§3.5 旋量场双线性协变量

一. 旋量场双线性协变量

1. $\bar{\Psi}(x)$的洛伦兹变换

$x^\mu \longrightarrow x'^\mu = a^\mu{}_\nu x^\nu$

$\psi(x) \longrightarrow \psi'(x') \equiv S\psi(x) = e^{-\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}}\psi(x)$

$$\bar{\Psi}(x) \longrightarrow \bar{\Psi}'(x') \equiv \psi^+(x')\gamma^0 = \psi^+(x)\gamma^0\gamma^0 e^{\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}^+}\gamma^0$$

$$= \bar{\Psi}(x)\gamma^0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{4}\omega^{\rho\sigma}\right)^n(\sigma_{\rho\sigma}^+)^n\gamma^0$$

$$= \bar{\Psi}(x)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{4}\omega^{\rho\sigma}\right)^n\underbrace{(\gamma_0\sigma_{\rho\sigma}^+\gamma_0)(\gamma_0\sigma_{\rho\sigma}^+\gamma_0)\cdots(\gamma_0\sigma_{\rho\sigma}^+\gamma_0)}_{n个相乘}$$

$\because \gamma_i^+ = -\gamma_i,\ \gamma_0^+ = \gamma_0,$

$\therefore \gamma_0\gamma_i^+\gamma_0 = -\gamma_0\gamma_i\gamma_0 = \gamma_i\gamma_0\gamma_0 = \gamma_i,\quad \gamma_0\gamma_0^+\gamma_0 = \gamma_0\gamma_0\gamma_0 = \gamma_0$，即 $\gamma_0\gamma_\rho^+\gamma_0 = \gamma_\rho$

$$\therefore \gamma_0\sigma_{\rho\sigma}^+\gamma_0 = \gamma_0\left(\frac{i}{2}[\gamma_\rho,\gamma_\sigma]\right)^+\gamma_0 = -\frac{i}{2}\gamma_0(\gamma_\rho\gamma_\sigma-\gamma_\sigma\gamma_\rho)^+\gamma_0 = -\frac{i}{2}(\underline{\gamma_0\gamma_\sigma^+\gamma_0}\,\underline{\gamma_0\gamma_\rho^+\gamma_0} - \underline{\gamma_0\gamma_\rho^+\gamma_0}\,\underline{\gamma_0\gamma_\sigma^+\gamma_0})$$

$$= -\frac{i}{2}(\gamma_\sigma\gamma_\rho - \gamma_\rho\gamma_\sigma) = \frac{i}{2}(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho) = \sigma_{\rho\sigma}$$

则 $\bar{\Psi}'(x') = \bar{\Psi}(x)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{4}\omega^{\rho\sigma}\right)^n(\sigma_{\rho\sigma})^n = \bar{\Psi}(x)e^{\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}} = \bar{\Psi}(x)S^{-1}$

$\therefore \bar{\Psi}(x) \longrightarrow \bar{\Psi}'(x') = \bar{\Psi}(x)S^{-1}$，其中 $S = e^{-\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}}$，$S^{-1} = e^{\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}}$

2. 由$\psi(x)$, $\bar{\Psi}(x)$可以构造的双线性协变量

(1) 洛伦兹标量：$\bar{\Psi}(x)\psi(x)$.

$x \longrightarrow x' = ax$

$\bar{\Psi}(x)\psi(x) \longrightarrow \bar{\Psi}'(x')\psi'(x') = \bar{\Psi}(x)S^{-1}S\psi(x) = \bar{\Psi}(x)\psi(x)$. 不变.

(2) 洛伦兹矢量：$\bar{\Psi}(x)\gamma^\mu\psi(x)$

① 推导. $\bar{\Psi}(x)\gamma^\mu\psi(x) \longrightarrow \bar{\Psi}'(x')\gamma^\mu\psi'(x') = \bar{\Psi}(x)S^{-1}\gamma^\mu S\psi(x)$

又对于洛伦兹变换，$\boxed{S\gamma^\mu S^{-1} = (a^{-1})^\mu{}_\nu\gamma^\nu,}$

左乘S^{-1}，右乘S，　$\gamma^\mu = (a^{-1})^\mu{}_\nu S^{-1}\gamma^\nu S$.

乘$a^\rho{}_\mu$并对μ求和.　$a^\rho{}_\mu\gamma^\mu = a^\rho{}_\mu(a^{-1})^\mu{}_\nu S^{-1}\gamma^\nu S = \delta^\rho{}_\nu S^{-1}\gamma^\nu S = S^{-1}\gamma^\rho S$

$\therefore \boxed{S^{-1}\gamma^\mu S = a^\mu{}_\nu\gamma^\nu}$

则 $\bar{\Psi}'(x')\gamma^\mu\psi'(x') = a^\mu{}_\nu\bar{\Psi}(x)\gamma^\nu\psi(x)$

$\therefore \bar{\Psi}(x)\gamma^\mu\psi(x) \longrightarrow \bar{\Psi}'(x')\gamma^\mu\psi'(x') = a^\mu{}_\nu\bar{\Psi}(x)\gamma^\nu\psi(x)$　类似矢量变换

3. 空间反射

1) ~~②~~ 旋量场的空间反射变换.

$$(\vec{x},t) \longrightarrow (\vec{x}',t') = (-\vec{x},t)$$

$$\psi(x,t) \longrightarrow \psi'(\vec{x}',t') = \psi'(-\vec{x},t) \equiv \eta_p P \psi(\vec{x},t), \quad \bar{\psi}(x,t) \to \bar{\psi}'(x',t') = \eta_p^* \psi^+(\vec{x},t) P^+ \gamma^0$$

其中 η_p 是相因子, P 是 4×4 矩阵, 由拉氏密度在洛变下不变来确定.

旋量场 Dirac 方程.
$$\begin{cases} (i\gamma^\mu\partial_\mu - m)\psi(x) = 0 & (*) \\ \bar{\psi}(x)(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0 & (**) \end{cases}$$

相应拉氏密度 $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$, 拉方 $\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0. (\star)$

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0, \overset{\star}{\Longrightarrow} (*)$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \bar{\psi}i\gamma^\mu \quad \overset{\star}{\Longrightarrow} (**)$$

在空间反射变换下.

$$\mathcal{L}(x) = \bar{\psi}(\vec{x},t)(i\gamma^\mu\partial_\mu - m)\psi(\vec{x},t) = \bar{\psi}(\vec{x},t)(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\nabla - m)\psi(\vec{x},t)$$
$$\downarrow$$
$$\mathcal{L}'(x') = \bar{\psi}'(-\vec{x},t)(i\gamma^\mu\partial'_\mu - m)\psi'(-\vec{x},t) = \bar{\psi}'(-\vec{x},t)(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\nabla - m)\psi'(-\vec{x},t)$$
$$= \psi'^+(-\vec{x},t)\gamma_0(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\nabla - m)\eta_p P\psi(\vec{x},t)$$
$$= \psi^+(\vec{x},t)P^+\eta_p^+\gamma_0(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\nabla - m)\eta_p P\psi(\vec{x},t)$$
$$= |\eta_p|^2\bar{\psi}(\vec{x},t)\gamma_0 P^+\gamma_0(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\nabla - m)P\psi(\vec{x},t)$$
$$\overset{令}{=} \mathcal{L}(x) = \bar{\psi}(\vec{x},t)(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\nabla - m)\psi(\vec{x},t)$$

$\therefore |\eta_p|^2 = 1 \Rightarrow \eta_p = \pm 1$, η_p 称为宇称. $\eta_p = +1$ 为正宇称, 如质子, 中子. $\eta_p = -1$ 为负宇称.

