

§4.3 复标量场量子化

一. 经典场

实标量场, 由于是实数性的 $\phi^* = \phi$, 因此在量子化以后只能描述不带电荷的中性介子, 比如 π^0 介子。为了描述带电荷的介子, 比如 $\pi^+ \pi^-$ 介子, 必须引入复标量场。此时, $\phi^* \neq \phi$, 从而在量子化以后能描述带电荷的正粒子和反粒子, 比如 π^+, π^- 。

1. 经典复标量场.

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi, \quad \mathcal{L}(\phi, \phi^*)$$

$$\text{拉氏方程: } (\partial_\mu \partial^\mu + m^2) \phi(x) = 0, \quad (\partial_\mu \partial^\mu + m^2) \phi^*(x) = 0.$$

2. 正则量子化

引入正则动量

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}} (\partial_0 \phi^* \dot{\phi} - \nabla \phi^* \cdot \nabla \phi) = \dot{\phi}^*(x)$$

$$\pi^*(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \frac{\partial}{\partial \dot{\phi}^*} (\partial_0 \phi^* \dot{\phi} - \nabla \phi^* \cdot \nabla \phi) = \dot{\phi}(x)$$

1) 哈密顿量

$$H \equiv \int (\pi \dot{\phi} + \pi^* \dot{\phi}^* - \mathcal{L}) d^3x = \int [\dot{\phi}^* \dot{\phi} + \dot{\phi} \dot{\phi}^* - \dot{\phi}^* \dot{\phi} + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi] d^3x$$

$$= \int [\pi \pi^* + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi] d^3x$$

$$\therefore H = \int d^3x \mathcal{H}$$

$$\mathcal{H} = \pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi$$

假定正则对易关系.

$$[\phi(\vec{x}, t), \pi(\vec{x}', t)] = i \delta^3(\vec{x} - \vec{x}')$$

$$[\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0.$$

$$[\phi^*(\vec{x}, t), \pi^*(\vec{x}', t)] = i \delta^3(\vec{x} - \vec{x}')$$

$$[\phi^*(\vec{x}, t), \phi^*(\vec{x}', t)] = [\pi^*(\vec{x}, t), \pi^*(\vec{x}', t)] = 0$$

$$[\phi(\vec{x}, t), \phi^*(\vec{x}', t)] = [\phi(\vec{x}, t), \pi^*(\vec{x}', t)] = [\pi(\vec{x}, t), \pi^*(\vec{x}', t)] = [\phi^*(\vec{x}, t), \pi(\vec{x}', t)] = 0$$

则有海森堡方程

$$[H, \phi(\vec{x}, t)] = \int d^3x' [\pi^*(\vec{x}', t) \pi(\vec{x}, t) + \nabla \phi^*(\vec{x}', t) \cdot \nabla \phi(\vec{x}, t) + m^2 \phi^*(\vec{x}', t) \phi(\vec{x}, t)]$$

$$= \int d^3x' \{ [\pi^*(\vec{x}', t), \phi(\vec{x}, t)] \pi(\vec{x}, t) + \pi^*(\vec{x}', t) [\pi(\vec{x}', t), \phi(\vec{x}, t)] \}$$

$$= \int d^3x' \pi^*(\vec{x}', t) (-i \delta^3(\vec{x} - \vec{x}')) = -i \pi^*(\vec{x}, t) = -i \dot{\phi}(x)$$

同理,海森堡方程为

$$\begin{cases} \dot{\phi}(\vec{x}, t) = i[H, \phi(\vec{x}, t)] \\ \dot{\pi}(\vec{x}, t) = i[H, \pi(\vec{x}, t)] \end{cases}$$

$$\begin{cases} \dot{\phi}^*(\vec{x}, t) = i[H, \phi^*(\vec{x}, t)] \\ \dot{\pi}^*(\vec{x}, t) = i[H, \pi^*(\vec{x}, t)] \end{cases}$$

