$H\psi = i\frac{\partial}{\partial t} e^{-iEt+i\vec{p}\cdot\vec{x}} u(p) = E e^{-iEt+i\vec{p}\cdot\vec{x}} u(p)$

三. 自由Dirac场的解

1. 平面波解:

Dirac方程的场满足K-G方程，故解 $\psi(x) \sim e^{\pm ipx}$.

$$\begin{cases} \psi^{(+)}(x) = e^{-ipx} u(p) & (E=\sqrt{\vec{p}^2+m^2},\ \vec{p}) \quad \text{正能解} \\ \psi^{(-)}(x) = e^{+ipx} v(p) & (E=-\sqrt{\vec{p}^2+m^2},\ +\vec{p}) \quad \text{负能解} \end{cases}$$

利用 $\not{a}\not{a} = a^\mu a^\nu \gamma_\mu \gamma_\nu = a^\mu a^\nu \frac{1}{2}\{\gamma_\mu, \gamma_\nu\} = a^\mu a^\nu g_{\mu\nu} = a^\mu a_\mu = a^2$

$$(i\not{\partial}+m)(i\not{\partial}-m)\psi(x) = (-\partial^\mu\partial_\mu - m^2)\psi(x) = 0.$$

$$\therefore (\Box + m^2)\psi(x) = 0$$

将 $\psi^\pm(x)$ 代入, $i\not{\partial} e^{\mp ipx} = \pm \not{p} e^{\mp ipx}$, $\Box e^{\mp ipx} = -p^2 e^{\mp ipx}$

$i\gamma_\mu \partial^\mu e^{-ipx} = i\gamma_\mu(-ip^\mu) e^{-ipx} = \not{p} e^{-ipx}$

$\therefore p^2 = m^2$,

又 $p^\mu = (p^0, \vec{p})$, $p^2 = p^{02} - \vec{p}^2 = m^2$,

$\therefore p^{02} = \vec{p}^2 + m^2$, 或 $p^0 = E = \pm\sqrt{\vec{p}^2+m^2} \equiv \pm E_p$

将正、负能解代入Dirac方程，得旋量部分 $u(p)$, $v(p)$ 满足的动量空间方程

$$\begin{cases} (\not{p}-m)u(p) = 0 \\ (\not{p}+m)v(p) = 0 \end{cases}$$

2. $u(p)$, $v(p)$ 的解 (旋量场本征方程求解法).

将 $\psi^{(\pm)}$ 分别代入Dirac方程， $px = Et - \vec{p}\cdot\vec{x}$, $i\frac{\partial}{\partial t} e^{-ipx} = E e^{-ipx}$, $i\vec{\alpha}\cdot\nabla e^{-ipx} = -\vec{\alpha}\cdot\vec{p} e^{-ipx}$

$(i\frac{\partial}{\partial t} + i\vec{\alpha}\cdot\nabla - \beta m) e^{-ipx} u(p) = (E - \vec{\alpha}\cdot\vec{p} - \beta m) u(p) e^{-ipx} = 0$

$(i\frac{\partial}{\partial t} + i\vec{\alpha}\cdot\nabla - \beta m) e^{ipx} v(p) = (-E + \vec{\alpha}\cdot\vec{p} + \beta m) v(p) e^{+ipx} = 0$

形式相同!

由于 α, β 为 4×4 矩阵，故 $u(p)$, $v(p)$ 为 4×1 矩阵. 令其为 $\begin{pmatrix}\chi \\ \phi\end{pmatrix}$.

1) 正能解

① 对 $(E - \vec{\alpha}\cdot\vec{p} - \beta m) u(p) = 0$, 即.

$$\left[\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}E - \begin{pmatrix}0 & \vec{\sigma} \\ \vec{\sigma} & 0\end{pmatrix}\cdot\vec{p} - \begin{pmatrix}I & 0 \\ 0 & -I\end{pmatrix}m\right]\begin{pmatrix}\chi \\ \phi\end{pmatrix} = 0 \quad \text{或} \begin{pmatrix}E-m & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & E+m\end{pmatrix}\begin{pmatrix}\chi \\ \phi\end{pmatrix} = 0$$

$$\therefore \begin{cases} (E-m)\chi - \vec{\sigma}\cdot\vec{p}\,\phi = 0. & ※ \\ -\vec{\sigma}\cdot\vec{p}\,\chi + (E+m)\phi = 0. & ※※ \end{cases}$$

当正能时 $E = E_p = \sqrt{\vec{p}^2+m^2}$, 由※※ $\phi = \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi$, $u(p) = \begin{pmatrix}\chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi\end{pmatrix}$ 正能解

2) 负能解

② 对 $(E - \vec{\alpha}\cdot\vec{p} - \beta m) v(p) = 0$.

$$\therefore \begin{cases} (E-m)\chi - \vec{\sigma}\cdot\vec{p}\,\phi = 0 & ※ \\ -\vec{\sigma}\cdot\vec{p}\,\chi + (E+m)\phi = 0 & ※※ \end{cases}$$

当负能时 $E = -E_p = -\sqrt{\vec{p}^2+m^2}$, 由※, $\chi = \frac{-\vec{\sigma}\cdot\vec{p}}{E_p+m}\phi$, $v(p) = \begin{pmatrix}\frac{-\vec{\sigma}\cdot\vec{p}}{E_p+m}\phi \\ \phi\end{pmatrix}$ 负能解.

其中 χ, ϕ 为 2×1 矩阵，它们形式任意，不能由Dirac方程确定. 通常取为 $\vec{\sigma}\cdot\frac{\vec{p}}{|\vec{p}|}$ 之本征态.

3. $u(p)$, $v(p)$ 在 $\vec\sigma\cdot\frac{\vec p}{|\vec p|}$ 本征态上的解

1) $\vec\sigma\cdot\vec n$ 的本征态

~~2)~~ χ_λ 中若取为 $\vec\sigma\cdot\frac{\vec p}{|\vec p|}$ 的本征态

引入沿 $\vec p$ 方向的单位矢量 $\vec n=\frac{\vec p}{|\vec p|}=(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$,

由 $\vec\sigma\cdot\vec n=\sigma_1 n_1+\sigma_2 n_2+\sigma_3 n_3=\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\theta\cos\phi+\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\theta\sin\phi+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos\theta=\begin{pmatrix}\cos\theta&\sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi}&-\cos\theta\end{pmatrix}$

得 $\vec\sigma\cdot\vec n$ 的本征方程为 $\vec\sigma\cdot\vec n f=\varepsilon f$, 令 $f=\begin{pmatrix}a\\b\end{pmatrix}$, 则

$$\begin{pmatrix}\cos\theta&\sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi}&-\cos\theta\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0.$$

解得本征值 $\varepsilon_\lambda=\pm1$, 即 $\varepsilon_1=1$, $\varepsilon_2=-1$

当 $\varepsilon_1=1$ 时, $\begin{pmatrix}\cos\theta-1&\sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi}&-\cos\theta-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0$, 解得 $f_1=\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}$ 或 $\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}$

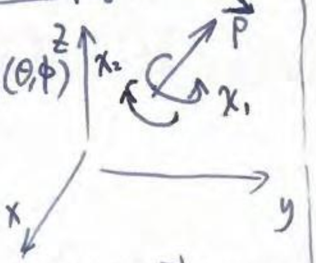
当 $\varepsilon_2=-1$ 时 $\begin{pmatrix}\cos\theta+1&\sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi}&-\cos\theta+1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0$, 解得 $f_2=\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi}\\ \cos\frac{\theta}{2}\end{pmatrix}$ 或 $\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}$

$f_\lambda^+ f_{\lambda'}=\delta_{\lambda\lambda'}$

2) ~~3)~~ 进一步, 考虑到旋量波函数的正交归一条件, 得旋量场的解为

① 正能解:

$$u(p,\lambda)\equiv u_\lambda(p)=\sqrt{\frac{E_p+m}{2m}}\begin{pmatrix}\chi_\lambda\\ \frac{\vec\sigma\cdot\vec p}{E_p+m}\chi_\lambda\end{pmatrix},\quad\begin{cases}\varepsilon_1=1\text{ 时},\ \chi_1(\theta,\phi)=\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}=\phi_1(\theta,\phi)\\ \varepsilon_2=-1\text{ 时},\ \chi_2(\theta,\phi)=\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}=\phi_2\end{cases}$$



~~正能解~~: 四动量为 $(E_p,\vec p)$。

电子能量 E_p, 电子动量为 $\vec p$, 自旋沿 $\vec p$ 方向。$\lambda=1$ 时, 自旋沿 $\vec p$ 方向, $\lambda=2$ 时, 自旋沿 $\vec p$ 反方向.

