

Problems Sets from Dynamics by Kane

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Part I

Kinematics

1 Problem Set 1

1.1 1(a) Connected parallelepipeds

Problem: Four rectangular parallelepipeds, A, B, C, and D, are arranged as shown in Figure 1. $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ designate unit vectors respectively parallel to the edges of A; $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are unit vectors respectively parallel to the edges of B, and so forth, and ϕ, θ and ψ denote the radian measures of angles that determine the relative orientation of the bodies. The configuration shown is one in which ϕ, θ, ψ are regarded positive. Determine the magnitude of each of the following derivatives:

$$\frac{{}^B \partial \mathbf{a}_1}{\partial \phi}, \frac{{}^B \partial \mathbf{b}_1}{\partial \phi}, \frac{{}^B \partial \mathbf{a}_3}{\partial \phi}, \frac{{}^B \partial \mathbf{b}_2}{\partial \theta}, \frac{{}^C \partial \mathbf{b}_2}{\partial \theta}, \frac{{}^D \partial \mathbf{b}_2}{\partial \theta}, \frac{{}^C \partial \mathbf{b}_2}{\partial \psi}, \frac{{}^D \partial \mathbf{b}_2}{\partial \psi}, \frac{{}^D \partial \mathbf{a}_1}{\partial \psi}$$

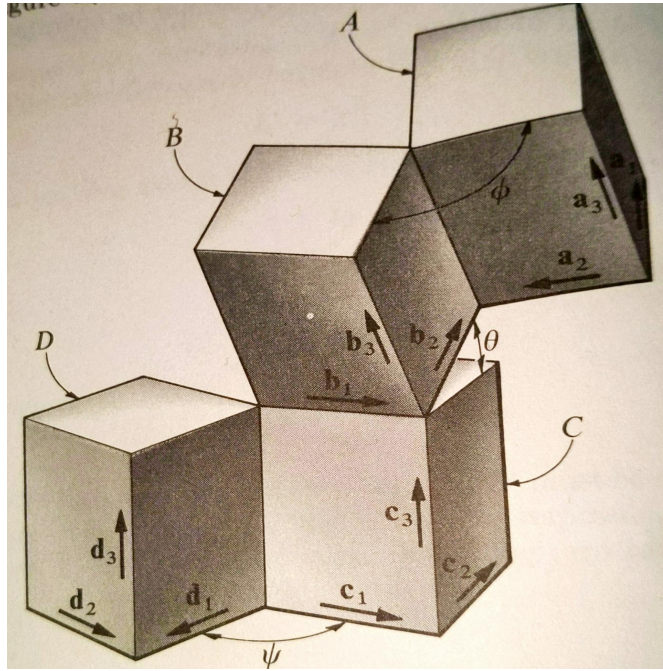


Figure 1

sol.

We have the following rotation matrices:

$${}^B \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3(\phi)} {}^B \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}; \quad {}^C \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}}_{R_1(\theta)} {}^C \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix}; \quad {}^D \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3(\psi)} {}^D \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix}$$

Let, $\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}$ $\mathbf{c} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix}$ $\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix}$

Thus, we have,

1.

$$\begin{aligned} {}^B \frac{\partial \mathbf{a}_1}{\partial \phi} &= \frac{\partial}{\partial \phi} \left(R_3(\phi)[1, :] \times {}^B \mathbf{b} \right) = \frac{\partial R_3(\phi)[1, :]}{\partial \phi} {}^B \mathbf{b} \quad [\because \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \text{ are fixed in } B] \\ &= \begin{bmatrix} -\sin \phi & \cos \phi & 0 \end{bmatrix} \times {}^B \mathbf{b} \\ &\implies {}^B \left| \frac{\partial \mathbf{a}_1}{\partial \phi} \right| = 1 \end{aligned}$$

2.

$${}^B \frac{\partial \mathbf{b}_1}{\partial \phi} = 0 \implies {}^B \left| \frac{\partial \mathbf{b}_1}{\partial \phi} \right| = 0 \quad [\because \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \text{ are fixed in } B]$$

3.

$$\begin{aligned} {}^B \frac{\partial \mathbf{a}_3}{\partial \phi} &= \frac{\partial}{\partial \phi} \left(R_3(\phi)[3, :] \times {}^B \mathbf{b} \right) = \frac{\partial \mathbf{b}_3}{\partial \phi} = 0 \quad [\because \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \text{ are fixed in } B] \\ &\implies {}^B \left| \frac{\partial \mathbf{a}_3}{\partial \phi} \right| = 0 \end{aligned}$$

4.

$${}^B \frac{\partial \mathbf{b}_2}{\partial \theta} = 0 \implies {}^B \left| \frac{\partial \mathbf{b}_2}{\partial \theta} \right| = 0 \quad [\because \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \text{ are fixed in } B]$$

5.

$$\begin{aligned} {}^C \frac{\partial \mathbf{b}_2}{\partial \theta} &= \frac{\partial}{\partial \theta} \left(R_1(\theta)[2, :] \times {}^C \mathbf{c} \right) = \frac{\partial R_1(\theta)[2, :]}{\partial \theta} \times {}^C \mathbf{c} \quad [\because \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \text{ are fixed in } C] \\ &= \begin{bmatrix} 0 & -\sin \theta & \cos \theta \end{bmatrix} \times {}^C \mathbf{c} \\ &\implies {}^C \left| \frac{\partial \mathbf{b}_2}{\partial \theta} \right| = 1 \end{aligned}$$

6.

$$\begin{aligned} {}^D \frac{\partial \mathbf{b}_2}{\partial \theta} &= \frac{\partial}{\partial \theta} \left((R_1(\theta)R_3(\psi))[2, :] \times {}^D \mathbf{d} \right) \quad [\because \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 \text{ are fixed in } D] \\ &= \frac{\partial (R_1(\theta)R_3(\psi))[2, :]}{\partial \theta} \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix} = \left(\frac{\partial R_1(\theta)}{\partial \theta} R_3(\psi) \right) [2, :] {}^D \mathbf{d} = \left(\frac{\partial R_1(\theta)}{\partial \theta} [2, :] R_3(\psi) \right) {}^D \mathbf{d} \\ &= \left(\begin{bmatrix} 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) {}^D \mathbf{d} = \begin{bmatrix} \sin \theta \sin \psi & -\sin \theta \cos \psi & \cos \theta \end{bmatrix} {}^D \mathbf{d} \\ &\implies {}^D \left| \frac{\partial \mathbf{b}_2}{\partial \theta} \right| = 1 \end{aligned}$$

7.

$${}^C \frac{\partial \mathbf{b}_2}{\partial \psi} = 0 \implies {}^C \left| \frac{\partial \mathbf{b}_2}{\partial \psi} \right| = 0 \quad [\because \text{Any vector defined in } C \text{ is independent of } \psi]$$