2) 动量算符

$$\vec{P} = \int d^3x \vec{T} = \int d^3x (-\partial^0 \mathcal{L} + \partial^0 \phi^* \partial^i \phi + \partial^i \phi^* \partial^0 \phi) = -\int d^3x (\pi \nabla \phi + \pi^* \nabla \phi^*)$$

$$\begin{aligned} R) [\vec{P}, \phi(\vec{x}, t)] &= \int d^3x' [\pi(\vec{x}', t) \nabla' \phi(\vec{x}, t) + \pi^*(\vec{x}', t) \nabla' \phi^*(\vec{x}, t), \phi(\vec{x}, t)] \\ &= \int d^3x' \{ [\pi(\vec{x}', t), \phi(\vec{x}, t)] \nabla' \phi(\vec{x}', t) + [\pi^*(\vec{x}', t), \phi(\vec{x}, t)] \nabla' \phi^*(\vec{x}', t) \} \\ &= \int d^3x' i \delta(\vec{x} - \vec{x}') \nabla' \phi(\vec{x}', t) = i \nabla \phi(\vec{x}, t) \end{aligned}$$

$$\text{同理上} \begin{cases} \nabla \phi(\vec{x}, t) = i[-\vec{P}, \phi(\vec{x}, t)] \\ \nabla \pi(\vec{x}, t) = i[-\vec{P}, \pi(\vec{x}, t)] \end{cases}, \begin{cases} \nabla \phi^*(\vec{x}, t) = i[-\vec{P}, \phi^*(\vec{x}, t)] \\ \nabla \pi^*(\vec{x}, t) = i[-\vec{P}, \pi^*(\vec{x}, t)] \end{cases}$$

$$\text{推广的海森堡方程} \begin{cases} \frac{\partial}{\partial x^\mu} \phi(\vec{x}, t) = i[P_\mu, \phi(\vec{x}, t)] \\ \frac{\partial}{\partial x^\mu} \pi(\vec{x}, t) = i[P_\mu, \pi(\vec{x}, t)] \end{cases}, \begin{cases} \frac{\partial}{\partial x^\mu} \phi^*(\vec{x}, t) = i[P_\mu, \phi^*(\vec{x}, t)] \\ \frac{\partial}{\partial x^\mu} \pi^*(\vec{x}, t) = i[P_\mu, \pi^*(\vec{x}, t)] \end{cases}$$

从而解为(复标量场在时空平移下的变化性质)

$$\phi(x) = e^{iP \cdot x} \phi(0) e^{-iP \cdot x}, \quad \phi^*(x) = e^{iP \cdot x} \phi^*(0) e^{-iP \cdot x}$$

3) 守恒荷 在物理中存在许多守恒律,如电荷守恒,重子数守恒,轻子数守恒,奇异数守恒,同位旋守恒。但却找不到与之相应的局域对称性。为解决这一问题,人们引入“内禀空间”。

复标量场满足规范变换(1)规范变换,即在坐标不变,仅改变复标量场中(x)

时, \mathcal{L} 保持不变:

$$\phi \rightarrow \phi'(x) = e^{i\alpha} \phi(x), \quad \delta\phi = i\delta\alpha \phi$$

$$\phi^* \rightarrow \phi'^*(x) = e^{-i\alpha} \phi^*(x), \quad \delta\phi^* = -i\delta\alpha \phi^*$$

并认为是内部空间的对称性导致了上面的守恒律。

利用作用量变分原理 $\delta S = 0$, 得 $\partial_\mu j^\mu = 0$, 从而引入流密度。

$$\begin{aligned} \text{守恒流} \quad j^\mu &= -i \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \phi(x) - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} \phi^*(x) \right) \\ &= -i (\partial^\mu \phi^* \phi - \partial^\mu \phi \phi^*) \equiv i \phi^* \overleftrightarrow{\partial}^\mu \phi \quad (\text{其中 } \overleftrightarrow{\partial}^\mu = \overrightarrow{\partial}^\mu - \overleftarrow{\partial}^\mu) \end{aligned}$$

$$\begin{aligned} \text{守恒荷} \quad Q &\equiv \int j^0 d^3x = -i \int (\partial^0 \phi^* \phi - \partial^0 \phi \phi^*) d^3x = -i \int (\dot{\phi}^* \phi - \dot{\phi} \phi^*) d^3x \\ &= -i \int (\pi \phi - \pi^* \phi^*) d^3x \end{aligned}$$