② 负能解:

$$v(p,\lambda)\equiv v_\lambda(p)=\sqrt{\frac{E_p+m}{2m}}\begin{pmatrix}\frac{+\vec\sigma\cdot\vec p}{E_p+m}\phi_\lambda\\ \phi_\lambda\end{pmatrix},\quad\begin{cases}\varepsilon_1=1\text{ 时},\ \phi_1=\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}=\chi_1\\ \varepsilon_2=-1\text{ 时},\ \phi_2=\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}=\chi_2\end{cases}$$

四动量 $(-E_p, \cancel{+\vec p}\ -\vec p)$

反电子能量 $-E_p$, 动量 $+\vec p$, 自旋沿 $\vec p$ 方向。$\lambda=1$ 时, 自旋沿 $\vec p$ 方向, $\lambda=2$ 时, 自旋沿 $\vec p$ 反方向

作业一

3) ~~4)~~ 归一化条件:

① $\bar u_\lambda(p)u_{\lambda'}(p)\equiv\bar u(p,\lambda)u(p,\lambda')=\delta_{\lambda\lambda'}$　　② $\bar u_\lambda(p)v_{\lambda'}(p)=0$

③ $\bar v_\lambda(p)v_{\lambda'}(p)\equiv\bar v(p,\lambda)v(p,\lambda')=-\delta_{\lambda\lambda'}$　　④ $\bar v_\lambda(p)u_{\lambda'}(p)=0$

证明: ① $\bar u_\lambda(p)u_{\lambda'}(p)=u_\lambda^+(p)\gamma_0 u_{\lambda'}(p)=\frac{E_p+m}{2m}\left(\chi_\lambda^+,\ \frac{\vec\sigma\cdot\vec p}{E_p+m}\chi_\lambda^+\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\chi_{\lambda'}\\ \frac{\vec\sigma\cdot\vec p}{E_p+m}\chi_{\lambda'}\end{pmatrix}$

$$=\frac{E_p+m}{2m}\left(\chi_\lambda^+,\ -\frac{\vec\sigma\cdot\vec p}{E_p+m}\chi_\lambda^+\right)\begin{pmatrix}\chi_{\lambda'}\\ \frac{\vec\sigma\cdot\vec p}{E_p+m}\chi_{\lambda'}\end{pmatrix}=\frac{E_p+m}{2m}\left(\chi_\lambda^+\chi_{\lambda'}-\chi_\lambda^+\left(\frac{\vec\sigma\cdot\vec p}{E_p+m}\right)^2\chi_{\lambda'}\right)$$

$$=\frac{E_p+m}{2m}\left(1-\frac{E_p-m}{E_p+m}\right)\delta_{\lambda\lambda'}=\delta_{\lambda\lambda'}$$

② $\bar u_\lambda(p)v_{\lambda'}(p)=\frac{E_p+m}{2m}\left(\chi_\lambda^+,\ -\frac{\vec\sigma\cdot\vec p}{E_p+m}\chi_\lambda^+\right)\begin{pmatrix}\frac{+\vec\sigma\cdot\vec p}{E_p+m}\chi_{\lambda'}\\ \chi_{\lambda'}\end{pmatrix}\cancel{=\frac{E_p+m}{2m}\left(\frac{\vec\sigma\cdot\vec p}{E_p+m}\chi_\lambda^+\chi_{\lambda'}-\frac{\vec\sigma\cdot\vec p}{E_p+m}\chi_\lambda^+\chi_{\lambda'}\right)}$

$$=\frac{E_p+m}{2m}\left(\chi_\lambda^+,\ \frac{-|\vec p|}{E_p+m}\varepsilon_\lambda\chi_\lambda^+\right)\begin{pmatrix}\frac{|\vec p|}{E_p+m}\varepsilon_{\lambda'}\chi_{\lambda'}\\ \chi_{\lambda'}\end{pmatrix}=\frac{E_p+m}{2m}\delta_{\lambda\lambda'}\frac{-|\vec p|}{E_p+m}(-\varepsilon_{\lambda'}+\varepsilon_\lambda)=0$$

37'

2) χ,ϕ 若取为 σ_3 的本征态。静止系中 $p=(E,\vec{p})=(E,\vec{0})$, $\not{p}=\gamma^\mu p_\mu=\gamma^0 E=\gamma^0 m$, $E=\sqrt{\vec{p}^2+m^2}=m$

① $(\not{p}-m)u(p)=0 \longrightarrow (\gamma^0-1)u(m,\vec{0})=0$, 其中 $\gamma^0=\beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\begin{pmatrix}1-1&0\\0&-1-1\end{pmatrix}\begin{pmatrix}\chi\\\phi\end{pmatrix}=0 \longrightarrow \begin{cases}\chi 任意\\ \phi=0\end{cases}$ 或 $u_\lambda(m,\vec{0})=\begin{pmatrix}\chi\\0\end{pmatrix}\sim\begin{pmatrix}\chi_\lambda\\0\end{pmatrix}$

χ,ϕ 取为 σ_3 本征态 $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

② $(\not{p}+m)v(p)=0 \longrightarrow (\gamma^0+1)v(-m,\vec{0})=0$

$\begin{pmatrix}1+1&0\\0&-1+1\end{pmatrix}\begin{pmatrix}\chi\\\phi\end{pmatrix}=0 \longrightarrow \begin{cases}\chi=0\\ \phi 任意\end{cases}$ 或 $v_\lambda(-m,\vec{0})=\begin{pmatrix}0\\\phi\end{pmatrix}\sim\begin{pmatrix}0\\\phi_\lambda\end{pmatrix}$

满足 $\begin{cases}\sigma^3\chi_\lambda=\varepsilon_\lambda\chi_\lambda, & \varepsilon_1=-\varepsilon_2=1.\\ \sigma^3\phi_\lambda=\varepsilon_\lambda\phi_\lambda, & \varepsilon_1=-\varepsilon_2=1\end{cases}$

则旋量场解为 $\begin{cases}\psi^{(+)}=e^{-ipx}u_\lambda(p)=e^{-ipx}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}_{\substack{\lambda=1\\ \varepsilon_1=1}}, \quad e^{-ipx}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}_{\substack{\lambda=2\\ \varepsilon_2=-1}}\\ \psi^{(-)}=e^{ipx}v_\lambda(p)=e^{ipx}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}_{\substack{\lambda=1\\ \varepsilon_1=1}}, \quad e^{ipx}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}_{\substack{\lambda=2\\ \varepsilon_2=-1}}\end{cases}$

3) χ,ϕ 若取为 $\vec{\sigma}\cdot\frac{\vec{p}}{|\vec{p}|}$ 的本征态.