由含 ∂_0 的项相等

$$\gamma_0 P^+\gamma_0\gamma^0 P = \gamma^0, \text{即} \ \gamma_0 P^+ P = \gamma^0, \quad \text{得} \ \underline{P^+ = P^{-1}}, \ P \text{为幺正矩阵}.$$

由含 m 的项相等

$$\gamma_0 P^+\gamma_0 P = 1, \quad \text{即} \ \underline{\gamma_0 P^+\gamma_0 = P^{-1}}$$

由含 ∇ 的项相等

$$\gamma_0 P^+\gamma_0\vec{\gamma}P = -\vec{\gamma}, \ \text{结合上式, 得} \ \underline{P^{-1}\vec{\gamma}P = -\vec{\gamma}}.$$

综上, $P = \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\therefore$ 空间反射: $\psi(\vec{x},t) \longrightarrow \psi'(\vec{x}',t') = \cancel{\pm\gamma_0\psi(-\vec{x},t)}\ \eta_p\gamma_0\psi(\vec{x},t) = \pm\gamma_0\psi(\vec{x},t)$

$$\bar{\psi}(\vec{x},t) \longrightarrow \bar{\psi}'(\vec{x}',t') = \cancel{\bar{\psi}(-\vec{x},t)} = \eta_p^*\bar{\psi}(\vec{x},t)\gamma_0 = \pm\bar{\psi}(\vec{x},t)\gamma_0$$

$\bar{\psi}'(-\vec{x},t)$

2) ~~(2)~~ 洛伦兹矢量的空间反射变换

在空间反射变换下.

$$\bar{\psi}(\vec{x},t)\gamma^{\mu}\psi(\vec{x},t)\longrightarrow\bar{\psi}'(\vec{x}',t')\gamma^{\mu}\psi'(\vec{x}',t')$$

$$=\eta_p^{*}\eta_p\bar{\psi}(\vec{x},t)\gamma_0\gamma^{\mu}\gamma_0\psi(\vec{x},t)=\begin{cases}\bar{\psi}(x)\gamma^{\mu}\psi(x) & \mu=0.\\ -\bar{\psi}(x)\gamma^{\mu}\psi(x). & \mu=1,2,3\end{cases}$$

故 $\bar{\psi}(\vec{x},t)\gamma^{\mu}\psi(\vec{x},t)$ 为洛伦兹矢量

3) ~~(3)~~ 洛伦兹赝标量.

$$\bar{\psi}(x)\gamma^5\psi(x)\longrightarrow\bar{\psi}'(x')\gamma^5\psi'(x')$$

$$=\bar{\psi}(x)\gamma_0\gamma^5\gamma_0\psi(x)=-\bar{\psi}(x)\gamma^5\psi(x)$$

故 $\bar{\psi}(x)\gamma^5\psi(x)$ 为洛伦兹赝标量.

4) ~~(4)~~ 洛伦兹轴矢量.

$$\bar{\psi}(x)\gamma^{\mu}\gamma^5\psi(x)\rightarrow\bar{\psi}'(x')\gamma^{\mu}\gamma^5\psi'(x')$$

$$=\bar{\psi}(x)\gamma_0\gamma^{\mu}\gamma^5\gamma_0\psi(x)=\begin{cases}-\bar{\psi}(x)\gamma^{\mu}\gamma^5\psi(x) & \mu=0\\ \bar{\psi}(x)\gamma^{\mu}\gamma^5\psi(x) & \mu=1,2,3\end{cases}$$

故 $\bar{\psi}(x)\gamma^{\mu}\gamma^5\psi(x)$ 为洛伦兹轴矢量.

4 ~~3.~~ 一般双线性协变量

一般地讲,在 $\bar{\psi}$ 与 ψ 之间插入任意的 4×4 矩阵可以构成双线性协变量。可以证明,旋量空间的 4×4 矩阵可以按照 16个基矩阵展开(因为 4×4 矩阵含16个元素,故 4×4 矩阵一定可以用16个基矩阵展开),所以独立的双线性协变量有16个。

(1) ~~*~~ 下面证明,这16个基矩阵可由 γ 矩阵生成。

引入16个 γ 矩阵

$$\begin{cases}\Gamma^S\equiv 1,\\ \Gamma^V_{\mu}\equiv\gamma_{\mu}.\\ \Gamma^T_{\mu\nu}\equiv\sigma_{\mu\nu}=\frac{i}{2}[\gamma_{\mu},\gamma_{\nu}]\\ \Gamma^A_{\mu}\equiv\gamma_5\gamma_{\mu}.\\ \Gamma^P\equiv i\gamma_5\end{cases}$$

易证 Γ^a ($a=S,V,T,A,P$) 有如下性质:

(1) $(\Gamma^a)^2=\pm1$.

(2) 对任意 Γ^a ($\Gamma^a\neq\Gamma^S=1$),存在一个 Γ^b,使 $\Gamma^a\Gamma^b=-\Gamma^b\Gamma^a$

(3) 由此,除 Γ^S 外,所有 Γ^a 的迹为0。因为 $\forall\Gamma^a$,可找到 Γ^b,

$$T_r(\Gamma^a(\Gamma^b)^2)=T_r(\Gamma^b\Gamma^a\Gamma^b)=-T_r((\Gamma^b)^2\Gamma^a)$$

无论 $(\Gamma^b)^2=\pm1$,皆有 $T_r\Gamma^a=0$

(4) 封闭性。对任意一对 (Γ^a,Γ^b) ($a\neq b$),存在 $\Gamma^c\neq\Gamma^S=1$,使得 $\Gamma^a\Gamma^b=\Gamma^c$ (前面可差相因子 $\pm1,\pm i$).

(5) 集合 $\{\Gamma^a\}$ ($a=S,V,T,A,P$) 线性无关。

设 $\sum_a\lambda_a\Gamma^a=0$ (λ_a 为常数),由 Γ^a 相继乘积求迹,可证 $\lambda_a=0$.

$$\sum_a\lambda_a T_r(\Gamma^a\Gamma^b)$$

$$=\lambda_S T_r(\Gamma^S\Gamma^b)+\lambda_V T_r(\Gamma^V\Gamma^b)+\lambda_T T_r(\Gamma^T\Gamma^b)+\lambda_A T_r(\Gamma^A\Gamma^b)+\lambda_P T_r(\Gamma^P\Gamma^b)$$

① 当 $\Gamma^b=\Gamma^S=1$ 时,后4项为0,第1项为 $4\lambda_S$,故 $\lambda_S=0$.

② 当 $\Gamma^b\neq\Gamma^S=1$ 时, 对于 $b\neq a$ 情况时,存在 $\Gamma^c\neq1$,使 $\Gamma^a\Gamma^b=\Gamma^c$, $T_r\Gamma^c=0$, 对于 $b=a$ 情况时, $T_r(\Gamma^b\Gamma^a)\neq0$. 故由此可得此时的 $\lambda_a=0$,改变 Γ^b,可得所有 $\lambda_a=0$.

综上, $\lambda_a\equiv0$, $\{\Gamma^a\}$ 线性无关.

(2) 16个独立的双线性协变量

S: $\bar{\psi}(x)\psi(x)$; V: $\bar{\psi}(x)\gamma^\mu\psi(x)$; T: $\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$; A: $\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$; P: $i\bar{\psi}(x)\gamma_5\psi(x)$.

在洛伦兹变换下，上述双线性协变量变换为

$$x^\mu \longrightarrow x'^\mu = a^\mu{}_\nu x^\nu$$

$\bar{\psi}(x)\psi(x) \longrightarrow \bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x)$ 标量

$\bar{\psi}(x)\gamma^\mu\psi(x) \longrightarrow \bar{\psi}'(x')\gamma^\mu\psi'(x') = a^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\psi(x)$ 矢量.

$\bar{\psi}(x)\sigma^{\mu\nu}\psi(x) \longrightarrow \bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = a^\mu{}_\rho a^\nu{}_\sigma\bar{\psi}(x)\sigma^{\rho\sigma}\psi(x)$ 反对称张量

$\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x) \longrightarrow \bar{\psi}'(x')\gamma_5\gamma^\mu\psi'(x') = \det(a)\,a^\mu{}_\nu\bar{\psi}(x)\gamma_5\gamma^\nu\psi(x)$ 轴矢量.

$i\bar{\psi}(x)\gamma_5\psi(x) \longrightarrow i\bar{\psi}'(x')\gamma_5\psi'(x') = i\det(a)\bar{\psi}(x)\gamma_5\psi(x)$ 赝标量.

空间反射时 $\det(a) = -1$

基于双线性协变量，可写出Dirac旋量场构造的洛伦兹不变量：

$$\bar{\psi}\psi,\quad \bar{\psi}\gamma^\mu\partial_\mu\psi,\quad \partial_\mu\bar{\psi}\gamma^\mu\psi,\ \cdots$$

二. ~~拉氏密度~~旋量场的拉氏量、守恒流、守恒荷.

