2

Principles of Steady-State Converter Analysis

2.1 INTRODUCTION

In the previous chapter, the buck converter was introduced as a means of reducing the dc voltage, using only nondissipative switches, inductors, and capacitors. The switch produces a rectangular waveform $v_s(t)$ as illustrated in Fig. 2.1. The voltage $v_s(t)$ is equal to the dc input voltage V_g when the switch is in position 1, and is equal to zero when the switch is in position 2. In practice, the switch is realized using

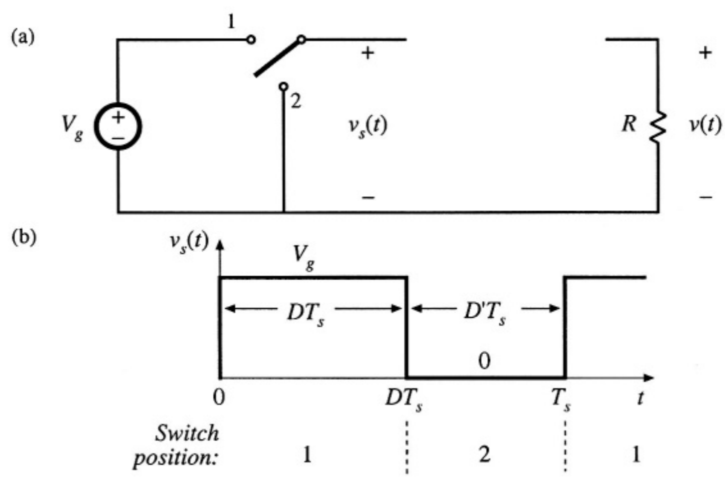


Fig. 2.1 Ideal switch, (a), used to reduce the voltage dc component, and (b) its output voltage waveform $v_s(t)$.

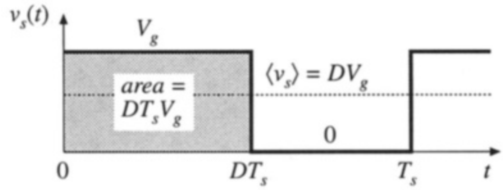


Fig. 2.2 Determination of the switch output voltage dc component, by integrating and dividing by the switching period.

power semiconductor devices, such as transistors and diodes, which are controlled to turn on and off as required to perform the function of the ideal switch. The switching frequency f_s, equal to the inverse of the switching period T_s, generally lies in the range of 1 kHz to 1 MHz, depending on the switching speed of the semiconductor devices. The duty ratio D is the fraction of time that the switch spends in position 1, and is a number between zero and one. The complement of the duty ratio, D', is defined as $(1 - D)$.

The switch reduces the dc component of the voltage: the switch output voltage $v_s(t)$ has a dc component that is less than the converter dc input voltage V_g. From Fourier analysis, we know that the dc component of $v_s(t)$ is given by its average value $\langle v_s \rangle$, or

$$\langle v_s \rangle = \frac{1}{T_s}\int_0^{T_s} v_s(t)dt \tag{2.1}$$

As illustrated in Fig. 2.2, the integral is given by the area under the curve, or DT_sV_g. The average value is therefore

$$\langle v_s \rangle = \frac{1}{T_s}\left(DT_sV_g\right) = DV_g \tag{2.2}$$

So the average value, or dc component, of $v_s(t)$ is equal to the duty cycle times the dc input voltage V_g. The switch reduces the dc voltage by a factor of D.

What remains is to insert a low-pass filter as shown in Fig. 2.3. The filter is designed to pass the dc component of $v_s(t)$, but to reject the components of $v_s(t)$ at the switching frequency and its harmonics. The output voltage $v(t)$ is then essentially equal to the dc component of $v_s(t)$:

$$v \approx \langle v_s \rangle = DV_g \tag{2.3}$$

The converter of Fig. 2.3 has been realized using lossless elements. To the extent that they are ideal, the inductor, capacitor, and switch do not dissipate power. For example, when the switch is closed, its voltage drop is zero, and the current is zero when the switch is open. In either case, the power dissipated by the switch is zero. Hence, efficiencies approaching 100% can be obtained. So to the extent that the components are ideal, we can realize our objective of changing dc voltage levels using a lossless network.

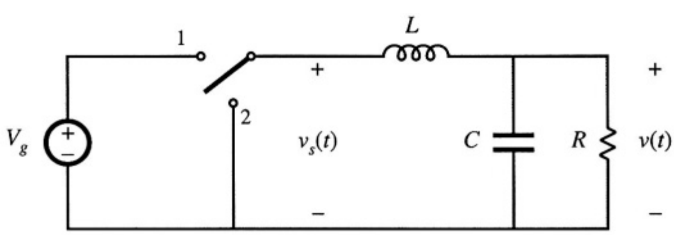


Fig. 2.3 Insertion of low-pass filter, to remove the switching harmonics and pass only the dc component of $v_s(t)$ to the output.

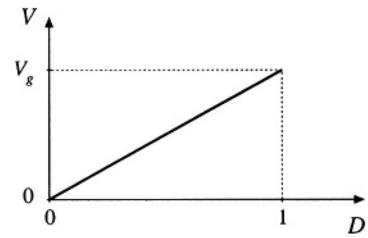


Fig. 2.4 Buck converter dc output voltage V vs. duty cycle D.

The network of Fig. 2.3 also allows control of the output. Figure 2.4 is the control characteristic of the converter. The output voltage, given by Eq. (2.3), is plotted vs. duty cycle. The buck converter has a linear control characteristic. Also, the output voltage is less than or equal to the input voltage, since $0 \le D \le 1$. Feedback systems are often constructed that adjust the duty cycle D to regulate the converter output voltage. Inverters or power amplifiers can also be built, in which the duty cycle varies slowly with time and the output voltage follows.

The buck converter is just one of many possible switching converters. Two other commonly used converters, which perform different voltage conversion functions, are illustrated in Fig. 2.5. In the boost converter, the positions of the inductor and switch are reversed. It is shown later in this chapter that the boost converter steps the voltage up: $V \ge V_g$. Another converter, the buck-boost converter, can either increase or decrease the magnitude of the voltage, but the polarity is inverted. So with a positive input voltage, the ideal buck-boost converter can produce a negative output voltage of any magnitude. It may at first be surprising that dc output voltages can be produced that are greater in magnitude than the input, or that have opposite polarity. But it is indeed possible to produce any desired dc output voltage using a passive network of only inductors, capacitors, and embedded switches.

In the above discussion, it was possible to derive an expression for the output voltage of the buck converter, Eq. (2.3), using some simple arguments based on Fourier analysis. However, it may not be immediately obvious how to directly apply these arguments to find the dc output voltage of the boost, buck–boost, or other converters. The objective of this chapter is the development of a more general method for analyzing any switching converter comprised of a network of inductors, capacitors, and switches [1-8].

The principles of *inductor volt-second balance* and *capacitor charge balance* are derived; these can be used to solve for the inductor currents and capacitor voltages of switching converters. A useful approximation, the *small–ripple* or *linear–ripple approximation,* greatly facilitates the analysis. Some simple methods for selecting the filter element values are also discussed.

2.2 INDUCTOR VOLT-SECOND BALANCE, CAPACITOR CHARGE BALANCE, AND THE SMALL-RIPPLE APPROXIMATION

Let us more closely examine the inductor and capacitor waveforms in the buck converter of Fig. 2.6. It is impossible to build a perfect low-pass filter that allows the dc component to pass but completely removes the components at the switching frequency and its harmonics. So the low-pass filter must allow at least some small amount of the high-frequency harmonics generated by the switch to reach the output. Hence, in practice the output voltage waveform $v(t)$ appears as illustrated in Fig. 2.7, and can be expressed as

$$v(t) = V + v_{ripple}(t) \tag{2.4}$$

So the actual output voltage $v(t)$ consists of the desired dc component V, plus a small undesired ac com-

