§4.2 实标量场量子化

一. 经典场的正则量子化

1. 经典标量场

$$\mathcal{L}_\phi = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\dot{\phi}^2 - (\nabla\phi)^2) - \frac{1}{2}m^2\phi^2$$

拉氏方程. $(\Box + m^2)\phi(x) = 0$. 或 $(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\phi(x) = 0$.

2. 正则量子化.

引入正则动量
$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)} = \dot{\phi}(x),$$

哈密顿量
$$H \equiv \int(\pi\dot{\phi} - \mathcal{L})d^3x = \int\left[\pi^2 - \frac{1}{2}(\pi^2 - (\nabla\phi)^2) + \frac{1}{2}m^2\phi^2\right]d^3x$$

$$\therefore H = \int \mathcal{H}\, d^3x, \quad \mathcal{H} = \frac{1}{2}\left[\pi^2(\vec{x},t) + (\nabla\phi(\vec{x},t))^2 + m^2\phi^2(\vec{x},t)\right]$$

假定ϕ, π为厄米算符, 满足正则对易关系

$$\boxed{\begin{aligned}&[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}')\\ &[\phi(\vec{x},t), \phi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0.\end{aligned}}$$

则可推出海森堡方程 (H为哈密顿算符)

$$[H, \phi(\vec{x},t)] = \int d^3\vec{x}'\, \frac{1}{2}\left[\pi^2(\vec{x}',t) + (\nabla'\phi(\vec{x}',t))^2 + m^2\phi^2(\vec{x}',t), \phi(\vec{x},t)\right]$$

$$= \int d^3\vec{x}'\, \pi(\vec{x}',t)[\pi(\vec{x}',t), \phi(\vec{x},t)] = -i\pi(\vec{x},t) = -i\dot{\phi}(\vec{x},t).$$

$$\therefore \dot{\phi}(\vec{x},t) = i[H, \phi(\vec{x},t)]$$

$$[H, \pi(\vec{x},t)] = \int d^3\vec{x}'\, \frac{1}{2}\left[\pi^2(\vec{x}',t) + (\nabla'\phi(\vec{x}',t))^2 + m^2\phi^2(\vec{x}',t), \pi(\vec{x},t)\right]$$

$$= \int d^3\vec{x}'\left\{\nabla'\phi(\vec{x}',t)[\nabla'\phi(\vec{x}',t), \pi(\vec{x},t)] + m^2\phi(\vec{x}',t)[\phi(\vec{x}',t), \pi(\vec{x},t)]\right\}$$

$$= \int d^3\vec{x}'\, \nabla'\phi(\vec{x}',t)\nabla'[\phi(\vec{x}',t), \pi(\vec{x},t)] + \int d^3\vec{x}'\, m^2\phi(\vec{x}',t)\, i\delta(\vec{x}-\vec{x}')$$

$$= \int d^3\vec{x}'\, \nabla'\phi(\vec{x}',t)\nabla'(i\delta^3(\vec{x}'-\vec{x})) + im^2\phi(\vec{x},t)$$

$$= -\nabla\int d^3\vec{x}'\, \nabla'\phi(\vec{x}',t)\, i\delta^3(\vec{x}'-\vec{x}) + im^2\phi(\vec{x},t)$$

$$= -\nabla\, i\nabla\phi(\vec{x},t) + im^2\phi(\vec{x},t) = -i\nabla^2\phi(\vec{x},t) + im^2\phi(\vec{x},t)$$ 利用K-G方程

$$= -i\ddot{\phi}(\vec{x},t) = -i\dot{\pi}(\vec{x},t) \qquad \therefore \dot{\pi}(\vec{x},t) = i[H, \pi(\vec{x},t)]$$

综上，海森堡方程为 $\boxed{\begin{cases}\dot\phi(\vec{x},t)=i[H,\phi(\vec{x},t)]\\ \dot\pi(\vec{x},t)=i[H,\pi(\vec{x},t)]\end{cases}}$

又动量算符

$$\vec{P}=\int d^3x\,T^{0i}=\int d^3x\,(-g^{0i}\mathcal{L}+\partial^0\phi\,\partial^i\phi)\overset{\partial^\mu=(\frac{\partial}{\partial t},-\nabla)}{=}\int d^3x\,(+\dot\phi(\nabla\phi))=\int\pi(\nabla\phi)\,d^3x.$$

