

§4.3 复标量场量子化

一. 经典场

实标量场，由于是实数性的 $\phi^* = \phi$ ，因此在量子化以后只能描述不带电荷的中性介子，比如 π^0 介子。为了描述带电荷的介子，比如 $\pi^+ \pi^-$ 介子，必须引入复标量场。此时， $\phi^* \neq \phi$ ，从而在量子化以后能描述带电荷的正粒子和反粒子，比如 π^+, π^- 。

1. 经典复标量场.

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi, \quad \mathcal{L}(\phi, \phi^*)$$

$$\text{拉氏方程: } (\partial_\mu \partial^\mu + m^2) \phi(x) = 0, \quad (\partial_\mu \partial^\mu + m^2) \phi^*(x) = 0.$$

2. 正则量子化

引入正则动量

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}} (\partial_0 \phi^* \partial^0 \phi - \nabla \phi^* \cdot \nabla \phi) = \dot{\phi}^*(x)$$

$$\pi^*(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \frac{\partial}{\partial \dot{\phi}^*} (\partial_0 \phi^* \partial^0 \phi - \nabla \phi^* \cdot \nabla \phi) = \dot{\phi}(x)$$

3) 哈密顿量

$$H \equiv \int (\pi \dot{\phi} + \pi^* \dot{\phi}^* - \mathcal{L}) d^3x = \int [\dot{\phi}^* \dot{\phi} + \dot{\phi} \dot{\phi}^* - \dot{\phi}^* \dot{\phi} + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi] d^3x \\ = \int [\cancel{\pi \dot{\phi}^*} \pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi] d^3x$$

$$\therefore H = \int d^3x \mathcal{H}$$

$$\mathcal{H} = \pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi$$

假定正则对易关系.

$$[\phi(\vec{x}, t), \pi(\vec{x}', t)] = i \delta^3(\vec{x} - \vec{x}')$$

$$[\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0.$$

$$[\phi^*(\vec{x}, t), \pi^*(\vec{x}', t)] = i \delta^3(\vec{x} - \vec{x}')$$

$$[\phi^*(\vec{x}, t), \phi^*(\vec{x}', t)] = [\pi^*(\vec{x}, t), \pi^*(\vec{x}', t)] = 0$$

$$[\phi(\vec{x}, t), \phi^*(\vec{x}', t)] = [\phi(\vec{x}, t), \pi^*(\vec{x}', t)] = [\pi(\vec{x}, t), \pi^*(\vec{x}', t)] = [\phi^*(\vec{x}, t), \pi(\vec{x}', t)] = 0$$

则有海森堡方程

$$[H, \phi(\vec{x}, t)] = \int d^3x' [\pi^*(\vec{x}', t) \pi(\vec{x}, t) + \nabla \phi^*(\vec{x}', t) \cdot \nabla \phi(\vec{x}, t) + m^2 \phi^*(\vec{x}', t) \phi(\vec{x}, t)] \\ = \int d^3x' \{ [\pi^*(\vec{x}', t), \phi(\vec{x}, t)] \pi(\vec{x}', t) + \pi^*(\vec{x}', t) [\pi(\vec{x}', t), \phi(\vec{x}, t)] \} \\ = \int d^3x' \pi^*(\vec{x}', t) (-i) \delta^3(\vec{x}' - \vec{x}) = -i \pi^*(\vec{x}, t) = -i \dot{\phi}(x)$$

同理，海森堡方程为

$$\begin{cases} \dot{\phi}(\vec{x}, t) = i[H, \phi(\vec{x}, t)] \\ \dot{\pi}(\vec{x}, t) = i[H, \pi(\vec{x}, t)] \end{cases}$$

$$\begin{cases} \dot{\phi^*}(\vec{x}, t) = i[H, \phi^*(\vec{x}, t)] \\ \dot{\pi^*}(\vec{x}, t) = i[H, \pi^*(\vec{x}, t)] \end{cases}$$

