Analysis of Symmetrical Component Phasors in Rotating Reference Frames

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Symmetrical component phasors

One of the most commonly used concepts in power engineering is the symmetrical components (SCs). The concept of symmetrical components, introduced by Fortescue in 1918 (in phasor form) [1], is extensively used to study unbalanced systems. According to SC transformation, an unbalanced $3-\phi$ signal (say $\mathbf{V_a} = V_a\angle\delta_a$, $\mathbf{V_b} = V_b\angle\delta_b$ and $\mathbf{V_c} = V_c\angle\delta_c$) can be resolved into 3 sets of balanced phasors, namely positive, negative, and zero sequence components.

- The positive sequence component phasors (denoted by $\mathbf{V_a^+}$, $\mathbf{V_b^+}$ and $\mathbf{V_c^+}$) have the same phase sequence as that of original $3-\phi$ signal and are displaced by $\frac{2\pi}{3}$ i.e.,

$$\begin{aligned} \mathbf{V_b^+} &= \mathbf{V_a^+} e^{-j\frac{2\pi}{3}} \text{ and,} \\ \mathbf{V_c^+} &= \mathbf{V_a^+} e^{j\frac{2\pi}{3}} \end{aligned} \tag{1}$$

- The negative sequence component phasors (denoted by $\mathbf{V_a^-}$, $\mathbf{V_b^-}$ and $\mathbf{V_c^-}$) have the reversed phase sequence as that of original $3-\phi$ signal and are displaced by $\frac{2\pi}{3}$ i.e.,

$$\begin{aligned} \mathbf{V_b^-} &= \mathbf{V_a^-} e^{j\frac{2\pi}{3}} \text{ and,} \\ \mathbf{V_c^-} &= \mathbf{V_a^-} e^{-j\frac{2\pi}{3}} \end{aligned} \tag{2}$$

- The zero sequence component phasors (denoted by $\mathbf{V_a^0}$, $\mathbf{V_b^0}$ and $\mathbf{V_c^0}$) that are in phase with each other i.e.,

$$\mathbf{V_b^0} = \mathbf{V_c^0} = \mathbf{V_a^0} \tag{3}$$

The phasors of the unbalanced $3-\phi$ signal can be written in terms of the SC phasors (using the power invariant SC phasors) as

$$\begin{aligned} \mathbf{V_a} &= \frac{1}{\sqrt{3}}\left(\mathbf{V_a^0} + \mathbf{V_a^+} + \mathbf{V_a^-}\right) \\ \mathbf{V_b} &= \frac{1}{\sqrt{3}}\left(\mathbf{V_b^0} + \mathbf{V_b^+} + \mathbf{V_b^-}\right) \\ \mathbf{V_c} &= \frac{1}{\sqrt{3}}\left(\mathbf{V_c^0} + \mathbf{V_c^+} + \mathbf{V_c^-}\right) \end{aligned} \tag{4}$$

Equations (1) - (4) indicate that, the SC phasors for phases b and c can be computed if the corresponding components of phase a are known. For the purpose of simplicity, we denote the positive, negative and zero sequence component phasors of phase a using $\mathbf{V^+} = V^+\angle\delta^+$, $\mathbf{V^-} = V^-\angle\delta^-$ and, $\mathbf{V^0} = V^0\angle\delta^0$ respectively. The instantaneous values of the symmetrical components (i.e. $v^0(t)$, $v^+(t)$ and $v^-(t)$) can be written in terms from the SC phasors as

$$v^0(t) = \sqrt{2}V^0\cos\left(\omega_s t + \delta^0\right),\ v^+(t) = \sqrt{2}V^+\cos\left(\omega_s t + \delta^+\right) \text{ and, } v^-(t) = \sqrt{2}V^-\cos\left(\omega_s t + \delta^-\right) \tag{5}$$

