If g,f are surjections then (g o f) is a surjection.

Let y be an element of C. Then, since g from B to C is a surjection, there exists $u \in B$ such that $g(u) = y$. Since f from A to B is a surjection and $u \in B$, there exists $v \in A$ such that $f(v) = u$. We would like to find $x \in A$ s.t. $g(f(x)) = y$.

L1

H1. f from A to B is a surjection
H2. g from B to C is a surjection

T1. $g \circ f$ from A to C is a surjection

1. Expand pre-universal target T1.

L1

H1. f from A to B is a surjection
H2. g from B to C is a surjection

T2. $\forall y.(y \in C \Rightarrow \exists x.(x \in A \wedge g(f(x)) = y))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let y be an element of C.

L1

y

H1. f from A to B is a surjection
H2. g from B to C is a surjection
H3. $y \in C$

T3. $\exists x.(x \in A \wedge g(f(x)) = y)$

3. Forwards reasoning using H2 with H3.

Since g from B to C is a surjection and $y \in C$, there exists $u \in B$ such that $g(u) = y$.

L1

$y \ u[y]$

H1. f from A to B is a surjection
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$
H5. $g(u[y]) = y$

T3. $\exists x.(x \in A \wedge g(f(x)) = y)$

4. Deleted H3, as this unexpandable atomic statement is unmatchable.

L1

$y \ u[y]$

H1. f from A to B is a surjection
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$
H5. $g(u[y]) = y$

T3. $\exists x.(x \in A \wedge g(f(x)) = y)$

5. Delete H2 as no other statement mentions C.

L1

$y \ u[y]$

H1. f from A to B is a surjection
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$
H5. $g(u[y]) = y$

T3. $\exists x.(x \in A \wedge g(f(x)) = y)$

6. Forwards reasoning using H1 with H4.

Since f from A to B is a surjection and $u \in B$, there exists $v \in A$ such that $f(v) = u$.

L1

$y \; u[y] \; v[u[y]]$

H1. f from A to B is a surjection [Vuln.; Used with H4.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$ [Vuln.]
H5. $g(u[y]) = y$
H6. $v[u[y]] \in A$
H7. $f(v[u[y]]) = u[y]$

T3. $\exists x.(x \in A \wedge g(f(x)) = y)$

7. Deleted H4, as this unexpandable atomic statement is unmatchable.

L1

$y \; u[y] \; v[u[y]]$

H1. f from A to B is a surjection [Vuln.; Used with H4.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$ [Vuln.]
H5. $g(u[y]) = y$
H6. $v[u[y]] \in A$
H7. $f(v[u[y]]) = u[y]$

T3. $\exists x.(x \in A \wedge g(f(x)) = y)$

8. Delete H1 as no other statement mentions B.

L1

$y \; u[y] \; v[u[y]]$

H1. f from A to B is a surjection [Vuln.; Used with H4.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$ [Vuln.]
H5. $g(u[y]) = y$
H6. $v[u[y]] \in A$
H7. $f(v[u[y]]) = u[y]$

T3. $\boldsymbol{\exists x.(x \in A \wedge g(f(x)) = y)}$

9. Unlock existential target T3.

We would like to find $x \in A$ s.t. $g(f(x)) = y$.

L1

$y \; u[y] \; v[u[y]]$

H1. f from A to B is a surjection [Vuln.; Used with H4.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$ [Vuln.]
H5. $g(u[y]) = y$
H6. $v[u[y]] \in A$
H7. $f(v[u[y]]) = u[y]$

L2$^{\blacklozenge}$

$x^{\blacklozenge}$

T4. $x^{\blacklozenge} \in A$
T5. $g(f(x^{\blacklozenge})) = y$

No moves possible.

If f is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of $f(A) \cap f(B)$. Then $x \in f(A)$ and $x \in f(B)$. That is, there exists $y \in A$ such that $f(y) = x$ and there exists $z \in B$ such that $f(z) = x$. Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$. We would like to find $u \in A \cap B$ s.t. $f(u) = x$. But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$. Since $y = z$, we have that $y \in B$. Therefore, setting $u = y$, we are done.

L1

H1. f is an injection

T1. $\boldsymbol{f(A) \cap f(B) \subset f(A \cap B)}$

1. Expand pre-universal target T1.

L1

H1. f is an injection

T2. $\boldsymbol{\forall x.(x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(A) \cap f(B)$.

L1

x

H1. f is an injection

H2. $\boldsymbol{x \in f(A) \cap f(B)}$

T3. $x \in f(A \cap B)$

3. Quantifier-free expansion of hypothesis H2.

Since $x \in f(A) \cap f(B)$, $x \in f(A)$ and $x \in f(B)$.

L1

x

H1. f is an injection

H3. $\boldsymbol{x \in f(A)}$

H4. $x \in f(B)$

T3. $x \in f(A \cap B)$

4. Expand pre-existential hypothesis H3.

By definition, since $x \in f(A)$, there exists $y \in A$ such that $f(y) = x$.

L1

$x \ y$

H1. f is an injection

H5. $y \in A$

H6. $f(y) = x$

H4. $\boldsymbol{x \in f(B)}$

T3. $x \in f(A \cap B)$

5. Expand pre-existential hypothesis H4.

By definition, since $x \in f(B)$, there exists $z \in B$ such that $f(z) = x$.

L1

$x \ y \ z$

H1. $\boldsymbol{f}$ **is an injection**

H5. $y \in A$

H6. $\boldsymbol{f(y) = x}$

H7. $z \in B$

H8. $\boldsymbol{f(z) = x}$

T3. $x \in f(A \cap B)$

6. Forwards reasoning using H1 with (H6,H8).

Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

T3. $\boldsymbol{x \in f(A \cap B)}$

7. Expand pre-existential target T3.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

T4. $\boldsymbol{\exists u.(u \in A \cap B \wedge f(u) = x)}$

8. Unlock existential target T4.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

L2$^{\blacklozenge}$

$u^{\blacklozenge}$

T5. $\boldsymbol{u^{\blacklozenge} \in A \cap B}$
T6. $f(u^{\blacklozenge}) = x$

9. Quantifier-free expansion of target T5.

But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $\boldsymbol{y = z}$

L2$^{\blacklozenge}$

$u^{\blacklozenge}$

T7. $u^{\blacklozenge} \in A$
T8. $u^{\blacklozenge} \in B$
T6. $f(u^{\blacklozenge}) = x$

10. Rewrite z as y throughout the tableau using hypothesis H9.

Since $y = z$, we have that $y \in B$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H10. $y \in B$

L2$^{\blacklozenge}$

$u^{\blacklozenge}$

T7. $u^{\blacklozenge} \in A$
T8. $u^{\blacklozenge} \in B$
T6. $f(u^{\blacklozenge}) = x$

11. Moved H6 down, as x can only be utilised by T6.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H10. $y \in B$

L2$^{\blacklozenge}$

$u^{\blacklozenge}$

T7. $u^{\blacklozenge} \in A$
T8. $u^{\blacklozenge} \in B$

L3

H6. $f(y) = x$ [Vuln.; From L1.]

T6. $f(u^{\blacklozenge}) = x$

12. Choosing $u^{\blacklozenge} = y$ matches all targets inside L2$^{\blacklozenge}$ against hypotheses, so L2$^{\blacklozenge}$ is done.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H10. $y \in B$

L2$^{\blacklozenge}$ Done

13. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.

Therefore, setting $u = y$, we are done.