2) 动量算符 黄P41(3.85)

$$\vec{p} = \int d^3x T^{0i} = \int d^3x (-g^{0i} L + \partial^0 \phi^* \partial^i \phi + \partial^i \phi^* \partial^0 \phi) = - \int d^3x (\pi \nabla \phi + \pi^* \nabla \phi^*)$$

R] $\langle \vec{p}, \phi(\vec{x}, t) \rangle = \int d^3x' [\pi(\vec{x}, t) \nabla' \phi(\vec{x}', t) + \pi^*(\vec{x}, t) \nabla' \phi^*(\vec{x}', t), \phi(\vec{x}, t)]$
 $= \int d^3x' \{[\pi(\vec{x}, t), \phi(\vec{x}', t)] \nabla' \phi(\vec{x}', t) + [\pi^*(\vec{x}, t), \phi(\vec{x}', t)] \nabla' \phi^*(\vec{x}', t)\}$
 $= \int d^3x' i \delta(\vec{x}' - \vec{x}) \nabla' \phi(\vec{x}', t) = i \nabla \phi(\vec{x}, t)$

综上 $\begin{cases} \nabla \phi(\vec{x}, t) = i[-\vec{p}, \phi(\vec{x}, t)] \\ \nabla \pi(\vec{x}, t) = i[-\vec{p}, \pi(\vec{x}, t)] \end{cases}, \quad \begin{cases} \nabla \phi^*(\vec{x}, t) = i[-\vec{p}, \phi^*(\vec{x}, t)] \\ \nabla \pi^*(\vec{x}, t) = i[-\vec{p}, \pi^*(\vec{x}, t)] \end{cases}$

3) 推广的海森堡方程 $\begin{cases} \frac{\partial}{\partial x^m} \phi(\vec{x}, t) = i[P_m, \phi(\vec{x}, t)] \\ \frac{\partial}{\partial x^m} \pi(\vec{x}, t) = i[P_m, \pi(\vec{x}, t)] \end{cases} \quad \begin{cases} \frac{\partial}{\partial x^m} \phi^*(\vec{x}, t) = i[P_m, \phi^*(\vec{x}, t)] \\ \frac{\partial}{\partial x^m} \pi^*(\vec{x}, t) = i[P_m, \pi^*(\vec{x}, t)] \end{cases}$

从而角解为（复标量场在时空平移下的变化性质）

(1) $\phi(x) = e^{iPx} \phi(0) e^{-iPx}, \quad \phi^*(x) = e^{iPx} \phi^*(0) e^{-iPx}$

在物理中存在许多守恒律，如电荷守恒，重子数守恒，轻子数守恒，奇偶数守恒，同位旋守恒。但却找不到与之相对应的对称性。为解决这一问题，人们引入“内部空间”。

复标量场 L 满足规范变换，即在坐标不变，仅改变复标量场 $\phi(x)$ 时， L 保持不变：

$$\phi \rightarrow \phi(x) = e^{i\alpha} \phi(x), \quad \delta\phi = i\delta\alpha \phi$$

$$\phi^* \rightarrow \phi^*(x) = e^{-i\alpha} \phi^*(x), \quad \delta\phi^* = -i\delta\alpha \phi^*$$

利用作用量变分原理 $\delta S = 0$ ，得 $\partial_\mu j^\mu = 0$ ，从而引入流密度。

守恒流 $j^\mu = -i \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \phi(x) - \frac{\partial L}{\partial (\partial_\mu \phi^*)} \phi^*(x) \right)$
 $= -i (\partial^\mu \phi^* \phi - \partial^\mu \phi \phi^*) \equiv i \phi^* \overleftrightarrow{\partial}^\mu \phi \quad (\text{其中 } \overleftrightarrow{\partial}^\mu = \overrightarrow{\partial}^\mu - \overleftarrow{\partial}^\mu)$

守恒荷 $Q = \int j^0 d^3x = -i \int (\partial^0 \phi^* \phi - \partial^0 \phi \phi^*) d^3x = -i \int (\phi^* \phi - \phi \phi^*) d^3x$

$$= -i \int (\pi \phi - \pi^* \phi^*) d^3x \quad \left\{ \begin{array}{l} \text{若复标量场，存在守恒荷如电荷，奇偶数，轻子数，} \\ \text{这些守恒量是相加性守恒量。} \end{array} \right.$$

