# Problems Sets from Dynamics by Kane

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## Contents

# Part I
# Kinematics

## 1 Problem Set 1

### 1.1 1(a) Connected parallelopipeds

***Problem***: Four rectangular parallelopipeds, A, B, C, and D, are arranged as shown in Figure 1. $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ designate unit vedtors respectively parallel to the edges of $A$: $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ are unit vectors respectively parallel to the edges of $B$, and so forth, and $\phi, \theta$ and $\psi$ denote the radian measures of angles that determine the relative orientiation of the bodies. The configuration shown is one in which $\phi, \theta, \psi$ are regarded positive. Determine the maginitude of each of the following derivatives:

$$^{B}\frac{\partial \boldsymbol{a}_1}{\partial \phi}, \quad ^{B}\frac{\partial \boldsymbol{b}_1}{\partial \phi}, \quad ^{B}\frac{\partial \boldsymbol{a}_3}{\partial \phi}, \quad ^{B}\frac{\partial \boldsymbol{b}_2}{\partial \theta}, \quad ^{C}\frac{\partial \boldsymbol{b}_2}{\partial \theta}, \quad ^{D}\frac{\partial \boldsymbol{b}_2}{\partial \theta}, \quad ^{C}\frac{\partial \boldsymbol{b}_2}{\partial \psi}, \quad ^{D}\frac{\partial \boldsymbol{b}_2}{\partial \psi}, \quad ^{D}\frac{\partial \boldsymbol{a}_1}{\partial \psi}$$

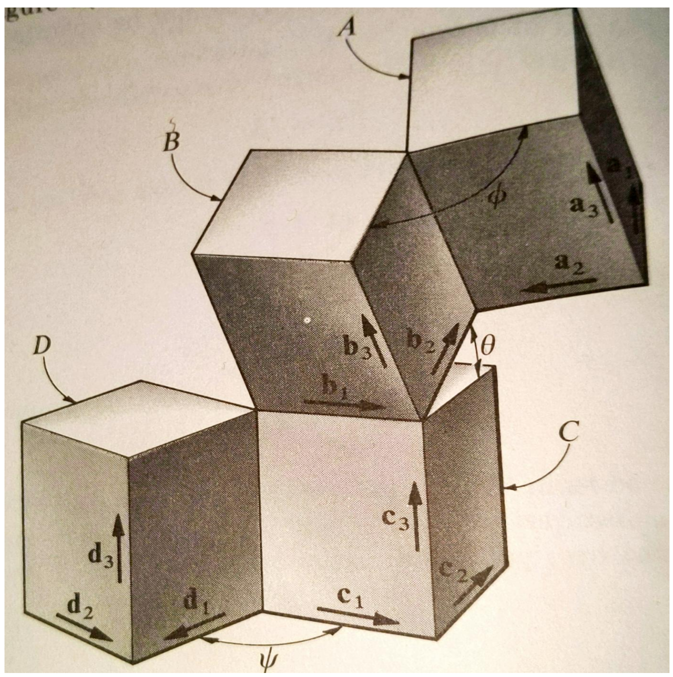


Figure 1

***sol.***

We have the following rotation matrices:

$$^{B}\begin{bmatrix}\boldsymbol{a}_1\\ \boldsymbol{a}_2\\ \boldsymbol{a}_3\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}}_{R_3(\phi)} {}^{B}\begin{bmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\\ \boldsymbol{b}_3\end{bmatrix}; \quad ^{C}\begin{bmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\\ \boldsymbol{b}_3\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}}_{R_1(\theta)} {}^{C}\begin{bmatrix}\boldsymbol{c}_1\\ \boldsymbol{c}_2\\ \boldsymbol{c}_3\end{bmatrix}; \quad ^{D}\begin{bmatrix}\boldsymbol{c}_1\\ \boldsymbol{c}_2\\ \boldsymbol{c}_3\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}}_{R_3(\psi)} {}^{D}\begin{bmatrix}\boldsymbol{d}_1\\ \boldsymbol{d}_2\\ \boldsymbol{d}_3\end{bmatrix}$$

Let, $\quad \boldsymbol{a} = \begin{bmatrix}\boldsymbol{a}_1\\ \boldsymbol{a}_2\\ \boldsymbol{a}_3\end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix}\boldsymbol{b}_1\\ \boldsymbol{b}_2\\ \boldsymbol{b}_3\end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix}\boldsymbol{c}_1\\ \boldsymbol{c}_2\\ \boldsymbol{c}_3\end{bmatrix} \quad \boldsymbol{d} = \begin{bmatrix}\boldsymbol{d}_1\\ \boldsymbol{d}_2\\ \boldsymbol{d}_3\end{bmatrix}$

Thus, we have,

1.

$$
\begin{aligned}
{}^{B}\frac{\partial \boldsymbol{a}_1}{\partial \phi} &= \frac{\partial}{\partial \phi}\left(R_3(\phi)[1,:] \times {}^{B}\boldsymbol{b}\right) = \frac{\partial R_3(\phi)[1,:]}{\partial \phi}\ {}^{B}\boldsymbol{b} \qquad [\because \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3 \text{ are fixed in } B] \\
&= \begin{bmatrix} -\sin\phi & cos\phi & 0 \end{bmatrix} \times {}^{B}\boldsymbol{b} \\
&\implies {}^{B}\left|\frac{\partial \boldsymbol{a}_1}{\partial \phi}\right| = 1
\end{aligned}
$$

2.

$$
{}^{B}\frac{\partial \boldsymbol{b}_1}{\partial \phi} = 0 \implies {}^{B}\left|\frac{\partial \boldsymbol{b}_1}{\partial \phi}\right| = 0 \qquad [\because \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3 \text{ are fixed in } B]
$$

3.

$$
\begin{aligned}
{}^{B}\frac{\partial \boldsymbol{a}_3}{\partial \phi} &= \frac{\partial}{\partial \phi}\left(R_3(\phi)[3,:] \times {}^{B}\boldsymbol{b}\right) = \frac{\partial \boldsymbol{b}_3}{\partial \phi} = 0 \qquad [\because \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3 \text{ are fixed in } B] \\
&\implies {}^{B}\left|\frac{\partial \boldsymbol{a}_3}{\partial \phi}\right| = 0
\end{aligned}
$$

4.

$$
{}^{B}\frac{\partial \boldsymbol{b}_2}{\partial \theta} = 0 \implies {}^{B}\left|\frac{\partial \boldsymbol{b}_2}{\partial \theta}\right| = 0 \qquad [\because \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3 \text{ are fixed in } B]
$$

5.

$$
\begin{aligned}
{}^{C}\frac{\partial \boldsymbol{b}_2}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(R_1(\theta)[2,:] \times {}^{C}\boldsymbol{c}\right) = \frac{\partial R_1(\theta)[2,:]}{\partial \theta} \times {}^{C}\boldsymbol{c} \qquad [\because \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3 \text{ are fixed in } C] \\
&= \begin{bmatrix} 0 & -\sin\theta & \cos\theta \end{bmatrix} \times {}^{C}\boldsymbol{c} \\
&\implies {}^{C}\left|\frac{\partial \boldsymbol{b}_2}{\partial \theta}\right| = 1
\end{aligned}
$$

6.

