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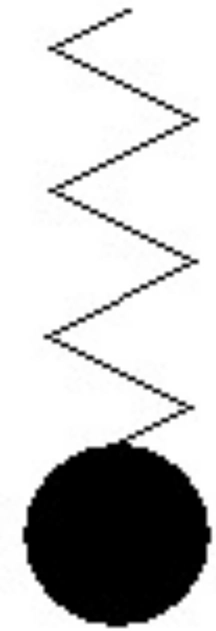
Oscillations: Mass on a Spring

Learning Outcomes

By the end of this lesson, you should be able to:

- Recognize that a mass attached to a spring will go through an oscillatory motion like the SHM.
- Define the SHM equation in second-order differential form.
- Apply the SHM equation to solve examples with masses on springs.
- Understand and apply the relation between the mass m, the spring constant k, and the angular frequency ω of the SHM.

A mass on a spring



- Let's consider a simple oscillating system: a mass hanging from a spring.
- Start by considering the mass at rest. The mass stretches the spring by an amount x_0. What are the forces acting on the mass?

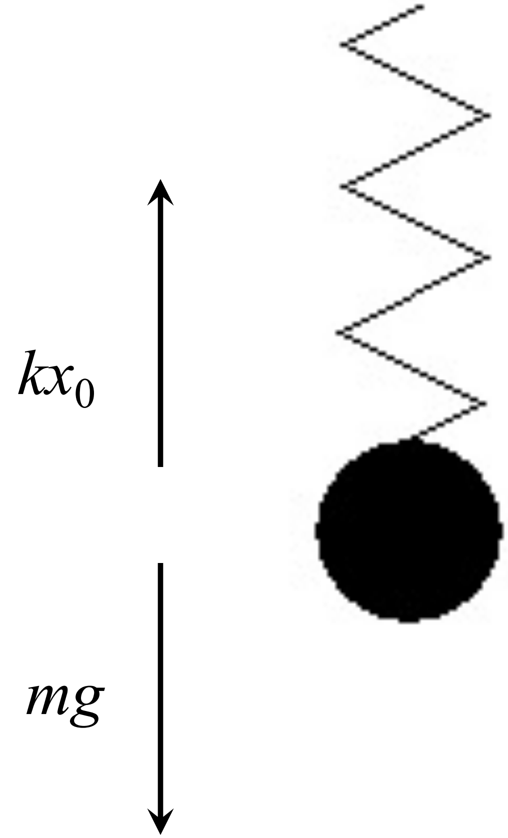
A mass on a spring

- The spring's restoring force acts upwards
  - from **Hooke's Law**

$$F_r = -kx_0$$

- Gravity acts downwards

$$F_g = mg$$



- When the mass is at rest, these forces balance:

$$mg = kx_0$$

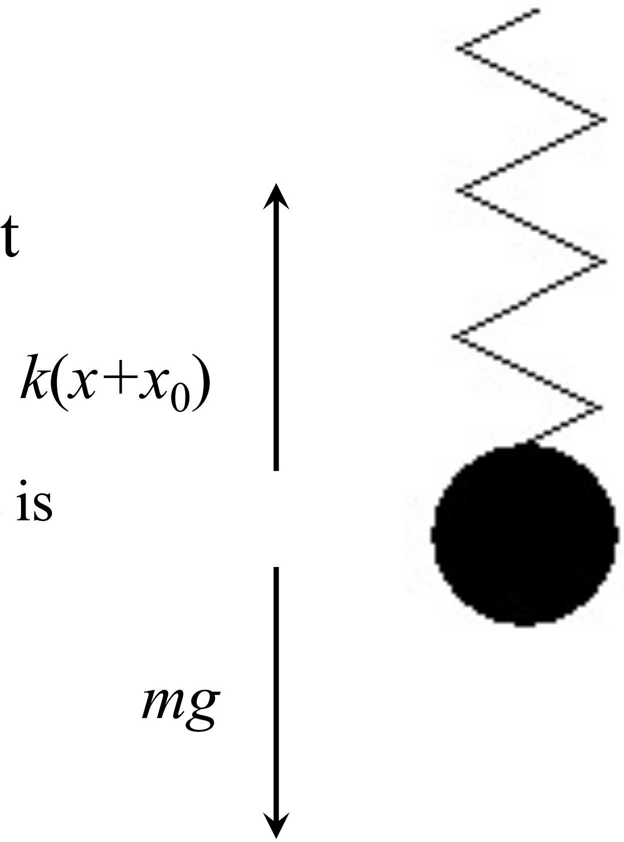
A mass on a spring

- Now we displace the mass downwards a bit and let go. What are the forces acting on it now?
  - The spring's restoring force acts upwards, but now the displacement is $x+x_0$. From Hooke's Law:

$$F_r = -k(x+x_0)$$

  - Gravity acts downwards:

$$F_g = mg$$



A mass on a spring

The net force on the mass is

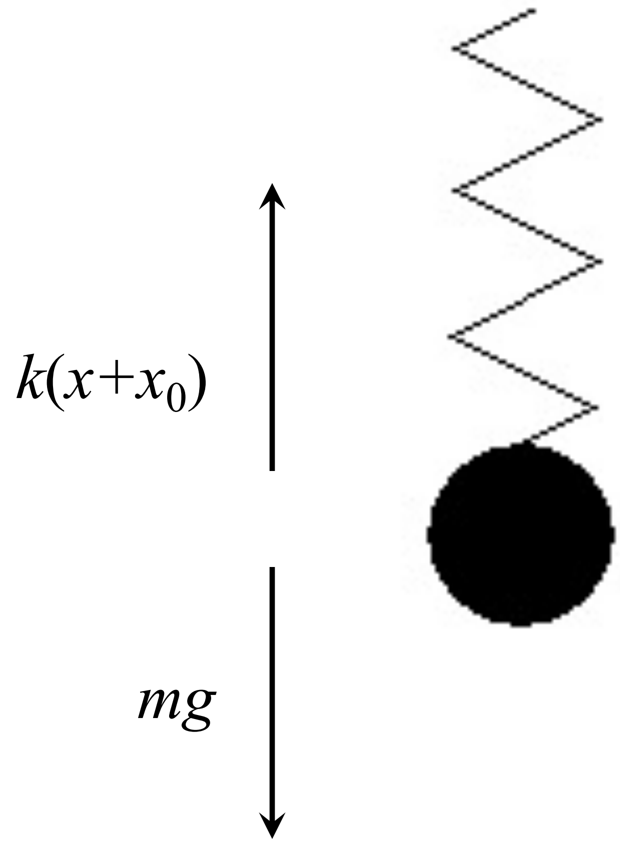
$$F = mg - k(x + x_0)$$

but we know from when the mass was at rest that

$$mg = kx_0$$

so we end up with

$$F = -kx$$



Whenever we have a linear restoring force, like Hooke's Law, we have SMH

- The net force acting on our mass is $F = -kx$
- Newton's 2^{nd} law says $F = ma = m\dfrac{d^2x}{dt^2}$
- Therefore $m\dfrac{d^2x}{dt^2} = -kx$
- This is Newton's second law for our mass on a spring system. This is a differential equation which can be solved to give $x(t)$.

Simple Harmonic Motion Equation in differential form

$$m\frac{d^2x}{dt^2} = -kx$$

There are many techniques for solving equations like this, which you can learn about later…. One method is to guess. Since we expect the system to oscillate in time, let's *assume* that the solution looks like:

$$x(t) = A\cos(\omega t + \varphi)$$

We'll plug this into our differential equation and see if it works!

