta m)S_{\vec p}(t) = \sum_{\vec p} E_{\vec p}\, S_{\vec p}(t),$$ 其中 $E_{\vec p} = \sqrt{\vec p^2+m^2}$

$\therefore\ i\frac{\partial}{\partial t}S_{\vec p}(t) = E_{\vec p}S_{\vec p}(t)$，　$S_{\vec p}(t) = Ce^{-iE_{\vec p}t}$，从而有 $\begin{cases} a_{\vec p,s}(t) = a_{\vec p,s}e^{-iE_p t} \\ b^+_{\vec p,s}(t) = b^+_{\vec p,s}e^{iE_p t}\end{cases}$

故
$$\boxed{\begin{cases}\psi(\vec x,t) = \frac{1}{\sqrt V}\sum_{\vec p,s}\left(a_{\vec p,s}u_{\vec p,s}e^{-ipx} + b^+_{\vec p,s}v_{\vec p,s}e^{+ipx}\right) \\ \psi^+(\vec x,t) = \frac{1}{\sqrt V}\sum_{\vec p,s}\left(a^+_{\vec p,s}u^+_{\vec p,s}e^{+ipx} + b_{\vec p,s}v^+_{\vec p,s}e^{-ipx}\right)\end{cases}}$$

其中 $px = E_p t - \vec p\cdot\vec x$
$\vec p = \vec p_n = \vec n\frac{2\pi}{L}$,
$\vec n = 0, \pm1, \pm2, \ldots$

考虑归一化，狄拉克场改写为

$$\boxed{\begin{aligned}\psi(\vec x,t) &= \sum_{\vec p,s}\sqrt{\frac{m}{VE_p}}\left(C_{\vec p,s}u_{\vec p,s}e^{-ipx} + d^+_{\vec p,s}v_{\vec p,s}e^{ipx}\right) \\ \psi^+(\vec x,t) &= \sum_{\vec p,s}\sqrt{\frac{m}{VE_p}}\left(C^+_{\vec p,s}u^+_{\vec p,s}e^{ipx} + d_{\vec p,s}v^+_{\vec p,s}e^{-ipx}\right)\end{aligned}}$$

姜志进
$\vec p = \vec p_n = \vec n\frac{2\pi}{L}$
动量分立

当 $L\to\infty$，$V\to\infty$ 时，动量由分立 $\longrightarrow$ 连续

$\sum_{\vec p} \longrightarrow \int\frac{V}{(2\pi)^3}d^3p$，　$u^{姜}_{\vec p,s} = \frac{1}{\sqrt{2m}}u^{黄}_s(\vec p)$，$v^{姜}_{\vec p,s} = \frac{1}{\sqrt{2m}}v^{黄}_s(\vec p)$

引入 $C^{黄}_s(\vec p) = \sqrt{2E_pV}\,C^{姜}_{\vec p,s}$，　$d^{黄}_s(\vec p) = \sqrt{2E_pV}\,d^{姜}_{\vec p,s}$

$$\begin{aligned}\psi(\vec x,t) &= \int\frac{d^3p}{(2\pi)^3}\sum_s V\sqrt{\frac{m}{VE_p}}\left(C^{姜}_{\vec p,s}\frac{1}{\sqrt{2m}}u^{黄}_s(\vec p)e^{-ipx} + d^{+姜}_{\vec p,s}\frac{1}{\sqrt{2m}}v^{黄}_s(\vec p)e^{ipx}\right) \\ &= \int\frac{d^3p}{(2\pi)^3}\sum_s\sqrt{\frac{V}{2E_p}}\left(C^{姜}_{\vec p,s}u^{黄}_s(\vec p)e^{-ipx} + d^{+姜}_{\vec p,s}v^{黄}_s(\vec p)e^{ipx}\right) \\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s\left(\sqrt{2E_pV}\left(C^{姜}_{\vec p,s}u^{黄}_s(\vec p)e^{-ipx} + d^{+姜}_{\vec p,s}v^{黄}_s(\vec p)e^{ipx}\right)\right)\end{aligned}$$