引入 $\vec{n}=\frac{\vec{p}}{|\vec{p}|}=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\vec{\sigma}\cdot\vec{n}$ 的本征方程为

由 $\vec{\sigma}\cdot\vec{n}=\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}$, 得本征方程 $\begin{pmatrix}\cos\theta-\varepsilon & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta-\varepsilon\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0$, 本征值 $\varepsilon=\pm1$

当 $\varepsilon_1=1$ 时, $\begin{pmatrix}\cos\theta-1 & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0$, 解得 $\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}$ 或 $\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}$

当 $\varepsilon_2=-1$ 时 $\begin{pmatrix}\cos\theta+1 & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta+1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0$ 解得 $\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi}\\ \cos\frac{\theta}{2}\end{pmatrix}$ 或 $\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}$

满足 $\vec{\sigma}\cdot\frac{\vec{p}}{|\vec{p}|}\begin{pmatrix}a\\b\end{pmatrix}=\varepsilon_\lambda\begin{pmatrix}a\\b\end{pmatrix}$, $\varepsilon_1=-\varepsilon_2=+1$, $\varepsilon_1=1$ 时, $\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}$,

$\varepsilon_2=-1$ 时, $\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi}\\ \cos\frac{\theta}{2}\end{pmatrix}$

则旋量场解为

正能解 $u_\lambda(p)=\begin{pmatrix}\chi_\lambda\\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\lambda\end{pmatrix}$, $\lambda=1$ 时, $\varepsilon_1=1$, $\chi_1=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}$, $\lambda=2$ 时, $\varepsilon_2=-1$, $\chi_2=\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi}\\ \cos\frac{\theta}{2}\end{pmatrix}$

负能解 $v_\lambda(p)=\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\phi_\lambda\\ \phi_\lambda\end{pmatrix}$, $\lambda=1$ 时, $\varepsilon_1=1$, $\phi_1=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}$, $\lambda=2$ 时, $\varepsilon_2=-1$, $\phi_2=\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi}\\ \cos\frac{\theta}{2}\end{pmatrix}$

在静止系中σ_3本征态上的解

4) ~~3.~~ $u(p)$, $v(p)$ ~~的解（静止系σ_3本征态再经洛伦兹推动变换法）~~.

1). 静止系中，在σ_3本征态下求解.

$$p^\mu=(E_p,\vec{0}) \quad E_p=\sqrt{\vec{p}^2+m^2}=m, \quad \not{p}=\gamma^\mu p_\mu=g_{\mu\nu}\gamma^\mu p^\nu=\gamma^0p^0-\vec{\gamma}\cdot\vec{p}=\gamma^0 m$$

① $(\not{p}-m)u(p)=0 \longrightarrow (\gamma^0-1)u(m,\vec{0})=0$. 其中 $\gamma^0=\beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

即 $\begin{pmatrix}1-1 & 0\\ 0 & -1-1\end{pmatrix}\begin{pmatrix}\chi\\ \phi\end{pmatrix}=0 \longrightarrow u_\lambda(m,\vec{0})=\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix}$.

② $(\not{p}+m)v(p)=0 \longrightarrow (\gamma^0+1)v(m,\vec{0})=0$.

即 $\begin{pmatrix}1+1 & 0\\ 0 & -1+1\end{pmatrix}\begin{pmatrix}\chi\\ \phi\end{pmatrix}=0 \longrightarrow v_\lambda(m,\vec{0})=\begin{pmatrix}0\\ \phi_\lambda\end{pmatrix}$

③ χ,ϕ 选σ_3本征态，$\begin{cases}\sigma^3\chi_\lambda=\varepsilon_\lambda\chi_\lambda, & \varepsilon_1=-\varepsilon_2=1\\ \sigma^3\phi_\lambda=\varepsilon_\lambda\phi_\lambda, & \varepsilon_1=-\varepsilon_2=1.\end{cases}$

④ 则旋量场解为

1° 正能解：$u_\lambda(m,\vec{0})=\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix}$ $\begin{cases}\varepsilon_1=1\text{时}, & \chi_1=\begin{pmatrix}1\\0\end{pmatrix}\\ \varepsilon_2=-1\text{时}, & \chi_2=\begin{pmatrix}0\\1\end{pmatrix}\end{cases}$

电子能量m，静止，自旋沿z轴。$\lambda=1$时，自旋沿z轴正方向，$\lambda=2$时，自旋沿z轴反方向.

2° 负能解：$v_\lambda(m,\vec{0})=\begin{pmatrix}0\\ \phi_\lambda\end{pmatrix}$，$\begin{cases}\varepsilon_1=1\text{时}, & \phi_1=\begin{pmatrix}1\\0\end{pmatrix}\\ \varepsilon_2=-1\text{时}, & \phi_2=\begin{pmatrix}0\\1\end{pmatrix}\end{cases}$

反电子能量m，静止，自旋沿z轴，$\lambda=1$时，自旋沿z轴正方向，$\lambda=2$时，自旋沿z轴反方向

★运动系中，解可以由静止系中的解经洛伦兹变换而得到.

2) 洛伦兹变换 旋量场.

$$x^\mu \longrightarrow x'^\mu=a^\mu{}_\nu x^\nu$$

$$\psi(x)\longrightarrow\psi'(x)=e^{-\frac{i}{2}w^{\rho\sigma}M_{\rho\sigma}}\psi(x).$$

其中 $\begin{cases}M_{\rho\sigma}=S_{\rho\sigma}+L_{\rho\sigma}.\\ S_{\rho\sigma}=\frac{1}{2}\sigma_{\rho\sigma}=\frac{i}{4}[\gamma_\rho,\gamma_\sigma]\\ L_{\rho\sigma}=i(x_\rho\partial_\sigma-x_\sigma\partial_\rho).\end{cases}$

① 转动(rotation)

引入 $\tau_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$,

$\tau_1 \equiv \frac{1}{2}\varepsilon_{123}M_{23}+\frac{1}{2}\varepsilon_{132}M_{32}=M_{23}$, $\tau_2\equiv M_{31}$, $\tau_3\equiv M_{12}$.

对有自旋粒子 $M_{ij}=S_{ij}=\frac{1}{2}\sigma_{ij}=\frac{i}{4}[\gamma_i,\gamma_j]=\frac{i}{4}\left[\begin{pmatrix}0&\sigma_i\\ \sigma_i&0\end{pmatrix},\begin{pmatrix}0&\sigma_j\\ \sigma_j&0\end{pmatrix}\right]=\frac{i}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}[\sigma_i,\sigma_j]=-\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}\sigma_k$.

$\therefore \tau_1=i\begin{pmatrix}\sigma_1&0\\0&\sigma_1\end{pmatrix}$, $\tau_2=i\begin{pmatrix}\sigma_2&0\\0&\sigma_2\end{pmatrix}$ $\tau_3=i\begin{pmatrix}\sigma_3&0\\0&\sigma_3\end{pmatrix}$ 即 $\vec\tau=i\begin{pmatrix}\vec\sigma&0\\0&\vec\sigma\end{pmatrix}$?

令 $\tau_1=\begin{pmatrix}\sigma_1&0\\0&\sigma_1\end{pmatrix}$, $\tau_2=\begin{pmatrix}\sigma_2&0\\0&\sigma_2\end{pmatrix}$, $\tau_3=\begin{pmatrix}\sigma_3&0\\0&\sigma_3\end{pmatrix}$, 即 $\vec\tau=\begin{pmatrix}\vec\sigma&0\\0&\vec\sigma\end{pmatrix}$.

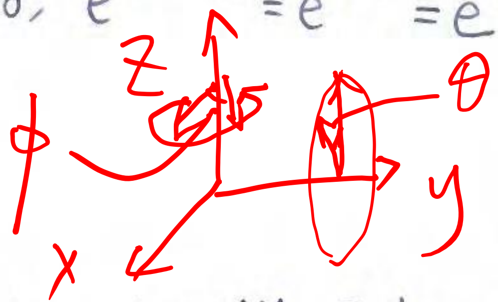
$\theta_1=\psi\equiv w^{23}$, $\theta_2=\theta\equiv w^{31}$, $\theta_3=\phi\equiv w^{12}$

则 $e^{-\frac{i}{2}w^{\rho\sigma}M_{\rho\sigma}}=e^{-\frac{i}{2}w^{ij}M_{ij}}=e^{-\frac{i}{2}(\tau_1\theta_1+\tau_2\theta_2+\tau_3\theta_3)}$

对绕y轴转动θ时, $e^{-\frac{i}{2}w^{\rho\sigma}M_{\rho\sigma}}=e^{-\frac{i}{2}\tau_2\theta_2}=e^{-\frac{i}{2}\tau_2\theta}=\begin{pmatrix}e^{-\frac{i}{2}\sigma_2\theta}&0\\0&e^{-\frac{i}{2}\sigma_2\theta}\end{pmatrix}$

对绕z轴转动ϕ时, $e^{-\frac{i}{2}w^{\rho\sigma}M_{\rho\sigma}}=e^{-\frac{i}{2}\tau_3\theta_3}=e^{-\frac{i}{2}\tau_3\phi}=\begin{pmatrix}e^{-\frac{i}{2}\sigma_3\phi}&0\\0&e^{-\frac{i}{2}\sigma_3\phi}\end{pmatrix}$

~~对绕$\vec n$轴转动~~



$$\begin{cases}\cosh x=\dfrac{e^{x}+e^{-x}}{2}\\ \sinh x=\dfrac{e^{x}-e^{-x}}{2}\end{cases}$$

② 推动(boost)

引入 $K_i\equiv -iM_{0i}$

$K_1=-iM_{01}\equiv\alpha_1$, $K_2=-iM_{02}\equiv\alpha_2$, $K_3=-iM_{03}\equiv\alpha_3$.

