AM II 2010/2011 2ª F / EXAME 27-6-2011

I

① Integral de $f(x,y,z) = xy + y^2 + x^2 - xyz$

ao longo circunferência $x^2+y^2=1$ no plano xy.

$$\int_C f \, d\alpha$$

1º Parametrizar a curva

$$\alpha(t) = (1\cos t, 1 \operatorname{sen} t, 0)$$

$$0 \leq t \leq 2\pi$$

$$\alpha'(t) = (-\operatorname{sen} t, \cos t, 0)$$

$$\|\alpha'\| = \sqrt{(-\operatorname{sen} t)^2 + \cos^2 t} = 1$$

$$\int f \, d\alpha = \int_0^{2\pi} f(\alpha(t)) \cdot \|\alpha'(t)\| \, dt =$$

$$= \int_0^{2\pi} f(\cos t, \operatorname{sen} t, 0) \times 1 \, dt = \int_0^{2\pi} \cos t \operatorname{sen} t + \overbrace{\operatorname{sen}^2 t + \cos^2 t}^{1} - 0 \, dt$$

$$= \int_0^{2\pi} \cos t \operatorname{sen} t + 1 \, dt = \left[\frac{\operatorname{sen}^2 t}{2} + t \right]_0^{2\pi} = \frac{\operatorname{sen}^2(2\pi)}{2} + 2\pi - \frac{\operatorname{sen}^2 0}{2} + 0$$

$= 2\pi$ //

(2) Integral de linha do gradiente de

$f(x,y,z) = \operatorname{sen} x + \operatorname{sen} y + \operatorname{sen} z$

ao longo da linha $\alpha(t) = \left(-\frac{\pi}{2}\cos t, -\frac{\pi}{2}\operatorname{sen} t, t\right)$

$0 \leq t \leq \pi$

O gradiente de f é $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$= (\cos x, \cos y, \cos z)$

Vectorial →

$$\int_C \overbrace{(\cos x, \cos y, \cos z)}^{F} \, d\alpha = \quad \boxed{\alpha' = \left(\frac{\pi}{2}\operatorname{sen} t, -\frac{\pi}{2}\cos t, 1\right)}$$

$$= \int_0^\pi F(\alpha(t)) \cdot \alpha'(t) \, dt = \int_0^\pi F\left(-\frac{\pi}{2}\cos t, -\frac{\pi}{2}\operatorname{sen} t, t\right) \cdot \left(\frac{\pi}{2}\operatorname{sen} t, -\frac{\pi}{2}\cos t, 1\right)$$

$$= \int_0^\pi \left(\cos\left(-\frac{\pi}{2}\cos t\right), \cos\left(-\frac{\pi}{2}\operatorname{sen} t\right), \cos t\right) \cdot \left(\frac{\pi}{2}\operatorname{sen} t, -\frac{\pi}{2}\cos t, 1\right) dt$$

... muito complicado

ou outra forma ... pois trata-se de um campo conservativo, gradiente

$$\int_C \nabla f \, d\alpha = f(\text{ponto final}) - f(\text{ponto inicial})$$

$$\begin{cases} \alpha(0) = \left(-\frac{\pi}{2}\cos 0, -\frac{\pi}{2}\text{sen}\, 0, 0\right) = \left(-\frac{\pi}{2}, 0, 0\right) \\ \alpha(\pi) = \left(-\frac{\pi}{2}\cos(\pi), -\frac{\pi}{2}\text{sen}\,\pi, \pi\right) = \left(\frac{\pi}{2}, 0, \pi\right) \end{cases}$$

Fica $= f\left(\frac{\pi}{2}, 0, \pi\right) - f\left(-\frac{\pi}{2}, 0, 0\right) = \text{sen}\frac{\pi}{2} + \text{sen}\, 0 + \text{sen}\,\pi - \left(\text{sen}\left(-\frac{\pi}{2}\right) + \text{sen}\, 0 + \text{sen}\, 0\right)$

$$= 1 + 0 + 0 - (-1 + 0 + 0) = 2$$

③ Verifique que $F(x,y) = (2xy^3, 1 + 3x^2y^2)$ é conservativo e determine um potencial.

