# Random Calculations

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# 1 | Exercise 1-2-3-4 Solutions [Baumann]

## 1.1 Thermal Distributions

### 1.1.1 Relativistic Limit (part a)

$T \gg m$ and $T \gg |\mu|$,

$$\begin{aligned}
n &= \frac{g}{(2\pi)^3}\int d^3p f(p) && (1.1)\\
&= \frac{g}{2\pi^2}\int_0^\infty dp p^2 \frac{1}{e^{(E-\mu)/T}\pm 1} && (1.2)\\
&= \{\xi = p/T \quad x = m/T\} && (1.3)\\
&= \left\{E = \left(p^2+m^2\right)^{1/2} = T\left(\xi^2+x^2\right)^{1/2} = \{x \ \to \ 0\} \approx T\xi\right\} && (1.4)\\
&= \frac{g}{2\pi^2}\int_0^\infty T d\xi \xi^2 T^2 \frac{1}{e^{\xi}\pm 1} && (1.5)\\
&= \frac{gT^3}{2\pi^2}\int_0^\infty d\xi \frac{\xi^2}{e^{\xi}\pm 1} && (1.6)
\end{aligned}$$

For bosons,

$$n = \frac{gT^3}{2\pi^2}\int_0^\infty d\xi \frac{\xi^2}{e^{\xi}-1} = \frac{gT^3}{2\pi^2}\zeta(3)\,\Gamma(3) = \frac{gT^3}{\pi^2}\zeta(3) \qquad (1.7)$$

For fermions,

$$\begin{aligned}
n &= \frac{gT^3}{2\pi^2}\int_0^\infty d\xi \frac{\xi^2}{e^{\xi}+1} && (1.8)\\
&= \frac{gT^3}{2\pi^2}\int_0^\infty d\xi \left(\frac{\xi^2}{e^{\xi}-1} - \frac{2\xi^2}{e^{2\xi}-1}\right) && (1.9)\\
&= \frac{gT^3}{2\pi^2}2\zeta(3) - \frac{gT^3}{2\pi^2}\frac{1}{4}2\zeta(3) && (1.10)\\
&= \frac{gT^3}{\pi^2}\zeta(3)\,\frac{3}{4} && (1.11)
\end{aligned}$$

where I used the $2\xi = u$ transformation while calculating the second integral. Therefore, we have,

$$n = \frac{gT^3}{\pi^2}\zeta(3)\begin{cases} 1 & \text{bosons} \\ \frac{3}{4} & \text{fermions} \end{cases} \qquad (1.12)$$

$$\rho = \frac{g}{2\pi^2}\int_0^\infty dp p^2 f\left(p\right) E\left(p\right) \tag{1.13}$$

$$= \frac{g}{2\pi^2}\int_0^\infty dp p^3 \frac{1}{e^{p/T} \pm 1} \tag{1.14}$$

$$= \frac{gT^4}{2\pi^2}\int_0^\infty d\xi \frac{\xi^3}{e^{\xi} \pm 1} \tag{1.15}$$

For bosons,

$$\rho = \frac{gT^4}{2\pi^2}\int_0^\infty d\xi \frac{\xi^3}{e^{\xi} - 1} = \tag{1.16}$$

## 2 | Variation of Einstein-Hilbert Action

Einstein-Hilbert action,

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R \text{ where } \kappa = \frac{8\pi G}{c^4} \tag{2.1}$$

Taking the variation of the action with respect to the inverse metric $g^{\mu\nu}$,

$$\begin{aligned} \delta S_{EH} &= \frac{1}{2\kappa} \int d^4x \frac{\delta\left(\sqrt{-g}R\right)}{\delta g^{\mu\nu}} \\ &= \frac{1}{2\kappa} \int d^4x \left( \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} R + \frac{\delta R}{\delta g^{\mu\nu}} \sqrt{-g} \right) \end{aligned} \tag{2.2}$$