1. 拉氏密度: $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi = \bar{\psi}(i\not{\partial} - m)\psi$

$[\mathcal{L}] = 4$, $[\partial_\mu] = 1$, 故 $[\psi] = \frac{3}{2}$, 代入E-L方程，得 $\begin{cases}(i\not{\partial} - m)\psi = 0 \\ \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0\end{cases}$

2. 洛伦兹变换下的守恒流、守恒荷.

(1) 时空平移无穷小变换.

$$x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon^\mu$$

作业

$\boxed{\therefore\ T_{\mu\nu} = i\bar{\psi}\gamma_\mu\partial_\nu\psi,\ \text{满足}\ \partial^\mu T_{\mu\nu} = 0,\ T_{\mu\nu}\ \text{称为旋量场能量-动量张量密度}}$

相应守恒荷

$\boxed{P_\nu = \int d^3x\, T_{0\nu} = i\int d^3x\, \psi^\dagger\partial_\nu\psi \qquad P_\nu\ \text{称为旋量场的四动量.}}$

(2) 无穷小洛伦兹变换.

$$x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon^{\mu\nu} x_\nu$$

$$\psi(x) \to \psi'(x') = \left(1 - \frac{i}{4} \varepsilon^{\rho\sigma} \sigma_{\rho\sigma}\right)\psi(x).$$

$$\boxed{J_{\mu\nu\rho} = x_\nu T_{\mu\rho} - x_\rho T_{\mu\nu} + \bar\psi \gamma_\mu \frac{\sigma_{\nu\rho}}{2} \psi}$$ ，满足 $\partial^\mu J_{\mu\nu\rho} = 0$，$J_{\mu\nu\rho}$ 称为广义角动量张量密度

相应守恒荷.

$$\boxed{M_{\nu\rho} = \int d^3x\, J_{0\nu\rho} = \int d^3x \big(\underbrace{x_\nu T_{0\rho} - x_\rho T_{0\nu}}_{\delta x_\lambda \text{轨道角动量}} + \underbrace{\bar\psi \gamma_0 \frac{\sigma_{\nu\rho}}{2} \psi}_{\delta\psi \text{之自旋角动量}}\big)}$$，$M_{\nu\rho}$ 称为广义角动量

(3) 整体规范变换（坐标 x 不变）

$$\psi(x) \longrightarrow \psi'(x) = e^{i\alpha}\psi, \quad \delta\psi = +i\alpha\psi$$

$$\bar\psi(x) \longrightarrow \bar\psi'(x) = \bar\psi e^{-i\alpha}, \quad \delta\bar\psi = -\bar\psi i\alpha$$

$$j^\mu_a = -\left[\mathcal{L} g^\mu{}_\rho - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi\right]\frac{\partial x^\rho}{\partial\theta^a} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\delta\theta^a},$$

对整体规范变换，$\frac{\partial x^\rho}{\partial\theta^a} = 0$，$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \bar\psi i\gamma^\mu$，$\frac{\delta\psi}{\delta\theta^a} = \frac{\delta\psi}{\delta\alpha} = i\psi$.

$$j^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\delta\psi}{\delta\theta^a} = -\bar\psi i\gamma^\mu i\psi = \bar\psi\gamma^\mu\psi.$$

$\therefore J^\mu = \bar\psi\gamma^\mu\psi$，满足 $\partial_\mu J^\mu = 0$.　　J^μ 称为旋量场矢量流。如 eJ^μ 为电荷电流密度.

相应守恒荷.

$$Q = \int d^3x\, J^0 = \int d^3x\, \bar\psi\gamma^0\psi.$$

Q 称为守恒荷。如 eQ 为电荷，Q 对应费米数守恒

§3.6 零质量旋量场

上节讨论了自旋为$\frac{1}{2}$的粒子，当这类粒子质量为0时，它们具有一些特殊的性质。这类粒子如：中微子，电子，轻质量夸克。

一. 零质量粒子的手征变换

1. 手征变换

自旋$\frac{1}{2}$粒子：$\mathcal{L}=\bar{\psi}(x)(i\gamma^\mu\partial_\mu-m)\psi(x)$,

$m=0$时：$\mathcal{L}=\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)$.

考虑整体手征变换(x不变)

$$\psi(x)\longrightarrow\psi'(x)=e^{i\theta\gamma_5}\psi(x)$$

$\{\gamma_5,\gamma^0\}=0$

$$\bar{\psi}(x)\longrightarrow\bar{\psi}'(x)=\psi'^+(x)\gamma^0\overset{\gamma_5^+=\gamma_5}{=\!=}\psi^+(x)e^{-i\theta\gamma_5}\gamma^0=\psi^+(x)\gamma^0e^{i\theta\gamma_5}$$

$$\therefore\bar{\psi}'(x)=\bar{\psi}(x)e^{i\theta\gamma_5}.$$

$$\mathcal{L}(x)=\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)\longrightarrow\mathcal{L}'(x)=\bar{\psi}'(x)i\gamma^\mu\partial_\mu\psi'(x)$$
$$=\bar{\psi}(x)e^{i\theta\gamma_5}i\gamma^\mu\partial_\mu e^{i\theta\gamma_5}\psi(x)\leftarrow\{\gamma^\mu,\gamma_5\}=0$$
$$=\bar{\psi}(x)i\gamma^\mu\partial_\mu e^{-i\theta\gamma_5}e^{i\theta\gamma_5}\psi(x)$$
$$=\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)$$

$\therefore\mathcal{L}'(x)=\mathcal{L}(x)$. 拉氏量具有手征对称性

(但质量项$m\bar{\psi}\psi$破坏手征对称性) $\bar{\psi}\psi\to\bar{\psi}'\psi'=\bar{\psi}e^{2i\theta\gamma_5}\psi$

2. 手征变换下的守恒流，守恒荷

$$j^\mu{}_a=-\left[\mathcal{L}g^\mu{}_\rho-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi\right]\frac{\partial x^\rho}{\partial\theta^a}-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\delta\theta^a}.$$

1) 对手征变换，$\frac{\partial x^\rho}{\partial\theta^a}=0$，$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}=\bar{\psi}(x)i\gamma^\mu$,

$$\psi(x)\to\psi'(x)=e^{i\theta\gamma_5}\psi(x)$$

$\therefore$ 无穷小变换时 $\delta\psi(x)=\psi'(x)-\psi(x)=i\theta\gamma_5\psi(x)$, $\frac{\delta\psi(x)}{\delta\theta}=i\gamma_5\psi(x)$

$\therefore$ 守恒流 $j_5^\mu=-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\delta\psi}{\delta\theta^a}=-\bar{\psi}(x)i\gamma^\mu i\gamma_5\psi(x)=\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$

满足 $\partial_\mu j_5^\mu=0$. "轴矢量流守恒" ~~质量项~~

2) 守恒荷

$$Q_5=\int d^3x\, j_5^0=\int d^3x\,\bar{\psi}\gamma^0\gamma_5\psi=\int d^3x\,\psi^+\gamma_5\psi$$

二. 手征变换与螺旋度变换的关系.

1. $\vec{\Sigma}\cdot\hat{p}$ 与 γ_5 有共同本征态.