则

$$[\vec{P},\phi(\vec{x},t)]=\int d^3\vec{x}'\,[\pi(\vec{x}',t)\nabla'\phi(\vec{x}',t),\phi(\vec{x},t)]=-\int d^3\vec{x}'\,[\pi(\vec{x}',t),\phi(\vec{x},t)]\nabla'\phi(\vec{x}',t)$$

$$=-\int d^3\vec{x}'\,(-i\delta(\vec{x}'-\vec{x}))\nabla'\phi(\vec{x}',t)=i\nabla\phi(\vec{x},t)$$

$$[\vec{P},\pi(\vec{x},t)]=-\int d^3\vec{x}'\,[\pi(\vec{x}',t)\nabla'\phi(\vec{x}',t),\pi(\vec{x},t)]=-\int d^3\vec{x}'\,\pi(\vec{x}',t)\nabla'[\phi(\vec{x}',t),\pi(\vec{x},t)]$$

$$=-\int d^3\vec{x}'\,\pi(\vec{x}',t)\nabla'(i\delta(\vec{x}'-\vec{x}))=i\nabla\int d^3\vec{x}'\,\pi(\vec{x}',t)\delta(\vec{x}'-\vec{x})=i\nabla\pi(\vec{x},t)$$

综上，$\boxed{\begin{cases}\nabla\phi(\vec{x},t)=i[-\vec{P},\phi(\vec{x},t)]\\ \nabla\pi(\vec{x},t)=i[-\vec{P},\pi(\vec{x},t)]\end{cases}}$

考虑到 $P_\mu=i\frac{\partial}{\partial x^\mu}=i\partial_\mu=i(\frac{\partial}{\partial t},\nabla)$，且 $P_\mu=(H,-\vec{P})$，故

$\therefore\ \boxed{\begin{cases}\frac{\partial}{\partial x^\mu}\phi(\vec{x},t)=i[P_\mu,\phi(\vec{x},t)]\\ \frac{\partial}{\partial x^\mu}\pi(\vec{x},t)=i[P_\mu,\pi(\vec{x},t)]\end{cases}}$ 标量场在时空平移下变化性质

可证，上式的解为 $\boxed{\phi(x)=e^{iPx}\phi(0)e^{-iPx},\quad \phi(x+b)=e^{iPb}\phi(x)e^{-iPb}}$

证明：

法一（直接法）：利用 $e^A B e^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots$

将 $\phi(x)$ 在 $x=0$ 附近做泰勒展开

$$\phi(x)=\phi(0)=\frac{\partial\phi(x)}{\partial x^\mu}\Big|_{x^\mu=0}x^\mu+\frac{1}{2!}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\phi(x)\Big|_{\substack{x^\mu=0\\x^\nu=0}}x^\mu x^\nu+\cdots$$

$$=\phi(0)+[iP_\mu x^\mu,\phi(x)]\Big|_{\phi(x)=\phi(0)}+\frac{1}{2!}[iP_\mu x^\mu,[iP_\nu x^\nu,\phi(x)]]\Big|_{\phi(x)=\phi(0)}+\cdots$$

$$=e^{iPx}\phi(0)e^{-iPx}$$

法二（反向检验法）

$$\frac{\partial}{\partial x^\mu}\phi(x)=\partial_\mu\phi(x)=\partial_\mu(e^{iPx}\phi(0)e^{-iPx})=iP_\mu e^{iPx}\phi(0)e^{-iPx}+e^{iPx}\phi(0)e^{-iPx}(-iP_\mu)$$

$$=i(P_\mu\phi(x)-\phi(x)P_\mu)=i[P_\mu,\phi(x)]$$

3.系统的动力学不变量.

Nother定理告诉我们,物理系统具有连续变换下的不变性时,一定存在相应的守恒量。如时间平移不变时具有能量守恒,空间平移不变时具有动量守恒,空间转动不变时具有角动量守恒。因而,对于一个具有四维时空平移不变性的系统,其四维动量P^{μ}守恒;具有正洛伦兹变换下的不变性的系统,其广义角动量$M^{\nu\rho}$守恒。而这些力学量不是别的,正是上述量子化过程中力学量算符;如能量对应于哈密顿量算符,动量对应于动量算符,类似,广义角动量对应于广义角动量算符,四动量对应于四动量算符,

$$\begin{cases} P^{\mu} = \int d^3x\, T^{0\mu} = \int d^3x\,(-g^{0\mu}\mathcal{L} + \partial^0\phi\,\partial^{\mu}\phi), \\ M^{\nu\rho} = \int d^3x\,(x^{\nu}T^{0\rho} - x^{\rho}T^{0\nu}). \end{cases}$$

类似于场量在四维时空平移下的变化,场量在四维时空广义转动下的变化为

$$\begin{cases} P_{\mu} = i\partial_{\mu}, \\ M_{\nu\rho} = i(x_{\nu}\partial_{\rho} - x_{\rho}\partial_{\nu}), \end{cases} \quad \begin{cases} \dfrac{\partial}{\partial x^{\mu}}\phi(\vec{x},t) = i[P_{\mu}, \phi(\vec{x},t)] \\ (x_{\nu}\partial_{\rho} - x_{\rho}\partial_{\nu})\phi(\vec{x},t) = i[M_{\nu\rho}, \phi(\vec{x},t)]. \end{cases} \quad \text{解}\begin{cases} \phi(x) = e^{ipx}\phi(0)e^{-ipx} \\ \end{cases}$$