注：① 对不同标量场， j^μ 和 Q 的表达式与含义不同。

$\left. \begin{array}{l} \text{若实标量场 } \phi^* = \phi, \text{ 或电磁场 } A_\mu^* = A_\mu; \text{ 则 } j^\mu = 0, Q = 0. \\ \text{表明不存在标量场的守恒荷。} \end{array} \right\}$

② 对复标量场， j^0 相当于电荷密度。

在量子化后， j^0 中包含了正电荷粒子和负电荷粒子。

二. 傅立叶展开 (未满元) P_{32}

利用空间 V 内的系统中, 复标量场 $\phi(\vec{x})$, $\phi^*(\vec{x})$ 满足的周期性边界条件, 非厄米性和 k -G 方程, 可得复标量场的傅立叶展开形式的解。

1. 在周期性边界条件下, 得满足正交性、完备性的本征函数

$$\left\{ e^{i\vec{k}_n \cdot \vec{x}} \mid, \vec{k}_n = \vec{n} \frac{2\pi}{L}, \vec{n} = 0, \pm 1, \pm 2, \dots \right\}$$

2. 复标量场满足周期边界, 可傅立叶展开如下.

$$\begin{cases} \phi(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{i\vec{k} \cdot \vec{x}} g_{\vec{k}}(t) \\ \pi(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{-i\vec{k} \cdot \vec{x}} p_{\vec{k}}(t) \end{cases} \quad \text{其中} \quad \begin{cases} g_{\vec{k}}(t) = \frac{1}{\sqrt{V}} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \phi(\vec{x}, t) \\ p_{\vec{k}}(t) = \frac{1}{\sqrt{V}} \int d^3x e^{i\vec{k} \cdot \vec{x}} \pi(\vec{x}, t) \end{cases}$$

$$\begin{cases} \phi^*(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{-i\vec{k} \cdot \vec{x}} g_{\vec{k}}^*(t) \\ \pi^*(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{i\vec{k} \cdot \vec{x}} p_{\vec{k}}^*(t) \end{cases} \quad \text{其中} \quad \begin{cases} g_{\vec{k}}^*(t) = \frac{1}{\sqrt{V}} \int d^3x e^{i\vec{k} \cdot \vec{x}} \phi^*(\vec{x}, t) \\ p_{\vec{k}}^*(t) = \frac{1}{\sqrt{V}} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \pi^*(\vec{x}, t) \end{cases}$$

经过傅立叶展开, 场量成为动量空间函数, 独立变量为 $\{g_{\vec{k}}(t), p_{\vec{k}}(t), g_{\vec{k}}^*(t), p_{\vec{k}}^*(t)\}$

① 动量空间的产生湮灭算符表示

$$\begin{cases} a_{\vec{k}}(t) = \frac{\sqrt{w}}{2} (g_{\vec{k}}(t) + i \frac{1}{w} p_{\vec{k}}^*(t)) \\ b_{-\vec{k}}^*(t) = \frac{\sqrt{w}}{2} (g_{\vec{k}}(t) - i \frac{1}{w} p_{\vec{k}}^*(t)) \\ a_{\vec{k}}^*(t) = \frac{\sqrt{w}}{2} (g_{\vec{k}}^*(t) - i \frac{1}{w} p_{\vec{k}}(t)) \\ b_{-\vec{k}}(t) = \frac{\sqrt{w}}{2} (g_{\vec{k}}^*(t) + i \frac{1}{w} p_{\vec{k}}(t)) \end{cases}$$

$$\begin{cases} g_{\vec{k}}(t) = \frac{1}{\sqrt{2w}} (a_{\vec{k}}(t) + b_{-\vec{k}}^*(t)) \\ p_{\vec{k}}^*(t) = \frac{-iw}{\sqrt{2w}} (a_{\vec{k}}(t) - b_{-\vec{k}}^*(t)) \\ g_{\vec{k}}^*(t) = \frac{1}{\sqrt{2w}} (a_{\vec{k}}^*(t) + b_{-\vec{k}}(t)) \\ p_{\vec{k}}(t) = \frac{+iw}{\sqrt{2w}} (a_{\vec{k}}^*(t) - b_{-\vec{k}}(t)) \end{cases}$$