$$\therefore\ \boxed{\begin{aligned}\psi(\vec x,t) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s\left(C_s(\vec p)u_s(\vec p)e^{-ipx} + d^+_s(\vec p)v_s(\vec p)e^{ipx}\right) \\ \psi^+(\vec x,t) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s\left(C^+_s(\vec p)u^+_s(\vec p)e^{ipx} + d_s(\vec p)v^+_s(\vec p)e^{-ipx}\right)\end{aligned}}$$

$\vec p = -\vec\infty, \vec\infty$
动量连续
黄涛

3.动量空间的振幅算子 $\{ C_{\vec p,s},\ d^{+}_{\vec p,s},\ C^{+}_{\vec p,s},\ d_{\vec p,s} \}$.

$$\boxed{\begin{aligned}&\begin{cases} C_{\vec p,s} = \sqrt{\dfrac{m}{VE_{\vec p}}}\displaystyle\int e^{ipx} u^{+}_{\vec p,s}\psi(x)\, d^3x \\ d^{+}_{\vec p,s} = \sqrt{\dfrac{m}{VE_{\vec p}}}\displaystyle\int e^{-ipx} v^{+}_{\vec p,s}\psi(x)\, d^3x \end{cases} \qquad \text{K分立 姜志进} \\ &\begin{cases} C^{+}_{\vec p,s} = \sqrt{\dfrac{m}{VE_{\vec p}}}\displaystyle\int e^{-ipx} \psi^{+}(x)\, u_{\vec p,s}\, d^3x \\ d_{\vec p,s} = \sqrt{\dfrac{m}{VE_{\vec p}}}\displaystyle\int e^{ipx} \psi^{+}(x)\, v_{\vec p,s}\, d^3x \end{cases}\end{aligned}}$$

当 $L\to\infty$, $V\to\infty$ 时.

引入 $C^{黄}_{s}(\vec p) = \sqrt{2E_pV}\, C^{姜}_{\vec p,s}$, $d^{黄}_{s}(\vec p) = \sqrt{2E_pV}\, d^{姜}_{\vec p,s}$

$$\cancel{C_s(\vec p)}\quad C_{\vec p,s} = \sqrt{\frac{m}{VE_{\vec p}}}\int e^{ipx}\frac{1}{\sqrt{2m}}u^{黄\,+}_{s}(\vec p)\psi(x)\,d^3x = \frac{1}{\sqrt{2E_pV}}\int e^{ipx}u^{黄\,+}_{s}(\vec p)\psi(x)\,d^3x \equiv \frac{C^{黄}_{s}(\vec p)}{\sqrt{2E_pV}}$$

$$\cancel{C_s(\vec p) = \int d^3x\, e^{-ipx}\,\psi^{+}(x)\,u_s(\vec p)}$$

$$\boxed{\begin{aligned}&\begin{cases} C_s(\vec p) = \displaystyle\int d^3x\, e^{+ipx}\, u^{+}_{s}(\vec p)\psi(x) \\ d^{+}_{s}(\vec p) = \displaystyle\int d^3x\, e^{-ipx}\, v^{+}_{s}(\vec p)\psi(x) \end{cases} \qquad \text{K连续 黄涛} \\ &\begin{cases} C^{+}_{s}(\vec p) = \displaystyle\int d^3x\, e^{-ipx}\, \psi^{+}(x)\, u_s(\vec p) \\ d_s(\vec p) = \displaystyle\int d^3x\, e^{ipx}\, \psi^{+}(x)\, v_s(\vec p) \end{cases}\end{aligned}}$$

四. 动量空间的场算符.