注: ① 对不同的场, j^μ 和 Q 的表达式与含义不同。

② 对复标量场, j^0 相当于电荷密度。

在量子化后, j^0 中包含了正电荷粒子和负电荷粒子。

若复标量场, 存在守恒荷如电荷, 奇异数, 轻子数, 底数, 这些守恒量是相加性守恒量。 $\pi^\pm, K^\pm, \bar{K}^\pm$ 若实标量场 $\phi^* = \phi$, 或电磁场 $A^\mu = A_\mu$, 则 $j^\mu = 0, Q = 0$ 表明不存在标量场的守恒流, 守恒荷和电磁场, 从而这些粒子电中性, 如 π, η, γ 。

二. 傅立叶展开 (朱洪元 P32)

利用空间 V 内的系统中, 复标量场 $\phi(\vec{x})$, $\phi^*(\vec{x})$ 满足的周期性边界条件, 非厄米性和 K-G 方程, 求得复标量场的傅立叶展开形式的解。

1. 在周期性边界条件下, 得满足正交归一、完备性的本征函数为

$$\left\{ e^{i\vec{k}_n \cdot \vec{x}} \mid \vec{k}_n = \vec{n} \frac{2\pi}{L}, \vec{n} = 0, \pm 1, \pm 2, \dots \right\}$$

2. 复标量场满足周期边界, 可傅立叶展开如下。

$$\begin{cases} \phi(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{i\vec{k} \cdot \vec{x}} q_{\vec{k}}(t) \\ \pi(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{-i\vec{k} \cdot \vec{x}} p_{\vec{k}}(t) \end{cases}$$

$$\text{其中} \begin{cases} q_{\vec{k}}(t) = \frac{1}{\sqrt{V}} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \phi(\vec{x}, t) \\ p_{\vec{k}}(t) = \frac{1}{\sqrt{V}} \int d^3x e^{i\vec{k} \cdot \vec{x}} \pi(\vec{x}, t) \end{cases}$$

$$\begin{cases} \phi^*(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{-i\vec{k} \cdot \vec{x}} q_{\vec{k}}^*(t) \\ \pi^*(\vec{x}, t) = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{i\vec{k} \cdot \vec{x}} p_{\vec{k}}^*(t) \end{cases}$$

$$\text{其中} \begin{cases} q_{\vec{k}}^*(t) = \frac{1}{\sqrt{V}} \int d^3x e^{i\vec{k} \cdot \vec{x}} \phi^*(\vec{x}, t) \\ p_{\vec{k}}^*(t) = \frac{1}{\sqrt{V}} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \pi^*(\vec{x}, t) \end{cases}$$

经过傅立叶展开, 场量成为动量空间函数, 独立变量为 $\{q_{\vec{k}}(t), p_{\vec{k}}(t), q_{\vec{k}}^*(t), p_{\vec{k}}^*(t)\}$

① 动量空间的产生湮灭算符表示。

$$\text{定义} \begin{cases} a_{\vec{k}}(t) = \sqrt{\frac{\omega}{2}} (q_{\vec{k}}(t) + \frac{i}{\omega} p_{\vec{k}}^*(t)) \\ b_{-\vec{k}}^{\dagger}(t) = \sqrt{\frac{\omega}{2}} (q_{\vec{k}}(t) - \frac{i}{\omega} p_{\vec{k}}^*(t)) \end{cases}$$

$$\begin{cases} a_{\vec{k}}^{\dagger}(t) = \sqrt{\frac{\omega}{2}} (q_{\vec{k}}^*(t) - \frac{i}{\omega} p_{\vec{k}}(t)) \\ b_{-\vec{k}}(t) = \sqrt{\frac{\omega}{2}} (q_{\vec{k}}^*(t) + \frac{i}{\omega} p_{\vec{k}}(t)) \end{cases}$$

