3

Steady-State Equivalent Circuit Modeling, Losses, and Efficiency

Let us now consider the basic functions performed by a switching converter, and attempt to represent these functions by a simple equivalent circuit. The designer of a converter power stage must calculate the network voltages and currents, and specify the power components accordingly. Losses and efficiency are of prime importance. The use of equivalent circuits is a physical and intuitive approach which allows the well-known techniques of circuit analysis to be employed. As noted in the previous chapter, it is desirable to ignore the small but complicated switching ripple, and model only the important dc components of the waveforms.

The dc transformer is used to model the ideal functions performed by a dc-dc converter [1–4]. This simple model correctly represents the relationships between the dc voltages and currents of the converter. The model can be refined by including losses, such as semiconductor forward voltage drops and on-resistances, inductor core and copper losses, etc. The resulting model can be directly solved, to find the voltages, currents, losses, and efficiency in the actual nonideal converter.

3.1 THE DC TRANSFORMER MODEL

As illustrated in Fig. 3.1, any switching converter contains three ports: a power input, a power output, and a control input. The input power is processed as specified by the control input, and then is output to the load. Ideally, these functions are performed with 100% efficiency, and hence

$$P_{in} = P_{out} \tag{3.1}$$

or,

$$V_g I_g = VI \tag{3.2}$$

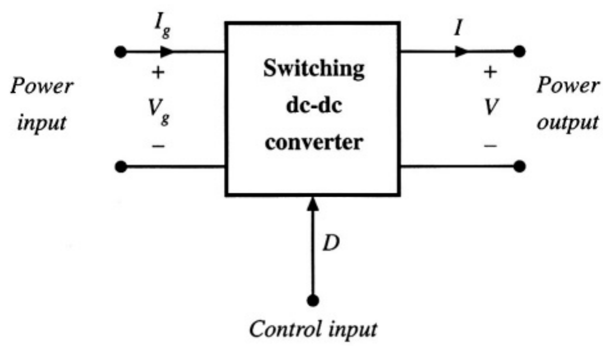


Fig. 3.1 Switching converter terminal quantities.

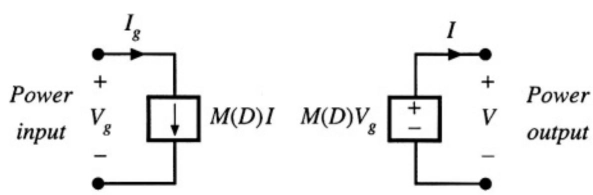


Fig. 3.2 A switching converter equivalent circuit using dependent sources, corresponding to Eqs. (3.3) and (3.4).

These relationships are valid only under equilibrium (dc) conditions: during transients, the net stored energy in the converter inductors and capacitors may change, causing Eqs. (3.1) and (3.2) to be violated.

In the previous chapter, we found that we could express the converter output voltage in an equation of the form

$$V = M(D)V_g \tag{3.3}$$

where $M(D)$ is the equilibrium conversion ratio of the converter. For example, $M(D) = D$ for the buck converter, and $M(D) = 1/(1 - D)$ for the boost converter. In general, for ideal PWM converters operating in the continuous conduction mode and containing an equal number of independent inductors and capacitors, it can be shown that the equilibrium conversion ratio M is a function of the duty cycle D and is independent of load.

Substitution of Eq. (3.3) into Eq. (3.2) yields

$$I_g = M(D)I \tag{3.4}$$

Hence, the converter terminal currents are related by the same conversion ratio.

Equations (3.3) and (3.4) suggest that the converter could be modeled using dependent sources, as in Fig. 3.2. An equivalent but more physically meaningful model (Fig. 3.3) can be obtained through the realization that Eqs. (3.1) to (3.4) coincide with the equations of an ideal transformer. In an ideal transformer, the input and output powers are equal, as stated in Eqs. (3.1) and (3.2). Also, the output voltage is equal to the turns ratio times the input voltage. This is consistent with Eq. (3.3), with the turns ratio taken to be the equilibrium conversion ratio $M(D)$. Finally, the input and output currents should be related by the same turns ratio, as in Eq. (3.4).

Thus, we can model the ideal dc-dc converter using the ideal dc transformer model of Fig. 3.3.

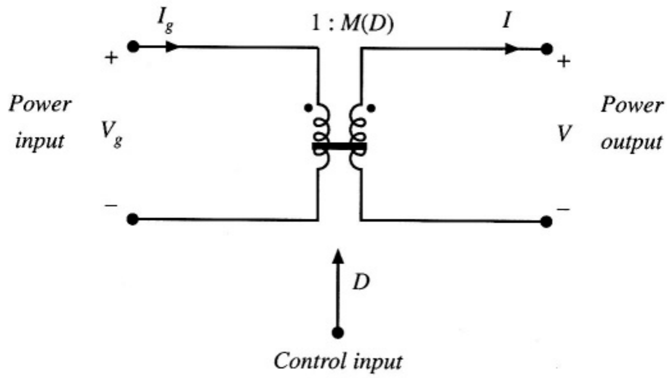


Fig. 3.3 Ideal dc transformer model of a dc-dc converter operating in continuous conduction mode, corresponding to Eqs. (3.1) to (3.4).

This symbol represents the first-order dc properties of any switching dc-dc converter: transformation of dc voltage and current levels, ideally with 100% efficiency, controllable by the duty cycle D. The solid horizontal line indicates that the element is ideal and capable of passing dc voltages and currents. It should be noted that, although standard magnetic-core transformers cannot transform dc signals (they saturate when a dc voltage is applied), we are nonetheless free to define the idealized model of Fig. 3.3 for the purpose of modeling dc-dc converters. Indeed, the absence of a physical dc transformer is one of the reasons for building a dc-dc switching converter. So the properties of the dc-dc converter of Fig. 3.1 can be modeled using the equivalent circuit of Fig. 3.3. An advantage of this equivalent circuit is that, for constant duty cycle, it is time invariant: there is no switching or switching ripple to deal with, and only the important dc components of the waveforms are modeled.

The rules for manipulating and simplifying circuits containing transformers apply equally well to circuits containing dc-dc converters. For example, consider the network of Fig. 3.4(a), in which a resistive load is connected to the converter output, and the power source is modeled by a Thevenin-equivalent voltage source V_1 and resistance R_1. The converter is replaced by the dc transformer model in Fig. 3.4(b). The elements V_1 and R_1 can now be pushed through the dc transformer as in Fig. 3.4(c); the volt-

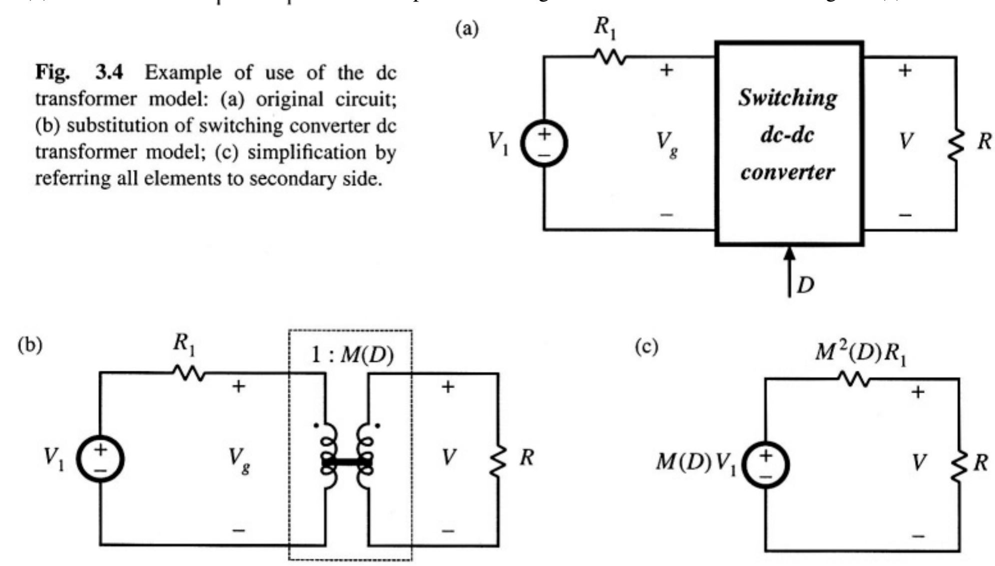


Fig. 3.4 Example of use of the dc transformer model: (a) original circuit; (b) substitution of switching converter dc transformer model; (c) simplification by referring all elements to secondary side.

age source V_1 is multiplied by the conversion ratio $M(D)$, and the resistor R_1 is multiplied by $M^2(D)$. This circuit can now be solved using the voltage divider formula to find the output voltage:

$$V = M(D)V_1 \frac{R}{R + M^2(D)R_1} \tag{3.5}$$

It should be apparent that the dc transformer/equivalent circuit approach is a powerful tool for understanding networks containing converters.