8.

$$\begin{aligned} {}^D \frac{\partial \mathbf{b}_2}{\partial \psi} &= \frac{\partial}{\partial \psi} \left((R_1(\theta)R_3(\psi))[2, :] \times {}^D \mathbf{d} \right) \quad [\because \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 \text{ are fixed in } D] \\ &= R_1(\theta)[2, :] \times \frac{\partial R_3(\psi)}{\partial \psi} \times {}^D \mathbf{d} = \begin{bmatrix} 0 & \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} -\sin \psi & \cos \psi & 0 \\ -\cos \psi & -\sin \psi & 0 \\ 0 & 0 & 0 \end{bmatrix} \times {}^D \mathbf{d} \\ &= \begin{bmatrix} -\cos \theta \cos \psi & -\cos \theta \sin \psi & 0 \end{bmatrix} \times {}^D \mathbf{d} \\ &\implies {}^D \left| \frac{\partial \mathbf{b}_2}{\partial \psi} \right| = |\cos \theta| \end{aligned}$$

9.

$$\begin{aligned} {}^D \frac{\partial \mathbf{a}_1}{\partial \psi} &= \frac{\partial}{\partial \psi} \left((R_3(\phi)R_1(\theta)R_3(\psi))[2, :] \times {}^D \mathbf{d} \right) = R_3(\phi)[1, :] \times R_1(\theta) \times \frac{\partial R_3(\psi)}{\partial \psi} \times {}^D \mathbf{d} \\ &= \begin{bmatrix} \cos \phi & \sin \phi & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} -\sin \psi & \cos \psi & 0 \\ -\cos \psi & -\sin \psi & 0 \\ 0 & 0 & 0 \end{bmatrix} \times {}^D \mathbf{d} \\ &= \begin{bmatrix} -\cos \phi \sin \psi - \sin \phi \cos \theta \cos \psi & \cos \phi \cos \psi - \sin \phi \cos \theta \sin \psi & 0 \end{bmatrix} \times {}^D \mathbf{d} \\ \implies {}^D \left| \frac{\partial \mathbf{a}_1}{\partial \psi} \right| &= \sqrt{(-\cos \phi \sin \psi - \sin \phi \cos \theta \cos \psi)^2 + (\cos \phi \cos \psi - \sin \phi \cos \theta \sin \psi)^2} \\ &= (\cos^2 \phi + \sin^2 \phi \sin^2 \theta)^{1/2} \end{aligned}$$

1.2 1(b) ω in connected parallelopipeds

Problem: Referring to Problem 1(a), determine w_1, w_2 and w_3 such that

$${}^C \frac{\partial \mathbf{a}_1}{\partial \theta} = w_1 \mathbf{a}_1 + w_2 \mathbf{a}_2 + w_3 \mathbf{a}_3$$

sol.

We have,

$$\begin{aligned} {}^C \frac{\partial \mathbf{a}_1}{\partial \theta} &= \frac{\partial}{\partial \theta} [R_3(\phi)[1, :] R_1(\theta)] \times {}^C \mathbf{c} = R_3(\phi)[1, :] \frac{\partial R_1(\theta)}{\partial \theta} \times {}^C \mathbf{c} = R_3(\phi)[1, :] \frac{\partial R_1(\theta)}{\partial \theta} \times [R_3(\phi)R_1(\theta)]^{-1} \times {}^C \mathbf{a} \\ &= R_3(\phi)[1, :] \frac{\partial R_1(\theta)}{\partial \theta} \times R_1^T(\theta) R_3^T(\phi) \times {}^C \mathbf{a} \quad [\because R_i^{-1} = R_i^T] \\ &= \begin{bmatrix} \cos \phi & \sin \phi & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin \theta & \cos \theta \\ 0 & -\cos \theta & -\sin \theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \times {}^C \mathbf{a} \\ &= \begin{bmatrix} \cos \phi & \sin \phi & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \times {}^C \mathbf{a} = \begin{bmatrix} 0 & 0 & \sin \phi \end{bmatrix} \times {}^C \mathbf{a} \end{aligned}$$

Hence,

$$w_1 = w_2 = 0, \quad w_3 = \sin \phi$$

1.3 1(c) α in connected parallelopipeds

Problem: Referring to Problem 1(a), and assuming that θ, ϕ and ψ are functions of the time t such that, at a certain instant t^* , $\phi = \theta = \psi = \pi/6 \text{ rad}$, $\dot{\phi} = 4 \text{ rad/sec}$, and $\dot{\theta} = \dot{\psi} = 6 \text{ rad/sec}$, show that at time t^* ,

$${}^C \frac{\partial \mathbf{a}_1}{\partial t} = 4\mathbf{a}_2 + 3\mathbf{a}_3$$

sol.

We have,

$${}^C \mathbf{a}_1 = R_3(\phi)[1, :] R_1(\theta) \times {}^C \mathbf{c}$$

$$\begin{aligned} \Rightarrow {}^C \frac{d\mathbf{a}_1}{dt} &= \left(\frac{\partial R_3(\phi)[1, :]}{\partial \phi} R_1(\theta) \dot{\phi} + R_3(\phi)[1, :] \frac{\partial R_1(\theta)}{\partial \theta} \dot{\theta} \right) \times {}^C \mathbf{c} \\ &= \left(\frac{\partial R_3(\phi)[1, :]}{\partial \phi} R_1(\theta) \dot{\phi} + R_3(\phi)[1, :] \frac{\partial R_1(\theta)}{\partial \theta} \dot{\theta} \right) \times R_1^T(\theta) R_3^T(\phi) \times {}^C \mathbf{a} \\ &= \left[\left(\frac{\partial R_3(\phi)[1, :]}{\partial \phi} R_3^T(\phi) \right) \dot{\phi} + \left(R_3(\phi)[1, :] \frac{\partial R_1(\theta)}{\partial \theta} R_1^T(\theta) R_3^T(\phi) \right) \dot{\theta} \right] \times {}^C \mathbf{a} \\ &= \left[\begin{pmatrix} -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \dot{\phi} + ([0 \ 0 \ \sin \phi]) \dot{\theta} \times {}^C \mathbf{a} \quad [\text{From 1(b)}] \\ &= ([0 \ 1 \ 0] \dot{\phi} + [0 \ 0 \ \sin \phi] \dot{\theta}) \times {}^C \mathbf{a} \end{aligned}$$

Substituting $\phi = \frac{\pi}{6}$ $\dot{\phi} = 4$ $\dot{\theta} = 6$

$$\Rightarrow {}^C \frac{d\mathbf{a}_1}{dt} = 4\mathbf{a}_2 + 3\mathbf{a}_3$$

1.4 1(d) Holonomic constraints in a rotating plane

Problem: In Figure 2, N designates a plane that is made to rotate with constant angular speed ω about a line Z fixed in N and in a reference frame R . The unit vectors $\mathbf{n}_x, \mathbf{n}_y$ and \mathbf{n}_z are mutually perpendicular and fixed in R , and \mathbf{n} is a unit vector normal to N and equal to \mathbf{n}_x at time $t = 0$. Finally, P_1 and P_2 represent particles connected to each other by a rigid rod of length L , these particles remaining at all times in contact with N .