$$\text{反变换} \begin{cases} q_{\vec{k}}(t) = \frac{1}{\sqrt{2\omega}} (a_{\vec{k}}(t) + b_{-\vec{k}}^{\dagger}(t)) \\ p_{\vec{k}}(t) = \frac{-i\omega}{\sqrt{2\omega}} (a_{\vec{k}}(t) - b_{-\vec{k}}^{\dagger}(t)) \end{cases}$$

$$\begin{cases} q_{\vec{k}}^*(t) = \frac{1}{\sqrt{2\omega}} (a_{\vec{k}}^{\dagger}(t) + b_{-\vec{k}}(t)) \\ p_{\vec{k}}^*(t) = \frac{+i\omega}{\sqrt{2\omega}} (a_{\vec{k}}^{\dagger}(t) - b_{-\vec{k}}(t)) \end{cases}$$

代入场展开式 (为推广, 将 $\vec{k} \rightarrow +$)

$$\begin{cases} \phi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2\omega V}} (a_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}} + b_{-\vec{k}}^{\dagger}(t) e^{-i\vec{k} \cdot \vec{x}}) \\ \pi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{+i\omega}{\sqrt{2\omega V}} (a_{\vec{k}}^{\dagger}(t) e^{-i\vec{k} \cdot \vec{x}} - b_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}}) \end{cases}$$

$$\begin{cases} \phi^*(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2\omega V}} (a_{\vec{k}}^{\dagger}(t) e^{-i\vec{k} \cdot \vec{x}} + b_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}}) \\ \pi^*(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{-i\omega}{\sqrt{2\omega V}} (a_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}} - b_{-\vec{k}}^{\dagger}(t) e^{-i\vec{k} \cdot \vec{x}}) \end{cases}$$

② ϕ, ϕ^* 满足 K-G 方程的场量解。

$$(\square + m^2)\phi(x) = 0, \quad (\square + m^2)\phi^*(x) = 0, \quad \text{引入 } \omega = \sqrt{\vec{k}^2 + m^2}$$

$$\text{得 } (\omega^2 + \partial_t^2)g_{\vec{k}}(t) = 0, \quad (\omega^2 + \partial_t^2)g_{\vec{k}}^*(t) = 0$$

$$\text{解为 } g_{\vec{k}}(t), \quad g_{\vec{k}}^*(t) \sim e^{i\omega t}, \quad e^{-i\omega t}$$

$$\text{结合场算符的产生湮灭表示, 得 } \begin{cases} a_{\vec{k}}(t) = a_{\vec{k}} e^{-i\omega t} \\ b_{+\vec{k}}^+(t) = b_{\vec{k}}^+ e^{i\omega t} \end{cases} \quad \begin{cases} a_{\vec{k}}^+(t) = a_{\vec{k}}^+ e^{i\omega t} \\ b_{\vec{k}}(t) = b_{\vec{k}} e^{-i\omega t} \end{cases}$$

$$\therefore \begin{cases} \phi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2\omega V}} (a_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + b_{\vec{k}}^+ e^{i\vec{k}\cdot\vec{x}}) \\ \pi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{i\omega}{\sqrt{2\omega V}} (a_{\vec{k}}^+ e^{i\vec{k}\cdot\vec{x}} - b_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}) \\ \phi^*(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2\omega V}} (a_{\vec{k}}^+ e^{i\vec{k}\cdot\vec{x}} + b_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}) \\ \pi^*(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{-i\omega}{\sqrt{2\omega V}} (a_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} - b_{\vec{k}}^+ e^{i\vec{k}\cdot\vec{x}}) \end{cases}$$

$$\text{其中 } kx \equiv \omega t - \vec{k} \cdot \vec{x}$$

$$\vec{k} = \vec{k}_n = \vec{n} \frac{2\pi}{L}, \quad \text{动量分立}$$

$$\vec{n} = 0, \pm 1, \pm 2, \dots$$

复共轭

当 $L \rightarrow \infty, V \rightarrow \infty$ 时, 动量由分立 \rightarrow 连续, $\sum_{\vec{k}} \rightarrow \int \frac{V}{(2\pi)^3} d^3k$, 则