Extending the phasor definition of symmetrical components (Eq. 4), the unbalanced $3-\phi$ signal in time domain can be expressed in terms of the instantaneous values of symmetrical components as

$$\begin{aligned} v_a(t) &= \sqrt{\frac{2}{3}}\left[V^0\cos\left(\omega_s t+\delta^0\right)+V^+\cos\left(\omega_s t+\delta^+\right)+V^-\cos\left(\omega_s t+\delta^-\right)\right] \\ v_b(t) &= \sqrt{\frac{2}{3}}\left[V^0\cos\left(\omega_s t+\delta^0\right)+V^+\cos\left(\omega_s t+\delta^+-\frac{2\pi}{3}\right)+V^-\cos\left(\omega_s t+\delta^-+\frac{2\pi}{3}\right)\right] \\ v_c(t) &= \sqrt{\frac{2}{3}}\left[V^0\cos\left(\omega_s t+\delta^0\right)+V^+\cos\left(\omega_s t+\delta^++\frac{2\pi}{3}\right)+V^-\cos\left(\omega_s t+\delta^--\frac{2\pi}{3}\right)\right] \end{aligned} \tag{6}$$

The purpose of this document is to determine the components of SC phasors in synchronously rotating reference frames. [1]

Reference frame transformation

The concept of reference frame transformation is extensively used in modeling of machines and control of inverter systems. A commonly used reference frame is the synchronously rotating reference frame (also referred to as $dq0$ reference frame). The transformation of a $3-\phi$ time domain signal (i.e. instantaneous value) to the rotating reference frame (RRF) can be achieved through the Park transformation matrix. The elements of the Park transformation matrix depends on the direction of rotation of the RRF. Let (v_d^+, v_q^+) denote the d and q components respectively of a RRF that is rotating with a speed of ω_n in counter clockwise direction (referred to as positive RRF) with q-axis leading the d-axis by $\frac{\pi}{2}$. The components of the $3-\phi$ signal in a positive RRF can be obtained (using Park transformation matrix) as

$$\begin{bmatrix} v_d^+ \\ v_q^+ \end{bmatrix} = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos\left(\omega_n t\right) & \cos\left(\omega_n t-\frac{2\pi}{3}\right) & \cos\left(\omega_n t+\frac{2\pi}{3}\right) \\ -\sin\left(\omega_n t\right) & -\sin\left(\omega_n t-\frac{2\pi}{3}\right) & -\sin\left(\omega_n t+\frac{2\pi}{3}\right) \end{bmatrix}\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \tag{7}$$

Using Eq. 7 and Eq. 6, the components of SC phasors in the positive RRF can be computed as (steps detailed in Appendix A)

$$\begin{aligned} v_d^+ &= \left[V^+\cos\left(\Delta\omega t+\delta^+\right)+V^-\cos\left(\omega_1 t+\delta^-\right)\right] \\ v_q^+ &= \left[V^+\sin\left(\Delta\omega t+\delta^+\right)-V^-\sin\left(\omega_1 t+\delta^-\right)\right] \end{aligned} \tag{8}$$

where $\Delta\omega = \omega_s - \omega_n$ and $\omega_1 = \omega_s + \omega_n$. Equation 8 indicates that the positive sequence components appears in the positive RRF as a low frequency component (ideally DC, i.e., when $\omega_s = \omega_n$). Similarly, let (v_d^-, v_q^-) denote the d and q components respectively of a RRF that is rotating with a speed of ω_n in clockwise direction (referred to as negative RRF) with q-axis leading the d-axis by $\frac{\pi}{2}$. In the negative RRF, the speed of the RRF is numerically negative i.e. $\omega = -\omega_n = -2\pi f_n$. The components of the $3-\phi$ signal in a negative RRF can be obtained (using Park transformation matrix bearing in mind that $\omega = -\omega_n$ is numerically negative) as

$$\begin{bmatrix} v_d^- \\ v_q^- \end{bmatrix} = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos\left(\omega_n t\right) & \cos\left(\omega_n t+\frac{2\pi}{3}\right) & \cos\left(\omega_n t-\frac{2\pi}{3}\right) \\ \sin\left(\omega_n t\right) & \sin\left(\omega_n t+\frac{2\pi}{3}\right) & \sin\left(\omega_n t-\frac{2\pi}{3}\right) \end{bmatrix}\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \tag{9}$$