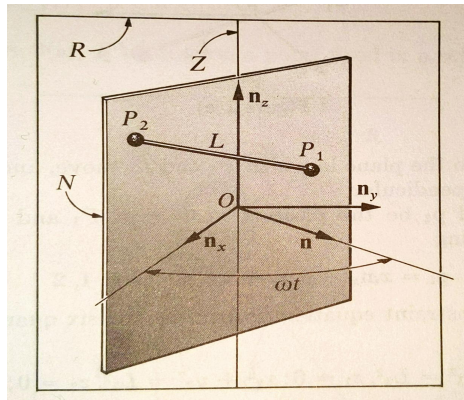


Figure 2

Letting \mathbf{p}_1 and \mathbf{p}_2 be the position vectors of P_1 and P_2 relative to a point O fixed in line Z , and taking

$$\mathbf{p}_i = x_i \mathbf{n}_x + y_i \mathbf{n}_y + z_i \mathbf{n}_z \quad i = 1, 2$$

determine functions $f_j(x_1, y_1, z_1, x_2, y_2, z_2, t)$, for $j = 1, 2, 3$, such that the requirements that P_1 and P_2 remain in N and be separated by distance L can be expressed as $f_j = 0$, $j = 1, 2, 3$.

Sol.:

1. For P_1 and P_2 to be attached to N at all times,

$$\mathbf{p}_i \cdot \mathbf{n} = 0 \quad \forall t, \quad i = 1, 2$$

We have,

$$\mathbf{n}(t) = \mathbf{n}_x \sin \omega t + \mathbf{n}_y \cos \omega t$$

$$\implies \mathbf{p}_i \cdot \mathbf{n} = x_i \sin \omega t + y_i \cos \omega t = 0 \quad i = 1, 2$$

$$\therefore f_1 = x_1 \sin \omega t + y_1 \cos \omega t$$

$$f_2 = x_2 \sin \omega t + y_2 \cos \omega t$$

2. For the distance between P_1 and P_2 to remain L :

$$|\mathbf{p}_1 - \mathbf{p}_2| = L$$

$$\implies f_3 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - L^2$$

Note: f_1, f_2, f_3 are holonomic constraints. f_1, f_2 are rheonomic and f_3 is scleronomic.

A **holonomic** constraint is a **kinematic constraint** equations that only involves position vectors (**scleronomic**) or can be integrated to position vectors and time (**rheonomic**) with none of its derivatives. If a kinematic constraint equations has non-integrable derivatives of position vectors, then they are **non-holonomic** constraints.

1.5 1(e) Holonomic constraints in a rotating bar

Two particles, P_1 and P_2 , are connected by a rigid rod that is free to rotate about an axis parallel to a unit vector \mathbf{n}_z and passing through a point O of the rod, as shown in Figure 3, where \mathbf{n}_x and \mathbf{n}_y are unit vectors parallel to the plane in which P_1 and P_2 move and $\mathbf{n}_x, \mathbf{n}_y$ and \mathbf{n}_z are mutually perpendicular.

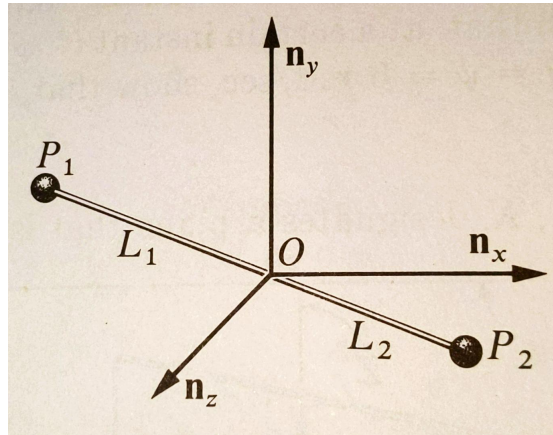


Figure 3

Letting \mathbf{p}_1 and \mathbf{p}_2 be the position vectors of P_1 and P_2 relative to point O , and taking,

$$\mathbf{p}_i = x_i \mathbf{n}_x + y_i \mathbf{n}_y + z_i \mathbf{n}_z \quad i = 1, 2$$

construct five constraint equations governing the six quantities x_i, y_i, z_i for $i = 1, 2$.

Sol.:

- The system rotates in only x-y plane.

$$z_1 = z_2 = 0 \quad \dots 1, 2$$

- The distance from origin remain the same.

$$x_1^2 + y_1^2 = L_1^2 \quad \dots 3$$

$$x_2^2 + y_2^2 = L_2^2 \quad \dots 4$$

- P_1 and P_2 lie on the same straight line, i.e., the slopes are equal. Let,

$$\sin \theta = \frac{-y_1}{L_1} = \frac{y_2}{L_2} = \sqrt{\frac{y_1 y_2}{-L_1 L_2}}$$

$$\cos \theta = \frac{x_1}{L_1} = \frac{x_2}{L_2} = \sqrt{\frac{x_1 x_2}{-L_1 L_2}}$$

$$\text{Substituting into: } \sin^2 \theta + \cos^2 \theta = 1$$

$$\implies x_1 x_2 + y_1 y_2 = -L_1 L_2 \quad \dots 5$$

1.6 1(f) Relative degrees of freedom

Problem: Referring to Problem 1(d), and letting S be the set of particles P_1 and P_2 , determine the number of degrees of freedom of S in R .

Sol.:

We have 3 constraint equations ($M = 3$) and 2 particles ($N = 2$). The number of degrees of freedom:

$$3N - M = 3 \times 2 - 3 = 3$$

1.7 1(g) Generalized coordinates.

Problem: Referring to the Problem 1(e), express the six quantities x_i, y_i, z_i with $i = 1, 2$, each as a function of a single quantity q in such a way that the five constraint equations found previously are satisfied for all values of q . (Suspension: Let q be the radian measure of the angle between \mathbf{n}_x and \mathbf{p}_2 .)

