AM1 ◆ 2009/2010 ◆ E. Recurso ◆ 5-02-2010

(1) Convergência

a) $x_n = \left(\sqrt{n^2+1} - n\right) \operatorname{sen}(n^2)$ $\quad +\infty - \infty$ indeterminação

$$\lim_n \operatorname{sen}(n^2) \frac{\left(\sqrt{n^2+1} - n\right)\left(\sqrt{n^2+1} + n\right)}{\sqrt{n^2+1} + n} =$$

$$= \lim_n \operatorname{sen}(n^2) \frac{\left(n^2+1 - n^2\right)}{\sqrt{n^2+1} + n} = \lim \underbrace{\operatorname{sen}(n^2)}_{\text{limitada}} \underbrace{\frac{1}{\sqrt{n^2+1} + n}}_{\to 0}$$

$= 0$ // (converge)

b) $y_n = \sum_{k=1}^{2n} \frac{1}{\sqrt[3]{8n^3 - k}} = \frac{1}{\sqrt[3]{8n^3-1}} + \dots + \frac{1}{\sqrt{8n^3-2n}}$

$$2n \frac{1}{\sqrt{8n^3-2n}} \leq \frac{1}{\sqrt[3]{8n^3-1}} + \dots + \frac{1}{\sqrt[3]{8n^3-2n}} \leq \frac{1}{\sqrt[3]{8n^3-1}} \times 2n$$

$\to$ nº termos

$$\frac{2n}{\sqrt[3]{8n^3-2n}} \qquad\qquad \frac{2n}{\sqrt[3]{8n^3-1}}$$

$$\lim \frac{2n}{\sqrt[3]{8n^3-2n}} = \lim \frac{2n}{\sqrt[3]{8}\,\sqrt[3]{n^3}} = \lim \frac{2n}{2n} = 1$$

$$\lim \frac{2n}{\sqrt[3]{8n^3-1}} = \dots = 1$$

∴ Pelo teorema das sucessões enquadradas o limite de y_n é 1 (converge)

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② • $\sum_{n=1}^{\infty} \frac{3^{n-1}}{5^n} = \sum \frac{3^n \times 3}{5^n} = 3 \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$

é uma série geométrica de razão $\frac{3}{5}$.

Como a razão verifica $-1 < \frac{3}{5} < 1$ é convergente.

A soma é $S = \frac{1^o}{1-R} = \frac{\frac{3^0}{5^1}}{1-\frac{3}{5}} = \frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{2}$ //

• $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$

$$\lim \left(\frac{n}{n+1}\right)^n = \lim \left(\frac{n+1}{n+1} + \frac{-1}{n+1}\right)^n = \lim \left[\left(1 + \frac{-1}{n+1}\right)^{n+1}\right]^{\frac{n}{n+1}}$$

$$= \left(e^{-1}\right)^1 = e^{-1}$$

∴ Como o termo geral $\not\to 0$ a série é divergente (a soma é $+\infty$) //

③ $\sum u_n$ é convergente de termos positivos.

Mostre que $\sum (-1)^n \frac{u_n^2}{u_n+1}$ é convergente

1º Ver a convergência absoluta

$$\sum \left| (-1)^n \frac{u_n^2}{u_n+1} \right| = \sum \frac{u_n^2}{u_n+1}$$

Vou dividir pela mesma série que sei que converge

$$\lim \frac{\frac{u_n^2}{u_n+1}}{u_n} = \lim \frac{u_n}{u_n+1} = \lim \frac{u_n}{u_n} = 1$$

Pelo C. de comparação as séries $\sum \frac{u_n^2}{u_n+1}$ e $\sum u_n$ têm a mesma natureza. Como é dito que $\sum u_n$ converge, logo $\sum \frac{u_n^2}{u_n+1}$ tb converge