$$\text{引入 } a(\vec{k}) = \sqrt{2\omega V} a_{\vec{k}}, \quad b(\vec{k}) = \sqrt{2\omega V} b_{\vec{k}}$$

$$\begin{cases} \phi(\vec{x}, t) = \int \frac{d^3k}{2\omega(2\pi)^3} (a(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} + b^+(\vec{k}) e^{i\vec{k}\cdot\vec{x}}) \\ \pi(\vec{x}, t) = \int \frac{d^3k (i\omega)}{2\omega(2\pi)^3} (a^+(\vec{k}) e^{i\vec{k}\cdot\vec{x}} - b(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}) \\ \phi^*(\vec{x}, t) = \int \frac{d^3k}{2\omega(2\pi)^3} (a^+(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + b(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}) \\ \pi^*(\vec{x}, t) = \int \frac{d^3k (-i\omega)}{2\omega(2\pi)^3} (a(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} - b^+(\vec{k}) e^{i\vec{k}\cdot\vec{x}}) \end{cases}$$

$$\text{其中 } kx = \omega t - \vec{k} \cdot \vec{x}$$

$$\vec{k} = -\infty, \infty, \text{ 动量连续}$$

黄涛

③ 下面, 确定场量 $\{\phi, \pi, \phi^*, \pi^*\}$ 中不含时的振幅算子 $\{a_{\vec{k}}, b_{\vec{k}}^+, a_{\vec{k}}^+, b_{\vec{k}}\}$.

$$\begin{aligned} a_{\vec{k}}(t) &= a_{\vec{k}} e^{-i\omega t} = \sqrt{\frac{\omega}{2}} \left(q_{\vec{k}}(t) + \frac{i}{\omega} p_{\vec{k}}^*(t) \right) = \sqrt{\frac{\omega}{2}} \frac{1}{\sqrt{V}} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \left[\phi(\vec{x}, t) + \frac{i}{\omega} \pi^*(\vec{x}, t) \right] \\ &= \frac{1}{\sqrt{2\omega V}} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \left[\omega \phi(\vec{x}, t) + i\pi^*(\vec{x}, t) \right] \end{aligned}$$

$$\therefore a_{\vec{k}} = \frac{1}{\sqrt{2\omega V}} \int d^3x e^{i\vec{k} \cdot \vec{x}} \left[\omega \phi(\vec{x}, t) + i\pi^*(\vec{x}, t) \right]$$

$$\begin{aligned} b_{\vec{k}}^+(t) &= b_{\vec{k}}^+ e^{i\omega t} = \sqrt{\frac{\omega}{2}} \left(q_{-\vec{k}}(t) - \frac{i}{\omega} p_{-\vec{k}}^*(t) \right) = \sqrt{\frac{\omega}{2}} \frac{1}{\sqrt{V}} \int d^3x e^{i\vec{k} \cdot \vec{x}} \left[\phi(\vec{x}, t) - \frac{i}{\omega} \pi^*(\vec{x}, t) \right] \\ &= \frac{1}{\sqrt{2\omega V}} \int d^3x e^{i\vec{k} \cdot \vec{x}} \left[\omega \phi(\vec{x}, t) - i\pi^*(\vec{x}, t) \right] \end{aligned}$$

$$\therefore b_{\vec{k}}^+ = \frac{1}{\sqrt{2\omega V}} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \left[\omega \phi(\vec{x}, t) - i\pi^*(\vec{x}, t) \right]$$

$$\therefore \begin{cases} a_{\vec{k}} = \frac{1}{\sqrt{2\omega V}} \int d^3x e^{i\vec{k} \cdot \vec{x}} \left[\omega \phi(\vec{x}, t) + i\pi^*(\vec{x}, t) \right] & \vec{k} \text{ 分立} \\ b_{\vec{k}}^+ = \frac{1}{\sqrt{2\omega V}} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \left[\omega \phi(\vec{x}, t) - i\pi^*(\vec{x}, t) \right] \\ a_{\vec{k}}^+ = \frac{1}{\sqrt{2\omega V}} \int d^3x e^{-i\vec{k} \cdot \vec{x}} \left[\omega \phi^*(\vec{x}, t) - i\pi(\vec{x}, t) \right] \\ b_{\vec{k}} = \frac{1}{\sqrt{2\omega V}} \int d^3x e^{i\vec{k} \cdot \vec{x}} \left[\omega \phi^*(\vec{x}, t) + i\pi(\vec{x}, t) \right] & \text{姜志进} \end{cases}$$