First term in the paranthesis in the Eq. 2.2,

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}} \delta g \\ &= \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} \end{aligned} \tag{2.3}$$

where we used the Jacobi's formula,

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} \tag{2.4}$$

To transform the $g^{\mu\nu}\delta g_{\mu\nu}$ in the above formula to $g_{\mu\nu}\delta g^{\mu\nu}$, we can do some index manipulation as,

$$\begin{aligned} g^{\mu\nu}\delta g_{\mu\nu} &= g^{\mu\nu}\delta\left(g_{\mu\alpha}g_{\nu\beta}g^{\alpha\beta}\right) \\ &= g^{\mu\nu}\left[g_{\nu\beta}g^{\alpha\beta}\delta g_{\mu\alpha} + g_{\mu\alpha}g^{\alpha\beta}\delta g_{\nu\beta} + g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}\right] \\ &= g^{\mu\nu}\left[\delta^\alpha_\nu \delta g_{\mu\alpha} + \delta^\beta_\mu \delta g_{\nu\beta} + g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}\right] \\ &= g^{\mu\nu}\delta^\alpha_\nu \delta g_{\mu\alpha} + g^{\mu\nu}\delta^\beta_\mu \delta g_{\nu\beta} + g^{\mu\nu}g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta} \\ &= g^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu} + \delta^\nu_\alpha g_{\nu\beta}\delta g^{\alpha\beta} \\ &= g^{\mu\nu}\delta g_{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu} + g_{\mu\nu}\delta g^{\mu\nu} \end{aligned} \tag{2.5}$$

Therefore, we have

$$g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu} \tag{2.6}$$

Inserting the above result to Eq. 2.4,

$$\delta g = -g g_{\mu\nu} \delta g^{\mu\nu} \tag{2.7}$$

Now we can write the first term in Eq. 2.2 by using the Eq. 2.3 and Eq. **??** as,

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \tag{2.8}$$

Second term in Eq. 2.2 which contains the variation of Ricci scalar with respect to inverse metric can be calculated from the variation of Riemann tensor as,

$$\begin{aligned}\delta R^{\rho}{}_{\sigma\mu\nu} &= \delta\left(\partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\sigma\mu}\right)\\ &= \partial_{\mu}\left(\delta\Gamma^{\rho}{}_{\sigma\nu}\right) - \partial_{\nu}\left(\delta\Gamma^{\rho}{}_{\sigma\mu}\right) + \Gamma^{\lambda}{}_{\sigma\nu}\delta\Gamma^{\rho}{}_{\mu\nu} + \Gamma^{\rho}{}_{\mu\lambda}\delta\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\lambda}{}_{\sigma\mu}\delta\Gamma^{\rho}{}_{\nu\lambda} - \Gamma^{\rho}{}_{\nu\lambda}\delta\Gamma^{\lambda}{}_{\sigma\mu}\end{aligned} \tag{2.9}$$

Covariant derivative of the $\delta\Gamma^{\rho}{}_{\sigma\nu}$,

$$\begin{aligned}\nabla_{\mu}\left(\delta\Gamma^{\rho}{}_{\sigma\nu}\right) &= \partial_{\mu}\delta\Gamma^{\rho}{}_{\sigma\nu} + \Gamma^{\rho}{}_{\mu\lambda}\delta\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\lambda}{}_{\mu\sigma}\delta\Gamma^{\rho}{}_{\lambda\nu} - \Gamma^{\lambda}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\lambda\sigma}\\ \nabla_{\mu}\left(\delta\Gamma^{\rho}{}_{\sigma\nu}\right) + \Gamma^{\lambda}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\lambda\sigma} &= \partial_{\mu}\delta\Gamma^{\rho}{}_{\sigma\nu} + \Gamma^{\rho}{}_{\mu\lambda}\delta\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\lambda}{}_{\mu\sigma}\delta\Gamma^{\rho}{}_{\lambda\nu}\end{aligned} \tag{2.10}$$