代入场展开式 (为推广, 将 \rightarrow \rightarrow)

$$\begin{cases} \phi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2wV}} (a_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}} + b_{\vec{k}}^*(t) e^{-i\vec{k} \cdot \vec{x}}) \\ \pi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{+iw}{\sqrt{2wV}} (a_{\vec{k}}^*(t) e^{-i\vec{k} \cdot \vec{x}} - b_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}}) \end{cases}$$

$$\begin{cases} \phi^*(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2wV}} (a_{\vec{k}}^*(t) e^{-i\vec{k} \cdot \vec{x}} + b_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}}) \\ \pi^*(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{-iw}{\sqrt{2wV}} (a_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}} - b_{\vec{k}}^*(t) e^{-i\vec{k} \cdot \vec{x}}) \end{cases}$$

② ϕ, ϕ^* 满足 K-G 方程的场量解.

$$(\square + m^2)\phi(x) = 0, \quad (\square + m^2)\phi^*(x) = 0, \quad \text{且 } w = \sqrt{k^2 + m^2}$$

$$\Rightarrow (w^2 + \partial_t^2) g_{\vec{k}}(t) = 0, \quad (w^2 + \partial_t^2) g_{\vec{k}}^*(t) = 0$$

解为 $g_{\vec{k}}(t), g_{\vec{k}}^*(t) \sim e^{int}, e^{-int}$

结合场算符的产生湮灭表示, 得 $\begin{cases} a_{\vec{k}}(t) = a_{\vec{k}} e^{-int} \\ b_{\vec{k}}^+(t) = b_{\vec{k}}^+ e^{int} \end{cases}$

$$\begin{cases} a_{\vec{k}}^+(t) = a_{\vec{k}}^+ e^{int} \\ b_{\vec{k}}(t) = b_{\vec{k}}^- e^{-int} \end{cases}$$

$$\therefore \left\{ \begin{array}{l} \phi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2wV}} (a_{\vec{k}} e^{-ikx} + b_{\vec{k}}^+ e^{ikx}) \\ \pi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{iw}{\sqrt{2wV}} (a_{\vec{k}}^+ e^{ikx} - b_{\vec{k}}^- e^{-ikx}) \end{array} \right.$$

其中 $kx = wt - \vec{k} \cdot \vec{x}$

$$\vec{k} = \vec{k}_n = \vec{n} \frac{2\pi}{L}$$

$$\vec{n} = 0, \pm 1, \pm 2, \dots$$

$$\left\{ \begin{array}{l} \phi^*(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2wV}} (a_{\vec{k}}^+ e^{ikx} + b_{\vec{k}}^- e^{-ikx}) \\ \pi^*(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{-iw}{\sqrt{2wV}} (a_{\vec{k}}^- e^{-ikx} - b_{\vec{k}}^+ e^{ikx}) \end{array} \right.$$

麦克斯

当 $L \rightarrow \infty, V \rightarrow \infty$ 时, 动量由分立 \rightarrow 连续. $\sum_{\vec{k}} \rightarrow \int \frac{V}{(2\pi)^3} d^3 k$, 及

$$\boxed{a(\vec{k}) = \sqrt{2wV} a_{\vec{k}}, \quad b(\vec{k}) = \sqrt{2wV} b_{\vec{k}}}$$

$$\left\{ \begin{array}{l} \phi(\vec{x}, t) = \int \frac{d^3 k}{2w(2\pi)^3} (a(\vec{k}) e^{-ikx} + b^+(\vec{k}) e^{ikx}) \end{array} \right.$$