系统的动力学量P_{μ}, $M_{\nu\rho}$,作为系统相应变换下的守恒量,同时也是相应变换的生成元,因而满足生成元代数关系。而目前,当场量子化后,这些用场来表示的力学量成为了力学量算符,可以证明,此时的力学量算符仍然满足前述力学量生成元代数关系。因此说,我们所进行的量子化正则(包括ϕ,π的量子化假定,海森堡方程)与动力学量生成元P_{μ}, $M_{\nu\rho}$是自洽的,进而正则量子化满足了庞加莱不变性。

$$\underset{(\text{庞氏})}{\text{不变性}} \xleftrightarrow[\text{代数}]{\text{生成元}} \underset{(P_{\mu},M_{\nu\rho})}{\text{动力学量}} \xleftrightarrow[\text{子化}]{\text{场量}} \text{力学量算符} \longleftrightarrow \text{海森堡方程}$$

二. 傅立叶变换/展开

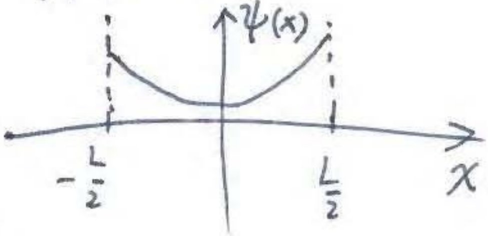
下面，利用在空间 V 内的系统中的场量 $\phi(\vec{x},t)$ 满足的周期性边界条件、厄米性和 Klein-Gordon 方程，获得标量场 $\phi(\vec{x},t)$ 的傅立叶展开形式的解。

1. 波函数周期性边界条件下的平面波解

1). 一维平面波

对一维自由粒子波，$H\psi = k^2\psi$，$-\dfrac{d^2}{dx^2}\psi(x) = k^2\psi(x)$，解 $\psi(x) \sim e^{ikx}$

2) 周期性边界条件



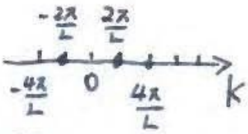
函数周期为 L，$f(x) = f(x+L)$

波函数满足周期为 L，则 $e^{-ik\frac{L}{2}} = e^{ik\frac{L}{2}}$，即 $e^{ikL} = 1$.

$\therefore\ kL = 2n\pi$，$k = n\dfrac{2\pi}{L}$ $(n=0, \pm 1, \pm 2, \cdots)$

3) 本征函数

$\therefore$ 波函数 $\psi(x) \sim e^{ik_n x}$，$k_n = n\dfrac{2\pi}{L}$，x 连续，k_n 分立.



本征值 $H = k^2$：$\left\{0, \left(\frac{2\pi}{L}\right)^2, \left(\frac{4\pi}{L}\right)^2, \left(\frac{6\pi}{L}\right)^2, \cdots\right\}$，有下限无上限

↓ 数学上可证

本征函数 $\left\{ e^{in\frac{2\pi}{L}x} \mid n=0, \pm 1, \pm 2, \cdots \right\}$，正交归一，完备.

① 正交归一：$\boxed{\int_{-L/2}^{L/2} dx\, e^{i(k-k')x} = L\delta_{nn'}}$ $\xrightarrow{\text{三维}}$ $\boxed{\int_{-L/2}^{L/2} d^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = V\delta_{\vec{n}\vec{n}'}}$

证明：
$$\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, e^{ikx} \cdot e^{-ik'x} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, e^{i(k-k')x} = \frac{e^{i(k-k')\frac{L}{2}} - e^{-i(k-k')\frac{L}{2}}}{i(k-k')} = 2\frac{\sin\left[(k-k')\frac{L}{2}\right]}{k-k'}$$

$$\xlongequal{\text{周期边条}} 2\frac{\sin\left[(n-n')\frac{2\pi}{L}\frac{L}{2}\right]}{(n-n')\frac{2\pi}{L}} = L\frac{\sin(n-n')\pi}{(n-n')\pi} = \begin{cases} L & \text{当 } n=n' \\ 0 & \text{当 } n\neq n' \end{cases} = L\delta_{nn'}$$

② 完备性：$\boxed{\sum_{k=-\infty}^{\infty} e^{ik(x-x')} = L\delta(x-x')}$ $\xrightarrow{\text{三维}}$ $\boxed{\sum_{\vec{n}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = V\delta(\vec{x}-\vec{x}')}$ $\displaystyle\lim_{N\to\infty} S_N$