Covariant derivative of the $\delta\Gamma^{\rho}{}_{\sigma\mu}$,

$$\begin{aligned}\nabla_{\nu}\left(\delta\Gamma^{\rho}{}_{\sigma\mu}\right) &= \partial_{\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\nu\lambda}\delta\Gamma^{\lambda}{}_{\sigma\mu} - \Gamma^{\lambda}{}_{\nu\sigma}\delta\Gamma^{\rho}{}_{\lambda\mu} - \Gamma^{\lambda}{}_{\nu\mu}\delta\Gamma^{\rho}{}_{\lambda\sigma}\\ -\partial_{\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\lambda}{}_{\nu\sigma}\delta\Gamma^{\rho}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\nu\lambda}\delta\Gamma^{\lambda}_{\sigma\mu} &= -\nabla_{\nu}\left(\delta\Gamma^{\rho}{}_{\sigma\mu}\right) - \Gamma^{\lambda}{}_{\nu\mu}\Gamma^{\rho}{}_{\lambda\sigma}\end{aligned} \tag{2.11}$$

Inserting the rearranged covariant derivatives into the Eq. 2.9 we have the variation of Riemann tensor as,

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \nabla_{\mu}\left(\delta\Gamma^{\rho}{}_{\sigma\nu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\rho}{}_{\sigma\mu}\right) \tag{2.12}$$

Contaction of the first and third indices gives us the variation of the Ricci tensor as,

$$\delta R^{\mu}{}_{\sigma\mu\nu} = \delta R_{\sigma\nu} = \nabla_{\mu}\left(\delta\Gamma^{\mu}{}_{\sigma\nu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\mu}{}_{\sigma\mu}\right) \tag{2.13}$$

Variation of Ricci scalar is therefore,

$$\begin{aligned}\delta\left(g^{\sigma\nu}R_{\sigma\nu}\right) = \delta R &= R_{\sigma\nu}\delta g^{\sigma\nu} + g^{\sigma\nu}\delta R_{\sigma\nu}\\ &= R_{\sigma\nu}\delta g^{\sigma\nu} + g^{\sigma\nu}\left(\nabla_{\mu}\left(\delta\Gamma^{\mu}{}_{\sigma\nu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\mu}{}_{\sigma\mu}\right)\right)\\ &= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left(g^{\sigma\nu}\delta\Gamma^{\mu}{}_{\sigma\nu}\right) - \nabla_{\nu}\left(g^{\sigma\nu}\delta\Gamma^{\mu}{}_{\sigma\mu}\right)\\ &= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left(g^{\sigma\nu}\delta\Gamma^{\mu}{}_{\sigma\nu}\right) - \nabla_{\mu}\left(g^{\sigma\mu}\delta\Gamma^{\nu}{}_{\sigma\nu}\right)\\ &= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left(g^{\sigma\nu}\delta\Gamma^{\mu}{}_{\sigma\nu} - g^{\sigma\mu}\delta\Gamma^{\nu}{}_{\sigma\nu}\right)\end{aligned} \tag{2.14}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} \tag{2.15}$$

From the integration covariant derivative will not be contributing the variation therefore now we can write the second term in Eq. 2.2 as,

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \tag{2.16}$$

Combining the two terms we calculated in Eq. 2.8 and Eq. 2.16 we have the variation of the

Einstein-Hilbert action as,

$$\begin{aligned}\delta S_{EH} &= \frac{1}{2\kappa}\int d^4x\left(\frac{1}{2}\sqrt{-g}g_{\mu\nu}R + R_{\mu\nu}\sqrt{-g}\right)\\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\end{aligned} \tag{2.17}$$

where the term in the paranthesis is the Einstein tensor $G_{\mu\nu}$,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{2.18}$$

# 3 | Variation of Relativistic Point Particle Action

Relativistic point particle action,

$$S_{PP} = -m \int ds = -m \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau \tag{3.1}$$