Simple Harmonic Motion Equation in differential form

We obtain: $x(t) = A\cos(\omega t + \varphi)$ … as we know for SHM

$$\frac{dx}{dt} = -A\omega\sin(\omega t + \varphi)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \varphi)$$

Now, the SHM equation was: $$m\frac{d^2x}{dt^2} = -kx$$

Substituting for $x(t)$ into the above equation gives

$$-mA\omega^2\cos(\omega t + \varphi) = -kA\cos(\omega t + \varphi)$$

This is an SHM as long as $$m\omega^2 = k$$

Simple Harmonic Motion Equation in differential form

- This tells us that our guess for the form of $x(t)$ was right – it solves the SMH equation and satisfies Newton's 2nd Law…
- …but only if

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}}$$

- The angular frequency ω depends on the characteristics of our system: the mass m and the spring constant k.
- To get A and φ we have to know the initial conditions, i.e., $x(0)$ and $v(0)$.

Example 1

- A 200 g mass is attached to a spring of spring constant $k = 5.6$ N/m and set into oscillations on a horizontal frictionless surface with amplitude $A = 25$ cm. Determine:
  (a) The frequency in Hz
  (b) The period
  (c) The maximum velocity
  (d) The maximum force in the spring

Example 1: solution

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(a) The mass-on-a spring setup means we'll have SHM.

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{\frac{k}{m}}}{2\pi} = \frac{\sqrt{\frac{5.6\frac{\mathrm{N}}{\mathrm{m}}}{0.2\ \mathrm{kg}}}}{2\pi} = \mathbf{0.84\ Hz}.$$

(b) $T = \frac{1}{f} = \mathbf{1.2\ sec}$

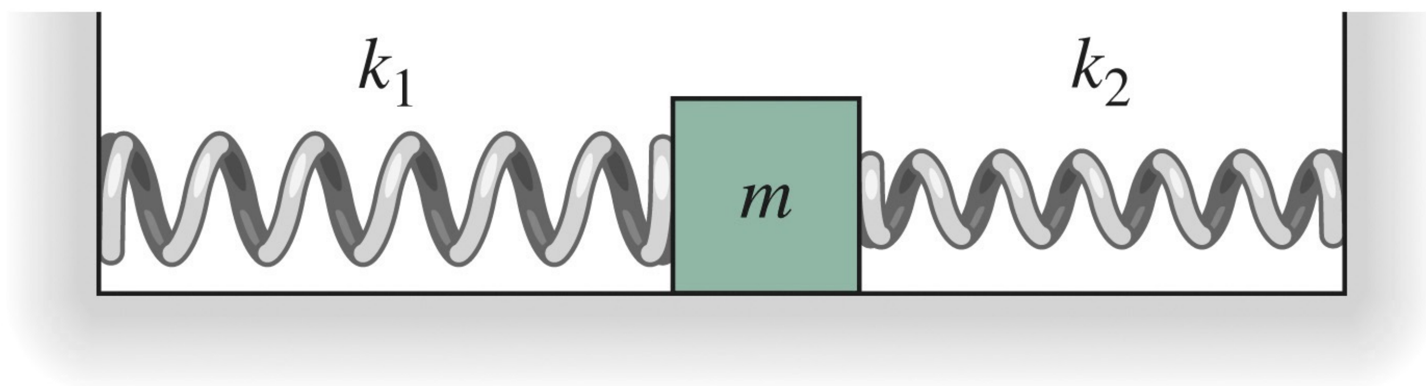
(c) $v(t) = -A\omega \sin(\omega t + \varphi)$, so $v_{max} = A\omega = A2\pi f = \mathbf{1.32\ m/s}$

