Parametric Model Identification for Motor-Propeller Actuator Dynamics

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Contents

1 Brushless DC motor model

1.1 Brushless DC motor speed-troque characteristics [1]

The torque and speed characteristics can be determined by the balance between motor's mechanical output power adn electrical input power over a conduction period:

$$
\begin{aligned}
P &= \omega_m T_e = 2e_p I \\
e_p &= N_p B_g \pi r l \omega_m
\end{aligned}
$$

Where,

$$
\begin{aligned}
\omega_m &- \text{Mechanical rpm} \\
T_e &- \text{Electromagnetic torque} \\
e_p &- \text{Back emf}
\end{aligned}
$$

The factor 2 is the result of current flowing through 2-motor phases (trapesoidal wave form).

We have, electromagnetic torque:

$$T_e = 4N_p B_g l r I \qquad [\because \text{Lenz law}]$$

let, $E = 2e_p$. We have,

$$
\begin{aligned}
E &= k\psi\omega_m = K_v\omega_m \\
T_e &= k\psi I = K_T I
\end{aligned}
$$

Where,

$$
\begin{aligned}
k &= 4N_p && \text{(Armature Constant)} \\
\psi &= B_g \pi r l && \text{(Flux)}
\end{aligned}
$$

Thus, ideally back-emf constant and torque constants are same.

Using the above equations, the following steady-state torque speed characteristics can be drived. We have, the instantaneous voltage equation:

$$V_s = E + IR$$

Where,

$$
\begin{aligned}
V_s &- \text{Supply voltage} \\
I &- \text{Total DC current} \\
R &- \text{Sum of the terminal phase ressistances}
\end{aligned}
$$

We have torque speed relationship:

$$\omega_m = \omega_0 \left(1 - \frac{T}{T_0}\right)$$

Where,

$$
\begin{aligned}
\omega_0 &= \frac{V_s}{k\psi} && \text{(No-load Speed)} \\
T_0 &= k\psi I_0 && \text{(Stall Torque)} \\
I_0 &= \frac{V_s}{R} && \text{(Stall Current)}
\end{aligned}
$$

1.2 Dynamic Model (without Propeller)

We have the dynamic model of BLDC motor using moment balance:

$$J_m\dot{\omega}_m = T_e - b_f\omega_m - M_f$$

where,

$$\begin{aligned} J_m &- \text{Moment of inertia of the motor} \\ b_f &- \text{lumped parameter for viscous friction} \\ M_f &- \text{lumped parameter for coulomb friction} \end{aligned}$$

Friction:

1. Viscous friction: $-b_f\omega$.
2. Columb friction: $-M_f\text{sign}(\omega) = -M_f$ $\quad$ [$\because$ the motor turns in only one direction].

From the speed-torque characteristics of the BLDC motor:

$$T_e = K_T I = K_T\frac{(V_s - E)}{R} = \frac{K_T}{R}(V_s - K_v\omega_m) \qquad [\because K_v = K_T = k\psi]$$

Let,

$$K_r = \frac{K_T}{R}$$

From the definition of Input to ESC:

$$\begin{aligned} V_s &= uV_{in} \\ \therefore T_e &= uK_rV_{in} - K_rK_v\omega_m \end{aligned}$$

Substituting:

$$J_m\dot{\omega}_m = uK_rV_{in} - K_rK_v\omega_m - b_f\omega_m - M_f \qquad \textit{Let}$$
$$b_m = K_rK_v + b_f$$

$$\boxed{J_m\dot{\omega}_m + b_m\omega_m + M_f = uK_rV_{in}} \tag{1}$$

1.3 BLDC Motor with Propeller

Adding propeller moment of inertia and the moment due to propeller drag into the BLDC motor model.

$$(J_m + J_p)\dot{\omega} + b_m\omega + M_f = uK_rV_{in} - C_D\omega^2$$

Where, C_D is the aerodynamic drag. Let, $J_m + J_p = J$

$$\boxed{J\dot{\omega} + b_m\omega + C_D\omega^2 + M_f = uK_rV_{in}} \tag{2}$$

1.4 Propeller Aerodynamics

Aerodynamics are assumed to be faster than mechanical dynamics of the actuator. The thrust generation process due to the propagation of pressure wave is assumed to be instantaneous. This assumption is inherent to the standard models that use potential flow theory (lifting-line, blade-element and momentum-disk theories), as they assume incompressible flow.

