Given (for all proofs below)

$V = IR + E_{\text{mf}}$ by Ohm's Law with back EMF for a DC motor

$E_{\text{mf}} = \omega\kappa_e$ by the Back EMF equation of a motor

$\tau = I\kappa_t$ by the Motor torque equation

$\tau = J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)$ by the Rotational equation of motion

The value of $\omega$ and $\alpha$ (Only for the proof directly below this)

Find $V$ voltage

$V = IR + E_{\text{mf}}$, $E_{\text{mf}} = \omega\kappa_e$, $\tau = I\kappa_t$, $\tau = J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)$, and the value of $\omega$ and $\alpha$ because they are given

$I\kappa_t = J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)$ By substitution

$I = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}$ By Division

$-IR = E_{\text{mf}} - V$ By Subtraction

$I = \frac{E_{\text{mf}} - V}{-R}$ By Division

$\frac{E_{\text{mf}} - V}{-R} = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}$ By substitution

$\frac{\omega\kappa_e - V}{-R} = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}$ By substitution

$\omega\kappa_e - V = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}(-R)$ By multiplication

$-V = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}(-R) - \omega\kappa_e$ By subtraction

$V = -\left(\frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}(-R) - \omega\kappa_e\right)$ By division

$V = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}(R) + \omega\kappa_e$ By substitution

$V = \frac{J\alpha R}{\kappa_t} + \frac{B\omega R}{\kappa_t} + \frac{\tau_{\text{load}}\ \text{sign}(\omega)R}{\kappa_t} + \frac{\omega\kappa_e\kappa_t}{\kappa_t}$ By substitution

$V = \frac{J\alpha R}{\kappa_t} + \frac{B\omega R + \omega\kappa_e\kappa_t}{\kappa_t} + \frac{\tau_{\text{load}}\ \text{sign}(\omega)R}{\kappa_t}$ By substitution

$V = \frac{J\alpha R}{\kappa_t} + \frac{\omega(BR + \kappa_e\kappa_t)}{\kappa_t} + \frac{\tau_{\text{load}}\ \text{sign}(\omega)R}{\kappa_t}$ By substitution

Define $K_v$ by $K_v = \frac{BR + \kappa_e\kappa_t}{\kappa_t}$

Define $K_a$ by $K_a = \frac{JR}{\kappa_t}$

Define $K_s$ by $K_s = \frac{\tau_{\text{load}}R}{\kappa_t}$

$V = K_a\alpha + K_v\omega + K_s\ \text{sign}(\omega)$

Define a function $V$ by $V(\omega, \alpha) = K_a\alpha + K_v\omega + K_s\ \text{sign}(\omega)$

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Given (Only for the proof directly below this)

The value of $\omega$

Find the maximum $\alpha$ angular acceleration at some angular velocity $\omega$

$V = IR + E_{\text{mf}}$, $E_{\text{mf}} = \omega\kappa_e$, $\tau = I\kappa_t$, $\tau = J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)$, and the value of $\omega$ because they are given

Note: The VEX motor controller has current limits so we must also account for those.

Define $I_{\text{min}}$ and $I_{\text{max}}$ as the current limits and define $I_{\text{clamp}}$ as the clamped value of I

Define $V_{\text{max}}$ as the maximum voltage and set it to $V$ when finding the max acceleration

Define $\alpha_{\text{max}}$ as the maximum voltage and set it to $\alpha$ when finding the max acceleration

$I_{\text{clamp}} = \min(\max(I, I_{\text{min}}), I_{\text{max}})$ by the definition of clamping

$I_{\text{clamp}} = \min\left(\max\left(\left(\frac{E_{\text{mf}} - V_{\text{max}}}{-R}\right), I_{\text{min}}\right), I_{\text{max}}\right)$ by substitution

$I_{\text{clamp}} = \min\left(\max\left(\left(\frac{\omega\kappa_e - V_{\text{max}}}{-R}\right), I_{\text{min}}\right), I_{\text{max}}\right)$ by substitution

Define $V_{\text{max}} = V$ as the max voltage

$I_{\text{clamp}} = \min\left(\max\left(\left(\frac{\omega\kappa_e - V_{\text{max}}}{-R}\right), I_{\text{min}}\right), I_{\text{max}}\right)$ by substitution

Restrict I by $I_{\text{clamp}} = I$ by the note

$-J\alpha_{\text{max}} = B\omega + \tau_{\text{load}}\ \text{sign}(\omega) - \tau$ by Subtraction

$\alpha_{\text{max}} = \frac{B\omega + \tau_{\text{load}}\ \text{sign}(\omega) - \tau}{-J}$ by Division

$\alpha_{\text{max}} = \frac{B\omega + \tau_{\text{load}}\ \text{sign}(\omega) - I\kappa_t}{-J}$ by Substitution