[1] The mathematical relations outlined in this document are used in some of the publications from our lab. See References [2,3]

Using Eq. 9 and Eq. 6, the components of SC phasors in the negative RRF can be computed as (steps detailed in Appendix B)

$$\begin{aligned} v_d^- &= \left[V^- \cos\left(\Delta\omega t + \delta^+\right) + V^+ \cos\left(\omega_1 t + \delta^-\right)\right] \\ v_q^- &= \left[V^- \sin\left(\Delta\omega t + \delta^+\right) - V^+ \sin\left(\omega_1 t + \delta^-\right)\right] \end{aligned} \tag{10}$$

where $\Delta\omega = \omega_s - \omega_n$ and $\omega_1 = \omega_s + \omega_n$.

Appendix A : Sequence components in Positive Rotating Reference Frame

Using Eq. 7 and Eq. 6, the components of SC phasors in positive RRF can be computed as

$$
\begin{aligned}
v_d^+ &= \frac{2}{3}V^0\cos\left(\omega_s t+\delta^0\right)\underbrace{\left\{\cos\left(\omega_n t\right)+\cos\left(\omega_n t-\frac{2\pi}{3}\right)+\cos\left(\omega_n t+\frac{2\pi}{3}\right)\right\}}_{0}\;+\\
&\quad \frac{2}{3}V^+\left\{\cos(\omega_s t+\delta^+)\cos\left(\omega_n t\right)+\cos\left(\omega_s t+\delta^+-\frac{2\pi}{3}\right)\cos\left(\omega_n t-\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\delta^++\frac{2\pi}{3}\right)\cos\left(\omega_n t+\frac{2\pi}{3}\right)\right\}\;+\\
&\quad \frac{2}{3}V^-\left\{\cos(\omega_s t+\delta^-)\cos\left(\omega_n t\right)+\cos\left(\omega_s t+\delta^-+\frac{2\pi}{3}\right)\cos\left(\omega_n t-\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\delta^+-\frac{2\pi}{3}\right)\cos\left(\omega_n t+\frac{2\pi}{3}\right)\right\}\\
&= \frac{2}{3}V^+\left[\frac{3}{2}\cos\left(\omega_s t-\omega_n t+\delta^+\right)+\frac{1}{2}\underbrace{\left\{\cos\left(\omega_s t+\omega_n t+\delta^+\right)+\cos\left(\omega_s t+\omega_n t+\delta^++\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\omega_n t+\delta^+-\frac{2\pi}{3}\right)\right\}}_{0}\right]\\
&\quad +\frac{2}{3}V^-\left[\frac{3}{2}\cos\left(\omega_s t+\omega_n t+\delta^-\right)+\frac{1}{2}\underbrace{\left\{\cos\left(\omega_s t+\delta^--\omega_n t\right)+\cos\left(\omega_s t+\delta^--\omega_n t-\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\delta^--\omega_n t+\frac{2\pi}{3}\right)\right\}}_{0}\right]\\
v_d^+ &= \left[V^+\cos\left(\omega_s t-\omega_n t+\delta^+\right)+V^-\cos\left(\omega_s t+\omega_n t+\delta^-\right)\right] \;=\; \left[V^+\cos\left(\Delta\omega t+\delta^+\right)+V^-\cos\left(\omega_1 t+\delta^-\right)\right]
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
v_q^+ &= -\frac{2}{3}V^0\cos\left(\omega_s t+\delta^0\right)\underbrace{\left\{\sin\left(\omega_n t\right)+\sin\left(\omega_n t-\frac{2\pi}{3}\right)+\sin\left(\omega_n t+\frac{2\pi}{3}\right)\right\}}_{0}\;-\\
&\quad \frac{2}{3}V^+\left\{\cos(\omega_s t+\delta^+)\sin\left(\omega_n t\right)+\cos\left(\omega_s t+\delta^+-\frac{2\pi}{3}\right)\sin\left(\omega_n t-\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\delta^++\frac{2\pi}{3}\right)\sin\left(\omega_n t+\frac{2\pi}{3}\right)\right\}\;-\\
&\quad \frac{2}{3}V^-\left\{\cos(\omega_s t+\delta^-)\sin\left(\omega_n t\right)+\cos\left(\omega_s t+\delta^-+\frac{2\pi}{3}\right)\sin\left(\omega_n t-\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\delta^+-\frac{2\pi}{3}\right)\sin\left(\omega_n t+\frac{2\pi}{3}\right)\right\}\\
&= -\frac{2}{3}V^+\left[\frac{1}{2}\underbrace{\left\{\sin\left(\omega_s t+\omega_n t+\delta^+\right)+\sin\left(\omega_s t+\omega_n t+\delta^++\frac{2\pi}{3}\right)+\sin\left(\omega_s t+\omega_n t+\delta^+-\frac{2\pi}{3}\right)\right\}}_{0}-\frac{3}{2}\sin\left(\omega_s t-\omega_n t+\delta^+\right)\right]\\
&\quad -\frac{2}{3}V^-\left[\frac{3}{2}\sin\left(\omega_s t+\omega_n t+\delta^-\right)-\frac{1}{2}\underbrace{\left\{\sin\left(\omega_s t+\delta^--\omega_n t\right)+\sin\left(\omega_s t+\delta^--\omega_n t-\frac{2\pi}{3}\right)+\sin\left(\omega_s t+\delta^--\omega_n t+\frac{2\pi}{3}\right)\right\}}_{0}\right]\\
v_q^+ &= \left[V^+\sin\left(\omega_s t-\omega_n t+\delta^+\right)-V^-\sin\left(\omega_s t+\omega_n t+\delta^-\right)\right] \;=\; \left[V^+\sin\left(\Delta\omega t+\delta^+\right)-V^-\sin\left(\omega_1 t+\delta^-\right)\right]
\end{aligned}
\tag{12}
$$

