

## If $f$ is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let  $x$  be an element of  $f(A) \cap f(B)$ . Then  $x \in f(A)$  and  $x \in f(B)$ . That is, there exists  $y \in A$  such that  $f(y) = x$  and there exists  $z \in B$  such that  $f(z) = x$ . Since  $f$  is an injection,  $f(y) = x$  and  $f(z) = x$ , we have that  $y = z$ . We would like to find  $u \in A \cap B$  s.t.  $f(u) = x$ . But  $u \in A \cap B$  if and only if  $u \in A$  and  $u \in B$ . Since  $y = z$ , we have that  $y \in B$ . Therefore, setting  $u = y$ , we are done.

L1  
H1.  $f$  is an injection  


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T1.  $f(A) \cap f(B) \subset f(A \cap B)$

1. Expand pre-universal target T1.

L1  
H1.  $f$  is an injection  


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T2.  $\forall x. (x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let  $x$  be an element of  $f(A) \cap f(B)$ .

L1  
 $x$   
H1.  $f$  is an injection  
H2.  $x \in f(A) \cap f(B)$   


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T3.  $x \in f(A \cap B)$

3. Quantifier-free expansion of hypothesis H2.

Since  $x \in f(A) \cap f(B)$ ,  
 $x \in f(A)$  and  $x \in f(B)$ .

L1  
 $x$   
H1.  $f$  is an injection  
H3.  $x \in f(A)$   
H4.  $x \in f(B)$   


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T3.  $x \in f(A \cap B)$

4. Expand pre-existential hypothesis H3.

By definition, since  $x \in f(A)$ , there exists  $y \in A$  such that  $f(y) = x$ .

L1  
 $x \ y$   
H1.  $f$  is an injection  
H5.  $y \in A$   
H6.  $f(y) = x$   
H4.  $x \in f(B)$   


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T3.  $x \in f(A \cap B)$

5. Expand pre-existential hypothesis H4.

By definition, since  $x \in f(B)$ , there exists  $z \in B$  such that  $f(z) = x$ .

L1  
 $x \ y \ z$   
H1.  $f$  is an injection  
H5.  $y \in A$   
H6.  $f(y) = x$   
H7.  $z \in B$   
H8.  $f(z) = x$   


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T3.  $x \in f(A \cap B)$

6. Forwards reasoning using H1 with (H6,H8).

Since  $f$  is an injection,  
 $f(y) = x$  and  $f(z) = x$ ,  
we have that  $y = z$ .

L1	$x \ y \ z$
H1. $f$ is an injection	[Vuln.; Used with (H6,H8).]
H5. $y \in A$	
H6. $f(y) = x$	[Vuln.]
H7. $z \in B$	
H8. $f(z) = x$	[Vuln.]
H9. $y = z$	
<hr/>	
<b>T3. <math>x \in f(A \cap B)</math></b>	

7. Expand pre-existential target T3.

We would like to find  $u \in A \cap B$  s.t.  $f(u) = x$ .

L1	$x \ y \ z$
H1. $f$ is an injection	[Vuln.; Used with (H6,H8).]
H5. $y \in A$	
H6. $f(y) = x$	[Vuln.]
H7. $z \in B$	
H8. $f(z) = x$	[Vuln.]
H9. $y = z$	
<hr/>	
<b>T4. <math>\exists u. (u \in A \cap B \wedge f(u) = x)</math></b>	

8. Unlock existential target T4.

We would like to find  $u \in A \cap B$  s.t.  $f(u) = x$ .

L1	$x \ y \ z$
H1. $f$ is an injection	[Vuln.; Used with (H6,H8).]
H5. $y \in A$	
H6. $f(y) = x$	[Vuln.]
H7. $z \in B$	
H8. $f(z) = x$	[Vuln.]
H9. $y = z$	
<hr/>	
L2♦	$u^\diamond$
<hr/>	
<b>T5. <math>u^\diamond \in A \cap B</math></b>	
T6. $f(u^\diamond) = x$	

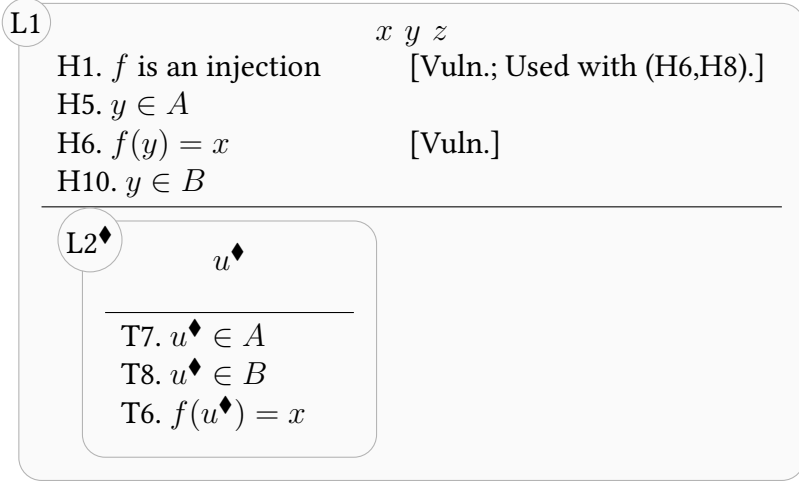
9. Quantifier-free expansion of target T5.

But  $u \in A \cap B$  if and only if  $u \in A$  and  $u \in B$ .

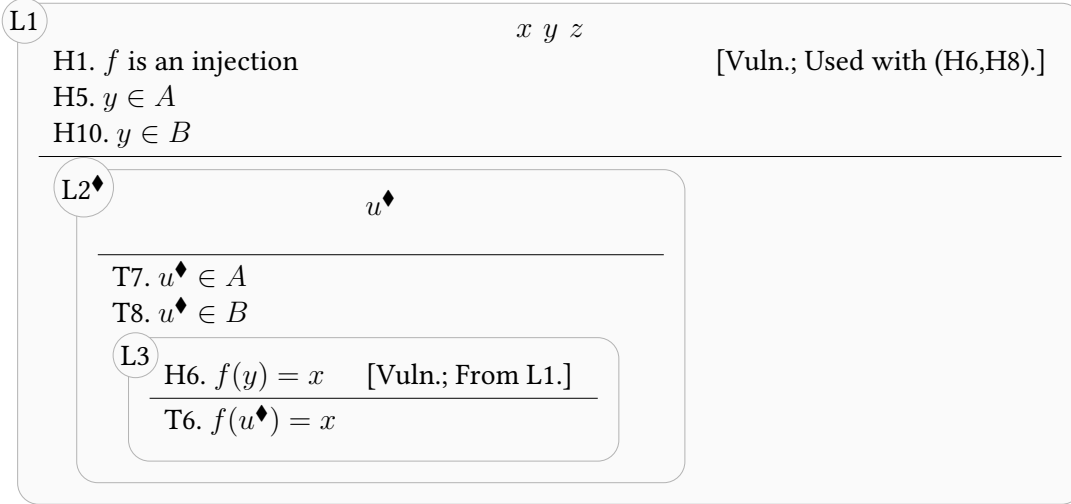
L1	$x \ y \ z$
H1. $f$ is an injection	[Vuln.; Used with (H6,H8).]
H5. $y \in A$	
H6. $f(y) = x$	[Vuln.]
H7. $z \in B$	
H8. $f(z) = x$	[Vuln.]
<b>H9. <math>y = z</math></b>	
<hr/>	
L2♦	$u^\diamond$
<hr/>	
T7. $u^\diamond \in A$	
T8. $u^\diamond \in B$	
T6. $f(u^\diamond) = x$	

10. Rewrite  $z$  as  $y$  throughout the tableau using hypothesis H9.

Since  $y = z$ , we have that  $y \in B$ .

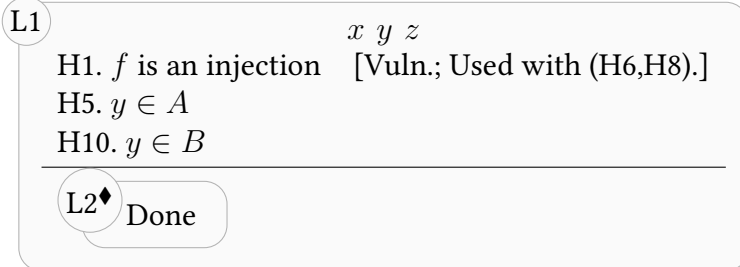


11. Moved H6 down, as  $x$  can only be utilised by T6.



12. Choosing  $u^\blacklozenge = y$  matches all targets inside L2 $\blacklozenge$  against hypotheses, so L2 $\blacklozenge$  is done.

Therefore, setting  $u = y$ , we are done.



13. All targets of L1 are 'Done', so L1 is itself done.



Problem solved.



## If $g, f$ are injections then $(g \circ f)$ is an injection.

Let  $x, y$  and  $z$  be such that  $g(f(x)) = z$  and  $g(f(y)) = z$ . Then, since  $g$  is an injection, we have that  $f(x) = f(y)$ . Therefore, since  $f$  is an injection,  $x = y$  if  $f(x) = f(y)$ . Since  $g$  is an injection and  $g(f(x)) = z, f(x) = f(y)$  if  $g(f(x)) = z$ . But this is clearly the case, so we are done.

L1  
H1.  $f$  is an injection  
H2.  $g$  is an injection  


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**T1.  $g \circ f$  is an injection**

1. Expand pre-universal target T1.

L1  
H1.  $f$  is an injection  
H2.  $g$  is an injection  


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**T2.  $\forall x, y, z. (g(f(x)) = z \wedge g(f(y)) = z \Rightarrow x = y)$**

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

L1  
 $x \ y \ z$   
H1.  $f$  is an injection  
**H2.  $g$  is an injection**  
**H3.  $g(f(x)) = z$**   
**H4.  $g(f(y)) = z$**   


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**T3.  $x = y$**

Let  $x, y$  and  $z$  be such that  $g(f(x)) = z$  and  $g(f(y)) = z$ .

3. Forwards reasoning using H2 with (H3,H4).

L1  
 $x \ y \ z$   
**H1.  $f$  is an injection**  
H2.  $g$  is an injection [Vuln.; Used with (H3,H4).]  
H3.  $g(f(x)) = z$  [Vuln.]  
H4.  $g(f(y)) = z$  [Vuln.]  
**H5.  $f(x) = f(y)$**   


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**T3.  $x = y$**

Since  $g$  is an injection,  $g(f(x)) = z$  and  $g(f(y)) = z$ , we have that  $f(x) = f(y)$ .

4. Backwards reasoning using H1 with (T3,H5).

L1  
 $x \ y \ z$   
H1.  $f$  is an injection [Vuln.]  
**H2.  $g$  is an injection** [Vuln.; Used with (H3,H4).]  
H3.  $g(f(x)) = z$  [Vuln.]  
**H4.  $g(f(y)) = z$**  [Vuln.]  
H5.  $f(x) = f(y)$  [Vuln.]  


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**T4.  $f(y) = f(y)$**

Since  $f$  is an injection and  $f(x) = f(y), x = y$  if  $f(x) = f(y)$ .

5. Backwards reasoning using H2 with (T4,H4).

Since  $g$  is an injection and  $g(f(y)) = z, f(y) = f(y)$  if  $g(f(y)) = z$ .

L1		$x$	$y$	$z$
	H1. $f$ is an injection	[Vuln.]		
	H2. $g$ is an injection	[Vuln.; Used with (H3,H4).]		
	H3. $g(f(x)) = z$	[Vuln.]		
	<b>H4. <math>g(f(y)) = z</math></b>	<b>[Vuln.]</b>		
	H5. $f(x) = f(y)$	[Vuln.]		
<hr/>				
	<b>T5. <math>g(f(y)) = z</math></b>			

6. Hypothesis H4 matches target T5, so L1 is done.

L1 Done

Problem solved.

But this is clearly the case,  
so we are done.

## Prove that $A \subseteq f^{-1}(f(A))$

Let  $x$  be an element of  $A$ . We would like to show that  $x \in f^{-1}(f(A))$ , i.e. that  $f(x) \in f(A)$ . But this is clearly the case, so we are done.

L1

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**T1.**  $A \subset f^{-1}(f(A))$

1. Expand pre-universal target T1.

L1

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**T2.**  $\forall x.(x \in A \Rightarrow x \in f^{-1}(f(A)))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let  $x$  be an element of  $A$ .

L1

$x$

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**H1.**  $x \in A$

**T3.**  $x \in f^{-1}(f(A))$

3. Quantifier-free expansion of target T3.

We would like to show that  $x \in f^{-1}(f(A))$ , i.e. that  $f(x) \in f(A)$ .

L1

$x$

---

**H1.**  $x \in A$

**T4.**  $f(x) \in f(A)$

4. All conjuncts of T4 (after expansion) can be simultaneously matched against H1 or rendered trivial by choosing  $y = x$ , so L1 is done.

We would like to show that  $f(x) \in f(A)$ . But this is clearly the case, so we are done.

L1

Done

Problem solved.





## Prove that $f(f^{-1}(A)) \subset A$

Let  $x$  be an element of  $f(f^{-1}(A))$ . Then there exists  $y \in f^{-1}(A)$  such that  $f(y) = x$ . Since  $y \in f^{-1}(A)$ , we have that  $f(y) \in A$ . Since  $f(y) = x$ , we have that  $x \in A$ . But this is clearly the case, so we are done.

L1

**T1.**  $f(f^{-1}(A)) \subset A$

1. Expand pre-universal target T1.

L1

**T2.**  $\forall x. (x \in f(f^{-1}(A)) \Rightarrow x \in A)$

2. Apply ‘let’ trick and move premise of universal-conditional target T2 above the line.

Let  $x$  be an element of  $f(f^{-1}(A))$ .

L1

**H1.**  $x \in f(f^{-1}(A))$   
**T3.**  $x \in A$

3. Expand pre-existential hypothesis H1.

By definition, since  $x \in f(f^{-1}(A))$ , there exists  $y \in f^{-1}(A)$  such that  $f(y) = x$ .

L1

**H2.**  $y \in f^{-1}(A)$   
**H3.**  $f(y) = x$   
**T3.**  $x \in A$

4. Quantifier-free expansion of hypothesis H2.

Since  $y \in f^{-1}(A)$ , we have that  $f(y) \in A$ .

L1

**H4.**  $f(y) \in A$   
**H3.**  $f(y) = x$   
**T3.**  $x \in A$

5. Rewrite  $f(y)$  as  $x$  throughout the tableau using hypothesis H3.

Since  $f(y) = x$ , we have that  $x \in A$ .

L1

**H5.**  $x \in A$   
**T3.**  $x \in A$

6. Hypothesis H5 matches target T3, so L1 is done.

But this is clearly the case, so we are done.

L1

Done

Problem solved.



## Prove that $f(A \cap B) \subset f(A) \cap f(B)$

By definition, since  $y \in f(A \cap B)$ , there exists  $z \in A \cap B$  such that  $f(z) = y$ . Since  $z \in A \cap B$ ,  $z \in A$  and  $z \in B$ . We would like to show that  $y \in f(A) \cap f(B)$ , i.e. that  $y \in f(A)$  and  $y \in f(B)$ . We would like to show that  $y \in f(A)$ . But this is clearly the case, so we are done. Thus  $y \in f(B)$  and we are done.

L1  

$$\frac{\text{H1. } y \in f(A \cap B)}{\text{T1. } y \in f(A) \cap f(B)}$$

1. Expand pre-existential hypothesis H1.

L1  

$$\frac{\begin{array}{l} \text{H2. } z \in A \cap B \\ \text{H3. } f(z) = y \end{array}}{\text{T1. } y \in f(A) \cap f(B)}$$

2. Quantifier-free expansion of hypothesis H2.

L1  

$$\frac{\begin{array}{l} \text{H4. } z \in A \\ \text{H5. } z \in B \\ \text{H3. } f(z) = y \end{array}}{\text{T1. } y \in f(A) \cap f(B)}$$

3. Quantifier-free expansion of target T1.

L1  

$$\frac{\begin{array}{l} \text{H4. } z \in A \\ \text{H5. } z \in B \\ \text{H3. } f(z) = y \end{array}}{\begin{array}{l} \text{T2. } y \in f(A) \\ \text{T3. } y \in f(B) \end{array}}$$

4. All conjuncts of T2 (after expansion) can be simultaneously matched against H4 and H3 or rendered trivial by choosing  $u = z$ , so we can remove T2.

L1  

$$\frac{\begin{array}{l} \text{H4. } z \in A \\ \text{H5. } z \in B \\ \text{H3. } f(z) = y \end{array}}{\text{T3. } y \in f(B)}$$

5. All conjuncts of T3 (after expansion) can be simultaneously matched against H5 and H3 or rendered trivial by choosing  $u = z$ , so L1 is done.

L1 Done

Problem solved.

By definition, since  $y \in f(A \cap B)$ , there exists  $z \in A \cap B$  such that  $f(z) = y$ .

Since  $z \in A \cap B$ ,  $z \in A$  and  $z \in B$ .

We would like to show that  $y \in f(A) \cap f(B)$ , i.e. that  $y \in f(A)$  and  $y \in f(B)$ .

We would like to show that  $y \in f(A)$ . But this is clearly the case, so we are done.

We would like to show that  $y \in f(B)$ . But this is clearly the case, so we are done.



**Prove that**  $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$

Since  $x \in f^{-1}(A \cap B)$ , we have that  $f(x) \in A \cap B$ . Then  $f(x) \in A$  and  $f(x) \in B$ . We would like to show that  $x \in f^{-1}(A) \cap f^{-1}(B)$ , i.e. that  $x \in f^{-1}(A)$  and  $x \in f^{-1}(B)$ . We would like to show that  $x \in f^{-1}(A)$ , i.e. that  $f(x) \in A$ . We would like to show that  $x \in f^{-1}(B)$ , i.e. that  $f(x) \in B$ . But this is clearly the case, so we are done.

L1  $\frac{\text{H1. } x \in f^{-1}(A \cap B)}{\text{T1. } x \in f^{-1}(A) \cap f^{-1}(B)}$

1. Quantifier-free expansion of hypothesis H1.

Since  $x \in f^{-1}(A \cap B)$ , we have that  $f(x) \in A \cap B$ .

L1  $\frac{\text{H2. } f(x) \in A \cap B}{\text{T1. } x \in f^{-1}(A) \cap f^{-1}(B)}$

2. Quantifier-free expansion of hypothesis H2.

Since  $f(x) \in A \cap B$ ,  $f(x) \in A$  and  $f(x) \in B$ .

L1  $\frac{\begin{array}{l} \text{H3. } f(x) \in A \\ \text{H4. } f(x) \in B \end{array}}{\text{T1. } x \in f^{-1}(A) \cap f^{-1}(B)}$

3. Quantifier-free expansion of target T1.

We would like to show that  $x \in f^{-1}(A) \cap f^{-1}(B)$ , i.e. that  $x \in f^{-1}(A)$  and  $x \in f^{-1}(B)$ .

L1  $\frac{\begin{array}{l} \text{H3. } f(x) \in A \\ \text{H4. } f(x) \in B \end{array}}{\begin{array}{l} \text{T2. } x \in f^{-1}(A) \\ \text{T3. } x \in f^{-1}(B) \end{array}}$

4. Quantifier-free expansion of target T2.

We would like to show that  $x \in f^{-1}(A)$ , i.e. that  $f(x) \in A$ .

L1  $\frac{\begin{array}{l} \text{H3. } f(x) \in A \\ \text{H4. } f(x) \in B \end{array}}{\begin{array}{l} \text{T4. } f(x) \in A \\ \text{T3. } x \in f^{-1}(B) \end{array}}$

5. Hypothesis H3 matches target T4, so we can remove T4.

L1  $\frac{\begin{array}{l} \text{H3. } f(x) \in A \\ \text{H4. } f(x) \in B \end{array}}{\text{T3. } x \in f^{-1}(B)}$

6. Quantifier-free expansion of target T3.

We would like to show that  $x \in f^{-1}(B)$ , i.e. that  $f(x) \in B$ .

L1  $\frac{\begin{array}{l} \text{H3. } f(x) \in A \\ \text{H4. } f(x) \in B \end{array}}{\text{T5. } f(x) \in B}$

7. Hypothesis H4 matches target T5, so L1 is done.

But this is clearly the case, so we are done.

L1 Done

Problem solved.



**Prove that  $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$**

Let  $x$  be an element of  $f^{-1}(A) \cap f^{-1}(B)$ . Then  $x \in f^{-1}(A)$  and  $x \in f^{-1}(B)$ . Then  $f(x) \in A$  and  $f(x) \in B$ . We would like to show that  $x \in f^{-1}(A \cap B)$ , i.e. that  $f(x) \in A \cap B$ . We would like to show that  $f(x) \in A \cap B$ , i.e. that  $f(x) \in A$  and  $f(x) \in B$ . But this is clearly the case, so we are done.

L1

**T1.  $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$**

1. Expand pre-universal target T1.

L1

**T2.  $\forall x.(x \in f^{-1}(A) \cap f^{-1}(B) \Rightarrow x \in f^{-1}(A \cap B))$**

2. Apply ‘let’ trick and move premise of universal-conditional target T2 above the line.

Let  $x$  be an element of  $f^{-1}(A) \cap f^{-1}(B)$ .

L1

**H1.  $x \in f^{-1}(A) \cap f^{-1}(B)$**   
**T3.  $x \in f^{-1}(A \cap B)$**

3. Quantifier-free expansion of hypothesis H1.

Since  $x \in f^{-1}(A) \cap f^{-1}(B)$ ,  $x \in f^{-1}(A)$  and  $x \in f^{-1}(B)$ .

L1

**H2.  $x \in f^{-1}(A)$**   
**H3.  $x \in f^{-1}(B)$**   
**T3.  $x \in f^{-1}(A \cap B)$**

4. Quantifier-free expansion of hypothesis H2.

Since  $x \in f^{-1}(A)$ , we have that  $f(x) \in A$ .

L1

**H4.  $f(x) \in A$**   
**H3.  $x \in f^{-1}(B)$**   
**T3.  $x \in f^{-1}(A \cap B)$**

5. Quantifier-free expansion of hypothesis H3.

Since  $x \in f^{-1}(B)$ , we have that  $f(x) \in B$ .

L1

**H4.  $f(x) \in A$**   
**H5.  $f(x) \in B$**   
**T3.  $x \in f^{-1}(A \cap B)$**

6. Quantifier-free expansion of target T3.

We would like to show that  $x \in f^{-1}(A \cap B)$ , i.e. that  $f(x) \in A \cap B$ .

L1

**H4.  $f(x) \in A$**   
**H5.  $f(x) \in B$**   
**T4.  $f(x) \in A \cap B$**

7. Quantifier-free expansion of target T4.

We would like to show that  $f(x) \in A \cap B$ , i.e. that  $f(x) \in A$  and  $f(x) \in B$ .

L1

**H4.  $f(x) \in A$**   
**H5.  $f(x) \in B$**   
**T5.  $f(x) \in A$**   
**T6.  $f(x) \in B$**

8. Hypothesis H4 matches target T5, so we can remove T5.

L1

$$\begin{array}{l} \text{H4. } f(x) \in A \\ \text{H5. } f(x) \in B \\ \hline \text{T6. } f(x) \in B \end{array}$$

9. Hypothesis H5 matches target T6, so L1 is done.

L1 Done

Problem solved.

But this is clearly the case,  
so we are done.



