Epidemic-Type Aftershock Sequence model approximation with Inlabru

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The only form of point process supported by Inlabru are Log-Gaussian Cox process (LGCP, Møller, Syversveen, and Waagepetersen 1998) models. A point process is an LGCP if its intensity fulfils two conditions:

1. $\lambda(\mathbf{x}) = \exp(S(\mathbf{x}))$ where $S(\mathbf{x}), \mathbf{x} \in \mathcal{X}$ is a Gaussian process on $\mathcal{X}$
2. Conditional on a realization of $S(\mathbf{x})$ the process is an inhomogeneous Poisson process with intensity $\lambda(\mathbf{x})$.

To fulfil the above conditions we consider the log-intensity to be a linear function of parameters and the parameters to be Gaussian. The Gaussianity of the parameters can be removed using a transformation as we have done for θ_6 and θ_7 in Section 1.1 of the theory document.

The main difficulty of this approach is that the components of the likelihood of the ETAS model are non-linear and we have to consider them separately to obtain an acceptable approximation. The decomposition that we use is based on how Inlabru approximate LGCP models which is also based on Poisson Count models. Specifically, having observed a set of points $\mathbf{x}_1, ..., \mathbf{x}_n$, the log-likelihood of any Poisson process model is given by

$$\mathcal{L}_{PP} = -\Lambda(\mathcal{X}) + \sum_{i=1}^{n} \log \lambda(\mathbf{x}),$$

where

$$\Lambda(\mathcal{X}) = \int_{\mathcal{X}} \lambda(\mathbf{x}) d\mathbf{x}.$$

The log-likelihood of a Poisson Counts model with exposures $E_1, ..., E_k$ and having observed counts $N_1, ..., N_k$ at locations $\mathbf{x}_1, ..., \mathbf{x}_k$ is

$$\mathcal{L}_{PC} \propto -\sum_{i=1}^{k} \lambda(\mathbf{x}_i) E_i + \sum_{i=1}^{k} \log \lambda(\mathbf{x}_i) N_i$$

Assuming $\log \lambda$ is linear in the parameters (we are going to soften this assumption), the idea is to use the first summation to approximate the integral and the second summation to approximate the sum of the observed points log-intensity.

We can approximate the integral part considering a triangulation of the space $\mathcal{X}$. Specifically, considering $\mathbf{p}_1, ..., \mathbf{p}_m$ representing the vertices of the triangulation and $w_1, ..., w_m$ weights representing the area of the voronoi-tasselation associated with the triangulation, we can approximate the integral as

$$\int_{\mathcal{X}} \lambda(\mathbf{x}) d\mathbf{x} \approx \sum_{i=1}^{m} \lambda(\mathbf{p}_i) w_i$$

Thus, considering $k = m$, the triangulation vertices as locations, and the voronoi-tasselation weights as exposures we can approximate the integral part of the likelihood.

Considering, $k = n$, the observed points as locations, and the counts $N_1, ..., N_k$ equal to 1 the second summation is identical to the sum of the observed points' log-intensity.

To represent both part of the Poisson process likelihood we have to consider an extended set of points $k = m + n$, with locations given by the union of the mesh points $\mathbf{p}_1, ..., \mathbf{p}_m$ and the observed points $x_1, ..., x_n$, exposures equal to the the voronoi-tasselation weights for the mesh points and to 0 for the observed points, and counts equal 0 for the mesh points and 1 for the observed points. The obtained Poisson Counts likelihood in this case is:

$$\mathcal{L}_{PC} = -\sum_{i=1}^{m} \lambda(\mathbf{p}_i) w_i + \sum_{i=1}^{n} \log \lambda(\mathbf{x}_i)$$

If the log-intensity is linear the reliability of the approximation depends only on the reliability of the integral approximation. Considering a finer triangulation provides better results increasing the computational costs.

If the log-intensity is not linear in the parameters $\boldsymbol{\theta}$, a linearised version of it is considered. Specifically, the linearised log-intensity around a point $\boldsymbol{\theta}_0$, is given by the first term of its Taylor expansion, namely

$$\overline{\log \lambda}(\mathbf{x}, \boldsymbol{\theta}) = \log \lambda(\mathbf{x}, \boldsymbol{\theta}_0) + \sum_i (\theta_i - \theta_{0i}) \frac{\partial}{\partial \theta_i} \log \lambda(\mathbf{x}, \boldsymbol{\theta}) \bigg|_{\boldsymbol{\theta} = \boldsymbol{\theta}_0}$$

The quantity $\boldsymbol{\theta}_0$ is considered to be the posterior mode, in this way the approximation is reliable at the posterior mode of the parameters and becomes less reliable moving away from it. In this case, the considered Poisson Counts likelihood is

$$\mathcal{L}_{PC} = -\sum_{i=1}^{m} \exp\left(\overline{\log \lambda}(\mathbf{p}_i)\right) w_i + \sum_{i=1}^{n} \overline{\log \lambda}(\mathbf{x}_i)$$

2 ETAS model Approximation using Poisson Counts models

The approximation described in the previous section is still problematic in the case of an ETAS model. The approximation of the integral is the main problem of such method, indeed, the ETAS intensity presents huge peaks at the observed points and we need a large number of points in the triangulation to have a reliable representation. This fact slows down the computation and makes the triangulation dependent on the observed points. Neither of this consequences are desirable.