Propeller Thrust:

$$F_T = C_T \omega^2$$

Propeller moment due to drag:

$$M_D = C_D \omega^2$$

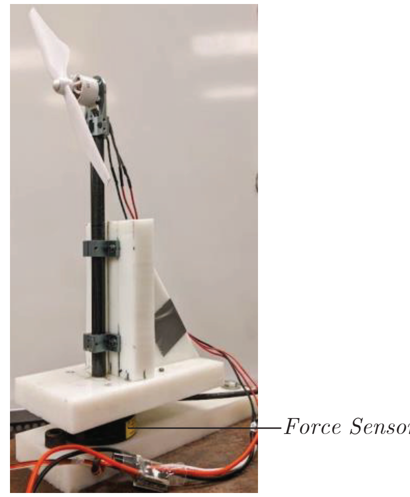


Figure 1: Experimental Setup

Aeroelasticity of the propeller: It is assumed that the aeroelasticity of the propeller produces high-frequency oscillations in the thrust and torque of the propller which are assumed to be very fast and roll off w.r.t the mechanical dyanmics dyanmics of the actuator as well as the transmission through the propller shaft. The constant bias in the torque due to flutter is captured in the drag coefficient and it's parameter uncertainity.

1.4.1 Parameter estimation form the static data

In the experimental setup (Figure 1), the total moment measured is the result of aerodynamic moment and the friction of the BLDC motor. Thus the total moment becomes:

$$M = C_D \omega^2 + b_f \omega + M_f$$

The aerodynamic coefficients are estimated from the static measuremnts using least-squares estimation.

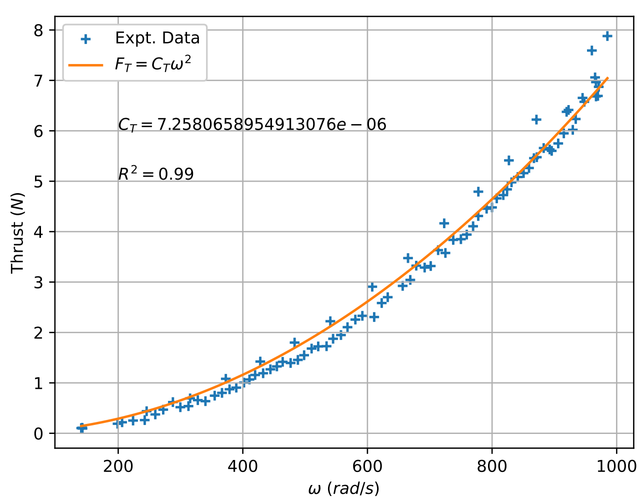


Figure 2: Variation of thrust with rpm

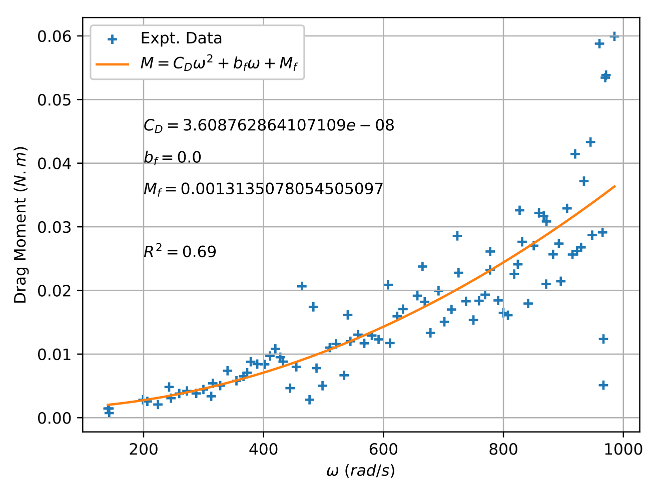


Figure 3: Variation of drag moment with rpm

The moment data has more variation from model due to the unmodelled aerodynamic effects such as aerodynamic-flutter. Thus we have the estimates of force coefficients:

Parameter	Value
C_T	$7.2581 \times 10^{-06}\ N/(rad/s)^2$
C_D	$3.6088 \times 10^{-08}\ N.m/(rad/s)^2$
b_f	$0.0\ N.m/(rad/s)$
M_f	$1.3135 \times 10^{-3}\ N.m$

Table 1: Estimates Force coefficients

2 RPM Measurement

An is interrupt triggered for every commutation high and the ISR gets the counter value of an independently runing timer. Using this value, the RPM is calcluated at every interrupt trigger as follows:

$$rpm = \frac{60 f_t}{N_p \times T_c}$$

Where,

$$\begin{aligned} f_t &- \text{Frequency of the timer counts (here, 1 MHz)} \\ N_p &- \text{Number of pole-pairs in the BLDC motor (here, 7)} \\ T_c &- \text{Timer counts between the two interrupts} \end{aligned}$$

The above method of measurement is verified against the tach-meter reading. The readings are in agreement with each other, validating the measurement method.

