CHAPTER 0: ON SET THEORY

1. Introduction

There are essentially two ways to specify a particular set. One way, if possible, is by listing its members. For example, $A = \{a, e, i, o, u\}$ denotes the set *A* whose elements are the letters *a, e, i, o and u*. The other way is by stating those properties which characterize the elements in the set. For example, $B = \{x: x\ is\ an\ integer,\ x > 0\}$ denotes the set *B* whose elements are the positive integers.

Example 1.1.

The set *B* above can also be written as $B = \{1, 2, 3, . . .\}$.

Note that $-6 \notin B,\ 33 \in B\ \ and\ \pi \notin B$.

Example1.2.

Intervals on the real line, defined below, appear very often in mathematics. Here *a* and *b* are real numbers with $a < b$.

Open interval

$$(a, b) = \{x \in \mathbb{R}: a < x < b\}.$$

Closed interval

$$[a, b] = \{x \in \mathbb{R}: a \leq x \leq b\}.$$

Open-closed interval

$$]a, b] = \{x \in \mathbb{R}: a < x \leq b\}.$$

Closed-open interval

$$[a, b[= \{x \in \mathbb{R}: a \leq x < b\}.$$

Definition 1.3.

Two sets *A* and *B* are equal, ($A = B$), if they consists of the same elements, i.e., if each member of *A* belongs to *B* and each member of *B* belongs to *A*. The negation of $A = B$ is written as: $A \neq B$.

Example 1.4.

Let $E = \{x \in \mathbb{R}: x^2 - 3x + 2 = 0\}, F = \{2, 1\}$ ***and*** $G = \{1, 2, 2, 1\}$.

Then:$E = F = G$.

Observe that the set does not depend on the way in which its elements are displayed. A set remains the same if its elements are repeated or rearranged.

Definition 1.5.

A set is finite if it consists of *n* different elements, where *n* is some positive integer, otherwise a set is infinite. In particular, a set which consists of exactly one element is called a singleton set.

Definition 1.6.

A set *A* is a subset of a set *B* or, equivalently *B* is a superset of *A*, written

$A \subset B$, ***or*** $B \supset A$ **if each element in *A* is also in *B*; i.e.** $(x \in A \Rightarrow x \in B)$.

The negation of $A \subset B$ **is written** $A \not\subset B$ **and states that:**

$(\exists \ x \in A \ \ s.t. \ \ x \notin B)$.

Example 1.7.

We will let $\mathbb{N}$ denote the set of positive integers, $\mathbb{Z}$ denote the set of integers, $\mathbb{Q}$ denote the set of rational numbers and $\mathbb{R}$ denote the set of real numbers. Accordingly, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. **Observe that** $A \subset B$ **does not exclude the possibility that** $A = B$. **In case that** $A \subset B$ **but** $A \neq B$, **we say that A is a proper subset of *B* or *B* contains *A* properly.**

The following proposition follows from the preceding definitions.

Proposition 1.8.

Let *A*, *B and C* be any sets. Then:

(i) $A \subset A$.

(ii) $(A \subset B \text{ and } B \subset A) \Leftrightarrow (A = B)$.

(iii) $(A \subset B \text{ and } B \subset C) \Rightarrow A \subset C$.

In any application of the theory of sets, all sets under investigation are subsets of a fixed set. We call this set <u>the universal set</u> and denote it, in this chapter, by *U*. It is also convenient to introduce the concept of the <u>empty set</u>, i.e. a set which contains no elements. This set, (denoted by $\emptyset$), is considered finite and a subset of every other set. Thus: For any set *A*, *we have*

$$\emptyset \subset A \subset U.$$

<u>Example 1.9.</u>

In plane geometry, the universal set consists of all the points in the plane.

<u>Example 1.10.</u>

Let $A = \{x: x^2 = 4,\ x\ is\ odd\}$. Then *A* is empty, i.e. $A = \emptyset$.

<u>Example 1.11.</u>

Let $B = \{\emptyset\}$. Then B $\neq \emptyset$; for *B* contains one element, namely, $\emptyset$.

Frequently, the members of a set are sets themselves. For example, each line in a set of lines is a set of points. To help clarity these situations, we use the words "family", "class" and "collection" synonymously with set. The words "subfamily", "subclass" and "sub-collection" have meanings analogous to "subset".

<u>Example 1.12.</u>

The members of the Family $\{\{2,3\},\{2\},\{5,6\}\}$ are the sets $\{2,3\},\{2\}\ and\ \{5,6\}$.

<u>Example 1.13.</u>

Consider any set *A*. The power set of *A*, denoted by $P(A)$ or 2^A, is the family of all subsets of *A*. In particular, if $A = \{a, b, c\}$, then

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}, A\}.$$

In general if *A* is finite, say *A* has *n* elements, then *P(A)* will have 2^n members.