$$
\begin{aligned}
{}^{D}\frac{\partial \boldsymbol{b}_2}{\partial \theta} &= \frac{\partial}{\partial \theta}\left((R_1(\theta)R_3(\psi))[2,:] \times {}^{D}\boldsymbol{d}\right) \qquad [\because \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3 \text{ are fixed in } D] \\
&= \frac{\partial (R_1(\theta)R_3(\psi))[2,:]}{\partial \theta}\ {}^{D}\begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \boldsymbol{d}_3 \end{bmatrix} = \left(\frac{\partial R_1(\theta)}{\partial \theta} R_3(\psi)\right)[2,:]\ {}^{D}\boldsymbol{d} = \left(\frac{\partial R_1(\theta)[2,:]}{\partial \theta} R_3(\psi)\right) {}^{D}\boldsymbol{d} \\
&= \left(\begin{bmatrix} 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) {}^{D}\boldsymbol{d} = \begin{bmatrix} \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{bmatrix} {}^{D}\boldsymbol{d} \\
&\implies {}^{D}\left|\frac{\partial \boldsymbol{b}_2}{\partial \theta}\right| = 1
\end{aligned}
$$

7.

$$ {}^{C}\frac{\partial \boldsymbol{b}_2}{\partial \psi} = 0 \implies {}^{C}\left|\frac{\partial \boldsymbol{b}_2}{\partial \psi}\right| = 0 \qquad [\because \text{Any vector defined in C is independent of } \psi] $$

8.

$$ \begin{aligned} {}^{D}\frac{\partial \boldsymbol{b}_2}{\partial \psi} &= \frac{\partial}{\partial \psi}\Big((R_1(\theta)R_3(\psi))[2,:] \times {}^{D}\boldsymbol{d}\Big) \qquad [\because \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3 \text{ are fixed in } D] \\ &= R_1(\theta)[2,:] \times \frac{\partial R_3(\psi)}{\partial \psi} \times {}^{D}\boldsymbol{d} = \begin{bmatrix} 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\psi & \cos\psi & 0 \\ -\cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix} \times {}^{D}\boldsymbol{d} \\ &= \begin{bmatrix} -\cos\theta\cos\psi & -\cos\theta\sin\psi & 0 \end{bmatrix} \times {}^{D}\boldsymbol{d} \\ &\implies {}^{D}\left|\frac{\partial \boldsymbol{b}_2}{\partial \psi}\right| = |\cos\theta| \end{aligned} $$

9.

$$ \begin{aligned} {}^{D}\frac{\partial \boldsymbol{a}_1}{\partial \psi} &= \frac{\partial}{\partial \psi}\Big((R_3(\phi)R_1(\theta)R_3(\psi))[2,:] \times {}^{D}\boldsymbol{d}\Big) = R_3(\phi)[1,:] \times R_1(\theta) \times \frac{\partial R_3(\psi)}{\partial \psi} \times {}^{D}\boldsymbol{d} \\ &= \begin{bmatrix} \cos\phi & \sin\phi & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -\sin\psi & \cos\psi & 0 \\ -\cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix} {}^{D}\boldsymbol{d} \\ &= \begin{bmatrix} -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & 0 \end{bmatrix} {}^{D}\boldsymbol{d} \\ \implies {}^{D}\left|\frac{\partial \boldsymbol{a}_1}{\partial \psi}\right| &= \sqrt{(-\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi)^2 + (\cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi)^2} \\ &= (\cos^2\phi + \sin^2\phi\sin^2\theta)^{1/2} \end{aligned} $$

## 1.2 1(b) $\omega$ in connected parallelopipeds

***Problem***: Referring to Problem 1($a$), determine $w_1, w_2$ and $w_3$ such that

$$ {}^{C}\frac{\partial \boldsymbol{a}_1}{\partial \theta} = w_1\boldsymbol{a}_1 + w_2\boldsymbol{a}_2 + w_3\boldsymbol{a}_3 $$

***sol.***
We have,

$$ \begin{aligned} {}^{C}\frac{\partial \boldsymbol{a}_1}{\partial \theta} &= \frac{\partial}{\partial \theta}[R_3(\phi)[1,:]R_1(\theta)] \times {}^{C}\boldsymbol{c} = R_3(\phi)[1,:]\frac{\partial R_1(\theta)}{\partial \theta} \times {}^{C}\boldsymbol{c} = R_3(\phi)[1,:]\frac{\partial R_1(\theta)}{\partial \theta} \times [R_3(\phi)R_1(\theta)]^{-1} \times {}^{C}\boldsymbol{a} \\ &= R_3(\phi)[1,:]\frac{\partial R_1(\theta)}{\partial \theta} \times R_1^T(\theta)R_3^T(\phi) \times {}^{C}\boldsymbol{a} \qquad [\because R_i^{-1} = R_i^T] \\ &= \begin{bmatrix} \cos\phi & \sin\phi & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \\ 0 & -\cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \times {}^{c}\boldsymbol{a} \\ &= \begin{bmatrix} \cos\phi & \sin\phi & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \times {}^{c}\boldsymbol{a} = \begin{bmatrix} 0 & 0 & \sin\phi \end{bmatrix} \times {}^{c}\boldsymbol{a} \end{aligned} $$

Hence,

$w_1 = w_2 = 0,\ w_3 = \sin\phi$

### 1.3 1(c) $\alpha$ in connected parallelopipeds

***Problem***: Referring to Problem 1($a$), and assuming that $\theta, \phi$ and $\psi$ are functions of the time $t$ such that, at a certain instant $t^*$, $\phi = \theta = \psi = \pi/6\ rad$, $\dot{\phi} = 4\ rad/sec$, and $\dot{\theta} = \dot{\psi} = 6\ rad/sec$, show that at time $t^*$,

$$\frac{{}^{C}\partial \boldsymbol{a}_1}{\partial t} = 4\boldsymbol{a}_2 + 3\boldsymbol{a}_3$$

***sol.***
We have,

$${}^{C}\boldsymbol{a}_1 = R_3(\phi)[1,:]R_1(\theta) \times {}^{C}\boldsymbol{c}$$

$$\begin{aligned}
\implies \quad \frac{{}^{C}d\boldsymbol{a}_1}{dt} &= \left(\frac{\partial R_3(\phi)[1,:]}{\partial \phi}R_1(\theta)\dot{\phi} + R_3(\phi)[1,:]\frac{\partial R_1(\theta)}{\partial \theta}\dot{\theta}\right) \times {}^{C}\boldsymbol{c} \\
&= \left(\frac{\partial R_3(\phi)[1,:]}{\partial \phi}R_1(\theta)\dot{\phi} + R_3(\phi)[1,:]\frac{\partial R_1(\theta)}{\partial \theta}\dot{\theta}\right) \times R_1^T(\theta)R_3^T(\phi) \times {}^{C}\boldsymbol{a} \\
&= \left[\left(\frac{\partial R_3(\phi)[1,:]}{\partial \phi}R_3^T(\phi)\right)\dot{\phi} + \left(R_3(\phi)[1,:]\frac{\partial R_1(\theta)}{\partial \theta}R_1^T(\theta)R_3^T(\phi)\right)\dot{\theta}\right] \times {}^{C}\boldsymbol{a} \\
&= \left[\left(\begin{bmatrix} -\sin\phi & \cos\phi & 0 \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\dot{\phi} + \left(\begin{bmatrix} 0 & 0 & \sin\phi \end{bmatrix}\right)\dot{\theta}\right] \times {}^{C}\boldsymbol{a} \qquad \text{[From 1(}b\text{)]} \\
&= \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\dot{\phi} + \begin{bmatrix} 0 & 0 & \sin\phi \end{bmatrix}\dot{\theta}\right){}^{C}\boldsymbol{a}
\end{aligned}$$

Substituting $\phi = \frac{\pi}{6}\ \dot{\phi} = 4\ \dot{\theta} = 6$

$$\implies \quad \frac{{}^{C}d\boldsymbol{a}_1}{dt} = 4\boldsymbol{a}_2 + 3\boldsymbol{a}_3$$

### 1.4 1(d) Holonomic constraints in a rotating plane

***Problem***: In Figure 2, N designates a plane that is made to rotate with constant angular speed $\omega$ about a line $Z$ fixed in $N$ and in a reference frame $R$. The unit vectors $\boldsymbol{n}_x, \boldsymbol{n}_y$ and $\boldsymbol{n}_z$ are mutually perpendicular and fixed in R, and $\boldsymbol{n}$ is a unit vector normal to $N$ and equal to $\boldsymbol{n}_x$ at time $t = 0$. Finally, $P_1$ and $P_2$ represent particles connected to each other by a rigid rod of lenghth $L$, these particles remaining at all times in contact with $N$.