Taking the variation of the action above,

$$\begin{aligned}
&\delta S_{PP} = \\
&= -m \int d\tau \delta \left( \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \right) = 0 \\
&= -m \int d\tau \frac{1}{2} \frac{1}{\sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}} \delta \left( g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) \\
&= -m \int d\tau \frac{1}{2} \frac{1}{\sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}} \left( \delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{d\delta x^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d\delta x^\nu}{d\tau} \right) \\
&= -\frac{m}{2} \int \frac{d\tau}{\sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}} \left( \delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + 2 g_{\mu\nu} \frac{d\delta x^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) \\
&= -\frac{m}{2} \int \frac{d\tau}{\sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}} \left( \delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + 2 \left[ \cancelto{0}{\frac{d}{d\tau} \left( g_{\mu\nu} \frac{dx^\nu}{d\tau} \delta x^\mu \right)} - \frac{dg_{\mu\nu}}{d\tau} \frac{dx^\nu}{d\tau} \delta x^\mu - g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} \delta x^\mu \right] \right) \\
&= -\frac{m}{2} \int \frac{d\tau}{\sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}} \left( \delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - 2 \partial_\alpha g_{\mu\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} \delta x^\mu - g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} \delta x^\mu \right) \\
&= -\frac{m}{2} \int \frac{d\tau}{\sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}} \left( \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \partial_\alpha g_{\mu\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} - \partial_\nu g_{\mu\alpha} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} - g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} \right) \delta x^\mu \\
&= -\frac{m}{2} \int \frac{d\tau}{\sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}} \left( \partial_\mu g_{\alpha\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} - \partial_\alpha g_{\mu\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} - \partial_\nu g_{\mu\alpha} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} - g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} \right) \delta x^\mu
\end{aligned} \tag{3.2}$$

Since the right hand side is zero, the term in the paranthesis is must be zero. Multiplying the term with $-\frac{1}{2} g^{\mu\beta}$ we have,

$$\partial_\mu g_{\alpha\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} - \partial_\alpha g_{\mu\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} - \partial_\nu g_{\mu\alpha} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} - g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} = 0$$

$$\frac{d^2 x^\beta}{d\tau^2} = -\frac{1}{2} g^{\mu\beta} \left( \partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu} \right) \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} \tag{3.3}$$

$$\frac{d^2 x^\beta}{d\tau^2} = -\Gamma^\beta{}_{\alpha\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau}$$

# 4 | Variation of Scalar Field Action

Scalar field action,

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L} \tag{4.1}$$

where the Lagrangian of the scalar field is,

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \tag{4.2}$$

Taking the variation of the action with respect to the inverse metric $g^{\mu\nu}$,

$$\begin{aligned}
\delta S_\phi &= \int d^4x \frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta g^{\mu\nu}} \\
&= \int d^4x \left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\mathcal{L} + \sqrt{-g}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\right) \\
&= \int d^4x \left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\mathcal{L}\delta g^{\mu\nu} + \sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\delta g^{\mu\nu}\right)\right) \\
&= \int d^4x\sqrt{-g}\left(-\frac{1}{2}\right)\left(\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}\right)
\end{aligned} \tag{4.3}$$

where the term in the paranthesis is the energy-momentum tensor of the scalar field,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L} \tag{4.4}$$

Taking the variation of the action with respect to the scalar field $\phi$ by,

$$\begin{aligned}
\phi &\rightarrow \phi + \delta\phi \\
S &\rightarrow S + \delta S
\end{aligned} \tag{4.5}$$