其中 $kx = wt - \vec{k} \cdot \vec{x}$

$$\left\{ \begin{array}{l} \pi(\vec{x}, t) = \int \frac{d^3 k (iw)}{2w(2\pi)^3} (a(\vec{k})^+ e^{ikx} - b(\vec{k}) e^{-ikx}) \end{array} \right.$$

$\vec{k} = -\infty, \infty$, 动量连续

$$\left\{ \begin{array}{l} \phi^*(\vec{x}, t) = \int \frac{d^3 k}{2w(2\pi)^3} (a(\vec{k})^+ e^{ikx} + b(\vec{k}) e^{-ikx}) \end{array} \right.$$

$$\left\{ \begin{array}{l} \pi^*(\vec{x}, t) = \int \frac{d^3 k (-iw)}{2w(2\pi)^3} (a(\vec{k}) e^{-ikx} - b(\vec{k})^+ e^{ikx}) \end{array} \right.$$

黄涛

③下面确定场量 $\{\phi, \pi, \phi^*, \pi^*\}$ 中不含时的振幅算子 $\{a_{\vec{k}}, b_{\vec{k}}^+, a_{\vec{k}}^+, b_{\vec{k}}^-\}$.

$$a_{\vec{k}}(t) = a_{\vec{k}} e^{-int} = \sqrt{\frac{w}{2}} (q_{\vec{k}}(t) + \frac{i}{w} p_{\vec{k}}^*(t)) = \sqrt{\frac{w}{2}} \frac{1}{\sqrt{V}} \int d^3x e^{-i\vec{k}\cdot\vec{x}} [\phi(\vec{x}, t) + \frac{i}{w} \pi^*(\vec{x}, t)] \\ = \frac{1}{\sqrt{2wV}} \int d^3x e^{-i\vec{k}\cdot\vec{x}} [w\phi(\vec{x}, t) + i\pi^*(\vec{x}, t)]$$

$$\therefore a_{\vec{k}} = \frac{1}{\sqrt{2wV}} \int d^3x e^{ikx} [w\phi(\vec{x}, t) + i\pi^*(\vec{x}, t)]$$

$$b_{\vec{k}}^+(t) = b_{\vec{k}}^+ e^{int} = \sqrt{\frac{w}{2}} (q_{-\vec{k}}(t) - \frac{i}{w} p_{-\vec{k}}^*(t)) = \sqrt{\frac{w}{2}} \frac{1}{\sqrt{V}} \int d^3x e^{i\vec{k}\cdot\vec{x}} [\phi(\vec{x}, t) - \frac{i}{w} \pi^*(\vec{x}, t)] \\ = \frac{1}{\sqrt{2wV}} \int d^3x e^{i\vec{k}\cdot\vec{x}} [w\phi(\vec{x}, t) - i\pi^*(\vec{x}, t)]$$

$$\therefore b_{\vec{k}}^+ = \frac{1}{\sqrt{2wV}} \int d^3x e^{-ikx} [w\phi(\vec{x}, t) - i\pi^*(\vec{x}, t)]$$

$$\left\{ \begin{array}{l} a_{\vec{k}} = \frac{1}{\sqrt{2wV}} \int d^3x e^{ikx} [w\phi(\vec{x}, t) + i\pi^*(\vec{x}, t)] \quad \vec{k} \text{ 分立} \\ b_{\vec{k}}^+ = \frac{1}{\sqrt{2wV}} \int d^3x e^{-ikx} [w\phi(\vec{x}, t) - i\pi^*(\vec{x}, t)] \\ a_{\vec{k}}^+ = \frac{1}{\sqrt{2wV}} \int d^3x e^{-ikx} [w\phi^*(\vec{x}, t) - i\pi(\vec{x}, t)] \\ b_{\vec{k}} = \frac{1}{\sqrt{2wV}} \int d^3x e^{ikx} [w\phi^*(\vec{x}, t) + i\pi(\vec{x}, t)] \end{array} \right. \quad \begin{array}{l} \text{姜志进} \end{array}$$