The word space means a non-empty set which possesses some type of mathimatical structure, e.g. vector space, metric space or topological space. In such a situation, we call the elements in a space "points"

Definition 1.14.

(1) The union of two sets A and B, (denoted by $A \cup B$) is the set of all elements which belong to A or B, i.e.:

$$A \cup B = \{x : x \in A \ \ or \ \ x \in B\}.$$

Here "or" is used in the sense of "and / or".

(2) The intersection of two sets A and B, (denoted by $A \cap B$) is the set of all elements which belong to A and B, i.e.:

$$A \cap B = \{x : x \in A \ \ and \ \ x \in B\}.$$

If $A \cap B = \emptyset$, i. e., if A and B do not have any elements in common, then A and B are said to be disjoint or non-intersecting. A family α of sets is called a disjoint family of sets if each pair of distinct sets in α is disjoint.

(3) The complement of a set A, (denoted by A^c), is the set of elements which do not belong to A, i.e.

$$A^c = \{x \in U : x \notin A\}.$$

Sets under the above operations satisfy various laws, or identities which are listed in the following proposition.

Proposition 1.15.

Sets satisfy the following laws.

Laws of the algebra of sets

1- Idempotent Laws: (i) $A \cup A = A$. (ii) $A \cap A = A$.

2- Commutative Laws: (i) $A \cup B = B \cup A$. (ii) $A \cap B = B \cap A$.

3- Associative Laws:

(i) $(A \cup B) \cup C = A \cup (B \cup C)$. (ii) $(A \cap B) \cap C = A \cap (B \cap C)$.

4- Distributive Laws:

$$(i) A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

$$(ii)\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

5- Identity Laws:

$(i) A \cup \emptyset = A \text{ and } A \cap U = A.$ $(ii)\ A \cup U = U \text{ and } A \cap \emptyset = \emptyset.$

6- Complement Laws:

$(i) A \cup A^c = U \text{ and } A \cap A^c = \emptyset.$ $(ii) (A^c)^c = A,$ $(iii) U^c = \emptyset \text{ and } \emptyset^c = U.$

The relationship between set inclusion and the above set operations are given in the following proposition.

Proposition 1.16.

The following conditions are equivalent:

$(i)\ A \subset B.$ $(ii)\ A \cap B = A.$ $(iii)\ A \cup B = B.$

$(iv)\ B^c \subset A^c$ $(v)\ A \cap B^c = \emptyset$ $(vi)\ B \cup A^c = U.$

2. Indexed family of sets.

Definition 2.1.

An indexed family of sets, (denoted by $\{A_i: i \in I\}$, $\{A_i\}_{i\in I}$ *or* $\{A_i\}$), assigns a set A_i to each $i \in I$. The set I is called the index set, the sets A_i are called indexed sets, and each $i \in I$ is called an index.

When the index set I is the set of positive integers $\mathbb{N}$, the indexed family $\{A_1, A_2, A_3, \ . \ . \ .\}$ is called a sequence of sets.

Example 2.2.

$D_n = \{x \in \mathbb{N}: x \text{ is a multipul of } n\},\ n \in \mathbb{N} \Rightarrow$

$D_1 = \{1, 2, 3, . \ . \ .\} = \mathbb{N}$, $D_2 = \{2, 4, 6, . \ . \ .\}$, $D_3 = \{3, 6, 9, . \ . \ .\}$. . . .

Definition 2.3. Let $\alpha \subset \mathcal{P}(U)$, i.e., α is a family of subsets of U:

(a) The union of the sets in α, (denoted by $\cup\, \alpha$ *or* $\cup\, \{A: A \in \alpha\}$) is the set:

$$\cup\, \alpha = \cup\, \{A: A \in \alpha\} = \{x \in U: \exists\ A \in \alpha\ \ s.t.\ \ x \in A\}.$$

(b) The intersection of the sets in α, (denoted by $\cap\, \{A: A \in \alpha\}$) is the set:

$$\cap\, \alpha = \cap\, \{A: A \in \alpha\} = \{x \in U: \forall\ A \in \alpha\ ,\ x \in A\}.$$

Definition 2.4.