证明：
$$\sum_{k=-\infty}^{\infty} \langle x|k\rangle\langle k|x'\rangle = \sum_{n=-\infty}^{\infty} e^{ikx} e^{-ikx'} = \sum_{n=-\infty}^{\infty} e^{in\frac{2\pi}{L}(x-x')} = \lim_{N\to\infty}\sum_{n=-N}^{N}\left(e^{i\frac{2\pi}{L}(x-x')}\right)^n \equiv \lim_{N\to\infty}\sum_{n=-N}^{N} A^n$$

记 $S_N = A^{-N} + A^{-N+1} + \cdots + A^{-1} + 1 + A + \cdots + A^N$

$A S_N = A^{-N+1} + A^{-N+2} + \cdots + 1 + A + A^2 + \cdots + A^{N+1}$，$\therefore (A-1)S_N = A^{N+1} - A^{-N}$

$$\therefore S_N = \frac{A^{N+1} - A^{-N}}{A-1} = \frac{A^{N+\frac{1}{2}} - A^{-(N+\frac{1}{2})}}{A^{\frac{1}{2}} - A^{-\frac{1}{2}}} = \frac{\left(e^{i\frac{2\pi}{L}(x-x')}\right)^{N+\frac{1}{2}} - \left(e^{i\frac{2\pi}{L}(x-x')}\right)^{-(N+\frac{1}{2})}}{e^{i\frac{\pi}{L}(x-x')} - e^{-i\frac{\pi}{L}(x-x')}} = \frac{\sin\left[(2N+1)\frac{\pi}{L}(x-x')\right]}{\sin\left[\frac{\pi}{L}(x-x')\right]}$$

$$\because \sum_{k=-\infty}^{\infty} e^{ikx} e^{ikx'} = \lim_{N\to\infty} \frac{\pi(x-x')}{\sin\left[\frac{\pi}{L}(x-x')\right]} \frac{\sin\left[(2N+1)\frac{\pi}{L}(x-x')\right]}{\pi(x-x')}$$

讨论：$\frac{\pi(x-x')}{\sin\frac{\pi}{L}(x-x')} = L\frac{\frac{\pi}{L}(x-x')}{\sin\frac{\pi}{L}(x-x')} = \begin{cases} L & x=x' \\ \text{自身有限}, & x\neq x' \end{cases}$

由 $\lim_{W\to\infty}\frac{\sin W(x-x')}{\pi(x-x')} = \delta(x-x')$，得 $\lim_{N\to\infty}\frac{\sin(2N+1)\frac{\pi}{L}(x-x')}{\pi(x-x')} = \delta(x-x') = \begin{cases} \infty & x=x' \\ 0 & x\neq x' \end{cases}$

综上. $\sum_{k=-\infty}^{\infty} e^{ikx} e^{ikx'} = L\,\delta(x-x')$

4). 满足周期性边界条件时. $f(x)$ 可傅立叶展开如下（其实，不满足时也可展开）.

傅氏展开定理 $\begin{cases} f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\frac{2\pi}{L}x}/\sqrt{L} \\ C_n = \frac{1}{\sqrt{L}}\int_{-L/2}^{L/2} dx\; e^{-in\frac{2\pi}{L}x} f(x) \end{cases}$

2. 5) 动量空间的本征函数性质.

上面在坐标空间中考察了满足周期性边界条件时的本征函数 $\{e^{ik_n x}|_{n=0,\pm1,\ldots}\}$ 这些函数在动量上是分立的 $k_n = n\frac{2\pi}{L}$, $(n=0,\pm1,\pm2,\ldots)$，这一方法形式上较简单。另一种方法是令上述空间体积 $V\to\infty$（从而 $L\to\infty$），则此时本征函数的动量间隔 $\Delta k\Rightarrow 0$，从而动量成为连续的。这一方法的优点是，此时动力学力学量能量.动量的形式更为简洁清晰，从而便于显示场的粒子特性。

~~当 $L\to\infty$ 时~~ 动量取值：$k_1 = n_1\frac{2\pi}{L}$， $k_2 = n_2\frac{2\pi}{L}$，$k_3 = n_3\frac{2\pi}{L}$.

动量空间的最小线元 $\Delta k_1 = \frac{2\pi}{L}$，$\Delta k_2 = \frac{2\pi}{L}$，$\Delta k_3 = \frac{2\pi}{L}$.