令 $\alpha_1=\begin{pmatrix}0&\sigma_1\\ \sigma_1&0\end{pmatrix}$, $\alpha_2=\begin{pmatrix}0&\sigma_2\\ \sigma_2&0\end{pmatrix}$, $\alpha_3=\begin{pmatrix}0&\sigma_3\\ \sigma_3&0\end{pmatrix}$, 即 $\vec\alpha=\begin{pmatrix}0&\vec\sigma\\ \vec\sigma&0\end{pmatrix}$.

$\epsilon_1\equiv w^{01}$, $\epsilon_2\equiv w^{02}$, $\epsilon_3=w^{03}$, 即 $\vec\epsilon=(\epsilon_1,\epsilon_2,\epsilon_3)\equiv\epsilon\vec n$

则 $e^{-\frac{i}{2}w^{\rho\sigma}M_{\rho\sigma}}=e^{-\frac{i}{2}w^{0i}M_{0i}}=e^{-\frac{1}{2}(K_1\epsilon_1+K_2\epsilon_2+K_3\epsilon_3)}=e^{\frac{1}{2}\vec\alpha\cdot\vec\epsilon}=e^{\vec\alpha\cdot\frac{\vec\epsilon}{2}}$

对沿z轴推动. $e^{-\frac{i}{2}w^{\rho\sigma}M_{\rho\sigma}}=e^{\frac{1}{2}\alpha_3\epsilon_3}=e^{\alpha_3\frac{\epsilon_3}{2}}=\cosh\frac{\epsilon}{2}+\alpha_3\sinh\frac{\epsilon}{2}=\begin{pmatrix}\cosh\frac{\epsilon}{2}&\sigma_3\sinh\frac{\epsilon}{2}\\ \sigma_3\sinh\frac{\epsilon}{2}&\cosh\frac{\epsilon}{2}\end{pmatrix}$

对沿$\vec n$轴推动. $e^{-\frac{i}{2}w^{\rho\sigma}M_{\rho\sigma}}=e^{\vec\alpha\cdot\frac{\vec\epsilon}{2}}=e^{\vec\alpha\cdot\vec n\frac{\epsilon}{2}}=\cosh\frac{\epsilon}{2}+\vec\alpha\cdot\vec n\sinh\frac{\epsilon}{2}$

若引入 $\begin{cases}\cosh\frac{\epsilon}{2}=\sqrt{\frac{E_p+m}{2m}}\\ \sinh\frac{\epsilon}{2}=\sqrt{\frac{E_p-m}{2m}}\\ \tanh\frac{\epsilon}{2}=\sqrt{\frac{E_p-m}{E_p+m}}\end{cases}$

$=\sqrt{\frac{E_p+m}{2m}}+\vec\alpha\cdot\vec n\sqrt{\frac{E_p-m}{2m}}=\frac{E_p+m}{\sqrt{2m(E_p+m)}}+\vec\alpha\cdot\vec n\frac{\sqrt{E_p^2-m^2}}{\sqrt{2m(E_p+m)}}$ $|\vec p|$

$=\frac{E_p+\vec\alpha\cdot\vec p+m}{\sqrt{2m(E_p+m)}}$

3) $\vec n = \vec p/|\vec p|$

③ 将静系中沿z轴解，先沿z轴推动ε，再转动至 $\vec n$ 方向，得 $u(p), v(p)$ 的解.

① 静系中沿z轴解: $u_\lambda(m,\vec 0)=\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix}$, $v_\lambda(m,\vec 0)=\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix}$

↓ 沿z轴

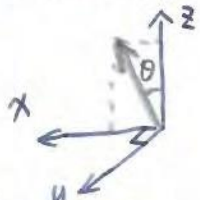
② 将解推动至动量 $|\vec p|$:

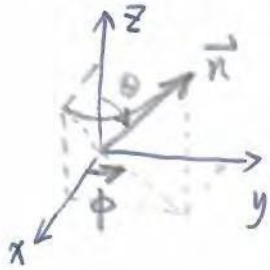
$$u_\lambda(E_p,\vec p_z)=e^{\alpha_3\frac{\epsilon}{2}}u_\lambda(m,\vec 0)=\begin{pmatrix}ch\frac{\epsilon}{2} & \sigma_3 sh\frac{\epsilon}{2}\\ \sigma_3 sh\frac{\epsilon}{2} & ch\frac{\epsilon}{2}\end{pmatrix}\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix}=\begin{pmatrix}ch\frac{\epsilon}{2}\chi_\lambda\\ \sigma_3 sh\frac{\epsilon}{2}\chi_\lambda\end{pmatrix}=\begin{pmatrix}\chi_\lambda\sqrt{\frac{E_p+m}{2m}}\\ \epsilon_\lambda\chi_\lambda\sqrt{\frac{E_p-m}{2m}}\end{pmatrix}$$

$$v_\lambda(E_p,\vec p_z)=e^{\alpha_3\frac{\epsilon}{2}}v_\lambda(m,\vec 0)=\begin{pmatrix}ch\frac{\epsilon}{2} & \sigma_3 sh\frac{\epsilon}{2}\\ \sigma_3 sh\frac{\epsilon}{2} & ch\frac{\epsilon}{2}\end{pmatrix}\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix}=\begin{pmatrix}\sigma_3 sh\frac{\epsilon}{2}\chi_\lambda\\ ch\frac{\epsilon}{2}\chi_\lambda\end{pmatrix}=\begin{pmatrix}\epsilon_\lambda\chi_\lambda\sqrt{\frac{E_p-m}{2m}}\\ \chi_\lambda\sqrt{\frac{E_p+m}{2m}}\end{pmatrix}$$

③ 将解再转动至方向 $\vec n=\hat p$ 向 (先绕y轴转θ，再绕z轴转φ)

$$u_\lambda(E_p,\vec p)=e^{-\frac{i}{2}\tau_3\phi}e^{-\frac{i}{2}\tau_2\theta}u_\lambda(E_p,\vec p_z)$$





$$\cancel{e^{-i\tau_3\frac{\phi}{2}}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\tau_3\frac{\phi}{2}\right)^n=}$$