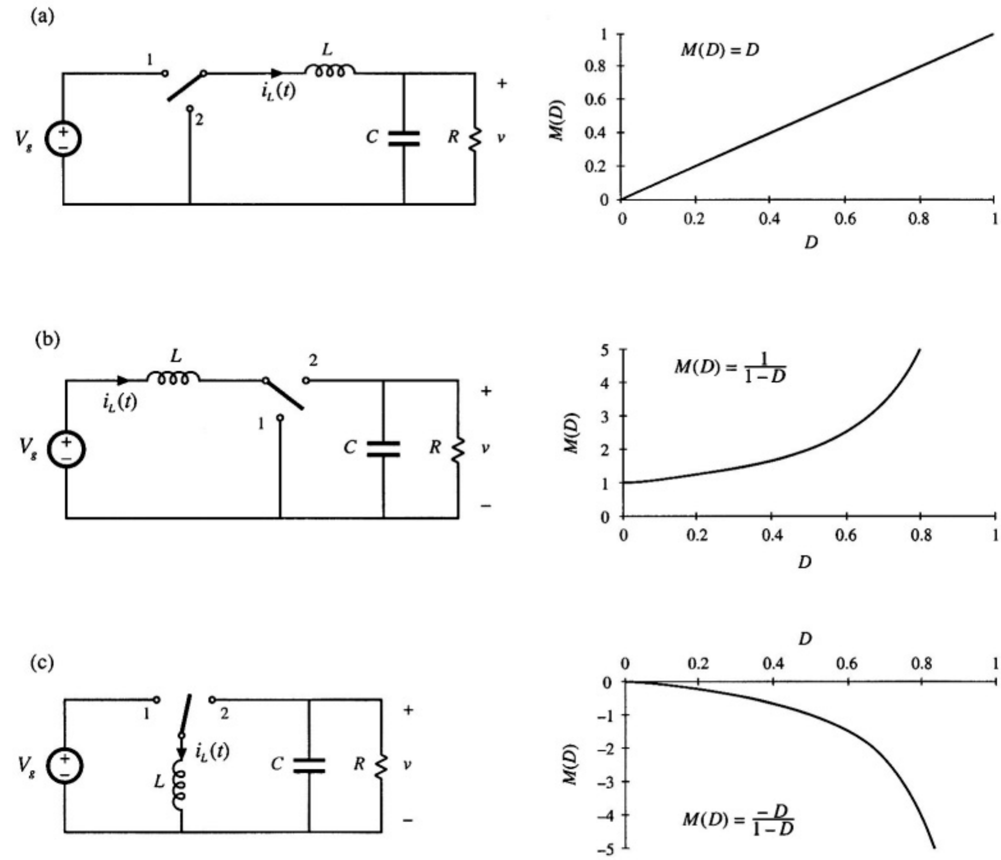
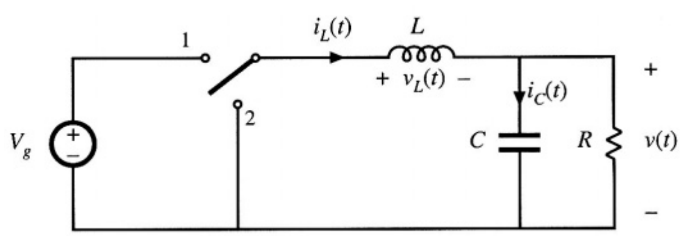


Fig. 2.5 Three basic converters and their dc conversion ratios $M(D) = V/V_g$: (a) buck, (b) boost, (c) buck–boost.

Fig. 2.6 Buck converter circuit, with the inductor voltage $v_L(t)$ and capacitor current $i_C(t)$ waveforms specifically identified.



ponent $v_{ripple}(t)$ arising from the incomplete attenuation of the switching harmonics by the low-pass filter. The magnitude of $v_{ripple}(t)$has been exaggerated in Fig. 2.7.

The output voltage switching ripple should be small in any well-designed converter, since the object is to produce a dc output. For example, in a computer power supply having a 3.3 V output, the switching ripple is normally required to be less than a few tens of millivolts, or less than 1% of the dc component V. So it is nearly always a good approximation to assume that the magnitude of the switching

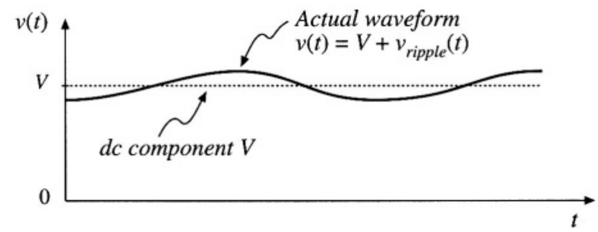


Fig. 2.7 Output voltage waveform $v(t)$, consisting of dc component V and switching ripple $v_{ripple}(t)$.

ripple is much smaller than the dc component:

$$\left\| v_{ripple} \right\| \ll V \tag{2.5}$$

Therefore, the output voltage $v(t)$ is well approximated by its dc component V, with the small ripple term $v_{ripple}(t)$ neglected:

$$v(t) \approx V \tag{2.6}$$

This approximation, known as the small-ripple approximation, or the linear-ripple approximation, greatly simplifies the analysis of the converter waveforms and is used throughout this book.

Next let us analyze the inductor current waveform. We can find the inductor current by integrating the inductor voltage waveform. With the switch in position 1, the left side of the inductor is connected to the input voltage V_g, and the circuit reduces to Fig. 2.8(a). The inductor voltage $v_L(t)$ is then given by

$$v_L = V_g - v(t) \tag{2.7}$$

As described above, the output voltage $v(t)$ consists of the dc component V, plus a small ac ripple term $v_{ripple}(t)$. We can make the small ripple approximation here, Eq. (2.6), to replace $v(t)$ with its dc component V:

$$v_L \approx V_g - V \tag{2.8}$$

So with the switch in position 1, the inductor voltage is essentially constant and equal to $V_g - V$, as shown in Fig. 2.9. By knowledge of the inductor voltage waveform, the inductor current can be found by use of the definition

$$v_L(t) = L \frac{di_L(t)}{dt} \tag{2.9}$$

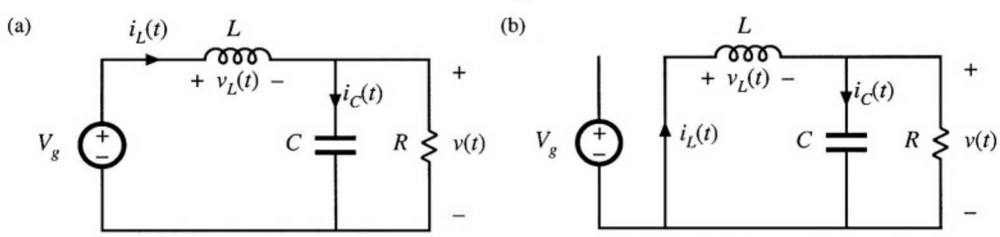


Fig. 2.8 Buck converter circuit: (a) while the switch is in position 1, (b) while the switch is in position 2.

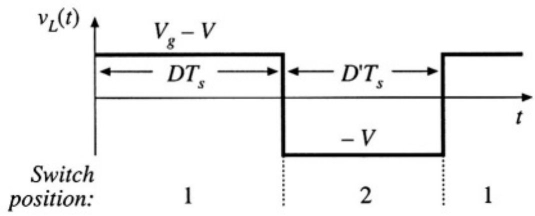


Fig. 2.9 Steady-state inductor voltage waveform, buck converter.

Thus, during the first interval, when $v_L(t)$ is approximately $(V_g - V)$, the slope of the inductor current waveform is

$$\frac{di_L(t)}{dt} = \frac{v_L(t)}{L} \approx \frac{V_g - V}{L} \tag{2.10}$$

which follows by dividing Eq. (2.9) by L, and substituting Eq. (2.8). Since the inductor voltage $v_L(t)$ is essentially constant while the switch is in position 1, the inductor current slope is also essentially constant and the inductor current increases linearly.

Similar arguments apply during the second subinterval, when the switch is in position 2. The left side of the inductor is then connected to ground, leading to the circuit of Fig. 2.8(b). It is important to consistently define the polarities of the inductor current and voltage; in particular, the polarity of $v_L(t)$ is defined consistently in Figs. 2.7, 2.8(a), and 2.8(b). So the inductor voltage during the second subinterval is given by

$$v_L(t) = -v(t) \tag{2.11}$$

Use of the small ripple approximation, Eq. (2.6), leads to

$$v_L(t) \approx -V \tag{2.12}$$

So the inductor voltage is also essentially constant while the switch is in position 2, as illustrated in Fig. 2.9. Substitution of Eq. (2.12) into Eq. (2.9) and solution for the slope of the inductor current yields

$$\frac{di_L(t)}{dt} \approx -\frac{V}{L} \tag{2.13}$$

Hence, during the second subinterval the inductor current changes with a negative and essentially constant slope.

We can now sketch the inductor current waveform (Fig. 2.10). The inductor current begins at some initial value $i_L(0)$. During the first subinterval, with the switch in position 1, the inductor current increases with the slope given in Eq. (2.10). At time $t = DT_s$, the switch changes to position 2. The current then decreases with the constant slope given by Eq. (2.13). At time $t = T_s$, the switch changes back to

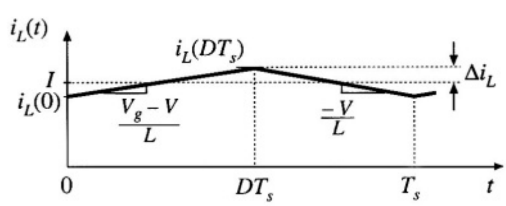


Fig. 2.10 Steady-state inductor current waveform, buck converter.

position 1, and the process repeats.