3.2 INCLUSION OF INDUCTOR COPPER LOSS

The dc transformer model of Fig. 3.3 can be extended, to model other properties of the converter. Nonidealities, such as sources of power loss, can be modeled by adding resistors as appropriate. In later chapters, we will see that converter dynamics can be modeled as well, by adding inductors and capacitors to the equivalent circuit.

Let us consider the inductor copper loss in a boost converter. Practical inductors exhibit power loss of two types: (1) *copper loss,* originating in the resistance of the wire, and (2) *core loss,* due to hysteresis and eddy current losses in the magnetic core. A suitable model that describes the inductor copper loss is given in Fig. 3.5, in which a resistor R_L is placed in series with the inductor. The actual inductor then consists of an ideal inductor, L, in series with the copper loss resistor R_L.

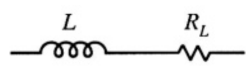


Fig. 3.5 Modeling inductor copper loss via series resistor R_L.

The inductor model of Fig. 3.5 is inserted into the boost converter circuit in Fig. 3.6. The circuit can now be analyzed in the same manner as used for the ideal lossless converter, using the principles of inductor volt-second balance, capacitor charge balance, and the small-ripple approximation. First, we draw the converter circuits during the two subintervals, as in Fig. 3.7.

For $0 < t < DT_s$, the switch is in position 1 and the circuit reduces to Fig. 3.7(a). The inductor voltage $v_L(t)$, across the ideal inductor L, is given by

$$v_L(t) = V_g - i(t)R_L \tag{3.6}$$

and the capacitor current $i_C(t)$ is

$$i_C(t) = -\frac{v(t)}{R} \tag{3.7}$$

Next, we simplify these equations by assuming that the switching ripples in $i(t)$ and $v(t)$ are small compared to their respective dc components I and V. Hence, $i(t) \approx I$ and $v(t) \approx V$, and Eqs. (3.6) and (3.7)

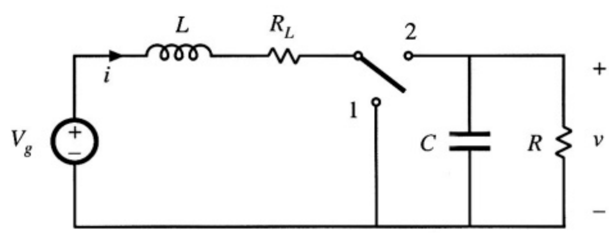


Fig. 3.6 Boost converter circuit, including inductor copper resistance R_L.

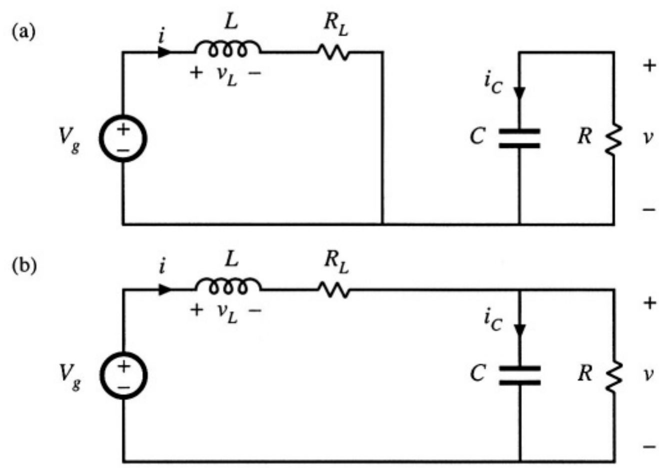


Fig. 3.7 Boost converter circuits during the two subintervals, including inductor copper loss resistance R_L: (a) with the switch in position 1, (b) with the switch in position 2.

become

$$\begin{aligned} v_L(t) &= V_g - IR_L \\ i_C(t) &= -\frac{V}{R} \end{aligned} \tag{3.8}$$

For $DT_s < t < T_s$, the switch is in position 2 and the circuit reduces to Fig. 3.7(b). The inductor current and capacitor voltage are then given by

$$\begin{aligned} v_L(t) &= V_g - i(t)R_L - v(t) \approx V_g - IR_L - V \\ i_C(t) &= i(t) - \frac{v(t)}{R} \approx I - \frac{V}{R} \end{aligned} \tag{3.9}$$

We again make the small-ripple approximation.

The principle of inductor volt-second balance can now be invoked. Equations (3.8) and (3.9) are used to construct the inductor voltage waveform $v_L(t)$ in Fig. 3.8. The dc component, or average value, of the inductor voltage $v_L(t)$ is

$$\left\langle v_L(t) \right\rangle = \frac{1}{T_s}\int_0^{T_s} v_L(t)dt = D\left(V_g - IR_L\right) + D'\left(V_g - IR_L - V\right) \tag{3.10}$$

By setting $\langle v_L \rangle$ to zero and collecting terms, one obtains

$$0 = V_g - IR_L - D'V \tag{3.11}$$

(recall that $D + D' = 1$). It can be seen that the inductor winding resistance R_L adds another term to the inductor volt-second balance equation. In the ideal boost converter ($R_L = 0$) example of Chapter 2, we were able to solve this equation directly for the voltage conversion ratio V/V_g. Equation (3.11) cannot be immediately solved in this manner, because the inductor current I is unknown. A second equation is needed, to eliminate I.

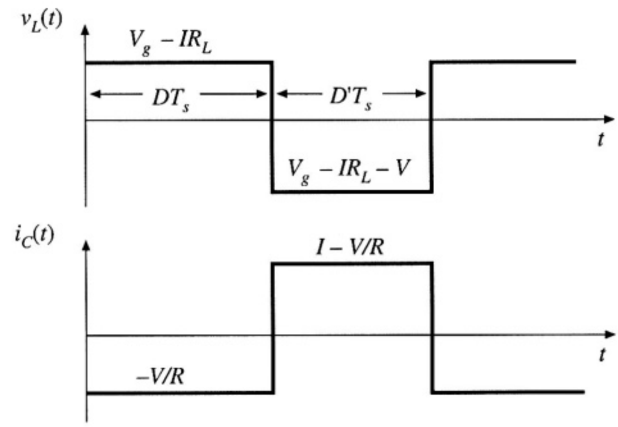


Fig. 3.8 Inductor voltage and capacitor current waveforms, for the nonideal boost converter of Fig. 3.6.

The second equation is obtained using capacitor charge balance. The capacitor current $i_C(t)$ waveform is given in Fig. 3.8. The dc component, or average value, of the capacitor current waveform is

$$\langle i_C(t) \rangle = D\left(-\frac{V}{R}\right) + D'\left(I - \frac{V}{R}\right) \tag{3.12}$$

By setting $\langle i_C \rangle$ to zero and collecting terms, one obtains

$$0 = D'I - \frac{V}{R} \tag{3.13}$$

We now have two equations, Eqs. (3.11) and (3.13), and two unknowns, V and I. Elimination of I and solution for V yields

$$\frac{V}{V_g} = \frac{1}{D'} \frac{1}{\left(1 + \frac{R_L}{D'^2 R}\right)} \tag{3.14}$$

This is the desired solution for the converter output voltage V. It is plotted in Fig. 3.9 for several values of R_L/R. It can be seen that Eq. (3.14) contains two terms. The first, $1/D'$, is the ideal conversion ratio, with $R_L = 0$. The second term, $1/(1 + R_L/D'^2R)$, describes the effect of the inductor winding resistance. If R_L is much less than D'^2R, then the second term is approximately equal to unity and the conversion ratio is approximately equal to the ideal value $1/D'$. However, as R_L is increased in relation to D'^2R, then the second term is reduced in value, and V/V_g is reduced as well.