**Prove that  $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$**

Let  $x$  be an element of  $f^{-1}(A \cup B)$ . Then  $f(x) \in A \cup B$ . Then  $f(x) \in A$  or  $f(x) \in B$ . We would like to show that  $x \in f^{-1}(A) \cup f^{-1}(B)$ , i.e. that  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$ . We would like to show that  $x \in f^{-1}(A)$ , i.e. that  $f(x) \in A$ . But this is clearly the case, so we are done. We would like to show that  $x \in f^{-1}(A) \cup f^{-1}(B)$ , i.e. that  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$ . We would like to show that  $x \in f^{-1}(A)$ , i.e. that  $f(x) \in A$ . We would like to show that  $x \in f^{-1}(B)$ , i.e. that  $f(x) \in B$ . But this is clearly the case, so we are done.

L1

$$\text{T1. } f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$$

1. Expand pre-universal target T1.

L1

$$\text{T2. } \forall x. (x \in f^{-1}(A \cup B) \Rightarrow x \in f^{-1}(A) \cup f^{-1}(B))$$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let  $x$  be an element of  $f^{-1}(A \cup B)$ .

L1

$$\begin{array}{l} \text{H1. } x \in f^{-1}(A \cup B) \\ \hline \text{T3. } x \in f^{-1}(A) \cup f^{-1}(B) \end{array}$$

3. Quantifier-free expansion of hypothesis H1.

Since  $x \in f^{-1}(A \cup B)$ , we have that  $f(x) \in A \cup B$ .

L1

$$\begin{array}{l} \text{H2. } f(x) \in A \cup B \\ \hline \text{T3. } x \in f^{-1}(A) \cup f^{-1}(B) \end{array}$$

4. Quantifier-free expansion of hypothesis H2.

Since  $f(x) \in A \cup B$ ,  $f(x) \in A$  or  $f(x) \in B$ .

L1

$$\begin{array}{l} \text{H3. } f(x) \in A \vee f(x) \in B \\ \hline \text{T3. } x \in f^{-1}(A) \cup f^{-1}(B) \end{array}$$

5. Split into cases to handle disjunctive hypothesis H3.

L1

$x$

L2

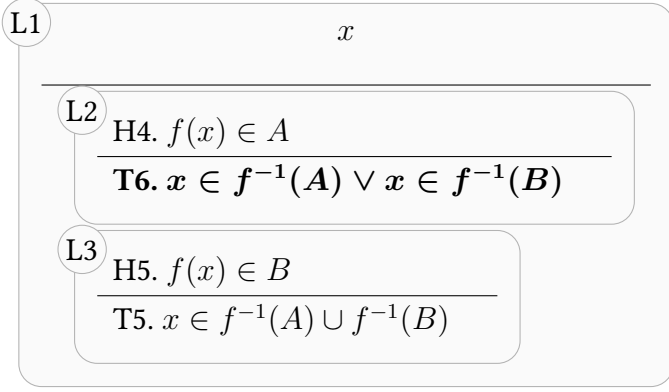
$$\begin{array}{l} \text{H4. } f(x) \in A \\ \hline \text{T4. } x \in f^{-1}(A) \cup f^{-1}(B) \end{array}$$

L3

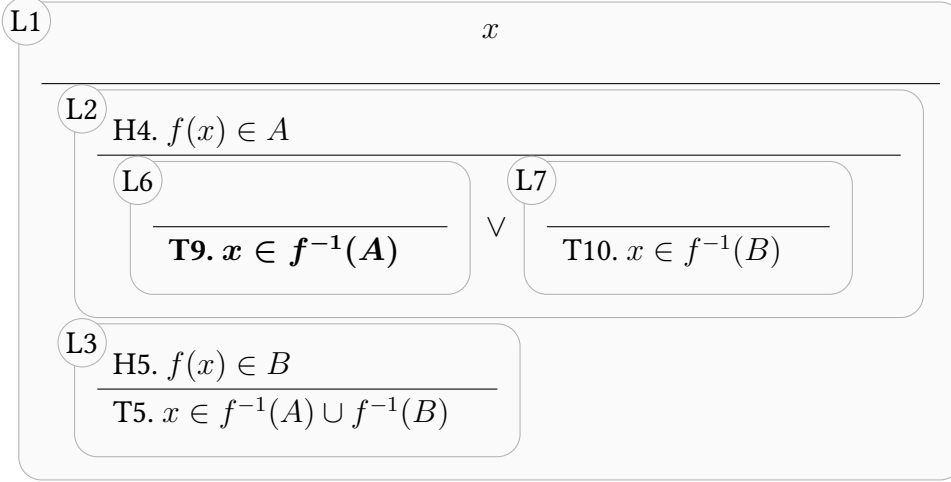
$$\begin{array}{l} \text{H5. } f(x) \in B \\ \hline \text{T5. } x \in f^{-1}(A) \cup f^{-1}(B) \end{array}$$

6. Quantifier-free expansion of target T4.

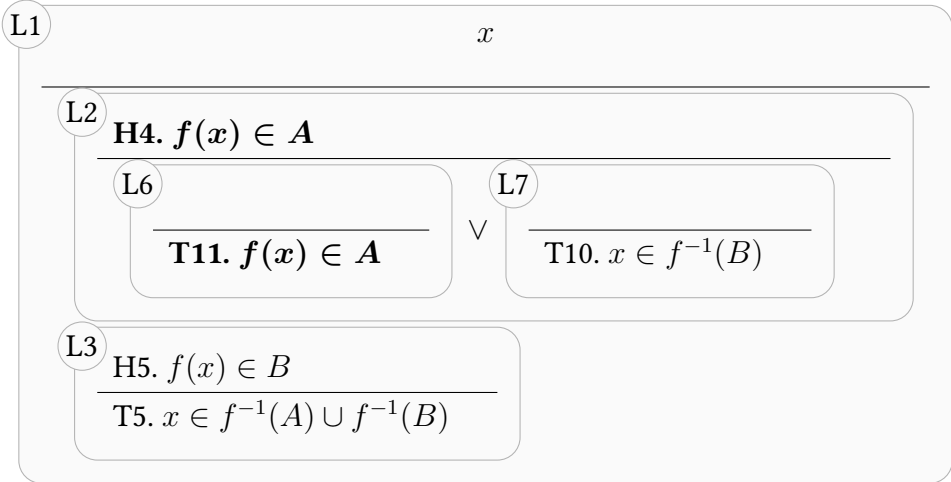
We would like to show that  $x \in f^{-1}(A) \cup f^{-1}(B)$ , i.e. that  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$ .



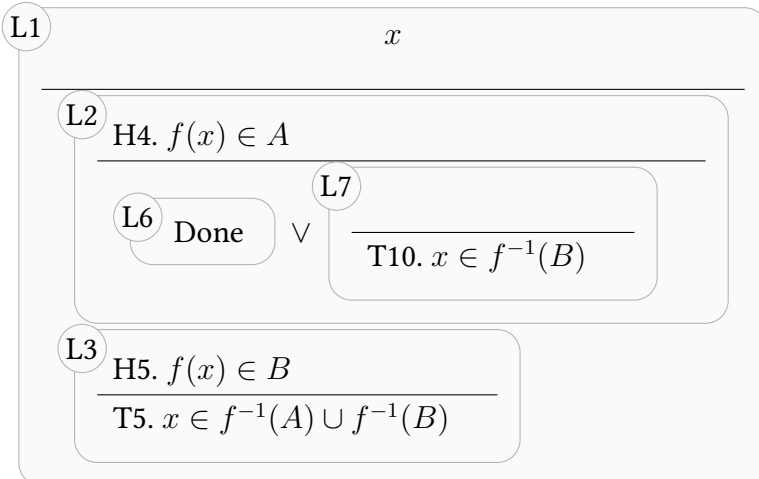
7. Split up disjunctive target T6.



8. Quantifier-free expansion of target T9.



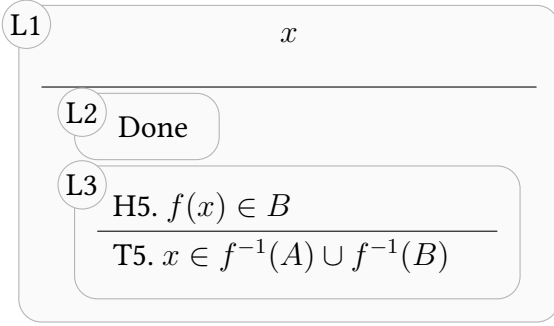
9. Hypothesis H4 matches target T11, so L6 is done.



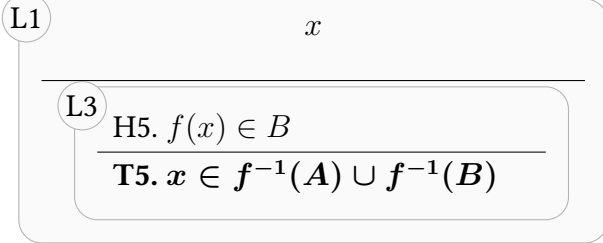
10. Some disjunct of the target of L2 is 'Done', so L2 is itself 'Done'.

We would like to show that  $x \in f^{-1}(A)$ , i.e. that  $f(x) \in A$ .

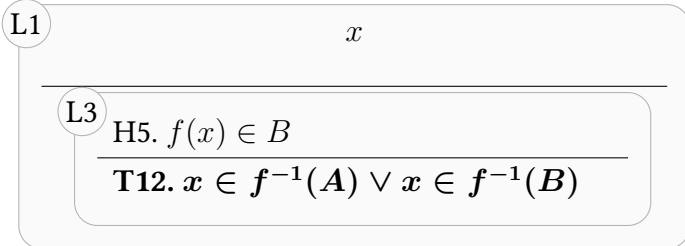
But this is clearly the case, so we are done.



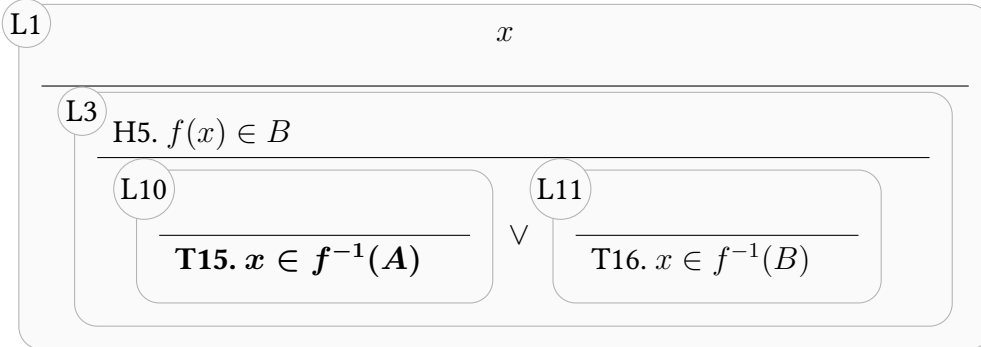
11. Remove 'Done' targets of L1.



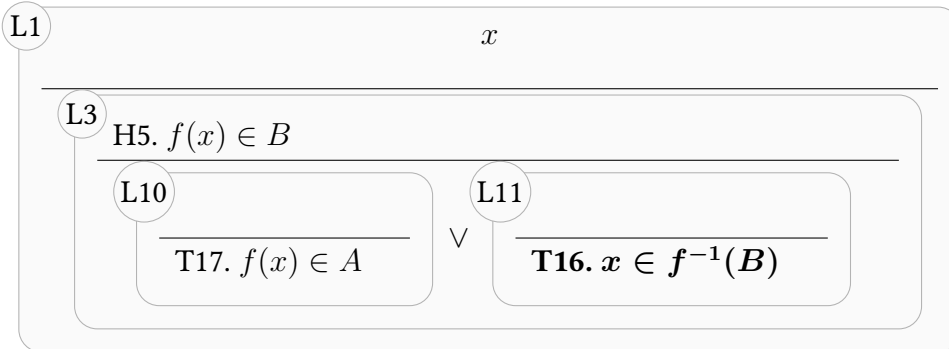
12. Quantifier-free expansion of target T5.



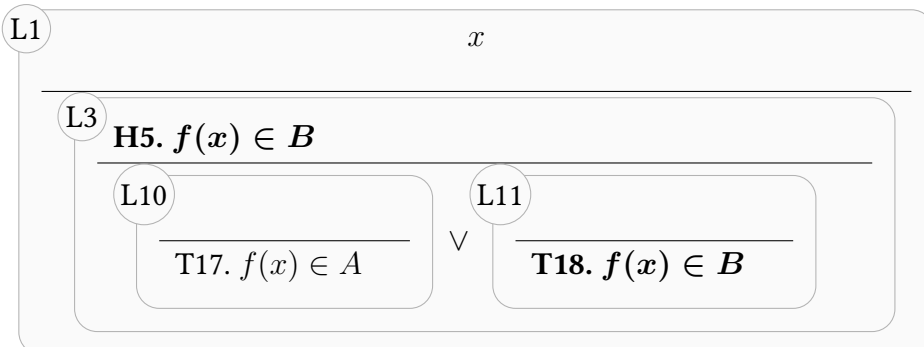
13. Split up disjunctive target T12.



14. Quantifier-free expansion of target T15.



15. Quantifier-free expansion of target T16.



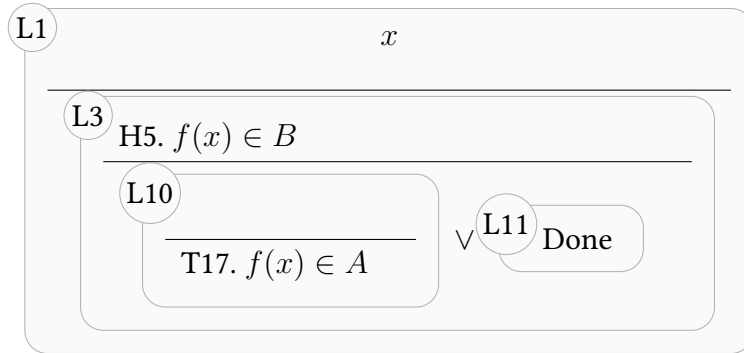
We would like to show that  $x \in f^{-1}(A) \cup f^{-1}(B)$ , i.e. that  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$ .

We would like to show that  $x \in f^{-1}(A)$ , i.e. that  $f(x) \in A$ .

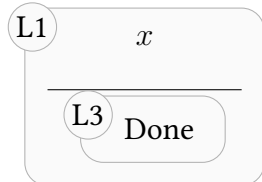
We would like to show that  $x \in f^{-1}(B)$ , i.e. that  $f(x) \in B$ .

16. Hypothesis H5 matches target T18, so L11 is done.

But this is clearly the case,  
so we are done.



17. Some disjunct of the target of L3 is 'Done', so L3 is itself 'Done'.



18. All targets of L1 are 'Done', so L1 is itself done.



Problem solved.

**Prove that  $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$**

Let  $x$  be an element of  $f^{-1}(A) \cup f^{-1}(B)$ . Then  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$ . Since  $x \in f^{-1}(A)$ , we have that  $f(x) \in A$ . Since  $x \in f^{-1}(B)$ , we have that  $f(x) \in B$ . We would like to show that  $x \in f^{-1}(A \cup B)$ , i.e. that  $f(x) \in A \cup B$ . We would like to show that  $f(x) \in A \cup B$ , i.e. that  $f(x) \in A$  or  $f(x) \in B$ . But this is clearly the case, so we are done. We would like to show that  $x \in f^{-1}(A \cup B)$ , i.e. that  $f(x) \in A \cup B$ . We would like to show that  $f(x) \in A \cup B$ , i.e. that  $f(x) \in A$  or  $f(x) \in B$ . But this is clearly the case, so we are done.

L1

**T1.**  $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$

1. Expand pre-universal target T1.

L1

**T2.**  $\forall x. (x \in f^{-1}(A) \cup f^{-1}(B) \Rightarrow x \in f^{-1}(A \cup B))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let  $x$  be an element of  $f^{-1}(A) \cup f^{-1}(B)$ .

L1

**H1.**  $x \in f^{-1}(A) \cup f^{-1}(B)$   
**T3.**  $x \in f^{-1}(A \cup B)$

3. Quantifier-free expansion of hypothesis H1.

Since  $x \in f^{-1}(A) \cup f^{-1}(B)$ ,  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$ .

L1

**H2.**  $x \in f^{-1}(A) \vee x \in f^{-1}(B)$   
**T3.**  $x \in f^{-1}(A \cup B)$

4. Split into cases to handle disjunctive hypothesis H2.

L1

$x$

L2

**H3.**  $x \in f^{-1}(A)$

**T4.**  $x \in f^{-1}(A \cup B)$

L3

**H4.**  $x \in f^{-1}(B)$

**T5.**  $x \in f^{-1}(A \cup B)$

5. Quantifier-free expansion of hypothesis H3.

Since  $x \in f^{-1}(A)$ , we have that  $f(x) \in A$ .

L1

$x$

L2

**H5.**  $f(x) \in A$

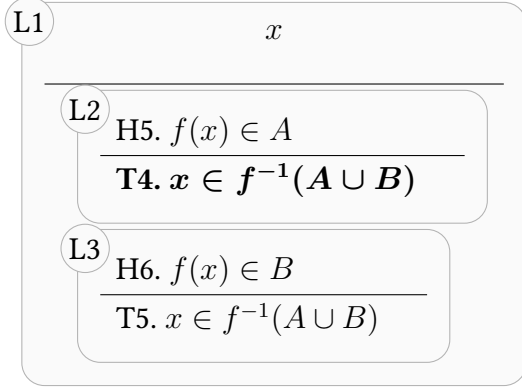
**T4.**  $x \in f^{-1}(A \cup B)$

L3

**H4.**  $x \in f^{-1}(B)$

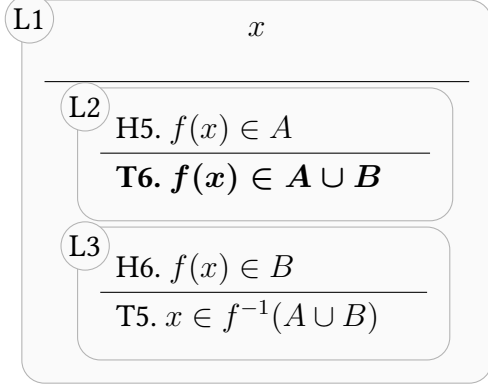
**T5.**  $x \in f^{-1}(A \cup B)$

6. Quantifier-free expansion of hypothesis H4.



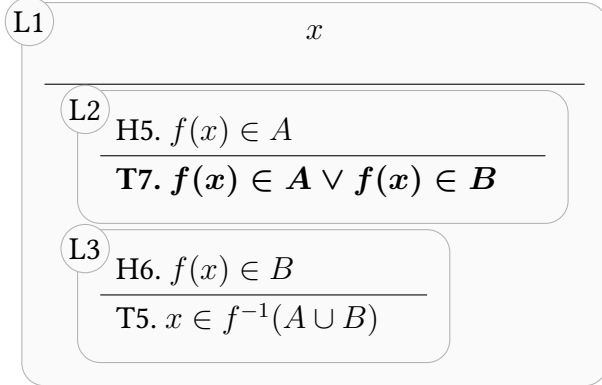
Since  $x \in f^{-1}(B)$ , we have that  $f(x) \in B$ .

7. Quantifier-free expansion of target T4.



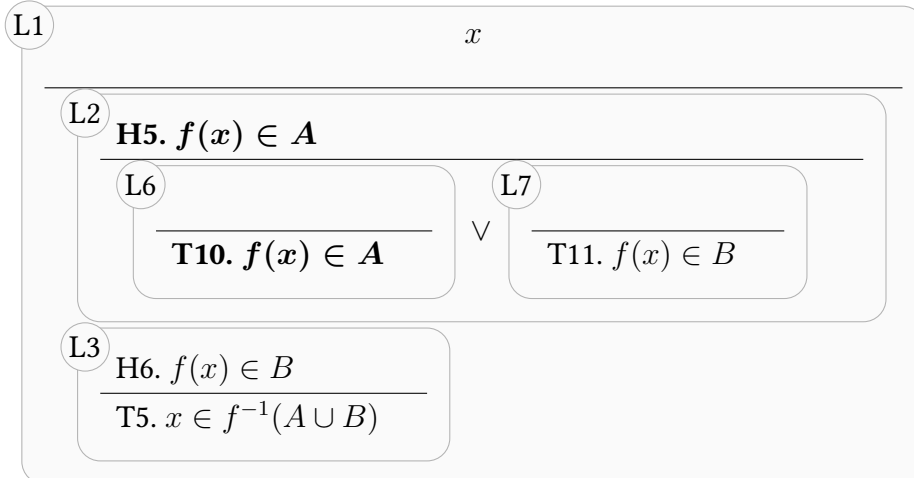
We would like to show that  $x \in f^{-1}(A \cup B)$ , i.e. that  $f(x) \in A \cup B$ .

8. Quantifier-free expansion of target T6.



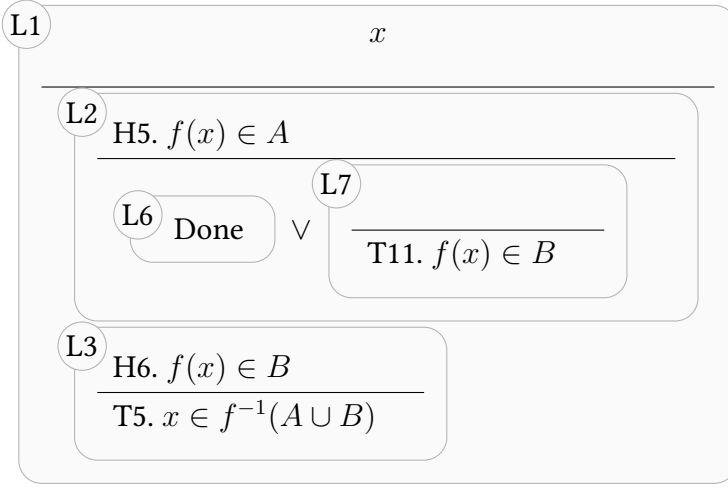
We would like to show that  $f(x) \in A \cup B$ , i.e. that  $f(x) \in A$  or  $f(x) \in B$ .

9. Split up disjunctive target T7.

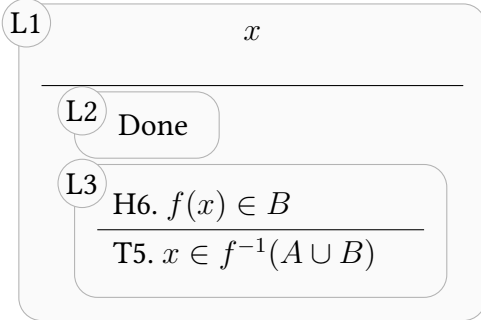


10. Hypothesis H5 matches target T10, so L6 is done.

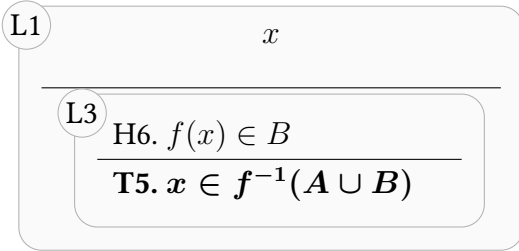
But this is clearly the case, so we are done.