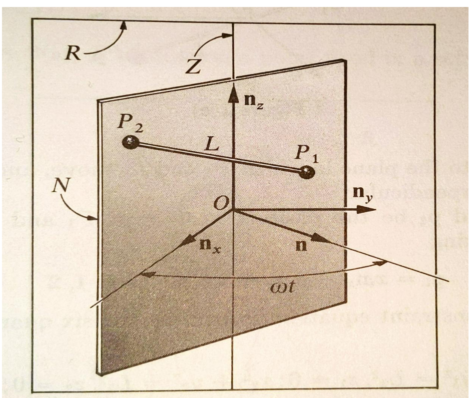


Figure 2

Letting $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ be the position vectors of $P_1$ and $P_2$ relative to a point $O$ fixed in line $Z$, and taking

$$\boldsymbol{p}_i = x_i\boldsymbol{n}_x + y_i\boldsymbol{n}_y + z_i\boldsymbol{n}_z \qquad i = 1,2$$

detarmine functions $f_j(x_1, y_1, z_1, x_2, y_2, z_2, t)$, for $j = 1,2,3$, such that the requirements that $P_1$ and $P_2$ remain in $N$ and be separated by distance $L$ can be experessed as $f_j = 0$, $j = 1,2,3$.

***Sol.***:

1. For $P_1$ and $P_2$ to be attached to $N$ at all times,

$$\boldsymbol{p}_i.\boldsymbol{n} = 0 \quad \forall\, t, \qquad i = 1,2$$

We have,

$$\boldsymbol{n}(t) = \boldsymbol{n}_x \sin\omega t + \boldsymbol{n}_y \cos\omega t$$

$$\implies \boldsymbol{p}_i.\boldsymbol{n} = x_i \sin\omega t + y_i \cos\omega t = 0 \qquad i = 1,2$$

$$\therefore f_1 = x_1 \sin\omega t + y_1 \cos\omega t$$
$$f_2 = x_2 \sin\omega t + y_2 \cos\omega t$$

2. For the distance between $P_1$ and $P_2$ to remain $L$:

$$|\boldsymbol{p}_1 - \boldsymbol{p}_2| = L$$

$$\implies f_3 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - L^2$$

**Note**: $f_1, f_2, f_3$ are holonomic constraints. $f_1, f_2$ are rheonomic and $f_3$ is scleronomic.

A ***holonomic*** constraint is a ***kinematic constraint*** equations that only involves position vectors (***scleronomic***) or can be integrated to position vectors and time (***rheonomic***) with none of it's derivatives. If a kinematic constraint equations has non-integrable derivatives of position vectors, then they are ***non-holonomic*** constraints.

### 1.5 1(e) Holonomic constraints in a rotating bar

Two particles, $P_1$ and $P_2$, are connected by a rigid rod that is free to rotate about an axis parallel to a unit vector $\boldsymbol{n}_z$ and passing through a point $O$ of the rod, as shown in Figure 3, where $\boldsymbol{n}_x$ and $\boldsymbol{n}_y$ are unit vectors parallel to the plane in which $P_1$ and $P_2$ move and $\boldsymbol{n}_x, \boldsymbol{n}_y$ and $\boldsymbol{n}_z$ are mutually perpendicular.

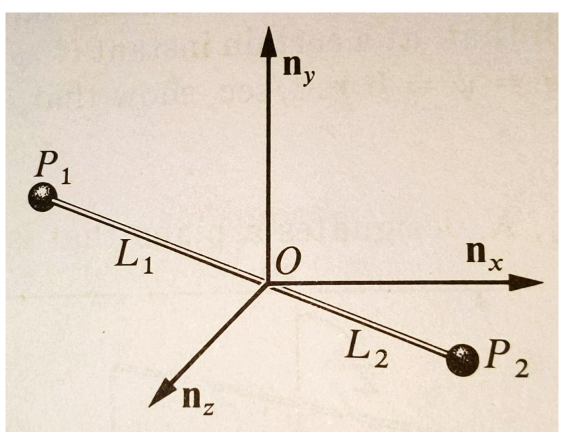


Figure 3

Letting $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ be the position vectors of $P_1$ and $P_2$ relative to point $O$, and taking,

$$\boldsymbol{p}_i = x_i\boldsymbol{n}_x + y_i\boldsymbol{n}_y + z_i\boldsymbol{n}_z \qquad i = 1, 2$$

construct five constraint eqautions governing the six quantities $x_i, y_i, z_i$ for $i = 1, 2$.
***Sol.***:

- The system rotates in only x-y plane.

$$z_1 = z_2 = 0 \qquad \ldots 1, 2$$

- The distance from origin remain the same.

$$x_1^2 + y_1^2 = L_1^2 \qquad \ldots 3$$
$$x_2^2 + y_2^2 = L_2^2 \qquad \ldots 4$$

- $P_1$ and $P_2$ lie on the same straight line, i.e., the slopes are equal. Let,

$$\sin\theta = \frac{-y_1}{L_1} = \frac{y_2}{L_2} = \sqrt{\frac{y_1 y_2}{-L_1 L_2}}$$
$$\cos\theta = \frac{x_1}{L_1} = \frac{x_2}{L_2} = \sqrt{\frac{x_1 x_2}{-L_1 L_2}}$$

Substituting into: $\sin^2\theta + \cos^2\theta = 1$

$$\Longrightarrow x_1x_2 + y_1y_2 = -L_1L_2 \qquad \ldots 5$$

## 1.6 1(f) Relative degrees of freedom

***Problem***: Referring to Problem $1(d)$, and letting S be the set of particles $P_1$ and $P_2$, determine the number of degrees of freedom of $S$ in $R$.

***Sol.***:

We have 3 constraint eqautions ($M = 3$) and 2 particles ($N = 2$). The number of degrees of freedom:

$$3N - M = 3 \times 2 - 3 = 3$$

## 1.7 1(g) Generalized coordinates.

***Problem***: Referring to the Problem $1(e)$, express the six quantities $x_i, y_i, z_i$ with $i = 1, 2$, each as a function of a single quantity $q$ in such a way that the five constraint equations found previously are satisfied for all values of $q$. (Suspension: Let $q$ be the radian measure of the angle between $\boldsymbol{n}_x$ and $\boldsymbol{p}_2$.)

