CHAPTER 1

TOPOLOGICAL SPACES

1.1. Topology, Open Sets, closed sets and clopen sets.

Definition 1.1.1.

$X \neq \emptyset, \tau \subset \mathcal{P}(X) \Rightarrow$

(a) τ *is a topology on* $X \Leftrightarrow \tau$ *satisfies* $[O1] - [O3]$.

Where:

$[O1]$ $X, \emptyset \in \tau$.

$[O2]$ $\forall\, G_i \in \tau, i \in I$; $(\cup_{i \in I} G_i) \in \tau$.

$[O3]$ *[O3]* $\forall\, G_1, G_2 \in \tau$; $(G_1 \cap G_2) \in \tau$.

(b) *Members of* τ *are the open sets*. [*i.e.*, G *is open set* $\Leftrightarrow G \in \tau$.]

(c) (X, τ) *is a topological space*.

(d) F *is a closed set*[*denoted* $F \in \tau^*$] $\Leftrightarrow F^c$ *is open set*.

Remark 1.1.2.

In (X, τ):

(1) $\tau \equiv \{all\ open\ sets\}$. **(2)** $\tau^* \equiv \{all\ closed\ sets\}$.

(3) $\tau \cap \tau^* \equiv \{all\ clopen\ sets\}$.

Example 1.1.3.

Show that:

(a) $X = \{a, b, c, d, e, f\}, \tau_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$

$\Rightarrow \tau_1$ *is a topology on* X.

(b) $X = \{a, b, c, d, e\}, \tau_2 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, e\}, \{b, c, d\}\}$

$\Rightarrow \boldsymbol{\tau_2}$ ***is not topology on X***.

(c) $\boldsymbol{X = \{a, b, c, d, e, f\}, \tau_3 = \{X, \emptyset, \{a\}, \{f\}, \{a, f\}, \{a, c, f\}, \{b, c, d, e, f\}\}}$

$\boldsymbol{\tau_3 = \{X, \emptyset, \{a\}, \{f\}, \{a, f\}, \{a, c, f\}, \{b, c, d, e, f\}\}}$.

$\Rightarrow \boldsymbol{\tau_3}$ ***is not topology on X***.

Answer.

(a) $\boldsymbol{\tau_1}$ ***satisfies*** $[\boldsymbol{O1}] - [\boldsymbol{O3}]$.

(b) $\boldsymbol{\tau_2}$ ***does not satisfy*** $[\boldsymbol{O2}]$.

[$\exists\ \boldsymbol{\{c, d\}, \{a, c, e\} \in \tau_2, s.t. \{c, d\} \cup \{a, c, e\} = \{a, c, d, e\} \notin \tau_2}$.**]**

(c) $\boldsymbol{\tau_3}$ ***does not satisfy*** $[\boldsymbol{O3}]$.

[$\exists\ \boldsymbol{\{a, c, f\}, \{b, c, d, e, f\} \in \tau_3, s.t. \{a, c, f\} \cap \{b, c, d, e, f\} = \{c, f\} \notin \tau_3}$.**]**

Remark 1.1.4.

In $(\boldsymbol{X}, \boldsymbol{\tau_1})$:

$$\boldsymbol{\tau_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\} \equiv \{all\ open\ sets\}}.$$

$$\boldsymbol{\tau_1^* = \{\emptyset, X, \{b, c, d, e, f\}\{a, b, e, f\}, \{b, e, f\}, \{a\}\} \equiv \{all\ closed\ sets\}}.$$

$$\boldsymbol{\tau_1 \cap \tau_1^* = \{X, \emptyset, \{a\}, \{b, c, d, e, f\}\} \equiv \{all\ clopen\ sets\}}.$$

Note that:

(i) $\boldsymbol{A = \{b, c, d, e, f\} \in \tau_1 \cap \tau_1^* \Rightarrow A\ clopen\ set}$.

(ii) $\boldsymbol{B = \{c, d\} \in \tau_1, B = \{c, d\} \notin \tau_1^* \Rightarrow B\ is\ open\ set\ but\ not\ closed}$.