It is of interest to calculate the inductor current ripple Δi_L. As illustrated in Fig. 2.10, the peak inductor current is equal to the dc component I plus the peak-to-average ripple Δi_L. This peak current flows through not only the inductor, but also through the semiconductor devices that comprise the switch. Knowledge of the peak current is necessary when specifying the ratings of these devices.

Since we know the slope of the inductor current during the first subinterval, and we also know the length of the first subinterval, we can calculate the ripple magnitude. The $i_L(t)$ waveform is symmetrical about I, and hence during the first subinterval the current increases by $2\Delta i_L$ (since Δi_L is the peak ripple, the peak-to-peak ripple is $2\Delta i_L$). So the change in current, $2\Delta i_L$, is equal to the slope (the applied inductor voltage divided by L) times the length of the first subinterval (DT_s):

$$\text{(change in } i_L) = \text{(slope)(length of subinterval)}$$
$$\left(2\Delta i_L\right) = \left(\frac{V_g - V}{L}\right)\left(DT_s\right) \tag{2.14}$$

Solution for Δi_L yields

$$\Delta i_L = \frac{V_g - V}{2L} DT_s \tag{2.15}$$

Typical values of Δi_L lie in the range of 10% to 20% of the full-load value of the dc component I. It is undesirable to allow Δi_L to become too large; doing so would increase the peak currents of the inductor and of the semiconductor switching devices, and would increase their size and cost. So by design the inductor current ripple is also usually small compared to the dc component I. The small-ripple approximation $i_L(t) \approx I$ is usually justified for the inductor current.

The inductor value can be chosen such that a desired current ripple Δi_L is attained. Solution of Eq. (2.15) for the inductance L yields

$$L = \frac{V_g - V}{2\Delta i_L} DT_s \tag{2.16}$$

This equation is commonly used to select the value of inductance in the buck converter.

It is entirely possible to solve converters exactly, without use of the small-ripple approximation. For example, one could use the Laplace transform to write expressions for the waveforms of the circuits of Figs. 2.8(a) and 2.8(b). One could then invert the transforms, match boundary conditions, and find the periodic steady-state solution of the circuit. Having done so, one could then find the dc components of the waveforms and the peak values. But this is a great deal of work, and the results are nearly always intractable. Besides, the extra work involved in writing equations that exactly describe the ripple is a waste of time, since the ripple is small and is undesired. The small-ripple approximation is easy to apply, and quickly yields simple expressions for the dc components of the converter waveforms.

The inductor current waveform of Fig. 2.10 is drawn under steady-state conditions, with the converter operating in equilibrium. Let's consider next what happens to the inductor current when the converter is first turned on. Suppose that the inductor current and output voltage are initially zero, and an input voltage V_g is then applied. As shown in Fig. 2.11, $i_L(0)$ is zero. During the first subinterval, with the switch in position 1, we know that the inductor current will increase, with a slope of $(V_g - v)/L$ and with v initially zero. Next, with the switch in position 2, the inductor current will change with a slope of $-v/L$; since v is initially zero, this slope is essentially zero. It can be seen that there is a net increase in inductor current over the first switching period, because $i_L(T_s)$ is greater than $i_L(0)$. Since the inductor current

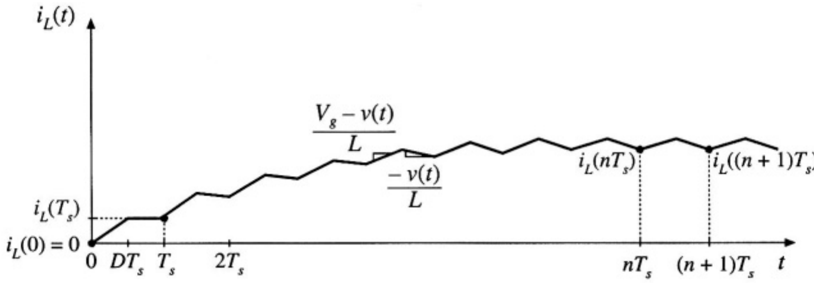


Fig. 2.11 Inductor current waveform during converter turn-on transient.

flows to the output, the output capacitor will charge slightly, and v will increase slightly. The process repeats during the second and succeeding switching periods, with the inductor current increasing during each subinterval 1 and decreasing during each subinterval 2.

As the output capacitor continues to charge and v increases, the slope during subinterval 1 decreases while the slope during subinterval 2 becomes more negative. Eventually, the point is reached where the increase in inductor current during subinterval 1 is equal to the decrease in inductor current during subinterval 2. There is then no net change in inductor current over a complete switching period, and the converter operates in steady state. The converter waveforms are periodic: $i_L(nT_s) = i_L((n+1)T_s)$. From this point on, the inductor current waveform appears as in Fig. 2.10.

The requirement that, in equilibrium, the net change in inductor current over one switching period be zero leads us to a way to find steady-state conditions in any switching converter: the principle of *inductor volt-second balance.* Given the defining relation of an inductor:

$$v_L(t) = L\frac{di_L(t)}{dt} \tag{2.17}$$

Integration over one complete switching period, say from $t = 0$ to T_s, yields

$$i_L(T_s) - i_L(0) = \frac{1}{L}\int_0^{T_s} v_L(t)dt \tag{2.18}$$

This equation states that the net change in inductor current over one switching period, given by the left-hand side of Eq. (2.18), is proportional to the integral of the applied inductor voltage over the interval. In steady state, the initial and final values of the inductor current are equal, and hence the left-hand side of Eq. (2.18) is zero. Therefore, in steady state the integral of the applied inductor voltage must be zero:

$$0 = \int_0^{T_s} v_L(t)dt \tag{2.19}$$

The right-hand side of Eq. (2.19) has the units of volt-seconds or flux-linkages. Equation (2.19) states that the total area, or net volt-seconds, under the $v_L(t)$ waveform must be zero.

An equivalent form is obtained by dividing both sides of Eq. (2.19) by the switching period T_s:

$$0 = \frac{1}{T_s}\int_0^{T_s} v_L(t)dt = \langle v_L \rangle \tag{2.20}$$

The right-hand side of Eq. (2.20) is recognized as the average value, or dc component, of $v_L(t)$. Equation

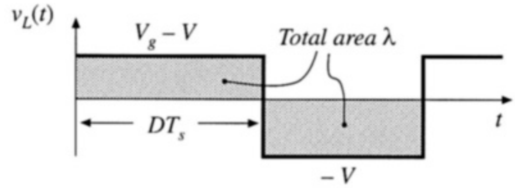


Fig. 2.12 The principle of inductor volt-second balance: in steady state, the net volt-seconds applied to an inductor (i.e., the total area λ) must be zero.

(2.20) states that, in equilibrium, the applied inductor voltage must have zero dc component.

The inductor voltage waveform of Fig. 2.9 is reproduced in Fig. 2.12, with the area under the $v_L(t)$ curve specifically identified. The total area λ is given by the areas of the two rectangles, or

$$\lambda = \int_0^{T_s} v_L(t)dt = \left(V_g - V\right)\left(DT_s\right) + \left(-V\right)\left(D'T_s\right) \tag{2.21}$$

The average value is therefore

$$\langle v_L \rangle = \frac{\lambda}{T_s} = D\left(V_g - V\right) + D'\left(-V\right) \tag{2.22}$$

By equating $\langle v_L \rangle$ to zero, and noting that $D + D' = 1$, one obtains

$$0 = DV_g - \left(D + D'\right)V = DV_g - V \tag{2.23}$$

Solution for V yields

$$V = DV_g \tag{2.24}$$

which coincides with the result obtained previously, Eq. (2.3). So the principle of inductor volt-second balance allows us to derive an expression for the dc component of the converter output voltage. An advantage of this approach is its generality—it can be applied to any converter. One simply sketches the applied inductor voltage waveform, and equates the average value to zero. This method is used later in this chapter, to solve several more complicated converters.

Similar arguments can be applied to capacitors. The defining equation of a capacitor is

$$i_C(t) = C\frac{dv_C(t)}{dt} \tag{2.25}$$

Integration of this equation over one switching period yields

$$v_C(T_s) - v_C(0) = \frac{1}{C}\int_0^{T_s} i_C(t)dt \tag{2.26}$$

In steady state, the net change over one switching period of the capacitor voltage must be zero, so that the left-hand side of Eq. (2.26) is equal to zero. Therefore, in equilibrium the integral of the capacitor current over one switching period (having the dimensions of amp-seconds, or charge) should be zero. There is no net change in capacitor charge in steady state. An equivalent statement is

$$0 = \frac{1}{T_s}\int_0^{T_s} i_C(t)dt = \langle i_C \rangle \tag{2.27}$$

The average value, or dc component, of the capacitor current must be zero in equilibrium.