***Sol.***:

$$x_1 = -L_1\cos q \qquad x_2 = L_2\cos q$$
$$y_1 = L_1\sin q \qquad y_2 = -L_2\sin q$$
$$z_1 = 0 \qquad z_2 = 0$$

### 1.8 1(h) Degrees of freedom

***Problem***: Determine the number of degrees of freedom of each of the following holonomic systems:
***Sol.***:

1. Two rigid bodies attached to each other by means of a ball-and-socket connection.

   –The position of both the bodies is constrained but not the orientations of the individual bodies.

   $$n = 9$$

2. An earth satellite carrying a rotor that is made to rotate at a prescribed rate about an axis fixed in the satellite.

   – All dof's of the rotor are constrained to that of satellite except the rotation about its axis which is also constrained as its rate is prescribed.

   $$n = 6$$

3. An earth satellite carrying a rotor that is made to rotate at a prescribed rate about an axis fixed in the satellite.

   – All dof's of the rotor are constrained to that of satellite except the rotation about its axis.

   $$n = 7$$

4. The particles $P_1, P_2$ of Problem $1(e)$.

   – The only degree of freedom is the rotation about $\boldsymbol{n}_z$.

   $$n = 1$$

## 2 Problem Set 2

### 2.1 2(a) ${}^R\boldsymbol{\omega}^B = {}^R\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B = \Omega A_3 + \dot{\theta}_1 X_1 + \dot{\theta}_2 X_2 + \dot{\theta}_3 X_3$

***Problem***: In Figure 4, P represents a point fixed in a reference frame R, and $B^*$ designates the mass center of a rigid body $B$ that moves on a circular orbit $C$ fixed in $R$ and centered at $P$. $A_1, A_2$ and $A_3$ are mutually perpendicular directed line segments, $A_1$ being the extension of line $PB^*$, $A_2$ pointing in the direction of motion of $B^*$ on $C$, and $A_3$ thus being normal to the plane of the orbit $B^*$.

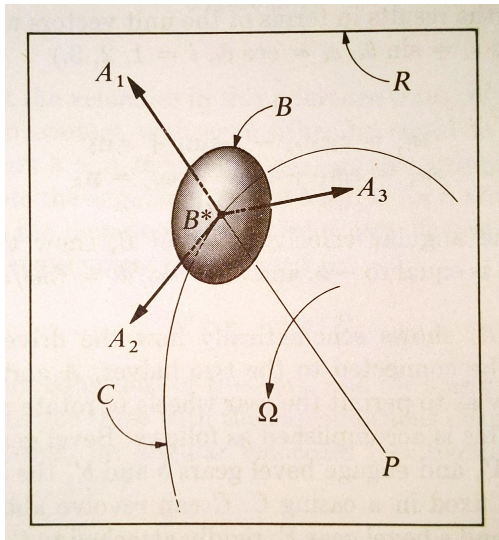


Figure 4

If $X_1, X_2$ and $X_3$ are mutually perpendicular directed line segments passing through $B^*$ and fixed in the body $B$, the "attitude" of $B$ relative to $A_1, A_2, A_3$ can be specified in terms of three angles $\theta_1, \theta_2$ and $\theta_3$, generated as follows: ***Align $X_i$ with $A_i$, for $i = 1, 2, 3$, and perform successive right-handed rotations of $B$, of amount $\theta_1$ about $X_1$, $\theta_2$ about $X_2$, and $\theta_3$ about $X_3$.***

The angular velocity $\boldsymbol{\omega}$ of $B$ in $R$ can be expressed as $\boldsymbol{\omega} = \omega_1 \boldsymbol{n}_1 + \omega_2 \boldsymbol{n}_2 + \omega_3 \boldsymbol{n}_3$, where $\boldsymbol{n}_i$ is a unit vector parallel to $X_i$, and $\omega_i$ is a function of $\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$, and the anglular speed $\Omega$ of the line $PB^*$ in R. Consequently, $\dot{\theta}_i$ can be expressed as a function $f_i$ of $\theta_1, \theta_2, \theta_3, \omega_1, \omega_2, \omega_3$, $i = 1, 2, 3$ and $\Omega$.

Determine the functions $f_1, f_2$ and $f_3$, using the abbrivations $s_i = \sin\theta_i, c_i = \cos\theta_i$ for $i = 1, 2, 3$, to state the results.

***Sol.***:

***Note***: The above explanation of $X_1, X_2$ and $X_3$ is the loose definition of Euler angles used to define the orientation of a rigid body.

***Rotation Matrices***:

The given description of obtaining the attitude of the body can be put mathematically using rotation matrices as follows:

1. Right-handed rotation about $X_1$ by $\theta_1$:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix}}_{R_1} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

2. Right-handed rotation about $X_2$ by $\theta_2$:

$$\begin{bmatrix} X_1' \\ X_2 \\ X_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix}}_{R_2} \begin{bmatrix} X_1 \\ X_2 \\ X_3' \end{bmatrix}$$

3. Right-handed rotation about $X_3$ by $\theta_3$:

$$\begin{bmatrix} X_1'' \\ X_2' \\ X_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3} \begin{bmatrix} X_1' \\ X_2 \\ X_3 \end{bmatrix}$$

***Interpretation***: $\boldsymbol{n}_i$ are the unit vectors parallel to $X_i$ after the above transformation sequence. Thus,

$$\begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{bmatrix} = \begin{bmatrix} X_1'' \\ X_2' \\ X_3 \end{bmatrix}$$

We have, the (instantaneous) angular velocity of the body in $A$ frame:

$${}^A\boldsymbol{\omega}^B = \dot{\theta}_1 X_1 + \dot{\theta}_2 X_2 + \dot{\theta}_3 X_3$$

The angular velocity of $A$ in $R$:

$${}^R\boldsymbol{\omega}^A = \Omega A_3$$

The angular velocity of the body in reference frame:

$$\begin{aligned} lx{}^R\boldsymbol{\omega}^B &= {}^R\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B \\ &= \Omega A_3 + \dot{\theta}_1 X_1 + \dot{\theta}_2 X_2 + \dot{\theta}_3 X_3 \end{aligned}$$

Requried form:

$${}^R\boldsymbol{\omega}^B = \omega_1 \boldsymbol{n}_1 + \omega_2 \boldsymbol{n}_2 + \omega_3 \boldsymbol{n}_3$$

Thus we need to write $A_3, X_1, X_2$ and $X_3$ in terms of $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$.

From the third transformation:

$$\begin{bmatrix} X_1' \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{bmatrix}$$

$$\implies X_2 = s_3 \boldsymbol{n}_1 + c_3 \boldsymbol{n}_2 \qquad X_3 = \boldsymbol{n}_3$$

$$X_1' = c_3 \boldsymbol{n}_1 - s_3 \boldsymbol{n}_2$$

From the second transformation:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3' \end{bmatrix} = \begin{bmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{bmatrix} \begin{bmatrix} X_1' \\ X_2 \\ X_3 \end{bmatrix}$$

$$\implies X_1 = c_2 X_1' + s_2 X_3$$

$$\implies X_1 = c_2(c_3 \boldsymbol{n}_1 - s_3 \boldsymbol{n}_2) + s_2 \boldsymbol{n}_3$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T} \begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{bmatrix}$$

From the first transformation:

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3' \end{bmatrix} \quad \textit{and} \quad X_3' = -s_2 X_1' + c_2 X_3 = -s_2(c_3 \boldsymbol{n}_1 - s_3 \boldsymbol{n}_2) + c_2 \boldsymbol{n}_3$$

$$\begin{aligned} \implies A_3 &= s_1 X_2 + c_1 X_3' = s_1(s_3 \boldsymbol{n}_1 + c_3 \boldsymbol{n}_2) + c_1(-s_2(c_3 \boldsymbol{n}_1 - s_3 \boldsymbol{n}_2) + c_2 \boldsymbol{n}_3) \\ &= \underbrace{\begin{bmatrix} s_1 s_3 - c_1 s_2 c_3 & s_1 c_3 + c_1 s_2 s_3 & c_1 c_2 \end{bmatrix}}_{P^T} \begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{bmatrix} \end{aligned}$$