As the duty cycle D approaches one, the inductor winding resistance R_L causes a major qualitative change in the V/V_g curve. Rather than approaching infinity at $D = 1$, the curve tends to zero. Of course, it is unreasonable to expect that the converter can produce infinite voltage, and it should be comforting to the engineer that the prediction of the model is now more realistic. What happens at $D = 1$ is that the switch is always in position 1. The inductor is never connected to the output, so no energy is transferred to the output and the output voltage tends to zero. The inductor current tends to a large value, limited only by the inductor resistance R_L. A large amount of power is lost in the inductor winding resistance, equal to V_g^2/R_L, while no power is delivered to the load; hence, we can expect that the converter efficiency tends to zero at $D = 1$.

Another implication of Fig. 3.9 is that the inductor winding resistance R_L limits the maximum

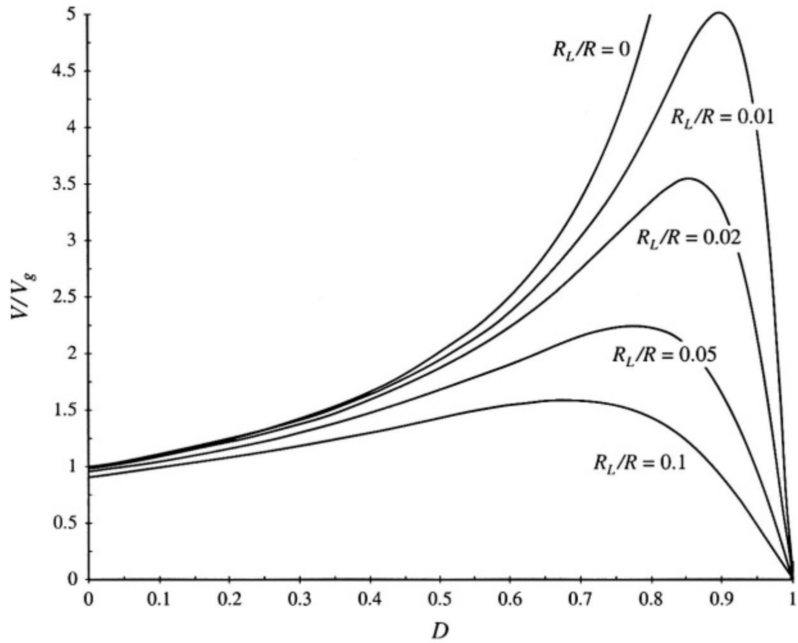


Fig. 3.9 Output voltage vs. duty cycle, boost converter with inductor copper loss.

voltage that the converter can produce. For example, with $R_L/R = 0.02$, it can be seen that the maximum V/V_g is approximately 3.5. If it is desired to obtain $V/V_g = 5$, then according to Fig. 3.9 the inductor winding resistance R_L must be reduced to less than 1% of the load resistance R. The only problem is that decreasing the inductor winding resistance requires building a larger, heavier, more expensive inductor. So it is usually important to optimize the design, by correctly modeling the effects of loss elements such as R_L, and choosing the smallest inductor that will do the job. We now have the analytical tools needed to do this.

3.3 CONSTRUCTION OF EQUIVALENT CIRCUIT MODEL

Next, let us refine the dc transformer model, to account for converter losses. This will allow us to determine the converter voltages, currents, and efficiency using well-known techniques of circuit analysis.

In the previous section, we used the principles of inductor volt-second balance and capacitor charge balance to write Eqs. (3.11) and (3.13), repeated here:

$$\begin{aligned} \langle v_L \rangle &= 0 = V_g - IR_L - D'V \\ \langle i_C \rangle &= 0 = D'I - \frac{V}{R} \end{aligned} \tag{3.15}$$

These equations state that the dc components of the inductor voltage and capacitor current are equal to zero. Rather than algebraically solving the equations as in the previous section, we can reconstruct a circuit model based on these equations, which describes the dc behavior of the boost converter with inductor copper loss. This is done by constructing a circuit whose Kirchoff loop and node equations are

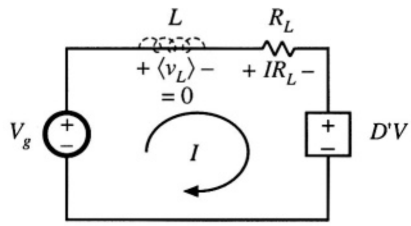


Fig. 3.10 Circuit whose loop equation is identical to Eq. (3.16), obtained by equating the average inductor voltage $\langle v_L \rangle$ to zero.

identical to Eqs. (3.15).

3.3.1 Inductor Voltage Equation

$$\langle v_L \rangle = 0 = V_g - IR_L - D'V \tag{3.16}$$

This equation was derived by use of Kirchoff's voltage law to find the inductor voltage during each subinterval. The results were averaged and set to zero. Equation (3.16) states that the sum of three terms having the dimensions of voltage are equal to $\langle v_L \rangle$, or zero. Hence, Eq. (3.16) is of the same form as a loop equation; in particular, it describes the dc components of the voltages around a loop containing the inductor, with loop current equal to the dc inductor current I.

So let us construct a circuit containing a loop with current I, corresponding to Eq. (3.16). The first term in Eq. (3.16) is the dc input voltage V_g, so we should include a voltage source of value V_g as shown in Fig. 3.10. The second term is a voltage drop of value IR_L, which is proportional to the current I in the loop. This term corresponds to a resistance of value R_L. The third term is a voltage $D'V$, dependent on the converter output voltage. For now, we can model this term using a dependent voltage source, with polarity chosen to satisfy Eq. (3.16).

3.3.2 Capacitor Current Equation

$$\langle i_C \rangle = 0 = D'I - \frac{V}{R} \tag{3.17}$$

This equation was derived using Kirchoff's current law to find the capacitor current during each subinterval. The results were averaged, and the average capacitor current was set to zero.

Equation (3.17) states that the sum of two dc currents are equal to $\langle i_C \rangle$, or zero. Hence, Eq. (3.17) is of the same form as a node equation; in particular, it describes the dc components of currents

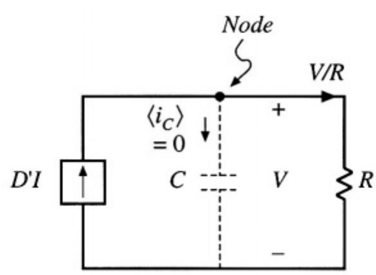


Fig. 3.11 Circuit whose node equation is identical to Eq. (3.17), obtained by equating the average capacitor current $\langle i_C \rangle$ to zero.

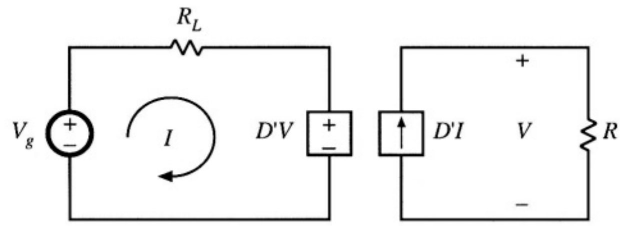


Fig. 3.12 The circuits of Figs. 3.10 and 3.11, drawn together.