Sol.:

$$x_1 = -L_1 \cos q$$

$$y_1 = L_1 \sin q$$

$$z_1 = 0$$

$$x_2 = L_2 \cos q$$

$$y_2 = -L_2 \sin q$$

$$z_2 = 0$$

1.8 1(h) Degrees of freedom

Problem: Determine the number of degrees of freedom of each of the following holonomic systems:

Sol.:

1. Two rigid bodies attached to each other by means of a ball-and-socket connection.
– The position of both the bodies is constrained but not the orientations of the individual bodies.

$$n = 9$$

2. An earth satellite carrying a rotor that is made to rotate at a prescribed rate about an axis fixed in the satellite.
– All dof's of the rotor are constrained to that of satellite except the rotation about its axis which is also constrained as its rate is prescribed.

$$n = 6$$

3. An earth satellite carrying a rotor that is made to rotate at a prescribed rate about an axis fixed in the satellite.
– All dof's of the rotor are constrained to that of satellite except the rotation about its axis.

$$n = 7$$

4. The particles P_1, P_2 of Problem 1(e).
– The only degree of freedom is the rotation about \mathbf{n}_z .

$$n = 1$$

2 Problem Set 2

2.1 2(a) ${}^R\omega^B = {}^R\omega^A + {}^A\omega^B = \Omega A_3 + \dot{\theta}_1 X_1 + \dot{\theta}_2 X_2 + \dot{\theta}_3 X_3$

Problem: In Figure 4, P represents a point fixed in a reference frame R, and B^* designates the mass center of a rigid body B that moves on a circular orbit C fixed in R and centered at P. A_1, A_2 and A_3 are mutually perpendicular directed line segments, A_1 being the extension of line PB^* , A_2 pointing in the direction of motion of B^* on C, and A_3 thus being normal to the plane of the orbit B^* .

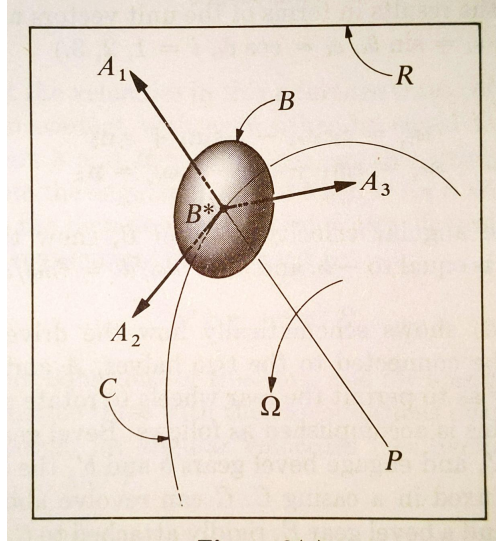


Figure 4

If X_1, X_2 and X_3 are mutually perpendicular directed line segments passing through B^* and fixed in the body B, the "attitude" of B relative to A_1, A_2, A_3 can be specified in terms of three angles θ_1, θ_2 and θ_3 , generated as follows: **Align X_i with A_i , for $i = 1, 2, 3$, and perform successive right-handed rotations of B, of amount θ_1 about X_1 , θ_2 about X_2 , and θ_3 about X_3 .**

The angular velocity ω of B in R can be expressed as $\omega = \omega_1 \mathbf{n}_1 + \omega_2 \mathbf{n}_2 + \omega_3 \mathbf{n}_3$, where \mathbf{n}_i is a unit vector parallel to X_i , and ω_i is a function of $\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$, and the angular speed Ω of the line PB^* in R. Consequently, $\dot{\theta}_i$ can be expressed as a function f_i of $\theta_1, \theta_2, \theta_3, \omega_1, \omega_2, \omega_3$, $i = 1, 2, 3$ and Ω .

Determine the functions f_1, f_2 and f_3 , using the abbreviations $s_i = \sin \theta_i, c_i = \cos \theta_i$ for $i = 1, 2, 3$, to state the results.

Sol.:

Note: The above explanation of X_1, X_2 and X_3 is the loose definition of Euler angles used to define the orientation of a rigid body.

Rotation Matrices:

The given description of obtaining the attitude of the body can be put mathematically using rotation matrices as follows:

1. Right-handed rotation about X_1 by θ_1 :

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix}}_{R_1} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

2. Right-handed rotation about X_2 by θ_2 :

$$\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} = \underbrace{\begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix}}_{R_2} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Interpretation: \mathbf{n}_i are the unit vectors parallel to X_i after the above transformation sequence.
Thus,

$$\begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix} = \begin{bmatrix} X_1'' \\ X_2' \\ X_3 \end{bmatrix}$$

3. Right-handed rotation about X_3 by θ_3 :

$$\begin{bmatrix} X_1'' \\ X_2' \\ X_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3} \begin{bmatrix} X_1' \\ X_2' \\ X_3 \end{bmatrix}$$

We have, the (instantaneous) angular velocity of the body in A frame:

$${}^A\boldsymbol{\omega}^B = \dot{\theta}_1 X_1 + \dot{\theta}_2 X_2 + \dot{\theta}_3 X_3$$

The angular velocity of A in R :

$${}^R\boldsymbol{\omega}^A = \Omega A_3$$

The angular velocity of the body in reference frame:

$$\begin{aligned} l_x {}^R\boldsymbol{\omega}^B &= {}^R\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B \\ &= \Omega A_3 + \dot{\theta}_1 X_1 + \dot{\theta}_2 X_2 + \dot{\theta}_3 X_3 \end{aligned}$$

Required form:

$${}^R\boldsymbol{\omega}^B = \omega_1 \mathbf{n}_1 + \omega_2 \mathbf{n}_2 + \omega_3 \mathbf{n}_3$$

Thus we need to write A_3, X_1, X_2 and X_3 in terms of $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$.

From the third transformation:

$$\begin{aligned} \begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} &= \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix} \\ \Rightarrow X_2 &= s_3 \mathbf{n}_1 + c_3 \mathbf{n}_2 & X_3 &= \mathbf{n}_3 \\ X_1' &= c_3 \mathbf{n}_1 - s_3 \mathbf{n}_2 \end{aligned}$$

From the second transformation:

$$\begin{aligned} \begin{bmatrix} X_1 \\ X_2 \\ X_3' \end{bmatrix} &= \begin{bmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \\ X_3 \end{bmatrix} \\ \Rightarrow X_1 &= c_2 X_1' + s_2 X_3 \\ \Rightarrow X_1 &= c_2 (c_3 \mathbf{n}_1 - s_3 \mathbf{n}_2) + s_2 \mathbf{n}_3 \end{aligned}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_T \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix}$$

From the first transformation:

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \text{and} \quad X_3' = -s_2 X_1' + c_2 X_3 = -s_2(c_3 \mathbf{n}_1 - s_3 \mathbf{n}_2) + c_2 \mathbf{n}_3$$

$$\begin{aligned} \Rightarrow A_3 &= s_1 X_2 + c_1 X_3' = s_1(s_3 \mathbf{n}_1 + c_3 \mathbf{n}_2) + c_1(-s_2(c_3 \mathbf{n}_1 - s_3 \mathbf{n}_2) + c_2 \mathbf{n}_3) \\ &= \underbrace{\begin{bmatrix} s_1 s_3 - c_1 s_2 c_3 & s_1 c_3 + c_1 s_2 s_3 & c_1 c_2 \end{bmatrix}}_{P^T} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix} \end{aligned}$$