we have,

$$\begin{aligned}
\cancelto{}{S} + \delta S &= \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\left(\phi+\delta\phi\right)\partial_\nu\left(\phi+\delta\phi\right) - V\left(\phi+\delta\phi\right)\right) \\
&= \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi + \partial_\mu\delta\phi\right)\left(\partial_\nu\phi + \partial_\nu\delta\phi\right) - V(\phi) - V'(\phi)\,\delta\phi\right) \\
&= \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\delta\phi - \frac{1}{2}\partial_\mu\delta\phi\partial_\nu\phi - V'(\phi)\,\delta\phi\right) \\
&= \cancelto{}{S} + \int d^4x\sqrt{-g}\left(-g^{\mu\nu}\partial_\mu\phi\partial_\nu\delta\phi - V'(\phi)\,\delta\phi\right) \\
&= \int d^4x\sqrt{-g}\left(\cancelto{0}{-\partial_\nu\left(g^{\mu\nu}\partial_\mu\phi\delta\phi\right)} + \cancelto{0}{\partial_\nu g^{\mu\nu}\partial_\mu\phi\delta\phi} + g^{\mu\nu}\partial_\mu\partial_\nu\phi\delta\phi - V'(\phi)\,\delta\phi\right) \\
&= \int d^4x\sqrt{-g}\left(\Box\phi - V'(\phi)\right)\delta\phi
\end{aligned} \tag{4.6}$$

# 5 | Variation of Scalar Field Coupled EH Action

The action is given as,

$$S = \int d^4x \sqrt{-g}\mathcal{L} \tag{5.1}$$

where $\mathcal{L}$,

$$\mathcal{L} = f(\phi) R - \frac{h(\phi)}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \tag{5.2}$$

Taking the variation wrt inverse metric $g^{\mu\nu}$,

$$\begin{aligned}
\delta S &= \int d^4x \frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta g^{\mu\nu}} \delta g^{\mu\nu} && (5.3)\\
&= \int d^4x \left\{ \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \sqrt{-g} \right\} \delta g^{\mu\nu} && (5.4)
\end{aligned}$$

From above variation we already know the first term as,

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \tag{5.5}$$

Second term can be calculated as,

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} &= \frac{\delta\left(f(\phi) R - \frac{h(\phi)}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)\right)}{\delta g^{\mu\nu}} && (5.6)\\
&= f(\phi) \frac{\delta R}{\delta g^{\mu\nu}} - \frac{h(\phi)}{2} \partial_\mu \phi \partial_\nu \phi \frac{\delta g^{\mu\nu}}{\delta g^{\mu\nu}} && (5.7)\\
& && (5.8)
\end{aligned}$$

Variation of the Ricci scalar can be calculated as,

$$\begin{aligned}
\delta R^\rho{}_{\sigma\mu\nu} &= \delta\left(\partial_\mu \Gamma^\rho{}_{\sigma\nu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\sigma\mu}\right) && (5.9)\\
&= \partial_\mu \delta\Gamma^\rho{}_{\sigma\nu} - \partial_\nu \delta\Gamma^\rho{}_{\sigma\mu} + \Gamma^\lambda{}_{\sigma\nu}\delta\Gamma^\rho{}_{\mu\lambda} + \Gamma^\rho{}_{\mu\lambda}\delta\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\lambda{}_{\sigma\mu}\delta\Gamma^\rho{}_{\nu\lambda} - \Gamma^\rho{}_{\nu\lambda}\delta\Gamma^\lambda{}_{\sigma\mu} && (5.10)\\
& && (5.11)
\end{aligned}$$

Covariant derivative of the $\delta\Gamma^\rho{}_{\sigma\nu}$,

$$\begin{aligned}
\nabla_\mu\left(\delta\Gamma^\rho{}_{\sigma\nu}\right) &= \partial_\mu \delta\Gamma^\rho{}_{\sigma\nu} + \Gamma^\rho{}_{\mu\lambda}\delta\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\lambda{}_{\mu\sigma}\delta\Gamma^\rho{}_{\lambda\nu} - \Gamma^\lambda{}_{\mu\nu}\delta\Gamma^\rho{}_{\lambda\sigma}\\
\nabla_\mu\left(\delta\Gamma^\rho{}_{\sigma\nu}\right) + \Gamma^\lambda{}_{\mu\nu}\delta\Gamma^\rho{}_{\lambda\sigma} &= \partial_\mu \delta\Gamma^\rho{}_{\sigma\nu} + \Gamma^\rho{}_{\mu\lambda}\delta\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\lambda{}_{\mu\sigma}\delta\Gamma^\rho{}_{\lambda\nu}
\end{aligned} \tag{5.12}$$