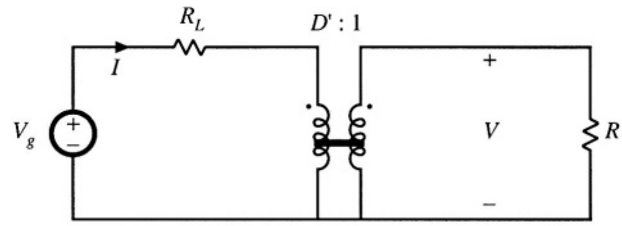


Fig. 3.13 Equivalent circuit model of the boost converter, including a D':1 dc transformer and the inductor winding resistance R_L.

flowing into a node connected to the capacitor. The dc capacitor voltage is V.

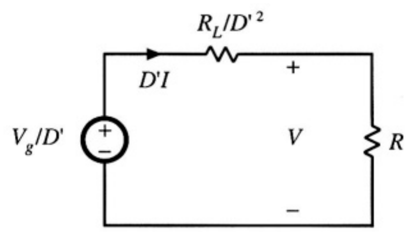
So now let us construct a circuit containing a node connected to the capacitor, as in Fig. 3.11, whose node equation satisfies Eq. (3.17). The second term in Eq. (3.17) is a current of magnitude V/R, proportional to the dc capacitor voltage V. This term corresponds to a resistor of value R, connected in parallel with the capacitor so that its voltage is V and hence its current is V/R. The first term is a current $D'I$, dependent on the dc inductor current I. For now, we can model this term using a dependent current source as shown. The polarity of the source is chosen to satisfy Eq. (3.17).

3.3.3 Complete Circuit Model

The next step is to combine the circuits of Figs. 3.10 and 3.11 into a single circuit, as in Fig. 3.12. This circuit can be further simplified by recognizing that the dependent voltage and current sources constitute an ideal dc transformer, as discussed in Section 3.1. The $D'V$ dependent voltage source depends on V, the voltage across the dependent current source. Likewise, the $D'I$ dependent current source depends on I, the current flowing through the dependent voltage source. In each case, the coefficient is D'. Hence, the dependent sources form a circuit similar to Fig. 3.2; the fact that the voltage source appears on the primary rather than the secondary side is irrelevant, owing to the symmetry of the transformer. They are therefore equivalent to the dc transformer model of Fig. 3.3, with turns ratio D':1. Substitution of the ideal dc transformer model for the dependent sources yields the equivalent circuit of Fig. 3.13.

The equivalent circuit model can now be manipulated and solved to find the converter voltages and currents. For example, we can eliminate the transformer by referring the V_g voltage source and R_L resistance to the secondary side. As shown in Fig. 3.14, the voltage source value is divided by the effective turns ratio D', and the resistance R_L is divided by the square of the turns ratio, D'^2. This circuit can be solved directly for the output voltage V, using the voltage divider formula:

Fig. 3.14 Simplification of the equivalent circuit of Fig. 3.13, by referring all elements to the secondary side of the transformer.



$$V = \frac{V_g}{D'} \frac{R}{R + \frac{R_L}{D'^2}} = \frac{V_g}{D'} \frac{1}{1 + \frac{R_L}{D'^2 R}} \tag{3.18}$$

This result is identical to Eq. (3.14). The circuit can also be solved directly for the inductor current I, by referring all elements to the transformer primary side. The result is:

$$I = \frac{V_g}{D'^2 R + R_L} = \frac{V_g}{D'^2 R} \frac{1}{1 + \frac{R_L}{D'^2 R}} \tag{3.19}$$

3.3.4 Efficiency

The equivalent circuit model also allows us to compute the converter efficiency η. Figure 3.13 predicts that the converter input power is

$$P_{in} = (V_g)(I) \tag{3.20}$$

The load current is equal to the current in the secondary of the ideal dc transformer, or $D'I$. Hence, the model predicts that the converter output power is

$$P_{out} = (V)(D'I) \tag{3.21}$$

Therefore, the converter efficiency is

$$\eta = \frac{P_{out}}{P_{in}} = \frac{(V)(D'I)}{(V_g)(I)} = \frac{V}{V_g} D' \tag{3.22}$$

Substitution of Eq. (3.18) into Eq. (3.22) to eliminate V yields

$$\eta = \frac{1}{1 + \frac{R_L}{D'^2 R}} \tag{3.23}$$

This equation is plotted in Fig. 3.15, for several values of R_L/R. It can be seen from Eq. (3.23) that, to obtain high efficiency, the inductor winding resistance R_L should be much smaller that D'^2R, the load resistance referred to the primary side of the ideal dc transformer. This is easier to do at low duty cycle, where D' is close to unity, than at high duty cycle where D' approaches zero. It can be seen from Fig.

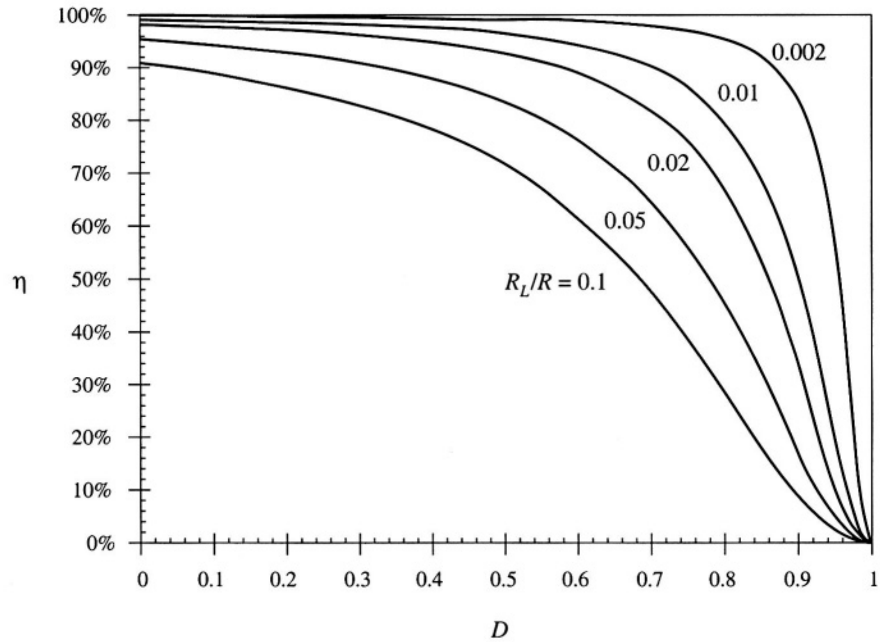


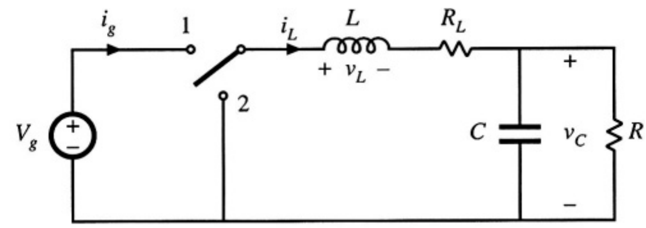
Fig. 3.15 Efficiency vs. duty cycle, boost converter with inductor copper loss.

3.15 that the efficiency is typically high at low duty cycles, but decreases rapidly to zero near $D = 1$.

Thus, the basic dc transformer model can be refined to include other effects, such as the inductor copper loss. The model describes the basic properties of the converter, including (a) transformation of dc voltage and current levels, (b) second-order effects such as power losses, and (c) the conversion ratio M. The model can be solved to find not only the output voltage V, but also the inductor current I and the efficiency η. All of the well-known techniques of circuit analysis can be employed to solve the model, making this a powerful and versatile approach.

The example considered so far is a relatively simple one, in which there is only a single loss element, R_L Of course, real converters are considerably more complicated, and contain a large number of loss elements. When solving a complicated circuit to find the output voltage and efficiency, it behooves the engineer to use the simplest and most physically meaningful method possible. Writing a large number of simultaneous loop or node equations is not the best approach, because its solution typically requires several pages of algebra, and the engineer usually makes algebra mistakes along the way. The practicing engineer often gives up before finding the correct solution. The equivalent circuit approach avoids this situation, because one can simplify the circuit via well-known circuit manipulations such as pushing the circuit elements to the secondary side of the transformer. Often the answer can then be written by inspection, using the voltage divider rule or other formulas. The engineer develops confidence that the result is correct, and does not contain algebra mistakes.