当 $L \rightarrow \infty$, 从而 $V \rightarrow \infty$ 时

$$\begin{cases} a(\vec{k}) = \sqrt{2\omega V} a_{\vec{k}} = \int d^3x e^{i\vec{k} \cdot \vec{x}} \left[\omega \phi(\vec{x}, t) + i\pi^*(\vec{x}, t) \right] & \vec{k} \text{ 连续} \\ b^+(\vec{k}) = \sqrt{2\omega V} b_{\vec{k}}^+ = \int d^3x e^{-i\vec{k} \cdot \vec{x}} \left[\omega \phi(\vec{x}, t) - i\pi^*(\vec{x}, t) \right] \end{cases}$$

$$\begin{cases} a^+(\vec{k}) = \sqrt{2\omega V} a_{\vec{k}}^+ = \int d^3x e^{-i\vec{k} \cdot \vec{x}} \left[\omega \phi^*(\vec{x}, t) - i\pi(\vec{x}, t) \right] \\ b(\vec{k}) = \sqrt{2\omega V} b_{\vec{k}} = \int d^3x e^{i\vec{k} \cdot \vec{x}} \left[\omega \phi^*(\vec{x}, t) + i\pi(\vec{x}, t) \right] & \text{黄涛} \end{cases}$$

三. 动量空间的(场)算符.

$$\begin{cases} \phi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2\omega V}} (a_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + b_{\vec{k}}^{\dagger} e^{i\vec{k}\cdot\vec{x}}) \\ \pi(\vec{x}, t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{i\omega}{\sqrt{2\omega V}} (a_{\vec{k}}^{\dagger} e^{i\vec{k}\cdot\vec{x}} - b_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}) \end{cases}, \quad T$$

其中, $\begin{cases} a_{\vec{k}} = \frac{1}{\sqrt{2\omega V}} \int d^3x e^{i\vec{k}\cdot\vec{x}} [\omega\phi(\vec{x}, t) + i\pi^*(\vec{x}, t)] \\ b_{\vec{k}}^{\dagger} = \frac{1}{\sqrt{2\omega V}} \int d^3x e^{-i\vec{k}\cdot\vec{x}} [\omega\phi(\vec{x}, t) - i\pi^*(\vec{x}, t)] \end{cases}$

也可以量子化.

由于 $\phi(\vec{x}, t), \pi(\vec{x}, t), \phi^*(\vec{x}, t), \pi^*(\vec{x}, t)$ 可以量子化, 故相应的 $a_{\vec{k}}, b_{\vec{k}}, a_{\vec{k}}^{\dagger}, b_{\vec{k}}^{\dagger}, H, \vec{P}$ 也可以量子化.

可证: $[a_{\vec{k}}, a_{\vec{k}'}^{\dagger}] = \delta_{\vec{k}\vec{k}'}$

\vec{k} 分立 $\begin{cases} [b_{\vec{k}}, b_{\vec{k}'}^{\dagger}] = \delta_{\vec{k}\vec{k}'} \text{ 其它对易子为0.} \end{cases}$

$\frac{1}{V}$ 对易子 $\begin{cases} H = \sum_{\vec{k}} \omega [a_{\vec{k}}^{\dagger} a_{\vec{k}} + b_{\vec{k}}^{\dagger} b_{\vec{k}} + 1] \end{cases}$

$\vec{P} = \sum_{\vec{k}} \vec{k} [a_{\vec{k}}^{\dagger} a_{\vec{k}} + b_{\vec{k}}^{\dagger} b_{\vec{k}}]$

$Q = \sum_{\vec{k}} [a_{\vec{k}}^{\dagger} a_{\vec{k}} - b_{\vec{k}}^{\dagger} b_{\vec{k}}]$ 正反粒子 电荷相反

\vec{k} 连续 $\begin{cases} [a(\vec{k}), a(\vec{k}')^{\dagger}] = (2\pi)^3 \omega_k \delta^3(\vec{k}-\vec{k}') \end{cases}$

$\begin{cases} [b(\vec{k}), b(\vec{k}')^{\dagger}] = (2\pi)^3 \omega_k \delta^3(\vec{k}-\vec{k}') \end{cases}$ 其它对易子为0.