$\because \vec\tau=\begin{pmatrix}\vec\sigma & 0\\ 0 & \vec\sigma\end{pmatrix}$, $\tau_1^n=\begin{pmatrix}\sigma_1^n & 0\\ 0 & \sigma_1^n\end{pmatrix}$, $\tau_2^n=\begin{pmatrix}\sigma_2^n & 0\\ 0 & \sigma_2^n\end{pmatrix}$ $\tau_3^n=\begin{pmatrix}\sigma_3^n & 0\\ 0 & \sigma_3^n\end{pmatrix}$

$$\therefore e^{-i\tau_3\frac{\phi}{2}}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\tau_3\frac{\phi}{2}\right)^n=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix}\left(-i\sigma_3\frac{\phi}{2}\right)^n & 0\\ 0 & \left(-i\sigma_3\frac{\phi}{2}\right)^n\end{pmatrix}=\begin{pmatrix}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\sigma_3\frac{\phi}{2}\right)^n & 0\\ 0 & \sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\sigma_3\frac{\phi}{2}\right)^n\end{pmatrix}=\begin{pmatrix}e^{-i\sigma_3\frac{\phi}{2}} & 0\\ 0 & e^{-i\sigma_3\frac{\phi}{2}}\end{pmatrix}$$

$$故\ e^{-i\tau_3\frac{\phi}{2}}=\begin{pmatrix}e^{-i\sigma_3\frac{\phi}{2}} & 0\\ 0 & e^{-i\sigma_3\frac{\phi}{2}}\end{pmatrix},\quad e^{-i\tau_2\frac{\theta}{2}}=\begin{pmatrix}e^{-i\sigma_2\frac{\theta}{2}} & 0\\ 0 & e^{-i\sigma_2\frac{\theta}{2}}\end{pmatrix}$$

$$u_\lambda(E_p,\vec p)=e^{-\frac{i}{2}\tau_3\phi}e^{-\frac{i}{2}\tau_2\theta}u_\lambda(E_p,\vec p_z)$$

$$=\begin{pmatrix}e^{-i\sigma_3\frac{\phi}{2}} & 0\\ 0 & e^{-i\sigma_3\frac{\phi}{2}}\end{pmatrix}\begin{pmatrix}e^{-i\sigma_2\frac{\theta}{2}} & 0\\ 0 & e^{-i\sigma_2\frac{\theta}{2}}\end{pmatrix}u_\lambda(E_p,\vec p_z)=\begin{pmatrix}e^{-i\sigma_3\frac{\phi}{2}}e^{-i\sigma_2\frac{\theta}{2}} & 0\\ 0 & e^{-i\sigma_3\frac{\phi}{2}}e^{-i\sigma_2\frac{\theta}{2}}\end{pmatrix}\begin{pmatrix}\chi_\lambda\sqrt{\frac{E_p+m}{2m}}\\ \epsilon_\lambda\chi_\lambda\sqrt{\frac{E_p-m}{2m}}\end{pmatrix}$$

$$=\begin{pmatrix}e^{-i\sigma_3\frac{\phi}{2}}e^{-i\sigma_2\frac{\theta}{2}}\chi_\lambda\sqrt{\frac{E_p+m}{2m}}\\ e^{-i\sigma_3\frac{\phi}{2}}e^{-i\sigma_2\frac{\theta}{2}}\epsilon_\lambda\chi_\lambda\sqrt{\frac{E_p-m}{2m}}\end{pmatrix}$$

$$\cancel{\chi_1=\begin{pmatrix}1\\0\end{pmatrix},\ \chi_2=\begin{pmatrix}0\\1\end{pmatrix},\ e^{-i\sigma_3\frac{\phi}{2}}e^{-i\sigma_2\frac{\theta}{2}}=}$$

$$e^{-\frac{i}{2}\sigma_3\phi}=\sum_{n=0}^{\infty}\frac{\left(-i\sigma_3\frac{\phi}{2}\right)^n}{n!}=\sum_{n=0}^{\infty}\frac{\left(-i\sigma_3\frac{\phi}{2}\right)^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\frac{\left(-i\sigma_3\frac{\phi}{2}\right)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\phi}{2}\right)^{2n}}{(2n)!}-i\sigma_3\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\phi}{2}\right)^{2n+1}}{(2n+1)!}=\cos\frac{\phi}{2}-i\sigma_3\sin\frac{\phi}{2}=\begin{pmatrix}e^{-i\frac{\phi}{2}} & 0\\ 0 & e^{i\frac{\phi}{2}}\end{pmatrix}$$

$$e^{-i\sigma_2\frac{\theta}{2}}=\cos\frac{\theta}{2}-i\sigma_2\sin\frac{\theta}{2}=\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$$

$$\cancel{\therefore e^{-i\sigma_3\frac{\phi}{2}}e^{-i\sigma_2\frac{\theta}{2}}=\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}}$$

$$\therefore e^{-i\sigma_3\frac{\phi}{2}}e^{-i\sigma_2\frac{\theta}{2}}=\begin{pmatrix}e^{-i\frac{\phi}{2}} & 0\\ 0 & e^{i\frac{\phi}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}, & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2}, & \cos\frac{\theta}{2}\end{pmatrix}=\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}, & -\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}, & \cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}$$

$$\cancel{\therefore u_\lambda(E_p,\vec p)=}$$

又 $e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\chi_1 = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} & -\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} & \cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix} = \chi_1(\theta,\phi)$

$e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\chi_2 = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} & -\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} & \cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix} = \chi_2(\theta,\phi)$

故 $e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\chi_\lambda = \chi_\lambda(\theta,\phi)$

$\therefore u_\lambda(E_p,\vec{p}) = \begin{pmatrix} e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\chi_\lambda\sqrt{\frac{E_p+m}{2m}} \\ e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\varepsilon_\lambda\chi_\lambda\sqrt{\frac{E_p-m}{2m}}\end{pmatrix} = \begin{pmatrix}\chi_\lambda(\theta,\phi)\sqrt{\frac{E_p+m}{2m}} \\ \varepsilon_\lambda\chi_\lambda(\theta,\phi)\sqrt{\frac{E_p-m}{2m}}\end{pmatrix}$

利用 $\sqrt{\frac{E_p-m}{2m}}\cdot\sqrt{\frac{2m}{E_p+m}} = \frac{\sqrt{E_p^2-m^2}}{E_p+m} = \frac{|\vec{p}|}{E_p+m}$，得 $u_\lambda(E_p,\vec{p}) = \begin{pmatrix}\chi_\lambda(\theta,\phi) \\ \frac{|\vec{p}|}{E_p+m}\varepsilon_\lambda\chi_\lambda(\theta,\phi)\end{pmatrix}\sqrt{\frac{E_p+m}{2m}}$

$\therefore u_\lambda(E_p,\vec{p}) = \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix}\chi_\lambda(\theta,\phi) \\ \frac{\vec{\sigma}\cdot\vec{n}|\vec{p}|}{E_p+m}\chi_\lambda(\theta,\phi)\end{pmatrix} = \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix}\chi_\lambda(\theta,\phi) \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\lambda(\theta,\phi)\end{pmatrix}$

2° 同理，对于 $v_\lambda(-E_p,+\vec{p})$，

$v_\lambda(-E_p,+\vec{p}) = e^{-i\tau_3\frac{\phi}{2}} e^{-i\tau_2\frac{\theta}{2}} v_\lambda(-E_p,\vec{p}_z)$

$= \begin{pmatrix} e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\varepsilon_\lambda\chi_\lambda\sqrt{\frac{E_p-m}{2m}} \\ e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\chi_\lambda\sqrt{\frac{E_p+m}{2m}}\end{pmatrix} = \begin{pmatrix}\varepsilon_\lambda\chi_\lambda(\theta,\phi)\sqrt{\frac{E_p-m}{2m}} \\ \chi_\lambda(\theta,\phi)\sqrt{\frac{E_p+m}{2m}}\end{pmatrix} = \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix}\frac{|\vec{p}|}{E_p+m}\varepsilon_\lambda\chi_\lambda(\theta,\phi) \\ \chi_\lambda(\theta,\phi)\end{pmatrix}$

$= \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\lambda(\theta,\phi) \\ \chi_\lambda(\theta,\phi)\end{pmatrix}$ $\vec{p}/|\vec{p}|$

4)

④ 将静系中沿z轴解，先转动至 $\vec{n}=\vec{p}$ 方向，再推动至 $\vec{p}$，得 $u(p)$, $v(p)$ 的解.

① 静系中沿z轴解： $u_\lambda(m,\vec{0}) = \begin{pmatrix}\chi_\lambda\\0\end{pmatrix}$， $v_\lambda(-m,\vec{0}) = \begin{pmatrix}0\\\chi_\lambda\end{pmatrix}$.