Let $\alpha = \{A_i: i \in I\} \subset \mathcal{P}(U)$ i.e., α is an indexed family of subsets of U:

(a) The union of the sets in α, (denoted by $\cup\, \alpha$, $\cup\, \{A_i: i \in I\}$, $\cup_{i\in I} A_i$, *or* $\cup_i A_i$) is the set:

$$\cup\, \alpha = \cup\, \{A_i: i \in I\} = \{x \in U: \exists\ i \in I,\ \ s.t.\ \ x \in A_i\}.$$

(b) The intersection of the sets in α, (denoted by $\cap\, \{A_i: i \in I\}$, $\cap_{i\in I} A_i$, *or* $\cap_i A_i$) is the set:

$$\cap\, \alpha = \cap\, \{A_i: i \in I\} = \{x \in U: \forall\ i \in I\ \ ,\ x \in A_i\}.$$

Definition 2.5. Let $\alpha = \{A_i: i \in \mathbb{N}\} \subset \mathcal{P}(U)$:

(a) The union of the sets in α, (denoted by $\cup\, \alpha$, $\cup\, \{A_i: i \in \mathbb{N}\}$, $\cup_{i\in \mathbb{N}} A_i$, $\cup_i A_i$ *or* $A_1 \cup A_2 \cup A_3 \cup\ .\ .\ .$) is the set: $\cup\, \alpha = \cup\, \{A_i: i \in \mathbb{N}\} = \{x \in U: \exists\ i \in \mathbb{N}\ s.t.\ \ x \in A_i\}$.

(b) <u>The intersection of the sets in α</u>, (denoted by $\cap\, \boldsymbol{\alpha}$, $\cap\, \{\boldsymbol{A_i} : \boldsymbol{i} \in \mathbb{N}\}$, $\cap_{i\in\mathbb{N}} \boldsymbol{A_i}$, $\cap_i \boldsymbol{A_i}$ ***or*** $\boldsymbol{A_1} \cap \boldsymbol{A_2} \cap \boldsymbol{A_3} \cap\ .\ .\ .$**) is the set:**

$$\cap\, \boldsymbol{\alpha} = \cap\, \{\boldsymbol{A_i} : \boldsymbol{i} \in \mathbb{N}\} = \{\boldsymbol{x} \in \boldsymbol{U} : \forall\, \boldsymbol{i} \in \mathbb{N},\ \boldsymbol{x} \in \boldsymbol{A_i}\}.$$

<u>Example 2.6.</u>

$\boldsymbol{\alpha} = \{\boldsymbol{D_n} : \boldsymbol{n} \in \mathbb{N}\}, \boldsymbol{where}\ \boldsymbol{\alpha} = \boldsymbol{D_n} = \{\boldsymbol{x} \in \mathbb{N} : \boldsymbol{x\ is\ a\ multipul\ of\ n}\} \Rightarrow$

$\boldsymbol{D_1} = \{\boldsymbol{x} \in \mathbb{N} : \boldsymbol{x\ is\ a\ multipul\ of\ 1}\} = \{\boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, .\ .\ .\} = \mathbb{N}.$

$\boldsymbol{D_2} = \{\boldsymbol{x} \in \mathbb{N} : \boldsymbol{x\ is\ a\ multipul\ of\ 2}\} = \{\boldsymbol{2}, \boldsymbol{4}, \boldsymbol{6}, .\ .\ .\}.$

$\boldsymbol{D_n} = \{\boldsymbol{x} \in \mathbb{N} : \boldsymbol{x\ is\ a\ multipul\ of\ n}\} = \{\boldsymbol{n}, \boldsymbol{2n}, \boldsymbol{3n}, .\ .\ .\}$**. . . . etc.**

Consequently,

$$\cup_{n=1}^{\infty} \boldsymbol{D_n} = \mathbb{N},\ \boldsymbol{and}\ \ \cap_{n=1}^{\infty} \boldsymbol{D_n} = \emptyset. \textbf{ (Verify.)}$$

<u>Example 2.7.</u>

$\forall\, \boldsymbol{i} \in \boldsymbol{I} = [\boldsymbol{0}, \boldsymbol{1}], \boldsymbol{A_i} = [\boldsymbol{0}, \boldsymbol{i}] \Rightarrow \cup_i \boldsymbol{A_i} = [\boldsymbol{0}, \boldsymbol{1}]$ **and** $\cap_i \boldsymbol{A_i} = \{\boldsymbol{0}\}$. **(Verify.)**

<u>Proposition 2.8</u>. (Generalized distributive laws)

$\boldsymbol{\alpha} = \{\boldsymbol{A_i}\} \subset \mathcal{P}(\boldsymbol{U}), \boldsymbol{B} \subset \boldsymbol{U} \Rightarrow$

(i) $\boldsymbol{B} \cup (\cap_i \boldsymbol{A_i}) = \cap_i (\boldsymbol{B} \cup \boldsymbol{A_i})$, ***(ii)*** $\boldsymbol{B} \cap (\cup_i \boldsymbol{A_i}) = \cup_i (\boldsymbol{B} \cap \boldsymbol{A_i})$.