Substituting, and writing in matrix form:

$$\begin{aligned} \omega_1 \mathbf{n}_1 + \omega_2 \mathbf{n}_2 + \omega_3 \mathbf{n}_3 &= \Omega A_3 + \dot{\theta}_1 X_1 + \dot{\theta}_2 X_2 + \dot{\theta}_3 X_3 \\ \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix} &= \Omega P^T \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix} + \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{bmatrix} T \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} &= \Omega P + T^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} &= [T^T]^{-1} \left(\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} - \Omega P \right) \end{aligned}$$

Symbolically solving (using sympy), we get:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} \frac{\Omega \sin(\theta_2) \cos(\theta_1) + \omega_1 \cos(\theta_3) - \omega_2 \sin(\theta_3)}{\cos(\theta_2)} \\ -\Omega \sin(\theta_1) + \omega_1 \sin(\theta_3) + \omega_2 \cos(\theta_3) \\ -\frac{\Omega \cos(\theta_1)}{\cos(\theta_2)} - \omega_1 \cos(\theta_3) \tan(\theta_2) + \omega_2 \sin(\theta_3) \tan(\theta_2) + \omega_3 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} (\omega_1 c_3 - \omega_2 s_3 + \Omega s_2 c_1)/c_2 \\ \omega_1 s_3 + \omega_2 c_3 - \Omega s_1 \\ [(\omega_2 s_3 - \omega_1 c_3)s_2 + \omega_3 c_2 - \Omega c_1]/c_2 \end{bmatrix}$$

Sympy Code:

```
import sympy as sp

th1, th2, th3 = sp.symbols("theta_1, theta_2, theta_3")
th1_d, th2_d, th3_d = sp.symbols("theta__'1 theta__'2 theta__'3")
Omega = sp.symbols("Omega")
omega_1, omega_2, omega_3 = sp.symbols("omega_1 omega_2 omega_3")

#R_1
R_1 = sp.Matrix([[1, 0, 0], [0, sp.cos(th1), sp.sin(th1)], [0, -sp.sin(th1), sp.cos(th1)]])

# R_2
R_2 = sp.Matrix([[sp.cos(th2), 0, -sp.sin(th2)], [0, 1, 0], [sp.sin(th2), 0, sp.cos(th2)]])

# R_3
```

```

R_3 = sp.Matrix([[sp.cos(th3), sp.sin(th3), 0], [-sp.sin(th3), sp.cos(th3), 0],
[0, 0, 1]])
#sp.pprint([R_1, R_2, R_3])

P = ((R_1.T @ R_2.T @ R_3.T)[2, :]).T
X_1 = (R_2.T @ R_3.T)[0, :]
X_2 = (R_3.T)[1, :]
X_3 = sp.Matrix([0, 0, 1]).T
Tr = sp.Matrix([X_1, X_2, X_3])
omega = sp.Matrix([[omega_1], [omega_2], [omega_3]])
invTr_T = sp.simplify(Tr.inv()).T
dots = sp.simplify( invTr_T @ (omega - Omega * P) )
sp.print_latex(dots)

```

2.2 2(b) $\omega = \sum \omega_{\dot{q}_i} \dot{q}_i + \omega_t$

Sol.

Given,

The motion of B^* on C is prescribed (predetermined) $\implies \Omega(t)$ is give.

We have angular velocity written interms of partial rates:

$$\omega = \sum \omega_{\dot{q}_i} \dot{q}_i + \omega_t$$

We have from 2(a),

$${}^R\omega^B = \Omega(t)P^T \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix} + \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{bmatrix} T \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix}$$

$$P = \begin{bmatrix} s_1 s_3 - c_1 s_2 c_3 \\ s_1 c_3 + c_1 s_2 s_3 \\ c_1 c_2 \end{bmatrix} \quad T = \begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let, $\mathbf{n} = [\mathbf{n}_1 \quad \mathbf{n}_2 \quad \mathbf{n}_3]^T$

Thus, we have the partial rates:

$${}^R\omega_t^B = \frac{\partial {}^R\omega^B}{\partial t} = \dot{\Omega}P^T \mathbf{n}$$

$${}^R\omega_{\dot{\theta}_1}^B = \frac{\partial {}^R\omega^B}{\partial \dot{\theta}_1} = T[1, :] \mathbf{n} = \begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \end{bmatrix} \mathbf{n}$$

$${}^R\omega_{\dot{\theta}_2}^B = \frac{\partial {}^R\omega^B}{\partial \dot{\theta}_2} = T[2, :] \mathbf{n} = \begin{bmatrix} s_3 & c_3 & 0 \end{bmatrix} \mathbf{n}$$

$${}^R\omega_{\dot{\theta}_3}^B = \frac{\partial {}^R\omega^B}{\partial \dot{\theta}_3} = T[3, :] \mathbf{n} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{n}$$

2.3 2(c) Prove that ${}^B\omega^A = -{}^A\omega^B$ but, ${}^A d\omega \backslash dt = {}^B d\omega \backslash dt$

Given, ${}^B\omega^A = \omega$.

Let, \mathbf{v} be a vector, then

$${}^B \frac{d\mathbf{v}}{dt} = {}^A \frac{d\mathbf{v}}{dt} + \boldsymbol{\omega} \times \mathbf{v} \implies {}^A \frac{d\mathbf{v}}{dt} = {}^B \frac{d\mathbf{v}}{dt} - \boldsymbol{\omega} \times \mathbf{v}$$

but,

$${}^A \frac{d\mathbf{v}}{dt} = {}^B \frac{d\mathbf{v}}{dt} + {}^A \boldsymbol{\omega}^B \times \mathbf{v}$$

By comparison,

$${}^A \boldsymbol{\omega}^B = -\boldsymbol{\omega}$$

$$\therefore {}^B \boldsymbol{\omega}^A = \boldsymbol{\omega} \implies {}^A \boldsymbol{\omega}^B = -\boldsymbol{\omega} \quad q.e.d$$

Using the operator definition of angular velocity vector on itself:

$${}^B \frac{d\boldsymbol{\omega}}{dt} = {}^A \frac{d\boldsymbol{\omega}}{dt} + \underbrace{\boldsymbol{\omega} \times \boldsymbol{\omega}}_{=0}$$

$$\therefore {}^B \frac{d\boldsymbol{\omega}}{dt} = {}^A \frac{d\boldsymbol{\omega}}{dt} \quad q.e.d$$