(d) $F = -kx$, so $F_{max} = kA = \mathbf{1.4\ N}$

Example 2

- A mass m is mounted on a frictionless surface between two springs with spring constants k_1 and k_2 as shown. Show that the angular frequency of oscillation is

$$\omega = \sqrt{(k_1 + k_2)/m}$$



Example 2: solution

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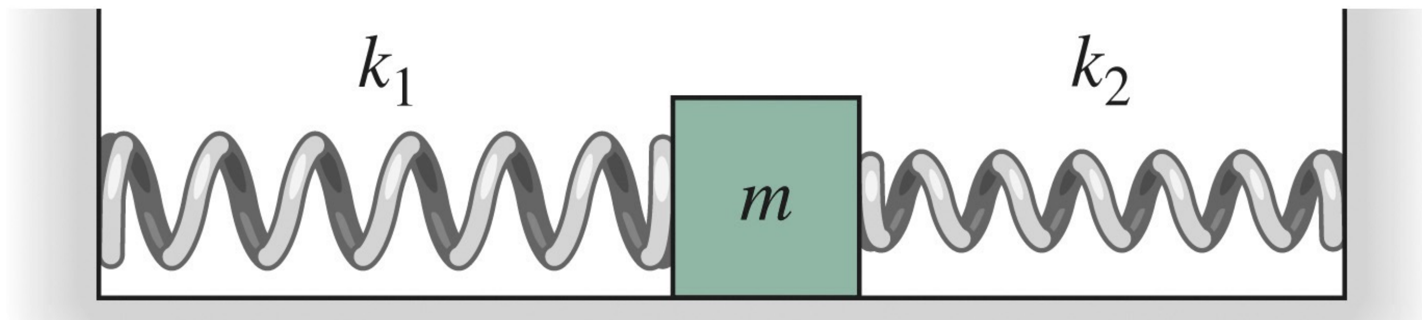

- The force on the mass is the sum of the forces from both springs. Note that the displacement $\vec{x}$ on the side of spring k_1 will be the same as on the side of spring k_2, and the forces of the two springs will act in the same direction:

$$\vec{F} = -k_1\vec{x} - k_2\vec{x} = -(k_1 + k_2)\vec{x}$$

- So, the combination of springs acts like a single spring of constant k_1+k_2. Hence, the angular frequency of oscillation is $\boldsymbol{\omega} = \sqrt{\frac{k_1+k_2}{m}}$.

Example 3

- Show that $x(t) = a\cos(\omega t) - b\sin(\omega t)$ represents simple harmonic motion with amplitude $A = \sqrt{a^2 + b^2}$ and phase $\varphi = \tan^{-1}(b/a)$.

Example 3: solution

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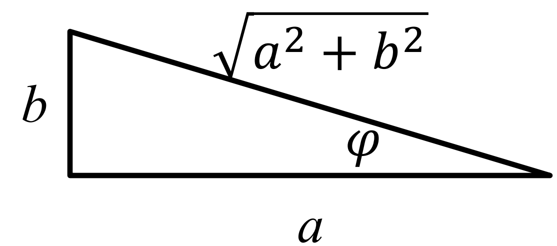
Let's see if we use the given *A* and φ in the SHM equation, if we then recover the expression for $x(t)$:

$$x(t) = A\cos(\omega t + \varphi) = A\,(\cos(\omega t)\cos\varphi - \sin(\omega t)\sin\varphi) =$$
$$= \sqrt{a^2+b^2}[\cos(\omega t)\cos(\tan^{-1}(b/a)) - \sin(\omega t)\sin(\tan^{-1}(b/a))].$$

Now, if $\varphi = \tan^{-1}(b/a)$, then *b* and *a* are the sides of a right-angle triangle, and φ is the angle opposite side *b*.



Then the hypotenuse is $\sqrt{a^2+b^2}$ and so

$$\sin\varphi = \frac{b}{\sqrt{a^2+b^2}}, \quad \cos\varphi = \frac{a}{\sqrt{a^2+b^2}}$$

Putting these in the expression for *x*(*t*) above:

$$x(t) = \sqrt{a^2+b^2}[\cos(\omega t)\frac{a}{\sqrt{a^2+b^2}} - \sin(\omega t)\frac{b}{\sqrt{a^2+b^2}}] =$$

$= \boldsymbol{a\cos(\omega t) - b\sin(\omega t)}$**, as we had to demonstrate.**

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