<u>Proposition 2.9.</u> (Generalized De Morgan's laws laws)

$\boldsymbol{\alpha} = \{\boldsymbol{A_i}\} \subset \mathcal{P}(\boldsymbol{U}) \Rightarrow$

(i) $(\cup_i \boldsymbol{A_i})^c = \cap_i \boldsymbol{A_i^c}$. ***(ii)*** $(\cap_i \boldsymbol{A_i})^c = \cup_i \boldsymbol{A_i^c}$.

<u>Proposition 2.10</u>.

$\boldsymbol{If}\ \forall\, \boldsymbol{p} \in \boldsymbol{A} \subset \boldsymbol{U}, \exists\, \boldsymbol{G_p} \subset \boldsymbol{A}, \boldsymbol{s.t.}\ \boldsymbol{p} \in \boldsymbol{G_p} \subset \boldsymbol{A}, \boldsymbol{then}\ \boldsymbol{A} = \cup \left\{\boldsymbol{G_p} : \boldsymbol{p} \in \boldsymbol{A}\right\}.$

<u>Proposition 2.11</u>.

$$\boldsymbol{A} \subset \boldsymbol{U} \Rightarrow \boldsymbol{A} = \cup \left\{\{\boldsymbol{p}\} : \boldsymbol{p} \in \boldsymbol{A}\right\}.$$

<u>Remark 2.12.</u>

$\boldsymbol{\alpha} = \{\boldsymbol{A_i} \subset \boldsymbol{U} : \boldsymbol{i} \in \emptyset\} = \emptyset \subset \mathcal{P}(\boldsymbol{U})$. **It is convenient to define:**

$$\cup\, \{\boldsymbol{A_i} \subset \boldsymbol{U} : \boldsymbol{i} \in \emptyset\} = \emptyset\ \boldsymbol{and}\ \ \cap\, \{\boldsymbol{A_i} \subset \boldsymbol{U} : \boldsymbol{i} \in \emptyset\} = \boldsymbol{U}.$$

3. Image and inverse image.

Definition 3.1.

Let $f: X \to Y, A \subset X, B \subset Y$.Then:

(a) The image of A is the set:

$$f(A) = \{f(x) \in Y: x \in A\} \subset Y.$$

(b) the inverse image of B is the set:

$$f^{-1}(B) = \{x \in X: f(x) \in B\} \subset X.$$

Example 3.2.

$f: \mathbb{R} \to \mathbb{R}$, **where** $f(x) = x^2 \Rightarrow$

$f(\{1,3,4,7\}) = \{1,9,16,49\}$ *and* $f([1,2] = [1,4])$.

Also, $f^{-1}(\{4,9\}) = \{-3,-2,2,3\}$, $f^{-1}([1,4]) = [1,2] \cup [-2,-1]$.

Proposition 3.3.

Let $f: X \to Y, \{A_i\} \subset \mathcal{P}(X) \Rightarrow$

(i) $f(\cup_i A_i) = \cup_i f(A_i)$. ***(ii)*** $A_{i_1} \subset A_{i_2} \Rightarrow f(A_{i_1}) \subset f(A_{i_2})$.

Remark 3.4.

In general, $f(\cap_i A_i) \neq \cap_i f(A_i)$.

Answer.

$$\pi_1: \mathbb{R}^2 \to \mathbb{R}, \textit{defined by } \pi_1((x,y)) = x.$$

$A_1 = [1,2] \times [1,2]$, $\quad A_2 = [1,2] \times [3,4] \subset \mathcal{P}(\mathbb{R}^2)$

$\Rightarrow \pi_1(A_1 \cap A_2) \neq \pi_1(A_1) \cap \pi_1(A_2)$. **(Verify.)**

Proposition 3.5.

Let $f: X \to Y, B, B_i \in \mathcal{P}(Y) \Rightarrow$

(i) $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$. ***(ii)*** $f^{-1}(\cap_i B_i) = \cap_i f^{-1}(B_i)$. ***(iii)***

$B_{i_1} \subset B_{i_2} \Rightarrow f^{-1}(B_{i_1}) \subset f^{-1}(B_{i_2})$. $(iv) f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$.

<u>Proposition 3.6</u>.

$\boldsymbol{f}: \boldsymbol{X} \rightarrow \boldsymbol{Y}, \boldsymbol{A} \subset \boldsymbol{X}, \boldsymbol{B} \subset \boldsymbol{Y} \Rightarrow$

(i) $\boldsymbol{A} \subset \boldsymbol{f}^{-1}\big(\boldsymbol{f}(\boldsymbol{A})\big)$.

(ii) $\boldsymbol{B} \supset \boldsymbol{f}\left(\boldsymbol{f}^{-1}(\boldsymbol{B})\right)$.