2.4 2(d) Differential gear and drive shaft kinematics

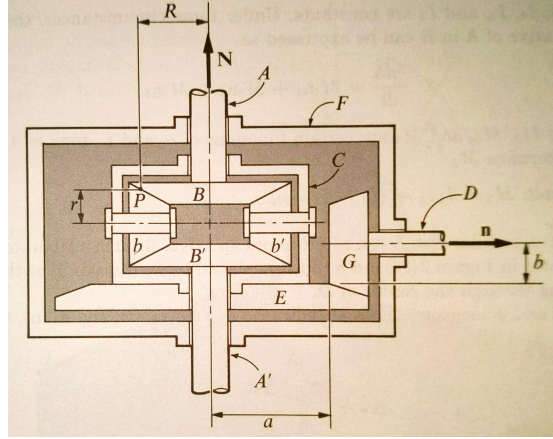


Figure 5

Mechanism: Bevel gears B and B' are keyed to A and A' , and engage bevel gears b and b' , the latter being free to rotate on pins fixed in a casing C . C can revolve about the common axis of A and A' , and a bevel gear E , rigidly attached to C , is driven by the gear G , which is keyed to the drive shaft D .

Given:

$${}^F \boldsymbol{\omega}^A = \Omega \mathbf{N}$$

$${}^F \boldsymbol{\omega}^{A'} = \Omega' \mathbf{N}$$

$${}^F \boldsymbol{\omega}^D = \omega \mathbf{n}$$

The gears have simple angular velocities w.r.t the frame attached to their shafts. And, the contact points should have same linear velocities. Writing only the magnitudes:

$${}^F \omega^D b = {}^F \omega^C a \implies {}^F \omega^C = \frac{b}{a} \omega$$

$$\therefore {}^F \boldsymbol{\omega}^C = \frac{b}{a} \omega \mathbf{N}$$

Also,

$$\begin{aligned} {}^C\boldsymbol{\omega}^B &= {}^F\boldsymbol{\omega}^B - {}^C\boldsymbol{\omega}^C = {}^F\boldsymbol{\omega}^A - {}^C\boldsymbol{\omega}^C = \left(\Omega - \frac{b}{a}\omega\right)\mathbf{N} \\ {}^C\boldsymbol{\omega}^{B'} &= {}^F\boldsymbol{\omega}^{B'} - {}^C\boldsymbol{\omega}^C = {}^F\boldsymbol{\omega}^{A'} - {}^C\boldsymbol{\omega}^C = \left(\Omega' - \frac{b}{a}\omega\right)\mathbf{N} \end{aligned}$$

$$\begin{aligned} \left. \begin{aligned} {}^C\boldsymbol{\omega}^{B'}R &= {}^C\boldsymbol{\omega}^{b'r} \\ {}^C\boldsymbol{\omega}^{B'}R &= -{}^C\boldsymbol{\omega}^{br} \\ {}^C\boldsymbol{\omega}^BR &= -{}^C\boldsymbol{\omega}^{b'r} \\ {}^C\boldsymbol{\omega}^BR &= {}^C\boldsymbol{\omega}^{br} \end{aligned} \right\} \implies {}^C\boldsymbol{\omega}^{B'} + {}^C\boldsymbol{\omega}^B = 0 \\ \implies \left(\Omega - \frac{b}{a}\omega\right) + \left(\Omega' - \frac{b}{a}\omega\right) = 0 \end{aligned}$$

$$\therefore \omega = \frac{a}{2b}(\Omega + \Omega')$$

2.5 2(e) Derivative of angular momentum

We have,

$$\begin{aligned} {}^R\frac{d\mathbf{A}}{dt} &= B\frac{d\mathbf{A}}{dt} + {}^R\boldsymbol{\omega}^B \times \mathbf{A} \\ &= \begin{bmatrix} \dot{\omega}_1 & \dot{\omega}_2 & \dot{\omega}_3 \end{bmatrix} \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix} + \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ A_1\omega_1 & A_2\omega_2 & A_3\omega_3 \end{vmatrix} \end{aligned}$$

$$\text{Let, } {}^R\frac{d\mathbf{A}}{dt} = M_1\mathbf{n}_1 + M_2\mathbf{n}_2 + M_3\mathbf{n}_3$$

We have,

$$\begin{aligned} M_1 &= \dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ M_2 &= \dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \\ M_3 &= \dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 \end{aligned}$$

2.6 2(f) Angular acceleration

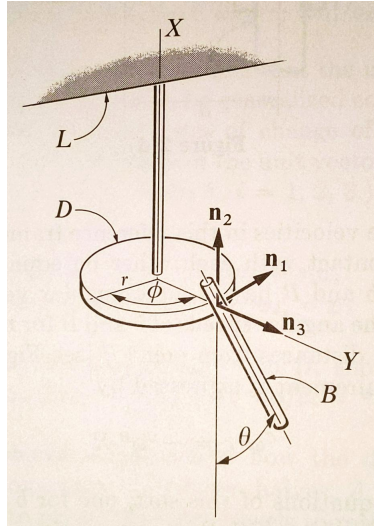


Figure 6

Sol.

$${}^L\boldsymbol{\omega}^D = \dot{\phi}\mathbf{n}_2 \quad \text{and} \quad {}^L\boldsymbol{\omega}^B = \dot{\phi}\mathbf{n}_2 + \dot{\theta}\mathbf{n}_3$$

$$\begin{aligned} {}^L\boldsymbol{\alpha}^B &= \frac{{}^L d\boldsymbol{\omega}^B}{dt} = \frac{{}^L d}{dt} (\dot{\phi}\mathbf{n}_2 + \dot{\theta}\mathbf{n}_3) \\ &= \ddot{\phi}\mathbf{n}_2 + \underbrace{\dot{\phi}({}^L\boldsymbol{\omega}^D \times \mathbf{n}_2)}_{=0} + \ddot{\theta}\mathbf{n}_3 + \underbrace{\dot{\theta}({}^L\boldsymbol{\omega}^D \times \mathbf{n}_3)}_{=\dot{\phi}n_1} \quad [\because \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \text{ are attached to D (not B).}] \\ \Rightarrow {}^L\boldsymbol{\alpha}^B &= \dot{\phi}\dot{\theta}\mathbf{n}_1 + \ddot{\phi}\mathbf{n}_2 + \ddot{\theta}\mathbf{n}_3 \\ \Rightarrow \alpha_1 &= \dot{\phi}\dot{\theta}, \quad \alpha_2 = \ddot{\phi}, \quad \alpha_3 = \ddot{\theta} \end{aligned}$$