$H = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2\omega_k} [a(\vec{k})^{\dagger} a(\vec{k}) + b(\vec{k})^{\dagger} b(\vec{k})]$

$\vec{P} = \int \frac{d^3k}{(2\pi)^3} \frac{\vec{k}}{2\omega_k} [a(\vec{k})^{\dagger} a(\vec{k}) + b(\vec{k})^{\dagger} b(\vec{k})]$

$Q = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} [a(\vec{k})^{\dagger} a(\vec{k}) - b(\vec{k})^{\dagger} b(\vec{k})]$

证明: 由实空间 $U(1)$ 变换为守恒荷

$Q = -i \int (\pi\phi - \pi^*\phi^*) d^3x$

$= -i \sum_{\vec{k}, \vec{k}'} \frac{1}{V} \int d^3x \left[\frac{i\omega}{\sqrt{2\omega V}} (a_{\vec{k}}^{\dagger} e^{i\vec{k}\cdot\vec{x}} - b_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}) (a_{\vec{k}'} e^{-i\vec{k}'\cdot\vec{x}} + b_{\vec{k}'}^{\dagger} e^{i\vec{k}'\cdot\vec{x}}) - \frac{i\omega}{\sqrt{2\omega V}} (a_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} - b_{\vec{k}}^{\dagger} e^{i\vec{k}\cdot\vec{x}}) (a_{\vec{k}'}^{\dagger} e^{i\vec{k}'\cdot\vec{x}} + b_{\vec{k}'} e^{-i\vec{k}'\cdot\vec{x}}) \right]$

对 x 积分, 利用 $\int d^3x e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = V \delta_{\vec{k}\vec{k}'}$ 对 \vec{k}' 求和.

正负归一

$\begin{cases} \sum_{\vec{k}'} \int d^3x e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = \sum_{\vec{k}'} \int d^3x e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} e^{i(\omega-\omega')t} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} = \sum_{\vec{k}'} V e^{\frac{i(\omega-\omega')t}{2}} \delta_{\vec{k}\vec{k}'} = V \end{cases}$

$\begin{cases} \sum_{\vec{k}'} \int d^3x e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} = \sum_{\vec{k}'} \int d^3x e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} e^{i(\omega+\omega')t} e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}} = \sum_{\vec{k}'} V e^{\frac{i(\omega+\omega')t}{2}} \delta_{\vec{k}, -\vec{k}'} = V e^{i\omega t} \end{cases}$

$2V\sqrt{\omega\omega'}$

$= \sum_{\vec{k}, \vec{k}'} \int d^3x \frac{\omega}{2\omega V} [a_{\vec{k}}^{\dagger} a_{\vec{k}'} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} - b_{\vec{k}}^{\dagger} b_{\vec{k}'} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} + a_{\vec{k}} a_{\vec{k}'}^{\dagger} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} - b_{\vec{k}} b_{\vec{k}'}^{\dagger} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}]$

$= \sum_{\vec{k}} \frac{1}{2} [a_{\vec{k}}^{\dagger} a_{\vec{k}} - b_{\vec{k}} b_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^{\dagger} - b_{\vec{k}} b_{\vec{k}}^{\dagger}] + a_{\vec{k}}^{\dagger} b_{-\vec{k}}^{\dagger} e^{2i\omega t} - b_{\vec{k}} a_{-\vec{k}} e^{-2i\omega t} + a_{\vec{k}} b_{-\vec{k}} e^{-2i\omega t} - b_{\vec{k}}^{\dagger} a_{-\vec{k}}^{\dagger} e^{2i\omega t}$

$= \sum_{\vec{k}} [a_{\vec{k}}^{\dagger} a_{\vec{k}} - b_{\vec{k}}^{\dagger} b_{\vec{k}}]$

$[a_{\vec{k}}, b_{-\vec{k}}] = [a_{\vec{k}}, b_{-\vec{k}}^{\dagger}] = 0$ 对易关系

$\vec{k} \leftrightarrow -\vec{k}$

四. 粒子数表象.

