Given (for all proofs below)

$V = IR + E_{\text{mf}}$ by Ohm's Law with back EMF for a DC motor

$E_{\text{mf}} = \omega\kappa_e$ by the Back EMF equation of a motor

$\tau = I\kappa_t$ by the Motor torque equation

$\tau = J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)$ by the Rotational equation of motion

The value of ω and α (Only for the proof directly below this)

Find V voltage

$V = IR + E_{\text{mf}}$, $E_{\text{mf}} = \omega\kappa_e$, $\tau = I\kappa_t$, $\tau = J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)$, and the value of ω and α because they are given

$I\kappa_t = J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)$ By substitution

$I = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}$ By Division

$-IR = E_{\text{mf}} - V$ By Subtraction

$I = \frac{E_{\text{mf}} - V}{-R}$ By Division

$\frac{E_{\text{mf}} - V}{-R} = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}$ By substitution

$\frac{\omega\kappa_e - V}{-R} = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}$ By substitution

$\omega\kappa_e - V = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}(-R)$ By multiplication

$-V = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}(-R) - \omega\kappa_e$ By subtraction

$V = -\left(\frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}(-R) - \omega\kappa_e\right)$ By division

$V = \frac{J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)}{\kappa_t}(R) + \omega\kappa_e$ By substitution

$V = \frac{J\alpha R}{\kappa_t} + \frac{B\omega R}{\kappa_t} + \frac{\tau_{\text{load}}\ \text{sign}(\omega)R}{\kappa_t} + \frac{\omega\kappa_e\kappa_t}{\kappa_t}$ By substitution

$V = \frac{J\alpha R}{\kappa_t} + \frac{B\omega R + \omega\kappa_e\kappa_t}{\kappa_t} + \frac{\tau_{\text{load}}\ \text{sign}(\omega)R}{\kappa_t}$ By substitution

$V = \frac{J\alpha R}{\kappa_t} + \frac{\omega(BR + \kappa_e\kappa_t)}{\kappa_t} + \frac{\tau_{\text{load}}\ \text{sign}(\omega)R}{\kappa_t}$ By substitution

Define K_v by $K_v = \frac{BR + \kappa_e\kappa_t}{\kappa_t}$

Define K_a by $K_a = \frac{JR}{\kappa_t}$

Define K_s by $K_s = \frac{\tau_{\text{load}}R}{\kappa_t}$

$V = K_a\alpha + K_v\omega + K_s\ \text{sign}(\omega)$

Define a function V by $V(\omega, \alpha) = K_a\alpha + K_v\omega + K_s\ \text{sign}(\omega)$

■

Given (Only for the proof directly below this)

The value of ω

Find the maximum α angular acceleration at some angular velocity ω

$V = IR + E_{\text{mf}}$, $E_{\text{mf}} = \omega\kappa_e$, $\tau = I\kappa_t$, $\tau = J\alpha + B\omega + \tau_{\text{load}}\ \text{sign}(\omega)$, and the value of ω because they are given

Note: The VEX motor controller has current limits so we must also account for those.

Define $I_{\min}$ and $I_{\max}$ as the current limits and define I_{clamp} as the clamped value of I

Define $V_{\max}$ as the maximum voltage and set it to V when finding the max acceleration

Define $\alpha_{\max}$ as the maximum voltage and set it to α when finding the max acceleration

$I_{\text{clamp}} = \min(\max(I, I_{\min}), I_{\max})$ by the definition of clamping

$I_{\text{clamp}} = \min\left(\max\left(\left(\frac{E_{\text{mf}}-V_{\max}}{-R}\right), I_{\min}\right), I_{\max}\right)$ by substitution

$I_{\text{clamp}} = \min\left(\max\left(\left(\frac{\omega\kappa_e-V_{\max}}{-R}\right), I_{\min}\right), I_{\max}\right)$ by substitution

Define $V_{\max} = V$ as the max voltage

$I_{\text{clamp}} = \min\left(\max\left(\left(\frac{\omega\kappa_e-V_{\max}}{-R}\right), I_{\min}\right), I_{\max}\right)$ by substitution

Restrict I by $I_{\text{clamp}} = I$ by the note

$-J\alpha_{\max} = B\omega + \tau_{\text{load}}\ \text{sign}(\omega) - \tau$ by Subtraction