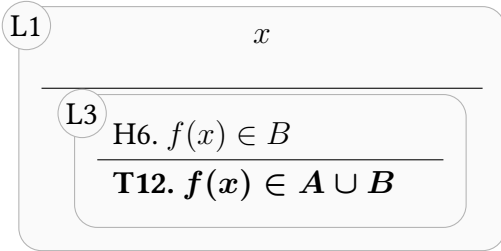
11. Some disjunct of the target of L2 is 'Done', so L2 is itself 'Done'.



12. Remove 'Done' targets of L1.

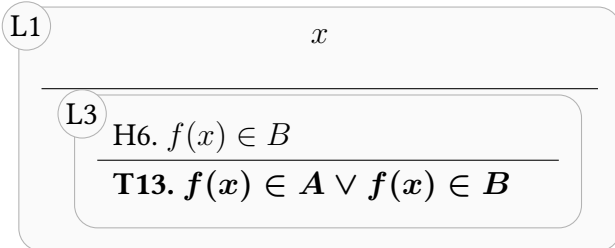


13. Quantifier-free expansion of target T5.



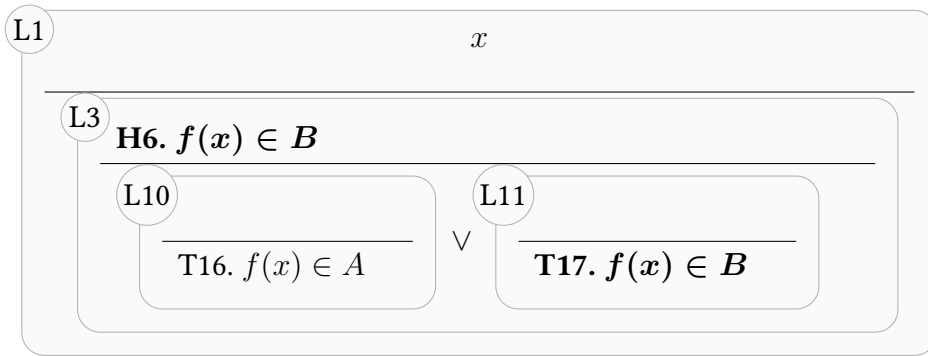
We would like to show that  $x \in f^{-1}(A \cup B)$ , i.e. that  $f(x) \in A \cup B$ .

14. Quantifier-free expansion of target T12.



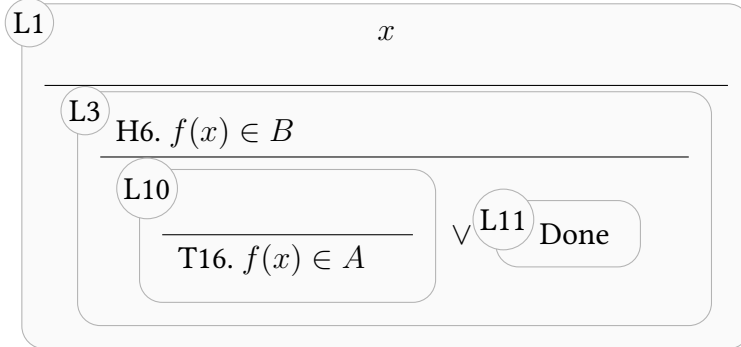
We would like to show that  $f(x) \in A \cup B$ , i.e. that  $f(x) \in A$  or  $f(x) \in B$ .

15. Split up disjunctive target T13.

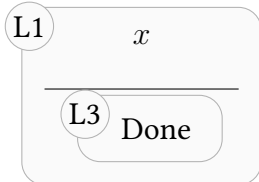


16. Hypothesis H6 matches target T17, so L11 is done.

But this is clearly the case, so we are done.



17. Some disjunct of the target of L3 is 'Done', so L3 is itself 'Done'.



18. All targets of L1 are 'Done', so L1 is itself done.



Problem solved.



**If  $A, B,$  and  $C$  are open sets, then  $A \cup (B \cup C)$  is also open.**

Let  $x$  be an element of  $A \cup B \cup C$ . Then  $x \in A$  or  $x \in B \cup C$ . Since  $A$  is open and  $x \in A$ , there exists  $\alpha > 0$  such that  $w \in A$  whenever  $d(x, w) < \alpha$ . Since  $x \in B \cup C$ ,  $x \in B$  or  $x \in C$ . Since  $B$  is open and  $x \in B$ , there exists  $\delta' > 0$  such that  $r \in B$  whenever  $d(x, r) < \delta'$ . Since  $C$  is open and  $x \in C$ , there exists  $\delta'' > 0$  such that  $s \in C$  whenever  $d(x, s) < \delta''$ . We would like to find  $\eta > 0$  s.t.  $z \in A \cup B \cup C$  whenever  $d(x, z) < \eta$ . But  $z \in A \cup B \cup C$  if and only if  $z \in A$  or  $z \in B \cup C$ . We know that  $z \in A$  if  $d(x, z) < \alpha$ . Therefore, setting  $\eta = \alpha$ , we are done. We would like to find  $\beta > 0$  s.t.  $v \in A \cup B \cup C$  whenever  $d(x, v) < \beta$ . But  $v \in A \cup B \cup C$  if and only if  $v \in A$  or  $v \in B \cup C$ . We would like to show that  $v \in B \cup C$ , i.e. that  $v \in B$  or  $v \in C$ . We know that  $v \in B$  if  $d(x, v) < \delta'$ . Therefore, setting  $\beta = \delta'$ , we are done. We would like to find  $\gamma > 0$  s.t.  $p \in A \cup B \cup C$  whenever  $d(x, p) < \gamma$ . But  $p \in A \cup B \cup C$  if and only if  $p \in A$  or  $p \in B \cup C$ . We would like to show that  $p \in B \cup C$ , i.e. that  $p \in B$  or  $p \in C$ . We know that  $p \in C$  if  $d(x, p) < \delta''$ . Therefore, setting  $\gamma = \delta''$ , we are done.

L1

H1.  $A$  is open  
H2.  $B$  is open  
H3.  $C$  is open

**T1.  $A \cup B \cup C$  is open**

1. Expand pre-universal target T1.

L1

H1.  $A$  is open  
H2.  $B$  is open  
H3.  $C$  is open

**T2.  $\forall x. (x \in A \cup B \cup C \Rightarrow \exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cup B \cup C)))$**

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let  $x$  be an element of  $A \cup B \cup C$ .

L1

$x$

H1.  $A$  is open  
H2.  $B$  is open  
H3.  $C$  is open  
**H4.  $x \in A \cup B \cup C$**

**T3.  $\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cup B \cup C))$**

3. Quantifier-free expansion of hypothesis H4.

Since  $x \in A \cup B \cup C$ ,  $x \in A$  or  $x \in B \cup C$ .

L1

$x$

H1.  $A$  is open  
H2.  $B$  is open  
H3.  $C$  is open  
**H5.  $x \in A \vee x \in B \cup C$**

**T3.  $\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cup B \cup C))$**

4. Split into cases to handle disjunctive hypothesis H5.

L1	$x$
H1. <b>A is open</b> H2. <i>B</i> is open H3. <i>C</i> is open	
L2	<b>H6. <math>x \in A</math></b> T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$
L3	H7. $x \in B \cup C$ T5. $\exists \theta. (\forall u. (d(x, u) < \theta \Rightarrow u \in A \cup B \cup C))$

5. Forwards reasoning using H1 with H6.

L1	$x$	[Vuln.; Used with H6.]
H1. <i>A</i> is open H2. <i>B</i> is open H3. <i>C</i> is open		
L2	$\alpha[x]$	[Vuln.]
H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$ T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$		
L3	H7. $x \in B \cup C$ T5. $\exists \theta. (\forall u. (d(x, u) < \theta \Rightarrow u \in A \cup B \cup C))$	

Since *A* is open and  $x \in A$ , there exists  $\alpha > 0$  such that  $w \in A$  whenever  $d(x, w) < \alpha$ .

6. Deleted H6, as this unexpandable atomic statement is unmatchable.

L1	$x$	[Vuln.; Used with H6.]
H1. <i>A</i> is open H2. <i>B</i> is open H3. <i>C</i> is open		
L2	$\alpha[x]$	[Vuln.]
H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$ T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$		
L3	H7. $x \in B \cup C$ T5. $\exists \theta. (\forall u. (d(x, u) < \theta \Rightarrow u \in A \cup B \cup C))$	

7. Deleted H1, as the conclusion of this implicative statement is unmatchable.

L1	$x$	[Vuln.; Used with H6.]
H1. <i>A</i> is open H2. <i>B</i> is open H3. <i>C</i> is open		
L2	$\alpha[x]$	[Vuln.]
H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$ T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$		
L3	<b>H7. <math>x \in B \cup C</math></b> T5. $\exists \theta. (\forall u. (d(x, u) < \theta \Rightarrow u \in A \cup B \cup C))$	

8. Quantifier-free expansion of hypothesis H7.

Since  $x \in B \cup C$ ,  $x \in B$  or  $x \in C$ .

L1	$x$	[Vuln.; Used with H6.]
	H1. $A$ is open H2. $B$ is open H3. $C$ is open	
L2	$\alpha[x]$	[Vuln.]
	H6. $x \in A$ H8. $\forall w.(d(x, w) < \alpha[x] \Rightarrow w \in A)$ T4. $\exists \eta.(\forall z.(d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$	
L3	<b>H9. <math>x \in B \vee x \in C</math></b>	
	T5. $\exists \theta.(\forall u.(d(x, u) < \theta \Rightarrow u \in A \cup B \cup C))$	

9. Split into cases to handle disjunctive hypothesis H9.

L1	$x$	[Vuln.; Used with H6.]
	H1. $A$ is open H2. $B$ is open H3. $C$ is open	
L2	$\alpha[x]$	[Vuln.]
	H6. $x \in A$ H8. $\forall w.(d(x, w) < \alpha[x] \Rightarrow w \in A)$ T4. $\exists \eta.(\forall z.(d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$	
L3		
L4	H10. $x \in B$	
	T6. $\exists \beta.(\forall v.(d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$	
L5	H11. $x \in C$	
	T7. $\exists \gamma.(\forall p.(d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$	

10. Collapsed subtableau L3 as it has no undeleted hypotheses.

L1	$x$	[Vuln.; Used with H6.]
	H1. $A$ is open <b>H2. <math>B</math> is open</b> H3. $C$ is open	
L2	$\alpha[x]$	[Vuln.]
	H6. $x \in A$ H8. $\forall w.(d(x, w) < \alpha[x] \Rightarrow w \in A)$ T4. $\exists \eta.(\forall z.(d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$	
L4	<b>H10. <math>x \in B</math></b>	
	T6. $\exists \beta.(\forall v.(d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$	
L5	H11. $x \in C$	
	T7. $\exists \gamma.(\forall p.(d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$	

11. Forwards reasoning using H2 with H10.

Since  $B$  is open and  $x \in B$ , there exists  $\delta' > 0$  such that  $r \in B$  whenever  $d(x, r) < \delta'$ .

L1	$x$	[Vuln.; Used with H6.] [Vuln.; Used with H10.]
H1. $A$ is open H2. $B$ is open H3. $C$ is open		
L2	$\alpha[x]$	[Vuln.]
H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$ T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$		
L4	$\delta'[x]$	[Vuln.]
H10. $x \in B$ H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$ T6. $\exists \beta. (\forall v. (d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$		
L5	$x \in C$	
T7. $\exists \gamma. (\forall p. (d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$		

12. Deleted H10, as this unexpandable atomic statement is unmatchable.

L1	$x$	[Vuln.; Used with H6.] [Vuln.; Used with H10.]
H1. $A$ is open H2. $B$ is open H3. $C$ is open		
L2	$\alpha[x]$	[Vuln.]
H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$ T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$		
L4	$\delta'[x]$	[Vuln.]
H10. $x \in B$ H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$ T6. $\exists \beta. (\forall v. (d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$		
L5	$x \in C$	
T7. $\exists \gamma. (\forall p. (d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$		

13. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1	$x$	[Vuln.; Used with H6.] [Vuln.; Used with H10.]
H1. $A$ is open H2. $B$ is open <b>H3. <math>C</math> is open</b>		
L2	$\alpha[x]$	[Vuln.]
H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$ T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$		
L4	$\delta'[x]$	[Vuln.]
H10. $x \in B$ H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$ T6. $\exists \beta. (\forall v. (d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$		
L5	<b>H11. <math>x \in C</math></b>	
T7. $\exists \gamma. (\forall p. (d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$		

14. Forwards reasoning using H3 with H11.

Since  $C$  is open and  $x \in C$ , there exists  $\delta'' > 0$  such that  $s \in C$  whenever

L1	$x$		
	H1. $A$ is open		[Vuln.; Used with H6.]
	H2. $B$ is open		[Vuln.; Used with H10.]
	H3. $C$ is open		[Vuln.; Used with H11.]
L2	$\alpha[x]$		
	H6. $x \in A$		[Vuln.]
	H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$		
	T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$		
L4	$\delta'[x]$		
	H10. $x \in B$		[Vuln.]
	H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$		
	T6. $\exists \beta. (\forall v. (d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$		
L5	$\delta''[x]$		
	H11. $x \in C$		[Vuln.]
	H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$		
	T7. $\exists \gamma. (\forall p. (d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$		

15. Deleted H11, as this unexpandable atomic statement is unmatchable.

L1	$x$		
	H1. $A$ is open		[Vuln.; Used with H6.]
	H2. $B$ is open		[Vuln.; Used with H10.]
	H3. $C$ is open		[Vuln.; Used with H11.]
L2	$\alpha[x]$		
	H6. $x \in A$		[Vuln.]
	H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$		
	T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$		
L4	$\delta'[x]$		
	H10. $x \in B$		[Vuln.]
	H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$		
	T6. $\exists \beta. (\forall v. (d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$		
L5	$\delta''[x]$		
	H11. $x \in C$		[Vuln.]
	H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$		
	T7. $\exists \gamma. (\forall p. (d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$		

16. Deleted H3, as the conclusion of this implicative statement is unmatchable.

L1	$x$	
	H1. $A$ is open H2. $B$ is open H3. $C$ is open	[Vuln.; Used with H6.] [Vuln.; Used with H10.] [Vuln.; Used with H11.]
L2	$\alpha[x]$	[Vuln.]
	H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$	
	<b>T4. <math>\exists \eta. (\forall z. (d(x, z) &lt; \eta \Rightarrow z \in A \cup B \cup C))</math></b>	
L4	$\delta'[x]$	[Vuln.]
	H10. $x \in B$ H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$	
	<b>T6. <math>\exists \beta. (\forall v. (d(x, v) &lt; \beta \Rightarrow v \in A \cup B \cup C))</math></b>	
L5	$\delta''[x]$	[Vuln.]
	H11. $x \in C$ H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$	
	<b>T7. <math>\exists \gamma. (\forall p. (d(x, p) &lt; \gamma \Rightarrow p \in A \cup B \cup C))</math></b>	

17. Unlock existential-universal-conditional target T4.

We would like to find  $\eta > 0$  s.t.  $z \in A \cup B \cup C$  whenever  $d(x, z) < \eta$ .

L1	$x$	
	H1. $A$ is open H2. $B$ is open H3. $C$ is open	[Vuln.; Used with H6.] [Vuln.; Used with H10.] [Vuln.; Used with H11.]
L2	$\alpha[x]$	[Vuln.]
	H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$	
L6 $\blacklozenge$	$\eta^\blacklozenge[\bar{z}] z$	[From L2.]
	H14. $d(x, z) < \eta^\blacklozenge[\bar{z}]$	
	<b>T8. <math>z \in A \cup B \cup C</math></b>	
L4	$\delta'[x]$	[Vuln.]
	H10. $x \in B$ H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$	
	<b>T6. <math>\exists \beta. (\forall v. (d(x, v) &lt; \beta \Rightarrow v \in A \cup B \cup C))</math></b>	
L5	$\delta''[x]$	[Vuln.]
	H11. $x \in C$ H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$	
	<b>T7. <math>\exists \gamma. (\forall p. (d(x, p) &lt; \gamma \Rightarrow p \in A \cup B \cup C))</math></b>	

18. Quantifier-free expansion of target T8.

But  $z \in A \cup B \cup C$  if and only if  $z \in A$  or  $z \in B \cup C$ .

L1	$x$ H1. $A$ is open H2. $B$ is open H3. $C$ is open	[Vuln.; Used with H6.] [Vuln.; Used with H10.] [Vuln.; Used with H11.]
L2	$\alpha[x]$ H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$	[Vuln.]
L6♦	$\eta^\diamond[\bar{z}] \ z$ H14. $d(x, z) < \eta^\diamond[\bar{z}]$ [From L2.] <b>T9. <math>z \in A \vee z \in B \cup C</math></b>	
L4	$\delta'[x]$ H10. $x \in B$ H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$ <b>T6. <math>\exists \beta. (\forall v. (d(x, v) &lt; \beta \Rightarrow v \in A \cup B \cup C))</math></b>	[Vuln.]
L5	$\delta''[x]$ H11. $x \in C$ H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$ <b>T7. <math>\exists \gamma. (\forall p. (d(x, p) &lt; \gamma \Rightarrow p \in A \cup B \cup C))</math></b>	[Vuln.]

### 19. Split up disjunctive target T9.

L1	$x$ H1. $A$ is open H2. $B$ is open H3. $C$ is open	[Vuln.; Used with H6.] [Vuln.; Used with H10.] [Vuln.; Used with H11.]
L2	$\alpha[x]$ H6. $x \in A$ <b>H8. <math>\forall w. (d(x, w) &lt; \alpha[x] \Rightarrow w \in A)</math></b>	[Vuln.]
L6♦	$\eta^\diamond[\bar{z}] \ z$ H14. $d(x, z) < \eta^\diamond[\bar{z}]$ [From L2.] <div style="display: flex; justify-content: space-around;"> <div style="border: 1px solid black; padding: 5px;"> L9  <b>T12. <math>z \in A</math></b> </div> <div style="text-align: center;"> <math>\vee</math> </div> <div style="border: 1px solid black; padding: 5px;"> L10  <b>T13. <math>z \in B \cup C</math></b> </div> </div>	
L4	$\delta'[x]$ H10. $x \in B$ H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$ <b>T6. <math>\exists \beta. (\forall v. (d(x, v) &lt; \beta \Rightarrow v \in A \cup B \cup C))</math></b>	[Vuln.]
L5	$\delta''[x]$ H11. $x \in C$ H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$ <b>T7. <math>\exists \gamma. (\forall p. (d(x, p) &lt; \gamma \Rightarrow p \in A \cup B \cup C))</math></b>	[Vuln.]

### 20. Backwards reasoning using H8 with T12.

We know that  $z \in A$  if  $d(x, z) < \alpha$ .

L1	$x$ H1. $A$ is open H2. $B$ is open H3. $C$ is open <div style="text-align: right;"> [Vuln.; Used with H6.]  [Vuln.; Used with H10.]  [Vuln.; Used with H11.] </div>
L2	$\alpha[x]$ H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$ <div style="text-align: right;"> [Vuln.]  [Vuln.] </div>
L6 $\blacklozenge$	$\eta^\blacklozenge[\bar{z}] \ z$ H14. $d(x, z) < \eta^\blacklozenge[\bar{z}]$ [From L2.] <div style="display: flex; justify-content: space-around;"> <div> L9  <hr/> T14. <math>d(x, z) &lt; \alpha[x]</math> </div> <div> L10  <hr/> T13. <math>z \in B \cup C</math> </div> </div>
L4	$\delta'[x]$ H10. $x \in B$ [Vuln.] H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$ <hr/> T6. $\exists \beta. (\forall v. (d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$
L5	$\delta''[x]$ H11. $x \in C$ [Vuln.] H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$ <hr/> T7. $\exists \gamma. (\forall p. (d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$

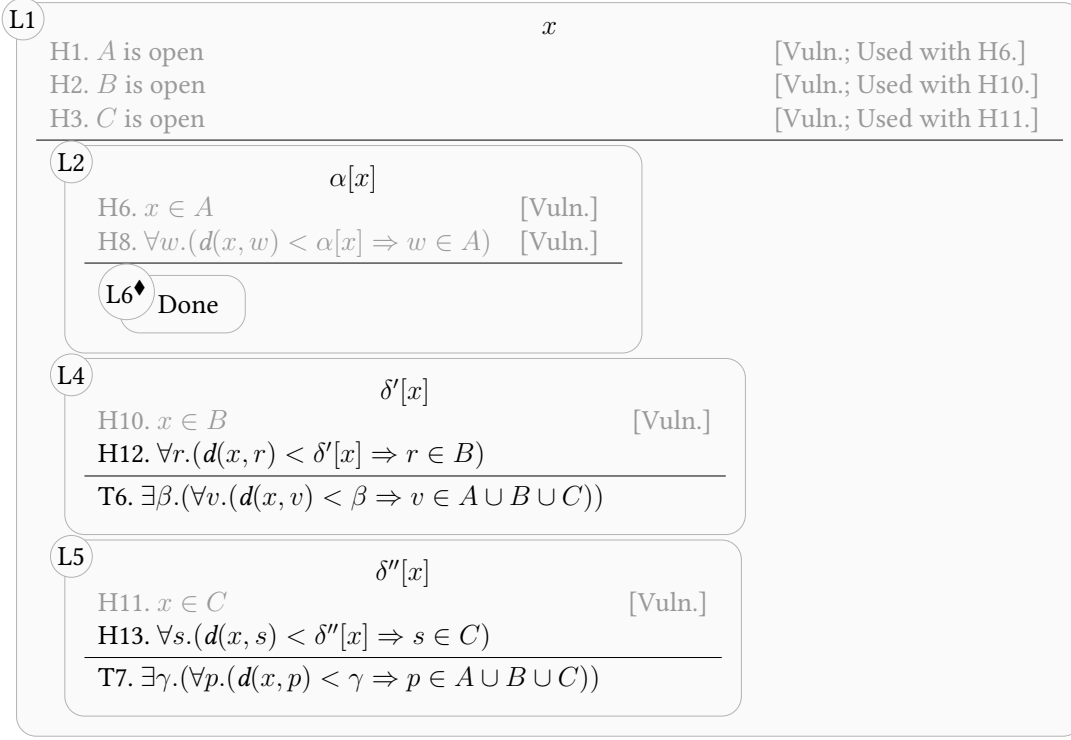
21. Delete H8 as no other statement mentions  $A$ .