$F = (F_1, F_2)$ é conservativo se $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

$\frac{\partial F_2}{\partial x} = 6xy^2$ $\quad \frac{\partial F_1}{\partial y} = 6xy^2$ ✓ é conservativo

f é tal que $F = \nabla f$

$$\begin{cases} \dfrac{\partial f}{\partial x} = F_1 \\ \dfrac{\partial f}{\partial y} = F_2 \end{cases} \Leftrightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 2xy^3 \\ \dfrac{\partial f}{\partial y} = 1+3x^2y^2 \end{cases} \Leftrightarrow \begin{cases} f = \int 2xy^3 \, dx \\ f = \int 1+3x^2y^2 \, dy \end{cases}$$

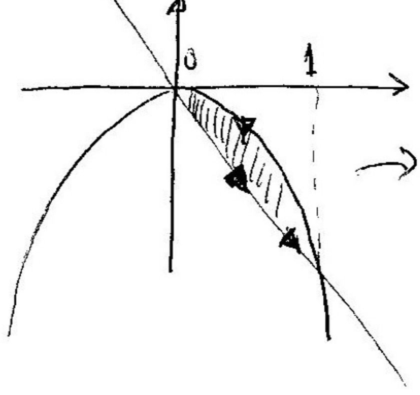
$$\Leftrightarrow \begin{cases} f = x^2y^3 + A(y) \\ f = y + x^2y^3 + B(x) \end{cases}$$

$$\therefore \; f(x,y) = x^2y^3 + y \; //$$

(4)

$$\oint_C \underbrace{x^2+y^2}_{F_1} dx + \underbrace{xy}_{F_2} dy$$

C é $y = -x^2$
$y = -x$



$\rightarrow$ $0 \leq x \leq 1$
$-x \leq y \leq -x^2$

Pelo Teorema de Green

$$= \int_0^1 \int_{-x}^{-x^2} y - 2y \, dy \, dx$$

$$\frac{\partial F_2}{\partial x} = y \qquad \frac{\partial F_1}{\partial y} = 2y$$

$$= \int_0^1 \int_{-x}^{-x^2} -y \, dy\, dx = \int_0^1 \left[\frac{-y^2}{2} \right]_{y=-x}^{y=-x^2} dx =$$

$$= \int_0^1 -\frac{(-x^2)^2}{2} - \left(-\frac{(-x)^2}{2} \right) dx = \int_0^1 -\frac{x^4}{2} + \frac{x^2}{2} \, dx$$

$$= \left[-\frac{x^5}{2\times 5} + \frac{x^3}{2\times 3} \right]_0^1 = -\frac{1}{10} + \frac{1}{6} - \left(-\frac{0}{10} + \frac{0}{6} \right)$$

$$= \frac{-6+10}{60} = \frac{4}{60} = \frac{1}{15}$$

⑤ Determine os pontos da esfera $x^2+y^2+z^2=4$ mais próximos e afastados de $(0,1,-1)$.

Função distância a $(0,1,-1)$

$$f(x,y,z) = \| (x,y,z) - (0,1,-1) \| = \sqrt{x^2+(y-1)^2+(z+1)^2}$$

⚠ A optimização da distância é igual à optimização do quadrado da distância

Vou usar o quadrado da distância

$$f(x,y,z) = x^2+(y-1)^2+(z+1)^2$$
$$= x^2+y^2+2y+1+z^2+2z+1$$

A restrição é a esfera $\underbrace{x^2+y^2+z^2-4}_{g}=0$

Multiplicador de Lagrange

$$L(x,y,z,\lambda) = x^2+y^2-2y+z^2+2z+2+\lambda(x^2+y^2+z^2-4)$$

$$\begin{cases} \dfrac{\partial L}{\partial x}=0 \\ \dfrac{\partial L}{\partial y}=0 \\ \dfrac{\partial L}{\partial z}=0 \\ \dfrac{\partial L}{\partial \lambda}=0 \end{cases} \Leftrightarrow \begin{cases} 2x+2\lambda x=0 \\ 2y-2+2\lambda y=0 \\ 2z+2+2\lambda z=0 \\ x^2+y^2+z^2-4=0 \end{cases} \Leftrightarrow \begin{cases} 2x(1+\lambda)=0 \\ \text{—} \\ \text{—} \\ \text{—} \end{cases}$$

$$\Leftrightarrow \begin{cases} x=0 \\ 2y(1+\lambda)=2 \\ 2z(1+\lambda)=-2 \\ y^2+z^2=4 \end{cases} \vee \cancel{\begin{cases} \lambda=-1 \\ -2=0 \end{cases}} \Leftrightarrow \begin{cases} x=0 \\ y=\dfrac{1}{1+\lambda} \\ z=\dfrac{-1}{1+\lambda} \\ \left(\dfrac{1}{1+\lambda}\right)^2+\left(\dfrac{-1}{1+\lambda}\right)^2=4 \end{cases} \Leftrightarrow \begin{cases} \text{—} \\ \text{—} \\ \text{—} \\ \dfrac{1}{(1+\lambda)^2}+\dfrac{1}{(1+\lambda)^2}=4 \end{cases}$$