Substituting, and writing in matrix form:

$$\omega_1 \boldsymbol{n}_1 + \omega_2 \boldsymbol{n}_2 + \omega_3 \boldsymbol{n}_3 = \Omega A_3 + \dot{\theta}_1 X_1 + \dot{\theta}_2 X_2 + \dot{\theta}_3 X_3$$

$$\begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix} \begin{bmatrix}\boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3\end{bmatrix} = \Omega P^T \begin{bmatrix}\boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3\end{bmatrix} + \begin{bmatrix}\dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3\end{bmatrix} T \begin{bmatrix}\boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3\end{bmatrix}$$

$$\implies \begin{bmatrix}\omega_1 \\ \omega_2 \\ \omega_3\end{bmatrix} = \Omega P + T^T \begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3\end{bmatrix}$$

$$\implies \begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3\end{bmatrix} = [T^T]^{-1} \left( \begin{bmatrix}\omega_1 \\ \omega_2 \\ \omega_3\end{bmatrix} - \Omega P \right)$$

Symbolically solving (using sympy), we get:

$$\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3\end{bmatrix} = \begin{bmatrix}\frac{\Omega \sin(\theta_2)\cos(\theta_1) + \omega_1 \cos(\theta_3) - \omega_2 \sin(\theta_3)}{\cos(\theta_2)} \\ -\Omega \sin(\theta_1) + \omega_1 \sin(\theta_3) + \omega_2 \cos(\theta_3) \\ -\frac{\Omega \cos(\theta_1)}{\cos(\theta_2)} - \omega_1 \cos(\theta_3)\tan(\theta_2) + \omega_2 \sin(\theta_3)\tan(\theta_2) + \omega_3\end{bmatrix}$$

Thus,

$$\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3\end{bmatrix} = \begin{bmatrix}(\omega_1 c_3 - \omega_2 s_3 + \Omega s_2 c_1)/c_2 \\ \omega_1 s_3 + \omega_2 c_3 - \Omega s_1 \\ [(\omega_2 s_3 - \omega_1 c_3) s_2 + \omega_3 c_2 - \Omega c_1]/c_2\end{bmatrix}$$

Sympy Code:

```python
import sympy as sp

th1, th2, th3 = sp.symbols("theta_1, theta_2, theta_3")
th1_d, th2_d, th3_d = sp.symbols("theta__'_1 theta__'_2 theta__'_3")
Omega = sp.symbols("Omega")
omega_1, omega_2, omega_3 = sp.symbols("omega_1 omega_2 omega_3")

#R_1
R_1 = sp.Matrix([[1, 0, 0], [0, sp.cos(th1), sp.sin(th1)], [0, -sp.sin(th1), sp
    .cos(th1)]])
# R_2
R_2 = sp.Matrix([[sp.cos(th2), 0, -sp.sin(th2)], [0, 1, 0], [sp.sin(th2), 0, sp
    .cos(th2)]])
# R_3
R_3 = sp.Matrix([[sp.cos(th3), sp.sin(th3), 0], [-sp.sin(th3), sp.cos(th3), 0],
     [0, 0, 1]])
#sp.pprint([R_1, R_2, R_3])

P = ((R_1.T @ R_2.T @ R_3.T)[2, :]).T
X_1 = (R_2.T @ R_3.T)[0, :]
X_2 = (R_3.T)[1, :]
X_3 = sp.Matrix([0, 0, 1]).T
Tr = sp.Matrix([X_1, X_2, X_3])
omega = sp.Matrix([[omega_1], [omega_2], [omega_3]])
invTr_T = sp.simplify(Tr.inv()).T
dots = sp.simplify( invTr_T @ (omega - Omega * P) )
sp.print_latex(dots)
```

## 2.2 2(b) $\boldsymbol{\omega} = \sum \boldsymbol{\omega}_{\dot{q}_i} \dot{q}_i + \boldsymbol{\omega}_t$

***Sol.***

Given,
The motion of $B^*$ on $C$ is prescribed (predeterimined) $\implies \Omega(t)$ is give.
We have angular velocity written interms of partial rates:

$$\boldsymbol{\omega} = \sum \boldsymbol{\omega}_{\dot{q}_i} \dot{q}_i + \boldsymbol{\omega}_t$$

We have from $2(a)$,

$$^R\boldsymbol{\omega}^B = \Omega(t) P^T \begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{bmatrix} + \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{bmatrix} T \begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{bmatrix}$$

$$P = \begin{bmatrix} s_1 s_3 - c_1 s_2 c_3 \\ s_1 c_3 + c_1 s_2 s_3 \\ c_1 c_2 \end{bmatrix} \qquad T = \begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let, $\boldsymbol{n} = \begin{bmatrix} \boldsymbol{n}_1 & \boldsymbol{n}_2 & \boldsymbol{n}_3 \end{bmatrix}^T$
Thus, we have the partial rates:

$$\begin{aligned}
^R\boldsymbol{\omega}^B_t &= \frac{^R\partial \boldsymbol{\omega}^B}{\partial t} = \dot{\Omega} P^T \boldsymbol{n} \\
^R\boldsymbol{\omega}^B_{\dot{\theta}_1} &= \frac{^R\partial \boldsymbol{\omega}^B}{\partial \dot{\theta}_1} = T[1,:]\boldsymbol{n} = \begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \end{bmatrix} \boldsymbol{n} \\
^R\boldsymbol{\omega}^B_{\dot{\theta}_2} &= \frac{^R\partial \boldsymbol{\omega}^B}{\partial \dot{\theta}_2} = T[2,:]\boldsymbol{n} = \begin{bmatrix} s_3 & c_3 & 0 \end{bmatrix} \boldsymbol{n} \\
^R\boldsymbol{\omega}^B_{\dot{\theta}_3} &= \frac{^R\partial \boldsymbol{\omega}^B}{\partial \dot{\theta}_3} = T[3,:]\boldsymbol{n} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \boldsymbol{n}
\end{aligned}$$

## 2.3 2(c) Prove that $^B\boldsymbol{\omega}^A = -\,^A\boldsymbol{\omega}^B$ but, $^A d\boldsymbol{\omega} \backslash dt = {}^B d\boldsymbol{\omega} \backslash dt$

Given, $^B\boldsymbol{\omega}^A = \boldsymbol{\omega}$.
Let, $\boldsymbol{v}$ be a vector, then

$$\begin{aligned}
\frac{^B d\boldsymbol{v}}{dt} &= \frac{^A d\boldsymbol{v}}{dt} + \boldsymbol{\omega} \times \boldsymbol{v} \implies \frac{^A d\boldsymbol{v}}{dt} = \frac{^B d\boldsymbol{v}}{dt} - \boldsymbol{\omega} \times \boldsymbol{v} \\
\text{but,} \qquad \frac{^A d\boldsymbol{v}}{dt} &= \frac{^B d\boldsymbol{v}}{dt} + {}^A\boldsymbol{\omega}^B \times v \\
\text{By comparision,} \qquad {}^A\boldsymbol{\omega}^B &= -\boldsymbol{\omega} \\
\therefore\ {}^B\boldsymbol{\omega}^A = \boldsymbol{\omega} &\implies {}^A\boldsymbol{\omega}^B = -\boldsymbol{\omega} \qquad \textit{q.e.d}
\end{aligned}$$