② 将解转动至 $\vec{n}=\vec{p}/|\vec{p}|$ 方向（先绕y轴转θ，再绕z轴转φ）

$u_\lambda(m,\vec{n}) = e^{-i\tau_3\frac{\phi}{2}} e^{-i\tau_2\frac{\theta}{2}} u_\lambda(m,\vec{0}) = \begin{pmatrix} e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}} & 0 \\ 0 & e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\end{pmatrix}\begin{pmatrix}\chi_\lambda\\0\end{pmatrix} = \begin{pmatrix} e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\chi_\lambda \\ 0\end{pmatrix}$,

$= \begin{pmatrix}\chi_\lambda(\theta,\phi)\\0\end{pmatrix}$

$v_\lambda(-m,\vec{n}) = \begin{pmatrix} e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}} & 0 \\ 0 & e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\end{pmatrix}\begin{pmatrix}0\\\chi_\lambda\end{pmatrix} = \begin{pmatrix}0 \\ e^{-i\sigma_3\frac{\phi}{2}} e^{-i\sigma_2\frac{\theta}{2}}\chi_\lambda\end{pmatrix} = \begin{pmatrix}0\\\chi_\lambda(\theta,\phi)\end{pmatrix}$

③ 将解再沿 $\vec{n}$ 方向推动至动量 $|\vec{p}|$.

接上页

将解再沿 $\vec{n}$ 方向推动至动量 $(\vec{p})$.

$$U_\lambda(E_p,\vec{p}) = e^{\vec{\alpha}\cdot\frac{\vec{\varepsilon}}{2}} U_\lambda(m,\vec{n}) = \frac{E_p+\vec{\alpha}\cdot\vec{p}+m}{\sqrt{2m(E_p+m)}}\begin{pmatrix}\chi_\lambda(\theta,\phi)\\ 0\end{pmatrix} \overset{※}{=\!=} \frac{\not{p}+m}{\sqrt{2m(E_p+m)}}\begin{pmatrix}\chi_\lambda(\theta,\phi)\\ 0\end{pmatrix}$$

$$V_\lambda(-E_p,-\vec{p}) = e^{\vec{\alpha}\cdot\frac{\vec{\varepsilon}}{2}} V_\lambda(-m,\vec{n}) = \frac{E_p+\vec{\alpha}\cdot\vec{p}+m}{\sqrt{2m(E_p+m)}}\begin{pmatrix}0\\ \chi_\lambda(\theta,\phi)\end{pmatrix} \overset{※※}{=\!=} \frac{-\not{p}+m}{\sqrt{2m(E_p+m)}}\begin{pmatrix}0\\ \chi_\lambda(\theta,\phi)\end{pmatrix}$$

证明

※ $\not{p} = \gamma^0 p^0 - \vec{\gamma}\cdot\vec{p}$, $\gamma^0 = \beta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\vec{\alpha} = \begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}$, $\vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}$.

$$\not{p}\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix} = \gamma^0 p^0\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix} - \vec{\gamma}\cdot\vec{p}\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix} = p^0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix} - \vec{p}\cdot\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix} = E_p\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix} + \vec{p}\cdot\begin{pmatrix}0\\ \vec{\sigma}\chi_\lambda\end{pmatrix}$$

而 $$(E_p+\vec{\alpha}\cdot\vec{p})\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix} = E_p\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix} + \vec{p}\cdot\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix} = E_p\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix} + \vec{p}\cdot\begin{pmatrix}0\\ \vec{\sigma}\chi_\lambda\end{pmatrix} = \not{p}\begin{pmatrix}\chi_\lambda\\ 0\end{pmatrix}$$ 证毕

※※ $$-\not{p}\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix} = -\gamma^0 p^0\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix} + \vec{\gamma}\cdot\vec{p}\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix} = -p^0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix} + \vec{p}\cdot\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix} = E_p\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix} + \vec{p}\cdot\begin{pmatrix}\vec{\sigma}\chi_\lambda\\ 0\end{pmatrix}$$

而 $$(E_p+\vec{\alpha}\cdot\vec{p})\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix} = E_p\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix} + \vec{p}\cdot\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix} = E_p\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix} + \vec{p}\cdot\begin{pmatrix}\vec{\sigma}\chi_\lambda\\ 0\end{pmatrix} = -\not{p}\begin{pmatrix}0\\ \chi_\lambda\end{pmatrix}$$ 证毕

还可以证明：此处 $$U_\lambda(E_p,\vec{p}) = \frac{\not{p}+m}{\sqrt{2m(E_p+m)}}\begin{pmatrix}\chi_\lambda(\theta,\phi)\\ 0\end{pmatrix} = \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix}\chi_\lambda(\theta,\phi)\\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\lambda(\theta,\phi)\end{pmatrix}.$$

$$V_\lambda(-E_p,-\vec{p}) = \frac{-\not{p}+m}{\sqrt{2m(E_p+m)}}\begin{pmatrix}0\\ \chi_\lambda(\theta,\phi)\end{pmatrix} = \sqrt{\frac{E_p+m}{2m}}\begin{pmatrix}-\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\lambda(\theta,\phi)\\ \chi_\lambda(\theta,\phi)\end{pmatrix}$$

作业二

四、螺旋度

1.螺旋度

狄拉克方程的解中，χ和φ中是2×1矩阵，它们不能由狄拉克方程确定。为了确定它们，引入螺旋度算符

$$h\equiv\vec{\Sigma}\cdot\hat{p}\equiv\vec{\Sigma}\cdot\vec{n}$$

其中，自旋极化算符 $\frac{1}{2}\vec{\Sigma}=\frac{1}{2}\gamma_5\gamma^0\vec{\gamma}=\frac{1}{2}\begin{pmatrix}\vec{\sigma}&0\\0&\vec{\sigma}\end{pmatrix}$，$\gamma_5\equiv\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3=-i\gamma_0\gamma_1\gamma_2\gamma_3=\begin{pmatrix}0&1\\1&0\end{pmatrix}$，

动量方向的单位矢量 $\hat{p}=\frac{\vec{p}}{|\vec{p}|}$。

由上可知，螺旋度是自旋极化算符在动量方向 $\hat{p}$ 上的投影。在实际应用中，人们通常将χ和φ统一取为螺旋度h的本征态。由于 $h^2=(\vec{\Sigma}\cdot\hat{p})^2=(I_{2\times2}\vec{\sigma}_{2\times2}\cdot\hat{p})^2=(\vec{\sigma}\cdot\vec{n})^2=\vec{n}^2=1$，故h的本征值为±1，以 χ_+ 表示h本征值为+1的本征态，χ_- 表示本征值为-1的本征态，

作业三

$$\begin{cases}h\chi_+=\chi_+\\h\chi_-=-\chi_-\end{cases}$$，将h代入解得 $\begin{cases}\chi_+(\theta,\phi)=\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\\sin\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}\\\chi_-(\theta,\phi)=\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\\\cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\end{pmatrix}\end{cases}$，正交归一 $\chi_r^+\chi_{r'}=\delta_{rr'}$ $(r,r'=\pm)$