This should be an intuitive result. If a dc current is applied to a capacitor, then the capacitor will charge continually and its voltage will increase without bound. Likewise, if a dc voltage is applied to an inductor, then the flux will increase continually and the inductor current will increase without bound. Equation (2.27), called the principle of *capacitor amp-second balance* or *capacitor charge balance,* can be used to find the steady-state currents in a switching converter.

2.3 BOOST CONVERTER EXAMPLE

The boost converter, Fig. 2.13(a), is another well-known switched-mode converter that is capable of producing a dc output voltage greater in magnitude than the dc input voltage. A practical realization of the switch, using a MOSFET and diode, is shown in Fig. 2.13(b). Let us apply the small-ripple approximation and the principles of inductor volt-second balance and capacitor charge balance to find the steady-state output voltage and inductor current for this converter.

With the switch in position 1, the right-hand side of the inductor is connected to ground, resulting in the network of Fig. 2.14(a). The inductor voltage and capacitor current for this subinterval are given by

$$\begin{aligned} v_L &= V_g \\ i_C &= -\frac{v}{R} \end{aligned} \tag{2.28}$$

Use of the linear ripple approximation, $v \approx V$, leads to

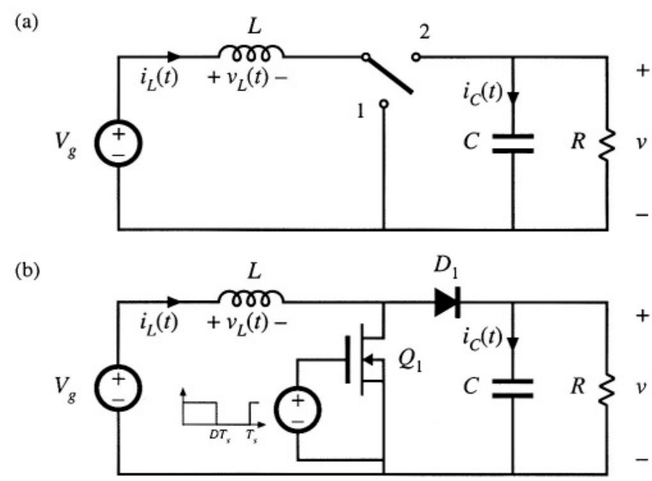


Fig. 2.13 Boost converter: (a) with ideal switch, (b) practical realization using MOSFET and diode.

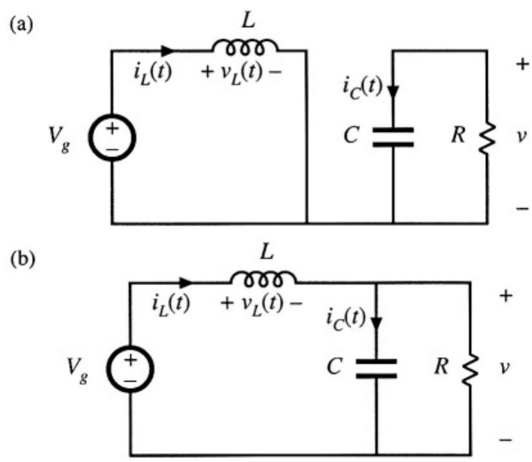


Fig. 2.14 Boost converter circuit, (a) while the switch is in position 1, (b) while the switch is in position 2.

$$\begin{aligned} v_L &= V_g \\ i_C &= -\frac{V}{R} \end{aligned} \tag{2.29}$$

With the switch in position 2, the inductor is connected to the output, leading to the circuit of Fig. 2.14(b). The inductor voltage and capacitor current are then

$$\begin{aligned} v_L &= V_g - v \\ i_C &= i_L - \frac{v}{R} \end{aligned} \tag{2.30}$$

Use of the small-ripple approximation, $v \approx V$ and $i_L \approx I$, leads to

$$\begin{aligned} v_L &= V_g - V \\ i_C &= I - \frac{V}{R} \end{aligned} \tag{2.31}$$

Equations (2.29) and (2.31) are used to sketch the inductor voltage and capacitor current waveforms of Fig. 2.15.

Fig. 2.15 Boost converter voltage and current waveforms.

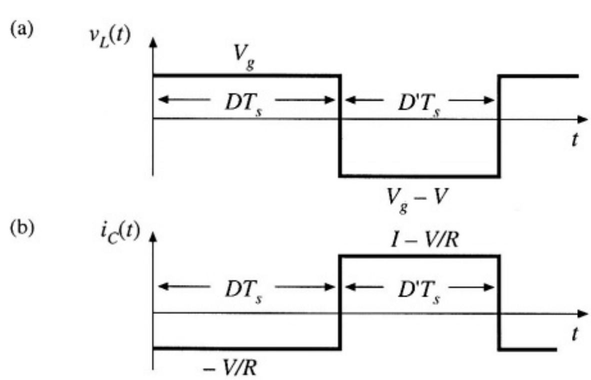
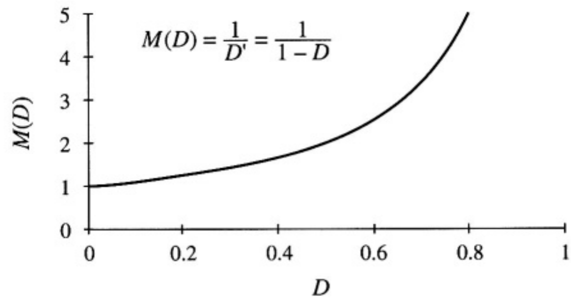


Fig. 2.16 Dc conversion ratio $M(D)$ of the boost converter.



It can be inferred from the inductor voltage waveform of Fig. 2.15(a) that the dc output voltage V is greater than the input voltage V_g. During the first subinterval, $v_L(t)$ is equal to the dc input voltage V_g, and positive volt-seconds are applied to the inductor. Since, in steady-state, the total volt-seconds applied over one switching period must be zero, negative volt-seconds must be applied during the second subinterval. Therefore, the inductor voltage during the second subinterval, $(V_g - V)$, must be negative. Hence, V is greater than V_g.

The total volt-seconds applied to the inductor over one switching period are:

$$\int_0^{T_s} v_L(t)dt = \left(V_g\right)DT_s + \left(V_g - V\right)D'T_s \tag{2.32}$$

By equating this expression to zero and collecting terms, one obtains

$$V_g(D + D') - VD' = 0 \tag{2.33}$$

Solution for V, and by noting that $(D + D') = 1$, yields the expression for the output voltage,

$$V = \frac{V_g}{D'} \tag{2.34}$$

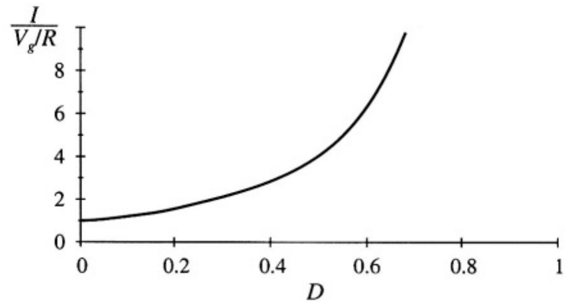
The voltage conversion ratio $M(D)$ is the ratio of the output to the input voltage of a dc-dc converter. Equation (2.34) predicts that the voltage conversion ratio is given by

$$M(D) = \frac{V}{V_g} = \frac{1}{D'} = \frac{1}{1-D} \tag{2.35}$$

This equation is plotted in Fig. 2.16. At $D = 0$, $V = V_g$. The output voltage increases as D increases, and in the ideal case tends to infinity as D tends to 1. So the ideal boost converter is capable of producing any output voltage greater than the input voltage. There are, of course, limits to the output voltage that can be produced by a practical boost converter. In the next chapter, component nonidealities are modeled, and it is found that the maximum output voltage of a practical boost converter is indeed limited. Nonetheless, very large output voltages can be produced if the nonidealities are sufficiently small.