(iii) $\boldsymbol{C = \{b, e, f\} \notin \tau_1, C = \{b, e, f\} \in \tau_1^* \Rightarrow C\ is\ closed\ set\ but\ not\ open}$.

(iv) $\boldsymbol{D = \{b\} \notin \tau_1, D = \{b\} \notin \tau_1^* \Rightarrow D\ is\ neither\ open\ set\ nor\ closed}$.

Example 1.1.5.

$$\boldsymbol{\tau_4 = \{\mathbb{N}, G \subset \mathbb{N}: G\ is\ a\ finite\ set\} \subset \mathcal{P}(\mathbb{N})}.$$

Show that: $\boldsymbol{\tau_4}$ ***is not topology on*** $\mathbb{N}$.

Answer.

$\exists \{2\},\{3\},\{4\},\{5\}, \ . \ . \ . \ \in \tau_4,\ s.t.\ \{2\} \cup \{3\} \cup \{4\} \cup \ \ . \ . \ . = (\mathbb{N} \setminus \{1\}) \notin \tau_4.$

τ_4 *does not satisfy* $[O2] \Rightarrow \tau_4$ *is not topology on* $\mathbb{N}$.

Example 1.1.6.

$$\tau_5 = \{\emptyset\ , G \subset \mathbb{Z}: G \text{ is an infinite set}\} \subset \mathcal{P}(\mathbb{N}).$$

Show that: τ_5 *is not topology on* $\mathbb{Z}$.

Answer.

$\exists\ G_1 = \{\dots, -2, -1, 0\}, G_2 = \{-2, -1, 0, 1, \dots\} \in \tau_5, s.t.,$

$G_1 \cap G_2 = \{-2, -1, 0\} \notin \tau_5.$

τ_5 *does not satisfy* $[O3] \Rightarrow \tau_5$ *is not topology on* $\mathbb{Z}$.

Definition 1.1.7.

$X \neq \emptyset, \mathcal{D} = \mathcal{P}(X) \Rightarrow \mathcal{D}$ *is a topology on* X.

[*It is named the discrete topology on X and*

$(X, \mathcal{D})$ *is a discrete space*.]

Remark 1.1.8.

In $(X, \mathcal{D})$: $\mathcal{D} = \mathcal{D}^* = \mathcal{P}(X) \Rightarrow \forall\ A \subset X, A$ *is a clopen set*.

Proposition 1.1.9.

In (X, τ): $\tau = \mathcal{D} \Leftrightarrow \forall\ x \in X, \{x\} \in \tau.$

Proof.

$(\Rightarrow)\ \tau = \mathcal{D}, x \in X \Rightarrow \{x\} \in \mathcal{P}(X) = \mathcal{D} = \tau \Rightarrow \forall\ x \in X, \{x\} \in \tau.$

$(\Leftarrow)\ \forall\ x \in X, \{x\} \in \tau, A \in \mathcal{D} = \mathcal{P}(X) \Rightarrow\ A = \cup_{x \in A}\ \{x\} \in \tau\ (By\ [O2])$

$\Rightarrow \forall\ A \in \mathcal{D}, A \in \tau \Rightarrow \mathcal{D} \subset \tau \subset \mathcal{P}(X) = \mathcal{D} \Rightarrow \tau = \mathcal{D}.$

Definition 1.1.10.

$X \neq \emptyset, \mathcal{I} = \{X, \emptyset\} \subset \mathcal{P}(X) \Rightarrow \mathcal{I}$ *is a topology on* X.

[*It is named the indiscrete topology on X and*

$(X, \mathcal{I})$ *is an indiscrete space*.]

Remark 1.1.11. In $(X, \mathcal{I})$: $\mathcal{I} = \mathcal{I}^* = \{X, \emptyset\} \Rightarrow$

(i) *The only clopen sets are* $X, \emptyset$.

(ii) $\forall\, A \subset X$ *s.t.* $\emptyset \neq A \neq X$; *A neither open set nor closed*.

Proposition 1.1.12.

$$\tau_1, \tau_2 \text{ two topologies on } X \Rightarrow (\tau_1 \subset \tau_2 \Leftrightarrow \tau_1^* \subset \tau_2^*).$$

Proof.