当 $L \rightarrow \infty$, 从而 $V \rightarrow \infty$ 时

$$\left\{ \begin{array}{l} a(\vec{k}) = \sqrt{2wV} a_{\vec{k}} = \int d^3x e^{ikx} [w\phi(\vec{x}, t) + i\pi^*(\vec{x}, t)] \\ b^+(\vec{k}) = \sqrt{2wV} b_{\vec{k}}^+ = \int d^3x e^{-ikx} [w\phi(\vec{x}, t) - i\pi^*(\vec{x}, t)] \end{array} \right. \quad \vec{k} \text{ 连续}$$

$$\left\{ \begin{array}{l} a_{\vec{k}}^+ = \sqrt{2wV} a_{\vec{k}}^+ = \int d^3x e^{-ikx} [w\phi^*(\vec{x}, t) - i\pi(\vec{x}, t)] \\ b(\vec{k}) = \sqrt{2wV} b_{\vec{k}} = \int d^3x e^{ikx} [w\phi^*(\vec{x}, t) + i\pi(\vec{x}, t)] \end{array} \right. \quad \begin{array}{l} \text{黄涛} \end{array}$$

三. 动量空间的(t)算符.

$$\begin{cases} \phi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\omega}} (a_{\vec{k}} e^{-i\vec{k}\vec{x}} + b_{\vec{k}}^+ e^{i\vec{k}\vec{x}}), \\ \pi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{i\omega}{\sqrt{2\pi\omega}} (a_{\vec{k}}^+ e^{i\vec{k}\vec{x}} - b_{\vec{k}} e^{-i\vec{k}\vec{x}}) \end{cases}, \quad T$$

其中, $\begin{cases} a_{\vec{k}} = \frac{1}{\sqrt{2\pi\omega}} \int d^3x e^{i\vec{k}\vec{x}} [\omega\phi(\vec{x}, t) + i\pi^*(\vec{x}, t)] \\ b_{\vec{k}}^+ = \frac{1}{\sqrt{2\pi\omega}} \int d^3x e^{-i\vec{k}\vec{x}} [\omega\phi(\vec{x}, t) - i\pi^*(\vec{x}, t)] \end{cases}$

也可以量子化。

由于 $\phi(\vec{x}, t), \pi(\vec{x}, t), \phi^*(\vec{x}, t), \pi^*(\vec{x}, t)$ 可以量子化, 故相应的 $a_{\vec{k}}, b_{\vec{k}}^+, a_{\vec{k}}^+, b_{\vec{k}}^-, H, \vec{P}$ ↓

可证: $[a_{\vec{k}}, a_{\vec{k}'}^+] = \delta_{\vec{k}\vec{k}'}$

\vec{k} 分 $\begin{cases} [b_{\vec{k}}, b_{\vec{k}'}^+] = \delta_{\vec{k}\vec{k}'} \text{ 其它对易子为0.} \\ H = \sum_{\vec{k}} \omega [a_{\vec{k}}^+ a_{\vec{k}} + b_{\vec{k}}^+ b_{\vec{k}} + 1] \end{cases}$

\vec{k} 对 $\begin{cases} \vec{P} = \sum_{\vec{k}} \vec{k} [a_{\vec{k}}^+ a_{\vec{k}} + b_{\vec{k}}^+ b_{\vec{k}}] \end{cases}$

$Q = \sum_{\vec{k}} [a_{\vec{k}}^+ a_{\vec{k}} - b_{\vec{k}}^+ b_{\vec{k}}]$ 正反粒子 电荷相反

连续 $\begin{cases} [a(\vec{k}), a^+(\vec{k}')] = (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}') \\ [b(\vec{k}), b^+(\vec{k}')] = (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}') \end{cases}$ 其它对易子为0.