$\therefore$ A série converge absolutamente //

(4) $f(x) = \begin{cases} \operatorname{arctg}(\operatorname{sen} x) & , x \geq 0 \\ e^{-x} + k & , x < 0 \end{cases}$

a) Determine k de modo que f seja contínua em 0

$$\lim_{x\to 0^-} e^{-x} + k = 1 + k$$

$$\lim_{x\to 0^+} \operatorname{arctg}(\operatorname{sen} x) = \operatorname{arctg} 0 = 0$$

$$\therefore \ 1+k = 0 \Leftrightarrow k = -1$$

b) Diferenciabilidade de f

$$(\operatorname{arctg}(\operatorname{sen} x))' = \frac{\cos x}{1+\operatorname{sen}^2 x}$$

$$(e^{-x}-1)' = -e^{-x}$$

$$f' = \begin{cases} \dfrac{\cos x}{1+\operatorname{sen}^2 x} & , x > 0 \\ -e^{-x} & , x < 0 \end{cases}$$

em $x=0$ por def.

$$f_e'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{e^{-x}+1-0}{x-0} \underset{R.C.}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0^-} \frac{-e^{-x}}{1}$$

$$= -1$$

$$f_d'(0) = \lim_{x\to 0^+} \frac{\operatorname{arctg}(\operatorname{sen} x) - 0}{x - 0} \overset{\frac{0}{0}}{\underset{R.C}{=}} \lim_{x\to 0^+} \frac{\frac{\cos x}{1+\operatorname{sen}^2 x}}{1} = 1$$

Como $f_e'(0) \neq f_d'(0)$, não existe $f'(0)$ //

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(5) $$\lim_{x\to 0^+} (1 - 2\operatorname{cotg} x)^{\operatorname{tg} x} \quad (\infty^0) \text{ indeterminação}$$

$$\boxed{\operatorname{cotg} x = \frac{\cos x}{\operatorname{sen} x}}$$

CA
$$e^{\ln(1-2\operatorname{cotg} x)^{\operatorname{tg} x}} = e^{\operatorname{tg} x \ln(1-2\operatorname{cotg} x)} =$$

$$= e^{\frac{\ln(1-2\operatorname{cotg} x)}{\frac{1}{\operatorname{tg} x}}} = e^{\frac{\ln(1-2\operatorname{cotg} x)}{\operatorname{cotg} x}}$$

$$e^{\lim_{x\to 0^+} \frac{\ln(1-2\operatorname{cotg} x)}{\operatorname{cotg} x}} \overset{\frac{\infty}{\infty}}{\underset{R.C.}{=}} e^{\lim_{x\to 0^+} \frac{\frac{\frac{2}{\operatorname{sen}^2 x}}{1-2\operatorname{cotg} x}}{\frac{-1}{\operatorname{sen}^2 x}}} =$$

$$(\operatorname{cotg} x)' = \left(\frac{\cos x}{\operatorname{sen} x}\right)' = \frac{-\operatorname{sen}^2 x - \cos^2 x}{\operatorname{sen}^2 x} = \frac{-1}{\operatorname{sen}^2 x}$$

$$= e^{\lim_{x\to 0^+} \frac{+2}{+1-2\operatorname{cotg} x}} = e^0 = 1 \ //$$

(6) $f(x) = e^x - x - 1$. Mostre que não há outra raiz além de $x = 0$

$f(0) = 0$

Como $f' = e^x - 1 > 0 \Leftrightarrow e^x > 1$ quando $x > 0$

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(7) $f' = \dfrac{6}{9-t^2}$ e $f(0) = 1$

C.A. $\dfrac{6}{9-t^2} = \dfrac{6}{(3-t)(3+t)} = \dfrac{A}{\underset{\times 3+t}{3-t}} + \dfrac{B}{\underset{\times 3-t}{3+t}} = \dfrac{3A + At + 3B - Bt}{(3-t)(3+t)}$

$\Rightarrow \begin{cases} 3A + 3B = 6 \\ A - B = 0 \end{cases} \Leftrightarrow \begin{cases} 6B = 6 \\ A = B \end{cases} \Leftrightarrow \begin{cases} B = 1 \\ A = 1 \end{cases}$

$f = \int f' dt = \int \dfrac{6}{9-t^2} dt = \int \dfrac{1}{3-t} dt + \int \dfrac{1}{3+t} dt =$

$= -\ln|3-t| + \ln|3+t| + C$

Como $f(0) = 1 \Leftrightarrow -\ln|3| + \ln|3| + C = 1 \Leftrightarrow C = 1$

$\therefore f(t) = -\ln|3-t| + \ln|3+t| + 1$

//

⑧ $\displaystyle\int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x^4}\,dx$ Subs $x = \frac{1}{t}$