$(\Rightarrow)$ $\tau_1 \subset \tau_2, F \in \tau_1^* \Rightarrow F^c \in \tau_1 \subset \tau_2 \Rightarrow F^c \in \tau_2 \Rightarrow F^{c\,c} = F \in \tau_2^* \Rightarrow \tau_1^* \subset \tau_2^*.$

$(\Leftarrow)$ $\tau_1^* \subset \tau_2^*, G \in \tau_1 \Rightarrow G^c \in \tau_1^* \subset \tau_2^* \Rightarrow G^c \in \tau_2^* \Leftrightarrow G^{c\,c} = G \in \tau_2 \Rightarrow \tau_1 \subset \tau_2.$

Remark 1.1.13.

In (X, τ): (i) $\mathcal{I} \subset \tau \subset \mathcal{D}$. (ii) $\mathcal{I}^* \subset \tau^* \subset \mathcal{D}^*$.

Proposition 1.1.14.

$X \neq \emptyset, \tau_f = \{\emptyset, G \subset X : G^c \text{ is finite}\} \subset \mathcal{P}(X) \Rightarrow \tau_f$ *is a topology on* X.

[*It is named the the co – finite topology on X and*

(X, τ_f) *is a co– finite space*.]

Proof.

It is required to prove that τ_f satisfies [O1] – [O3].

[O1] $\emptyset \in \tau_f$, *by definition and* $X \subset X$ *s.t.* $X^c = \emptyset$ *is finite* $\Rightarrow X, \emptyset \in \tau_f$.

[O2] $G_i \in \tau_f, i \in I \Rightarrow G_i^c$ *is finite*, $i \in I \Rightarrow (\cup_i G_i)^c = \cap_i G_i^c$ *is finite* $\Rightarrow$

$(\cup_i G_i) \in \tau_f$.

[O3] $G_1, G_2 \in \tau_f \Rightarrow G_1^c, G_2^c$ $are\ finite \Rightarrow (G_1, \cap G_2)^c = G_1^c \cup\ G_2^c\ is\ finite$
$\Rightarrow (G_1 \cap G_2) \in \tau_f$.

Remark 1.1.15.

In (X, τ_f): (a) $G \in \tau_f \Leftrightarrow G = \emptyset\ or\ G^c\ is\ finite$.

(b) $F \in \tau_f^* \Leftrightarrow F = X\ or\ F\ is\ finite$.

Example 1.1.16.

In the $co\text{–}finite\ space$ $(\mathbb{N}, \tau_f)$:

(i) $\tau_f \cap \tau_f^* = \{\mathbb{N}, \emptyset\} \Rightarrow$**The only clopen sets are** $\mathbb{N}$ and $\emptyset$**.**

(ii) $A = \{5, 6, 7, \dots\ \} \in \tau_f,\ \ A \notin \tau_f^* \Rightarrow A\ is\ an\ open\ set\ but\ not\ closed$ **.**

(iii) $B = \{2, 5, 13\} \notin \tau_f, B \in \tau_f^* \Rightarrow B\ is\ closed\ set\ but\ not\ open$.

(iv) $C = \{1, 3, 5, \dots\} \notin \tau_f, C \notin \tau_f^* \Rightarrow C\ is\ neither\ open\ set\ nor\ closed$.

Example 1.1.17.

Give an example for a topological space (X, τ) **in which:**

$G_i \in \tau, i \in I \not\Rightarrow\ (\cap_i G_i) \in \tau$.

Answer.

(a) Let $\tau = \{\emptyset, G_r = (-r, r) \subset \mathbb{R}: r > 0\}$**. Then** τ **is a topology on** $\mathbb{R}$**.**

In $(\mathbb{R}, \tau)$**:** $G_r \in \tau\ \forall\ r > 0$**. But** $\ \cap_r G_r = \{0\} \notin \tau$.

(b) In $(\mathbb{N}, \tau_f)$**: Let** $G_n = \{1\} \cup \{n + 1\} \cup \{n + 2\} \cup \{\ n + 3\} \cup\ .\ .\ .$

$[G_1 = \mathbb{N}, G_2 = \{1, 3, 4, \dots\}, G_3 = \{1, 4, 5, \dots\}, G_4 = \{1, 5, 6, \dots\}, .\ .\ .\ .]$

$\Rightarrow G_1^c = \emptyset, G_2^c = \{2\}, G_3^c = \{2, 3\}, G_4^c = \{2, 3, 4\}, \dots \dots\ \ finite\ sets$.