Covariant derivative of the $\delta\Gamma^\rho{}_{\sigma\mu}$,

$$\begin{aligned}
\nabla_\nu\left(\delta\Gamma^\rho{}_{\sigma\mu}\right) &= \partial_\nu \delta\Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\nu\lambda}\delta\Gamma^\lambda{}_{\sigma\mu} - \Gamma^\lambda{}_{\nu\sigma}\delta\Gamma^\rho{}_{\lambda\mu} - \Gamma^\lambda{}_{\nu\mu}\delta\Gamma^\rho{}_{\lambda\sigma}\\
-\partial_\nu \delta\Gamma^\rho{}_{\sigma\mu} + \Gamma^\lambda{}_{\nu\sigma}\delta\Gamma^\rho{}_{\lambda\mu} - \Gamma^\rho{}_{\nu\lambda}\delta\Gamma^\lambda_{\sigma\mu} &= -\nabla_\nu\left(\delta\Gamma^\rho{}_{\sigma\mu}\right) - \Gamma^\lambda{}_{\nu\mu}\Gamma^\rho{}_{\lambda\sigma}
\end{aligned} \tag{5.13}$$

Inserting the rearranged covariant derivatives into the Eq. 2.9 we have the variation of Riemann tensor as,

$$\delta R^{\rho}{}_{\sigma\mu\nu} = \nabla_{\mu}\left(\delta\Gamma^{\rho}{}_{\sigma\nu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\rho}{}_{\sigma\mu}\right) \tag{5.14}$$

Contaction of the first and third indices gives us the variation of the Ricci tensor as,

$$\delta R^{\mu}{}_{\sigma\mu\nu} = \delta R_{\sigma\nu} = \nabla_{\mu}\left(\delta\Gamma^{\mu}{}_{\sigma\nu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\mu}{}_{\sigma\mu}\right) \tag{5.15}$$

Variation of Ricci scalar is therefore,

$$\begin{aligned}
\delta\left(g^{\sigma\nu}R_{\sigma\nu}\right) = \delta R &= R_{\sigma\nu}\delta g^{\sigma\nu} + g^{\sigma\nu}\delta R_{\sigma\nu} \\
&= R_{\sigma\nu}\delta g^{\sigma\nu} + g^{\sigma\nu}\left(\nabla_{\mu}\left(\delta\Gamma^{\mu}{}_{\sigma\nu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\mu}{}_{\sigma\mu}\right)\right) \\
&= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left(g^{\sigma\nu}\delta\Gamma^{\mu}{}_{\sigma\nu}\right) - \nabla_{\nu}\left(g^{\sigma\nu}\delta\Gamma^{\mu}{}_{\sigma\mu}\right) \\
&= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left(g^{\sigma\nu}\delta\Gamma^{\mu}{}_{\sigma\nu}\right) - \nabla_{\mu}\left(g^{\sigma\mu}\delta\Gamma^{\nu}{}_{\sigma\nu}\right) \\
&= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left(g^{\sigma\nu}\delta\Gamma^{\mu}{}_{\sigma\nu} - g^{\sigma\mu}\delta\Gamma^{\nu}{}_{\sigma\nu}\right)
\end{aligned} \tag{5.16}$$

Variation of the connection can be calculated as,

$$\delta\Gamma^{\sigma}{}_{\mu\nu} = -\frac{1}{2}\left(g_{\lambda\mu}\nabla_{\nu}\left(\delta g^{\lambda\sigma}\right) + g_{\lambda\nu}\nabla_{\mu}\left(\delta g^{\lambda\sigma}\right) - g_{\mu\alpha}g_{\nu\beta}\nabla^{\sigma}\left(\delta g^{\alpha\beta}\right)\right) \tag{5.17}$$