当 $L\to\infty$ 时，$dk_{1,2,3} = \frac{2\pi}{L}\Big|_{L\to\infty}$，$d^3k = \frac{(2\pi)^3}{V}\Big|_{V\to\infty}$， $\therefore \int d^3k = \frac{(2\pi)^3}{V}\sum_{\vec{k}=-\infty}^{\infty}$

$\therefore$ $\boxed{\text{动量求和变为 } \sum_{\vec{k}} = \int\frac{V d^3k}{(2\pi)^3}}$

利用 $\delta(\vec{k}-\vec{k}')$ 性质：$\int d^3k\,\delta(\vec{k}-\vec{k}') = 1 = \frac{(2\pi)^3}{V}\sum_{\vec{k}}\frac{V}{(2\pi)^3}\delta_{\vec{k}\vec{k}'}$，得 $\delta(\vec{k}-\vec{k}') = \frac{V}{(2\pi)^3}\delta_{\vec{k}\vec{k}'}$

$\therefore$ $\boxed{\text{分立德尔塔函数变为 } \delta_{\vec{k}\vec{k}'} = \frac{(2\pi)^3}{V}\delta(\vec{k}-\vec{k}')}$

动量分立 $\xrightarrow{L\to\infty}$	动量连续
正交归一 $\int_{-L/2}^{L/2} d^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = V\delta_{\vec{k},\vec{k}'}$	正交归一 $\int_{-\infty}^{\infty} d^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = (2\pi)^3\delta(\vec{k}-\vec{k}')$
完备性 $\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = V\delta(\vec{x}-\vec{x}')$	完备性 $\int d^3k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = (2\pi)^3\delta(\vec{x}-\vec{x}')$

3. 场函数满足周期边条，可傅立叶展开如下：

$$\begin{cases}\phi(\vec{x},t)=\sum_{\vec{k}}\frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{x}}q_{\vec{k}}(t)\\ \pi(\vec{x},t)=\sum_{\vec{k}}\frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{x}}P_{-\vec{k}}(t)\end{cases}\quad \text{其中}\begin{cases}q_{\vec{k}}(t)=\frac{1}{\sqrt{V}}\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\phi(\vec{x},t)\\ P_{-\vec{k}}(t)=\frac{1}{\sqrt{V}}\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\pi(\vec{x},t)\end{cases}$$

由场量的厄米性，发现 $q_{\vec{k}}(t)$ 和 $P_{\vec{k}}(t)$ 并不是厄米的.

$$\begin{cases}\phi^{+}(\vec{x},t)=\phi(\vec{x},t)\\ \pi^{+}(\vec{x},t)=\pi(\vec{x},t)\end{cases}\xrightarrow{\text{厄米}}\begin{cases}q^{+}_{\vec{k}}(t)=q_{-\vec{k}}(t)\\ P^{+}_{\vec{k}}(t)=P_{-\vec{k}}(t)\end{cases}\ \text{不厄米}$$

经过傅立叶展开，场量成为动量空间的函数，独立变量为 $\{q_{\vec{k}}(t), P_{\vec{k}}(t)\}$

① 产生、湮灭算符动量~~算子~~空间的表示.（下面为了简单，$q_{\vec{k}}(t)$, $P_{\vec{k}}(t)$ 中的 t 省略）

定义 $a_{\vec{k}}(t)=\sqrt{\frac{\omega}{2}}\left(q_{\vec{k}}(t)+\frac{i}{\omega}P_{-\vec{k}}(t)\right)$，其中 $\omega=\sqrt{\vec{k}^2+m^2}$

得 $a^{+}_{k}(t)=\sqrt{\frac{\omega}{2}}\left(q_{-k}(t)-\frac{i}{\omega}P_{\vec{k}}(t)\right)$

从而有 $\begin{cases}a_{\vec{k}}(t)=\sqrt{\frac{\omega}{2}}\left(q_{\vec{k}}(t)+\frac{i}{\omega}P_{-\vec{k}}(t)\right)\\ a^{+}_{-\vec{k}}(t)=\sqrt{\frac{\omega}{2}}\left(q_{\vec{k}}(t)-\frac{i}{\omega}P_{-\vec{k}}(t)\right)\end{cases}$，反变换 $\begin{cases}q_{\vec{k}}(t)=\frac{1}{\sqrt{2\omega}}\left(a_{\vec{k}}(t)+a^{+}_{-\vec{k}}(t)\right)\\ P_{-\vec{k}}(t)=\frac{-i\omega}{\sqrt{2\omega}}\left(a_{\vec{k}}(t)-a^{+}_{-\vec{k}}(t)\right)\end{cases}$

代入场展开式.