The dc component of the inductor current is derived by use of the principle of capacitor charge balance. During the first subinterval, the capacitor supplies the load current, and the capacitor is partially discharged. During the second subinterval, the inductor current supplies the load and, additionally, recharges the capacitor. The net change in capacitor charge over one switching period is found by integrating the $i_C(t)$ waveform of Fig. 2.15(b),

Fig. 2.17 Variation of inductor current dc component I with duty cycle, boost converter.



$$\int_0^{T_s} i_C(t)dt = \left(-\frac{V}{R}\right)DT_s + \left(I - \frac{V}{R}\right)D'T_s \tag{2.36}$$

Collecting terms, and equating the result to zero, leads the steady-state result

$$-\frac{V}{R}\left(D + D'\right) + ID' = 0 \tag{2.37}$$

By noting that $(D + D') = 1$, and by solving for the inductor current dc component I, one obtains

$$I = \frac{V}{D'R} \tag{2.38}$$

So the inductor current dc component I is equal to the load current, V/R, divided by D'. Substitution of Eq. (2.34) to eliminate V yields

$$I = \frac{V_g}{D'^2 R} \tag{2.39}$$

This equation is plotted in Fig. 2.17. It can be seen that the inductor current becomes large as D approaches 1.

This inductor current, which coincides with the dc input current in the boost converter, is greater than the load current. Physically, this must be the case: to the extent that the converter elements are ideal, the converter input and output powers are equal. Since the converter output voltage is greater than the input voltage, the input current must likewise be greater than the output current. In practice, the inductor current flows through the semiconductor forward voltage drops, the inductor winding resistance, and other sources of power loss. As the duty cycle approaches one, the inductor current becomes very large and these component nonidealities lead to large power losses. In consequence, the efficiency of the boost converter decreases rapidly at high duty cycle.

Next, let us sketch the inductor current $i_L(t)$ waveform and derive an expression for the inductor current ripple Δi_L. The inductor voltage waveform $v_L(t)$ has been already found (Fig. 2.15), so we can sketch the inductor current waveform directly. During the first subinterval, with the switch in position 1, the slope of the inductor current is given by

$$\frac{di_L(t)}{dt} = \frac{v_L(t)}{L} = \frac{V_g}{L} \tag{2.40}$$

Likewise, when the switch is in position 2, the slope of the inductor current waveform is

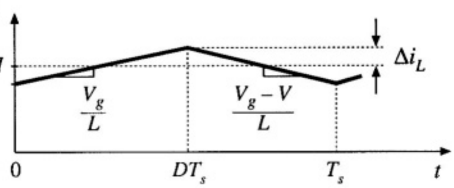


Fig. 2.18 Boost converter inductor current waveform $i_L(t)$.

$$\frac{di_L(t)}{dt} = \frac{v_L(t)}{L} = \frac{V_g - V}{L} \tag{2.41}$$

The inductor current waveform is sketched in Fig. 2.18. During the first subinterval, the change in inductor current, $2\Delta i_L$, is equal to the slope multiplied by the length of the subinterval, or

$$2\Delta i_L = \frac{V_g}{L} DT_s \tag{2.42}$$

Solution for Δi_L leads to

$$\Delta i_L = \frac{V_g}{2L} DT_s \tag{2.43}$$

This expression can be used to select the inductor value L such that a given value of Δi_L is obtained.

Likewise, the capacitor voltage $v(t)$ waveform can be sketched, and an expression derived for the output voltage ripple peak magnitude Δv. The capacitor current waveform $i_C(t)$ is given in Fig. 2.15. During the first subinterval, the slope of the capacitor voltage waveform $v(t)$ is

$$\frac{dv_C(t)}{dt} = \frac{i_C(t)}{C} = \frac{-V}{RC} \tag{2.44}$$

During the second subinterval, the slope is

$$\frac{dv_C(t)}{dt} = \frac{i_C(t)}{C} = \frac{I}{C} - \frac{V}{RC} \tag{2.45}$$

The capacitor voltage waveform is sketched in Fig. 2.19. During the first subinterval, the change in capacitor voltage, $-2\Delta v$, is equal to the slope multiplied by the length of the subinterval:

$$-2\Delta v = \frac{-V}{RC} DT_s \tag{2.46}$$

Solution for Δv yields

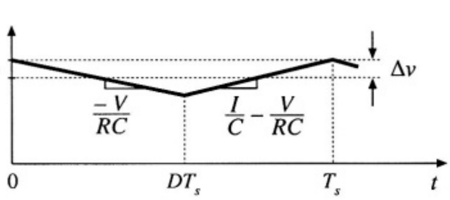


Fig. 2.19 Boost converter output voltage waveform $v(t)$.

$$\Delta v = \frac{V}{2RC} DT_s \tag{2.47}$$

This expression can be used to select the capacitor value C to obtain a given output voltage ripple peak magnitude Δv.

2.4 ĆUK CONVERTER EXAMPLE

As a second example, consider the Ćuk converter of Fig. 2.20(a). This converter performs a dc conversion function similar to the buck-boost converter: it can either increase or decrease the magnitude of the dc voltage, and it inverts the polarity. A practical realization using a transistor and diode is illustrated in Fig. 2.20(b).

This converter operates via capacitive energy transfer. As illustrated in Fig. 2.21, capacitor C_1 is connected through L_1 to the input source while the switch is in position 2, and source energy is stored in C_1. When the switch is in position 1, this energy is released through L_2 to the load.

The inductor currents and capacitor voltages are defined, with polarities assigned somewhat arbitrarily, in Fig. 2.20(a). In this section, the principles of inductor volt-second balance and capacitor charge balance are applied to find the dc components of the inductor currents and capacitor voltages. The voltage and current ripple magnitudes are also found.

During the first subinterval, while the switch is in position 1, the converter circuit reduces to Fig. 2.21 (a). The inductor voltages and capacitor currents are:

$$\begin{aligned} v_{L1} &= V_g \\ v_{L2} &= -v_1 - v_2 \\ i_{C1} &= i_2 \\ i_{C2} &= i_2 - \frac{v_2}{R} \end{aligned} \tag{2.48}$$

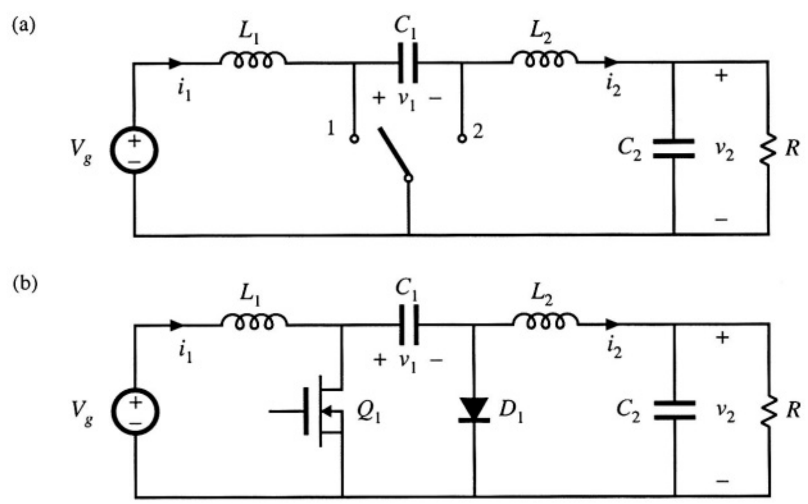


Fig. 2.20 Ćuk converter: (a) with ideal switch, (b) practical realization using MOSFET and diode.

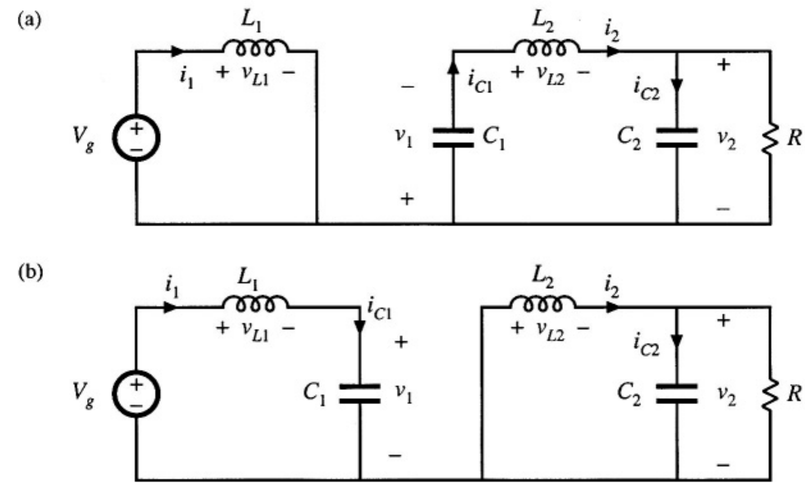


Fig. 2.21 Ćuk converter circuit: (a) while switch is in position 1, (b) while switch is in position 2.