$m=0$ 时运动方程 $\not{p}\psi=0$, 左乘 $\gamma_5\gamma^0$

$$\gamma_5\gamma^0\not{p}\psi=\gamma_5\gamma^0\gamma^\mu p_\mu\psi=\gamma_5\gamma^0\gamma^0p^0\psi-\gamma_5\gamma^0\vec{\gamma}\cdot\vec{p}\psi=0.$$

当 $m=0$ 时, $p^0=|\vec{p}|$ 而 $\vec{\Sigma}=\gamma_5\gamma^0\vec{\gamma}$, 故 $\vec{\Sigma}\cdot\vec{p}\psi=\gamma_5p^0\psi$, 故

$$\vec{\Sigma}\cdot\frac{\vec{p}}{|\vec{p}|}=\gamma_5\,\varepsilon(p^0),$$

这表明

$$\gamma_5=\begin{cases}+\vec{\Sigma}\cdot\frac{\vec{p}}{|\vec{p}|} & (\text{正能解 } p^0>0)\\ -\vec{\Sigma}\cdot\frac{\vec{p}}{|\vec{p}|} & (\text{负能解 } p^0<0)\end{cases}$$

即 γ_5 与 $\vec{\Sigma}\cdot\frac{\vec{p}}{|\vec{p}|}$ 具有共同的本征态，对于正能解时，二者本征值相同；对于负能解时，二者本征值相反。由此，可以由 $\vec{\Sigma}\cdot\frac{\vec{p}}{|\vec{p}|}$ 的本征态确定 γ_5 的本征态。

		螺旋度		手征性（修正）		
$\vec{\Sigma}\cdot\hat{p}\,u_+=u_+$	$u_+(p)$	$h_a=+1$	右旋态	$\varepsilon_a=$ ~~$+1$~~ -1	$\gamma_5u_+(p)=u_+(p)$	右手征态.
$\vec{\Sigma}\cdot\hat{p}\,u_-=-u_-$	$u_-(p)$	$h_a=-1$	左旋态	$\varepsilon_a=$ ~~-1~~ $+1$	$\gamma_5u_-(p)=-u_-(p)$	左手征态.
$\vec{\Sigma}\cdot\hat{p}\,v_+=v_+$	$v_+(p)$	$h_a=+1$	右旋态	$\varepsilon_a=-1$	$\gamma_5v_+(p)=-v_+(p)$	左手征态.
$\vec{\Sigma}\cdot\hat{p}\,v_-=-v_-$	$v_-(p)$	$h_a=-1$	左旋态	$\varepsilon_a=+1$	$\gamma_5v_-(p)=+v_-(p)$	右手征态.

上述性质概括为

$$\begin{cases}\vec{\Sigma}\cdot\hat{p}\,u_a(p)=h_a u_a(p)\\ \vec{\Sigma}\cdot\hat{p}\,v_a(p)=h_a v_a(p)\end{cases}\qquad \begin{cases}\gamma_5u_a(p)=\varepsilon_a u_a(p), & \varepsilon_a=+h_a\\ \gamma_5v_a(p)=\varepsilon_a v_a(p), & \varepsilon_a=-h_a\end{cases}$$

螺旋度与手征性的这种关系是正洛伦兹变换下不变的。

说明：①当 $m=0$ 时，在正洛伦兹变换下，粒子动量 $\vec{p}\to\vec{p}\,'$，但它的螺旋性保持不变。

②当 $m\neq0$ 时，对任意给定 $\vec{p}$，我们总可以沿 $\vec{p}$ 但以比 $\frac{\vec{p}}{E_p}$ 大的速度作一洛伦兹变换。于是粒子动量将改变方向，但自旋方向仍是一样的。结果，粒子的螺旋性 h_a 改变为 $-h_a$。（但这种洛伦兹变换在 $m=0$ 时是不可能的）李政道

③对于手征性，无论 $m=0$ 或 $m\neq0$，在正洛伦兹变换下，粒子的手征性均保持不变。

④从而对于 $m=0$ 粒子，粒子的螺旋性与手征性有上表关系，且在LT时保持不变。
$m\neq0$ 粒子，粒子的螺旋性与手征性没有上表关系，在LT时关系会改变。

2. u_a 与 v_a 互为电荷共轭态.

$u_a(p)$ 和 $v_a(p)$ 同为 γ_5 和 $\vec{\Sigma}\cdot\hat{p}$ 的本征态, 本征值相互联系, 故二者不独立.

1). 电荷共轭旋量

引入矩阵 $C \equiv i\gamma^2\gamma^0 = i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}$

定义电荷共轭旋量 $\psi^c \equiv C\bar{\psi}^T = i\gamma^2\psi^*$

由旋量场 Dirac 方程,

$$\begin{cases} (\not{p}-m)u(p)=0 \\ (\not{p}+m)v(p)=0 \end{cases}$$

可证 $u(p)$ 与 $v(p)$ 互为电荷共轭旋量. 作业二

$$\begin{cases} v(p)=C\bar{u}^T(p) \\ u(p)=C\bar{v}^T(p) \end{cases}$$

三. 手征旋量场.

1. 手征旋量场定义.

当 $m=0$ 时, 旋量场哈密顿量 $H=\vec{\alpha}\cdot\vec{p}+\beta m=-i\vec{\alpha}\cdot\nabla$, 且 $[H,\gamma_5]=0$, 故可以对旋量场进行变换.

$$\psi(x) \longrightarrow \psi'(x)=\gamma_5\psi(x).$$

利用 $\gamma_5=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, 将 ψ 分解为 $\psi=\psi_L+\psi_R$, 称 ψ_L、ψ_R 为左、右手手征旋量场

$$\begin{cases} \psi_L=\frac{1}{2}(1-\gamma_5)\psi, \\ \psi_R=\frac{1}{2}(1+\gamma_5)\psi. \end{cases}$$

它们是 γ_5 本征态 $\begin{cases} \gamma_5\psi_L=-\psi_L, & \text{本征值}-1 \\ \gamma_5\psi_R=+\psi_R, & \text{本征值}+1. \end{cases}$

引入投影算符 $\begin{cases} P_L\equiv\frac{1}{2}(1-\gamma_5) \\ P_R\equiv\frac{1}{2}(1+\gamma_5), \end{cases}$ 则 $\begin{cases} P_L\psi_L=\psi_L \\ P_R\psi_R=\psi_R \end{cases}$ 其它为 0.

2. Weyl 表象下的手征旋量场

1) Dirac 表象下

$\gamma^0=\beta\equiv\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^i=\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$, $\gamma^5=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\psi=\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ 4分量复旋量.

引入 $P_L\equiv\frac{1}{2}(1-\gamma_5)=\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $P_R\equiv\frac{1}{2}(1+\gamma_5)=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} \psi_L=\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} \psi_1-\psi_3 \\ \psi_2-\psi_4 \\ \psi_3-\psi_1 \\ \psi_4-\psi_2 \end{pmatrix} \\ \psi_R=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}=\begin{pmatrix} \psi_1+\psi_3 \\ \psi_2+\psi_4 \\ \psi_3+\psi_1 \\ \psi_4+\psi_2 \end{pmatrix} \end{cases}$$

4分量复旋量.

2). Weyl 表象下.

$\gamma^0=\beta\equiv\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^i=\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$, $\gamma_5=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\psi=\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$.

引入 $P_L\equiv\frac{1}{2}(1-\gamma_5)=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $P_R\equiv\frac{1}{2}(1+\gamma_5)=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Leftarrow \begin{cases} \psi_L=\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}=\begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix} & \psi\text{的前两个分量} \\ \psi_R=\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix} & \psi\text{的后两个分量.} \end{cases}$$

对 Dirac 表象做幺正变换,

$\gamma^\mu_{Weyl}=U\gamma^\mu_{Dirac}U$, $U=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

可得 Weyl 表象下的各种 γ 矩阵

3. 手征变换

$$\psi \to \psi' = e^{i\theta\gamma_5}\psi, \qquad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_5^n = \begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix}, \quad P_L\psi_L = \psi_L,\ P_R\psi_R = \psi_R$$

Weyl表象下.

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \to \begin{pmatrix} \psi_L' \\ \psi_R' \end{pmatrix} = e^{i\theta\gamma_5}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \sum \frac{(i\theta)^n}{n!}\gamma_5^n \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \sum \frac{(i\theta)^n}{n!}\begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \sum \frac{(i\theta)^n}{n!}\begin{pmatrix} (-1)^n\psi_L \\ \psi_R \end{pmatrix}$$

$$\therefore \begin{cases} \psi_L \to \psi_L' = \sum \frac{(-i\theta)^n}{n!}\psi_L = \sum \frac{(-i\theta)^n}{n!}\left(\frac{1-\gamma_5}{2}\right)^n\psi_L = e^{-\frac{i}{2}\theta(1-\gamma_5)}\psi_L. \\ \psi_R \to \psi_R' = \sum \frac{(i\theta)^n}{n!}\psi_R = \sum \frac{(i\theta)^n}{n!}\left(\frac{1+\gamma_5}{2}\right)^n\psi_R = e^{\frac{i}{2}\theta(1+\gamma_5)}\psi_R. \end{cases}$$

即 $\begin{cases} \psi_L \to \psi_L' = e^{\frac{i}{2}\theta_L(1-\gamma_5)}\psi_L. \\ \psi_R \to \psi_R' = e^{\frac{i}{2}\theta_R(1+\gamma_5)}\psi_R. \end{cases}$ ，可证此式在Dirac表象下也成立。

4. 运动方程

~~Weyl表象下, $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, 零质量旋量场拉氏密度~~.