Appendix B : Sequence components in Negative Rotating Reference Frame

Using Eq. 9 and Eq. 6, the components of SC phasors in positive RRF can be computed as

$$
\begin{aligned}
v_d^- &= \frac{2}{3}V^0\cos\left(\omega_s t+\delta^0\right)\underbrace{\left\{\cos\left(\omega_n t\right)+\cos\left(\omega_n t+\frac{2\pi}{3}\right)+\cos\left(\omega_n t-\frac{2\pi}{3}\right)\right\}}_{0} + \\
&\quad \frac{2}{3}V^+\left\{\cos(\omega_s t+\delta^+)\cos\left(\omega_n t\right)+\cos\left(\omega_s t+\delta^+-\frac{2\pi}{3}\right)\cos\left(\omega_n t+\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\delta^++\frac{2\pi}{3}\right)\cos\left(\omega_n t-\frac{2\pi}{3}\right)\right\} + \\
&\quad \frac{2}{3}V^-\left\{\cos(\omega_s t+\delta^-)\cos\left(\omega_n t\right)+\cos\left(\omega_s t+\delta^-+\frac{2\pi}{3}\right)\cos\left(\omega_n t+\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\delta^+-\frac{2\pi}{3}\right)\cos\left(\omega_n t-\frac{2\pi}{3}\right)\right\} \\
&= \frac{2}{3}V^+\left[\frac{3}{2}\cos\left(\omega_s t+\omega_n t+\delta^+\right)+\frac{1}{2}\underbrace{\left\{\cos\left(\omega_s t-\omega_n t+\delta^+\right)+\cos\left(\omega_s t-\omega_n t+\delta^+-\frac{2\pi}{3}\right)+\cos\left(\omega_s t-\omega_n t+\delta^++\frac{2\pi}{3}\right)\right\}}_{0}\right] \\
&\quad +\frac{2}{3}V^-\left[\frac{3}{2}\cos\left(\omega_s t-\omega_n t+\delta^-\right)+\frac{1}{2}\underbrace{\left\{\cos\left(\omega_s t+\delta^-+\omega_n t\right)+\cos\left(\omega_s t+\delta^-+\omega_n t-\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\delta^-+\omega_n t+\frac{2\pi}{3}\right)\right\}}_{0}\right] \\
v_d^- &= \left[V^+\cos\left(\omega_s t+\omega_n t+\delta^+\right)+V^-\cos\left(\omega_s t-\omega_n t+\delta^-\right)\right] \;=\; \left[V^-\cos\left(\Delta\omega t+\delta^-\right)+V^+\cos\left(\omega_1 t+\delta^+\right)\right]
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
v_q^- &= \frac{2}{3}V^0\cos\left(\omega_s t+\delta^0\right)\underbrace{\left\{\sin\left(\omega_n t\right)+\sin\left(\omega_n t+\frac{2\pi}{3}\right)+\sin\left(\omega_n t-\frac{2\pi}{3}\right)\right\}}_{0} + \\