Fig. 3.16 Buck converter example.



3.4 HOW TO OBTAIN THE INPUT PORT OF THE MODEL

Let's try to derive the model of the buck converter of Fig. 3.16, using the procedure of Section 3.3. The inductor winding resistance is again modeled by a series resistor R_L.

The average inductor voltage can be shown to be

$$\langle v_L \rangle = 0 = DV_g - I_L R_L - V_C \tag{3.24}$$

This equation describes a loop with the dc inductor current I_L. The dc components of the voltages around this loop are: (*i*) the DV_g term, modeled as a dependent voltage source, (*ii*) a voltage drop $I_L R_L$, modeled as resistor R_L, and (*iii*) the dc output voltage V_C.

The average capacitor current is

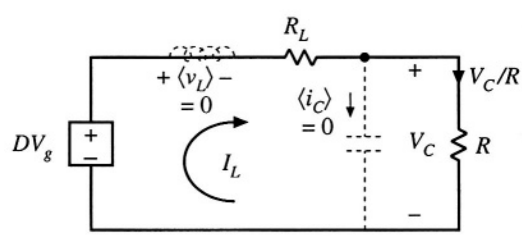
$$\langle i_C \rangle = 0 = I_L - \frac{V_C}{R} \tag{3.25}$$

This equation describes the dc currents flowing into the node connected to the capacitor. The dc component of inductor current, I_L, flows into this node. The dc load current V_C/R (i.e., the current flowing through the load resistor R) flows out of this node. An equivalent circuit that models Eqs. (3.24) and (3.25) is given in Fig. 3.17. This circuit can be solved to determine the dc output voltage V_C.

What happened to the dc transformer in Fig. 3.17? We expect the buck converter model to contain a dc transformer, with turns ratio equal to the dc conversion ratio, or 1:D. According to Fig. 3.2, the secondary of this transformer is equivalent to a dependent voltage source, of value DV_g. Such a source does indeed appear in Fig. 3.17. But where is the primary? From Fig. 3.2, we expect the primary of the dc transformer to be equivalent to a dependent current source. In general, to derive this source, it is necessary to find the dc component of the converter input current $i_g(t)$.

The converter input current waveform $i_g(t)$ is sketched in Fig. 3.18. When the switch is in position 1, $i_g(t)$ is equal to the inductor current. Neglecting the inductor current ripple, we have $i_g(t) \approx I_L$. When the switch is in position 2, $i_g(t)$ is zero. The dc component, or average value, of $i_g(t)$ is

Fig. 3.17 Equivalent circuit derived from Eqs. (3.24) and (3.25).



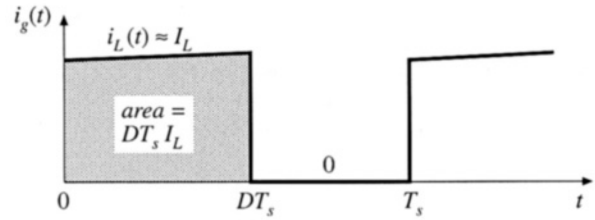


Fig. 3.18 Converter input current waveform $i_g(t)$.

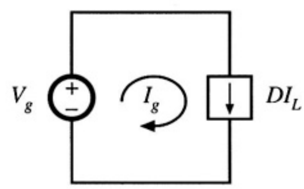


Fig. 3.19 Converter input port dc equivalent circuit.

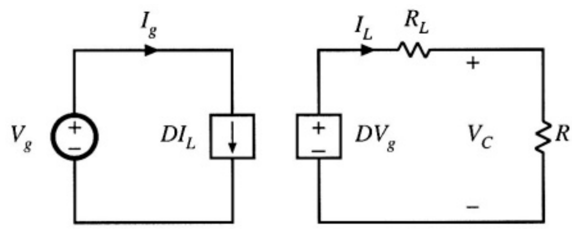


Fig. 3.20 The circuits of Figs. 3.17 and 3.19, drawn together.

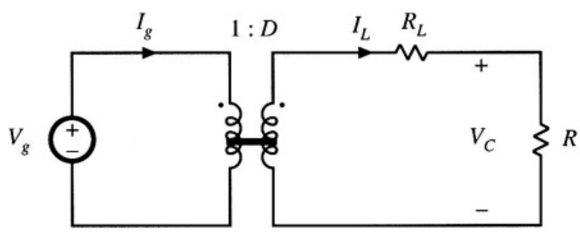


Fig. 3.21 Equivalent circuit of the buck converter, including a 1:D dc transformer and the inductor winding resistance R_L.

$$I_g = \frac{1}{T_s}\int_0^{T_s} i_g(t)dt = DI_L \tag{3.26}$$

The integral of $i_g(t)$ is equal to the area under the $i_g(t)$ curve, or DT_sI_L according to Fig. 3.18. The dc component I_g is therefore $(DT_sI_L)/T_s = DI_L$. Equation (3.26) states that I_g, the dc component of current drawn by the converter out of the V_g source, is equal to DI_L. An equivalent circuit is given in Fig. 3.19.

A complete model for the buck converter can now be obtained by combining Figs. 3.17 and 3.19 to obtain Fig. 3.20. The dependent current and voltage sources can be combined into a dc transformer, since the DV_g dependent voltage source has value D times the voltage V_g across the dependent current source, and the current source is the same constant D times the current I_L through the dependent voltage source. So, according to Fig. 3.2, the sources are equivalent to a dc transformer with turns ratio 1:D, as shown in Fig. 3.21.

In general, to obtain a complete dc equivalent circuit that models the converter input port, it is necessary to write an equation for the dc component of the converter input current. An equivalent circuit corresponding to this equation is then constructed. In the case of the buck converter, as well as in other converters having pulsating input currents, this equivalent circuit contains a dependent current source which becomes the primary of a dc transformer model. In the boost converter example of Section 3.3, it was unnecessary to explicitly write this equation, because the input current $i_g(t)$ coincided with the inductor current $i(t)$, and hence a complete equivalent circuit could be derived using only the inductor voltage and capacitor current equations.

3.5 EXAMPLE: INCLUSION OF SEMICONDUCTOR CONDUCTION LOSSES IN THE BOOST CONVERTER MODEL

As a final example, let us consider modeling semiconductor conduction losses in the boost converter of Fig. 3.22. Another major source of power loss is the conduction loss due to semiconductor device forward voltage drops. The forward voltage of a metal oxide semiconductor field-effect transistor (MOSFET) or bipolar junction transistor (BJT) can be modeled with reasonable accuracy as an on-resistance R_{on}. In the case of a diode, insulated-gate bipolar transistor (IGBT), or thyristor, a voltage source plus an on-resistance yields a model of good accuracy; the on-resistance may be omitted if the converter is being modeled at a single operating point.

When the gate drive signal is high, the MOSFET turns on and the diode is reverse-biased. The circuit then reduces to Fig. 3.23(a). In the conducting state, the MOSFET is modeled by the on-resistance R_{on}. The inductor winding resistance is again represented as in Fig. 3.5. The inductor voltage and capacitor current are given by

$$\begin{aligned} v_L(t) &= V_g - iR_L - iR_{on} \approx V_g - IR_L - IR_{on} \\ i_C(t) &= -\frac{v}{R} \approx -\frac{V}{R} \end{aligned} \tag{3.27}$$

The inductor current and capacitor voltage have again been approximated by their dc components.

When the gate drive signal is low, the MOSFET turns off. The diode becomes forward-biased by the inductor current, and the circuit reduces to Fig. 3.23(b). In the conducting state, the diode is modeled in this example by voltage source V_D and resistance R_D. The inductor winding resistance is again modeled by resistance R_L. The inductor voltage and capacitor current for this subinterval are

$$\begin{aligned} v_L(t) &= V_g - iR_L - V_D - iR_D - v \approx V_g - IR_L - V_D - IR_D - V \\ i_C(t) &= i - \frac{v}{R} \approx I - \frac{V}{R} \end{aligned} \tag{3.28}$$

The inductor voltage and capacitor current waveforms are sketched in Fig. 3.24.