~~$\mathcal{L} = \bar\psi i\gamma^\mu\partial_\mu\psi =$~~ 在Weyl表象下.,

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \bar\psi = \psi^\dagger\gamma^0 = (\psi_L^\dagger\ \psi_R^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\psi_R^\dagger\ \psi_L^\dagger).$$

$$\gamma^\mu\partial_\mu = \gamma^0\partial_0 + \vec\gamma\cdot\nabla = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\partial_0 + \begin{pmatrix} 0 & \vec\sigma \\ -\vec\sigma & 0 \end{pmatrix}\cdot\nabla = \begin{pmatrix} 0 & \partial_0 + \vec\sigma\cdot\nabla \\ \partial_0 - \vec\sigma\cdot\nabla & 0 \end{pmatrix}$$

则零质量旋量场拉氏密度

$$\mathcal{L} = \bar\psi i\gamma^\mu\partial_\mu\psi = (\psi_R^\dagger\ \psi_L^\dagger)\, i\begin{pmatrix} 0 & \partial_0 + \vec\sigma\cdot\nabla \\ \partial_0 - \vec\sigma\cdot\nabla & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = i(\psi_R^\dagger\ \psi_L^\dagger)\begin{pmatrix} (\partial_0 + \vec\sigma\cdot\nabla)\psi_R \\ (\partial_0 - \vec\sigma\cdot\nabla)\psi_L \end{pmatrix}$$

$$= i\psi_L^\dagger(\partial_0 - \vec\sigma\cdot\nabla)\psi_L + i\psi_R^\dagger(\partial_0 + \vec\sigma\cdot\nabla)\psi_R.$$

$$\bar\sigma^\mu \equiv (1, -\vec\sigma)$$

即 $\mathcal{L} = \mathcal{L}_L + \mathcal{L}_R$, 其中.

$$\begin{cases} \mathcal{L}_L = i\psi_L^\dagger\bar\sigma^\mu\partial_\mu\psi_L, & \cancel{\bar\sigma^\mu = (1, -\vec\sigma)} \\ \mathcal{L}_R = i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R, & \sigma^\mu = (1, +\vec\sigma) \end{cases}$$

二分量Weyl方程

运动方程若用ψ_L表示: $\frac{\partial\mathcal{L}}{\partial\psi_L^\dagger} = i\bar\sigma^\mu\partial_\mu\psi_L$, $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_L^\dagger)}\right) = 0$, $\Rightarrow$ $\boxed{i\bar\sigma^\mu\partial_\mu\psi_L = 0 \text{ 或 } (P^0 - \vec\sigma\cdot\vec p)\psi_L = 0}$

运动方程若用ψ_R表示: $\frac{\partial\mathcal{L}}{\partial\psi_R^\dagger} = i\sigma^\mu\partial_\mu\psi_R$, $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_R^\dagger)}\right) = 0$ $\Rightarrow$ $\boxed{i\sigma^\mu\partial_\mu\psi_R = 0 \text{ 或 } (P^0 + \vec\sigma\cdot\vec p)\psi_R = 0}$

二分量Weyl方程描述无质量.具有确定螺旋度的粒子:

① $\begin{cases} i\bar\sigma^\mu\partial_\mu\psi_L = 0. \\ \text{或}(P^0 - \vec\sigma\cdot\vec p)\psi_L = 0 \end{cases}$
- 若$P^0 > 0$(正能粒子), 左手性(手征性=螺旋度=−1)得左旋,左旋粒子,如$\boxed{\text{左旋中微子}}$
- 即若假设$m_\nu \simeq 0$, 则正中微子用左手场ψ_L描述, $\boxed{\gamma_5 = \vec\Sigma\cdot\frac{\vec p}{|\vec p|}, \text{本征值为}-1}$
- 若$P^0 < 0$, 左手性(手征性=−螺旋度=−1)得右旋, 右旋反粒子, $\boxed{\text{如右旋反中微子}, \gamma_5 = -\vec\Sigma\cdot\frac{\vec p}{|\vec p|}, \text{本征值}+1}$

②③ $\begin{cases} i\sigma^\mu\partial_\mu\psi_R = 0 \\ \text{或 } (P^0 + \vec\sigma\cdot\vec p)\psi_R = 0 \end{cases}$
- 若$P^0 > 0$(正能粒子), 右手性(手征性=+螺旋度=+1)得右旋, 右旋粒子,但与实验不符合.
- 若$P^0 < 0$.(负能粒子). 右手性(手征性=−螺旋度=+1)得左旋, 左旋反粒子, 但与实验不符合.

综上：

① Dirac方程对零质量粒子在Weyl表象中分解为一对二分量Weyl方程，左旋和右旋粒子分别遵从 $i\bar{\sigma}^\mu\partial_\mu\psi_L=0$ 和 $i\sigma^\mu\partial_\mu\psi_R=0$.

② 零质量粒子具有确定的螺旋度，即为左旋或右旋。

③ 由于左旋态和右旋态互为空间反射态，故具有确定螺旋度的态没有空间反射不变性，即不是宇称P的本征态。

④ 可以证明，上述一对二分量Weyl方程在空间反射P变换下互相转变，故每个方程不是~~宇称~~空间反射变换P不变的，同样它们也不是C不变的，但在CP变换下是不变的。

5. 手征变换的守恒流，守恒荷.

$$J_5^\mu=\bar{\psi}\gamma^\mu\gamma_5\psi.$$

$$Q_5=\int d^3x\,\bar{\psi}\gamma^0\gamma_5\psi=\int d^3x\,\psi^\dagger\gamma_5\psi=\begin{cases}\int d^3x\,(\psi_L^\dagger\psi_R+\psi_R^\dagger\psi_L) & \text{Dirac表象.}\\ \int d^3x\,(\psi_R^\dagger\psi_R-\psi_L^\dagger\psi_L) & \text{Weyl表象.}\end{cases}$$

四. Majorana旋量（马拉约纳旋量）.

1. Majorana旋量.

若电荷共轭旋量 ψ^c 满足 $\boxed{\psi_M^c=\psi_M\text{，则称 }\psi_M\text{ 为Majorana旋量}}$

由定义，它的自由度只有Dirac旋量的一半(?)，故可等价于二分量Weyl复旋量场，从而零质量自旋 $\frac{1}{2}$ 粒子也可用 ψ_M 描写。

动能项：
$$\mathcal{L}_M=\frac{i}{2}\bar{\psi}_M\gamma^\mu\partial_\mu\psi_M,\qquad (\psi_M)^c=\psi_M$$

质量项：
$$\mathcal{L}_M^m=-\frac{1}{2}m\bar{\psi}_M\psi_M$$

质量项能够由单一的手征场如 ψ_L 构成。在Weyl表象下，$C=i\gamma^2\gamma^0=i\begin{pmatrix}0&\sigma^2\\-\sigma^2&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}i\sigma^2&0\\0&-i\sigma^2\end{pmatrix}$

可以证明，形如 $\psi_M=\begin{pmatrix}\psi_L\\-i\sigma^2\psi_L^*\end{pmatrix}$ 满足Majorana旋量条件 $(\psi_M)^c=\psi_M$，质量项此时写为

$$\mathcal{L}_M^m=\frac{i}{2}(\psi_L^\dagger\sigma^2\psi_L^*-\psi_L^T\sigma^2\psi_L)\qquad \sigma_2^\dagger=\sigma_2$$