$$\begin{cases} \psi(\vec{x},t) = \sum_{\vec{p},s} \sqrt{\frac{m}{VE_p}} \left(C_{\vec{p},s} u_{\vec{p},s} e^{-ipx} + d^{+}_{\vec{p},s} v_{\vec{p},s} e^{ipx} \right) \\ \psi^{+}(\vec{x},t) = \sum_{\vec{p},s} \sqrt{\frac{m}{VE_p}} \left(C^{+}_{\vec{p},s} u^{+}_{\vec{p},s} e^{ipx} + d_{\vec{p},s} v^{+}_{\vec{p},s} e^{-ipx} \right) \end{cases}$$

其中 $\begin{cases} C_{\vec{p},s} = \sqrt{\frac{m}{VE_p}} \int e^{ipx} u^{+}_{\vec{p},s} \psi(x) d^3x \\ d^{+}_{\vec{p},s} = \sqrt{\frac{m}{VE_p}} \int e^{-ipx} v^{+}_{\vec{p},s} \psi(x) d^3x \end{cases}$

由于 $\psi(\vec{x},t)$, $\psi^{+}(\vec{x},t)$ 可以量子化, 故相应的 $C_{\vec{p},s}$, $d^{+}_{\vec{p},s}$, $C^{+}_{\vec{p},s}$, $d_{\vec{p},s}$ 也可以量子化.

可证:

$\vec{k}$ 分立
$$\begin{cases} \{C_{\vec{p},s}, C^{+}_{\vec{p}',s'}\} = \delta_{\vec{p}\vec{p}'}\delta_{ss'} \\ \{d_{\vec{p},s}, d^{+}_{\vec{p}',s'}\} = \delta_{\vec{p}\vec{p}'}\delta_{ss'} \text{ 其它反对易子为0.} \\ H = \sum_{\vec{p},s} E_p [C^{+}_{\vec{p},s} C_{\vec{p},s} + d^{+}_{\vec{p},s} d_{\vec{p},s} - 1] \\ \vec{P} = \sum_{\vec{p},s} \vec{p} [C^{+}_{\vec{p},s} C_{\vec{p},s} + d^{+}_{\vec{p},s} d_{\vec{p},s}] \\ Q = \sum_{\vec{p},s} [C^{+}_{\vec{p},s} C_{\vec{p},s} - d^{+}_{\vec{p},s} d_{\vec{p},s}] \end{cases}$$

真空能为负

$\vec{k}$ 连续
$$\begin{cases} \{C_s(\vec{p}), C^{+}_{s'}(\vec{p}')\} = (2\pi)^3 2E_p \delta_{ss'} \delta^3(\vec{p}-\vec{p}') \\ \{d_s(\vec{p}), d^{+}_{s'}(\vec{p}')\} = (2\pi)^3 2E_p \delta_{ss'} \delta^3(\vec{p}-\vec{p}') \\ H = \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2E_p} \sum_{s=1,2} [C^{+}_s(\vec{p}) C_s(\vec{p}) + d^{+}_s(\vec{p}) d_s(\vec{p})] \\ \vec{P} = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{2E_p} \sum_{s=1,2} [C^{+}_s(\vec{p}) C_s(\vec{p}) + d^{+}_s(\vec{p}) d_s(\vec{p})] \\ Q = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{s=1,2} [C^{+}_s(\vec{p}) C_s(\vec{p}) - d^{+}_s(\vec{p}) d_s(\vec{p})] \end{cases}$$

证明:

(1) $\{C_{p,s}, C^{+}_{p',s'}\} = \{\sqrt{\frac{m}{VE}} \int e^{ipx} u^{+}_{\alpha\,\vec{p},s} \psi^{\alpha}(\vec{x},t) d^3x, \sqrt{\frac{m}{VE'}} \int e^{-ip'x'} \psi^{+\beta}(\vec{x}',t) u_{\beta\,\vec{p}',s'} d^3x'\}$