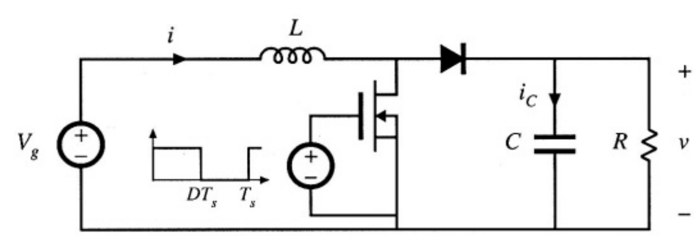


Fig. 3.22 Boost converter example

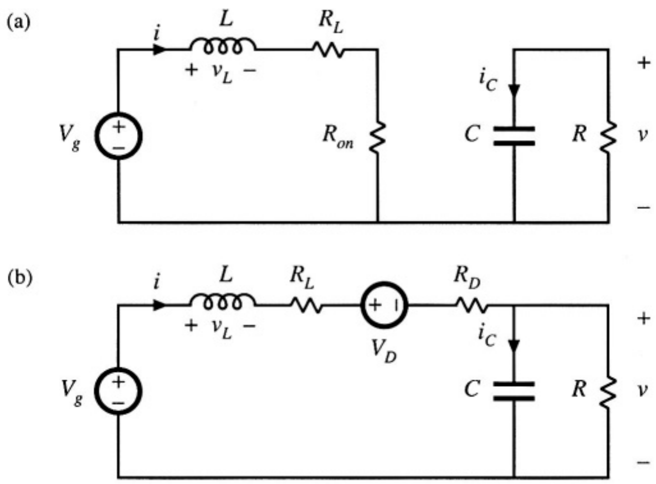


Fig. 3.23 Boost converter circuits: (a) when MOSFET conducts, (b) when diode conducts.

The dc component of the inductor voltage is given by

$$\langle v_L \rangle = D\left(V_g - IR_L - IR_{on}\right) + D'\left(V_g - IR_L - V_D - IR_D - V\right) = 0 \tag{3.29}$$

By collecting terms and noting that $D + D' = 1$, one obtains

$$V_g - IR_L - IDR_{on} - D'V_D - ID'R_D - D'V = 0 \tag{3.30}$$

This equation describes the dc components of the voltages around a loop containing the inductor, with loop current equal to the dc inductor current I. An equivalent circuit is given in Fig. 3.25.

Fig. 3.24 Inductor voltage $v_L(t)$ and capacitor current $i_C(t)$ waveforms, for the converter of Fig. 3.22.

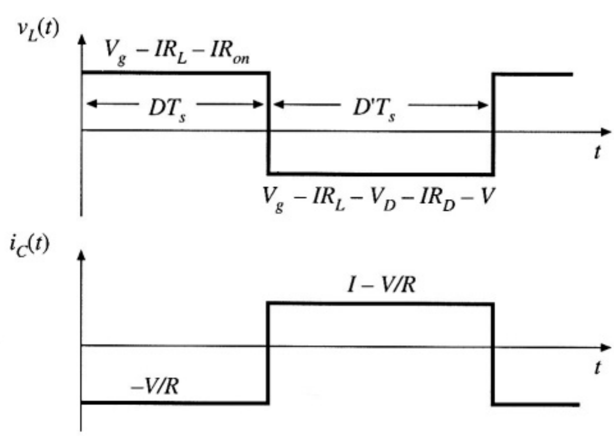


Fig. 3.25 Equivalent circuit corresponding to Eq. (3.30).

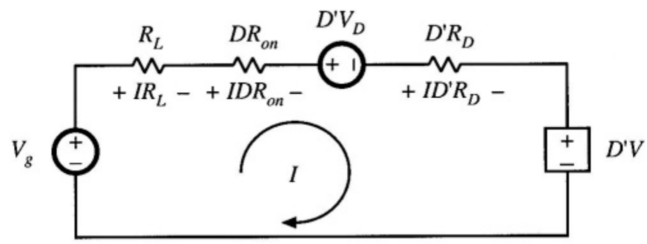
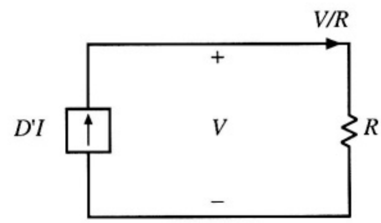


Fig. 3.26 Equivalent circuit corresponding to Eq. (3.32).



The dc component of the capacitor current is

$$\langle i_C \rangle = D\left(-\frac{V}{R}\right) + D'\left(I - \frac{V}{R}\right) = 0 \tag{3.31}$$

Upon collecting terms, one obtains

$$D'I - \frac{V}{R} = 0 \tag{3.32}$$

This equation describes the dc components of the currents flowing into a node connected to the capacitor, with dc capacitor voltage equal to V. An equivalent circuit is given in Fig. 3.26.

The two circuits are drawn together in 3.27. The dependent sources are combined into an ideal $D':1$ transformer in Fig. 3.28, yielding the complete dc equivalent circuit model.

Solution of Fig. 3.28 for the output voltage V yields

$$V = \left(\frac{1}{D'}\right)\left(V_g - D'V_D\right)\left(\frac{D'^2R}{D'^2R + R_L + DR_{on} + D'R_D}\right) \tag{3.33}$$

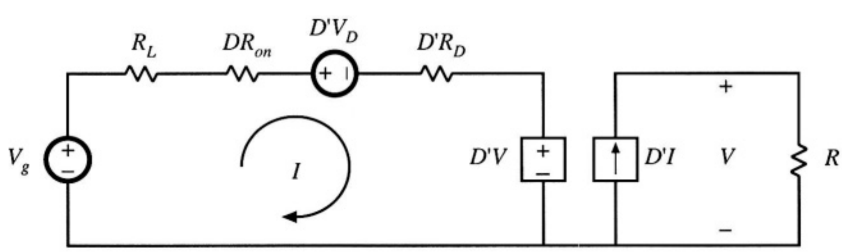


Fig. 3.27 The circuits of Figs. 3.25 and 3.26, drawn together.

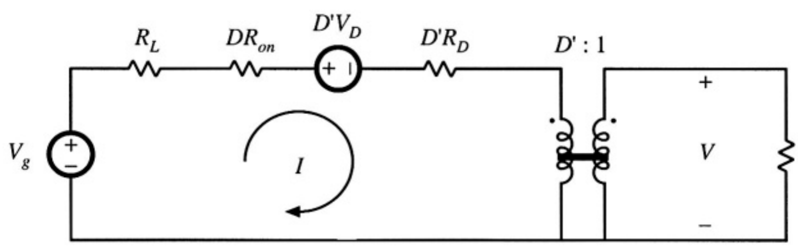


Fig. 3.28 Equivalent circuit model of the boost converter of Fig. 3.22, including ideal dc transformer, inductor winding resistance, and MOSFET and diode conduction losses.

Dividing by V_g gives the voltage conversion ratio:

$$\frac{V}{V_g} = \left(\frac{1}{D'}\right)\left(1 - \frac{D'V_D}{V_g}\right)\left(\frac{1}{1 + \dfrac{R_L + DR_{on} + D'R_D}{D'^2 R}}\right) \tag{3.34}$$

It can be seen that the effect of the loss elements V_D, R_D R_{on}, and R_D is to decrease the voltage conversion ratio below the ideal value $(1/D')$.