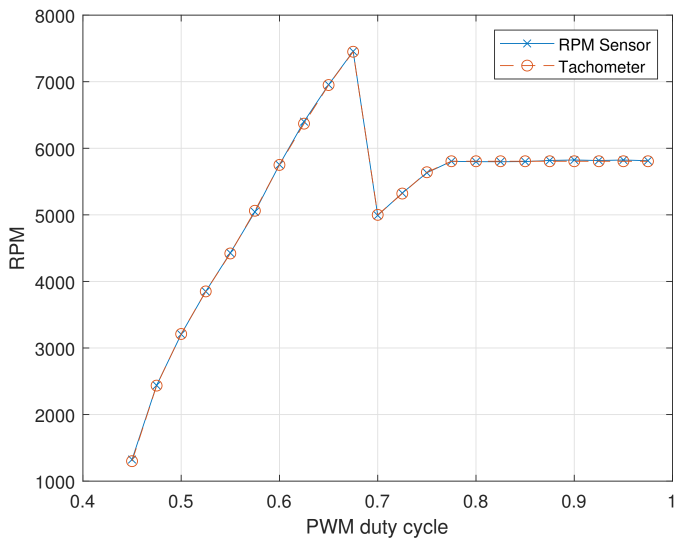


Figure 4: Rpm sensor and tachometer readings

The above method has an inherent flow at very low *rpm*s where there is no commutation at a given sampling instance which results in a zero rpm at that sample making the sensor noisy. This can be avoided by holding the rpm value from the previous measurement if there is no commutation in the sample instance.

Using external interrupt and timer

The commutations can be counted using external interrupts. The actual measurement of rpm involves a raising-edge triggered external interrupt and a timer interrupt. The ISR of the external interrupt updates a counter corresponding to the pulses (here, the electrical commutations). The timer-interrupt polls for this counter value at the specified frequency (f_s) and resets the counter. The rpm is calculated from this counter value as follows:

$$rpm = \frac{counter\_value}{N_P} \times f_s \times 60$$

where,
N_p- No. poles in the BLDC motor

The minimum rpm that can be measured depends on the sampling frequency and the poles in the BLDC motor ($= \frac{60 f_s}{N_P}$). And the maximum depends on the clock frequency of the micro-controller as the frequency of interrupts and ISR calls becomes the bottleneck in this case. The resolution of the sensor also depends on the sampling frequency and poles as the counter value is an integer. We have,

$$Sensor\, resolution = \frac{60 f_s}{N_P}$$

For the current system the range of rpm is $[2000, 10000]$. The number of poles in BLDC motor are 12. Assuming the acquisition frequency of $400\, Hz$, the resolution for the above sensor will be:

$$Sensor\, resolution = \frac{60 f_s}{N_P} = 2000\, rpm = 209.4395\, rad/s$$

For $100\, Hz$ acquisition rate:

$$Sensor\, resolution = 500\, rpm = 52.3599\, rad/s$$

The main source of sensor noise in this case is the latency of the external interrupt. The counter value will be oscillating around the actual value based on the timing of external and timer interrupts causing the measured rpm to fluctuate. Based on the resolution calculations above, the signal-to-noise ratio of the system will be very high if the current implementation is used.

This problem of resolution and signal-to-noise ratio is due to the interfacing method used. Alternatively, the following method is propsoed to overcome this problem.

Using two timers and an external interrupt

We use and additional timer as a high frequency counter of frequency f_t for calculating the rpm at every sampling period as follows:

$$rpm = \frac{counter\_value \times f_t}{N_P \times time\_counts} \times 60$$

where,
N_p- No. poles in the BLDC motor
$time\_counts-$ Number of timer interrupt counts during the sampling interval.
Hence, we have, maximum number of $time\_counts$ in a sampling interval is f_s/f_t

$$\implies Sensor\, resolution = \frac{60 f_s}{N_P f_t} = rpm_{min}$$

Therefore, the resolution of the sensor can be increased by arbitrarily increasing the frequency of the high frequency counter, limited only by the hardware.