$= \frac{m}{V\sqrt{EE'}} \int d^3x\, d^3x'\, e^{i(px-p'x')} u^{+}_{\alpha\,\vec{p},s} u_{\beta\,\vec{p}',s'} \{\psi^{\alpha}(\vec{x},t), \psi^{+\beta}(\vec{x}',t)\}$ $= \delta^{\alpha\beta}\delta(\vec{x}-\vec{x}')$

$= \frac{m}{V\sqrt{EE'}} \int d^3x\, e^{i(p-p')x} u^{+}_{\vec{p},s} u_{\vec{p}',s'} \overset{\text{正交}}{=} \frac{m}{\sqrt{EE'}} u^{+}_{\vec{p},s} u_{\vec{p}',s'} \delta_{\vec{p}\vec{p}'} e^{i(E-E')t}$

$u^{+}_{\vec{p},s} u_{\vec{p},s'} = \frac{E}{m}\delta_{ss'}$

① $\beta(\gamma\cdot p - m)u = \beta(\gamma^0 E - \vec{\gamma}\cdot\vec{p} - m)u = (E - \vec{\alpha}\cdot\vec{p} - \beta m)u = 0$, $\therefore (\vec{\alpha}\cdot\vec{p} + \beta m)u = E\,u$

$= \frac{m}{\sqrt{EE'}} e^{i(E-E')t} \delta_{\vec{p}\vec{p}'} u^{+}_{\vec{p},s} u_{\vec{p},s'} = \delta_{\vec{p}\vec{p}'}\delta_{ss'}$

② $\beta(\vec{\gamma}\cdot\vec{p} + m)v = \beta(\gamma^0 E - \vec{\gamma}\cdot\vec{p} + m)v = (E - \vec{\alpha}\cdot\vec{p} + \beta m)v = 0$, $\therefore (\vec{\alpha}\cdot\vec{p} - \beta m)v = E\,v$

(3) $H = \int \psi^{+}(x)(-i\vec{\alpha}\cdot\nabla + \beta m)\psi(x) d^3x = \int d^3x \sum_{\vec{p},s} \sqrt{\frac{m}{VE}} [C^{+}_{\vec{p},s} u^{+}_{\vec{p},s} e^{ipx} + d_{\vec{p},s} v^{+}_{\vec{p},s} e^{-ipx}] (-i\vec{\alpha}\cdot\nabla + \beta m) \sum_{\vec{p}',s'} \sqrt{\frac{m}{VE'}} [C_{\vec{p}',s'} u_{\vec{p}',s'} e^{-ip'x} + d^{+}_{\vec{p}',s'} v_{\vec{p}',s'} e^{ip'x}]$

利用 $(-i\vec{\alpha}\cdot\nabla + \beta m) u_{\vec{p}',s'} e^{-iE't + i\vec{p}'\cdot\vec{x}} = (\vec{\alpha}\cdot\vec{p}' + \beta m) u_{\vec{p}',s'} e^{-ip'x} = E_{p'} u_{\vec{p}',s} e^{-ip'x}$, $(-i\vec{\alpha}\cdot\nabla + \beta m) v_{\vec{p}',s'} e^{iE't - i\vec{p}'\cdot\vec{x}} = (-\vec{\alpha}\cdot\vec{p}' + \beta m) v_{\vec{p}',s'} e^{ip'x} = -E_{p'} v_{\vec{p}',s'} e^{ip'x} = -E_{p'} v_{\vec{p}',s'} e^{ip'x}$

$\therefore H = \int d^3x \sum_{\vec{p},s} \sum_{\vec{p}',s'} \frac{m E'}{V\sqrt{EE'}} [C^{+}_{\vec{p},s} u^{+}_{\vec{p},s} e^{ipx} + d_{\vec{p},s} v^{+}_{\vec{p},s} e^{-ipx}][C_{\vec{p}',s'} u_{\vec{p}',s'} e^{-ip'x} - d^{+}_{\vec{p}',s'} v_{\vec{p}',s'} e^{ip'x}]$