We next assume that the switching ripple magnitudes in $i_1(t)$, $i_2(t)$, $v_1(t)$, and $v_2(t)$ are small compared to their respective dc components I_1, I_2, V_1, and V_2. We can therefore make the small-ripple approximation, and Eq. (2.48) becomes

$$\begin{aligned} v_{L1} &= V_g \\ v_{L2} &= -V_1 - V_2 \\ i_{C1} &= I_2 \\ i_{C2} &= I_2 - \frac{V_2}{R} \end{aligned} \tag{2.49}$$

During the second subinterval, with the switch in position 2, the converter circuit elements are connected as in Fig. 2.21(b). The inductor voltages and capacitor currents are:

$$\begin{aligned} v_{L1} &= V_g - v_1 \\ v_{L2} &= -v_2 \\ i_{C1} &= i_1 \\ i_{C2} &= i_2 - \frac{v_2}{R} \end{aligned} \tag{2.50}$$

We again make the small-ripple approximation, and hence Eq. (2.50) becomes

$$\begin{aligned} v_{L1} &= V_g - V_1 \\ v_{L2} &= -V_2 \\ i_{C1} &= I_1 \\ i_{C2} &= I_2 - \frac{V_2}{R} \end{aligned} \tag{2.51}$$

Equations (2.49) and (2.51) are used to sketch the inductor voltage and capacitor current waveforms in Fig. 2.22.

The next step is to equate the dc components, or average values, of the waveforms of Fig. 2.22

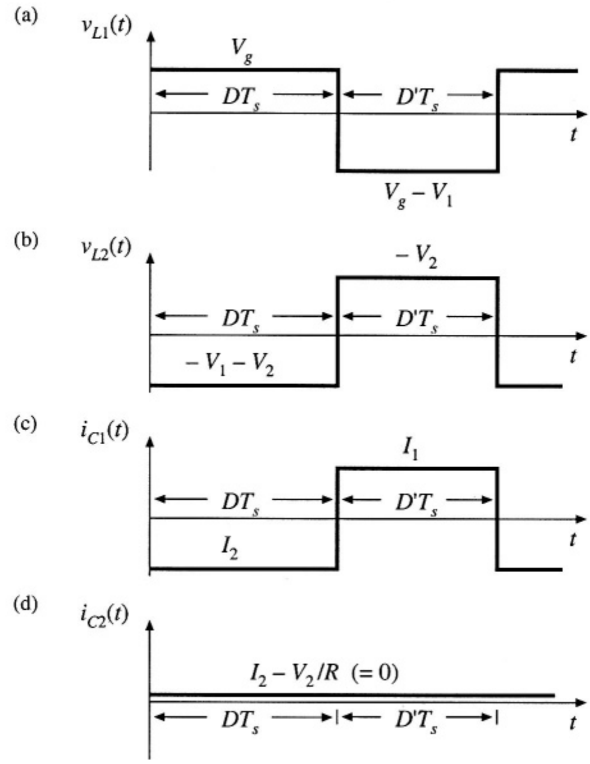


Fig. 2.22 Ćuk converter waveforms: (a) inductor voltage $v_{L1}(t)$, (b) inductor voltage $v_{L2}(t)$, (c) capacitor current $i_{C1}(t)$, (d) capacitor current $i_{C2}(t)$.

to zero, to find the steady-state conditions in the converter. The results are:

$$\begin{aligned}
\langle v_{L1}\rangle &= DV_g + D'\left(V_g - V_1\right) = 0\\
\langle v_{L2}\rangle &= D\left(-V_1 - V_2\right) + D'\left(-V_2\right) = 0\\
\langle i_{C1}\rangle &= DI_2 + D'I_1 = 0\\
\langle i_{C2}\rangle &= I_2 - \frac{V_2}{R} = 0
\end{aligned} \tag{2.52}$$

Solution of this system of equations for the dc components of the capacitor voltages and inductor currents leads to

$$\begin{aligned}
V_1 &= \frac{V_g}{D'}\\
V_2 &= -\frac{D}{D'}V_g\\
I_1 &= -\frac{D}{D'}I_2 = \left(\frac{D}{D'}\right)^2\frac{V_g}{R}\\
I_2 &= \frac{V_2}{R} = -\frac{D}{D'}\frac{V_g}{R}
\end{aligned} \tag{2.53}$$

The dependence of the dc output voltage V_2 on the duty cycle D is sketched in Fig. 2.23.

The inductor current waveforms are sketched in Fig. 2.24(a) and 2.24(b), and the capacitor C_1

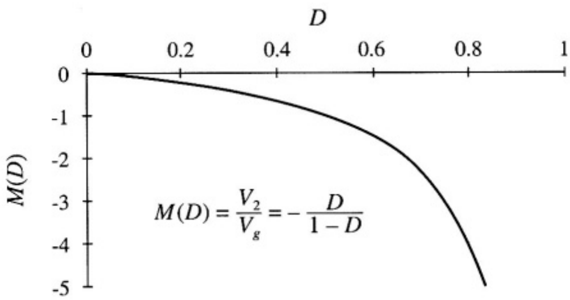


Fig. 2.23 Dc conversion ratio $M(D) = -V/V_g$ of the Ćuk converter.

voltage waveform $v_1(t)$ is sketched in Fig. 2.24(c). During the first subinterval, the slopes of the waveforms are given by

$$\begin{aligned}
\frac{di_1(t)}{dt} &= \frac{v_{L1}(t)}{L_1} = \frac{V_g}{L_1} \\
\frac{di_2(t)}{dt} &= \frac{v_{L2}(t)}{L_2} = \frac{-V_1 - V_2}{L_2} \\
\frac{dv_1(t)}{dt} &= \frac{i_{C1}(t)}{C_1} = \frac{I_2}{C_1}
\end{aligned} \tag{2.54}$$

Equation (2.49) has been used here to substitute for the values of v_{L1}, v_{L2}, and i_{C1} during the first subinterval. During the second interval, the slopes of the waveforms are given by

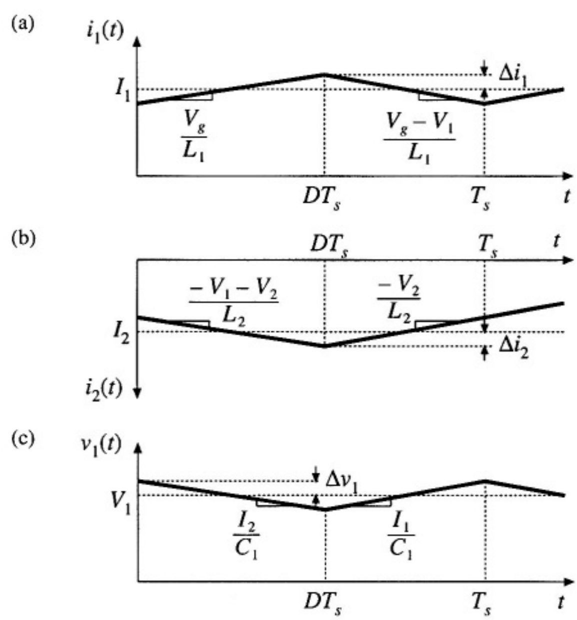


Fig. 2.24 Ćuk converter waveforms: (a) inductor current $i_1(t)$, (b) inductor current $i_2(t)$, (c) capacitor voltage $v_1(t)$.

$$\frac{di_1(t)}{dt} = \frac{v_{L1}(t)}{L_1} = \frac{V_g - V_1}{L_1}$$
$$\frac{di_2(t)}{dt} = \frac{v_{L2}(t)}{L_2} = \frac{-V_2}{L_2} \tag{2.55}$$
$$\frac{dv_1(t)}{dt} = \frac{i_{C1}(t)}{C_1} = \frac{I_1}{C_1}$$

Equation (2.51) was used to substitute for the values of v_{L1}, v_{L2}, and i_{C1} during the second subinterval.

During the first subinterval, the quantities $i_1(t)$, $i_2(t)$, and $v_1(t)$ change by $2\Delta i_1$, $-2\Delta i_2$, and $-2\Delta v_1$, respectively. These changes are equal to the slopes given in Eq. (2.54), multiplied by the subinterval length DT_s, yielding

$$\Delta i_1 = \frac{V_g DT_s}{2L_1}$$
$$\Delta i_2 = \frac{V_1 + V_2}{2L_2} DT_s \tag{2.56}$$
$$\Delta v_1 = \frac{-I_2 DT_s}{2C_1}$$

The dc relationships, Eq. (2.53), can now be used to simplify these expressions and eliminate V_1, V_2, and I_1, leading to

$$\Delta i_1 = \frac{V_g DT_s}{2L_1}$$
$$\Delta i_2 = \frac{V_g DT_s}{2L_2} \tag{2.57}$$
$$\Delta v_1 = \frac{V_g D^2 T_s}{2D'RC_1}$$

These expressions can be used to select values of L_1, L_2, and C_1, such that desired values of switching ripple magnitudes are obtained.