The efficiency is given by $\eta = P_{out}/P_{in}$. From Fig. 3.28, $P_{in} = V_g I$ and $P_{out} = VD'I$. Hence,

$$\eta = D'\frac{V}{V_g} = \frac{\left(1 - \dfrac{D'V_D}{V_g}\right)}{\left(1 + \dfrac{R_L + DR_{on} + D'R_D}{D'^2 R}\right)} \tag{3.35}$$

For high efficiency, we require

$$\begin{aligned} V_g/D' &\gg V_D \\ D'^2 R &\gg R_L + DR_{on} + D'R_D \end{aligned} \tag{3.36}$$

It may seem strange that the equivalent circuit model of Fig. 3.28 contains effective resistances DR_{on} and $D'R_D$, whose values vary with duty cycle. The reason for this dependence is that the semiconductor on-resistances are connected in the circuit only when their respective semiconductor devices conduct. For example, at $D = 0$, the MOSFET never conducts, and the effective resistance DR_{on} disappears from the model. These effective resistances correctly model the average power losses in the elements. For instance, the equivalent circuit predicts that the power loss in the MOSFET on-resistance is I^2DR_{on}. In the actual circuit, the MOSFET conduction loss is I^2R_{on} while the MOSFET conducts, and zero while the MOSFET is off. Since the MOSFET conducts with duty cycle D, the average conduction loss is DI^2R_{on}, which coincides with the prediction of the model.

In general, to predict the power loss in a resistor R, we must calculate the root-mean-square current I_{rms} through the resistor, rather than the average current. The average power loss is then given by I_{rms}^2R. Nonetheless, the average model of Fig. 3.28 correctly predicts average power loss, provided that the inductor current ripple is small. For example, consider the MOSFET conduction loss in the buck converter. The actual transistor current waveform is sketched in Fig. 3.29, for several values of inductor current ripple Δi. Case (a) corresponds to use of an infinite inductance L, leading to zero inductor current ripple. As shown in Table 3.1, the MOSFET conduction loss is then given by $I_{rms}^2R_{on} = DI^2R_{on}$, which

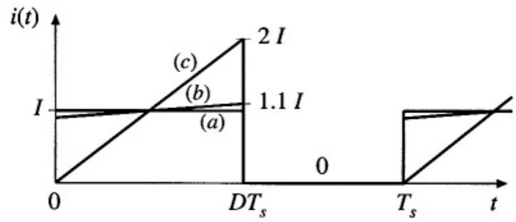


Fig. 3.29 Transistor current waveform, for various filter inductor values: (a) with a very large inductor, such that $\Delta i \approx 0$; (b) with a typical inductor value, such that $\Delta i = 0.1I$; (c) with a small inductor value, chosen such that $\Delta i = I$.

agrees exactly with the prediction of the average model. Case (b) is a typical choice of inductance L, leading to an inductor current ripple of $\Delta i = 0.1I$. The exact MOSFET conduction loss, calculated using the rms value of MOSFET current, is then only 0.33% greater than the prediction of the average model. In the extreme case (c) where $\Delta i = I$, the actual conduction loss is 33% greater than that predicted by the average model. Thus, the dc (average) model correctly predicts losses in the component nonidealities, even though rms currents are not calculated. The model is accurate provided that the inductor current ripple is small.

Table 3.1 Effect of inductor current ripple on MOSFET conduction loss

Inductor current ripple	MOSFET rms current	Average power loss in R_{on}
(a) $\Delta i = 0$	$I\sqrt{D}$	DI^2R_{on}
(b) $\Delta i = 0.1i$	$(1.00167)I\sqrt{D}$	$(1.0033)DI^2R_{on}$
(c) $\Delta i = I$	$(1.155)I\sqrt{D}$	$(1.3333)DI^2R_{on}$

3.6 SUMMARY OF KEY POINTS

1. The dc transformer model represents the primary functions of any dc-dc converter: transformation of dc voltage and current levels, ideally with 100% efficiency, and control of the conversion ratio M via the duty cycle D. This model can be easily manipulated and solved using familiar techniques of conventional circuit analysis.

2. The model can be refined to account for loss elements such as inductor winding resistance and semiconductor on-resistances and forward voltage drops. The refined model predicts the voltages, currents, and efficiency of practical nonideal converters.

3. In general, the dc equivalent circuit for a converter can be derived from the inductor volt-second balance and capacitor charge balance equations. Equivalent circuits are constructed whose loop and node equations coincide with the volt-second and charge balance equations. In converters having a pulsating input current, an additional equation is needed to model the converter input port; this equation may be obtained by averaging the converter input current.

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PROBLEMS

3.1 In the buck-boost converter of Fig. 3.30, the inductor has winding resistance R_L. All other losses can be ignored.

(a) Derive an expression for the nonideal voltage conversion ratio V/V_g.

(b) Plot your result of part (a) over the range $0 \le D \le 1$, for $R_L/R = 0, 0.01$, and 0.05.

(c) Derive an expression for the efficiency. Manipulate your expression into a form similar to Eq. (3.35)

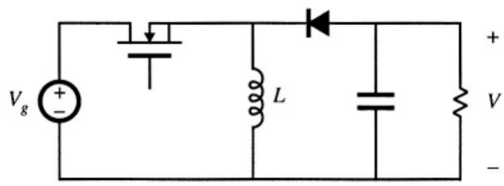

Fig. 3.30 Nonideal buck–boost converter, Problems 3.1 and 3.2.

3.2 The inductor in the buck-boost converter of Fig. 3.30 has winding resistance R_L. All other losses can be ignored. Derive an equivalent circuit model for this converter. Your model should explicitly show the input port of the converter, and should contain two dc transformers.

3.3 In the converter of Fig. 3.31, the inductor has winding resistance R_L. All other losses can be ignored. The switches operate synchronously: each is in position 1 for $0 < t < DT_s$, and in position 2 for $DT_s < t < T_s$.

(a) Derive an expression for the nonideal voltage conversion ratio V/V_g.

(b) Plot your result of part (a) over the range $0 \le D \le 1$, for $R_L/R = 0, 0.01$, and 0.05.

(c) Derive an expression for the efficiency. Manipulate your expression into a form similar to Eq. (3.35)

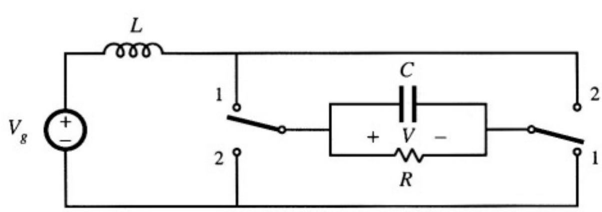

Fig. 3.31 Nonideal current-fed bridge converter, Problems 3.3 and 3.4.

3.4 The inductor in the converter of Fig. 3.31 has winding resistance R_L. All other losses can be ignored. Derive an equivalent circuit model for this converter.

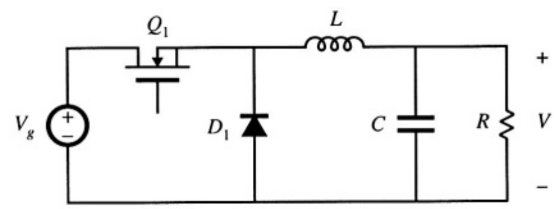

Fig. 3.32 Nonideal buck converter, Problem 3.5.

3.5 In the buck converter of Fig. 3.32, the MOSFET has on-resistance R_{on} and the diode forward voltage drop can be modeled by a constant voltage source V_D. All other losses can be neglected.

(a) Derive a complete equivalent circuit model for this converter.

(b) Solve your model to find the output voltage V.

(c) Derive an expression for the efficiency. Manipulate your expression into a form similar to Eq. (3.35).

3.6 To reduce the switching harmonics present in the input current of a certain buck converter, an input filter is added as shown in Fig. 3.33. Inductors L_1 and L_2 contain winding resistances R_{L1} and R_{L2}, respectively. The MOSFET has on-resistance R_{on}, and the diode forward voltage drop can be modeled by a constant voltage V_D plus a resistor R_D. All other losses can be ignored.