L1	$x$ H1. $A$ is open H2. $B$ is open H3. $C$ is open <div style="text-align: right;"> [Vuln.; Used with H6.]  [Vuln.; Used with H10.]  [Vuln.; Used with H11.] </div>
L2	$\alpha[x]$ H6. $x \in A$ H8. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$ <div style="text-align: right;"> [Vuln.]  [Vuln.] </div>
L6 $\blacklozenge$	$\eta^\blacklozenge[\bar{z}] \ z$ H14. $d(x, z) < \eta^\blacklozenge[\bar{z}]$ [From L2.] <div style="display: flex; justify-content: space-around;"> <div> L9  <hr/> T14. <math>d(x, z) &lt; \alpha[x]</math> </div> <div> L10  <hr/> T13. <math>z \in B \cup C</math> </div> </div>
L4	$\delta'[x]$ H10. $x \in B$ [Vuln.] H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$ <hr/> T6. $\exists \beta. (\forall v. (d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$
L5	$\delta''[x]$ H11. $x \in C$ [Vuln.] H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$ <hr/> T7. $\exists \gamma. (\forall p. (d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$

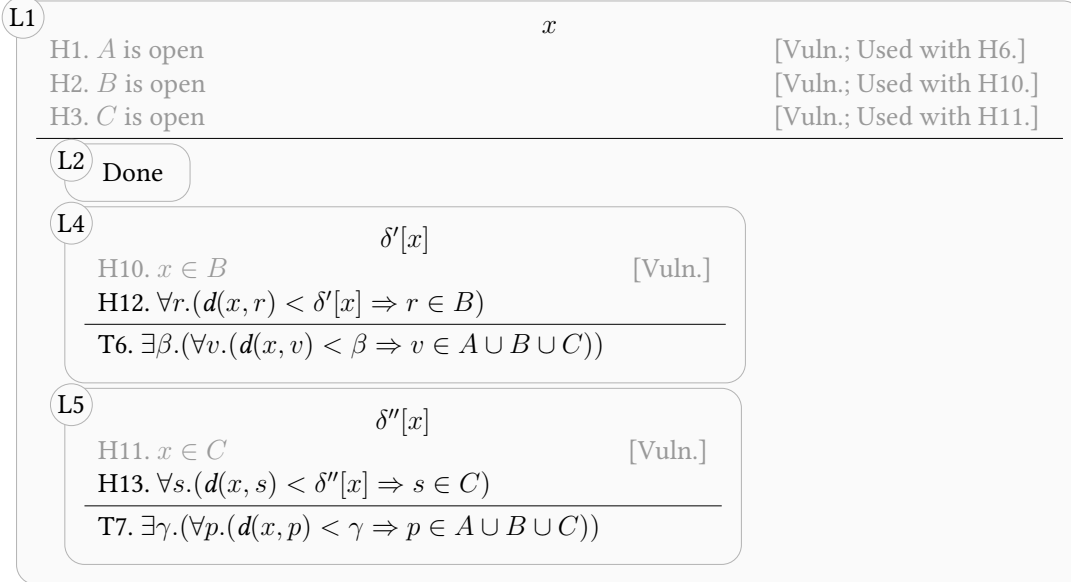
22. Hypothesis H14 matches target T14 after choosing  $\eta^\blacklozenge[\bar{z}] = \alpha[x]$ , so L6 $\blacklozenge$  is done.

Therefore, setting  $\eta = \alpha$ , we are done.

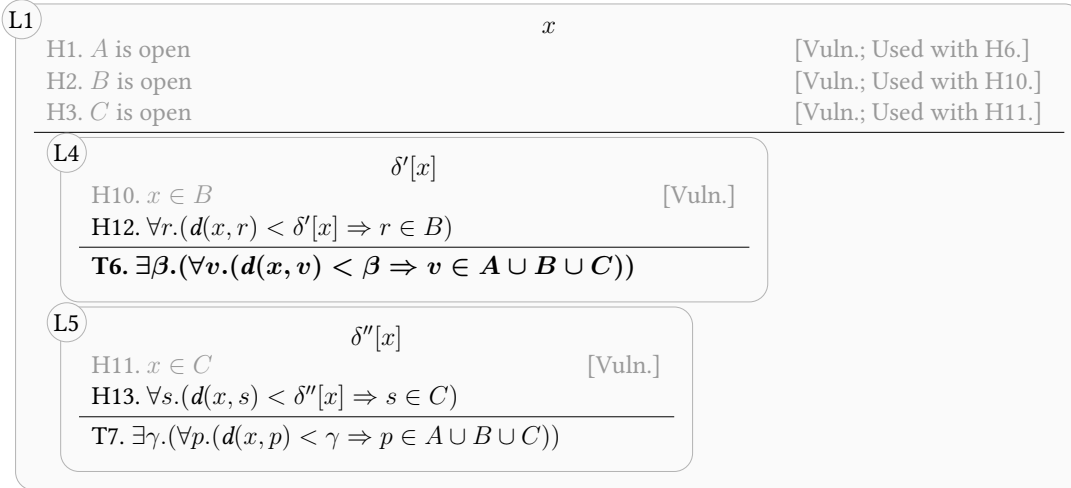




23. All targets of L2 are 'Done', so L2 is itself done.



24. Remove 'Done' targets of L1.



25. Unlock existential-universal-conditional target T6.

We would like to find  $\beta > 0$  s.t.  $v \in A \cup B \cup C$  whenever  $d(x, v) < \beta$ .

L1	$x$	
H1. $A$ is open H2. $B$ is open H3. $C$ is open		[Vuln.; Used with H6.] [Vuln.; Used with H10.] [Vuln.; Used with H11.]
L4	$\delta'[x]$	[Vuln.]
H10. $x \in B$ H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$		
L11 $\blacklozenge$	$\beta^\blacklozenge[\bar{v}] v$	[From L4.]
H15. $d(x, v) < \beta^\blacklozenge[\bar{v}]$		
T15. $v \in A \cup B \cup C$		
L5	$\delta''[x]$	[Vuln.]
H11. $x \in C$ H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$		
T7. $\exists \gamma. (\forall p. (d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$		

26. Quantifier-free expansion of target T15.

But  $v \in A \cup B \cup C$  if and only if  $v \in A$  or  $v \in B \cup C$ .

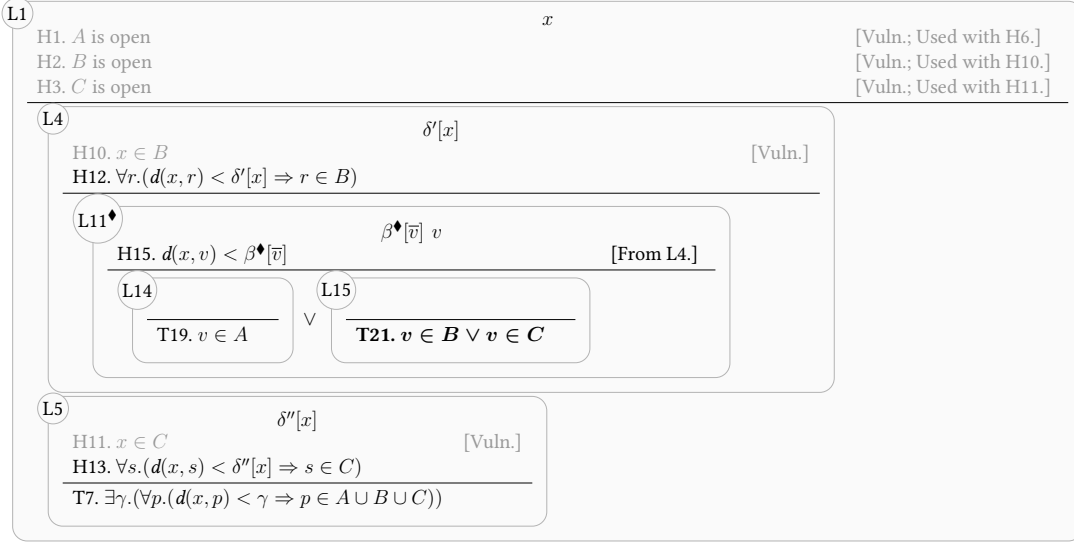
L1	$x$	
H1. $A$ is open H2. $B$ is open H3. $C$ is open		[Vuln.; Used with H6.] [Vuln.; Used with H10.] [Vuln.; Used with H11.]
L4	$\delta'[x]$	[Vuln.]
H10. $x \in B$ H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$		
L11 $\blacklozenge$	$\beta^\blacklozenge[\bar{v}] v$	[From L4.]
H15. $d(x, v) < \beta^\blacklozenge[\bar{v}]$		
T16. $v \in A \vee v \in B \cup C$		
L5	$\delta''[x]$	[Vuln.]
H11. $x \in C$ H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$		
T7. $\exists \gamma. (\forall p. (d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$		

27. Split up disjunctive target T16.

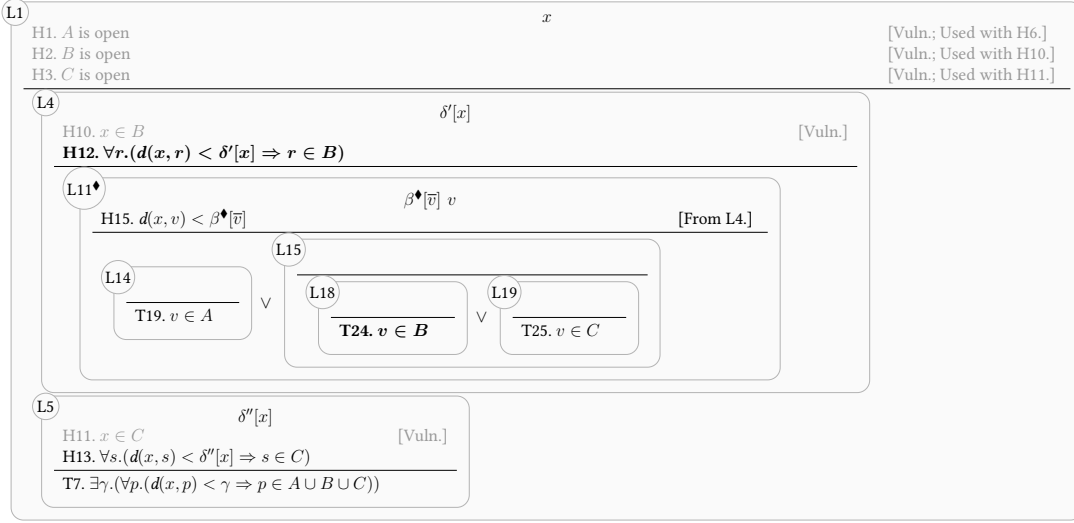
L1	$x$	
H1. $A$ is open H2. $B$ is open H3. $C$ is open		[Vuln.; Used with H6.] [Vuln.; Used with H10.] [Vuln.; Used with H11.]
L4	$\delta'[x]$	[Vuln.]
H10. $x \in B$ H12. $\forall r. (d(x, r) < \delta'[x] \Rightarrow r \in B)$		
L11 $\blacklozenge$	$\beta^\blacklozenge[\bar{v}] v$	[From L4.]
H15. $d(x, v) < \beta^\blacklozenge[\bar{v}]$		
L14	$\text{T19. } v \in A$	
L15	$\text{T20. } v \in B \cup C$	
L5	$\delta''[x]$	[Vuln.]
H11. $x \in C$ H13. $\forall s. (d(x, s) < \delta''[x] \Rightarrow s \in C)$		
T7. $\exists \gamma. (\forall p. (d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$		

28. Quantifier-free expansion of target T20.

We would like to show that  $v \in B \cup C$ , i.e. that  $v \in B$  or  $v \in C$ .

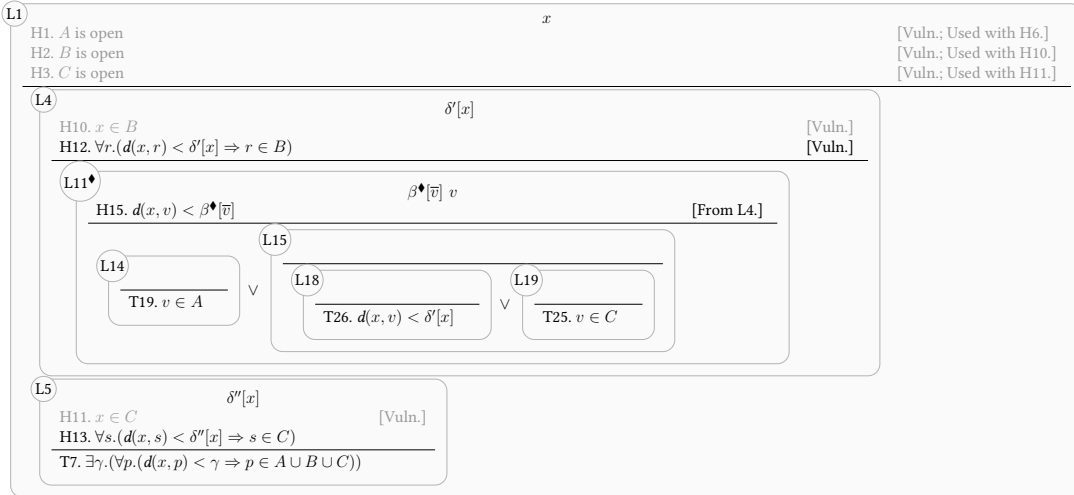


## 29. Split up disjunctive target T21.

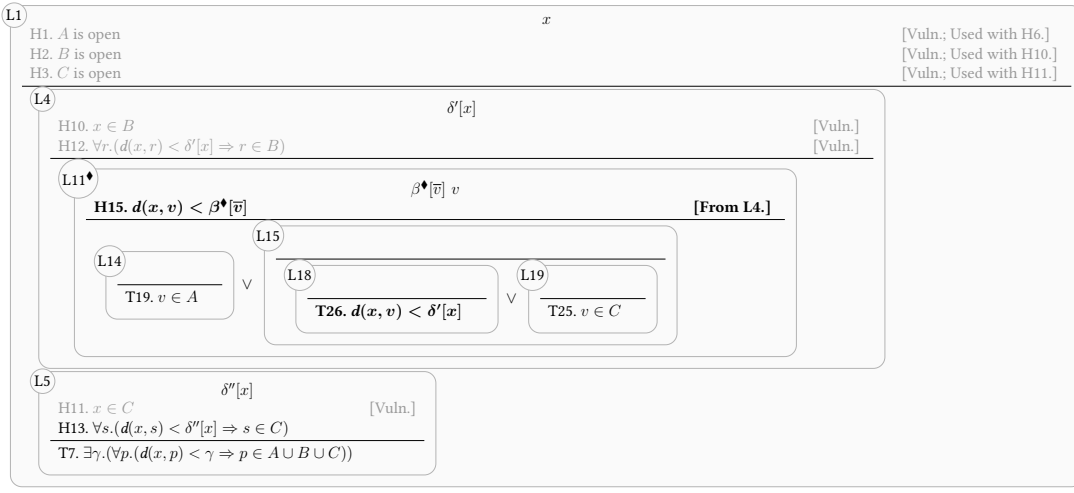


## 30. Backwards reasoning using H12 with T24.

We know that  $v \in B$  if  $d(x, v) < \delta'$ .

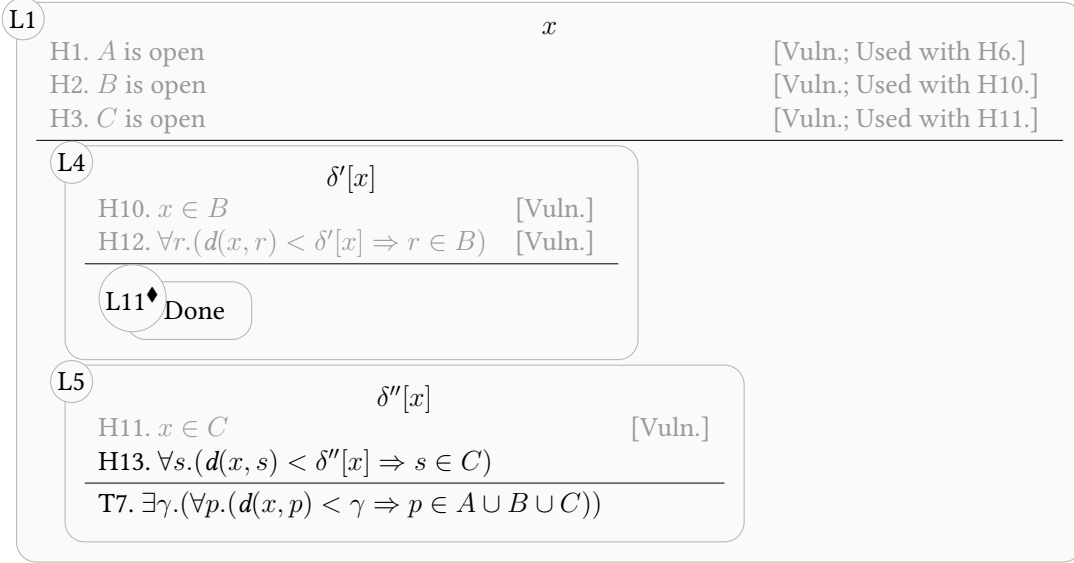


## 31. Delete H12 as no other statement mentions $B$ .

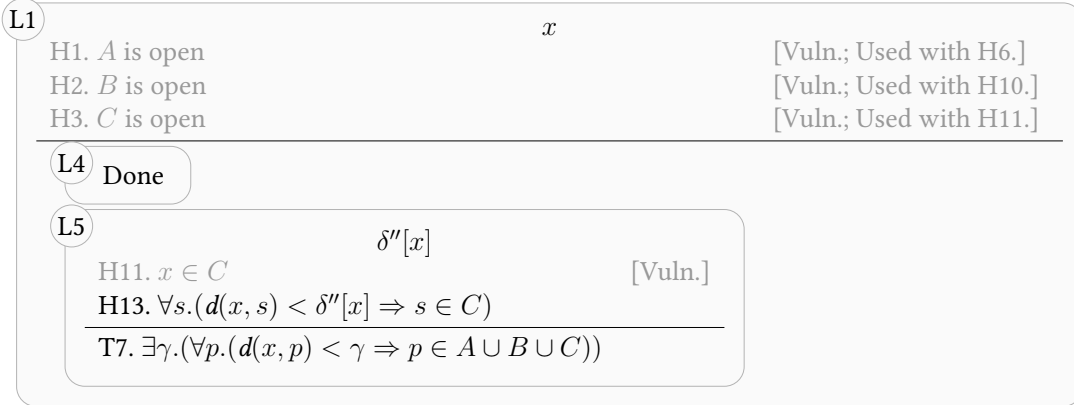


32. Hypothesis H15 matches target T26 after choosing  $\beta^\bullet[\bar{v}] = \delta'[x]$ , so L11<sup>♦</sup> is done.

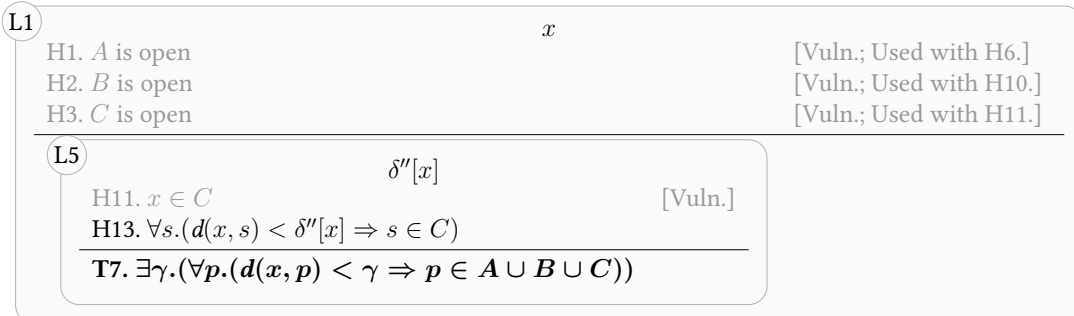
Therefore, setting  $\beta = \delta'$ , we are done.



33. All targets of L4 are 'Done', so L4 is itself done.

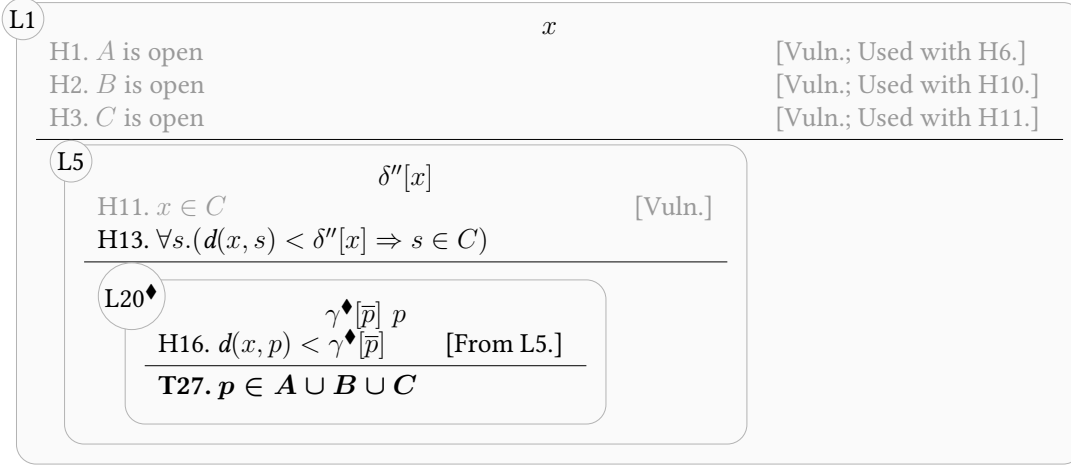


34. Remove 'Done' targets of L1.



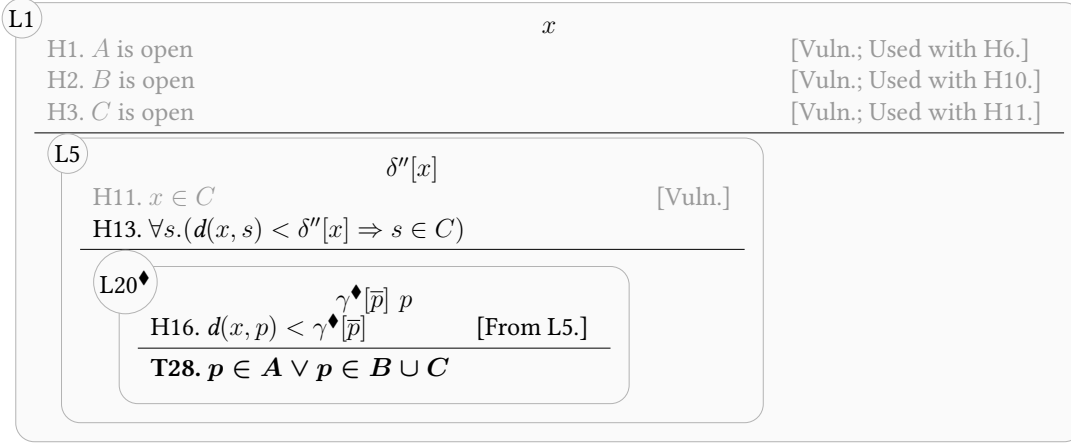
35. Unlock existential-universal-conditional target T7.

We would like to find  $\gamma > 0$  s.t.  $p \in A \cup B \cup C$  whenever  $d(x, p) < \gamma$ .

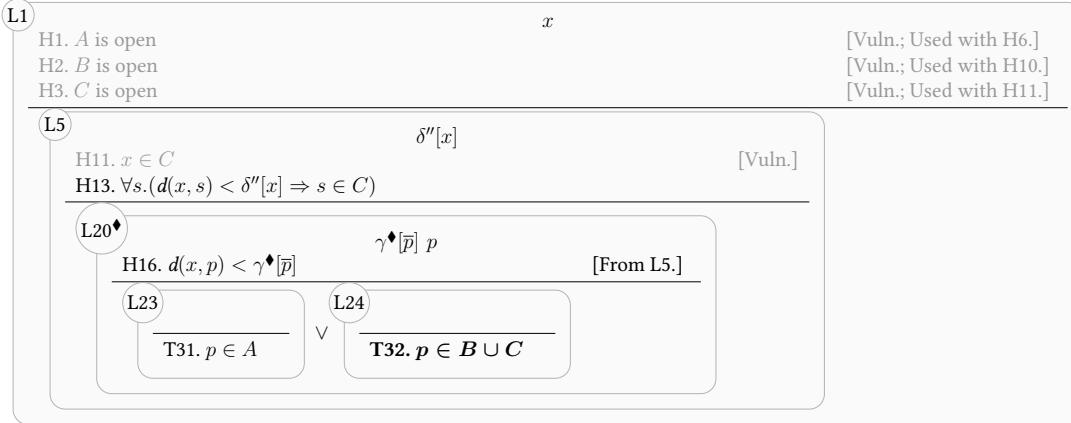


36. Quantifier-free expansion of target T27.

But  $p \in A \cup B \cup C$  if and only if  $p \in A$  or  $p \in B \cup C$ .