Inserting the variation of the connection into the variation of Ricci scalar we have,

$$\begin{aligned}
\delta R &= R_{\sigma\nu}\delta g^{\sigma\nu} \\
&+ \nabla_{\mu}\left[g^{\sigma\nu}\left(-\frac{1}{2}\left(g_{\lambda\sigma}\nabla_{\nu}\left(\delta g^{\lambda\mu}\right) + g_{\lambda\nu}\nabla_{\sigma}\left(\delta g^{\lambda\mu}\right) - g_{\sigma\alpha}g_{\nu\beta}\nabla^{\mu}\left(\delta g^{\alpha\beta}\right)\right)\right)\right] \\
&- \nabla_{\mu}\left[g^{\sigma\mu}\left(-\frac{1}{2}\left(g_{\lambda\sigma}\nabla_{\nu}\left(\delta g^{\lambda\nu}\right) + g_{\lambda\nu}\nabla_{\sigma}\left(\delta g^{\lambda\nu}\right) - g_{\sigma\alpha}g_{\nu\beta}\nabla^{\nu}\left(\delta g^{\alpha\beta}\right)\right)\right)\right] \\
&= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left[-\frac{1}{2}\left(g^{\sigma\nu}g_{\lambda\sigma}\nabla_{\nu}\left(\delta g^{\lambda\mu}\right) + g^{\sigma\nu}g_{\lambda\nu}\nabla_{\sigma}\left(\delta g^{\lambda\mu}\right) - g^{\sigma\nu}g_{\sigma\alpha}g_{\nu\beta}\nabla^{\mu}\left(\delta g^{\alpha\beta}\right)\right)\right] \\
&+ \nabla_{\mu}\left[-\frac{1}{2}\left(-g^{\sigma\mu}g_{\lambda\sigma}\nabla_{\nu}\left(\delta g^{\lambda\nu}\right) - g^{\sigma\mu}g_{\lambda\nu}\nabla_{\sigma}\left(\delta g^{\lambda\nu}\right) + g^{\sigma\mu}g_{\sigma\alpha}g_{\nu\beta}\nabla^{\nu}\left(\delta g^{\alpha\beta}\right)\right)\right] \\
&= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left[-\frac{1}{2}\left(\delta^{\nu}_{\lambda}\nabla_{\nu}\left(\delta g^{\lambda\mu}\right) + \delta^{\sigma}_{\lambda}\nabla_{\sigma}\left(\delta g^{\lambda\mu}\right) - \delta^{\nu}_{\alpha}g_{\nu\beta}\nabla^{\mu}\left(\delta g^{\alpha\beta}\right)\right)\right] \\
&+ \nabla_{\mu}\left[-\frac{1}{2}\left(-\delta^{\mu}_{\lambda}\nabla_{\nu}\left(\delta g^{\lambda\nu}\right) - g^{\sigma\mu}g_{\lambda\nu}\nabla_{\sigma}\left(\delta g^{\lambda\nu}\right) + \delta^{\mu}_{\alpha}g_{\nu\beta}\nabla^{\nu}\left(\delta g^{\alpha\beta}\right)\right)\right] \\
&= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left[-\frac{1}{2}\left(\nabla_{\nu}\left(\delta g^{\mu\nu}\right) + \nabla_{\sigma}\left(\delta g^{\sigma\mu}\right) - g_{\alpha\beta}\nabla^{\mu}\left(\delta g^{\alpha\beta}\right)\right)\right] \\
&+ \nabla_{\mu}\left[-\frac{1}{2}\left(-\nabla_{\nu}\left(\delta g^{\mu\nu}\right) - g^{\sigma\mu}g_{\lambda\nu}\nabla_{\sigma}\left(\delta g^{\lambda\nu}\right) + g_{\nu\beta}\nabla^{\nu}\left(\delta g^{\mu\beta}\right)\right)\right] \\
&= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left[-\frac{1}{2}\nabla_{\sigma}\left(\delta g^{\sigma\mu}\right) + \frac{1}{2}g_{\alpha\beta}\nabla^{\mu}\left(\delta g^{\alpha\beta}\right) + \frac{1}{2}g_{\lambda\nu}\nabla^{\mu}\left(\delta g^{\lambda\nu}\right) - \frac{1}{2}\nabla_{\beta}\left(\delta g^{\mu\beta}\right)\right] \\
&= R_{\sigma\nu}\delta g^{\sigma\nu} + \nabla_{\mu}\left[g_{\alpha\beta}\nabla^{\mu}\left(\delta g^{\alpha\beta}\right) - \nabla_{\sigma}\left(\delta g^{\mu\sigma}\right)\right] \\
&= \boxed{R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\nabla_{\lambda}\nabla^{\lambda}\delta g^{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\delta g^{\mu\nu}}
\end{aligned}$$