将 $-\vec{k}\to\vec{k}$

$$\begin{cases}\phi(\vec{x},t)=\sum_{\vec{k}=-\infty}^{\infty}\frac{1}{\sqrt{2\omega V}}\left(a_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}}+a^{+}_{\vec{k}}(t)e^{-i\vec{k}\cdot\vec{x}}\right)\\ \pi(\vec{x},t)=\sum_{\vec{k}=-\infty}^{\infty}\frac{-i\omega}{\sqrt{2\omega V}}\left(a_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}}-a^{+}_{\vec{k}}(t)e^{-i\vec{k}\cdot\vec{x}}\right)\end{cases}$$

②满足K-G方程的场量解

根据K-G方程，$(\Box+m^2)\phi(\vec{x},t)=0$，$\Box=\partial_t^2-\nabla^2$，

将 $\phi(\vec{x},t)=\frac{1}{\sqrt{V}}\sum_{\vec{k}}e^{i\vec{k}\cdot\vec{x}}q_{\vec{k}}(t)$ 代入，引入 $w^2=\vec{k}^2+m^2$，

$$\frac{1}{\sqrt{V}}\sum_{\vec{k}}(w^2+\partial_t^2)q_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}}=0.$$

两边乘 $\int_{-\frac{L}{2}}^{+\frac{L}{2}}d^3x\,e^{-i\vec{k}'\cdot\vec{x}}$，利用正交归一性 $\boxed{\text{分立}}$，得

$$\frac{1}{\sqrt{V}}\sum_{\vec{k}}(w^2+\partial_t^2)q_{\vec{k}}(t)V\delta_{\vec{k},\vec{k}'}=0$$

$\therefore\ (w^2+\partial_t^2)q_{\vec{k}}(t)=0$，解为 $q_{\vec{k}}(t)\sim e^{iwt},\ e^{-iwt}$

进一步，结合场算符的产生湮灭算符表示，得 $b_{\vec{k}}(t)=b_{\vec{k}}e^{-iwt}$，$b^+_{\vec{k}}(t)=b^+_{\vec{k}}e^{iwt}$，

$$\therefore\ \boxed{\begin{cases}\phi(\vec{x},t)=\sum_{\vec{k}=-\infty}^{\infty}\frac{1}{\sqrt{2wV}}\left(b_{\vec{k}}e^{-ikx}+b^+_{\vec{k}}e^{+ikx}\right)\\ \pi(\vec{x},t)=\sum_{\vec{k}=-\infty}^{\infty}\frac{-iw}{\sqrt{2wV}}\left(b_{\vec{k}}e^{-ikx}-b^+_{\vec{k}}e^{+ikx}\right)\end{cases}}$$

莫志进

其中 $kx\equiv wt-\vec{k}\cdot\vec{x}$，动量为分立量。$\vec{k}=\vec{k}_n=\vec{n}\frac{2\pi}{L}$，$\vec{n}=0,\pm1,\pm2,\cdots$

~~③下面确定展开式中的不含时振幅算子 $b_{\vec{k}}$，$b^+_{\vec{k}}$~~

~~利用 $L\to\infty$~~：

当 $L\to\infty$ 时，动量由分立 $\to$ 连续，将 $\sum_{\vec{k}}\longrightarrow\int\frac{V}{(2\pi)^3}d^3k$，

$$\phi(\vec{x},t)=\sum_{\vec{k}=-\infty}^{\infty}\frac{1}{\sqrt{2wV}}\left(b_{\vec{k}}e^{-ikx}+b^+_{\vec{k}}e^{ikx}\right)$$

$$\downarrow$$

$$\phi(\vec{x},t)=\int d^3k\frac{V}{(2\pi)^3}\left(\frac{1}{\sqrt{2wV}}b_{\vec{k}}e^{-ikx}+\frac{1}{\sqrt{2wV}}b^+_{\vec{k}}e^{ikx}\right)$$

$$=\int d^3k\frac{1}{2w(2\pi)^3}\left(\sqrt{2wV}\,b_{\vec{k}}e^{-ikx}+\sqrt{2wV}\,b^+_{\vec{k}}e^{ikx}\right)$$

引入 $\boxed{a^{\text{连}}_{\vec{k}}\equiv\sqrt{2wV}\,b^{\text{分}}_{\vec{k}}}$，得（记 $w_{\vec{k}}=w$）

$$\boxed{\begin{aligned}\phi(\vec{x},t)&=\int\frac{d^3k}{2w_{\vec{k}}(2\pi)^3}\left(a_{\vec{k}}e^{-ikx}+a^+_{\vec{k}}e^{ikx}\right)\\ \pi(\vec{x},t)&=\int\frac{d^3k(-iw)}{2w_{\vec{k}}(2\pi)^3}\left(a_{\vec{k}}e^{-ikx}-a^+_{\vec{k}}e^{ikx}\right)\end{aligned}}$$