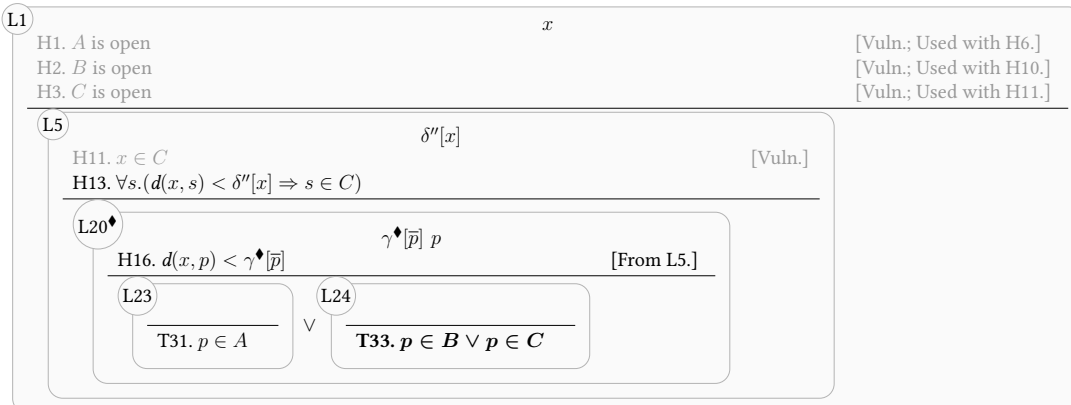


37. Split up disjunctive target T28.

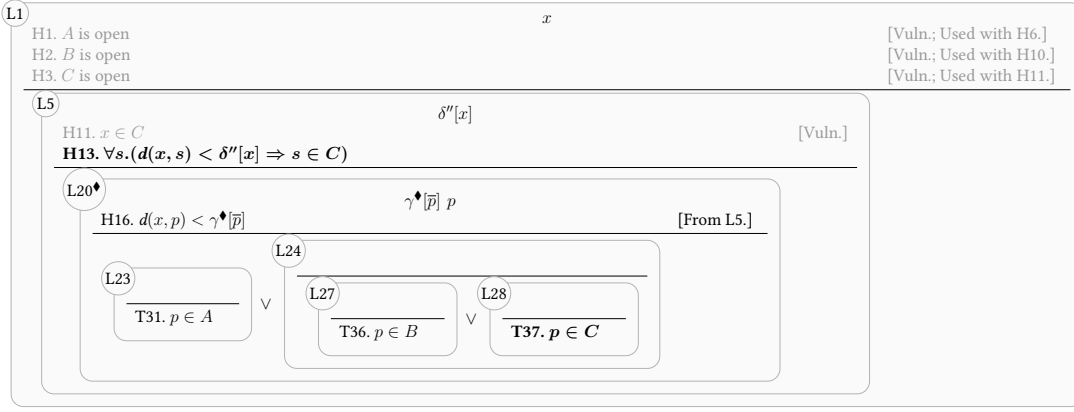


38. Quantifier-free expansion of target T32.

We would like to show that  $p \in B \cup C$ , i.e. that  $p \in B$  or  $p \in C$ .

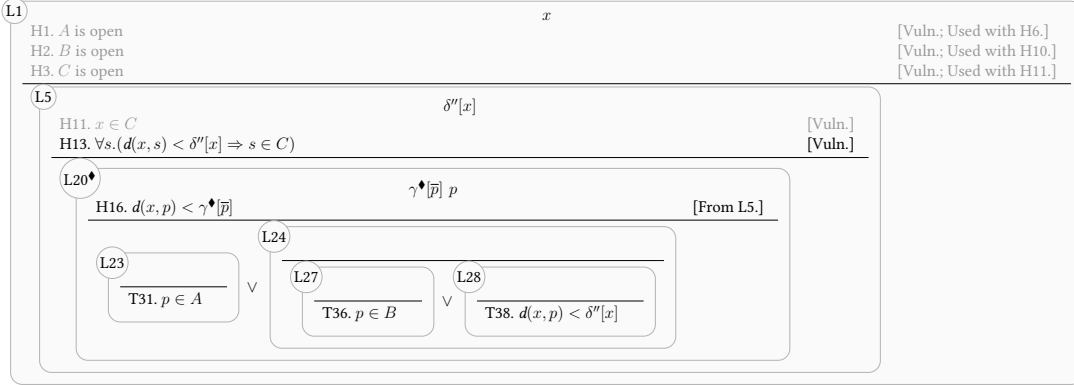


39. Split up disjunctive target T33.

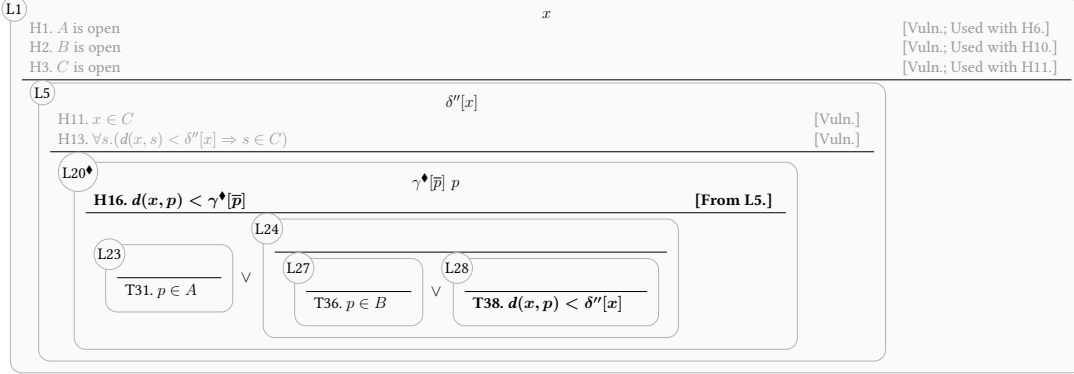


40. Backwards reasoning using H13 with T37.

We know that  $p \in C$  if  $d(x, p) < \delta''$ .

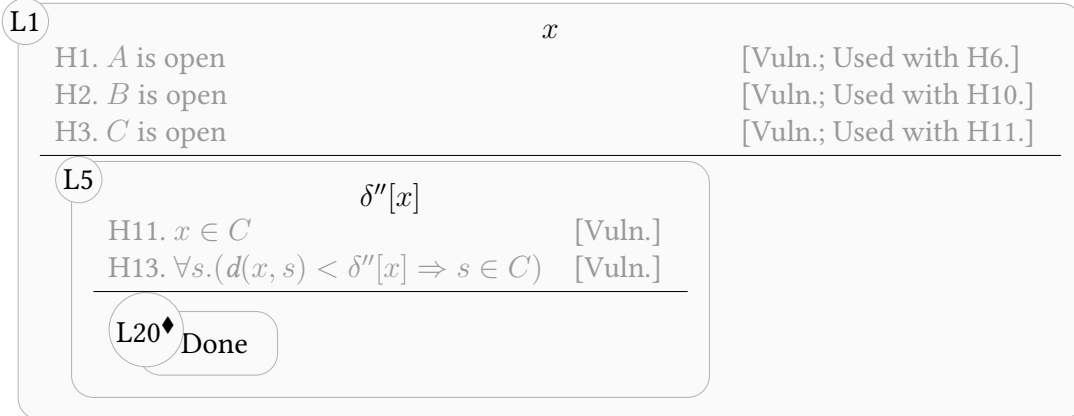


41. Delete H13 as no other statement mentions  $C$ .

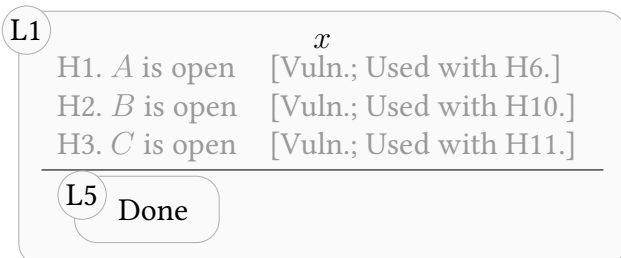


42. Hypothesis H16 matches target T38 after choosing  $\gamma^\blacklozenge[\bar{p}] = \delta''[x]$ , so L20 $\blacklozenge$  is done.

Therefore, setting  $\gamma = \delta''$ , we are done.



43. All targets of L5 are 'Done', so L5 is itself done.



44. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.





**If  $A$  and  $B$  are open sets, then  $A \cup B$  is also open.**

Let  $x$  be an element of  $A \cup B$ . Then  $x \in A$  or  $x \in B$ . Since  $A$  is open and  $x \in A$ , there exists  $\alpha > 0$  such that  $w \in A$  whenever  $d(x, w) < \alpha$ . Since  $B$  is open and  $x \in B$ , there exists  $\beta > 0$  such that  $p \in B$  whenever  $d(x, p) < \beta$ . We would like to find  $\eta > 0$  s.t.  $z \in A \cup B$  whenever  $d(x, z) < \eta$ . But  $z \in A \cup B$  if and only if  $z \in A$  or  $z \in B$ . We know that  $z \in A$  if  $d(x, z) < \alpha$ . Therefore, setting  $\eta = \alpha$ , we are done. We would like to find  $\theta > 0$  s.t.  $u \in A \cup B$  whenever  $d(x, u) < \theta$ . But  $u \in A \cup B$  if and only if  $u \in A$  or  $u \in B$ . We know that  $u \in B$  if  $d(x, u) < \beta$ . Therefore, setting  $\theta = \beta$ , we are done.

L1  
H1.  $A$  is open  
H2.  $B$  is open  


---

**T1.  $A \cup B$  is open**

1. Expand pre-universal target T1.

L1  
H1.  $A$  is open  
H2.  $B$  is open  


---

**T2.  $\forall x. (x \in A \cup B \Rightarrow \exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cup B)))$**

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let  $x$  be an element of  $A \cup B$ .

L1  
 $x$   
H1.  $A$  is open  
H2.  $B$  is open  
**H3.  $x \in A \cup B$**   


---

**T3.  $\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cup B))$**

3. Quantifier-free expansion of hypothesis H3.

Since  $x \in A \cup B$ ,  $x \in A$  or  $x \in B$ .

L1  
 $x$   
H1.  $A$  is open  
H2.  $B$  is open  
**H4.  $x \in A \vee x \in B$**   


---

**T3.  $\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cup B))$**

4. Split into cases to handle disjunctive hypothesis H4.

L1  
 $x$   
**H1.  $A$  is open**  
H2.  $B$  is open  


---

L2  
**H5.  $x \in A$**   


---

**T4.  $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B))$**   


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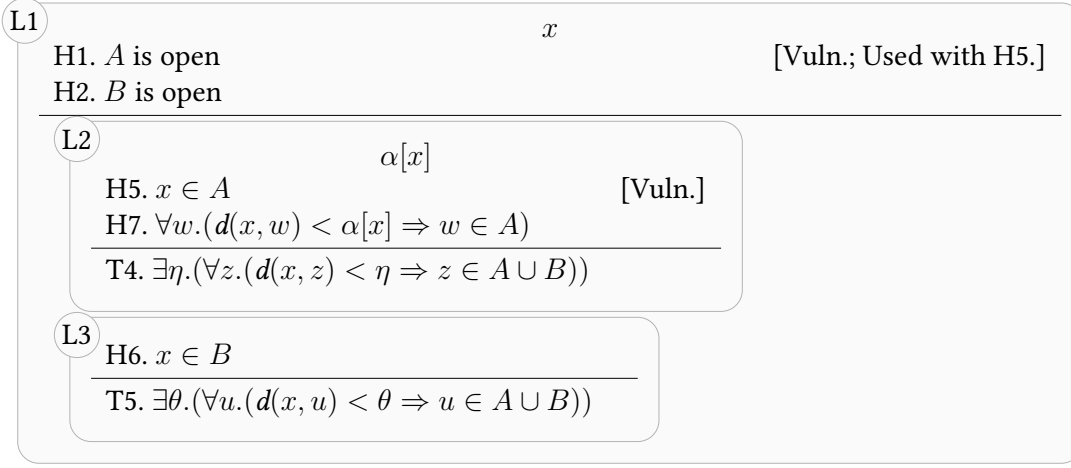
L3  
H6.  $x \in B$   


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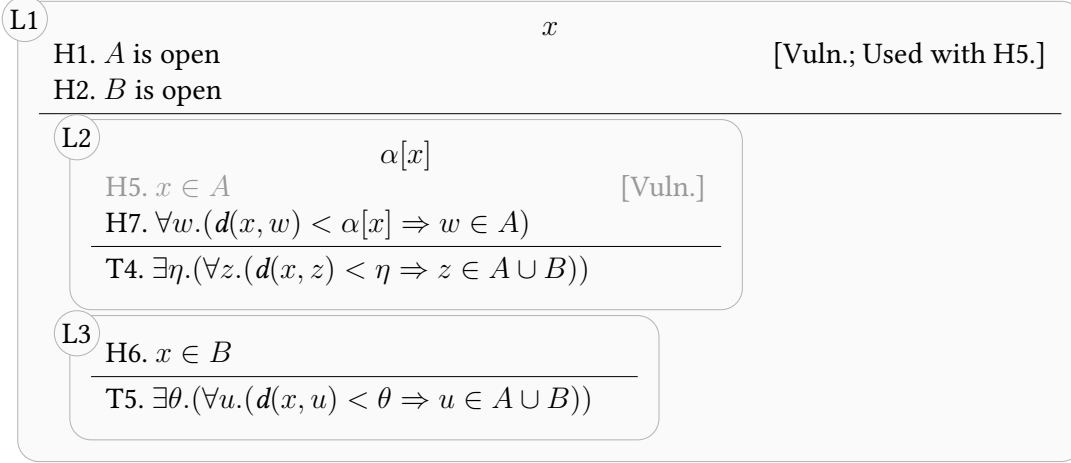
**T5.  $\exists \theta. (\forall u. (d(x, u) < \theta \Rightarrow u \in A \cup B))$**

5. Forwards reasoning using H1 with H5.

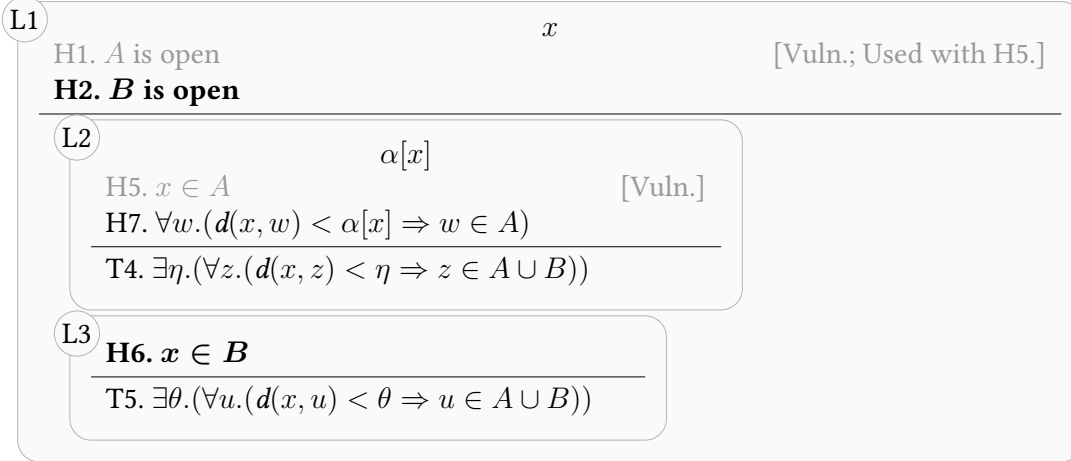
Since  $A$  is open and  $x \in A$ , there exists  $\alpha > 0$  such that  $w \in A$  whenever  $d(x, w) < \alpha$ .



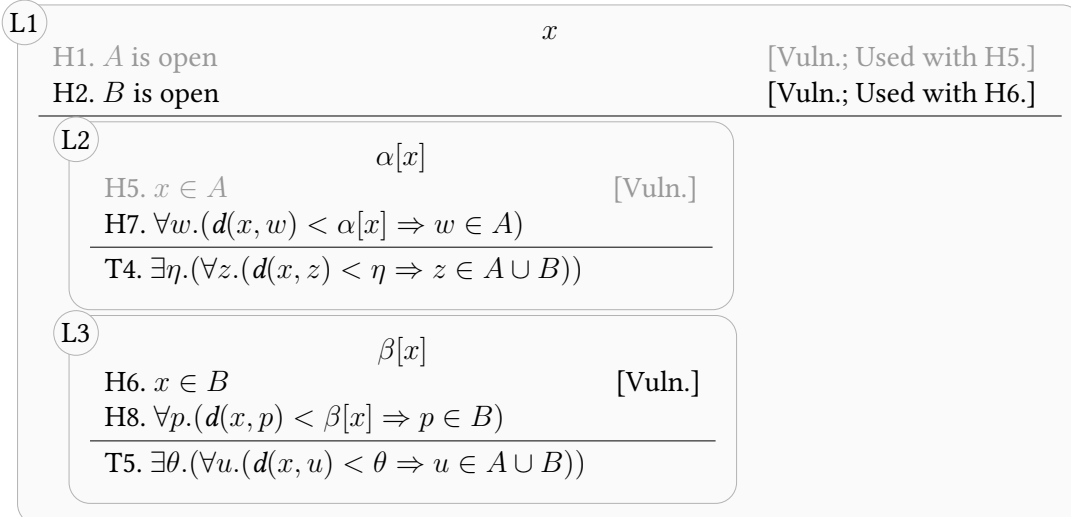
6. Deleted H5, as this unexpandable atomic statement is unmatchable.



7. Deleted H1, as the conclusion of this implicative statement is unmatchable.



8. Forwards reasoning using H2 with H6.



Since  $B$  is open and  $x \in B$ , there exists  $\beta > 0$  such that  $p \in B$  whenever  $d(x, p) < \beta$ .

9. Deleted H6, as this unexpandable atomic statement is unmatchable.

L1	$x$	
H1. $A$ is open		[Vuln.; Used with H5.]
H2. $B$ is open		[Vuln.; Used with H6.]
L2	$\alpha[x]$	[Vuln.]
H5. $x \in A$		
H7. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$		
T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B))$		
L3	$\beta[x]$	[Vuln.]
H6. $x \in B$		
H8. $\forall p. (d(x, p) < \beta[x] \Rightarrow p \in B)$		
T5. $\exists \theta. (\forall u. (d(x, u) < \theta \Rightarrow u \in A \cup B))$		

10. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1	$x$	
H1. $A$ is open		[Vuln.; Used with H5.]
H2. $B$ is open		[Vuln.; Used with H6.]
L2	$\alpha[x]$	[Vuln.]
H5. $x \in A$		
H7. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$		
T4. $\exists \eta. (\forall z. (d(x, z) < \eta \Rightarrow z \in A \cup B))$		
L3	$\beta[x]$	[Vuln.]
H6. $x \in B$		
H8. $\forall p. (d(x, p) < \beta[x] \Rightarrow p \in B)$		
T5. $\exists \theta. (\forall u. (d(x, u) < \theta \Rightarrow u \in A \cup B))$		

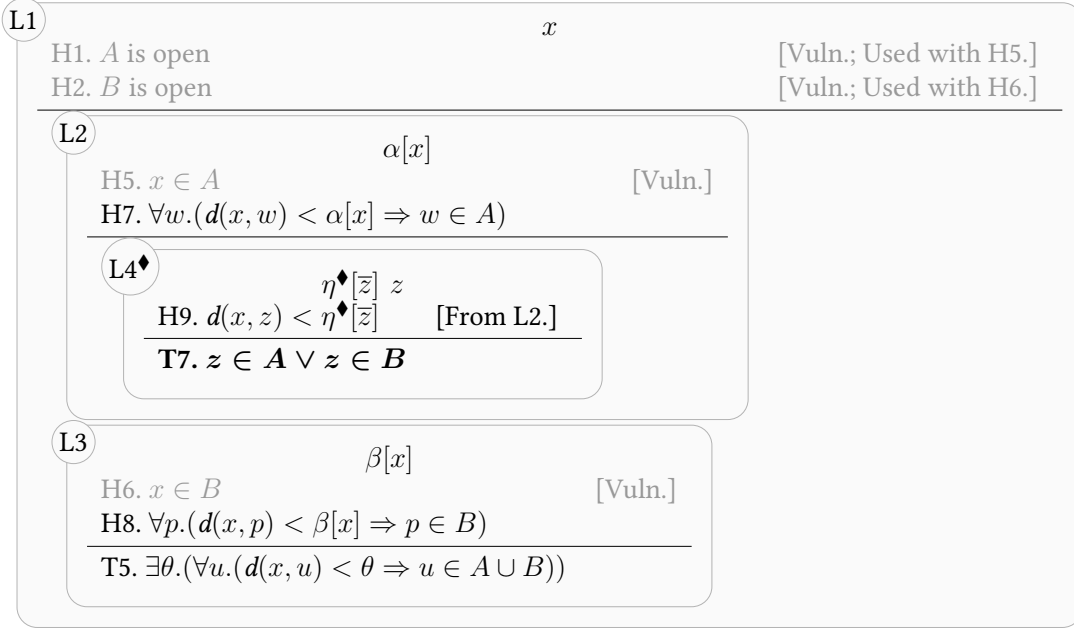
11. Unlock existential-universal-conditional target T4.

L1	$x$	
H1. $A$ is open		[Vuln.; Used with H5.]
H2. $B$ is open		[Vuln.; Used with H6.]
L2	$\alpha[x]$	[Vuln.]
H5. $x \in A$		
H7. $\forall w. (d(x, w) < \alpha[x] \Rightarrow w \in A)$		
L4 $\blacklozenge$	$\eta^{\blacklozenge}[\bar{z}] \ z$	[From L2.]
H9. $d(x, z) < \eta^{\blacklozenge}[\bar{z}]$		
T6. $z \in A \cup B$		
L3	$\beta[x]$	[Vuln.]
H6. $x \in B$		
H8. $\forall p. (d(x, p) < \beta[x] \Rightarrow p \in B)$		
T5. $\exists \theta. (\forall u. (d(x, u) < \theta \Rightarrow u \in A \cup B))$		

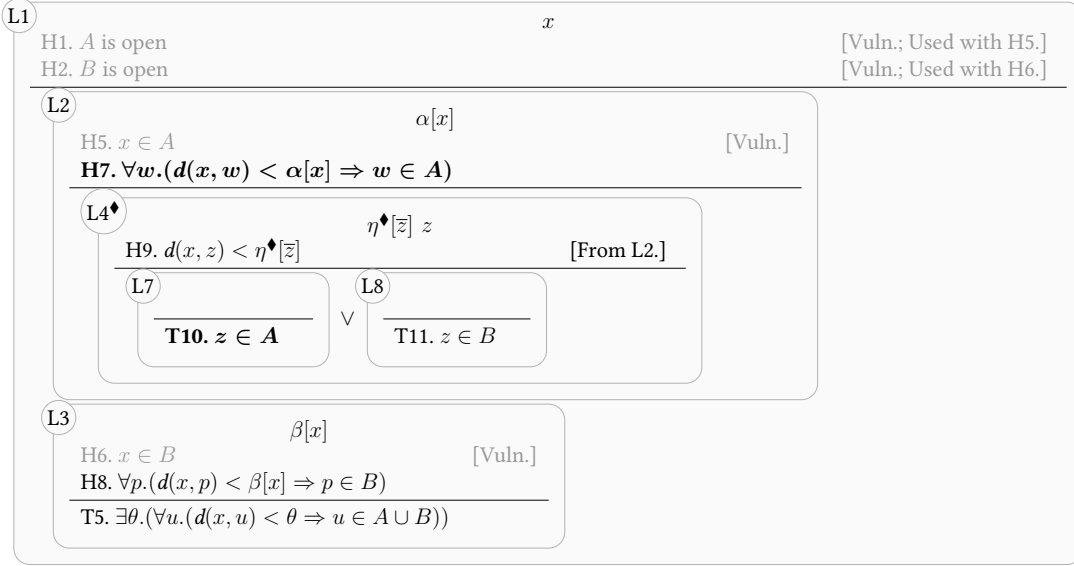
We would like to find  $\eta > 0$  s.t.  $z \in A \cup B$  whenever  $d(x, z) < \eta$ .