Another way to represent the integral part is to leverage on the integral decomposition valid for self-exciting processes. In fact, ignoring the magnitude domain which integrates to one we have that

$$
\begin{aligned}
\Lambda(\mathcal{X}) &= \int_{T_1}^{T_2}\int_W \Big(\mu + \sum_{h:t_h<t} g(\mathbf{x}-\mathbf{x}_h)\Big) d\mathbf{s}dt \\
&= \mu(T_2-T_1)|W| + \sum_{h=1}^{n}\int_{T_1}^{T_2}\int_W g(\mathbf{x}-\mathbf{x}_h)\mathbb{I}(t_h<t)d\mathbf{s}dt \\
&= \mu(T_2-T_1)|W| + \sum_{h=1}^{n}\int_{\max(T_1,t_h)}^{T_2}\int_W g(\mathbf{x}-\mathbf{x}_h)d\mathbf{s}dt \\
&= \Lambda_0(\mathcal{X}) + \sum_{h=1}^{n}\Lambda_h(\mathcal{X})
\end{aligned}
$$

Where, $|W|$ is the area of the spatial domain and $\mathbb{I}(\cdot)$ is an indicator function taking value 1 if the condition holds and 0 otherwise.

The quantity

$$\Lambda_0(\mathcal{X}) = \mu(T_2-T_1)|W|$$

is the integrated background rate and represents the expected number of background points.

The quantity

$$\Lambda_h(\mathcal{X}) = \int_{\max(T_1,t_h)}^{T_2}\int_W g(\mathbf{x}-\mathbf{x}_h)d\mathbf{s}dt$$

is the integrated triggering function and represents the expected number of point triggered by the h-th observation. Moreover, for each point $\mathbf{x}_h$ we can split the time domain in m_h bins $b_1,...,b_{m_h}$ and write down

$$\Lambda_h(\mathcal{X}) = \sum_{j=1}^{m_h}\Lambda_h(b_j\times W) = \sum_{j=1}^{m_h}\int_{b_j}\int_W g(\mathbf{x}-\mathbf{x}_h)d\mathbf{s}dt$$

The log-likelihood of the ETAS model becomes

$$\mathcal{L}_{PP} = -\Lambda_0(\mathcal{X}) - \sum_{h=1}^{n}\sum_{j=1}^{m_h}\Lambda_h(b_j) + \sum_{i=1}^{n}\log\lambda(\mathbf{x}_i)$$

This can be approximated considering three poisson counts model with different intensities. The first one has log-intensity equal to

$$\log\lambda_1(\mathbf{x}) = \overline{\log\Lambda_0}(\mathcal{X})$$

The log-intensity is homogeneous and equal to the linearised logarithm of the integrated background rate. Considering only one observation (it doesn't matter the space-time-magnitude location), exposure equal 1 and counts equal 0, the first log-likelihood is given by

$$\mathcal{L}_{PC}^{(1)} = -\exp\left(\overline{\log\Lambda_0}(\mathcal{X})\right)$$

The second Poisson Count model we are going to consider has log-intensity given by

$$\log\lambda_2(\mathbf{x},b) = \overline{\log\Lambda_{\mathbf{x}}}(b)$$

Which is a function of the observed point and the time bin. Considering each observation repeated by its number of bins (for a total of $\sum_h m_h$ observations), exposures equal one and counts equal 0, the log-likelihood of the second Poisson Count model is

$$\mathcal{L}_{PC}^{(2)} = -\sum_{h=1}^{n}\sum_{j=1}^{m_h} \exp(\overline{\log \Lambda_h}(b_j))$$

The third component has log-intensity given by the linearised version of the ETAS log-intensity

$$\log \lambda_3(\mathbf{x}) = \overline{\log \lambda}(\mathbf{x})$$

Considering the observed points as locations, exposures equal 0 and counts equal one, the log-likelihood of the third Poisson Count model is

$$\mathcal{L}_{PC}^{(3)} = \sum_{i=1}^{n} \overline{\log \lambda}(\mathbf{x}_i)$$

Merging the three Poisson Count models, we obtain a multi-likelihood model with $1 + \sum_h m_h + n$ observations with log-likelihood given by the sum of the three log-likelihoods above

$$\mathcal{L}_{PC}^{(1)} + \mathcal{L}_{PC}^{(2)} + \mathcal{L}_{PC}^{(3)} = -\exp\left(\overline{\log \Lambda_0}(\mathcal{X})\right) - \sum_{h=1}^{n}\sum_{j=1}^{m_h} \exp(\overline{\log \Lambda_h}(b_j)) + \sum_{i=1}^{n} \overline{\log \lambda}(\mathbf{x}_i)$$

Which approximates the target log-likelihood without relying on a mesh but just using the provided data.

Also, considering the parametrization based on $\boldsymbol{\theta}$ the quantity $\log \Lambda_0(\mathcal{X})$ is linear in θ_1 and thus, it does not need to be linearised. The approximation becomes

$$\mathcal{L}_{PC} = -\Lambda_0(\mathcal{X}) - \sum_{h=1}^{n}\sum_{j=1}^{m_h} \exp(\overline{\log \Lambda_h}(b_j)) + \sum_{i=1}^{n} \overline{\log \lambda}(\mathbf{x}_i)$$

References

Møller, Jesper, Anne Randi Syversveen, and Rasmus Plenge Waagepetersen. 1998. "Log Gaussian Cox Processes." *Scandinavian Journal of Statistics* 25 (3): 451–82.