2.旋量场与螺旋度关系

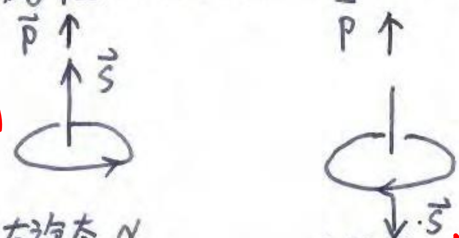
对于由Dirac方程描述的一次量子化系统，$h=\frac{1}{2}\vec{\Sigma}\cdot\hat{p}$ 与 $H=\vec{\alpha}\cdot\vec{p}+\beta m$ 彼此对易，因而可以用它的本征值来标记C数平面波解。又 $[\vec{\Sigma}\cdot\hat{p},\not{p}]=0$，则

$$\vec{\Sigma}\cdot\hat{p}\,u_\lambda(p)=\frac{\not{p}+m}{\sqrt{2m(p^0+m)}}\begin{pmatrix}\vec{\sigma}\cdot\hat{p}&0\\0&\vec{\sigma}\cdot\hat{p}\end{pmatrix}\begin{pmatrix}\chi_\lambda(\theta,\phi)\\0\end{pmatrix}=\varepsilon_\lambda\frac{\not{p}+m}{\sqrt{2m(p^0+m)}}\begin{pmatrix}\chi_\lambda(\theta,\phi)\\0\end{pmatrix}=\varepsilon_\lambda u_\lambda(p)$$

$$\vec{\Sigma}\cdot\hat{p}\,v_\lambda(p)=\frac{-\not{p}+m}{\sqrt{2m(p^0+m)}}\begin{pmatrix}\vec{\sigma}\cdot\hat{p}&0\\0&\vec{\sigma}\cdot\hat{p}\end{pmatrix}\begin{pmatrix}0\\\chi_\lambda(\theta,\phi)\end{pmatrix}=\varepsilon_\lambda\frac{-\not{p}+m}{\sqrt{2m(p^0+m)}}\begin{pmatrix}0\\\chi_\lambda(\theta,\phi)\end{pmatrix}=\varepsilon_\lambda v_\lambda(p)。$$

所以 $u_\lambda(p)$ 和 $v_\lambda(p)$ 是螺旋度h的本征值为±1的本征态，即是自旋极化算符在 $\hat{p}$ 方向投影 $\frac{1}{2}\vec{\Sigma}\cdot\hat{p}$ 的本征值为 $\pm\frac{1}{2}$ 的本征态。

可以证明，$\vec{\Sigma}\cdot\hat{p}$ 是正洛伦兹变换下协变的。通常将螺旋度算符的本征值 ε_λ 称为螺旋度，螺旋度为+1的粒子自旋与动量同向，称为右旋态，螺旋度为-1的粒子的自旋与动量反向，称为左旋态。



螺旋度+1　　右旋态 χ_+　　左旋态 χ_-　　螺旋度-1

§3.5 旋量场双线性协变量

一. 旋量场双线性协变量

1. $\bar{\Psi}(x)$的洛伦兹变换

$x^\mu \longrightarrow x'^\mu = a^\mu{}_\nu x^\nu$

$\psi(x) \longrightarrow \psi'(x') \equiv S\psi(x) = e^{-\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}}\psi(x)$

$\bar{\Psi}(x) \longrightarrow \bar{\Psi}'(x') \equiv \psi'^{+}(x')\gamma^0 = \psi^{+}(x)\gamma^0\gamma^0 e^{\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}^{+}}\gamma^0$

$= \bar{\Psi}(x)\gamma^0\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{i}{4}\omega^{\rho\sigma})^n(\sigma_{\rho\sigma}^{+})^n\gamma^0$

$= \bar{\Psi}(x)\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{i}{4}\omega^{\rho\sigma})^n\underbrace{(\gamma_0\sigma_{\rho\sigma}^{+}\gamma_0)(\gamma_0\sigma_{\rho\sigma}^{+}\gamma_0)\cdots(\gamma_0\sigma_{\rho\sigma}^{+}\gamma_0)}_{n\text{个相乘}}$

$\because \gamma_i^{+} = -\gamma_i,\ \gamma_0^{+} = \gamma_0,$

$\therefore \gamma_0\gamma_i^{+}\gamma_0 = -\gamma_0\gamma_i\gamma_0 = \gamma_i\gamma_0\gamma_0 = \gamma_i,\quad \gamma_0\gamma_0^{+}\gamma_0 = \gamma_0\gamma_0\gamma_0 = \gamma_0$, 即 $\gamma_0\gamma_\rho^{+}\gamma_0 = \gamma_\rho$

$\therefore \gamma_0\sigma_{\rho\sigma}^{+}\gamma_0 = \gamma_0\left(\frac{i}{2}[\gamma_\rho,\gamma_\sigma]\right)^{+}\gamma_0 = -\frac{i}{2}\gamma_0(\gamma_\rho\gamma_\sigma-\gamma_\sigma\gamma_\rho)^{+}\gamma_0 = -\frac{i}{2}(\underline{\gamma_0\gamma_\sigma^{+}\gamma_0}\,\underline{\gamma_0\gamma_\rho^{+}\gamma_0} - \underline{\gamma_0\gamma_\rho^{+}\gamma_0}\,\underline{\gamma_0\gamma_\sigma^{+}\gamma_0})$

$= -\frac{i}{2}(\gamma_\sigma\gamma_\rho - \gamma_\rho\gamma_\sigma) = \frac{i}{2}(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho) = \sigma_{\rho\sigma}$

则 $\bar{\Psi}'(x') = \bar{\Psi}(x)\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{i}{4}\omega^{\rho\sigma})^n(\sigma_{\rho\sigma})^n = \bar{\Psi}(x)e^{\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}} = \bar{\Psi}(x)S^{-1}$

$\therefore \bar{\Psi}(x) \longrightarrow \bar{\Psi}'(x') = \bar{\Psi}(x)S^{-1}$, 其中 $S = e^{-\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}}$, $S^{-1} = e^{\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}}$

2. 由$\psi(x)$, $\bar{\Psi}(x)$可以构造的双线性协变量

(1) 洛伦兹标量: $\bar{\Psi}(x)\psi(x)$.

$x \longrightarrow x' = ax$

$\bar{\Psi}(x)\psi(x) \longrightarrow \bar{\Psi}'(x')\psi'(x') = \bar{\Psi}(x)S^{-1}S\psi(x) = \bar{\Psi}(x)\psi(x)$. 不变.

(2) 洛伦兹矢量: $\bar{\Psi}(x)\gamma^\mu\psi(x)$

①證. $\bar{\Psi}(x)\gamma^\mu\psi(x) \longrightarrow \bar{\Psi}'(x')\gamma^\mu\psi'(x) = \bar{\Psi}(x)S^{-1}\gamma^\mu S\psi(x)$

又对于洛伦兹变换, $\boxed{S\gamma^\mu S^{-1} = (a^{-1})^\mu{}_\nu\gamma^\nu,}$

左乘S^{-1}, 右乘S, $\quad \gamma^\mu = (a^{-1})^\mu{}_\nu S^{-1}\gamma^\nu S$.

乘$a^\rho{}_\mu$并对μ求和. $\quad a^\rho{}_\mu\gamma^\mu = a^\rho{}_\mu(a^{-1})^\mu{}_\nu S^{-1}\gamma^\nu S = \delta^\rho{}_\nu S^{-1}\gamma^\nu S = S^{-1}\gamma^\rho S$

$\therefore \boxed{S^{-1}\gamma^\mu S = a^\mu{}_\nu\gamma^\nu}$

则 $\bar{\Psi}'(x')\gamma^\mu\psi'(x) = a^\mu{}_\nu\bar{\Psi}(x)\gamma^\nu\psi(x)$

$\therefore \bar{\Psi}(x)\gamma^\mu\psi(x) \longrightarrow \bar{\Psi}'(x')\gamma^\mu\psi'(x) = a^\mu{}_\nu\bar{\Psi}(x)\gamma^\nu\psi(x)$ 类似矢量变换

② 旋量场的空间反射变换.

$(\vec{x},t) \longrightarrow (\vec{x}',t') = (-\vec{x},t)$

$\psi(x,t) \longrightarrow \psi'(\vec{x}',t') = \psi'(-\vec{x},t) \equiv \eta_p P\, \psi(\vec{x},t)$, $\bar{\psi}(x,t) \to \bar{\psi}'(x',t') = \eta_p^* \psi^+(\vec{x},t) P^+ \gamma^0$

其中 η_p 是相因子, P 是 4×4 矩阵, 由拉氏密度在洛变下不变来确定.