12. Quantifier-free expansion of target T6.

But  $z \in A \cup B$  if and only if  $z \in A$  or  $z \in B$ .

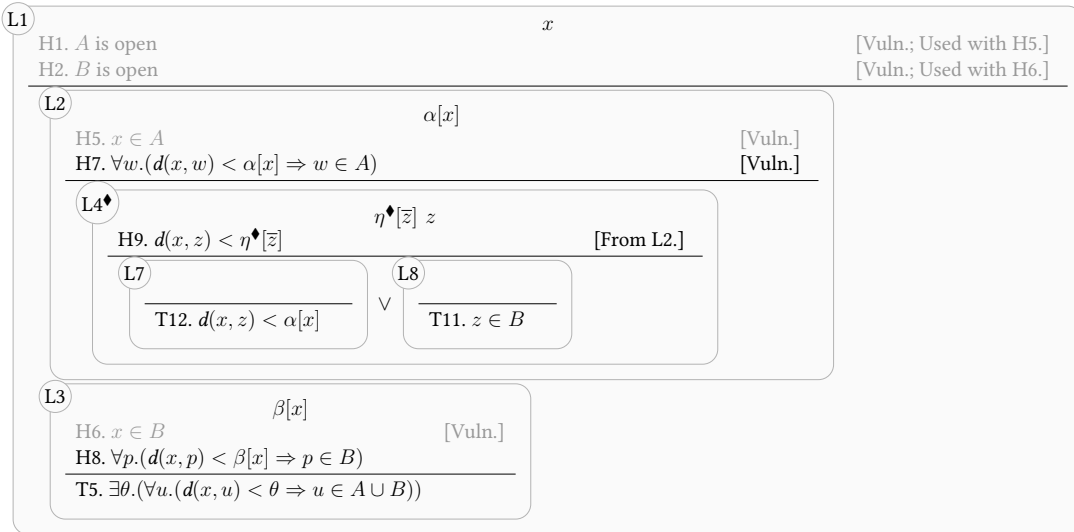


13. Split up disjunctive target T7.

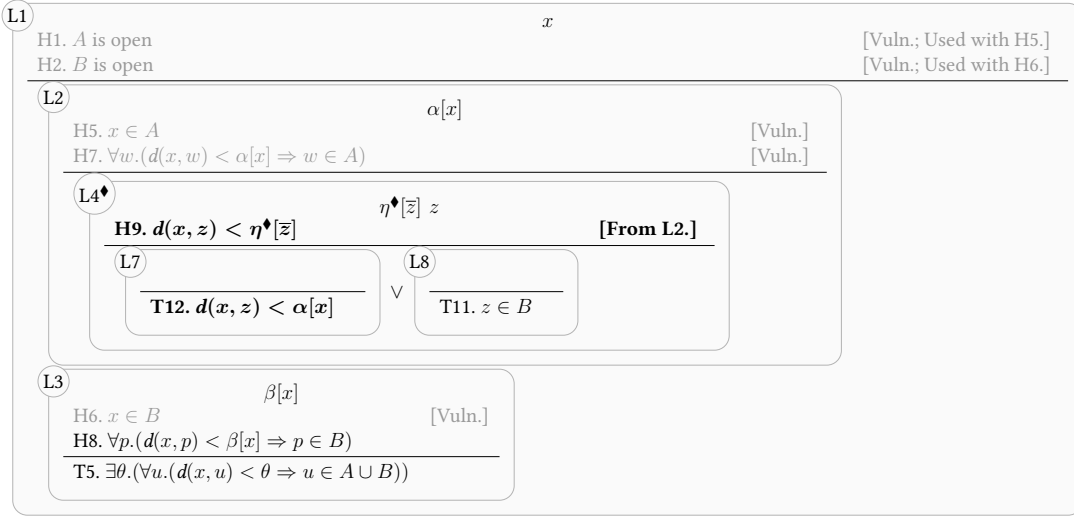


14. Backwards reasoning using H7 with T10.

We know that  $z \in A$  if  $d(x, z) < \alpha$ .

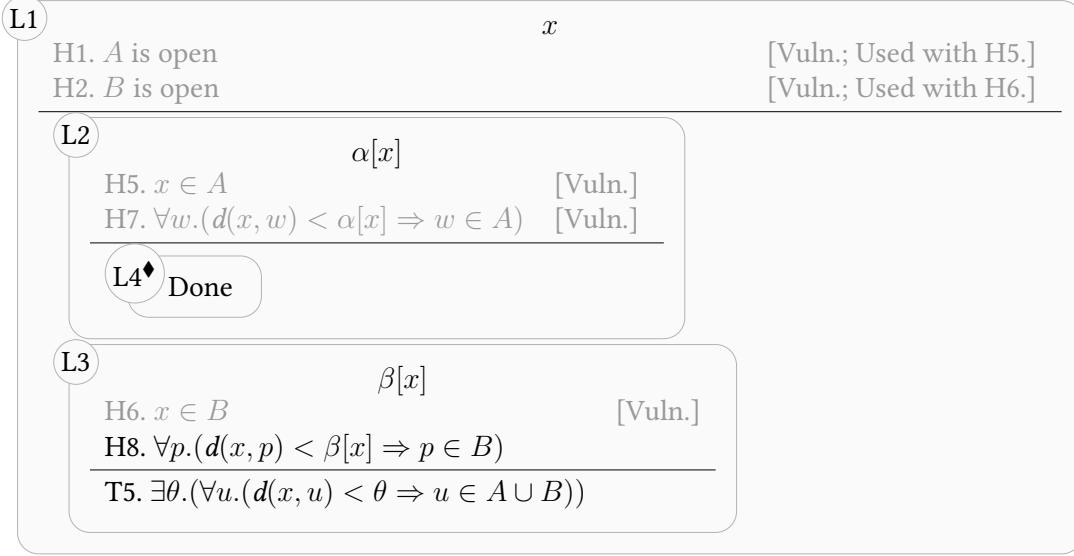


15. Delete H7 as no other statement mentions  $A$ .

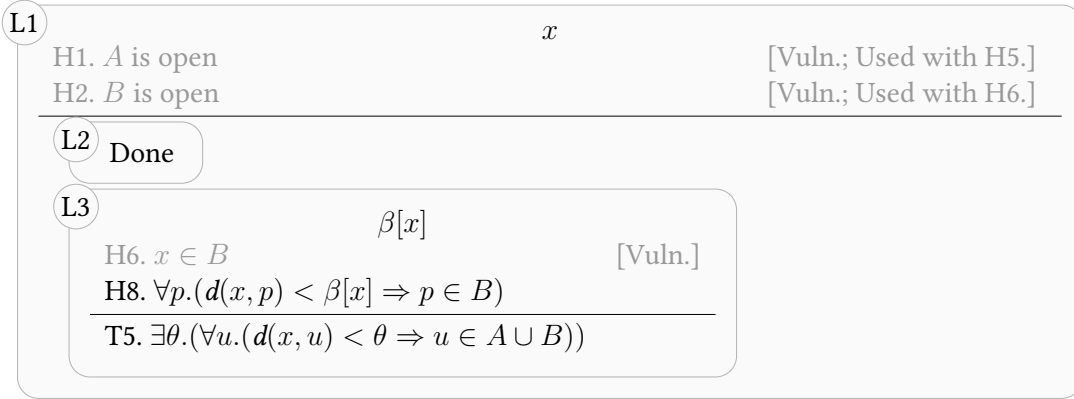


16. Hypothesis H9 matches target T12 after choosing  $\eta^\diamond[\bar{z}] = \alpha[x]$ , so L4<sup>♦</sup> is done.

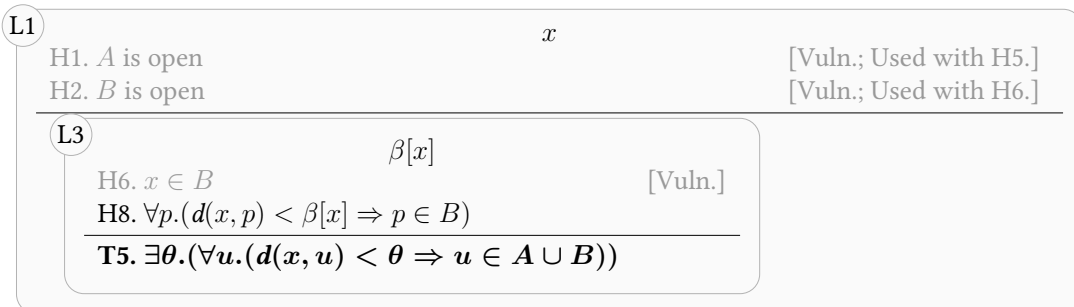
Therefore, setting  $\eta = \alpha$ , we are done.



17. All targets of L2 are ‘Done’, so L2 is itself done.

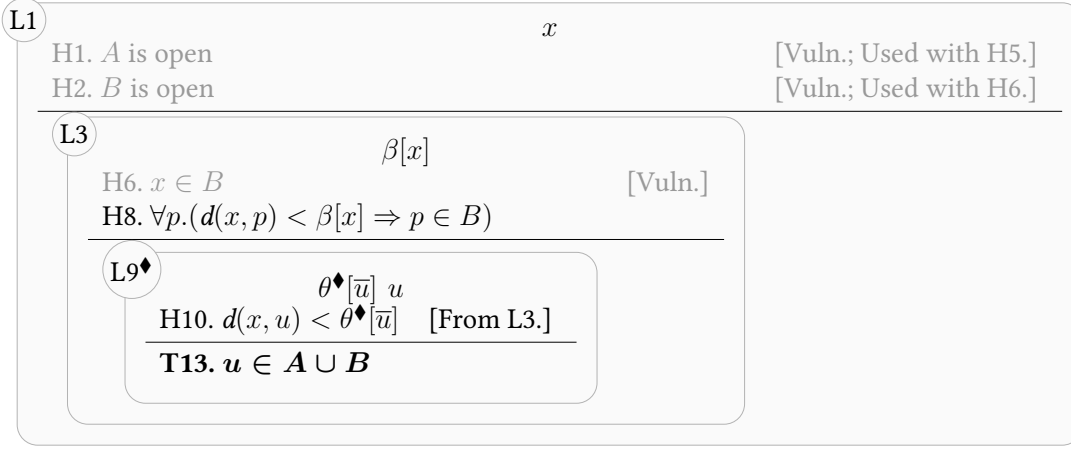


18. Remove ‘Done’ targets of L1.



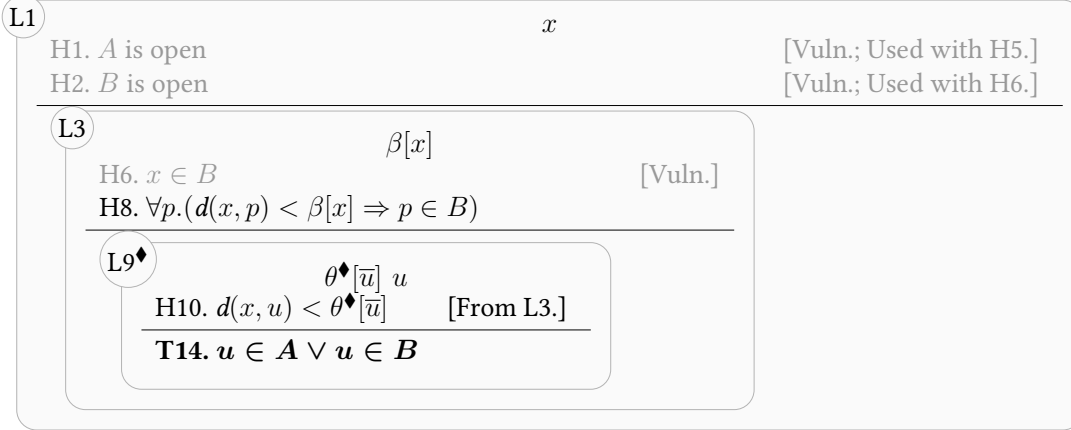
19. Unlock existential-universal-conditional target T5.

We would like to find  $\theta > 0$  s.t.  $u \in A \cup B$  whenever  $d(x, u) < \theta$ .

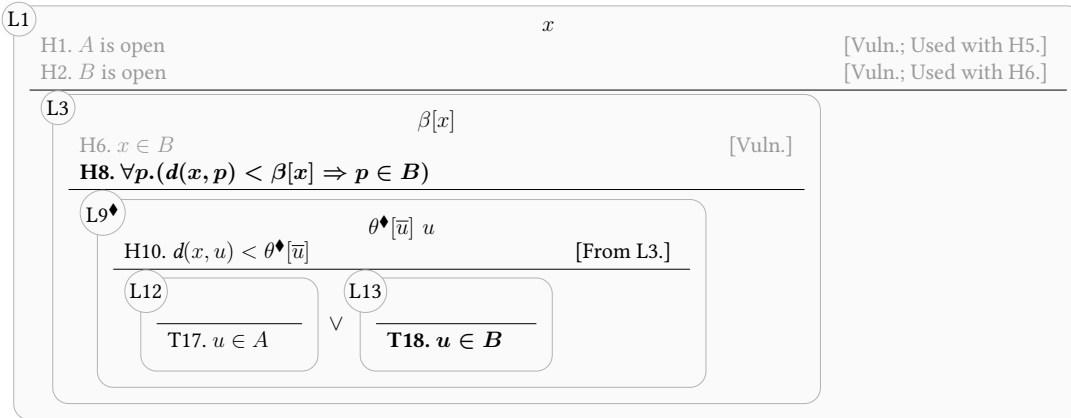


20. Quantifier-free expansion of target T13.

But  $u \in A \cup B$  if and only if  $u \in A$  or  $u \in B$ .

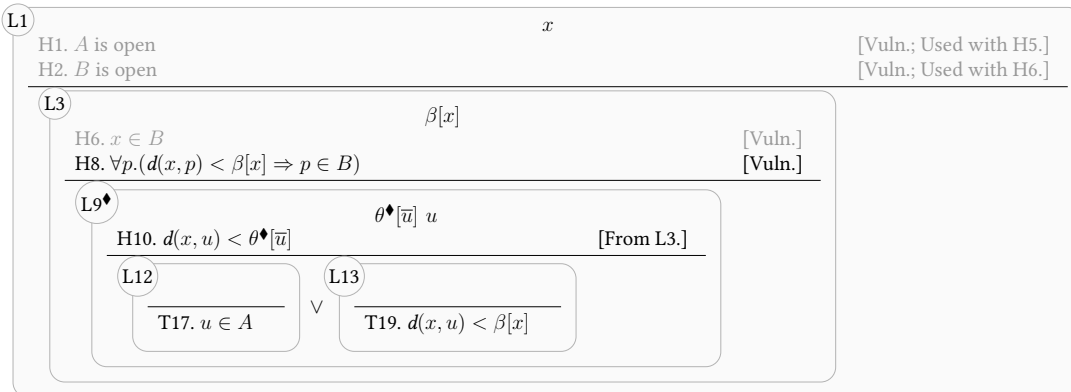


21. Split up disjunctive target T14.

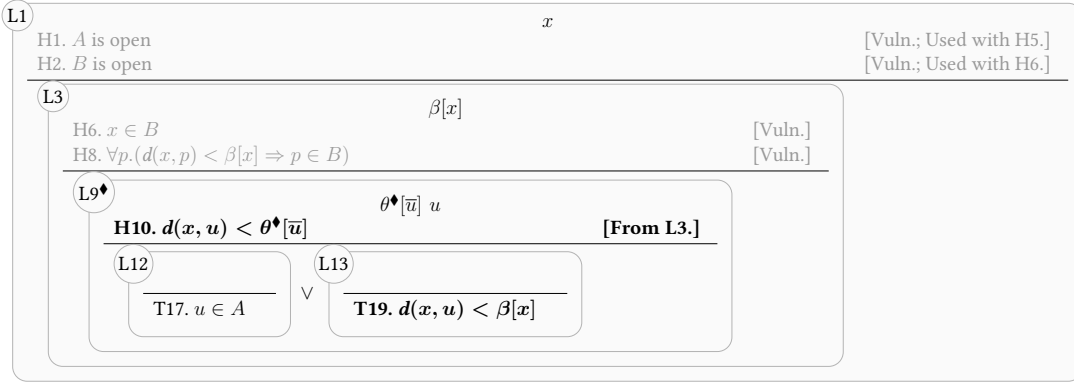


22. Backwards reasoning using H8 with T18.

We know that  $u \in B$  if  $d(x, u) < \beta$ .

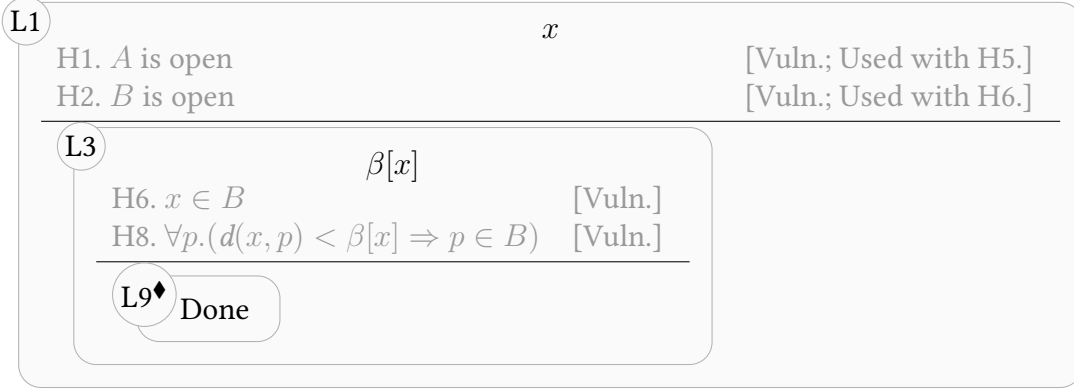


23. Delete H8 as no other statement mentions  $B$ .



24. Hypothesis H10 matches target T19 after choosing  $\theta^\blacklozenge[\bar{u}] = \beta[x]$ , so L9 $\blacklozenge$  is done.

Therefore, setting  $\theta = \beta$ ,  
we are done.



25. All targets of L3 are 'Done', so L3 is itself done.



26. All targets of L1 are 'Done', so L1 is itself done.



Problem solved.





**If  $A$  and  $B$  are open sets, then  $A \cap B$  is also open.**

Let  $x$  be an element of  $A \cap B$ . Then  $x \in A$  and  $x \in B$ . Therefore, since  $A$  is open, there exists  $\eta > 0$  such that  $u \in A$  whenever  $d(x, u) < \eta$  and since  $B$  is open, there exists  $\theta > 0$  such that  $v \in B$  whenever  $d(x, v) < \theta$ . We would like to find  $\delta > 0$  s.t.  $y \in A \cap B$  whenever  $d(x, y) < \delta$ . But  $y \in A \cap B$  if and only if  $y \in A$  and  $y \in B$ . We know that  $y \in A$  whenever  $d(x, y) < \eta$  and that  $y \in B$  whenever  $d(x, y) < \theta$ . Assume now that  $d(x, y) < \delta$ . Then  $d(x, y) < \eta$  if  $\delta \leq \eta$  and  $d(x, y) < \theta$  if  $\delta \leq \theta$ . We may therefore take  $\delta = \min(\eta, \theta)$  and we are done.

L1  
H1.  $A$  is open  
H2.  $B$  is open  


---

**T1.  $A \cap B$  is open**

1. Expand pre-universal target T1.

L1  
H1.  $A$  is open  
H2.  $B$  is open  


---

**T2.  $\forall x.(x \in A \cap B \Rightarrow \exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cap B)))$**

2. Apply ‘let’ trick and move premise of universal-conditional target T2 above the line.

Let  $x$  be an element of  $A \cap B$ .

L1  
 $x$   
H1.  $A$  is open  
H2.  $B$  is open  
**H3.  $x \in A \cap B$**   


---

**T3.  $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cap B))$**

3. Quantifier-free expansion of hypothesis H3.

Since  $x \in A \cap B$ ,  $x \in A$  and  $x \in B$ .

L1  
 $x$   
**H1.  $A$  is open**  
H2.  $B$  is open  
**H4.  $x \in A$**   
H5.  $x \in B$   


---

**T3.  $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cap B))$**

4. Forwards reasoning using H1 with H4.

Since  $A$  is open and  $x \in A$ , there exists  $\eta > 0$  such that  $u \in A$  whenever  $d(x, u) < \eta$ .

L1  
 $x \quad \eta[x]$   
H1.  $A$  is open [Vuln.; Used with H4.]  
H2.  $B$  is open  
H4.  $x \in A$  [Vuln.]  
H5.  $x \in B$   
H6.  $\forall u.(d(x, u) < \eta[x] \Rightarrow u \in A)$   


---

**T3.  $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cap B))$**

5. Deleted H4, as this unexpandable atomic statement is unmatchable.

L1	$x \ \eta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		
H4. $x \in A$		[Vuln.]
H5. $x \in B$		
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		
<hr/>		
T3. $\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cap B))$		

6. Deleted H1, as the conclusion of this implicative statement is unmatchable.

L1	$x \ \eta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
<b>H2. <math>B</math> is open</b>		
H4. $x \in A$		[Vuln.]
<b>H5. <math>x \in B</math></b>		
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		
<hr/>		
T3. $\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cap B))$		

7. Forwards reasoning using H2 with H5.

L1	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		
<hr/>		
T3. $\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cap B))$		

Since  $B$  is open and  $x \in B$ , there exists  $\theta > 0$  such that  $v \in B$  whenever  $d(x, v) < \theta$ .

8. Deleted H5, as this unexpandable atomic statement is unmatchable.

L1	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		
<hr/>		
T3. $\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cap B))$		

9. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		
<hr/>		
<b>T3. <math>\exists \delta. (\forall y. (d(x, y) &lt; \delta \Rightarrow y \in A \cap B))</math></b>		

10. Unlock existential-universal-conditional target T3.

We would like to find  $\delta > 0$  s.t.  $y \in A \cap B$  whenever  $d(x, y) < \delta$ .