证：$\bar{\psi}_M=\psi_M^\dagger\gamma^0=(\psi_L^\dagger\quad \psi_L^{*\dagger}i\sigma_2)\begin{pmatrix}0&1\\1&0\end{pmatrix}=(\psi_L^{*\dagger}i\sigma_2\quad \psi_L^\dagger)$

$$\cancel{-\frac{1}{2}m\bar{\psi}_M\psi_M=\left(-\frac{m}{2}\right)(\psi_L^\dagger\quad \psi_L^T i\sigma_2\psi_L^*)\begin{pmatrix}\psi_L\\-i\sigma^2\psi_L^*\end{pmatrix}=-\frac{m}{2}(\psi_L^\dagger\psi_L}$$

$$-\frac{1}{2}m\bar{\psi}_M\psi_M=-\frac{m}{2}(\psi_L^T i\sigma_2\quad \psi_L^\dagger)\begin{pmatrix}\psi_L\\-i\sigma_2\psi_L^*\end{pmatrix}=-\frac{m}{2}(\psi_L^T i\sigma_2\psi_L-\psi_L^\dagger i\sigma_2\psi_L^*)$$
$$=\frac{im}{2}(\psi_L^\dagger\sigma_2\psi_L^*-\psi_L^T\sigma_2\psi_L)$$

此式表明：① $\mathcal{L}_M^m$ 仅含 ψ_L 不含 ψ_R，因此仅有手征场并不能完全保证费米子无质量。仅有手征场将禁戒Dirac质量项但并不排除马拉约纳质量项。（仅保留 $\psi_L\to\psi_L'=-\psi_L$ 的对称性）

② $\mathcal{L}_M^m$ 破坏整体相位不变性 $\psi_L\to\psi_L'=e^{i\alpha}\psi_L$，从而手征费米子数（左手手征费米子数）不守恒.

§3.7. 自由电磁场

一. 麦克斯维方程

1867年，英国麦克斯韦(Maxwell)建立了描述电磁场运动的方程，后经德国赫兹(Hertz)提炼，得到我们今天的表示。1905年，爱因斯坦创立了狭义相对论实现了时间与空间、电场与磁场的统一，得到了麦克斯维方程的四维协变形式。1926年，量子力学问世，人们认识到麦方不但是经典电磁场的方程，也是光子的相对论性波动方程。1927年，狄拉克提出将电磁场作为一个具有无穷维自由度的系统进行量子化的方案，将电磁场傅里叶分解为一系列基本的振动模式。这一方案将经典电磁场量子化，从而统一描述了光子~~的产生和湮灭~~波动性和粒子性，由此，经典电磁场量子化的方案就成功地描述了光子的产生和湮灭的高速微观现象。

1. 经典麦克斯维方程

1) 经典麦方

$$\begin{cases}\nabla\cdot\vec{E}=\rho \\ \nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B}=0 \\ \nabla\times\vec{B}=\vec{j}+\dfrac{\partial\vec{E}}{\partial t}\end{cases}$$

引入 $\vec{B}=\nabla\times\underline{\vec{A}}$

$\nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\cdots\cdots\to\nabla\times\left(\vec{E}+\dfrac{\partial\vec{A}}{\partial t}\right)=0$

引入 $\vec{E}+\dfrac{\partial\vec{A}}{\partial t}=-\nabla\underline{\phi}$

势形式的麦方

$$\begin{cases}\nabla^2\vec{A}-\dfrac{\partial^2}{\partial t^2}\vec{A}-\nabla\left(\nabla\cdot\vec{A}+\dfrac{\partial\phi}{\partial t}\right)=-\vec{j} & (*) \\ \nabla^2\phi+\dfrac{\partial}{\partial t}(\nabla\cdot\vec{A})=-\rho & (**)\end{cases}$$

2) 引入四维势矢量. $A^\mu=(\phi,\vec{A})$ 或 $A_\mu=(\phi,-\vec{A})$；$j^\mu=(\rho,\vec{j})$ 或 $j_\mu=(\rho,-\vec{j})$

则

$$\boxed{\partial_\nu(\partial^\mu A^\nu-\partial^\nu A^\mu)=-j^\mu}$$

证明：

$$\partial_\nu(\partial^\mu A^\nu-\partial^\nu A^\mu)=\partial^\mu\partial_\nu A^\nu-\partial_\nu\partial^\nu A^\mu=\partial^\mu\left(\frac{\partial\phi}{\partial t}+\nabla\cdot\vec{A}\right)-\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)A^\mu=-j^\mu.$$

当 $\mu=0$ 时，$\dfrac{\partial}{\partial t}\left(\dfrac{\partial\phi}{\partial t}+\nabla\cdot\vec{A}\right)-\left(\dfrac{\partial^2}{\partial t^2}-\nabla^2\right)\phi=-\rho.$

即 $\nabla^2\phi+\dfrac{\partial}{\partial t}(\nabla\cdot\vec{A})=-\rho \qquad (**)$

当 $\mu=1,2,3$ 时，

$$-\nabla\left(\frac{\partial\phi}{\partial t}+\nabla\cdot\vec{A}\right)-\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)\vec{A}=-\vec{j}$$

即 $\nabla^2\vec{A}-\dfrac{\partial^2}{\partial t^2}\vec{A}-\nabla\left(\nabla\cdot\vec{A}+\dfrac{\partial\phi}{\partial t}\right)=-\vec{j} \qquad (*)$

3). 电磁场张量形式的麦方

引入 $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$，则

$$\boxed{\partial_\nu F^{\mu\nu}=-j^\mu}$$

$\partial^\mu=(\partial^0,-\nabla)$，$\partial_\mu=(\partial_0,\nabla)$

其中 $\begin{cases}F^{i0}=\partial^i A^0-\partial^0 A^i=-\nabla^i\phi-\frac{\partial}{\partial t}A^i=E^i \\ F^{ij}=\partial^i A^j-\partial^j A^i=-\nabla^i A^j-(-\nabla^j A^i)=-\varepsilon^{ijk}B^k\end{cases}$

其中 ~~$F_{i0}=\partial_i A_0-\partial_0 A_i=\partial_i\phi+\frac{\partial}{\partial t}A_i=-E_i\leftrightarrow F^{i0}=E^i$~~

~~$F_{ij}=\partial_i A_j-\partial_j A_i=\varepsilon_{ijk}B_k \quad\leftrightarrow F^{ij}=\varepsilon^{ijk}B^k$~~

$$F^{\mu\nu}=\begin{pmatrix}0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0\end{pmatrix}$$

2. 拉格朗日形式

由拉氏密度满足洛伦兹不变性，设

$$\mathcal{L} = -\frac{1}{2}\left[a\partial_\mu A^\nu \partial^\mu A_\nu + b\partial_\mu A^\nu \partial_\nu A^\mu + c(\partial_\mu A^\mu)^2 + dA_\mu A^\mu\right]$$

系数a.b.c.d为实数，待定。利用E-L方程与麦方比较可以定出。

E-L方程 $a\partial^\nu\partial_\nu A^\mu + (b+c)\partial^\mu\partial^\nu A_\nu - dA^\mu = 0$ (★)

真空中麦方. ① $\nabla\cdot\vec{E}=0$, $\nabla\times\vec{B}=\frac{\partial\vec{E}}{\partial t}$ 或 $\partial_\mu F^{\mu\nu}=0$. 其中 $F^{i0}=E^i$, $F^{ij}=-\varepsilon^{ijk}B^k$

② $\nabla\cdot\vec{B}=0$, $\nabla\times\vec{E}=-\frac{\partial}{\partial t}\vec{B}$ 或 $\partial_\mu\tilde{F}^{\mu\nu}=0$. 其中 $\tilde{F}^{\mu\nu}=\frac{-1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ $\begin{cases}\tilde{F}^{i0}=B^i\\ \tilde{F}^{ij}=+\varepsilon^{ijk}E^k\end{cases}$

满足 $F^{\mu\nu}\longrightarrow\tilde{F}^{\mu\nu}$ 时有 $\begin{cases}E^i\longleftrightarrow +B^i\\ B^i\longleftrightarrow -E^i\end{cases}$。且 $\partial_\mu\tilde{F}^{\mu\nu}=0$ 与 $\partial_\mu F^{\mu\nu}=0$ 并不独立.