旋量场 Dirac 方程. $\begin{cases} (i\gamma^\mu\partial_\mu - m)\psi(x) = 0 & (*) \\ \bar{\psi}(x)(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0 & (**) \end{cases}$

相应拉氏密度 $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$, 拉方 $\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0$. (★)

$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi$, $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0$, $\xRightarrow{★} (*)$

$\frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}$, $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \bar{\psi}\, i\gamma^\mu$ $\xRightarrow{★} (**)$

在空间反射变换下.

$\mathcal{L}(x) = \bar{\psi}(\vec{x},t)(i\gamma^\mu\partial_\mu - m)\psi(\vec{x},t) = \bar{\psi}(\vec{x},t)(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\nabla - m)\psi(\vec{x},t)$

$\downarrow$

$\mathcal{L}'(x') = \bar{\psi}'(-\vec{x},t)(i\gamma^\mu\partial'_\mu - m)\psi'(-\vec{x},t) = \bar{\psi}'(-\vec{x},t)(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\nabla - m)\psi'(-\vec{x},t)$

$= \psi'^+(-\vec{x},t)\gamma_0(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\nabla - m)\eta_p P\psi(\vec{x},t)$

$= \psi^+(\vec{x},t)P^+\eta_p^+\gamma_0(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\nabla - m)\eta_p P\psi(\vec{x},t)$

$= |\eta_p|^2\bar{\psi}(\vec{x},t)\gamma_0 P^+\gamma_0(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\nabla - m)P\psi(\vec{x},t)$

$\overset{令}{=} \mathcal{L}(x) = \bar{\psi}(\vec{x},t)(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\nabla - m)\psi(\vec{x},t)$

$\therefore |\eta_p|^2 = 1 \Rightarrow \eta_p = \pm1$, η_p 称为宇称. $\eta = +1$ 为正宇称, $\eta = -1$ 为负宇称.
如质子. 中子.

由含 ∂_0 的项相等

$\gamma_0 P^+\gamma_0\gamma^0 P = \gamma^0$, 即 $\gamma_0 P^+ P = \gamma^0$, 得 $\underline{P^+ = P^{-1}}$, P 为幺正矩阵.

由含 m 的项相等

$\gamma_0 P^+\gamma_0 P = 1$, 即 $\underline{\underline{\gamma_0 P^+\gamma_0 = P^{-1}}}$

由含 ∇ 的项相等

$\underline{\gamma_0 P^+\gamma_0\vec{\gamma}P = -\vec{\gamma}}$, 结合上式, 得 $\underline{\underline{P^{-1}\vec{\gamma}P = -\vec{\gamma}}}$.

综上, $P = \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\therefore$ 空间反射: $\psi(\vec{x},t) \longrightarrow \psi'(\vec{x}',t') = \eta_p\gamma_0\psi(\vec{x},t) = \pm\gamma_0\psi(\vec{x},t)$

$\bar{\psi}(\vec{x},t) \longrightarrow \bar{\psi}'(\vec{x}',t') = \bar{\psi}'(-\vec{x}',t') = \eta_p^*\bar{\psi}(\vec{x},t)\gamma_0 = \pm\bar{\psi}(\vec{x},t)\gamma_0$

③ 洛伦兹矢量.

在空间反射变换下.

$$\bar{\psi}(\vec{x},t)\gamma^\mu\psi(\vec{x},t)\longrightarrow\bar{\psi}'(\vec{x}',t')\gamma^\mu\psi'(\vec{x}',t')$$

$$=\eta_p^*\eta_p\bar{\psi}(\vec{x},t)\gamma_0\gamma^\mu\gamma_0\psi(\vec{x},t)=\begin{cases}\bar{\psi}(x)\gamma^\mu\psi(x) & \mu=0.\\ -\bar{\psi}(x)\gamma^\mu\psi(x). & \mu=1,2,3\end{cases}$$

故 $\bar{\psi}(\vec{x},t)\gamma^\mu\psi(\vec{x},t)$ 为洛伦兹矢量

(3) 洛伦兹赝标量.

$$\bar{\psi}(x)\gamma^5\psi(x)\longrightarrow\bar{\psi}'(x')\gamma^5\psi'(x')$$

$$=\bar{\psi}(x)\gamma_0\gamma^5\gamma_0\psi(x)=-\bar{\psi}(x)\gamma^5\psi(x)$$

故 $\bar{\psi}(x)\gamma^5\psi(x)$ 的洛伦兹赝标量.

(4) 洛伦兹轴矢量.

$$\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)\rightarrow\bar{\psi}'(x')\gamma^\mu\gamma^5\psi'(x')$$

$$=\bar{\psi}(x)\gamma_0\gamma^\mu\gamma^5\gamma_0\psi(x)=\begin{cases}-\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) & \mu=0\\ \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) & \mu=1,2,3\end{cases}$$

故 $\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$ 为洛伦兹轴矢量.

3. 一般双线性协变量.

一般地讲,在 $\bar{\psi}$ 与 ψ 之间插入任意的 4×4 矩阵可以构成双线性协变量。可以证明,旋量空间的4×4矩阵可以按照16个基矩阵展开(因为4×4矩阵含16个元素,故4×4矩阵一定可以用16个基矩阵展开),所以独立的双线性协变量有16个。

(1)※ 下面证明,这16个基矩阵可由 γ 矩阵生成。

引入16个 γ 矩阵

$$\begin{cases}\Gamma^S\equiv 1,\\ \Gamma^V_\mu\equiv\gamma_\mu.\\ \Gamma^T_{\mu\nu}\equiv\sigma_{\mu\nu}=\frac{i}{2}[\gamma_\mu,\gamma_\nu]\\ \Gamma^A_\mu\equiv\gamma_5\gamma_\mu.\\ \Gamma^P\equiv i\gamma_5\end{cases},$$

可证 Γ^a ($a=S,V,T,A,P$) 有如下性质:

(1) $(\Gamma^a)^2=\pm1$.

(2) 对任意 Γ^a ($\Gamma^a\neq\Gamma^S=1$) 存在一个 Γ^b,使 $\Gamma^a\Gamma^b=-\Gamma^b\Gamma^a$

(3) 由此,除 Γ^S 外,所有 Γ^a 的迹为0。因为 $\forall\Gamma^a$,可找到 Γ^b,

$$T_r(\Gamma^a(\Gamma^b)^2)=T_r(\Gamma^b\Gamma^a\Gamma^b)=-T_r((\Gamma^b)^2\Gamma^a)$$

无论 $(\Gamma^b)^2=\pm1$,皆有 $T_r\Gamma^a=0$

(4) 封闭性。对任意一对 (Γ^a,Γ^b) ($a\neq b$),存在 $\Gamma^c\neq\Gamma^S=1$,使得 $\Gamma^a\Gamma^b=\Gamma^c$ (前面可差相因子 $\pm1,\pm i$).

(5) 集合 $\{\Gamma^a\}$ ($a=S,V,T,A,P$) 线性无关。

设 $\sum_a\lambda_a\Gamma^a=0$ (λ_a 为常数),由 Γ^a 相继右乘并求迹,可证 $\lambda_a=0$.

$$\sum_a\lambda_a T_r(\Gamma^a\Gamma^b)$$

$$=\lambda_S T_r(\Gamma^S\Gamma^b)+\lambda_V T_r(\Gamma^V\Gamma^b)+\lambda_T T_r(\Gamma^T\Gamma^b)$$

$$+\lambda_A T_r(\Gamma^A\Gamma^b)+\lambda_P T_r(\Gamma^P\Gamma^b)$$

①当 $\Gamma^b=\Gamma^S=1$ 时,后4项为0,第1项为 λ_S,故 $\lambda_S=0$.

②当 $\Gamma^b\neq\Gamma^S=1$ 时,对于 $b\neq a$ 情况时,存在 $\Gamma^c\neq1$,使 $\Gamma^a\Gamma^b=\Gamma^c$,$T_r\Gamma^c=0$,故由此可得此时的 $\lambda_a=0$,改变 Γ^b,可得所有 $\lambda_a=0$.

对于 $b=a$ 情况时,$T_r(\Gamma^b\Gamma^a)\neq0$.

综上,$\lambda_a=0$,$\{\Gamma^a\}$ 线性无关.