$\Rightarrow G_1^c = \emptyset, G_2^c = \{2\}, G_3^c = \{2, 3\}, G_4^c = \{2, 3, 4\}, \dots \dots\ \ \in \tau_f^*$.

$\Rightarrow G_n \in \tau_f, \forall n \in \mathbb{N}. But\ G = (\cap_n G_n) = \{1\} \notin \tau_f.$

$[Since\ G^c = \mathbb{N} \setminus \{1\}\ not\ finite.]$

Proposition 1.1.18.

If (X, τ_f) $satisfying: \tau_f \cap \tau_f^*\ contains\ at\ least\ three\ clopen\ sets, then$

$(i)\ X\ is\ a\ finite\ set.$ $\qquad$ $(ii)\ \tau_f = \mathcal{D}.$

Proof.

$(i)\ \exists\ A \subset X, s.t. \emptyset \neq A \neq X\ \ and\ A \in \tau_f \cap \tau_f^* \Rightarrow A, A^c \in \tau_f \cap \tau_f^*$

$\Rightarrow A, A^c\ finite \Rightarrow X = A \cup\ A^c\ finite.$

$(ii)\ G \in \mathcal{D} = \mathcal{P}(X) \Rightarrow G, G^c\ finite\ \Rightarrow G^c \in \tau_f^* \Rightarrow G \in \tau_f$

$\Rightarrow \mathcal{D} \subset \tau_f \Rightarrow \tau_f = \mathcal{D}.$

Proposition 1.1.19.

In (X, τ)**;** τ^* **satisfies :**

[C1] $\emptyset, X \in \tau^*.$

[C2] $F_i \in \tau^*, i \in I\ \Rightarrow (\bigcap_{i \in I} F_i) \in \tau^*.$

[C3] $F_1, F_2 \in \tau^* \Rightarrow (F_1 \cup F_2) \in \tau^*.$

Proof:

[C1] $X, \emptyset \in \tau \Rightarrow X^c = \emptyset, \emptyset^c = X \in \tau^*.$

[C2] $F_i \in \tau^*, i \in I\ \Rightarrow F_i^c \in \tau,\ i \in I\ \Rightarrow (\cup_i F_i^c) = (\bigcap_i F_i)^c \in \tau \Rightarrow (\bigcap_i F_i) \in \tau^*.$

[C3] $F_1, F_2 \in \tau^* \Rightarrow F_1^c, F_2^c \in \tau \Rightarrow (F_1^c \cap F_2^c) = (F_1 \cup F_2)^c \in \tau$

$\Rightarrow\ (F_1 \cup F_2) \in \tau^*.$

Remark 1.1.20.

[Proposition1.1.13:$X \neq \emptyset, \tau_f = \{\emptyset, G \subset X: G^c \; is \; finite\} \subset \mathcal{P}(X) \Rightarrow$ $\tau_f \; is \; a \; topology \; on \; X.$ **] has another proof: We show that**

$$\tau_f^* = \{X, F \subset X: F \; is \; finite\}$$

satisfies [C1] - [C3].

[C1] $X \in \tau_f^*$ $, by \; definition \; and \; \emptyset \subset X \; s.t. \emptyset \; is \; finite.$

[C2] $F_i \in \tau_f^*, i \in I \Rightarrow F_i \; finite, \; i \in I \Rightarrow \; (\cap_i F_i) \; finite \Rightarrow (\cap_i F_i) \in \tau_f^*$**.**

[C3] $F_1, F_2 \in \tau^* \Rightarrow F_1, F_2 \; finite \Rightarrow (F_1 \cup F_2) \; finite \; \Rightarrow \; (F_1 \cup F_2) \in \tau^*$**.**

[C1] - [C3] $\Rightarrow \tau_f = \{\emptyset, G \subset X: G^c \; is \; finite\}$ **is a topology on** X**.**