For example, if $f_h = 1000\, Hz$, for the same values as above, the resolution of the sensor:

$$Sensor\, resolution = \frac{60 f_s}{N_P \times f_t} = \frac{2000}{1000} = 2\, rpm = \frac{1}{\pi}\, rad/s$$

This method will reduce the signal-to-noise ratio significantly.

2.1 Measurement Algorithm

In higher rpm cases there are more than one measurement instance wthin the sample time. Median of these measurements can be used to reduce sudden spikes in the data due to interrupts skips. Combining this with higher resolution algorithm, we have the algorithm for rpm measurement:

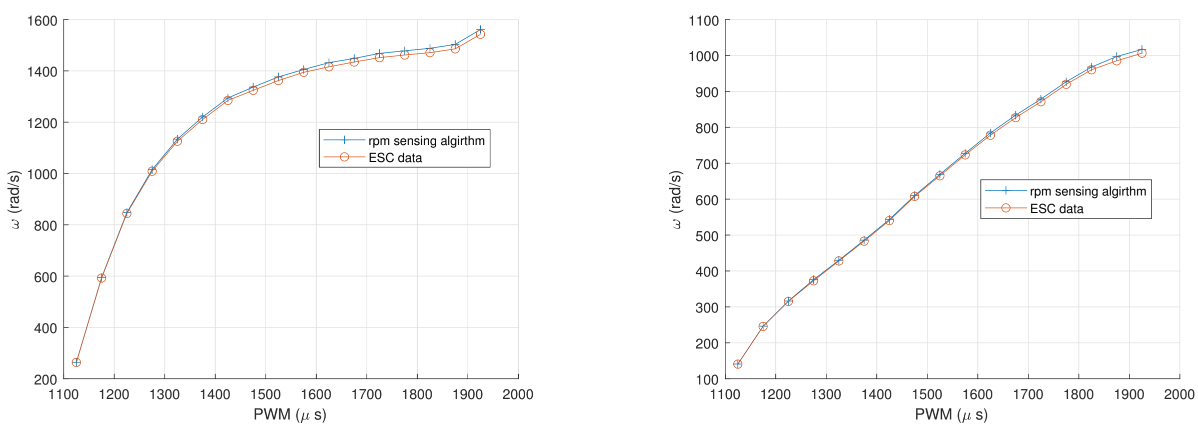
Let C_c be the current value of the 32-bit timer, P_c the previous value and n_C be the number of commutations within the sample.

At every interrupt trigger (in ISR)

n_C += 1;
$\delta t_k = C_c - P_c$;
if $\delta t_k \leq 0$ **then**
    δt_k += 2^{32} ; `/* Correcting for integer overflow */`
end
$\boldsymbol{\delta t_k}[n_C - 1] = \delta t_k$;
$P_c = C_c$;

At every sampling instance (in *get_rpm*()):

$N_p = 7$;
$f_t = 10^6$;
$M = \frac{2\pi}{N_p} \times f_t$;
if $n_C > 0$ *and* $n_C \leq n_{C_{max}}$ *and* $|n_C - n_{C_{old}}| \leq \delta n_{C_{max}}$ **then**
    $\omega = \frac{M}{\text{Median}(\boldsymbol{\delta t_k})}$; `/* Median removes spikes in the data */`
    $\omega_{old} = \omega$;
    $n_{C_{old}} = n_C$;
else
    $\omega = \omega_{old}$
end
$n_C = 0$;
$\boldsymbol{\delta t_k} = \mathbf{0}$;



Without Propeller   With Propeller

Figure 5: Validating the measurement algorithm

3 Input definition and Static Calibration

3.1 ESC and non-linear input compensation

The castle creations ESC has a micro-controller that non-lineraly scales the input PWM signal's duty-cycle to the duty-cycle of the $24\,kHz-$PWM signals to the inverter effectively scaling the source voltage to the motor by the duty-cycle. This is done to have a linear input to thrust curve instead of a quadratic one.

The input transmission can be described as follows: The $400\,Hz-$PWM duty cycle (u_p) islineraly scaled to a throttle ratio (p) (% power-out) between 0 and 1. Which is then filters using a non-linear function (g_u) to get the PWM duty-cycle input to the inverter (u) [2].

$$u_p \rightarrow \boxed{g_u(.)} \rightarrow u$$

and finally,

$$V_s = uV_{in} \qquad u \in [0, 1]$$

u is considered as the input to the motor-propeller system and the necessary invertion will be performed for transmitting the signals.