Using the operator definition of angular velocity vector on itself:

$$\begin{aligned}
\frac{^B d\boldsymbol{\omega}}{dt} &= \frac{^A d\boldsymbol{\omega}}{dt} + \underbrace{\boldsymbol{\omega} \times \boldsymbol{\omega}}_{=0} \\
\therefore \quad \frac{^B d\boldsymbol{\omega}}{dt} &= \frac{^A d\boldsymbol{\omega}}{dt} \qquad \textit{q.e.d}
\end{aligned}$$

## 2.4 2(d) Differential gear and drive shaft kinematics

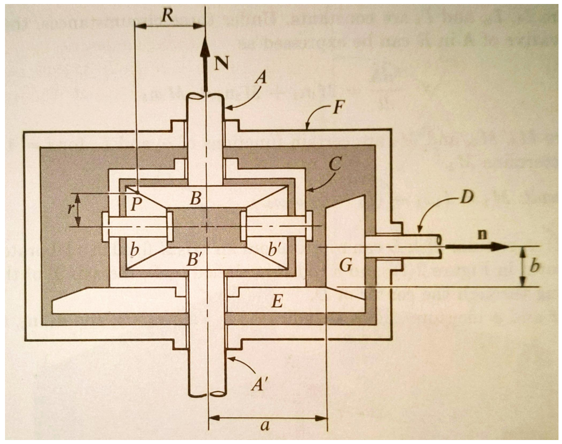


Figure 5

***Mechanism***: Bevel gears $B$ and $B'$ are keyed to $A$ and $A'$, and engage bevel gears $b$ and $b'$, the latter being free to rotate on pins fixed in a casing $C$. $C$ can revolve about the common axis of $A$ and $A'$, and a bevel gear $E$, rigidly attached to $C$, is driven by the gear $G$, which is keyed to the drive shaft $D$.

Given:

$$
\begin{aligned}
{}^F\boldsymbol{\omega}^A &= \Omega \boldsymbol{N} \\
{}^F\boldsymbol{\omega}^{A'} &= \Omega' \boldsymbol{N} \\
{}^F\boldsymbol{\omega}^D &= \omega \boldsymbol{n}
\end{aligned}
$$

The gears have simple angular velocities w.r.t the frame attached to their shafts. And, the contact points should have same linear velocities. Writing only the magnitudes:

$$
\begin{aligned}
&{}^F\omega^D b = {}^F\omega^C a \implies {}^F\omega^C = \frac{b}{a}\omega \\
&\therefore {}^F\boldsymbol{\omega}^C = \frac{b}{a}\omega \boldsymbol{N}
\end{aligned}
$$

Also,

$$
\begin{aligned}
{}^C\boldsymbol{\omega}^B &= {}^F\boldsymbol{\omega}^B - {}^C\boldsymbol{\omega}^C = {}^F\boldsymbol{\omega}^A - {}^C\boldsymbol{\omega}^C = \left(\Omega - \frac{b}{a}\omega\right)\boldsymbol{N} \\
{}^C\boldsymbol{\omega}^{B'} &= {}^F\boldsymbol{\omega}^{B'} - {}^C\boldsymbol{\omega}^C = {}^F\boldsymbol{\omega}^{A'} - {}^C\boldsymbol{\omega}^C = \left(\Omega' - \frac{b}{a}\omega\right)\boldsymbol{N}
\end{aligned}
$$

$$
\left.
\begin{aligned}
{}^C\omega^{B'} R &= {}^C\omega^{b'} r \\
{}^C\omega^{B'} R &= -\,{}^C\omega^{b} r \\
{}^C\omega^{B} R &= -\,{}^C\omega^{b'} r \\
{}^C\omega^{B} R &= {}^C\omega^{b} r
\end{aligned}
\right\} \implies {}^C\omega^{B'} + {}^C\omega^{B} = 0
$$

$$
\implies \left(\Omega - \frac{b}{a}\omega\right) + \left(\Omega' - \frac{b}{a}\omega\right) = 0
$$

$$
\therefore \omega = \frac{a}{2b}\left(\Omega + \Omega'\right)
$$

### 2.5 2(e) Derivative of angular momentum

We have,

$$
\begin{aligned}
\frac{{}^R d\boldsymbol{A}}{dt} &= B\frac{d\boldsymbol{A}}{dt} + {}^R\boldsymbol{\omega}^B \times A \\
&= \begin{bmatrix}\dot\omega_1 & \dot\omega_2 & \dot\omega_3\end{bmatrix}\begin{bmatrix}\boldsymbol{n}_1\\ \boldsymbol{n}_2\\ \boldsymbol{n}_3\end{bmatrix} + \begin{vmatrix}\boldsymbol{n}_1 & \boldsymbol{n}_2 & \boldsymbol{n}_3\\ \omega_1 & \omega_2 & \omega_3\\ A_1\omega_1 & A_2\omega_2 & A_3\omega_3\end{vmatrix}
\end{aligned}
$$

$$
\text{Let,}\qquad \frac{{}^R d\boldsymbol{A}}{dt} = M_1\boldsymbol{n}_1 + M_2\boldsymbol{n}_2 + M_3\boldsymbol{n}_3
$$

We have,

$$
\begin{aligned}
M_1 &= \dot\omega_1 + (I_3 - I_2)\omega_2\omega_3 \\
M_2 &= \dot\omega_2 + (I_1 - I_3)\omega_1\omega_3 \\
M_3 &= \dot\omega_3 + (I_2 - I_1)\omega_2\omega_1
\end{aligned}
$$

### 2.6 2(f) Angular acceleration

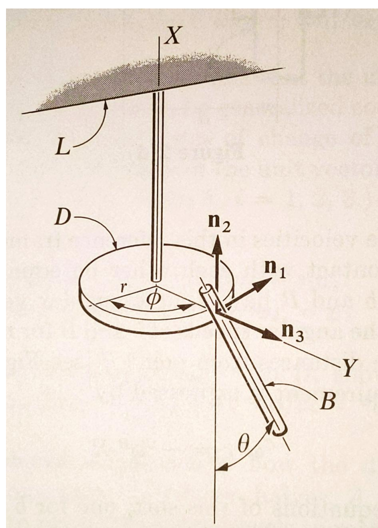


Figure 6

***Sol.***

$$
{}^L\boldsymbol{\omega}^D = \dot\phi\boldsymbol{n}_2 \qquad and \qquad {}^L\boldsymbol{\omega}^B = \dot\phi\boldsymbol{n}_2 + \dot\theta\boldsymbol{n}_3
$$

$$
\begin{aligned}
{}^L\boldsymbol{\alpha}^B &= \frac{{}^L d\boldsymbol{\omega}^B}{dt} = \frac{{}^L d}{dt}\left(\dot\phi\boldsymbol{n}_2 + \dot\theta\boldsymbol{n}_3\right) \\
&= \ddot\phi\boldsymbol{n}_2 + \dot\phi\underbrace{\left({}^L\boldsymbol{\omega}^D \times \boldsymbol{n}_2\right)}_{=0} + \ddot\theta\boldsymbol{n}_3 + \dot\theta\underbrace{\left({}^L\boldsymbol{\omega}^D \times \boldsymbol{n}_3\right)}_{=\dot\phi n_1} \quad [\because \boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3 \text{ are attached to D (not B).}]
\end{aligned}
$$

$$
\implies {}^L\boldsymbol{\alpha}^B = \dot\phi\dot\theta\boldsymbol{n}_1 + \ddot\phi\boldsymbol{n}_2 + \ddot\theta\boldsymbol{n}_2
$$

$$
\implies \alpha_1 = \dot\phi\dot\theta, \quad \alpha_2 = \ddot\phi, \quad \alpha_3 = \ddot\theta
$$