&\quad \frac{2}{3}V^+\left\{\cos(\omega_s t+\delta^+)\sin\left(\omega_n t\right)+\cos\left(\omega_s t+\delta^+-\frac{2\pi}{3}\right)\sin\left(\omega_n t+\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\delta^++\frac{2\pi}{3}\right)\sin\left(\omega_n t-\frac{2\pi}{3}\right)\right\} + \\
&\quad \frac{2}{3}V^-\left\{\cos(\omega_s t+\delta^-)\sin\left(\omega_n t\right)+\cos\left(\omega_s t+\delta^-+\frac{2\pi}{3}\right)\sin\left(\omega_n t+\frac{2\pi}{3}\right)+\cos\left(\omega_s t+\delta^+-\frac{2\pi}{3}\right)\sin\left(\omega_n t-\frac{2\pi}{3}\right)\right\} \\
&= \frac{2}{3}V^+\left[\frac{1}{2}\underbrace{\left\{\sin\left(\omega_s t-\omega_n t+\delta^+\right)+\sin\left(\omega_s t-\omega_n t+\delta^+-\frac{2\pi}{3}\right)+\sin\left(\omega_s t-\omega_n t+\delta^++\frac{2\pi}{3}\right)\right\}}_{0}-\frac{3}{2}\sin\left(\omega_s t+\omega_n t+\delta^+\right)\right] + \\
&\quad \frac{2}{3}V^-\left[\frac{3}{2}\sin\left(\omega_s t-\omega_n t+\delta^-\right)+\frac{1}{2}\underbrace{\left\{\sin\left(\omega_s t+\delta^-+\omega_n t\right)+\sin\left(\omega_s t+\delta^-+\omega_n t+\frac{2\pi}{3}\right)+\sin\left(\omega_s t+\delta^-+\omega_n t-\frac{2\pi}{3}\right)\right\}}_{0}\right] \\
v_q^- &= \left[V^-\sin\left(\omega_s t-\omega_n t+\delta^+\right)+V^+\sin\left(\omega_s t+\omega_n t+\delta^-\right)\right] \;=\; \left[V^-\sin\left(\Delta\omega t+\delta^+\right)-V^+\sin\left(\omega_1 t+\delta^-\right)\right]
\end{aligned} \tag{14}
$$

References

[1] C. L. Fortescue, "Method of symmetrical co-ordinates applied to the solution of polyphase networks," *Trans. of the American Institute of Electrical Engineers*, vol. 37, no. 2, 1918.

[2] C. Anirudh and V. S. S. Kumar, "Analysis of filter design approaches for extraction of instantaneous symmetrical components," in *2022 IEEE Texas Power and Energy Conference (TPEC)*, 2022, pp. 1–6.

[3] C. V. S. Anirudh and V. S. S. Kumar, "Estimation of symmetrical component phasors and frequency of three-phase voltage signals using transformations," *IEEE Transactions on Power Delivery*, pp. 1–10, 2022.