Scaling PWM Singal Duty cycle based on switching frequency: Pixhawk-4 uses a switching frequency of $400\,Hz$ for its PWM signals (can be swithed to $50\,Hz$ which is not that usefull in case of BLDC motors but usefull for servos). The controller thus scales the PWM duty cycle to the duration of on-time of the signal in its period in 'microseconds'. These inputs are handelled as integer types within the range [800, 2200][3].

The current ESC that has the rpm-feedback capabilites has an operating range between $1110\,\mu s$ and $1890\,\mu s$. After that, the ESC switchs to a constant power mode which sets the rpm to a constant.

$$\begin{aligned}
\text{Period of the PWM wave} &= \frac{1}{400} \times 10^6\,\mu s = 2500\,\mu s \\
\text{Minimum Operating Duty Cycle} &= \frac{1110}{2500} = 0.444 \\
\text{Maximum Operating Duty Cycle} &= \frac{1890}{2500} = 0.756
\end{aligned}$$

u can be considered to be the actual input to the system and system identification with the propeller. It turns out that the parameters of the above non-linear filter are not estimatable with the give information. To solve this problem, we chose angular velocity of the motor with propeller normalized with voltage as the input instead.

3.2 Normalized Angular Velocity Input

We have the no-load dynamic model for the BLDC motor with propeller:

$$J\dot{\omega} + b_m\omega + C_D\omega^2 + M_f = uK_rV_{in}$$

At steady state ($\dot{\omega} = 0$), the above equation becomes:

$$\begin{aligned}
b_m\omega + C_D\omega^2 + M_f &= uK_rV_{in} \\
\implies \frac{b_m}{K_r}\left(\frac{\omega_m}{V_{in}}\right) + \frac{V_{in}}{K_r}C_D\left(\frac{\omega}{V_{in}}\right)^2 + \frac{M_f}{K_rV_{in}} &= u
\end{aligned}$$

Definition: Let, u_ω be the angular velocity of the motor with propeller at unit supply voltage for the given pwm input (u_p). Also, let us call it "***Normalized angular velocity***".

$$u_\omega = \frac{\omega}{V_{in}} \text{ at } u = g_u(u_p)$$
$$\implies u = \underbrace{\frac{b_m}{K_r}u_\omega + \frac{\hat{V}_{in}}{K_r}C_D u_\omega^2 + \frac{M_f}{K_r\hat{V}_{in}}}_{g_\omega(u_\omega, \hat{V}_{in})}$$

The relationship between u_ω and u_p can be estimated from the staic measuremnt data.

We have:

$$u_\omega = au_p + b \qquad a = 0.0696 \qquad b = -64.3266$$

Also,

$$\because u = g_\omega(u_\omega, \hat{V}_{in})$$
$$\implies g_u(u_p) = g_\omega(au_p + b, \hat{V}_{in})$$

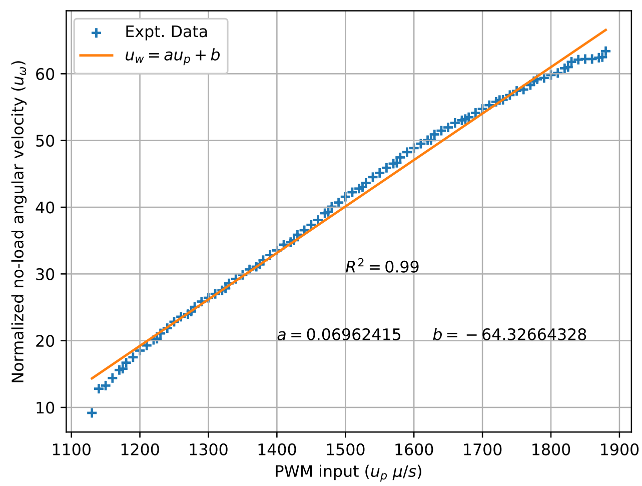


Figure 6: u_ω as a function of u_p

4 BLDC Motor with Propeller Model – Parameter Estimation

Intoducing the input definition into the BLDC-motor model with propeller:

$$
\begin{aligned}
J\dot{\omega} + b_m\omega + C_D\omega^2 + M_f &= uK_rV_{in} = K_rV_{in}g_\omega(u_\omega, \hat{V}_{in}) \\
\implies J\dot{\omega} + b_m\omega + C_D\omega^2 + M_f &= K_rV_{in}\left(\frac{b_m}{K_r}u_\omega + \frac{\hat{V}_{in}}{K_r}C_Du_\omega^2 + \frac{M_f}{K_r\hat{V}_{in}}\right) \\
J\dot{\omega} + b_m\omega + C_D\omega^2 + M_f\left(1 - \frac{V_{in}}{\hat{V}_{in}}\right) &= V_{in}b_mu_\omega + V_{in}\hat{V}_{in}C_Du_\omega^2
\end{aligned}
$$