2.7 2(g) Angular acceleration of a rolling disk: Choice of coordinates

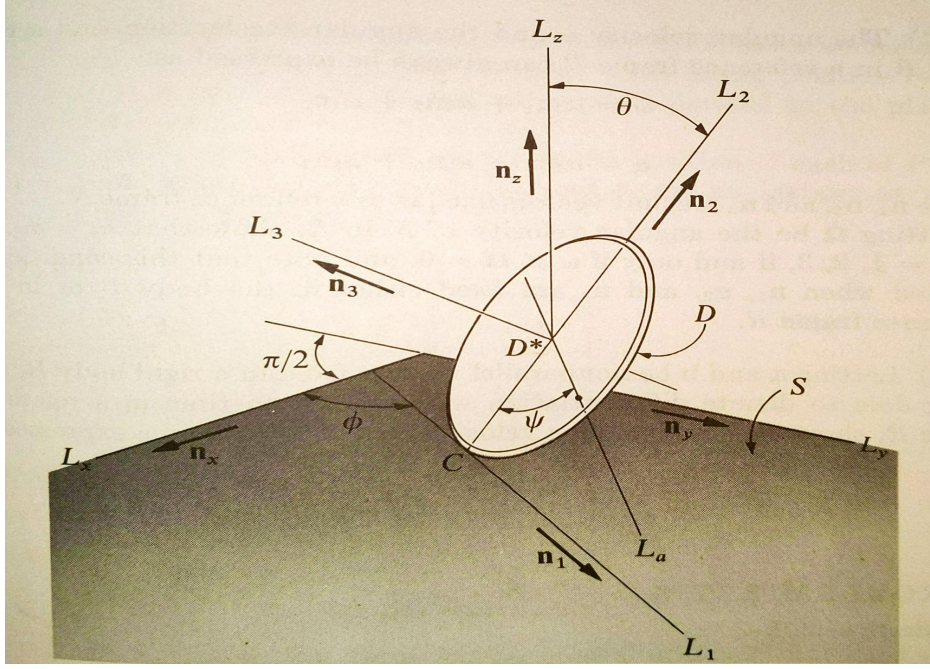


Figure 7

The transformation between the two coordinates $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ and $\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z$ can be written as:

$$\begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix} = \underbrace{R_1(90 - \theta)R_3(\phi)}_R \begin{bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{bmatrix}$$

$$R_1(90 - \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(90 - \theta) & \sin(90 - \theta) \\ 0 & -\sin(90 - \theta) & \cos(90 - \theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin \theta & \cos \theta \\ 0 & -\cos \theta & \sin \theta \end{bmatrix}$$

$$R_3(\phi) = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin \theta & \cos \theta \\ 0 & -\cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \theta \sin \phi & \sin \theta \cos \phi & \cos \theta \\ \cos \theta \sin \phi & -\cos \theta \cos \phi & \sin \theta \end{bmatrix}$$

$$R^{-1} = (R_1(90 - \theta)R_3(\phi))^{-1} = R_3^{-1}(\phi)R_1^{-1}(90 - \theta) = R_3^T(\phi)R_1^T(90 - \theta)$$

$$= \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin \theta & -\cos \theta \\ 0 & \cos \theta & \sin \theta \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \sin \theta & \sin \phi \cos \theta \\ \sin \phi & \cos \theta \cos \phi & -\cos \phi \cos \theta \\ 0 & \cos \theta & \sin \theta \end{bmatrix}$$

$${}^R\boldsymbol{\omega}^D = -\dot{\theta}\mathbf{n}_1 + \dot{\phi}\mathbf{n}_z + \dot{\psi}\mathbf{n}_3 = -\dot{\theta}\mathbf{n}_1 + \dot{\phi}\cos\theta\mathbf{n}_2 + (\dot{\phi}\sin\theta + \dot{\psi})\mathbf{n}_3$$

We have,

$${}^R\boldsymbol{\alpha}^B = \frac{{}^R d\boldsymbol{\omega}^B}{dt} = \frac{{}^R d}{dt} (-\dot{\theta}\mathbf{n}_1 + \dot{\phi}\mathbf{n}_z + \dot{\psi}\mathbf{n}_3) = -\ddot{\theta}\mathbf{n}_1 - \dot{\theta}\dot{\mathbf{n}}_1 + \ddot{\phi}\mathbf{n}_z + \dot{\phi}\dot{\mathbf{n}}_z + \ddot{\psi}\mathbf{n}_3 + \dot{\psi}\dot{\mathbf{n}}_3$$

also,

$$\begin{aligned}
{}^R \frac{d\mathbf{n}_1}{dt} &= {}^R \boldsymbol{\omega}^{L_1} \times \mathbf{n}_1 = \dot{\phi} \mathbf{n}_z \times \mathbf{n}_1 = \dot{\phi}(\cos \theta \mathbf{n}_2 + \sin \theta \mathbf{n}_3) \times \mathbf{n}_1 \\
&= \dot{\phi}(-\cos \theta \mathbf{n}_3 + \sin \theta \mathbf{n}_2) \\
{}^R \frac{d\mathbf{n}_z}{dt} &= 0 \quad [\because \mathbf{n}_z \text{ is fixed to the frame } R] \\
{}^R \frac{d\mathbf{n}_3}{dt} &= {}^R \boldsymbol{\omega}^{L_3} \times \mathbf{n}_3 = (\dot{\phi} \mathbf{n}_z - \dot{\theta} \mathbf{n}_1) \times \mathbf{n}_3 = (\dot{\phi}(\cos \theta \mathbf{n}_2 + \sin \theta \mathbf{n}_3) - \dot{\theta} \mathbf{n}_1) \times \mathbf{n}_3 \\
&= \dot{\phi} \cos \theta \mathbf{n}_1 + \dot{\theta} \mathbf{n}_2
\end{aligned}$$

Substituting,

$$\begin{aligned}
{}^R \boldsymbol{\alpha}^B &= -\ddot{\theta} \mathbf{n}_1 - \dot{\theta}(\dot{\phi}(-\cos \theta \mathbf{n}_3 + \sin \theta \mathbf{n}_2)) + \ddot{\phi}(\cos \theta \mathbf{n}_2 + \sin \theta \mathbf{n}_3) + \ddot{\psi} \mathbf{n}_3 + \dot{\psi}(\dot{\phi} \cos \theta \mathbf{n}_1 + \dot{\theta} \mathbf{n}_2) \\
&= \left[\underbrace{-\ddot{\theta} + \dot{\phi} \dot{\theta} \cos \theta}_{\alpha_1} \quad \underbrace{-\dot{\theta} \dot{\phi} \sin \theta + \ddot{\phi} \cos \theta + \dot{\psi} \dot{\theta}}_{\alpha_2} \quad \underbrace{\dot{\theta} \dot{\phi} \cos \theta + \ddot{\phi} \sin \theta + \ddot{\psi}}_{\alpha_3} \right] \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \end{bmatrix} \\
&= [\alpha_1 \quad \alpha_2 \quad \alpha_3] R \begin{bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{bmatrix} = [\alpha_1 \quad \alpha_2 \quad \alpha_3] \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \theta \sin \phi & \sin \theta \cos \phi & \cos \theta \\ \cos \theta \sin \phi & -\cos \theta \cos \phi & \sin \theta \end{bmatrix} \begin{bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 \cos \phi - \alpha_2 \sin \theta \sin \phi + \alpha_3 \cos \theta \cos \phi & (= \alpha_x) \\ \alpha_1 \sin \phi + \alpha_2 \sin \theta \cos \phi - \alpha_3 \cos \theta \cos \phi & (= \alpha_y) \\ \alpha_2 \cos \theta + \alpha_3 \sin \theta & (= \alpha_z) \end{bmatrix}^T \begin{bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{bmatrix}
\end{aligned}$$