$\alpha_{\max} = \frac{B\omega+\tau_{\text{load}}\ \text{sign}(\omega)-\tau}{-J}$ by Division

$\alpha_{\max} = \frac{B\omega+\tau_{\text{load}}\ \text{sign}(\omega)-I\kappa_t}{-J}$ by Substitution

$\alpha_{\max} = \frac{B\omega+\tau_{\text{load}}\ \text{sign}(\omega)-I_{\text{clamp}}\kappa_t}{-J}$ by Substitution

Define a function α by $\alpha_{\max}(\omega) = \frac{B\omega+\tau_{\text{load}}\ \text{sign}(\omega)-I_{\text{clamp}}\kappa_t}{-J}$

■

Find $\omega(t)$ given that the motor is constantly accelerating

$\alpha_{\max}(\omega) = \frac{V_{\max}-K_v\omega-\ \text{sgn}(\omega)K_s}{K_a}$

Let $s = \text{sgn}(\omega) \in \{1, -1\}$

$\frac{d\omega}{dt} = \frac{V_{\max}-K_v\omega-sK_s}{K_a}$

$\frac{d\omega}{dt} + \frac{K_v}{K_a}\omega = \frac{V_{\max}-sK_s}{K_a}$

$I(t) = e^{\int \frac{K_v}{K_a}dt}$

$I(t) = e^{\frac{K_v}{K_a}t}$

$\omega(t) = \frac{1}{I(t)}\left(\int_0^t I(\tau)Qd\tau + I(0)\omega_0\right)$

$\omega(t) = \frac{1}{e^{\frac{K_v}{K_a}t}}\left(\int_0^t e^{\frac{K_v}{K_a}\tau}\frac{V_{\max}-sK_s}{K_a}d\tau + \omega_0\right)$

$\omega(t) = \frac{1}{e^{\frac{K_v}{K_a}t}}\left(\frac{V_{\max}-sK_s}{K_a}\frac{K_a}{K_v}\left(e^{\frac{K_v}{K_a}\tau} - 1\right) + \omega_0\right)$

$\omega(t) = \frac{V_{\max}-sK_s}{K_v}\frac{e^{\frac{K_v}{K_a}\tau}-1}{e^{\frac{K_v}{K_a}t}} + \omega_0\left(e^{-\frac{K_v}{K_a}t}\right)$

$\omega(t) = \frac{V_{\max}-sK_s}{K_v} + \left(\omega_0 - \frac{V_{\max}-sK_s}{K_v}\right)e^{-\frac{K_v}{K_a}t}$

■

Find a continuous state-space model in the form

$\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u$

$y = \boldsymbol{C}x + \boldsymbol{D}u$

$v = K_a\alpha + K_v\omega + K_s\ \text{sign}(\omega)$ because it is given

let $u = v$ and $x = \begin{pmatrix} \omega \\ \text{sign}(\omega) \end{pmatrix}$

$\alpha = \dot{\omega}$ by the definition of acceleration

$\dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix} = \dot{\omega}$ by the derivative

$\alpha = \dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix}$ by substitution

$\text{sign}(\omega) = x\begin{pmatrix} 0 & 1 \end{pmatrix}$

$u = K_a \alpha + K_v \omega + K_s \ \text{sign}(\omega)$ by substitution

$K_a \dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix} = u - K_v x\begin{pmatrix} 1 & 0 \end{pmatrix} - K_s x\begin{pmatrix} 0 & 1 \end{pmatrix}$ by subtraction

$K_a \dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} -K_v & -K_s \end{pmatrix} x + u$ by substitution

$\dot{x}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{K_v}{K_a} & -\frac{K_s}{K_a} \end{pmatrix} x + \frac{1}{K_a} u$ by division

$\dot{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} -\frac{K_v}{K_a} & -\frac{K_s}{K_a} \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{K_a} u$ by multiplication

$\dot{x} = \begin{pmatrix} -\frac{K_v}{K_a} & -\frac{K_s}{K_a} \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} \frac{1}{K_a} \\ 0 \end{pmatrix} u$ by substitution

Let $\boldsymbol{A} = \begin{pmatrix} -\frac{K_v}{K_a} & -\frac{K_s}{K_a} \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} \frac{1}{K_a} \\ 0 \end{pmatrix}$