Note on Voltage: The battery voltage is assumed to be constant with small variations that can be introduced as uncertainities.

$$
\hat{V}_{in} = V_{in}(1+\delta v) \implies \frac{V_{in}}{\hat{V}_{in}} = 1 - \delta v \implies \left(1 - \frac{V_{in}}{\hat{V}_{in}}\right) = \delta v
$$

$$
J\dot{\omega} + b_m\omega + C_D\omega^2 + M_f\delta v = V_{in}b_mu_\omega + V_{in}^2(1+\delta v)C_Du_\omega^2 \tag{3}
$$

4.1 Parametric Identification model for BLDC motor - propeller dyanmcis

We have:

$$
J\dot{\omega} + b_m\omega + C_D\omega^2 + M_f\delta v = V_{in}b_mu_\omega + V_{in}^2(1+\delta v)C_Du_\omega^2
$$

4.1.1 Input singal (persistance of exitation and frequency limitation)

Note:

1. PE order of a square wave of half-period m is m+1.
2. PE order of a single sine wave is 2.

4.2 Small Perturbation Model

We get the linearised model using small perturbtation:

$$
\begin{aligned}
J\delta\dot{\omega} + b_m\delta\omega + 2C_D\omega_0\delta\omega &= \delta u_\omega\left(V_{in}b_m + 2V_{in}^2C_Du_{\omega_0}\right) \\
J\delta\dot{\omega} + (b_m + 2C_D\omega_0)\delta\omega &= \delta u_\omega\left(V_{in}b_m + 2V_{in}^2C_Du_{\omega_0}\right) \\
&\text{Laplace Transform:} \\
(Js + (b_m + 2C_D\omega_0))\,\delta\omega &= \delta u_\omega\left(V_{in}b_m + 2V_{in}^2C_Du_{\omega_0}\right)
\end{aligned}
$$

Thus we have the trannsfer function model:

$$
\frac{\delta\omega(s)}{\delta u_\omega(s)} = \frac{V_{in}b_m + 2V_{in}^2C_Du_{\omega_0}}{Js + (b_m + 2C_D\omega_0)}
$$

Thus both gain and time-constant increase with the nominal rpm. Getting the gain and cut-off frequency using the standard first-order model:

$$
\frac{\delta\omega(s)}{\delta u_w(s)} = \frac{k_m}{s + \omega_m} = \frac{V_{in}b_m + 2V_{in}^2C_Du_{\omega_0}}{Js + (b_m + 2C_D\omega_0)}
$$

Where,

$$
\begin{aligned}
k_p &= \frac{V_{in}}{J}\left(b_m + 2V_{in}C_Du_{\omega_0}\right) \\
\omega_p &= \frac{1}{J}\left(b_m + 2C_D\omega_0\right)
\end{aligned}
$$

This information will be used for establishing the validity of identified model.
In case of using u_p as input:

$$\begin{aligned} u_\omega &= a u_p + b \\ \implies \delta u_\omega &= a \delta u_p \\ \implies \frac{\delta\omega(s)}{\delta u_p(s)} &= \frac{1}{a}\frac{k_p}{s+\omega_p} \end{aligned}$$

Thus this results in variation of the static gain alone.

4.3 Non-linear Model Parameter Identification

We have:

$$J\dot{\omega} + b_m\omega + C_D\omega^2 + M_f\delta v = V_{in}b_m u_\omega + V_{in}^2(1+\delta v)C_D u_\omega^2$$

4.3.1 Input singal (persistance of exitation and frequency limitation)

Note:

1. PE order of a square wave of half-period m is m+1.
2. PE order of a single sine wave is 2.

5 Conclusion: Model and Model Parameters

5.1 Parametric Form

References

[1] Richard Crowder. *Electric drives and electromechanical systems: applications and control*. Butterworth-Heinemann, 2019.

[2] Sang-Hoon Kim. *Electric motor control: DC, AC, and BLDC motors*. Elsevier, 2017.

[3] Pixhawk-4 user manual. `https://docs.px4.io/main/en/peripheralspwm_escs_and_servo.html#pwm-servos-and-escs-motor-controllers`.