Similar arguments cannot be used to estimate the switching ripple magnitude in the output capacitor voltage $v_2(t)$. According to Fig. 2.22(d), the current $i_{C2}(t)$ is continuous: unlike v_{L1}, v_{L2}, and i_{C1}, the capacitor current $i_{C2}(t)$ is nonpulsating. If the switching ripple of $i_2(t)$ is neglected, then the capacitor current $i_{C2}(t)$ does not contain an ac component. The small-ripple approximation then leads to the conclusion that the output switching ripple Δv_2 is zero.

Of course, the output voltage switching ripple is not zero. To estimate the magnitude of the output voltage ripple in this converter, we must not neglect the switching ripple present in the inductor current $i_2(t)$, since this current ripple is the only source of ac current driving the output capacitor C_2. A simple way of doing this in the Ćuk converter and in other similar converters is discussed in the next section.

2.5 ESTIMATING THE OUTPUT VOLTAGE RIPPLE IN CONVERTERS CONTAINING TWO-POLE LOW-PASS FILTERS

A case where the small ripple approximation is not useful is in converters containing two-pole low-pass filters, such as in the output of the Ćuk converter (Fig. 2.20) or the buck converter (Fig. 2.25). For these

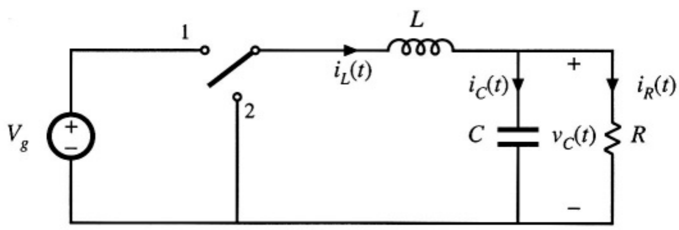


Fig. 2.25 The buck converter contains a two-pole output filter.

converters, the small-ripple approximation predicts zero output voltage ripple, regardless of the value of the output filter capacitance. The problem is that the only component of output capacitor current in these cases is that arising from the inductor current ripple. Hence, inductor current ripple cannot be neglected when calculating the output capacitor voltage ripple, and a more accurate approximation is needed.

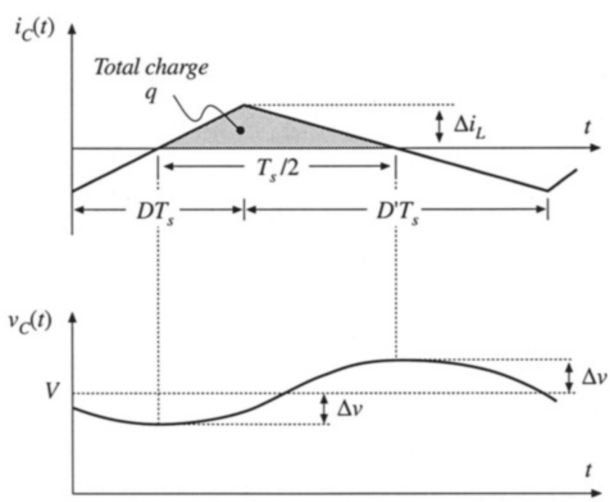
An improved approach that is useful for this case is to estimate the capacitor current waveform $i_C(t)$ more accurately, accounting for the inductor current ripple. The capacitor voltage ripple can then be related to the total charge contained in the positive portion of the $i_C(t)$ waveform.

Consider the buck converter of Fig. 2.25. The inductor current waveform $i_L(t)$ contains a dc component I and linear ripple of peak magnitude Δi_L, as shown in Fig. 2.10. The dc component I must flow entirely through the load resistance R (why?), while the ac switching ripple divides between the load resistance R and the filter capacitor C. In a well-designed converter, in which the capacitor provides significant filtering of the switching ripple, the capacitance C is chosen large enough that its impedance at the switching frequency is much smaller than the load impedance R. Hence nearly all of the inductor current ripple flows through the capacitor, and very little flows through the load. As shown in Fig. 2.26, the capacitor current waveform $i_C(t)$ is then equal to the inductor current waveform with the dc component removed. The current ripple is linear, with peak value Δi_L.

When the capacitor current $i_C(t)$ is positive, charge is deposited on the capacitor plates and the capacitor voltage $v_C(t)$ increases. Therefore, between the two zero-crossings of the capacitor current waveform, the capacitor voltage changes between its minimum and maximum extrema. The waveform is symmetrical, and the total change in v_C is the peak-to-peak output voltage ripple, or $2\Delta v$.

This change in capacitor voltage can be related to the total charge q contained in the positive

Fig. 2.26 Output capacitor voltage and current waveforms, for the buck converter in Fig. 2.25.



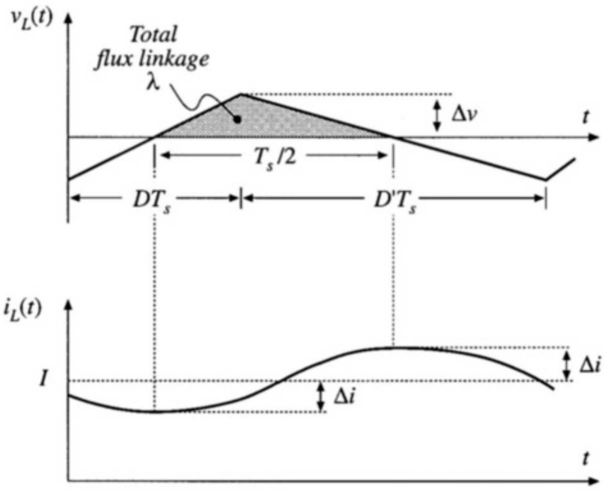


Fig. 2.27 Estimating inductor current ripple when the inductor voltage waveform is continuous.

portion of the capacitor current waveform. By the capacitor relation $Q = CV$,

$$q = C(2\Delta v) \tag{2.58}$$

As illustrated in Fig. 2.26, the charge q is the integral of the current waveform between its zero crossings. For this example, the integral can be expressed as the area of the shaded triangle, having a height Δi_L. Owing to the symmetry of the current waveform, the zero crossings occur at the centerpoints of the DT_s and $D'T_s$ subintervals. Hence, the base dimension of the triangle is $T_s/2$. So the total charge q is given by

$$q = \frac{1}{2} \Delta i_L \frac{T_s}{2} \tag{2.59}$$

Substitution of Eq. (2.58) into Eq. (2.59), and solution for the voltage ripple peak magnitude Δv yields

$$\Delta v = \frac{\Delta i_L T_s}{8C} \tag{2.60}$$

This expression can be used to select a value for the capacitance C such that a given voltage ripple Δv is obtained. In practice, the additional voltage ripple caused by the capacitor equivalent series resistance (esr) must also be included.

Similar arguments can be applied to inductors. An example is considered in Problem 2.9, in which a two-pole input filter is added to a buck converter as in Fig. 2.32. The capacitor voltage ripple cannot be neglected; doing so would lead to the conclusion that no ac voltage is applied across the input filter inductor, resulting in zero input current ripple. The actual inductor voltage waveform is identical to the ac portion of the input filter capacitor voltage, with linear ripple and with peak value, Δv as illustrated in Fig. 2.27. By use of the inductor relation $\lambda = Li$, a result similar to Eq. (2.60) can be derived. The derivation is left as a problem for the student.

2.6 SUMMARY OF KEY POINTS

1. The dc component of a converter waveform is given by its average value, or the integral over one switching period, divided by the switching period. Solution of a dc-dc converter to find its dc, or steady-state, voltages and currents therefore involves averaging the waveforms.

2. The linear- (or small-) ripple approximation greatly simplifies the analysis. In a well-designed converter, the switching ripples in the inductor currents and capacitor voltages are small compared to the respective dc components, and can be neglected.

3. The principle of inductor volt-second balance allows determination of the dc voltage components in any switching converter. In steady state, the average voltage applied to an inductor must be zero.

4. The principle of capacitor charge balance allows determination of the dc components of the inductor currents in a switching converter. In steady state, the average current applied to a capacitor must be zero.

5. By knowledge of the slopes of the inductor current and capacitor voltage waveforms, the ac switching ripple magnitudes may be computed. Inductance and capacitance values can then be chosen to obtain desired ripple magnitudes.

6. In converters containing multiple-pole filters, continuous (nonpulsating) voltages and currents are applied to one or more of the inductors or capacitors. Computation of the ac switching ripple in these elements can be done using capacitor charge and/or inductor flux-linkage arguments, without use of the small-ripple approximation.