CA
- $x = 1/t$
- $dx = -\frac{1}{t^2}\,dt$
- $t = 1/x$

$$\int \frac{\sqrt{1-1/t^2}}{\frac{1}{t^4}} \times \left(-\frac{1}{t^2}\right) dt =$$

$$= \int \frac{-\sqrt{\frac{t^2-1}{t^2}}}{\frac{t^2}{t^4}}\,dt = -\int \frac{\frac{\sqrt{t^2-1}}{t}}{\frac{1}{t^2}}\,dt =$$

$$= -\int \frac{t^2\sqrt{t^2-1}}{t}\,dt = -\frac{1}{2}\int 2t\,(t^2-1)^{1/2}\,dt =$$

$$= -\frac{1}{2}\,\frac{(t^2-1)^{3/2}}{3/2} = -\frac{1}{2}\times\frac{2}{3}\,(t^2-1)^{3/2} = -\frac{1}{3}\,(t^2-1)^{3/2}$$

$$-\frac{1}{3}\left(\left(\frac{1}{x}\right)^2-1\right)^{3/2}$$

$$\cancel{= \left[-\frac{1}{3}(t^2-1)^{3/2}\right]_{1/2}^{1} = -\frac{1}{3}\times 0 + \frac{1}{3}\left(\left(\frac{1}{2}\right)^2-1\right)^{3/2}} =$$

$$\cancel{= \frac{1}{3}\Big(}$$

$$= \left[-\frac{1}{3}\left(\left(\frac{1}{x}\right)^2-1\right)^{3/2}\right]_{1/2}^{1} = 0 + \frac{1}{3}\left(2^2-1\right)^{3/2} = \frac{1}{3}\,3^{3/2} = \frac{1}{3}\sqrt{3^2\times 3}$$

$$= \sqrt{3}$$ //

4/4

(9) $F(x) = \int_0^{x^2} \operatorname{sen}(2\pi t)\, dt$

a) Paridade de F

$$F(-x) = \int_0^{(-x)^2} \operatorname{sen}(2\pi t)\, dt = \int_0^{x^2} \operatorname{sen}(2\pi t)\, dt = F(x)$$

logo F é par

b) $$F'(x) = \left(\int_0^{x^2} \operatorname{sen}(2\pi t)\, dt \right)' = 2x \operatorname{sen}(2\pi x^2) - 0$$

$$= 2x \operatorname{sen}(2\pi x^2)$$

(10) $$\int \frac{\log x}{x^{\alpha}} = \int \underbrace{\frac{1}{x} \cdot \log x}_{\mu'} \times \underbrace{\frac{1}{x^a}}_{v} = \log^2 x \times \frac{1}{x^a} -$$

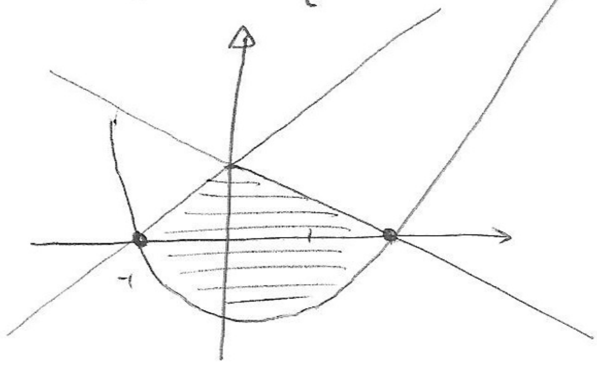
•••

(11) $x^2 - x - 2$ $\quad x+1 \quad$ $1 - \frac{x}{2}$

$x = \frac{1 \pm \sqrt{1+8}}{2}$

$x = \frac{1 \pm 3}{2}$

$x = 2 \vee x = -1$



$$\text{Área} = \int_{-1}^{0} x+1 - (x^2 - x - 2)\, dx$$

$$+ \int_0^2 1 - \frac{x}{2} - (x^2 - x - 2)\, dx$$

$$= \frac{5}{3} + \frac{13}{3} = \frac{18}{3} = 6$$