$\alpha_{\text{max}} = \frac{B\omega + \tau_{\text{load}}\ \text{sign}(\omega) - I_{\text{clamp}}\kappa_t}{-J}$ by Substitution

Define a function $\alpha$ by $\alpha_{\text{max}}(\omega) = \frac{B\omega + \tau_{\text{load}}\ \text{sign}(\omega) - I_{\text{clamp}}\kappa_t}{-J}$

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Find $\omega(t)$ given that the motor is constantly accelerating

$\alpha_{\text{max}}(\omega) = \frac{V_{\text{max}} - K_v\omega - \ \text{sgn}(\omega)K_s}{K_a}$

Let $s = \text{sgn}(\omega) \in \{1, -1\}$

$\frac{d\omega}{dt} = \frac{V_{\text{max}} - K_v\omega - sK_s}{K_a}$

$\frac{d\omega}{dt} + \frac{K_v}{K_a}\omega = \frac{V_{\text{max}} - sK_s}{K_a}$

$I(t) = e^{\int \frac{K_v}{K_a} dt}$

$I(t) = e^{\frac{K_v}{K_a} t}$

$\omega(t) = \frac{1}{I(t)}\left(\int_0^t I(\tau)Q d\tau + I(0)\omega_0\right)$

$\omega(t) = \frac{1}{e^{\frac{K_v}{K_a}t}}\left(\int_0^t e^{\frac{K_v}{K_a}\tau}\frac{V_{\text{max}} - sK_s}{K_a} d\tau + \omega_0\right)$

$\omega(t) = \frac{1}{e^{\frac{K_v}{K_a}t}}\left(\frac{V_{\text{max}} - sK_s}{K_a}\frac{K_a}{K_v}\left(e^{\frac{K_v}{K_a}\tau} - 1\right) + \omega_0\right)$

$\omega(t) = \frac{V_{\text{max}} - sK_s}{K_v}\frac{e^{\frac{K_v}{K_a}\tau} - 1}{e^{\frac{K_v}{K_a}t}} + \omega_0\left(e^{-\frac{K_v}{K_a}t}\right)$

$\omega(t) = \frac{V_{\text{max}} - sK_s}{K_v} + \left(\omega_0 - \frac{V_{\text{max}} - sK_s}{K_v}\right)e^{-\frac{K_v}{K_a}t}$

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Find a State-Space model in the form

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$V = K_a\alpha + K_v\omega + K_s\ \text{sign}(\omega)$ because it is given

Let $u = V$ and $x = \begin{pmatrix} \omega \\ \text{sign}(\omega) \end{pmatrix}$

$\alpha = \dot{\omega}$ by the definition of acceleration

$\dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix} = \dot{\omega}$ by the derivative

$\alpha = \dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix}$ by substitution

$\operatorname{sign}(\omega) = x\begin{pmatrix} 0 & 1 \end{pmatrix}$

$u = K_a \alpha + K_v \omega + K_s \operatorname{sign}(\omega)$ by substitution

$K_v \omega + K_s \operatorname{sign}(\omega) = -K_a \alpha + u$ by subtraction

$K_v x\begin{pmatrix} 1 & 0 \end{pmatrix} + K_s x\begin{pmatrix} 0 & 1 \end{pmatrix} = -K_a \dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix} + u$ by substitution

$x\left(K_v \begin{pmatrix} 1 & 0 \end{pmatrix} + K_s \begin{pmatrix} 0 & 1 \end{pmatrix}\right) = -K_a \dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix} + u$ by substitution

$x\begin{pmatrix} K_v & K_s \end{pmatrix} = -K_a \dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix} + u$ by substitution

$x = \begin{pmatrix} K_v & K_s \end{pmatrix}^+ \left(-K_a \dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix} + u\right)$ by multiplication

$x = \frac{\begin{pmatrix} K_v \\ K_s \end{pmatrix}}{K_v^2 + K_s^2}\left(-K_a \dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix} + u\right)$ by the psuedoinverse

$x = -\frac{K_a}{K_v^2 + K_s^2}\begin{pmatrix} K_v & 0 \\ K_s & 0 \end{pmatrix}\dot{x} + \frac{1}{K_v^2 + K_s^2}\begin{pmatrix} K_v \\ K_s \end{pmatrix} u$ by substitution

$x = \begin{pmatrix} \frac{-K_a K_v}{K_v^2 + K_s^2} & 0 \\ \frac{-K_a K_s}{K_v^2 + K_s^2} & 0 \end{pmatrix}\dot{x} + \begin{pmatrix} \frac{-K_a K_v}{K_v^2 + K_s^2} \\ \frac{-K_a K_s}{K_v^2 + K_s^2} \end{pmatrix} u$ by substitution

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