$$\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} = f\left(\phi\right) R_{\mu\nu} + g_{\mu\nu} \Box f\left(\phi\right) - \nabla_\mu \nabla_\nu f\left(\phi\right) - \frac{h\left(\phi\right)}{2} \partial_\mu \phi \partial_\nu \phi$$

Now putting all back to the variation of the action wrt inverse metric expression above, we have,

$$\boxed{\delta S = \int d^4x \sqrt{-g} \left[ f\left(\phi\right) \left\{ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right\} + g_{\mu\nu} \Box f\left(\phi\right) - \nabla_\mu \nabla_\nu f\left(\phi\right) + \frac{h\left(\phi\right)}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} V\left(\phi\right) \right] \delta g^{\mu\nu}} \tag{5.18}$$

Taking the variation wrt scalar field $\phi$,

$$\phi \rightarrow \phi + \delta\phi \tag{5.19}$$

$$S \rightarrow S + \delta S \tag{5.20}$$

$$\cancel{S} + \delta S = \int d^4x \sqrt{-g} \left( f\left(\phi + \delta\phi\right) R - \frac{h\left(\phi + \delta\phi\right)}{2} g^{\mu\nu} \partial_\mu \left(\phi + \delta\phi\right) \delta_\nu \left(\phi + \delta\phi\right) - V\left(\phi + \delta\phi\right) \right) \tag{5.21}$$

$$= \int d^4x \sqrt{-g} \left\{ \left(f\left(\phi\right) + f'\left(\phi\right)\delta\phi\right) R - \frac{1}{2} g^{\mu\nu} \left(h\left(\phi\right) + h'\left(\phi\right)\delta\phi\right) \left(\partial_\mu\phi + \partial_\mu \delta\phi\right) \left(\partial_\nu \phi + \partial_\nu \delta\phi\right) - V\left(\phi\right) - V'\left(\phi\right)\delta\phi \right\} \tag{5.22}$$

$$= \int d^4x \sqrt{-g} \left( \left( R f\left(\phi\right) - \frac{1}{2} g^{\mu\nu} h\left(\phi\right) \partial_\mu \phi \partial_\nu \phi - V\left(\phi\right) \right) \right. \tag{5.23}$$

$$\left. + R f'\left(\phi\right) \delta\phi - g^{\mu\nu} h\left(\phi\right) \partial_\mu \phi \partial_\nu \delta\phi - \frac{1}{2} g^{\mu\nu} h'\left(\phi\right) \partial_\mu \phi \partial_\nu \phi \delta\phi - V'\left(\phi\right) \delta\phi \right) \tag{5.24}$$

$$\boxed{\delta S = \int d^4x \sqrt{-g} \left\{ R f'\left(\phi\right) - g^{\mu\nu} \partial_\mu \phi \partial_\nu h\left(\phi\right) - \frac{1}{2} g^{\mu\nu} h'\left(\phi\right) \partial_\mu \phi \partial_\nu \phi - V'\left(\phi\right) \right\} \delta\phi} \tag{5.25}$$