7. Converters capable of increasing (boost), decreasing (buck), and inverting the voltage polarity (buck-boost and Ćuk) have been described. Converter circuits are explored more fully in the problems and in a later chapter.

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PROBLEMS

2.1 Analysis and design of a buck-boost converter: A buck-boost converter is illustrated in Fig. 2.28(a), and a practical implementation using a transistor and diode is shown in Fig. 2.28(b).

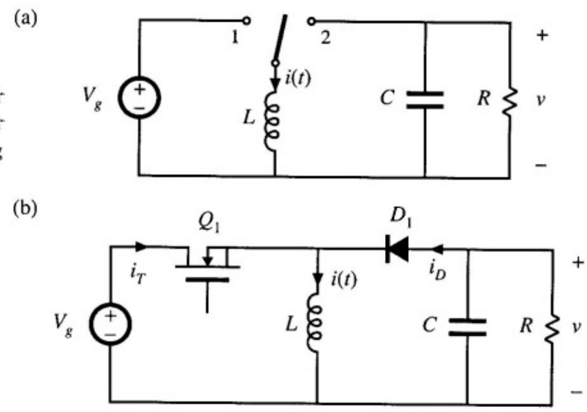

Fig. 2.28 Buck-boost converter of Problem 2.1: (a) ideal converter circuit, (b) implementation using MOSFET and diode.

(a) Find the dependence of the equilibrium output voltage V and inductor current I on the duty ratio D, input voltage V_g, and load resistance R. You may assume that the inductor current ripple and capacitor voltage ripple are small.

(b) Plot your results of part (a) over the range $0 \le D \le 1$.

(c) DC design: for the specifications

$$V_g = 30 \text{ V} \qquad V = -20 \text{ V}$$
$$R = 4\ \Omega \qquad f_s = 40 \text{ kHz}$$

(*i*) Find D and I

(*ii*) Calculate the value of L that will make the peak inductor current ripple Δi equal to ten percent of the average inductor current I.

(*iii*) Choose C such that the peak output voltage ripple Δv is 0.1 V.

(d) Sketch the transistor drain current waveform $i_T(t)$ for your design of part (c). Include the effects of inductor current ripple. What is the peak value of i_T? Also sketch $i_T(t)$ for the case when L is decreased such that Δi is 50% of I. What happens to the peak value of i_T in this case?

(e) Sketch the diode current waveform $i_D(t)$ for the two cases of part (d).

2.2 In a certain application, an unregulated dc input voltage can vary between 18 and 36 V. It is desired to produce a regulated output of 28 V to supply a 2 A load. Hence, a converter is needed that is capable of both increasing and decreasing the voltage. Since the input and output voltages are both positive, converters that invert the voltage polarity (such as the basic buck-boost converter) are not suited for this application.

One converter that is capable of performing the required function is the nonisolated SEPIC (single-ended primary inductance converter) shown in Fig. 2.29. This converter has a conversion ratio $M(D)$ that can both buck and boost the voltage, but the voltage polarity is not inverted. In the normal converter operating mode, the transistor conducts during the first subinterval $(0 < t < DT_s)$, and the diode conducts during the second subinterval $(DT_s < t < T_s)$. You may assume that all elements are ideal.

(a) Derive expressions for the dc components of each capacitor voltage and inductor current, as functions of the duty cycle D, the input voltage V_g, and the load resistance R.

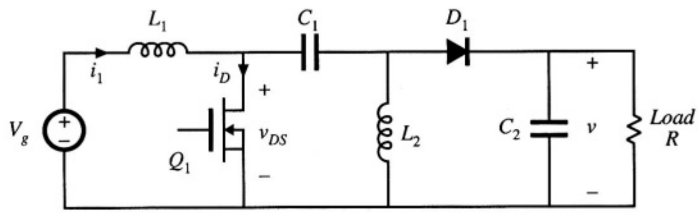

Fig. 2.29 SEPIC of Problems 2.2 and 2.3.

(b) A control circuit automatically adjusts the converter duty cycle D, to maintain a constant output voltage of $V = 28$ V. The input voltage slowly varies over the range $18\text{ V} \le V_g \le 36\text{ V}$. The load current is constant and equal to 2 A. Over what range will the duty cycle D vary? Over what range will the input inductor current dc component I_1 vary?

2.3 For the SEPIC of Problem 2.2,

(a) Derive expressions for each inductor current ripple and capacitor voltage ripple. Express these quantities as functions of the switching period T_s; the component values L_1, L_2, C_1, C_2; the duty cycle D; the input voltage V_g; and the load resistance R.

(b) Sketch the waveforms of the transistor voltage $v_{DS}(t)$ and transistor current $i_D(t)$, and give expressions for their peak values.

2.4 The switches in the converter of Fig. 2.30 operate synchronously: each is in position 1 for $0 < t < DT_s$, and in position 2 for $DT_s < t < T_s$. Derive an expression for the voltage conversion ratio $M(D) = V/V_g$ Sketch $M(D)$ vs. D.

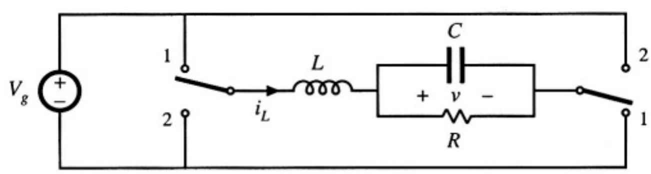

Fig. 2.30 H-bridge converter of Problems 2.4 and 2.6.

2.5 The switches in the converter of Fig. 2.31 operate synchronously: each is in position 1 for $0 < t < DT_s$, and in position 2 for $DT_s < t < T_s$. Derive an expression for the voltage conversion ratio $M(D) = V/V_g$. Sketch $M(D)$ vs. D.

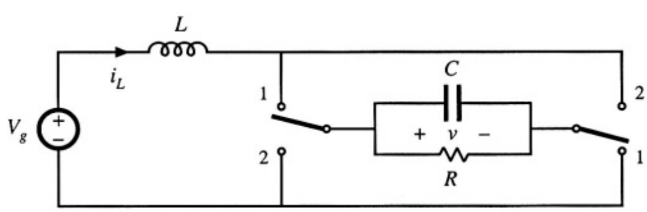

Fig. 2.31 Current-fed bridge converter of Problems 2.5, 2.7, and 2.8.

2.6 For the converter of Fig. 2.30, derive expressions for the inductor current ripple Δi_L and the capacitor voltage ripple Δv_C.

2.7 For the converter of Fig. 2.31, derive an analytical expression for the dc component of the inductor current, I, as a function of D, V_g, and R. Sketch your result vs. D.

2.8 For the converter of Fig. 2.31, derive expressions for the inductor current ripple Δi_L and the capacitor voltage ripple Δv_C.

2.9 To reduce the switching harmonics present in the input current of a certain buck converter, an input filter consisting of inductor L_1 and capacitor C_1 is added as shown in Fig. 2.32. Such filters are commonly used to meet regulations limiting conducted electromagnetic interference (EMI). For this problem, you may assume that all inductance and capacitance values are sufficiently large, such that all ripple magnitudes are small.

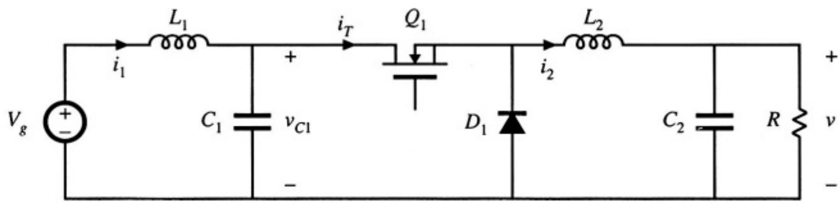

Fig. 2.32 Addition of L–C input filter to buck converter, Problem 2.9.

(a) Sketch the transistor current waveform $i_T(t)$

(b) Derive analytical expressions for the dc components of the capacitor voltages and inductor currents.

(c) Derive analytical expressions for the peak ripple magnitudes of the input filter inductor current and capacitor voltage.

(d) Given the following values:

Input voltage	$V_g = 48$ V
Output voltage	$V = 36$ V
Switching frequency	$f_s = 100$ kHz
Load resistance	$R = 6\ \Omega$

Select values for L_1 and C_1 such that (*i*) the peak voltage ripple on C_1, Δv_{C1}, is two percent of the dc component V_{C1}, and (*ii*) the input peak current ripple Δi_1 is 20 mA.

Extra credit problem: Derive exact analytical expressions for (*i*) the dc component of the output voltage, and (*ii*) the peak-to-peak inductor current ripple, of the ideal buck-boost converter operating in steady state. Do not make the small-ripple approximation.