$H = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2\omega_k} [a(\vec{k}) a^+(\vec{k}) + b(\vec{k}) b^+(\vec{k})]$

$\vec{P} = \int \frac{d^3k}{(2\pi)^3} \frac{\vec{k}}{2\omega_k} [a(\vec{k}) a^+(\vec{k}) + b(\vec{k}) b^+(\vec{k})]$

$Q = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} [a(\vec{k}) a^+(\vec{k}) - b(\vec{k}) b^+(\vec{k})]$

证明: 动量空间 U(1) 变换的守恒荷

$$Q = -i \int (\pi\phi - \pi^*\phi^*) d^3x$$

$$= -i \sum_{\vec{k}, \vec{k}'}^{\infty} \int d^3x \left[\frac{i\omega}{2\pi\omega} (a_{\vec{k}}^+ e^{i\vec{k}\vec{x}} - b_{\vec{k}} e^{-i\vec{k}\vec{x}}) (a_{\vec{k}'} e^{-i\vec{k}'\vec{x}} + b_{\vec{k}'}^+ e^{i\vec{k}'\vec{x}}) - \frac{-i\omega}{2\pi\omega} (a_{\vec{k}}^+ e^{-i\vec{k}\vec{x}} - b_{\vec{k}} e^{i\vec{k}\vec{x}}) (a_{\vec{k}'} e^{i\vec{k}'\vec{x}} + b_{\vec{k}'}^+ e^{-i\vec{k}'\vec{x}}) \right]$$

对 x 积分, 利用 $\int dx e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = \sqrt{\delta_{\vec{k}\vec{k}'}}$ 对 \vec{k}' 求和。 $\begin{cases} \int dx e^{i(\vec{k}-\vec{k}')\vec{x}} = \sqrt{\delta_{\vec{k}\vec{k}'}} \int dx e^{i(v-w')t} e^{-i(\vec{k}-\vec{k}')\vec{x}} = \sum_{\vec{k}'} V e^{i(v-w')t} \delta_{\vec{k}\vec{k}'} = V \\ \sum_{\vec{k}'} \int dx e^{i(v+w')t} e^{-i(\vec{k}+\vec{k}')\vec{x}} = \sum_{\vec{k}'} V e^{i(v+w')t} \delta_{\vec{k}\vec{k}'} = V e^{i(v+w')t} \end{cases}$

$2\sqrt{\omega\omega'}$ $= \sum_{\vec{k}, \vec{k}'} \int dx \frac{\omega}{2\pi\omega} \left[a_{\vec{k}}^+ a_{\vec{k}'} e^{i(\vec{k}-\vec{k}')\vec{x}} - b_{\vec{k}} b_{\vec{k}'}^+ e^{-i(\vec{k}-\vec{k}')\vec{x}} + a_{\vec{k}}^+ a_{\vec{k}'}^+ e^{-i(\vec{k}-\vec{k}')\vec{x}} - b_{\vec{k}}^+ b_{\vec{k}'} e^{i(\vec{k}-\vec{k}')\vec{x}} \right] + \left[a_{\vec{k}}^+ b_{\vec{k}'}^+ e^{i(\vec{k}+\vec{k}')\vec{x}} - b_{\vec{k}} a_{\vec{k}'}^+ e^{-i(\vec{k}+\vec{k}')\vec{x}} + a_{\vec{k}}^+ b_{\vec{k}'} e^{-i(\vec{k}+\vec{k}')\vec{x}} - b_{\vec{k}}^+ b_{\vec{k}'}^+ e^{i(\vec{k}+\vec{k}')\vec{x}} \right]$

 $= \sum_{\vec{k}} \frac{1}{2} \left[a_{\vec{k}}^+ a_{\vec{k}} - b_{\vec{k}} b_{\vec{k}}^+ + a_{\vec{k}}^+ a_{\vec{k}}^+ - b_{\vec{k}}^+ b_{\vec{k}} \right] + a_{\vec{k}}^+ b_{\vec{k}}^+ e^{2i\omega t} - b_{\vec{k}} a_{\vec{k}}^+ e^{-2i\omega t} + a_{\vec{k}}^+ b_{\vec{k}} e^{-2i\omega t} - b_{\vec{k}}^+ a_{\vec{k}}^+ e^{2i\omega t}$
 $= \sum_{\vec{k}} [a_{\vec{k}}^+ a_{\vec{k}} - b_{\vec{k}}^+ b_{\vec{k}}]$