alternately,

$$\begin{aligned}
{}^R \boldsymbol{\omega}^D &= -\dot{\theta} \mathbf{n}_1 + \dot{\phi} \mathbf{n}_z + \dot{\psi} \mathbf{n}_3 \\
&= -\dot{\theta}(\cos \phi \mathbf{n}_x + \sin \phi \mathbf{n}_y) + \dot{\phi} \mathbf{n}_z + \dot{\psi}(\cos \theta \sin \phi \mathbf{n}_x - \cos \theta \cos \phi \mathbf{n}_y + \sin \theta \mathbf{n}_z) \\
&= \begin{bmatrix} -\dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi & -\dot{\theta} \sin \phi - \dot{\psi} \cos \theta \cos \phi & \dot{\phi} + \dot{\psi} \sin \theta \end{bmatrix} \begin{bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{bmatrix}
\end{aligned}$$

Thus,

$$\begin{aligned}
{}^R \boldsymbol{\alpha}^B &= \frac{{}^R d\boldsymbol{\omega}^B}{dt} = \frac{{}^R d}{dt} \begin{bmatrix} -\dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi & -\dot{\theta} \sin \phi - \dot{\psi} \cos \theta \cos \phi & \dot{\phi} + \dot{\psi} \sin \theta \end{bmatrix} \begin{bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{bmatrix} \\
&= \begin{bmatrix} -\ddot{\theta} \cos \phi + \ddot{\psi} \cos \theta \sin \phi + \dot{\theta} \dot{\phi} \sin \phi + \dot{\psi} \dot{\phi} \cos \theta \cos \phi - \dot{\psi} \dot{\theta} \cos \theta \sin \phi & (= \alpha_x) \\ -\ddot{\theta} \sin \phi - \ddot{\psi} \cos \theta \cos \phi - \dot{\theta} \dot{\phi} \cos \phi + \dot{\psi} \dot{\theta} \sin \theta \cos \phi + \dot{\psi} \dot{\phi} \cos \theta \sin \phi & (= \alpha_y) \\ \ddot{\phi} + \ddot{\psi} \sin \theta + \dot{\psi} \dot{\theta} \cos \theta & (= \alpha_z) \end{bmatrix}^T \begin{bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \end{bmatrix}
\end{aligned}$$

Note:

- We can not just take $d\mathbf{n}_z/dt$ as $\boldsymbol{\omega} \times \mathbf{n}_z$ in $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ coordinates as \mathbf{n}_z is not rigidly fixed to the coordinates. This is only possible when \mathbf{n}_z doesn't change with time in $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ frame. But once, the differentiation is done, one vector in $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ frame can be written in $\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z$ frame using transformation matrices as they are instantaneous.
- Thus the rotation transformation can be used after differentiation as well.

2.8 2(h) α when body and frame have parallel ω 's

Given, $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\} \in N$ has an angular velocity Ω in R , and

$$\begin{aligned} {}^R\boldsymbol{\omega}^B &= \omega_1 \mathbf{n}_1 + \omega_2 \mathbf{n}_2 + \omega_3 \mathbf{n}_3 \\ {}^R\boldsymbol{\alpha}^B &= \alpha_1 \mathbf{n}_1 + \alpha_2 \mathbf{n}_2 + \alpha_3 \mathbf{n}_3 \\ \text{and, } \quad \alpha_i &= d\omega_i/dt = \dot{\omega}_i \end{aligned}$$

Consider,

$$\begin{aligned} {}^R\boldsymbol{\alpha}^B &= \frac{{}^R d\boldsymbol{\omega}^B}{dt} = \sum_{i=1}^3 \left(\dot{\omega}_i \mathbf{n}_i + \omega_i \frac{{}^R d\mathbf{n}_i}{dt} \right) = \sum_{i=1}^3 \left(\alpha_i \mathbf{n}_i + \omega_i \frac{{}^R d\mathbf{n}_i}{dt} \right) = {}^R\boldsymbol{\alpha}^B + \sum_{i=1}^3 \left(\omega_i \frac{{}^R d\mathbf{n}_i}{dt} \right) \\ &\iff \sum_{i=1}^3 \left(\omega_i \frac{{}^R d\mathbf{n}_i}{dt} \right) = 0 \end{aligned}$$

We have,

$$\sum_{i=1}^3 \left(\omega_i \frac{{}^R d\mathbf{n}_i}{dt} \right) = \sum_{i=1}^3 (\omega_i (\Omega \times \mathbf{n}_i)) = \Omega \times (\omega_1 \mathbf{n}_1 + \omega_2 \mathbf{n}_2 + \omega_3 \mathbf{n}_3) = \Omega \times \boldsymbol{\omega}$$

thus,

$$\alpha_i = \frac{d\omega_i}{dt}, \text{ for } i = 1, 2, 3 \iff \Omega \times \boldsymbol{\omega} = 0 \quad q.e.d$$

2.9 2(i) $\boldsymbol{\omega} = (\dot{\mathbf{a}} \times \dot{\mathbf{b}})/(\dot{\mathbf{a}} \cdot \dot{\mathbf{b}})$

Consider,

$$\begin{aligned} \dot{\mathbf{a}} \times \dot{\mathbf{b}} &= \dot{\mathbf{a}} \times (\boldsymbol{\omega} \times \mathbf{b}) = (\dot{\mathbf{a}} \cdot \mathbf{b}) \boldsymbol{\omega} - \underbrace{(\dot{\mathbf{a}} \cdot \boldsymbol{\omega})}_{=0} \mathbf{b} \quad \left[\begin{array}{l} \cdot \cdot \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \\ \cdot \cdot \quad (\boldsymbol{\omega} \times \mathbf{a}) \times \boldsymbol{\omega} = 0 \end{array} \right] \\ \implies \boldsymbol{\omega} &= \frac{\dot{\mathbf{a}} \times \dot{\mathbf{b}}}{(\dot{\mathbf{a}} \cdot \dot{\mathbf{b}})} \quad q.e.d \end{aligned}$$

3 Problem Set 3

KJDSAFF