L1	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		
<hr/>		
L2♦	$\delta^\diamond[\bar{y}] \ y$	
H8. $d(x, y) < \delta^\diamond[\bar{y}]$	[From L1.]	
<hr/>		
<b>T4. <math>y \in A \cap B</math></b>		

11. Quantifier-free expansion of target T4.

But  $y \in A \cap B$  if and only if  $y \in A$  and  $y \in B$ .

L1	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
<b>H6. <math>\forall u. (d(x, u) &lt; \eta[x] \Rightarrow u \in A)</math></b>		
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		
<hr/>		
L2♦	$\delta^\diamond[\bar{y}] \ y$	
H8. $d(x, y) < \delta^\diamond[\bar{y}]$	[From L1.]	
<hr/>		
<b>T5. <math>y \in A</math></b>		
T6. $y \in B$		

12. Backwards reasoning using H6 with T5.

We know that  $y \in A$  whenever  $d(x, y) < \eta$ .

L1	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		[Vuln.]
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		
<hr/>		
L2♦	$\delta^\diamond[\bar{y}] \ y$	
H8. $d(x, y) < \delta^\diamond[\bar{y}]$	[From L1.]	
<hr/>		
<b>T7. <math>d(x, y) &lt; \eta[x]</math></b>		
T6. $y \in B$		

13. Delete H6 as no other statement mentions  $A$ .

<b>L1</b>	$x \ \eta[x] \ \theta[x]$	
	H1. $A$ is open	[Vuln.; Used with H4.]
	H2. $B$ is open	[Vuln.; Used with H5.]
	H4. $x \in A$	[Vuln.]
	H5. $x \in B$	[Vuln.]
	H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$	[Vuln.]
	<b>H7. <math>\forall v. (d(x, v) &lt; \theta[x] \Rightarrow v \in B)</math></b>	

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<b>L2<math>\blacklozenge</math></b>	$\delta^{\blacklozenge}[\overline{y}] \ y$	
	H8. $d(x, y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]	

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		<b>T7. <math>d(x, y) &lt; \eta[x]</math></b> <b>T6. <math>y \in B</math></b>
--	--	---

14. Backwards reasoning using H7 with T6.

We know that  $y \in B$  whenever  $d(x, y) < \theta$ .

<b>L1</b>	$x \ \eta[x] \ \theta[x]$	
	H1. $A$ is open	[Vuln.; Used with H4.]
	H2. $B$ is open	[Vuln.; Used with H5.]
	H4. $x \in A$	[Vuln.]
	H5. $x \in B$	[Vuln.]
	H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$	[Vuln.]
	H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$	[Vuln.]

---

<b>L2<math>\blacklozenge</math></b>	$\delta^{\blacklozenge}[\overline{y}] \ y$	
	H8. $d(x, y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]	

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		<b>T7. <math>d(x, y) &lt; \eta[x]</math></b> <b>T8. <math>d(x, y) &lt; \theta[x]</math></b>
--	--	--

15. Delete H7 as no other statement mentions  $B$ .

<b>L1</b>	$x \ \eta[x] \ \theta[x]$	
	H1. $A$ is open	[Vuln.; Used with H4.]
	H2. $B$ is open	[Vuln.; Used with H5.]
	H4. $x \in A$	[Vuln.]
	H5. $x \in B$	[Vuln.]
	H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$	[Vuln.]
	H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$	[Vuln.]

---

<b>L2<math>\blacklozenge</math></b>	$\delta^{\blacklozenge}[\overline{y}] \ y$	
	H8. $d(x, y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]	

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		<b>T7. <math>d(x, y) &lt; \eta[x]</math></b> <b>T8. <math>d(x, y) &lt; \theta[x]</math></b>
--	--	--

16. Replacing diamonds with bullets in L2 $\blacklozenge$ .

Assume now that  $d(x, y) < \delta$ .

<b>L1</b>	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		[Vuln.]
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		[Vuln.]
<hr/>		
<b>L2</b>	$\delta^\bullet[\bar{y}] \ y$	
<b>H8. <math>d(x, y) &lt; \delta^\bullet[\bar{y}]</math></b>	<b>[From L1.]</b>	
<hr/>		
<b>T7. <math>d(x, y) &lt; \eta[x]</math></b>		
<b>T8. <math>d(x, y) &lt; \theta[x]</math></b>		

17. Backwards reasoning using library result “transitivity” with (T7,H8).

Since  $d(x, y) < \delta$ ,  
 $d(x, y) < \eta$  if  $\delta \leq \eta$ .

<b>L1</b>	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		[Vuln.]
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		[Vuln.]
<hr/>		
<b>L2</b>	$\delta^\bullet[\bar{y}] \ y$	
<b>H8. <math>d(x, y) &lt; \delta^\bullet[\bar{y}]</math></b>	<b>[From L1.; Vuln.]</b>	
<hr/>		
<b>T9. <math>\delta^\bullet[\bar{y}] \leq \eta[x]</math></b>		
<b>T8. <math>d(x, y) &lt; \theta[x]</math></b>		

18. Moved H8 down, as  $x$  can only be utilised by T8.

<b>L1</b>	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		[Vuln.]
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		[Vuln.]
<hr/>		
<b>L2</b>	$\delta^\bullet[\bar{y}] \ y$	
<hr/>		
<b>T9. <math>\delta^\bullet[\bar{y}] \leq \eta[x]</math></b>		
<hr/>		
<b>L3</b>	<b>H8. <math>d(x, y) &lt; \delta^\bullet[\bar{y}]</math></b>	<b>[From L1.; Vuln.; From L2.]</b>
<hr/>		
	<b>T8. <math>d(x, y) &lt; \theta[x]</math></b>	

19. Backwards reasoning using library result “transitivity” with (T8,H8).

Since  $d(x, y) < \delta$ ,  
 $d(x, y) < \theta$  if  $\delta \leq \theta$ .

<b>L1</b>	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		[Vuln.]
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		[Vuln.]
<hr/>		
<b>L2</b>	$\delta^\bullet[\bar{y}] \ y$	
<hr/>		
T9. $\delta^\bullet[\bar{y}] \leq \eta[x]$		
<b>L3</b>	H8. $d(x, y) < \delta^\bullet[\bar{y}]$ [From L1.; Vuln.; From L2.]	
<hr/>		
T10. $\delta^\bullet[\bar{y}] \leq \theta[x]$		

20. Delete H8 as no other statement mentions  $x$ .

<b>L1</b>	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		[Vuln.]
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		[Vuln.]
<hr/>		
<b>L2</b>	$\delta^\bullet[\bar{y}] \ y$	
<hr/>		
T9. $\delta^\bullet[\bar{y}] \leq \eta[x]$		
<b>L3</b>	H8. $d(x, y) < \delta^\bullet[\bar{y}]$ [From L1.; Vuln.; From L2.]	
<hr/>		
T10. $\delta^\bullet[\bar{y}] \leq \theta[x]$		

21. Collapsed subtableau L3 as it has no undeleted hypotheses.

<b>L1</b>	$x \ \eta[x] \ \theta[x]$	
H1. $A$ is open		[Vuln.; Used with H4.]
H2. $B$ is open		[Vuln.; Used with H5.]
H4. $x \in A$		[Vuln.]
H5. $x \in B$		[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$		[Vuln.]
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$		[Vuln.]
<hr/>		
<b>L2</b>	$\delta^\bullet[\bar{y}] \ y$	
<hr/>		
T9. $\delta^\bullet[\bar{y}] \leq \eta[x]$		
T10. $\delta^\bullet[\bar{y}] \leq \theta[x]$		

22. Taking  $\delta^\bullet[\bar{y}] = \min(\eta[x], \theta[x])$  matches all targets against a library solution, so L2 is done.

We may therefore take  $\delta = \min(\eta, \theta)$ . We are done.

L1

$x \ \eta[x] \ \theta[x]$

H1. $A$ is open	[Vuln.; Used with H4.]
H2. $B$ is open	[Vuln.; Used with H5.]
H4. $x \in A$	[Vuln.]
H5. $x \in B$	[Vuln.]
H6. $\forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)$	[Vuln.]
H7. $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$	[Vuln.]

---

L2

Done

23. All targets of L1 are 'Done', so L1 is itself done.

L1

Done

Problem solved.





**If  $A$  and  $B$  are closed sets, then  $A \cap B$  is also closed.**

Let  $(a_n)$  and  $a$  be such that  $(a_n)$  is a sequence in  $A \cap B$  and  $a_n \rightarrow a$ . Then  $(a_n)$  is a sequence in  $A$  and  $(a_n)$  is a sequence in  $B$ . Therefore, since  $A$  is closed and  $a_n \rightarrow a$ , we have that  $a \in A$  and since  $B$  is closed and  $a_n \rightarrow a$ , we have that  $a \in B$ . We would like to show that  $a \in A \cap B$ , i.e. that  $a \in A$  and  $a \in B$ . But this is clearly the case, so we are done.

L1  
H1.  $A$  is closed  
H2.  $B$  is closed  


---

**T1.  $A \cap B$  is closed**

1. Expand pre-universal target T1.

L1  
H1.  $A$  is closed  
H2.  $B$  is closed  


---

**T2.  $\forall(a_n), a. ((a_n) \text{ is a sequence in } A \cap B \wedge a_n \rightarrow a \Rightarrow a \in A \cap B)$**

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let  $(a_n)$  and  $a$  be such that  $(a_n)$  is a sequence in  $A \cap B$  and  $a_n \rightarrow a$ .

L1  
 $(a_n) \ a$   
H1.  $A$  is closed  
H2.  $B$  is closed  
**H3.  $(a_n)$  is a sequence in  $A \cap B$**   
H4.  $a_n \rightarrow a$   


---

**T3.  $a \in A \cap B$**

3. Quantifier-free expansion of hypothesis H3.

Since  $(a_n)$  is a sequence in  $A \cap B$ ,  $(a_n)$  is a sequence in  $A$  and  $(a_n)$  is a sequence in  $B$ .

L1  
 $(a_n) \ a$   
**H1.  $A$  is closed**  
H2.  $B$  is closed  
**H5.  $(a_n)$  is a sequence in  $A$**   
H6.  $(a_n)$  is a sequence in  $B$   
**H4.  $a_n \rightarrow a$**   


---

**T3.  $a \in A \cap B$**

4. Forwards reasoning using H1 with (H5,H4).

Since  $A$  is closed,  $(a_n)$  is a sequence in  $A$  and  $a_n \rightarrow a$ , we have that  $a \in A$ .

L1  
 $(a_n) \ a$   
H1.  $A$  is closed [Vuln.; Used with (H4,H5).]  
H2.  $B$  is closed  
H5.  $(a_n)$  is a sequence in  $A$  [Vuln.]  
H6.  $(a_n)$  is a sequence in  $B$   
H4.  $a_n \rightarrow a$  [Vuln.]  
H7.  $a \in A$   


---

**T3.  $a \in A \cap B$**

5. Deleted H1, as the conclusion of this implicative statement is unmatchable.

L1

 $(a_n) a$ 

H1.  $A$  is closed [Vuln.; Used with (H4,H5).]  
**H2.  $B$  is closed**  
H5.  $(a_n)$  is a sequence in  $A$  [Vuln.]  
**H6.  $(a_n)$  is a sequence in  $B$**   
**H4.  $a_n \rightarrow a$  [Vuln.]**  
H7.  $a \in A$   


---

T3.  $a \in A \cap B$

6. Forwards reasoning using H2 with (H6,H4).

L1

 $(a_n) a$ 

H1.  $A$  is closed [Vuln.; Used with (H4,H5).]  
H2.  $B$  is closed [Vuln.; Used with (H4,H6).]  
H5.  $(a_n)$  is a sequence in  $A$  [Vuln.]  
H6.  $(a_n)$  is a sequence in  $B$  [Vuln.]  
H4.  $a_n \rightarrow a$  [Vuln.]  
H7.  $a \in A$   
H8.  $a \in B$   


---

T3.  $a \in A \cap B$

Since  $B$  is closed,  $(a_n)$  is a sequence in  $B$  and  $a_n \rightarrow a$ , we have that  $a \in B$ .

7. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1

 $(a_n) a$ 

H1.  $A$  is closed [Vuln.; Used with (H4,H5).]  
H2.  $B$  is closed [Vuln.; Used with (H4,H6).]  
H5.  $(a_n)$  is a sequence in  $A$  [Vuln.]  
H6.  $(a_n)$  is a sequence in  $B$  [Vuln.]  
H4.  $a_n \rightarrow a$  [Vuln.]  
H7.  $a \in A$   
H8.  $a \in B$   


---

**T3.  $a \in A \cap B$**

8. Quantifier-free expansion of target T3.

L1

 $(a_n) a$ 

H1.  $A$  is closed [Vuln.; Used with (H4,H5).]  
H2.  $B$  is closed [Vuln.; Used with (H4,H6).]  
H5.  $(a_n)$  is a sequence in  $A$  [Vuln.]  
H6.  $(a_n)$  is a sequence in  $B$  [Vuln.]  
H4.  $a_n \rightarrow a$  [Vuln.]  
**H7.  $a \in A$**   
H8.  $a \in B$   


---

**T4.  $a \in A$**   
T5.  $a \in B$

We would like to show that  $a \in A \cap B$ , i.e. that  $a \in A$  and  $a \in B$ .

9. Hypothesis H7 matches target T4, so we can remove T4.

L1

$(a_n) \ a$

H1.  $A$  is closed [Vuln.; Used with (H4,H5).]

H2.  $B$  is closed [Vuln.; Used with (H4,H6).]

H5.  $(a_n)$  is a sequence in  $A$  [Vuln.]

H6.  $(a_n)$  is a sequence in  $B$  [Vuln.]

H4.  $a_n \rightarrow a$  [Vuln.]

H7.  $a \in A$

**H8.  $a \in B$**

---

**T5.  $a \in B$**

10. Hypothesis H8 matches target T5, so L1 is done.

L1

Done

Problem solved.

But this is clearly the case,  
so we are done.



## The pre-image of a closed set $U$ under a continuous function $f$ is closed.

Let  $(a_n)$  and  $a$  be such that  $(a_n)$  is a sequence in  $f^{-1}(U)$  and  $a_n \rightarrow a$ . Then  $f(a_n)$  is a sequence in  $U$ . We would like to show that  $a \in f^{-1}(U)$ , i.e. that  $f(a) \in U$  and since  $U$  is closed,  $f(a) \in U$  if  $f(a_n) \rightarrow f(a)$ . Let  $\epsilon > 0$ . We would like to find  $N$  s.t.  $d(f(a), f(a_n)) < \epsilon$  whenever  $n \geq N$ . Since  $f$  is continuous, there exists  $\delta > 0$  such that  $d(f(a), f(a_n)) < \epsilon$  whenever  $d(a, a_n) < \delta$ . Since  $a_n \rightarrow a$ , there exists  $N'$  such that  $d(a, a_n) < \delta$  whenever  $n \geq N'$ . Therefore, setting  $N = N'$ , we are done.

L1

H1.  $f$  is continuous

H2.  $U$  is closed

---

**T1.  $f^{-1}(U)$  is closed**

1. Expand pre-universal target T1.

L1

H1.  $f$  is continuous

H2.  $U$  is closed

---

**T2.  $\forall (a_n), a. ((a_n) \text{ is a sequence in } f^{-1}(U) \wedge a_n \rightarrow a \Rightarrow a \in f^{-1}(U))$**

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

L1

$(a_n) \ a$

H1.  $f$  is continuous

H2.  $U$  is closed

**H3.  $(a_n)$  is a sequence in  $f^{-1}(U)$**

H4.  $a_n \rightarrow a$

---

**T3.  $a \in f^{-1}(U)$**

Let  $(a_n)$  and  $a$  be such that  $(a_n)$  is a sequence in  $f^{-1}(U)$  and  $a_n \rightarrow a$ .

3. Quantifier-free expansion of hypothesis H3.

L1

$(a_n) \ a$

H1.  $f$  is continuous

H2.  $U$  is closed

H5.  $f(a_n)$  is a sequence in  $U$

H4.  $a_n \rightarrow a$

---

**T3.  $a \in f^{-1}(U)$**

Since  $(a_n)$  is a sequence in  $f^{-1}(U)$ , we have that  $f(a_n)$  is a sequence in  $U$ .

4. Quantifier-free expansion of target T3.

L1

$(a_n) \ a$

H1.  $f$  is continuous

**H2.  $U$  is closed**

**H5.  $f(a_n)$  is a sequence in  $U$**

H4.  $a_n \rightarrow a$

---

**T4.  $f(a) \in U$**

We would like to show that  $a \in f^{-1}(U)$ , i.e. that  $f(a) \in U$ .

5. Backwards reasoning using H2 with (T4,H5).

Since  $U$  is closed and  $f(a_n)$  is a sequence in  $U$ ,  $f(a) \in U$  if  $f(a_n) \rightarrow f(a)$ .

L1	$(a_n) \ a$
H1.	$f$ is continuous
H2.	$U$ is closed [Vuln.]
H5.	$f(a_n)$ is a sequence in $U$ [Vuln.]
H4.	$a_n \rightarrow a$
<hr/>	
T5.	$f(a_n) \rightarrow f(a)$

6. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1	$(a_n) \ a$
H1.	$f$ is continuous
H2.	$U$ is closed [Vuln.]
H5.	$f(a_n)$ is a sequence in $U$ [Vuln.]
H4.	$a_n \rightarrow a$
<hr/>	
T5.	$f(a_n) \rightarrow f(a)$

7. Delete H5 as no other statement mentions  $U$ .

L1	$(a_n) \ a$
H1.	$f$ is continuous
H2.	$U$ is closed [Vuln.]
H5.	$f(a_n)$ is a sequence in $U$ [Vuln.]
H4.	$a_n \rightarrow a$
<hr/>	
<b>T5.</b>	<b><math>f(a_n) \rightarrow f(a)</math></b>

8. Expand pre-universal target T5.

L1	$(a_n) \ a$
H1.	$f$ is continuous
H2.	$U$ is closed [Vuln.]
H5.	$f(a_n)$ is a sequence in $U$ [Vuln.]
H4.	$a_n \rightarrow a$
<hr/>	
<b>T6.</b>	<b><math>\forall \epsilon. (\exists N. (\forall n. (n \geq N \Rightarrow d(f(a), f(a_n)) &lt; \epsilon)))</math></b>

9. Apply 'let' trick and move premise of universal target T6 above the line.

Let  $\epsilon > 0$ .

L1	$(a_n) \ a \ \epsilon$
H1.	$f$ is continuous
H2.	$U$ is closed [Vuln.]
H5.	$f(a_n)$ is a sequence in $U$ [Vuln.]
H4.	$a_n \rightarrow a$
<hr/>	
<b>T7.</b>	<b><math>\exists N. (\forall n. (n \geq N \Rightarrow d(f(a), f(a_n)) &lt; \epsilon))</math></b>

10. Unlock existential-universal-conditional target T7.

We would like to find  $N$   
s.t.  $d(f(a), f(a_n)) < \epsilon$   
whenever  $n \geq N$ .

L1	$(a_n) \ a \ \epsilon$
<b>H1.</b>	<b><math>f</math> is continuous</b>
H2.	$U$ is closed [Vuln.]
H5.	$f(a_n)$ is a sequence in $U$ [Vuln.]
H4.	$a_n \rightarrow a$
<hr/>	
L2♦	$N^\diamond[\bar{n}] \ n$
H6.	$n \geq N^\diamond[\bar{n}]$ [From L1.]
<hr/>	
<b>T8.</b>	<b><math>d(f(a), f(a_n)) &lt; \epsilon</math></b>

11. Backwards reasoning using H1 with T8.

<b>L1</b>	$(a_n) \ a \in$	
	H1. $f$ is continuous	[Vuln.]
	H2. $U$ is closed	[Vuln.]
	H5. $f(a_n)$ is a sequence in $U$	[Vuln.]
	H4. $a_n \rightarrow a$	
<b>L2<math>\blacklozenge</math></b>	$N^\blacklozenge[\bar{n}] \ n \ \delta[x, \eta]$	
	H6. $n \geq N^\blacklozenge[\bar{n}]$	[From L1.]
	T9. $d(a, a_n) < \delta[a, \epsilon]$	

Since  $f$  is continuous, there exists  $\delta > 0$  such that  $d(f(a), f(a_n)) < \epsilon$  whenever  $d(a, a_n) < \delta$ .

12. Delete H1 as no other statement mentions  $f$ .

<b>L1</b>	$(a_n) \ a \in$	
	H1. $f$ is continuous	[Vuln.]
	H2. $U$ is closed	[Vuln.]
	H5. $f(a_n)$ is a sequence in $U$	[Vuln.]
	<b>H4. <math>a_n \rightarrow a</math></b>	
<b>L2<math>\blacklozenge</math></b>	$N^\blacklozenge[\bar{n}] \ n \ \delta[x, \eta]$	
	H6. $n \geq N^\blacklozenge[\bar{n}]$	[From L1.]
	<b>T9. <math>d(a, a_n) &lt; \delta[a, \epsilon]</math></b>	

13. Backwards reasoning using H4 with T9.

<b>L1</b>	$(a_n) \ a \in$	
	H1. $f$ is continuous	[Vuln.]
	H2. $U$ is closed	[Vuln.]
	H5. $f(a_n)$ is a sequence in $U$	[Vuln.]
	H4. $a_n \rightarrow a$	[Vuln.]
<b>L2<math>\blacklozenge</math></b>	$N^\blacklozenge[\bar{n}] \ n \ \delta[x, \eta] \ N'[\eta]$	
	H6. $n \geq N^\blacklozenge[\bar{n}]$	[From L1.]
	<b>T10. <math>n \geq N'[\delta[a, \epsilon]]</math></b>	

Since  $a_n \rightarrow a$ , there exists  $N'$  such that  $d(a, a_n) < \delta$  whenever  $n \geq N'$ .

14. Delete H4 as no other statement mentions  $a$ .

<b>L1</b>	$(a_n) \ a \in$	
	H1. $f$ is continuous	[Vuln.]
	H2. $U$ is closed	[Vuln.]
	H5. $f(a_n)$ is a sequence in $U$	[Vuln.]
	H4. $a_n \rightarrow a$	[Vuln.]
<b>L2<math>\blacklozenge</math></b>	$N^\blacklozenge[\bar{n}] \ n \ \delta[x, \eta] \ N'[\eta]$	
	<b>H6. <math>n \geq N^\blacklozenge[\bar{n}]</math></b>	[From L1.]
	<b>T10. <math>n \geq N'[\delta[a, \epsilon]]</math></b>	

15. Hypothesis H6 matches target T10 after choosing  $N^\blacklozenge[\bar{n}] = N'[\delta[a, \epsilon]]$ , so L2 $\blacklozenge$  is done.

Therefore, setting  $N = N'$ , we are done.

L1

$(a_n) \ a \in$

H1.  $f$  is continuous

[Vuln.]

H2.  $U$  is closed

[Vuln.]

H5.  $f(a_n)$  is a sequence in  $U$

[Vuln.]

H4.  $a_n \rightarrow a$

[Vuln.]

---

L2♦

Done

16. All targets of L1 are 'Done', so L1 is itself done.

L1

Done

Problem solved.