其中 $a_{\vec{k}}=\sqrt{2wV}\,b_{\vec{k}}$，$\vec{k}=-\infty,\infty$ 动量连续

黄涛

③ 下面,确定展开式中不含时的振幅算子 $\{b_{\vec{k}}, b_{\vec{k}}^{+}\}$ 或 $\{a_{\vec{k}}, a_{\vec{k}}^{+}\}$.

$$b_{\vec{k}}(t) = b_{\vec{k}} e^{-i\omega t} = \sqrt{\frac{\omega}{2}}\left(q_{\vec{k}}(t) + \frac{i}{\omega} p_{-\vec{k}}(t)\right) = \sqrt{\frac{\omega}{2}} \frac{1}{\sqrt{V}} \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \left[\phi(\vec{x},t) + \frac{i}{\omega}\pi(\vec{x},t)\right]$$

$$= \frac{1}{\sqrt{2\omega V}} \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \left[\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right]$$

$$\therefore \begin{cases} b_{\vec{k}} = \frac{1}{\sqrt{2\omega V}} \int d^3x\, e^{ikx} \left[\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right] & \vec{k}\text{ 分立} \\ b_{\vec{k}}^{+} = \frac{1}{\sqrt{2\omega V}} \int d^3x\, e^{-ikx} \left[\omega\phi(\vec{x},t) - i\pi(\vec{x},t)\right] & \end{cases}$$

姜志进

当 $L\to\infty$ 时,

$$\begin{cases} a_{\vec{k}} = \sqrt{2\omega V}\, b_{\vec{k}} = \int d^3x\, e^{ikx} \left[\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right] & \vec{k}\text{ 连续} \\ a_{\vec{k}}^{+} = \sqrt{2\omega V}\, b_{\vec{k}}^{+} = \int d^3x\, e^{-ikx} \left[\omega\phi(\vec{x},t) - i\pi(\vec{x},t)\right] & \end{cases}$$

黄涛

三. 动量空间的场算符

$$\phi(\vec{x},t) = \sum_{\vec{k}=-\infty}^{\infty} \frac{1}{\sqrt{2\omega V}} \left(b_{\vec{k}} e^{-ikx} + b_{\vec{k}}^{+} e^{ikx}\right), \quad \pi(\vec{x},t) = \dot{\phi}(\vec{x},t)$$

$$b_{\vec{k}} = \frac{1}{\sqrt{2\omega V}} \int d^3x\, e^{ikx} \left[\omega\phi(\vec{x},t) + i\pi(\vec{x},t)\right]$$

由于 $\phi(\vec{x},t)$, $\pi(\vec{x},t)$ 可以量子化,故由它们生成的 $b_{\vec{k}}$, $b_{\vec{k}}^{+}$, H, $\vec{p}$ 也可量子化.

可证: $\vec{k}$ 分立

$$\begin{cases} [b_{\vec{k}}, b_{\vec{k}'}^{+}] = \delta_{\vec{k},\vec{k}'} \\ [b_{\vec{k}}, b_{\vec{k}'}] = [b_{\vec{k}}^{+}, b_{\vec{k}'}^{+}] = 0 \\ H = \sum_{\vec{k}} \omega \left[b_{\vec{k}}^{+} b_{\vec{k}} + \frac{1}{2}\right] \\ \vec{p} = \sum_{\vec{k}} \vec{k}\, b_{\vec{k}}^{+} b_{\vec{k}} \end{cases}$$

简单.清晰

$\vec{k}$ 连续

$$\begin{cases} [a_{\vec{k}}, a_{\vec{k}'}^{+}] = (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}') \\ [a_{\vec{k}}, a_{\vec{k}'}] = [a_{\vec{k}}^{+}, a_{\vec{k}'}^{+}] = 0 \\ H = [redacted] \int \frac{d^3k}{(2\pi)^3} \frac{\omega_{\vec{k}}}{2\omega_{\vec{k}}} a_{\vec{k}}^{+} a_{\vec{k}} \\ \vec{p} = \int \frac{d^3k}{(2\pi)^3} \frac{\vec{k}}{2\omega_{\vec{k}}} a_{\vec{k}}^{+} a_{\vec{k}} \end{cases}$$

①本式若无粒子,则总能量为0, ②但由于已经将一个无穷大能量归入真空,故此时的真空能量为正。

引入 $P^{\mu} \equiv \sum_{\vec{k}} p^{\mu} b_{\vec{k}}^{+} b_{\vec{k}}$, $N \equiv \sum_{\vec{k}} b_{\vec{k}}^{+} b_{\vec{k}}$

自由场时 $[P^{\mu}, N] = 0$.

引入 $P^{\mu} \equiv \int \frac{d^3k}{(2\pi)^3} \frac{p^{\mu}}{2\omega_{\vec{k}}} a_{\vec{k}}^{+} a_{\vec{k}}$, $N \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} a_{\vec{k}}^{+} a_{\vec{k}}$

自由场时 $[P^{\mu}, N] = 0$

四.粒子数表象.