$\lambda \neq -1$

$$\Leftrightarrow \begin{cases} \text{---} \\ \text{---} \\ 1+1 = 4(1+\lambda)^2 \end{cases} \Leftrightarrow \begin{cases} \text{---} \\ \text{---} \\ (1+\lambda)^2 = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} \text{---} \\ \text{---} \\ 1+\lambda = \pm\sqrt{1/2} \end{cases} \Leftrightarrow \begin{cases} \text{---} \\ \text{---} \\ \lambda = \pm\sqrt{1/2} - 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = \frac{1}{\sqrt{1/2}} \\ z = -1/\sqrt{1/2} \\ \lambda = \sqrt{1/2} - 1 \end{cases} \vee \begin{cases} x = 0 \\ y = \frac{1}{-\sqrt{1/2}} \\ z = \frac{1}{\sqrt{1/2}} \\ \lambda = -\sqrt{1/2} - 1 \end{cases}$$

$$\left(0, \frac{1}{\sqrt{1/2}}, \frac{-1}{\sqrt{1/2}}\right) \text{ e } \left(0, \frac{-1}{\sqrt{1/2}}, \frac{1}{\sqrt{1/2}}\right)$$

$$\Downarrow$$

$$\left(0, \sqrt{2}, -\sqrt{2}\right) \text{ e } \left(0, -\sqrt{2}, \sqrt{2}\right)$$

$$f(0, \sqrt{2}, -\sqrt{2}) = 0^2 + (\sqrt{2})^2 - 2\sqrt{2} + (-\sqrt{2})^2 - 2\sqrt{2} + 2 = 6 - 4\sqrt{2}$$

$$f(0, -\sqrt{2}, \sqrt{2}) = 0^2 + (-\sqrt{2})^2 + 2\sqrt{2} + (\sqrt{2})^2 + 2\sqrt{2} + 2 = 6 + 4\sqrt{2}$$

$\Downarrow$

Ponto mais distante

Ponto mais próximo //

II

⚠ "é um dominant" ↗ no 1º octd

(1) Volume do sólido entre as esferas $x^2+y^2+z^2=1$ e $x^2+y^2+z^2=4$ no 1º octante

C. esféricas $\begin{cases} x = \rho \cos\theta \operatorname{sen}\varphi \\ y = \rho \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = \rho \cos\varphi \end{cases}$ O Jacobiano é $\rho^2 \operatorname{sen}\varphi$

$1 \leq \rho \leq 2$

$0 \leq \theta \leq \pi/2$ → 1º octante

$0 \leq \varphi \leq \pi/2$ → 1º octante

$$V = \iiint 1 \, dv = \int_1^2 \int_0^{\pi/2} \int_0^{\pi/2} \rho^2 \operatorname{sen}\varphi \, d\varphi \, d\theta \, d\rho =$$

$$= \int_1^2 \int_0^{\pi/2} \left[-\rho^2 \cos(\varphi) \right]_{\varphi=0}^{\varphi=\pi/2} d\theta \, d\rho = \int_1^2 \int_0^{\pi/2} -\rho^2 \underbrace{\cos(\pi/2)}_{0} - (-\rho^2 \cos 0) \, d\theta \, d\rho$$

$$= \int_1^2 \int_0^{\pi/2} \rho^2 \, d\theta \, d\rho = \int_1^2 \rho^2 \left(\frac{\pi}{2} - 0\right) d\rho =$$

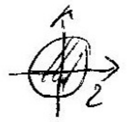
$$= \left[\frac{\pi}{2} \frac{\rho^3}{3} \right]_1^2 = \frac{\pi}{2} \frac{2^3}{3} - \frac{\pi}{2} \frac{1^3}{3} = \frac{8\pi}{6} - \frac{\pi}{6} = \frac{7\pi}{6}$$

② Calcule a área da porção de superfície do parabolóide $z = \frac{4-x^2-y^2}{4}$ acima de xy