对积分 $= \int d^3x \sum_{\vec{p},s} \sum_{\vec{p}',s'} \frac{m}{V}\sqrt{\frac{E'}{E}} [C^{+}_{\vec{p},s} C_{\vec{p}',s'} u^{+}_{\vec{p},s} u_{\vec{p}',s'} e^{i(p-p')x} - C^{+}_{\vec{p},s} d^{+}_{\vec{p}',s'} u^{+}_{\vec{p},s} v_{\vec{p}',s'} e^{i(p+p')x} + d_{\vec{p},s} C_{\vec{p}',s'} v^{+}_{\vec{p},s} u_{\vec{p}',s'} e^{-i(p+p')x} - d_{\vec{p},s} d^{+}_{\vec{p}',s'} v^{+}_{\vec{p},s} v_{\vec{p}',s'} e^{-i(p-p')x}]$

$= \sum_{\vec{p},s} \sum_{\vec{p}',s'} \frac{Vm}{V}\sqrt{\frac{E'}{E}} [C^{+}_{\vec{p},s} C_{\vec{p}',s'} u^{+}_{\vec{p},s} u_{\vec{p}',s'} e^{i(E-E')t} \delta_{\vec{p}\vec{p}'} - C^{+}_{\vec{p},s} d^{+}_{\vec{p}',s'} u^{+}_{\vec{p},s} v_{\vec{p}',s'} e^{i(E+E')t} \delta_{\vec{p},-\vec{p}'} + d_{\vec{p},s} C_{\vec{p}',s'} v^{+}_{\vec{p},s} u_{\vec{p}',s'} e^{-i(E+E')t} \delta_{\vec{p},-\vec{p}'} - d_{\vec{p},s} d^{+}_{\vec{p}',s'} v^{+}_{\vec{p},s} v_{\vec{p}',s'} e^{-i(E-E')t} \delta_{\vec{p}\vec{p}'}]$

求和 正交

$= \sum_{\vec{p},s,s'} m [C^{+}_{\vec{p},s} C_{\vec{p},s'} \underbrace{u^{+}_{\vec{p},s} u_{\vec{p},s'}}_{\frac{E}{m}\delta_{ss'}} - C^{+}_{\vec{p},s} d^{+}_{-\vec{p},s'} \underbrace{u^{+}_{\vec{p},s} v_{-\vec{p},s'}}_{0} e^{2iEt} + d_{\vec{p},s} C_{-\vec{p},s'} \underbrace{v^{+}_{\vec{p},s} u_{-\vec{p},s'}}_{0} e^{-2iEt} - d_{\vec{p},s} d^{+}_{\vec{p},s'} \underbrace{v^{+}_{\vec{p},s} v_{\vec{p},s'}}_{\frac{E}{m}\delta_{ss'}}] = \sum_{\vec{p},s} E [C^{+}_{\vec{p},s} C_{\vec{p},s} + d^{+}_{\vec{p},s} d_{\vec{p},s} - 1]$

五 粒子数表象

引入 $N = C^+_{\vec p,s} C_{\vec p,s}$ $\bar N = d^+_{\vec p,s} d_{\vec p,s}$

1.性质:

$[N, C^+] = [C^+C, C^+] = C^+CC^+ - C^+C^+C \overset{\text{反对易}}{=} C^+\underline{CC^+} = C^+(1-C^+C) = C^+ - \overset{=0}{\underline{C^+C^+}}C = C^+$

$[N, C] = [C^+C, C] = C^+\underbrace{CC}_{=0} - CC^+C = -C(1-CC^+) = -C + CCC^+ = -C$

$N^2 = C^+CC^+C = C^+(1-C^+C)C = C^+C = N$, $\bar N$亦有类似性质.

2.取N对角化表象$|n\rangle$, $\bar N$对角化表象$|\bar n\rangle$.