场的能量和动量都包含有算符 $b^+_{\vec{k}} b_{\vec{k}}$，故引入 $N_{\vec{k}} = b^+_{\vec{k}} b_{\vec{k}}$，$N=\sum_{\vec{k}} b^+_{\vec{k}} b_{\vec{k}}$

取 $N_{\vec{k}}$ 对角化的表象 $|n_{\vec{k}}\rangle$，N 对角化的表象 $|n_{\vec{k}_1}, n_{\vec{k}_2} \cdots n_{\vec{k}_n} \cdots\rangle$，

$$N_{\vec{k}} |n_{\vec{k}}\rangle = n_{\vec{k}} |n_{\vec{k}}\rangle, \quad N|n_{\vec{k}_1}, n_{\vec{k}_2} \cdots n_{\vec{k}_n} \cdots\rangle = \sum_{\vec{k}} n_{\vec{k}} |n_{\vec{k}_1} \cdots n_{\vec{k}_n}\cdots\rangle \equiv n|n_{\vec{k}_1} \cdots n_{\vec{k}_n}\cdots\rangle$$

则 $|n_{\vec{k}}\rangle$ 是 $N_{\vec{k}}$ 的本征值为 $n_{\vec{k}}$ 的本征态，$|n_{\vec{k}_1}, n_{\vec{k}_2}, \cdots n_{\vec{k}_n} \cdots\rangle$ 是 N 的本征值为 $\sum n_{\vec{k}}$ 的本征态。

上一节的粒子数表象已经表明，由 $b_{\vec{k}}$，$b^+_{\vec{k}}$ 可以产生粒子数表象：

1) 真空态　　$b_{\vec{k}} |0\rangle = 0$，　$\langle 0|0\rangle = 1$，（$\vec{k}$ 取所有分立动量）

2) 本征态　　$b^+_{\vec{k}} |0\rangle$　一个4动量为 k（或能量为 ω，动量为 $\vec{k}$ 的单粒子态）

$b^{+2}_{\vec{k}} |0\rangle$　两个4动量均为 k 的态 $\boxed{\text{双粒子}}$

$b^+_{k_1} b^+_{k_2} |0\rangle$　一个4动量为 k_1、另一个4动量为 k_2 的双粒子态.

⋮

$b^+_{k_1} b^+_{k_2} \cdots b^+_{k_n} |0\rangle$　4动量分别为 $k_1, k_2, \cdots k_n$ 的 n 粒子态

⋮

这些态构成 Hilbert 空间的基矢，称为 Fock 态。

这些态既是粒子数算符 $N=\sum_{\vec{k}} b^+_{\vec{k}} b_{\vec{k}}$ 的本征态，（由于 $[N, P^\mu]=0$）又是4动量 P^μ 的本征态。

3) 由于 N 的本征值非负，因此 $n_{\vec{k}_1}, n_{\vec{k}_2} \cdots n_{\vec{k}_n} \cdots = 0, 1, 2, \cdots$

从而 $E = \sum_{\vec{k}} \omega_{\vec{k}} (n_{\vec{k}} + \frac{1}{2})$ 非负，即能量 H 的本征值非负，故量子化后 K-G 场的负能量问题不复存在。不仅如此，对一个粒子也不存在的真空态，尽管场的动量为 0，但是其能量为 $\sum_{\vec{k}} \frac{1}{2} \omega_{\vec{k}} \neq 0$，这表明真空储存有能量，且是正的。真空具有能量，是量子场的一个基本属性。

4). 正交归一性.

记 $|k\rangle \equiv a_{\vec{k}} |0\rangle$，

$$\langle \vec{k}' | \vec{k}\rangle = \delta_{\vec{k}'\vec{k}}, \cdots, \langle \vec{k}_1' \vec{k}_2' \cdots \vec{k}_n' \cdots | \vec{k}_1 \vec{k}_2 \cdots \vec{k}_n \cdots\rangle = \delta_{\vec{k}_1'\vec{k}_1} \delta_{\vec{k}_2'\vec{k}_2} \cdots \delta_{\vec{k}_n\vec{k}_n'} \cdots$$

5) 完备性

$$\sum_{\vec{k}} |\vec{k}\rangle\langle\vec{k}| = 1, \cdots, \sum_{\vec{k}_1, \vec{k}_2, \cdots \vec{k}_n \cdots} |\vec{k}_1 \vec{k}_2 \cdots \vec{k}_n \cdots\rangle\langle \vec{k}_1 \vec{k}_2 \cdots \vec{k}_n \cdots| = 1$$