$z=0$

$0 = 4 - x^2 - y^2$

$x^2+y^2=4$



Polares

$0 \le \rho \le 2$

$0 \le \theta \le 2\pi$

$$A = \iint_S 1 = \iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} =$$

integral de superfície — integral duplo

$$= \iint_D \sqrt{1+(-2x)^2+(-2y)^2} = \iint_D \sqrt{1+\underbrace{4x^2+4y^2}_{4(x^2+y^2) = 4\rho^2}} = \text{Polares}$$

$$= \int_0^2 \int_0^{2\pi} \sqrt{1+4\rho^2} \times \underbrace{\rho}_{\text{Jacobiano}} \, d\theta \, d\rho =$$

$$= \int_0^2 \rho\sqrt{1+4\rho^2}\,(2\pi-0)\,d\rho = \frac{2\pi}{8}\int_0^2 8\rho(1+4\rho^2)^{1/2}\,d\rho$$

$$= \frac{\pi}{4}\left[\frac{(1+4\rho^2)^{3/2}}{3/2}\right]_0^2 = \frac{\pi}{4}\left(\frac{(1+4\times 2^2)^{3/2}}{3/2} - \frac{(1+4\times 0)^{3/2}}{3/2}\right)$$

$$= \frac{\pi}{4}\left(\frac{17^{3/2}}{3/2} - \frac{1}{3/2}\right) = \frac{\pi}{\cancel{4}_2}\left(\frac{\cancel{2}}{3}\sqrt{17^3} - \frac{\cancel{2}}{3}\right)$$

$$= \frac{\pi}{6}\left(17\sqrt{17}-1\right)$$

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③ Teorema Stokes $F(x,y,z)=(-yz, xz, xy)$

$$\iint_S \operatorname{rot} F \cdot n \, dS = \int_C F \, d\alpha$$

S parte da esfera $x^2+y^2+z^2=4$
dentro do cilindro $x^2+y^2=1$
acima de xy

A intersecção da esfera com o cilindro dá

$0 \leq t \leq 2\pi$

$$\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2=1 \end{cases} \Leftrightarrow \begin{cases} 1+z^2=4 \\ x^2+y^2=1 \end{cases}$$

$z^2=3 \Leftrightarrow z=\pm\sqrt{3}$

mas é acima de $z=0$

$\therefore$ isso $z=\sqrt{3}$

Parametrizar a curva C

$\alpha(t)=(1\cos t, 1\operatorname{sen} t, \sqrt{3})$

$\alpha'=(-\operatorname{sen} t, \cos t, 0)$

Fica $\displaystyle\int_0^{2\pi} F(\alpha)\cdot\alpha' \, dt$

$$= \int_0^{2\pi} F(\cos t, \operatorname{sen} t, \sqrt{3})\cdot(-\operatorname{sen} t, \cos t, 0)\, dt = \int_0^{2\pi} (-\sqrt{3}\operatorname{sen} t, \sqrt{3}\cos t, \cos t \operatorname{sen} t)\cdot(-\operatorname{sen} t, \cos t, 0)$$

$$= \int_0^{2\pi} \sqrt{3}\operatorname{sen}^2 t + \sqrt{3}\cos^2 t + 0 \, dt = \int_0^{2\pi} \sqrt{3}\, dt = \sqrt{3}(2\pi - 0)$$

$$= 2\pi\sqrt{3}$$

(4) O maior subconjunto onde $f(x,y,z) = \dfrac{xyz}{x^2+y^2-z}$ é contínua.

Como é um quociente entre funções polinomiais, f é contínua no seu domínio

$$D_f = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2-z \neq 0\}$$

(5) A derivada de $f(x,y) = e^{\sqrt{x^2+y^2}} + 1$ em $(-4,3)$ segundo o vetor $(-1,1)$.

1º normalizar o vetor $\|(-1,1)\| = \sqrt{1^2+1^2} = \sqrt{2} \rightsquigarrow \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\, 2x\,(x^2+y^2)^{-1/2} e^{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} e^{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\, 2y\,(x^2+y^2)^{-1/2} e^{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} e^{\sqrt{x^2+y^2}}$$

existem e são contínuas junto do $(-4,3)$ logo f é diferenciável em $(-4,3)$

$$f'_{\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}(-4,3) = \frac{\partial f}{\partial x}(-4,3) \times \left(\frac{-1}{\sqrt{2}}\right) + \frac{\partial f}{\partial y}(-4,3) \times \frac{1}{\sqrt{2}} =$$

$$= \frac{-4}{5} e^5 \times \left(-\frac{1}{\sqrt{2}}\right) + \frac{3}{5} e^5 \frac{1}{\sqrt{2}} = \frac{7}{5\sqrt{2}} e^5$$

FIM 11/11