$[a_{\vec{k}}, b_{-\vec{k}}] = [a_{\vec{k}}, b_{\vec{k}}^+] = 0$ \Rightarrow 易得 $\Rightarrow 0$

$\vec{k} \leftrightarrow -\vec{k}$ $\rightarrow 0$

四. 粒子数表象.

$$\text{引入 } N_a = a_{\vec{k}}^\dagger a_{\vec{k}}, \quad N_b = b_{\vec{k}}^\dagger b_{\vec{k}}.$$

$$N = \sum_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}} \quad \bar{N} = \sum_{\vec{k}} b_{\vec{k}}^\dagger b_{\vec{k}}$$

取 N_a 对角化表象 $|n_{a\vec{k}}\rangle$, N_b 对角化表象 $|n_{b\vec{k}}\rangle$, 则

N 对角化表象 $|n_{a\vec{k}_1}, n_{a\vec{k}_2}, \dots, n_{a\vec{k}_m}, \dots\rangle$, \bar{N} 对角化表象 $|n_{b\vec{k}_1}, n_{b\vec{k}_2}, \dots, n_{b\vec{k}_m}, \dots\rangle$.

$$\therefore N_a |n_{a\vec{k}}\rangle = n_{a\vec{k}} |n_{a\vec{k}}\rangle, \quad \bar{N}_b |n_{b\vec{k}}\rangle = n_{b\vec{k}} |n_{b\vec{k}}\rangle$$

即 $|n_{a\vec{k}}\rangle$ 是 N_a 的本征值为 $n_{a\vec{k}}$ 的本征态, $|n_{b\vec{k}}\rangle$ 是 N_b 的本征值为 $n_{b\vec{k}}$ 的本征态.

$$\text{而 } N |n_{a\vec{k}_1}, n_{a\vec{k}_2}, \dots\rangle = n |n_{a\vec{k}_1}, n_{a\vec{k}_2}, \dots\rangle = \sum_{\vec{k}} n_{a\vec{k}} |n_{a\vec{k}_1}, n_{a\vec{k}_2}, \dots\rangle$$

$$\bar{N} |n_{b\vec{k}_1}, n_{b\vec{k}_2}, \dots\rangle = \bar{n} |n_{b\vec{k}_1}, n_{b\vec{k}_2}, \dots\rangle = \sum_{\vec{k}} n_{b\vec{k}} |n_{b\vec{k}_1}, n_{b\vec{k}_2}, \dots\rangle$$

1) 真空态:

$$a_{\vec{k}} |0\rangle = 0, \quad b_{\vec{k}} |0\rangle, \quad \langle 0| 0 \rangle = 1, \quad (\vec{k} \text{ 取所有分立动量})$$

2) 本征态.

$a_{\vec{k}}^\dagger |0\rangle$ 一个4动量 \vec{k} 的粒子态.

$a_{\vec{k}_1}^\dagger a_{\vec{k}_2}^\dagger |0\rangle$, 两个4动量 \vec{k}_1, \vec{k}_2 的粒子态.

:

$b_{\vec{k}}^\dagger |0\rangle$ 一个4动量 \vec{k} 的反粒子态

$b_{\vec{k}_1}^\dagger b_{\vec{k}_2}^\dagger |0\rangle$ 两个4动量 \vec{k}_1, \vec{k}_2 的反粒子态.

3) 复标量场真空能量为正. 负能量问题消除, $E = \sum_{\vec{k}} w(n_{a\vec{k}} + n_{b\vec{k}} + 1)$

4) 正交归一性. 记 $|\vec{k}\rangle = a_{\vec{k}}^\dagger |0\rangle$, $\langle \vec{k}' | \vec{k} \rangle = \delta_{\vec{k}', \vec{k}}$.

5). 完备性.

$$\sum_{\vec{k}} |\vec{k}\rangle \langle \vec{k}| = 1, \quad \cdots$$