Let $y = \begin{pmatrix} \omega \end{pmatrix}$

$y = x\begin{pmatrix} 1 & 0 \end{pmatrix}$ by substitution

Let $\boldsymbol{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $\boldsymbol{D} = 0$

■

Given

$\boldsymbol{A}_d = e^{\boldsymbol{A}_c T}$

$\boldsymbol{B}_d = \int_0^T e^{\boldsymbol{A}_c \tau} d\tau \boldsymbol{B}_c$

$\boldsymbol{C}_d = \boldsymbol{C}_c$

$\boldsymbol{D}_d = \boldsymbol{D}_c$

$\boldsymbol{A}_{\boldsymbol{c}} = \begin{pmatrix} -\frac{K_v}{K_a} & -\frac{K_s}{K_a} \\ 0 & 0 \end{pmatrix}$

$\boldsymbol{B}_{\boldsymbol{c}} = \begin{pmatrix} \frac{1}{K_a} \\ 0 \end{pmatrix}$

$\boldsymbol{C}_{\boldsymbol{c}} = \begin{pmatrix} 1 & 0 \end{pmatrix}$

$\boldsymbol{D}_{\boldsymbol{c}} = 0$

Find a discrete state-space model in the form

$x_{k+1} = \boldsymbol{A} x_k + \boldsymbol{B} u_k$

$y_k = \boldsymbol{C} x_k + \boldsymbol{D} u_k$

$\boldsymbol{A}_d = e^{\begin{pmatrix} -\frac{K_v}{K_a} & -\frac{K_s}{K_a} \\ 0 & 0 \end{pmatrix} T}$ by substitution

$\boldsymbol{A}_d = e^{\begin{pmatrix} -T\frac{K_v}{K_a} & -T\frac{K_s}{K_a} \\ 0 & 0 \end{pmatrix}}$ by substitution

Let $a = -T\frac{K_v}{K_a}$ and $b = -T\frac{K_s}{K_a}$

$\boldsymbol{A}_d = e^{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}}$ by substitution

$\boldsymbol{A}_d = \begin{pmatrix} e^a & b\frac{e^a - e^0}{a - 0} \\ 0 & e^0 \end{pmatrix}$ by the exponential of an upper-triangular matrix formula

$\boldsymbol{A}_d = \begin{pmatrix} e^{-T\frac{K_v}{K_a}} & -T\frac{K_s}{K_a}\frac{e^{-T\frac{K_v}{K_a}-1}}{-T\frac{K_v}{K_a}} \\ 0 & 1 \end{pmatrix}$ by substitution

$\boldsymbol{A}_d = \begin{pmatrix} e^{-T\frac{K_v}{K_a}} & K_s\frac{e^{-T\frac{K_v}{K_a}-1}}{K_v} \\ 0 & 1 \end{pmatrix}$ by substitution

$\boldsymbol{B}_d = \int_0^T \begin{pmatrix} e^{-\tau\frac{K_v}{K_a}} & K_s\frac{e^{-\tau\frac{K_v}{K_a}-1}}{K_v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{K_a} \\ 0 \end{pmatrix} d\tau$ by substitution

$\boldsymbol{B}_d = \int_0^T \begin{pmatrix} \frac{e^{-\tau\frac{K_v}{K_a}}}{K_v} \\ 0 \end{pmatrix} d\tau$ by substitution

$\boldsymbol{B}_d = \begin{pmatrix} \frac{1}{K_a}\int_0^T e^{-\tau\frac{K_v}{K_a}} d\tau \\ 0 \end{pmatrix}$ by substitution

Let $I_d = \int_0^T e^{-\tau\frac{K_v}{K_a}} d\tau$

$\boldsymbol{B}_d = \begin{pmatrix} \frac{I_d}{K_v} \\ 0 \end{pmatrix}$ by substitution

$I_d = e^{-0\frac{K_v}{K_a}} - e^{-T\frac{K_v}{K_a}}$ by FTC

$I_d = 1 - e^{-T\frac{K_v}{K_a}}$ by substitution

$\boldsymbol{B}_d = \begin{pmatrix} \frac{1-e^{-T\frac{K_v}{K_a}}}{K_v} \\ 0 \end{pmatrix}$ by substitution

$\boldsymbol{C}_d = \begin{pmatrix} 1 & 0 \end{pmatrix}$ by substitution

$\boldsymbol{D}_d = 0$ by substitution