真空中麦方1为 $\partial_\nu F^{\nu\mu}=0$，即 $\partial_\nu(\partial^\nu A^\mu-\partial^\mu A^\nu)=0$ 或

$$\partial_\nu\partial^\nu A^\mu-\partial^\mu\partial_\nu A^\nu=0 \quad (※)$$

$$\tilde{F}^{\mu\nu}=\begin{pmatrix}0 & -B^1 & -B^2 & -B^3\\ B^1 & 0 & E^3 & -E^2\\ B^2 & -E^3 & 0 & E^1\\ B^3 & E^2 & -E^1 & 0\end{pmatrix}$$

(★)与(※)对比，可得

$$a=1,\quad b+c=-1,\quad d=0.$$

$$\therefore \mathcal{L}=-\frac{1}{2}(\partial_\mu A^\nu\partial^\mu A_\nu-\partial_\mu A^\nu\partial_\nu A^\mu)=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

拉氏密度中，A^μ的量纲为1，质量为0。

二. 规范变换下的不变性.

电磁理论中，$\vec{E}$、$\vec{B}$被认为是基本的，而ϕ、$\vec{A}$认为是辅助场；由于$F^{\mu\nu}$由$\vec{E}$,$\vec{B}$构成，所以是基本的，而A^μ是辅助的。

1. 规范变换

对于给定的$F(x)$，$A^\mu(x)$具有很大的任意性。对A^μ做变换（$\alpha(x)$为时空上标量函数）

$$A^\mu(x)\longrightarrow A'^\mu(x)=A^\mu(x)+\partial^\mu\alpha(x).$$

$$F^{\mu\nu}(x)\longrightarrow F'^{\mu\nu}(x)=\partial^\mu A'^\nu(x)-\partial^\nu A'^\mu(x)=\partial^\mu(A^\nu(x)+\partial^\nu\alpha(x))-\partial^\nu(A^\mu(x)+\partial^\mu\alpha(x))$$

$$=\partial^\mu A^\nu(x)-\partial^\nu A^\mu(x)=F^{\mu\nu}(x)$$ 保持不变，故$\mathcal{L}(x)$也不变。

由于$\frac{1}{2}m^2A_\mu A^\mu$不满足上述规范变换下的不变性，故电磁场质量为0。

2. 库仑规范（辐射规范）

由于规范变换带来的任意性，对A^μ可作不同的选择。不同的选择对应不同的规范。

$\boxed{取\begin{cases}\nabla\cdot\vec{A}=0\\ A^0=0\end{cases}}$　令 $\begin{cases}A^\mu(x)=A^\mu e^{ip\cdot x}\\ \vec{A}(x)=\vec{A}e^{-i\vec{p}\cdot\vec{x}}\end{cases}$ 则 $-i\vec{p}\cdot\vec{A}=0$ $\therefore \vec{A}\perp\vec{P}$, $A^0=0$. ⇒

电磁场A^μ中只留下垂直于运动方向的两个横向分量（左旋，右旋），没有非物理自由度。但库仑规范无洛伦兹协变性。

3. 洛伦兹规范

$\boxed{\partial_\mu A^\mu(x)=0}$ 即使最初A^μ不满足该条件，对A^μ作规范变换 $A^\mu\to A'^\mu=A^\mu-\partial^\mu\alpha$，只需选择$\alpha$使$\Box\alpha=\partial_\mu A^\mu$，则 A'^μ一定能满足条件 $\partial_\mu A'^\mu(x)=0$。

在洛伦兹规范下，麦方简化为$\Box A^\mu(x)=0$。~~由于~~此时$A^\mu(x)$仍有规范任意性。对A^μ再作规范变换 $A^\mu\to A'^\mu=A^\mu+\partial^\mu\alpha$，为使$A'^\mu$满足麦方$\Box A'^\mu(x)=0$，只需标量函数满足$\boxed{\Box\alpha=0}$即可，此时$\alpha$仍然有无限多。

洛伦兹规范$\partial_\mu A^\mu(x)=0$具有协变性，但它不能消除所有非物理的自由度

洛伦兹条件可以利用一个拉格朗日乘子入纳入拉氏密度和运动方程来实现。

$$\boxed{\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{\lambda}{2}(\partial_\rho A^\rho)^2}$$

一方面，$-\frac{\lambda}{2}(\partial_\rho A^\rho)^2$在规范变换$A^\mu\to A'^\mu=A^\mu+\partial^\mu\alpha$下仍然保持不变，只要其中$\alpha$满足$\Box\alpha=0$即可。

另一方面，相应E-L方程为

$$\boxed{\Box A^\mu+(\lambda-1)\partial^\mu(\partial_\rho A^\rho)=0}$$

可以证明，这个方程若在$t=0$时满足洛伦兹条件$\partial_\rho A^\rho=0$且$\frac{\partial}{\partial t}\partial_\rho A^\rho=0$，则$t$时刻也有$\partial_\rho A^\rho=0$。即运动方程也满足洛伦兹条件。

[证明：对E-L方程取四散度，$\Box\partial_\mu A^\mu+(\lambda-1)\Box(\partial\cdot A)=\lambda\Box(\partial\cdot A)=0$，

即 $\Box\partial_\mu A^\mu=0$ $(\lambda\neq0)$

若$t=0$时，$\chi\equiv\partial_\mu A^\mu=0$，$\partial_t\chi=0$，则$t$时刻时，将$\chi$对$t$作展开

$$\chi=\chi\Big|_{t=0}+\frac{\partial\chi}{\partial t}\Big|_{t=0}t+\frac{1}{2}\frac{\partial^2\chi}{\partial t^2}\Big|_{t=0}t^2+\cdots$$

由初始条件，第一、二项为0，由$\Box\partial_\mu A^\mu\equiv\Box\chi=0$，得$\chi\sim e^{i(k^0t-\vec{k}\cdot\vec{x})}$，是平面波之叠加，其中$\vec{x}$和$t$的依赖部分相互独立，从而

$$\frac{\partial^2}{\partial t^2}\chi\Big|_{t=0}=\nabla^2\chi\Big|_{t=0}=\nabla^2\left(\chi\Big|_{t=0}\right)=0.$$

$$\frac{\partial^3}{\partial t^3}\Big|_{t=0}=\nabla^2\frac{\partial\chi}{\partial t}\Big|_{t=0}=\nabla^2\left(\frac{\partial\chi}{\partial t}\Big|_{t=0}\right)=0$$

$\cdots$

故χ展开式中所有更高次项均为0。

综上，若$t=0$时，$\chi\equiv\partial_\mu A^\mu=0$，则$t$时刻洛伦兹条件$\chi\equiv\partial_\mu A^\mu=0$也满足。]

三. 守恒流、守恒荷.

$$j^{\mu,a} = -\left[\mathcal{L} g^{\mu\lambda} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial^\lambda A_\rho\right]\frac{\delta x_\lambda}{\delta\theta_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\frac{\delta A_\lambda}{\delta\theta_a}$$

$$\mathcal{L} = -\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} - \frac{\lambda}{2}(\partial_\rho A^\rho)^2 = -\frac{1}{4}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial^\rho A^\sigma - \partial^\sigma A^\rho) - \frac{\lambda}{2}(\partial_\rho A^\rho)(\partial_\sigma A^\sigma)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)}\left\{-\frac{1}{4}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial^\rho A^\sigma - \partial^\sigma A^\rho) - \frac{\lambda}{2}(\partial_\rho A^\rho)(\partial_\sigma A^\sigma)\right\}$$

$$= \frac{-1}{4}\left\{(\delta_\rho^\mu\delta_\sigma^\nu - \delta_\sigma^\mu\delta_\rho^\nu)(\partial^\rho A^\sigma - \partial^\sigma A^\rho) + (\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho})\right\}$$

$$-\frac{\lambda}{2}\left\{\delta_\rho^\mu\delta^{\nu\rho}\partial_\sigma A^\sigma + \partial_\rho A^\rho\delta_\sigma^\mu\delta^{\nu\sigma}\right\}$$

$$= -\frac{1}{4}\left\{[\partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\nu A^\mu + \partial^\mu A^\nu] + [\partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\nu A^\mu + \partial^\mu A^\nu]\right\} - \frac{2\lambda}{2}\left\{\delta^{\mu\nu}\partial_\sigma A^\sigma\right\}$$

$$= -(\partial^\mu A^\nu - \partial^\nu A^\mu) - \lambda g^{\mu\nu}\partial_\sigma A^\sigma = -F^{\mu\nu} - \lambda g^{\mu\nu}\partial_\rho A^\rho.$$

$$\therefore j^{\mu,\nu} = -\left[\mathcal{L} g^{\mu\lambda} + (F^{\mu\rho} + \lambda g^{\mu\rho}\partial\cdot A)\partial^\lambda A_\rho\right]\frac{\delta x_\lambda}{\delta\theta_\nu} + (F^{\mu\lambda} + \lambda g^{\mu\lambda}\partial\cdot A)\frac{\delta A_\lambda}{\delta\theta_\nu}$$