$N|n\rangle = n|n\rangle$, $\bar N|\bar n\rangle = \bar n|\bar n\rangle$.

则 $N^2|n\rangle = N|n\rangle$ $\bar N^2|\bar n\rangle = \bar N|\bar n\rangle$

故 $n^2 = n$ 即 $n = 0, 1$; $\bar n^2 = \bar n$, 即 $\bar n = 0, 1$

∴ N或$\bar N$的本征值只有两个: 0, 1, 这表明一个量子态上最多只能容纳一个粒子, 从而服从泡利不相容原理, 费米狄拉克统计.

3.产生湮灭算符.

$\begin{cases} NC^+|n\rangle = (C^+ + C^+N)|n\rangle = (n+1)C^+|n\rangle \\ NC|n\rangle = (-C + CN)|n\rangle = (n-1)C|n\rangle \end{cases}$

故$C^+|n\rangle$仍为N本征态, C^+称粒子产生算符. $C|n\rangle$仍为N本征态, C称粒子湮灭算符.

$\begin{cases} \bar N d^+|\bar n\rangle = (d^+ + d^+\bar N)|\bar n\rangle = (\bar n+1)d^+|\bar n\rangle \\ N d|\bar n\rangle = (-d + d\bar N)|\bar n\rangle = (\bar n-1)d|\bar n\rangle \end{cases}$

故$d^+|\bar n\rangle$仍为$\bar N$本征态, d^+称反粒子产生算符. $d|\bar n\rangle$仍为$\bar N$本征态, d称反粒子湮灭算符.

4.解释.

1) $\begin{cases} C^+_{\vec p,s}|0\rangle & \text{一个动量为}\vec p\text{, 自旋为S的单粒子本征态;} \\ d^+_{\vec p,s}|0\rangle & \text{一个动量为}\vec p\text{, 自旋为S的反粒子本征态;} \end{cases}$

2) $\begin{cases} C^+_{\vec p,s}C^+_{\vec p,s}|0\rangle = 0 \\ d^+_{\vec p,s}d^+_{\vec p,s}|0\rangle = 0 \end{cases}$ 不可能有两个相同粒子处于同一个量子态. "泡利不相容"

3) $C^+_{\vec p,s}C^+_{\vec p',s'}|0\rangle = -C^+_{\vec p',s'}C^+_{\vec p,s}|0\rangle$ 两个费米子交换位置时,系统状态出一个"负号", 费米狄拉克统计.

4) 由Q和动量空间表示. ~~$Q = \sum_k (\cdots)$~~ $Q = \sum_{\vec p,s}[C^+_{\vec p,s}C_{\vec p,s} - d^+d_{\vec p,s}]$,

$[Q, C^+_{\vec p,s}] = [\sum_{\vec p',s'}(C^+_{\vec p',s'}C_{\vec p',s'} - d^+_{\vec p',s'}d_{\vec p',s'}), C^+_{\vec p,s}] = \sum_{\vec p',s'} C^+_{\vec p',s'}\{C_{\vec p',s'}, C^+_{\vec p,s}\} = \sum_{\vec p',s'} C^+_{\vec p',s'}\delta_{\vec p\vec p'}\delta_{ss'} = C^+_{\vec p,s}$

$[Q, d^+_{\vec p,s}] = \cdots - d^+_{\vec p,s}$

∴ $QC^+_{\vec p,s}|0\rangle = [Q, C^+_{\vec p,s}]|0\rangle = C^+_{\vec p,s}|0\rangle$, $C^+_{\vec p,s}|0\rangle$是Q的本征值为+1的态 正粒子, Q表电荷

$Qd^+_{\vec p,s}|0\rangle = [Q, d^+_{\vec p,s}]|0\rangle = -d^+_{\vec p,s}|0\rangle$, $d^+_{\vec p,s}|0\rangle$是Q的本征值为-1的态 反粒子.