引入 $N_a = a_{\vec{k}}^{\dagger} a_{\vec{k}}$, $N_b = b_{\vec{k}}^{\dagger} b_{\vec{k}}$.

$$N = \sum_{\vec{k}} a_{\vec{k}}^{\dagger} a_{\vec{k}} \quad \bar{N} = \sum_{\vec{k}} b_{\vec{k}}^{\dagger} b_{\vec{k}}$$

取 N_a 对角化表象 $|n_{a\vec{k}}\rangle$, N_b 对角化表象 $|n_{b\vec{k}}\rangle$, 则

N 对角化表象 $|n_{a\vec{k}_1}, n_{a\vec{k}_2}, \dots, n_{a\vec{k}_m}, \dots\rangle$, \bar{N} 对角化表象 $|n_{b\vec{k}_1}, n_{b\vec{k}_2}, \dots, n_{b\vec{k}_m}, \dots\rangle$.

$$\therefore N_a |n_{a\vec{k}}\rangle = n_{a\vec{k}} |n_{a\vec{k}}\rangle, \quad \bar{N}_b |n_{b\vec{k}}\rangle = n_{b\vec{k}} |n_{b\vec{k}}\rangle$$

即 $|n_{a\vec{k}}\rangle$ 是 N_a 的本征值为 $n_{a\vec{k}}$ 的本征态, $|n_{b\vec{k}}\rangle$ 是 N_b 的本征值为 $n_{b\vec{k}}$ 的本征态.

而 $N |n_{a\vec{k}_1}, n_{a\vec{k}_2}, \dots\rangle = N |n_{a\vec{k}_1}, n_{a\vec{k}_2}, \dots\rangle = \sum_{\vec{k}} n_{a\vec{k}} |n_{a\vec{k}_1}, n_{a\vec{k}_2}, \dots\rangle$

$$\bar{N} |n_{b\vec{k}_1}, n_{b\vec{k}_2}, \dots\rangle = \bar{N} |n_{b\vec{k}_1}, n_{b\vec{k}_2}, \dots\rangle = \sum_{\vec{k}} n_{b\vec{k}} |n_{b\vec{k}_1}, n_{b\vec{k}_2}, \dots\rangle$$

1) 真空态: $a_{\vec{k}} |0\rangle = 0, b_{\vec{k}} |0\rangle = 0, \langle 0|0\rangle = 1, (\vec{k} \text{ 取所有分立动量})$

2) 本征态.

$a_{\vec{k}}^{\dagger} |0\rangle$ 一个4动量 \vec{k} 的 ~~单~~ 粒子态.

$a_{\vec{k}_1}^{\dagger} a_{\vec{k}_2}^{\dagger} |0\rangle$, 两个4动量 \vec{k}_1, \vec{k}_2 的 ~~双~~ 粒子态.

\vdots

$b_{\vec{k}}^{\dagger} |0\rangle$ 一个4动量 \vec{k} 的反粒子态

$b_{\vec{k}_1}^{\dagger} b_{\vec{k}_2}^{\dagger} |0\rangle$ 两个4动量 \vec{k}_1, \vec{k}_2 的反粒子态.

3). 复标量场真空能量为正, 负能量问题消除, $E = \sum_{\vec{k}} \omega (n_{a\vec{k}} + n_{b\vec{k}} + \frac{1}{2}) \geq 0$

4) 正交归一性. $i|\vec{k}\rangle = a_{\vec{k}}^{\dagger} |0\rangle, \langle \vec{k}' | \vec{k} \rangle = \delta_{\vec{k}', \vec{k}}$

5). 完备性.

$$\sum_{\vec{k}} |\vec{k}\rangle \langle \vec{k}| = 1, \dots$$