1. 平移无穷小变换 ($\delta A_\lambda = 0$)

$$\begin{cases} x_\mu \to x'_\mu = x_\mu + \varepsilon_\mu \\ \delta\theta_\nu = \varepsilon_\nu \end{cases} \longrightarrow \frac{\delta x_\mu}{\delta\theta_\nu} = g_\mu^\sigma\frac{\varepsilon_\sigma}{\varepsilon_\nu} = g_\mu^\sigma\delta_\sigma^\nu = g_\mu^\nu$$

$$T^{\mu\nu} = -\left[-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} - \frac{\lambda}{2}(\partial\cdot A)^2\right]g^{\mu\lambda}g_\lambda^\nu - (F^{\mu\rho} + \lambda g^{\mu\rho}\partial\cdot A)\partial^\lambda A_\rho g_\lambda^\nu$$

$\therefore$ 守恒流 $\boxed{T^{\mu\nu} = \left[\frac{1}{4}F^2 + \frac{\lambda}{2}(\partial\cdot A)^2\right]g^{\mu\nu} - \left[F^{\mu\rho} + \lambda g^{\mu\rho}(\partial\cdot A)\right]\partial^\nu A_\rho \text{ 能量动量张量密度}}$

守恒荷 $\boxed{P^\nu = \int d^3x\, T^{0\nu}, \text{ 四动量密度。}}$

由于 $T^{\mu\nu}$ 不满足对 μ, ν 的对称性和规范不变性，常对它作一个相差全散度的变换

$\boxed{T^{\mu\nu} \longrightarrow T^{\mu\nu} + \partial_\rho(F^{\mu\rho}A^\nu), \text{ 此时满足对 } \mu, \nu \text{ 的对称性和规范不变性,而}\text{P}\text{被}}$

2. 无穷小洛伦兹变换

$$\begin{cases} x_\mu \to x'_\mu = x_\mu + \varepsilon_{\mu\nu}x^\nu \\ \delta\theta_a \equiv \delta\theta_{\nu\rho} = \varepsilon_{\nu\rho} \end{cases} \longrightarrow \frac{\delta x_\lambda}{\delta\theta_{\nu\rho}} = \frac{1}{2}(g_\lambda^\nu x^\rho - g_\lambda^\rho x^\nu), \quad \text{另外 } A_\lambda \to A'_\lambda = A_\lambda + \varepsilon_{\lambda\nu}A^\nu$$

$$\delta A_\lambda = \varepsilon_{\lambda\nu}A^\nu$$

$$j^{\mu\nu\rho} = -\left[-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{\lambda}{2}(\partial\cdot A)^2\right]g^{\mu\lambda}\cdot\frac{1}{2}(g_\lambda^\nu x^\rho - g_\lambda^\rho x^\nu) - (F^{\mu\alpha} + \lambda g^{\mu\alpha}\partial\cdot A)\partial^\lambda A_\alpha\cdot\frac{1}{2}(g_\lambda^\nu x^\rho - g_\lambda^\rho x^\nu)$$

守恒流 $\boxed{j^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} + A^\nu(F^{\mu\rho} + \lambda g^{\mu\rho}\partial\cdot A) - A^\rho(F^{\mu\nu} + \lambda g^{\mu\nu}\partial\cdot A) \text{ 广义角动量张量密度}}$

守恒荷 $\boxed{M^{\nu\rho} = \int d^3x\, j^{0\nu\rho} \quad \text{广义角动量}}$

对空间各向同性情况，只考虑空间分量.

守恒流 $\boxed{M^{jk}=\int d^3x\, j^{0jk}=\int d^3x\left[x^jT^{0k}-x^kT^{0j}+A^j(F^{0k}+\lambda g^{0k}\partial\cdot A)-A^k(F^{0j}+\lambda g^{0j}\partial\cdot A)\right]}$

其中 $j,k=1,2,3$，进一步，可引入守恒的角动量.

$$J_i\equiv\frac{1}{2}\varepsilon_{ijk}M^{jk}=\int d^3x\,\frac{1}{2}\varepsilon_{ijk}j^{0jk}$$
$$=\int d^3x\,\frac{1}{2}\varepsilon_{ijk}\left[x^jT^{0k}-x^kT^{0j}+A^j(F^{0k}+\lambda g^{0k}\partial\cdot A)-A^k(F^{0j}+\lambda g^{0j}\partial\cdot A)\right]$$

定义 $J_i=L_i+S_i$.

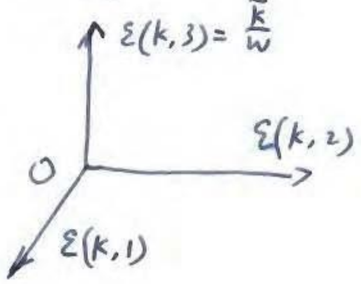
则由 δx_λ 改变引起的轨道角动量

$$\boxed{L_i=\frac{1}{2}\varepsilon_{ijk}\int d^3x\,(x^jT^{0k}-x^kT^{0j})}$$

由 δA_λ 改变引起的电磁场的自旋角动量.

$$\boxed{\begin{aligned}S_i&=\int d^3x\,\frac{1}{2}\varepsilon_{ijk}\left[A^j(F^{0k}+\lambda g^{0k}\partial\cdot A)-A^k(F^{0j}+\lambda g^{0j}\partial\cdot A)\right]\\&=\varepsilon_{ijk}\int d^3x\left[A^j(F^{0k}+\lambda g^{0k}\partial\cdot A)\right]\end{aligned}}$$

四. 电磁场的平面波解.



1. ~~三维极化矢量~~电磁波平面波解.

表示 $\Box A^\mu=0\longrightarrow\begin{cases}A_\mu^+(x)=A_\mu(k)e^{-ikx} & \text{正频解.}\\ A_\mu^-(x)=A_\mu(k)e^{ikx} & \text{负频解.}\end{cases}$

设电磁波频率 ω，波矢 $\vec{k}$，则

$$k^\mu=(\omega,\vec{k}),\quad k_\mu=(\omega,-\vec{k}).$$

由表示 $\Box A^\mu=0$ ~~由洛伦兹条件 $\partial_\mu A^\mu=0$~~，将正.负频解代入，得.

$k^2=0\Longrightarrow\omega^2-|\vec{k}|^2=0\Longrightarrow\omega=|\vec{k}|$，$E=\pm\omega$，能量有正负，在量子化后可消除.

2. 三维极化矢量

为描述光子的极化状态，取三维空间三个正交单位矢量(如上图)。$\varepsilon(\vec{k},3)$ 与 $\vec{k}$ 平行，$\varepsilon(k,1)$，$\varepsilon(k,2)$，$\varepsilon(k,3)$ 成右手螺旋关系，将它们记为 $\varepsilon(k,\lambda)$，($\lambda=1,2,3$)称极化矢量.

正交性 $\varepsilon(k,\lambda)\varepsilon(k,\lambda')=\delta_{\lambda\lambda'}$ $(\lambda,\lambda'=1,2,3)$.

完备性 $\sum_{\lambda=1}^{3}\varepsilon^i(k,\lambda)\varepsilon^j(k,\lambda)=\delta^{ij}$ $(i,j=1,2,3)$.

其中 $\varepsilon(k,\lambda)$ ($\lambda=1,2$)为横向极化矢量，$\varepsilon(k,3)$ 称纵向极化矢量。

库仑规范下.完备性. $\sum_{\lambda=1}^{2}\varepsilon^i(k,\lambda)\varepsilon^j(k,\lambda)=\delta^{ij}-\frac{k^ik^j}{|\vec{k}|^2}$