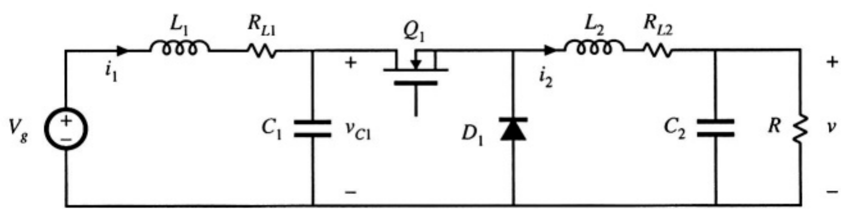

Fig. 3.33 Buck converter with input filter, Problem 3.6.

(a) Derive a complete equivalent circuit model for this circuit.

(b) Solve your model to find the dc output voltage V.

(c) Derive an expression for the efficiency. Manipulate your expression into a form similar to Eq. (3.35).

3.7 A 1.5 V battery is to be used to power a 5 V, 1 A load. It has been decided to use a buck-boost converter in this application. A suitable transistor is found with an on-resistance of 35 mΩ, and a Schottky diode is found with a forward drop of 0.5 V. The on-resistance of the Schottky diode may be ignored. The power stage schematic is shown in Fig. 3.34.

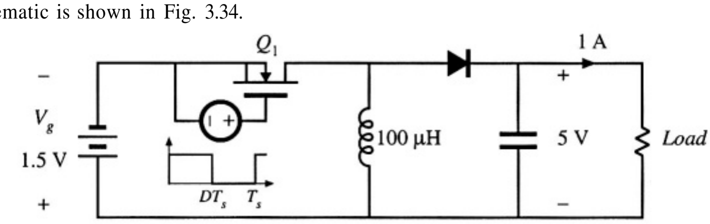

Fig. 3.34 Nonideal buck-boost converter powering a 5 V load from a 1.5 V battery, Problem 3.7

(a) Derive an equivalent circuit that models the dc properties of this converter. Include the transistor and diode conduction losses, as well as the inductor copper loss, but ignore all other sources of loss. Your model should correctly describe the converter dc input port.

(b) It is desired that the converter operate with at least 70% efficiency under nominal conditions (i.e., when the input voltage is 1.5 V and the output is 5 V at 1 A). How large can the inductor winding resistance be? At what duty cycle will the converter then operate? *Note*: there is an easy way and a not-so-easy way to analytically solve this part.

(c) For your design of part (b), compute the power loss in each element.

(d) Plot the converter output voltage and efficiency over the range $0 \le D \le 1$, using the value of inductor winding resistance which you selected in part (b).

(e) Discuss your plot of part (d). Does it behave as you expect? Explain.

For Problems 3.8 and 3.9, a transistor having an on-resistance of $0.5\ \Omega$ is used. To simplify the problems, you may neglect all losses other than the transistor conduction loss. You may also neglect the dependence of MOSFET on-resistance on rated blocking voltage. These simplifying assumptions reduce the differences between converters, but do not change the conclusions regarding which converter performs best in the given situations.

3.8 It is desired to interface a 500 V dc source to a 400 V, 10 A load using a dc-dc converter. Two possible approaches, using buck and buck-boost converters, are illustrated in Fig. 3.35. Use the assumptions described above to:

(a) Derive equivalent circuit models for both converters, which model the converter input and output ports as well as the transistor conduction loss.

(b) Determine the duty cycles that cause the converters to operate with the specified conditions.

(c) Compare the transistor conduction losses and efficiencies of the two approaches, and conclude which converter is better suited to the specified application.

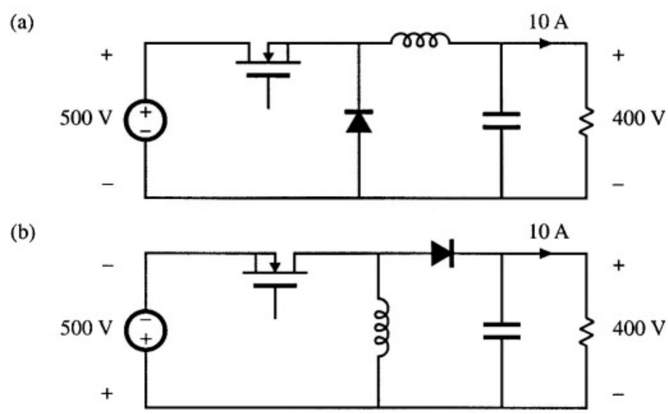

Fig. 3.35 Problem 3.8: interfacing a 500 V source to a 400 V load, using (a) a buck converter, (b) a buck-boost converter.

3.9 It is desired to interface a 300 V battery to a 400 V, 10 A load using a dc-dc converter. Two possible approaches, using boost and buck-boost converters, are illustrated in Fig. 3.36. Using the assumptions described above (before Problem 3.8), determine the efficiency and power loss of each approach. Which converter is better for this application?

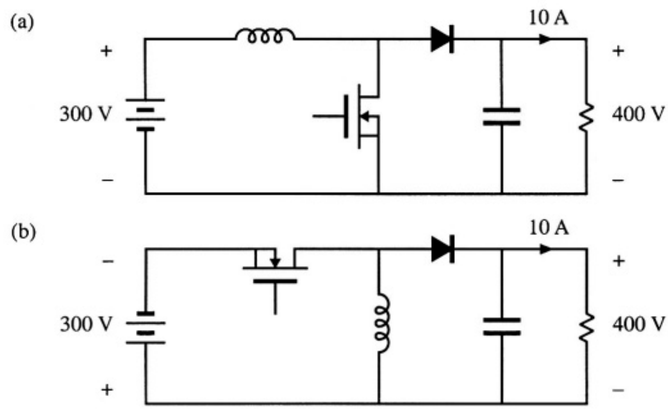

Fig. 3.36 Problem 3.9: interfacing a 300 V battery to a 400 V load, using: (a) a boost converter, (b) a buck–boost converter.

3.10 A buck converter is operated from the rectified 230 V ac mains, such that the converter dc input voltage is

$$V_g = 325 \text{ V} \pm 20\%$$

A control circuit automatically adjusts the converter duty cycle D, to maintain a constant dc output voltage of V = 240 V dc. The dc load current I can vary over a 10:1 range:

$$10 \text{ A} \leq I \leq 1 \text{ A}$$

The MOSFET has an on-resistance of 0.8 Ω. The diode conduction loss can be modeled by a 0.7 V source in series with a 0.2 Ω resistor. All other losses can be neglected.

(a) Derive an equivalent circuit that models the converter input and output ports, as well as the loss elements described above.

(b) Given the range of variation of V_g and I described above, over what range will the duty cycle vary?

(c) At what operating point (i.e., at what value of V_g and I) is the converter power loss the largest? What is the value of the efficiency at this operating point?

3.11 In the Ćuk converter of Fig. 3.37, the MOSFET has on-resistance R_{on} and the diode has a constant forward voltage drop V_D. All other losses can be neglected.

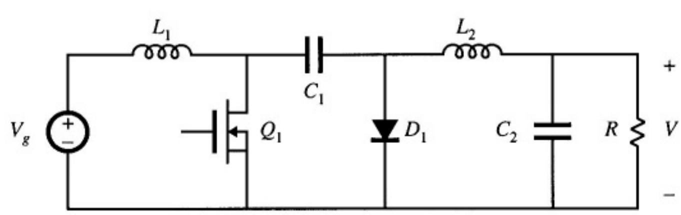

Fig. 3.37 Ćuk converter, Problem 3.11.

(a) Derive an equivalent circuit model for this converter. *Suggestion*: if you don't know how to handle some of the terms in your dc equations, then temporarily leave them as dependent sources. A more physical representation of these terms may become apparent once dc transformers are incorporated into the model.

(b) Derive analytical expressions for the converter output voltage and for the efficiency.

(c) For $V_D = 0$, plot V/V_g vs. D over the range $0 \le D \le 1$, for (*i*) $R_{on}/R = 0.01$, and (*ii*) $R_{on}/R = 0.05$.

(d) For $V_D = 0$, plot the converter efficiency over the range $0 \le D \le 1$, for (*i*) $R_{on}/R = 0.01$, and (*ii*) $R_{on}/R = 0.05$.