## The pre-image of an open set $U$ under a continuous function $f$ is open.

Let  $x$  be an element of  $f^{-1}(U)$ . Then  $f(x) \in U$ . Therefore, since  $U$  is open, there exists  $\eta > 0$  such that  $u \in U$  whenever  $d(f(x), u) < \eta$ . We would like to find  $\delta > 0$  s.t.  $y \in f^{-1}(U)$  whenever  $d(x, y) < \delta$ . But  $y \in f^{-1}(U)$  if and only if  $f(y) \in U$ . We know that  $f(y) \in U$  whenever  $d(f(x), f(y)) < \eta$ . Since  $f$  is continuous, there exists  $\theta > 0$  such that  $d(f(x), f(y)) < \eta$  whenever  $d(x, y) < \theta$ . Therefore, setting  $\delta = \theta$ , we are done.

L1  
H1.  $f$  is continuous  
H2.  $U$  is open  


---

**T1.  $f^{-1}(U)$  is open**

1. Expand pre-universal target T1.

L1  
H1.  $f$  is continuous  
H2.  $U$  is open  


---

**T2.  $\forall x.(x \in f^{-1}(U) \Rightarrow \exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U))))$**

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let  $x$  be an element of  $f^{-1}(U)$ .

L1  
 $x$   
H1.  $f$  is continuous  
H2.  $U$  is open  
**H3.  $x \in f^{-1}(U)$**   


---

**T3.  $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$**

3. Quantifier-free expansion of hypothesis H3.

Since  $x \in f^{-1}(U)$ , we have that  $f(x) \in U$ .

L1  
 $x$   
H1.  $f$  is continuous  
**H2.  $U$  is open**  
**H4.  $f(x) \in U$**   


---

**T3.  $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$**

4. Forwards reasoning using H2 with H4.

Since  $U$  is open and  $f(x) \in U$ , there exists  $\eta > 0$  such that  $u \in U$  whenever  $d(f(x), u) < \eta$ .

L1  
 $x \quad \eta[f(x)]$   
H1.  $f$  is continuous  
H2.  $U$  is open  
H4.  $f(x) \in U$   
H5.  $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$   


---

**T3.  $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$**

5. Deleted H4, as this unexpandable atomic statement is unmatched.

L1  
 $x \quad \eta[f(x)]$   
H1.  $f$  is continuous  
H2.  $U$  is open  
H4.  $f(x) \in U$   
H5.  $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$   


---

**T3.  $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$**

6. Deleted H2, as the conclusion of this implicative statement is unmatchable.

<b>L1</b>	$x \ \eta[f(x)]$	
	H1. $f$ is continuous	
	H2. $U$ is open	[Vuln.; Used with H4.]
	H4. $f(x) \in U$	[Vuln.]
	H5. $\forall u. (d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$	
<b>T3. <math>\exists \delta. (\forall y. (d(x, y) &lt; \delta \Rightarrow y \in f^{-1}(U)))</math></b>		

7. Unlock existential-universal-conditional target T3.

We would like to find  $\delta > 0$  s.t.  $y \in f^{-1}(U)$  whenever  $d(x, y) < \delta$ .

<b>L1</b>	$x \ \eta[f(x)]$							
	H1. $f$ is continuous							
	H2. $U$ is open	[Vuln.; Used with H4.]						
	H4. $f(x) \in U$	[Vuln.]						
	H5. $\forall u. (d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$							
<table border="0" style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 5%; vertical-align: top;"><b>L2♦</b></td> <td style="width: 40%; text-align: center; vertical-align: top;"> <math>\delta^\diamond[\bar{y}] \ y</math>  H6. <math>d(x, y) &lt; \delta^\diamond[\bar{y}]</math> [From L1.] </td> <td style="width: 55%;"></td> </tr> <tr> <td colspan="3" style="border-top: 1px solid black; padding-top: 5px;"> <b>T4. <math>y \in f^{-1}(U)</math></b> </td> </tr> </table>			<b>L2♦</b>	$\delta^\diamond[\bar{y}] \ y$ H6. $d(x, y) < \delta^\diamond[\bar{y}]$ [From L1.]		<b>T4. <math>y \in f^{-1}(U)</math></b>		
<b>L2♦</b>	$\delta^\diamond[\bar{y}] \ y$ H6. $d(x, y) < \delta^\diamond[\bar{y}]$ [From L1.]							
<b>T4. <math>y \in f^{-1}(U)</math></b>								

8. Quantifier-free expansion of target T4.

But  $y \in f^{-1}(U)$  if and only if  $f(y) \in U$ .

<b>L1</b>	$x \ \eta[f(x)]$							
	H1. $f$ is continuous							
	H2. $U$ is open	[Vuln.; Used with H4.]						
	H4. $f(x) \in U$	[Vuln.]						
	H5. $\forall u. (d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$							
<table border="0" style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 5%; vertical-align: top;"><b>L2♦</b></td> <td style="width: 40%; text-align: center; vertical-align: top;"> <math>\delta^\diamond[\bar{y}] \ y</math>  H6. <math>d(x, y) &lt; \delta^\diamond[\bar{y}]</math> [From L1.] </td> <td style="width: 55%;"></td> </tr> <tr> <td colspan="3" style="border-top: 1px solid black; padding-top: 5px;"> <b>T5. <math>f(y) \in U</math></b> </td> </tr> </table>			<b>L2♦</b>	$\delta^\diamond[\bar{y}] \ y$ H6. $d(x, y) < \delta^\diamond[\bar{y}]$ [From L1.]		<b>T5. <math>f(y) \in U</math></b>		
<b>L2♦</b>	$\delta^\diamond[\bar{y}] \ y$ H6. $d(x, y) < \delta^\diamond[\bar{y}]$ [From L1.]							
<b>T5. <math>f(y) \in U</math></b>								

9. Backwards reasoning using H5 with T5.

We know that  $f(y) \in U$  whenever  $d(f(x), f(y)) < \eta$ .

<b>L1</b>	$x \ \eta[f(x)]$							
	H1. $f$ is continuous							
	H2. $U$ is open	[Vuln.; Used with H4.]						
	H4. $f(x) \in U$	[Vuln.]						
	H5. $\forall u. (d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$	[Vuln.]						
<table border="0" style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 5%; vertical-align: top;"><b>L2♦</b></td> <td style="width: 40%; text-align: center; vertical-align: top;"> <math>\delta^\diamond[\bar{y}] \ y</math>  H6. <math>d(x, y) &lt; \delta^\diamond[\bar{y}]</math> [From L1.] </td> <td style="width: 55%;"></td> </tr> <tr> <td colspan="3" style="border-top: 1px solid black; padding-top: 5px;"> <b>T6. <math>d(f(x), f(y)) &lt; \eta[f(x)]</math></b> </td> </tr> </table>			<b>L2♦</b>	$\delta^\diamond[\bar{y}] \ y$ H6. $d(x, y) < \delta^\diamond[\bar{y}]$ [From L1.]		<b>T6. <math>d(f(x), f(y)) &lt; \eta[f(x)]</math></b>		
<b>L2♦</b>	$\delta^\diamond[\bar{y}] \ y$ H6. $d(x, y) < \delta^\diamond[\bar{y}]$ [From L1.]							
<b>T6. <math>d(f(x), f(y)) &lt; \eta[f(x)]</math></b>								

10. Delete H5 as no other statement mentions  $U$ .

L1	$x \eta[f(x)]$	
	H1. <b><math>f</math> is continuous</b>	
	H2. $U$ is open	[Vuln.; Used with H4.]
	H4. $f(x) \in U$	[Vuln.]
	H5. $\forall u. (d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$	[Vuln.]

---

L2♦	$\delta^\diamond[\bar{y}] y$	
	H6. $d(x, y) < \delta^\diamond[\bar{y}]$	[From L1.]
	<b>T6. <math>d(f(x), f(y)) &lt; \eta[f(x)]</math></b>	

11. Backwards reasoning using H1 with T6.

L1	$x \eta[f(x)]$	
	H1. $f$ is continuous	[Vuln.]
	H2. $U$ is open	[Vuln.; Used with H4.]
	H4. $f(x) \in U$	[Vuln.]
	H5. $\forall u. (d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$	[Vuln.]

---

L2♦	$\delta^\diamond[\bar{y}] y \theta[z, \epsilon]$	
	H6. $d(x, y) < \delta^\diamond[\bar{y}]$	[From L1.]
	<b>T7. <math>d(x, y) &lt; \theta[x, \eta[f(x)]]</math></b>	

Since  $f$  is continuous, there exists  $\theta > 0$  such that  $d(f(x), f(y)) < \eta$  whenever  $d(x, y) < \theta$ .

12. Delete H1 as no other statement mentions  $f$ .

L1	$x \eta[f(x)]$	
	H1. $f$ is continuous	[Vuln.]
	H2. $U$ is open	[Vuln.; Used with H4.]
	H4. $f(x) \in U$	[Vuln.]
	H5. $\forall u. (d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$	[Vuln.]

---

L2♦	$\delta^\diamond[\bar{y}] y \theta[z, \epsilon]$	
	<b>H6. <math>d(x, y) &lt; \delta^\diamond[\bar{y}]</math></b>	[From L1.]
	<b>T7. <math>d(x, y) &lt; \theta[x, \eta[f(x)]]</math></b>	

13. Hypothesis H6 matches target T7 after choosing  $\delta^\diamond[\bar{y}] = \theta[x, \eta[f(x)]]$ , so L2♦ is done.

Therefore, setting  $\delta = \theta$ , we are done.

L1	$x \eta[f(x)]$	
	H1. $f$ is continuous	[Vuln.]
	H2. $U$ is open	[Vuln.; Used with H4.]
	H4. $f(x) \in U$	[Vuln.]
	H5. $\forall u. (d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$	[Vuln.]

---

L2♦	Done
-----	------

14. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.



**If  $f$  and  $g$  are continuous functions, then  $g \circ f$  is continuous.**

Take  $x$  and  $\epsilon > 0$ . We would like to find  $\delta > 0$  s.t.  $d(g(f(x)), g(f(y))) < \epsilon$  whenever  $d(x, y) < \delta$ . Since  $g$  is continuous, there exists  $\eta > 0$  such that  $d(g(f(x)), g(f(y))) < \epsilon$  whenever  $d(f(x), f(y)) < \eta$ . Since  $f$  is continuous, there exists  $\theta > 0$  such that  $d(f(x), f(y)) < \eta$  whenever  $d(x, y) < \theta$ . Therefore, setting  $\delta = \theta$ , we are done.

L1  
H1.  $f$  is continuous  
H2.  $g$  is continuous  

---

T1.  $g \circ f$  is continuous

1. Expand pre-universal target T1.

L1  
H1.  $f$  is continuous  
H2.  $g$  is continuous  

---

T2.  $\forall x, \epsilon. (\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow d(g(f(x)), g(f(y))) < \epsilon)))$

2. Apply 'let' trick and move premise of universal target T2 above the line.

Take  $x$  and  $\epsilon > 0$ .

L1  
 $x \in$   
H1.  $f$  is continuous  
H2.  $g$  is continuous  

---

T3.  $\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow d(g(f(x)), g(f(y))) < \epsilon))$

3. Unlock existential-universal-conditional target T3.

We would like to find  $\delta > 0$  s.t.  $d(g(f(x)), g(f(y))) < \epsilon$  whenever  $d(x, y) < \delta$ .

L1  
 $x \in$   
H1.  $f$  is continuous  
H2.  $g$  is continuous  

---

L2 $\diamond$   
 $\delta^\diamond[\bar{y}] y$   
H3.  $d(x, y) < \delta^\diamond[\bar{y}]$  [From L1.]  

---

T4.  $d(g(f(x)), g(f(y))) < \epsilon$

4. Backwards reasoning using H2 with T4.

Since  $g$  is continuous, there exists  $\eta > 0$  such that  $d(g(f(x)), g(f(y))) < \epsilon$  whenever  $d(f(x), f(y)) < \eta$ .

L1  
 $x \in$   
H1.  $f$  is continuous  
H2.  $g$  is continuous [Vuln.]  

---

L2 $\diamond$   
 $\delta^\diamond[\bar{y}] y \eta[z, \theta]$   
H3.  $d(x, y) < \delta^\diamond[\bar{y}]$  [From L1.]  

---

T5.  $d(f(x), f(y)) < \eta[f(x), \epsilon]$

5. Delete H2 as no other statement mentions  $g$ .

L1  
 $x \in$   
H1.  $f$  is continuous  
H2.  $g$  is continuous [Vuln.]  

---

L2 $\diamond$   
 $\delta^\diamond[\bar{y}] y \eta[z, \theta]$   
H3.  $d(x, y) < \delta^\diamond[\bar{y}]$  [From L1.]  

---

T5.  $d(f(x), f(y)) < \eta[f(x), \epsilon]$

6. Backwards reasoning using H1 with T5.

L1	$x \in$										
	H1. $f$ is continuous	[Vuln.]									
	H2. $g$ is continuous	[Vuln.]									
<table style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 5%; text-align: center; vertical-align: top;">L2<math>\blacklozenge</math></td> <td style="width: 30%; text-align: center; vertical-align: top;"> <math>\delta^\blacklozenge[\bar{y}] \quad y \quad \eta[z, \theta] \quad \theta[z, \alpha]</math> </td> <td style="width: 65%;"></td> </tr> <tr> <td></td> <td>H3. <math>d(x, y) &lt; \delta^\blacklozenge[\bar{y}]</math></td> <td style="text-align: right;">[From L1.]</td> </tr> <tr> <td></td> <td colspan="2" style="border-top: 1px solid black; padding-top: 5px;"> T6. <math>d(x, y) &lt; \theta[x, \eta[f(x), \epsilon]]</math> </td> </tr> </table>			L2 $\blacklozenge$	$\delta^\blacklozenge[\bar{y}] \quad y \quad \eta[z, \theta] \quad \theta[z, \alpha]$			H3. $d(x, y) < \delta^\blacklozenge[\bar{y}]$	[From L1.]		T6. $d(x, y) < \theta[x, \eta[f(x), \epsilon]]$	
L2 $\blacklozenge$	$\delta^\blacklozenge[\bar{y}] \quad y \quad \eta[z, \theta] \quad \theta[z, \alpha]$										
	H3. $d(x, y) < \delta^\blacklozenge[\bar{y}]$	[From L1.]									
	T6. $d(x, y) < \theta[x, \eta[f(x), \epsilon]]$										

Since  $f$  is continuous, there exists  $\theta > 0$  such that  $d(f(x), f(y)) < \eta$  whenever  $d(x, y) < \theta$ .

7. Delete H1 as no other statement mentions  $f$ .

L1	$x \in$										
	H1. $f$ is continuous	[Vuln.]									
	H2. $g$ is continuous	[Vuln.]									
<table style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 5%; text-align: center; vertical-align: top;">L2<math>\blacklozenge</math></td> <td style="width: 30%; text-align: center; vertical-align: top;"> <math>\delta^\blacklozenge[\bar{y}] \quad y \quad \eta[z, \theta] \quad \theta[z, \alpha]</math> </td> <td style="width: 65%;"></td> </tr> <tr> <td></td> <td>H3. <math>d(x, y) &lt; \delta^\blacklozenge[\bar{y}]</math></td> <td style="text-align: right;">[From L1.]</td> </tr> <tr> <td></td> <td colspan="2" style="border-top: 1px solid black; padding-top: 5px;"> T6. <math>d(x, y) &lt; \theta[x, \eta[f(x), \epsilon]]</math> </td> </tr> </table>			L2 $\blacklozenge$	$\delta^\blacklozenge[\bar{y}] \quad y \quad \eta[z, \theta] \quad \theta[z, \alpha]$			H3. $d(x, y) < \delta^\blacklozenge[\bar{y}]$	[From L1.]		T6. $d(x, y) < \theta[x, \eta[f(x), \epsilon]]$	
L2 $\blacklozenge$	$\delta^\blacklozenge[\bar{y}] \quad y \quad \eta[z, \theta] \quad \theta[z, \alpha]$										
	H3. $d(x, y) < \delta^\blacklozenge[\bar{y}]$	[From L1.]									
	T6. $d(x, y) < \theta[x, \eta[f(x), \epsilon]]$										

8. Hypothesis H3 matches target T6 after choosing  $\delta^\blacklozenge[\bar{y}] = \theta[x, \eta[f(x), \epsilon]]$ , so L2 $\blacklozenge$  is done.

Therefore, setting  $\delta = \theta$ , we are done.

L1	$x \in$			
	H1. $f$ is continuous	[Vuln.]		
	H2. $g$ is continuous	[Vuln.]		
<table style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 5%; text-align: center; vertical-align: top;">L2<math>\blacklozenge</math></td> <td style="width: 95%; text-align: center; vertical-align: top;">Done</td> </tr> </table>			L2 $\blacklozenge$	Done
L2 $\blacklozenge$	Done			

9. All targets of L1 are 'Done', so L1 is itself done.

L1	Done
----	------

Problem solved.

**If  $f$  is a continuous function and  $(a_n) \rightarrow a$ , then  $(f(a_n)) \rightarrow f(a)$**

Let  $\epsilon > 0$ . We would like to find  $N$  s.t.  $d(f(a), f(a_n)) < \epsilon$  whenever  $n \geq N$ . Since  $f$  is continuous, there exists  $\delta > 0$  such that  $d(f(a), f(a_n)) < \epsilon$  whenever  $d(a, a_n) < \delta$ . Since  $a_n \rightarrow a$ , there exists  $N'$  such that  $d(a, a_n) < \delta$  whenever  $n \geq N'$ . Therefore, setting  $N = N'$ , we are done.

L1  
H1.  $f$  is continuous  
H2.  $a_n \rightarrow a$   


---

**T1.  $f(a_n) \rightarrow f(a)$**

1. Expand pre-universal target T1.

L1  
H1.  $f$  is continuous  
H2.  $a_n \rightarrow a$   


---

**T2.  $\forall \epsilon. (\exists N. (\forall n. (n \geq N \Rightarrow d(f(a), f(a_n)) < \epsilon)))$**

2. Apply ‘let’ trick and move premise of universal target T2 above the line.

Let  $\epsilon > 0$ .

L1  
 $\epsilon$   
H1.  $f$  is continuous  
H2.  $a_n \rightarrow a$   


---

**T3.  $\exists N. (\forall n. (n \geq N \Rightarrow d(f(a), f(a_n)) < \epsilon))$**

3. Unlock existential-universal-conditional target T3.

We would like to find  $N$  s.t.  $d(f(a), f(a_n)) < \epsilon$  whenever  $n \geq N$ .

L1  
 $\epsilon$   
**H1.  $f$  is continuous**  
H2.  $a_n \rightarrow a$   


---

L2 $\diamond$   
 $N^\diamond[\bar{n}] \ n$   
H3.  $n \geq N^\diamond[\bar{n}]$  [From L1.]  


---

**T4.  $d(f(a), f(a_n)) < \epsilon$**

4. Backwards reasoning using H1 with T4.

Since  $f$  is continuous, there exists  $\delta > 0$  such that  $d(f(a), f(a_n)) < \epsilon$  whenever  $d(a, a_n) < \delta$ .

L1  
 $\epsilon$   
H1.  $f$  is continuous [Vuln.]  
H2.  $a_n \rightarrow a$   


---

L2 $\diamond$   
 $N^\diamond[\bar{n}] \ n \ \delta[x, \eta]$   
H3.  $n \geq N^\diamond[\bar{n}]$  [From L1.]  


---

**T5.  $d(a, a_n) < \delta[a, \epsilon]$**

5. Delete H1 as no other statement mentions  $f$ .

L1  
 $\epsilon$   
H1.  $f$  is continuous [Vuln.]  
**H2.  $a_n \rightarrow a$**   


---

L2 $\diamond$   
 $N^\diamond[\bar{n}] \ n \ \delta[x, \eta]$   
H3.  $n \geq N^\diamond[\bar{n}]$  [From L1.]  


---

**T5.  $d(a, a_n) < \delta[a, \epsilon]$**

6. Backwards reasoning using H2 with T5.

<b>L1</b>	$\epsilon$										
		[Vuln.]									
		[Vuln.]									
<table border="1" style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 5%; text-align: center; vertical-align: top;"><b>L2<math>\blacklozenge</math></b></td> <td style="width: 15%; text-align: center; vertical-align: top;"> <math>N^\blacklozenge[\bar{n}] \quad n \quad \delta[x, \eta] \quad N'[\eta]</math> </td> <td style="width: 80%;"></td> </tr> <tr> <td></td> <td></td> <td style="text-align: right; vertical-align: top;">[From L1.]</td> </tr> <tr> <td colspan="3" style="border-top: 1px solid black; padding-top: 5px;"> <b>T6. <math>n \geq N'[\delta[a, \epsilon]]</math></b> </td> </tr> </table>			<b>L2<math>\blacklozenge</math></b>	$N^\blacklozenge[\bar{n}] \quad n \quad \delta[x, \eta] \quad N'[\eta]$				[From L1.]	<b>T6. <math>n \geq N'[\delta[a, \epsilon]]</math></b>		
<b>L2<math>\blacklozenge</math></b>	$N^\blacklozenge[\bar{n}] \quad n \quad \delta[x, \eta] \quad N'[\eta]$										
		[From L1.]									
<b>T6. <math>n \geq N'[\delta[a, \epsilon]]</math></b>											

Since  $a_n \rightarrow a$ , there exists  $N'$  such that  $d(a, a_n) < \delta$  whenever  $n \geq N'$ .

7. Delete H2 as no other statement mentions  $a$ .

<b>L1</b>	$\epsilon$										
		[Vuln.]									
		[Vuln.]									
<table border="1" style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 5%; text-align: center; vertical-align: top;"><b>L2<math>\blacklozenge</math></b></td> <td style="width: 15%; text-align: center; vertical-align: top;"> <math>N^\blacklozenge[\bar{n}] \quad n \quad \delta[x, \eta] \quad N'[\eta]</math> </td> <td style="width: 80%;"></td> </tr> <tr> <td></td> <td></td> <td style="text-align: right; vertical-align: top;">[From L1.]</td> </tr> <tr> <td colspan="3" style="border-top: 1px solid black; padding-top: 5px;"> <b>T6. <math>n \geq N'[\delta[a, \epsilon]]</math></b> </td> </tr> </table>			<b>L2<math>\blacklozenge</math></b>	$N^\blacklozenge[\bar{n}] \quad n \quad \delta[x, \eta] \quad N'[\eta]$				[From L1.]	<b>T6. <math>n \geq N'[\delta[a, \epsilon]]</math></b>		
<b>L2<math>\blacklozenge</math></b>	$N^\blacklozenge[\bar{n}] \quad n \quad \delta[x, \eta] \quad N'[\eta]$										
		[From L1.]									
<b>T6. <math>n \geq N'[\delta[a, \epsilon]]</math></b>											

8. Hypothesis H3 matches target T6 after choosing  $N^\blacklozenge[\bar{n}] = N'[\delta[a, \epsilon]]$ , so L2 $\blacklozenge$  is done.

Therefore, setting  $N = N'$ , we are done.

<b>L1</b>	$\epsilon$				
		[Vuln.]			
		[Vuln.]			
<table border="1" style="width: 100%; border-collapse: collapse;"> <tr> <td style="width: 5%; text-align: center; vertical-align: top;"><b>L2<math>\blacklozenge</math></b></td> <td colspan="2" style="text-align: center; vertical-align: top;">Done</td> </tr> </table>			<b>L2<math>\blacklozenge</math></b>	Done	
<b>L2<math>\blacklozenge</math></b>	Done				

9. All targets of L1 are 'Done', so L1 is itself done.

<b>L1</b>	Done
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Problem solved.