## 2.7 2(g) Angular acceleration of a rolling disk: Choice of coordinates

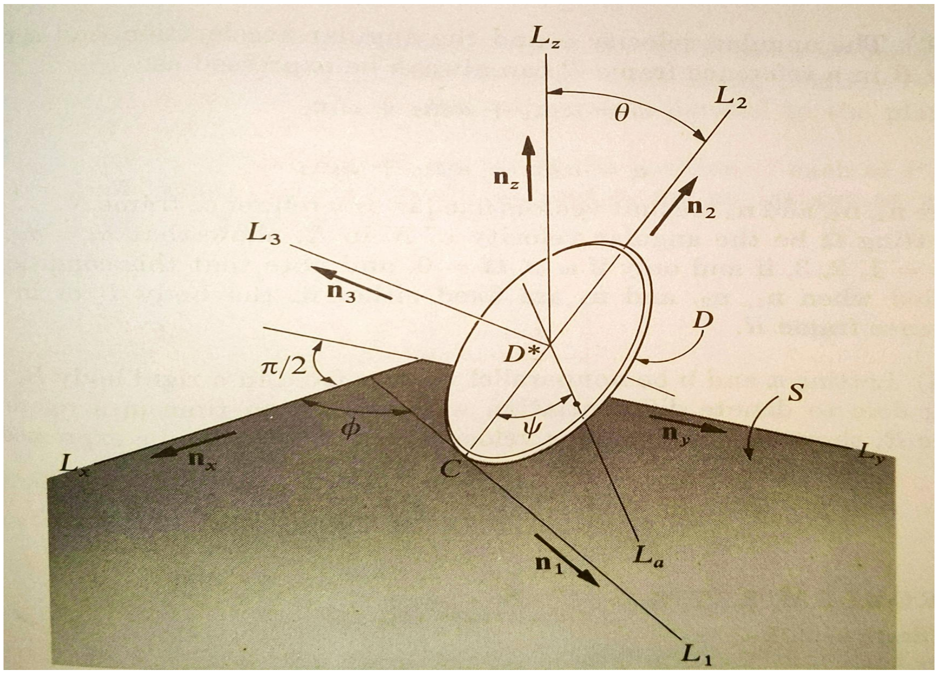


Figure 7

The transformation between the two coordinates $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$ and $\boldsymbol{n}_x, \boldsymbol{n}_y, \boldsymbol{n}_z$ can be written as:

$$\begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{bmatrix} = \underbrace{R_1(90-\theta)R_3(\phi)}_{R} \begin{bmatrix} \boldsymbol{n}_x \\ \boldsymbol{n}_y \\ \boldsymbol{n}_z \end{bmatrix}$$

$$R_1(90-\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(90-\theta) & \sin(90-\theta) \\ 0 & -\sin(90-\theta) & \cos(90-\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix}$$

$$R_3(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \\ \cos\theta\sin\phi & -\cos\theta\cos\phi & \sin\theta \end{bmatrix}$$

$$R^{-1} = (R_1(90-\theta)R_3(\phi))^{-1} = R_3^{-1}(\phi)R_1^{-1}(90-\theta) = R_3^T(\phi)R_1^T(90-\theta)$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & -\cos\theta \\ 0 & \cos\theta & \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi\sin\theta & \sin\phi\cos\theta \\ \sin\phi & \cos\theta\cos\phi & -\cos\phi\cos\theta \\ 0 & \cos\theta & \sin\theta \end{bmatrix}$$

$$^{R}\boldsymbol{\omega}^{D} = -\dot{\theta}\boldsymbol{n}_1 + \dot{\phi}\boldsymbol{n}_z + \dot{\psi}\boldsymbol{n}_3 = -\dot{\theta}\boldsymbol{n}_1 + \dot{\phi}\cos\theta\boldsymbol{n}_2 + (\dot{\phi}\sin\theta + \dot{\psi})\boldsymbol{n}_3$$

We have,

$$^{R}\boldsymbol{\alpha}^{B} = \frac{^{R}d\boldsymbol{\omega}^{B}}{dt} = \frac{^{R}d}{dt}\left(-\dot{\theta}\boldsymbol{n}_1 + \dot{\phi}\boldsymbol{n}_z + \dot{\psi}\boldsymbol{n}_3\right) = -\ddot{\theta}\boldsymbol{n}_1 - \dot{\theta}\dot{\boldsymbol{n}}_1 + \ddot{\phi}\boldsymbol{n}_z + \dot{\phi}\dot{\boldsymbol{n}}_z + \ddot{\psi}\boldsymbol{n}_3 + \dot{\psi}\dot{\boldsymbol{n}}_3$$

also,

$$
\begin{aligned}
\frac{{}^{R}d\boldsymbol{n}_1}{dt} &= {}^{R}\boldsymbol{\omega}^{L_1} \times \boldsymbol{n}_1 = \dot{\phi}\boldsymbol{n}_z \times \boldsymbol{n}_1 = \dot{\phi}(\cos\theta\boldsymbol{n}_2 + \sin\theta\boldsymbol{n}_3) \times \boldsymbol{n}_1 \\
&= \dot{\phi}(-\cos\theta\boldsymbol{n}_3 + \sin\theta\boldsymbol{n}_2) \\
\frac{{}^{R}d\boldsymbol{n}_z}{dt} &= 0 \qquad [\because \boldsymbol{n}_z \text{ is fixed to the frame } R] \\
\frac{{}^{R}d\boldsymbol{n}_3}{dt} &= {}^{R}\boldsymbol{\omega}^{L_3} \times \boldsymbol{n}_3 = (\dot{\phi}\boldsymbol{n}_z - \dot{\theta}\boldsymbol{n}_1) \times \boldsymbol{n}_3 = (\dot{\phi}(\cos\theta\boldsymbol{n}_2 + \sin\theta\boldsymbol{n}_3) - \dot{\theta}\boldsymbol{n}_1) \times \boldsymbol{n}_3 \\
&= \dot{\phi}\cos\theta\boldsymbol{n}_1 + \dot{\theta}\boldsymbol{n}_2
\end{aligned}
$$

