Exercise 9a

Proof the idempotence of the '`closing-opening`' alternated filter.

To prove the idempotence of the closing-opening ($\varphi_B\gamma_B$) we first need to assume some statements:

- Both γ_B (opening) and φ_B (closing) are increasing and idempotent.
- γ_B (opening) is anti-extensive.
- φ_B (closing) is extensive.

Theorem 1. *Idempotence of the 'closing-opening' alternated filter:*

$$\varphi_B\gamma_B\varphi_B\gamma_B(I) = \varphi_B\gamma_B(I)$$

Proof. The proof is done showing that both of the inequalities are satisfied.

$\boxed{\varphi_B\gamma_B\varphi_B\gamma_B(I) \leq \varphi_B\gamma_B(I)}$

$$\begin{aligned} \varphi_B\gamma_B(I) = \varphi_B\gamma_B(I) \Rightarrow \\ \varphi_B\varphi_B\varphi_B\gamma_B(I) = \varphi_B\gamma_B(I) \Rightarrow \\ \varphi_B\gamma_B\varphi_B\gamma_B(I) \leq \varphi_B\gamma_B(I) \end{aligned} \tag{1}$$

$\boxed{\varphi_B\gamma_B\varphi_B\gamma_B(I) \geq \varphi_B\gamma_B(I)}$

$$\begin{aligned} \varphi_B\gamma_B(I) = \varphi_B\gamma_B(I) \Rightarrow \\ \varphi_B\gamma_B\gamma_B\gamma_B(I) = \varphi_B\gamma_B(I) \Rightarrow \\ \varphi_B\gamma_B\varphi_B\gamma_B(I) \geq \varphi_B\gamma_B(I) \end{aligned} \tag{2}$$

By (1) and (2) it can be concluded that the equality $\varphi_B\gamma_B\varphi_B\gamma_B(I) = \varphi_B\gamma_B(I)$ holds. $\square$