Substituting,

$$
\begin{aligned}
{}^{R}\boldsymbol{\alpha}^{B} &= -\ddot{\theta}\boldsymbol{n}_1 - \dot{\theta}(\dot{\phi}(-\cos\theta\boldsymbol{n}_3 + \sin\theta\boldsymbol{n}_2)) + \ddot{\phi}(\cos\theta\boldsymbol{n}_2 + \sin\theta\boldsymbol{n}_3) + \ddot{\psi}\boldsymbol{n}_3 + \dot{\psi}(\dot{\phi}\cos\theta\boldsymbol{n}_1 + \dot{\theta}\boldsymbol{n}_2) \\
&= \begin{bmatrix} \underbrace{-\ddot{\theta} + \dot{\phi}\dot{\theta}\cos\theta}_{\alpha_1} & \underbrace{-\dot{\theta}\dot{\phi}\sin\theta + \ddot{\phi}\cos\theta + \dot{\psi}\dot{\theta}}_{\alpha_2} & \underbrace{\dot{\theta}\dot{\phi}\cos\theta + \ddot{\phi}\sin\theta + \ddot{\psi}}_{\alpha_3} \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} R \begin{bmatrix} \boldsymbol{n}_x \\ \boldsymbol{n}_y \\ \boldsymbol{n}_z \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \\ \cos\theta\sin\phi & -\cos\theta\cos\phi & \sin\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_x \\ \boldsymbol{n}_y \\ \boldsymbol{n}_z \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1\cos\phi - \alpha_2\sin\theta\sin\phi + \alpha_3\cos\theta\cos\phi & (=\alpha_x) \\ \alpha_1\sin\phi + \alpha_2\sin\theta\cos\phi - \alpha_3\cos\theta\cos\phi & (=\alpha_y) \\ \alpha_2\cos\theta + \alpha_3\sin\theta & (=\alpha_z) \end{bmatrix}^T \begin{bmatrix} \boldsymbol{n}_x \\ \boldsymbol{n}_y \\ \boldsymbol{n}_z \end{bmatrix}
\end{aligned}
$$

alternately,

$$
\begin{aligned}
{}^{R}\boldsymbol{\omega}^{D} &= -\dot{\theta}\boldsymbol{n}_1 + \dot{\phi}\boldsymbol{n}_z + \dot{\psi}\boldsymbol{n}_3 \\
&= -\dot{\theta}(\cos\phi\boldsymbol{n}_x + \sin\phi\boldsymbol{n}_y) + \dot{\phi}\boldsymbol{n}_z + \dot{\psi}(\cos\theta\sin\phi\boldsymbol{n}_x - \cos\theta\cos\phi\boldsymbol{n}_y + \sin\theta\boldsymbol{n}_z) \\
&= \begin{bmatrix} -\dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi & -\dot{\theta}\sin\phi - \dot{\psi}\cos\theta\cos\phi & \dot{\phi} + \dot{\psi}\sin\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_x \\ \boldsymbol{n}_y \\ \boldsymbol{n}_z \end{bmatrix}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
{}^{R}\boldsymbol{\alpha}^{B} = \frac{{}^{R}d\boldsymbol{\omega}^{B}}{dt} &= \frac{{}^{R}d}{dt}\left(\begin{bmatrix} -\dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi & -\dot{\theta}\sin\phi - \dot{\psi}\cos\theta\cos\phi & \dot{\phi} + \dot{\psi}\sin\theta \end{bmatrix}\right) \begin{bmatrix} \boldsymbol{n}_x \\ \boldsymbol{n}_y \\ \boldsymbol{n}_z \end{bmatrix} \\
&= \begin{bmatrix} -\ddot{\theta}\cos\phi + \ddot{\psi}\cos\theta sin\phi + \dot{\theta}\dot{\phi}\sin\phi + \dot{\psi}\dot{\phi}\cos\theta\cos\phi - \dot{\psi}\dot{\theta}\cos\theta\sin\phi & (=\alpha_x) \\ -\ddot{\theta}\sin\phi - \ddot{\psi}\cos\theta\cos\phi - \dot{\theta}\dot{\phi}\cos\phi + \dot{\psi}\dot{\theta}\sin\theta\cos\phi + \dot{\psi}\dot{\phi}\cos\theta\sin\phi & (=\alpha_y) \\ \ddot{\phi} + \ddot{\psi}\sin\theta + \dot{\psi}\dot{\theta}\cos\theta & (=\alpha_z) \end{bmatrix}^T \begin{bmatrix} \boldsymbol{n}_x \\ \boldsymbol{n}_y \\ \boldsymbol{n}_z \end{bmatrix}
\end{aligned}
$$

***Note:***

- We can not just take $d\boldsymbol{n}_z/dt$ as $\omega \times \boldsymbol{n}_z$ in $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$ coordinates as $\boldsymbol{n}_z$ is not rigidly fixed to the coordinates. This is only possible when $\boldsymbol{n}_z$ does't change with time in $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$ frame. But once, the differentiation is done, one vector in $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$ frame can be written in $\boldsymbol{n}_x, \boldsymbol{n}_y, \boldsymbol{n}_z$ frame using transformation matrices as they are instantaneous.
- Thus the rotation transformation can be used after differentiation as well.

### 2.8 2(h) $\alpha$ when body and frame have paraller $\omega$'s

Given, $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\} \in N$ has an angular velocity $\Omega$ in R, and

$$
\begin{aligned}
{}^R\boldsymbol{\omega}^B &= \omega_1\boldsymbol{n}_1 + \omega_2\boldsymbol{n}_2 + \omega_3\boldsymbol{n}_3 \\
{}^R\boldsymbol{\alpha}^B &= \alpha_1\boldsymbol{n}_1 + \alpha_2\boldsymbol{n}_2 + \alpha_3\boldsymbol{n}_3 \\
\text{and,} \qquad \alpha_i &= d\omega_i/dt = \dot{\omega}_i
\end{aligned}
$$

Consider,

$$
\begin{aligned}
{}^R\boldsymbol{\alpha}^B = \frac{{}^R d\boldsymbol{\omega}^B}{dt} = \sum_{i=1}^{3}\left(\dot{\omega}_i\boldsymbol{n}_i + \omega_i \frac{{}^R d\boldsymbol{n}_i}{dt}\right) = \sum_{i=1}^{3}\left(\alpha_i\boldsymbol{n}_i + \omega_i \frac{{}^R d\boldsymbol{n}_i}{dt}\right) = {}^R\boldsymbol{\alpha}^B + \sum_{i=1}^{3}\left(\omega_i \frac{{}^R d\boldsymbol{n}_i}{dt}\right) \\
\iff \sum_{i=1}^{3}\left(\omega_i \frac{{}^R d\boldsymbol{n}_i}{dt}\right) = 0
\end{aligned}
$$

We have,

$$
\sum_{i=1}^{3}\left(\omega_i \frac{{}^R d\boldsymbol{n}_i}{dt}\right) = \sum_{i=1}^{3}\left(\omega_i(\Omega \times \boldsymbol{n}_i)\right) = \Omega \times (\omega_1\boldsymbol{n}_1 + \omega_2\boldsymbol{n}_2 + \omega_3\boldsymbol{n}_3) = \Omega \times \boldsymbol{\omega}
$$

thus,

$$
\alpha_i = \frac{d\omega_i}{dt},\ for\ i = 1, 2, 3 \iff \Omega \times \boldsymbol{\omega} = 0 \qquad\qquad q.e.d
$$

### 2.9 2(i) $\boldsymbol{\omega} = (\dot{\boldsymbol{a}} \times \dot{\boldsymbol{b}})/(\dot{\boldsymbol{a}}.\boldsymbol{b})$

Consider,

$$
\dot{\boldsymbol{a}} \times \dot{\boldsymbol{b}} = \dot{\boldsymbol{a}} \times (\boldsymbol{\omega} \times \boldsymbol{b}) = (\dot{\boldsymbol{a}}.\boldsymbol{b})\boldsymbol{\omega} - \underbrace{(\dot{\boldsymbol{a}}.\boldsymbol{\omega})}_{=0}\boldsymbol{b} \qquad \begin{bmatrix} \because & \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c} \\ \because & (\boldsymbol{\omega} \times \boldsymbol{a}) \times \boldsymbol{\omega} = 0 \end{bmatrix}
$$

$$
\implies \boldsymbol{\omega} = \frac{\dot{\boldsymbol{a}} \times \dot{\boldsymbol{b}}}{(\dot{\boldsymbol{a}}.\boldsymbol{b})} \qquad\qquad q.e.d
$$

# 3 Problem Set 3

KJDSAFF