

If g, f are surjections then $(g \circ f)$ is a surjection.

Let y be an element of C . Then, since g from B to C is a surjection, there exists $u \in B$ such that $g(u) = y$. Since f from A to B is a surjection and $u \in B$, there exists $v \in A$ such that $f(v) = u$. We would like to find $x \in A$ s.t. $g(f(x)) = y$.

L1

| | |
|---|--|
| H1. f from A to B is a surjection | |
| H2. g from B to C is a surjection | |
| T1. $g \circ f$ from A to C is a surjection | |

1. Expand pre-universal target T1.

L1

| | |
|--|--|
| H1. f from A to B is a surjection | |
| H2. g from B to C is a surjection | |
| T2. $\forall y. (y \in C \Rightarrow \exists x. (x \in A \wedge g(f(x)) = y))$ | |

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let y be an element of C .

L1

| | |
|---|-----|
| H1. f from A to B is a surjection | y |
| H2. g from B to C is a surjection | |
| H3. $y \in C$ | |
| T3. $\exists x. (x \in A \wedge g(f(x)) = y)$ | |

3. Forwards reasoning using H2 with H3.

Since g from B to C is a surjection and $y \in C$, there exists $u \in B$ such that $g(u) = y$.

L1

| | |
|---|------------------------|
| H1. f from A to B is a surjection | $y \ u[y]$ |
| H2. g from B to C is a surjection | [Vuln.; Used with H3.] |
| H3. $y \in C$ | [Vuln.] |
| H4. $u[y] \in B$ | |
| H5. $g(u[y]) = y$ | |
| T3. $\exists x. (x \in A \wedge g(f(x)) = y)$ | |

4. Deleted H3, as this unexpandable atomic statement is unmatchable.

L1

| | |
|---|------------------------|
| H1. f from A to B is a surjection | $y \ u[y]$ |
| H2. g from B to C is a surjection | [Vuln.; Used with H3.] |
| H3. $y \in C$ | [Vuln.] |
| H4. $u[y] \in B$ | |
| H5. $g(u[y]) = y$ | |
| T3. $\exists x. (x \in A \wedge g(f(x)) = y)$ | |

5. Delete H2 as no other statement mentions C .

L1

| | |
|---|------------------------|
| H1. f from A to B is a surjection | $y \ u[y]$ |
| H2. g from B to C is a surjection | [Vuln.; Used with H3.] |
| H3. $y \in C$ | [Vuln.] |
| H4. $u[y] \in B$ | |
| H5. $g(u[y]) = y$ | |
| T3. $\exists x. (x \in A \wedge g(f(x)) = y)$ | |

6. Forwards reasoning using H1 with H4.

| | |
|-------|--|
| L1 | $y \ u[y] \ v[u[y]]$ |
| H1. | f from A to B is a surjection [Vuln.; Used with H4.] |
| H2. | g from B to C is a surjection [Vuln.; Used with H3.] |
| H3. | $y \in C$ [Vuln.] |
| H4. | $u[y] \in B$ [Vuln.] |
| H5. | $g(u[y]) = y$ |
| H6. | $v[u[y]] \in A$ |
| H7. | $f(v[u[y]]) = u[y]$ |
| <hr/> | |
| T3. | $\exists x.(x \in A \wedge g(f(x)) = y)$ |

Since f from A to B is a surjection and $u \in B$, there exists $v \in A$ such that $f(v) = u$.

7. Deleted H4, as this unexpandable atomic statement is unmatched.

| | |
|-------|--|
| L1 | $y \ u[y] \ v[u[y]]$ |
| H1. | f from A to B is a surjection [Vuln.; Used with H4.] |
| H2. | g from B to C is a surjection [Vuln.; Used with H3.] |
| H3. | $y \in C$ [Vuln.] |
| H4. | $u[y] \in B$ [Vuln.] |
| H5. | $g(u[y]) = y$ |
| H6. | $v[u[y]] \in A$ |
| H7. | $f(v[u[y]]) = u[y]$ |
| <hr/> | |
| T3. | $\exists x.(x \in A \wedge g(f(x)) = y)$ |

8. Delete H1 as no other statement mentions B .

| | |
|-------|--|
| L1 | $y \ u[y] \ v[u[y]]$ |
| H1. | f from A to B is a surjection [Vuln.; Used with H4.] |
| H2. | g from B to C is a surjection [Vuln.; Used with H3.] |
| H3. | $y \in C$ [Vuln.] |
| H4. | $u[y] \in B$ [Vuln.] |
| H5. | $g(u[y]) = y$ |
| H6. | $v[u[y]] \in A$ |
| H7. | $f(v[u[y]]) = u[y]$ |
| <hr/> | |
| T3. | $\exists x.(x \in A \wedge g(f(x)) = y)$ |

9. Unlock existential target T3.

We would like to find $x \in A$ s.t. $g(f(x)) = y$.

| | |
|-------|--|
| L1 | $y \ u[y] \ v[u[y]]$ |
| H1. | f from A to B is a surjection [Vuln.; Used with H4.] |
| H2. | g from B to C is a surjection [Vuln.; Used with H3.] |
| H3. | $y \in C$ [Vuln.] |
| H4. | $u[y] \in B$ [Vuln.] |
| H5. | $g(u[y]) = y$ |
| H6. | $v[u[y]] \in A$ |
| H7. | $f(v[u[y]]) = u[y]$ |
| <hr/> | |
| L2♦ | x^\diamond |
| <hr/> | |
| T4. | $x^\diamond \in A$ |
| T5. | $g(f(x^\diamond)) = y$ |

No moves possible.

If f is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of $f(A) \cap f(B)$. Then $x \in f(A)$ and $x \in f(B)$. That is, there exists $y \in A$ such that $f(y) = x$ and there exists $z \in B$ such that $f(z) = x$. Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$. We would like to find $u \in A \cap B$ s.t. $f(u) = x$. But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$. Since $y = z$, we have that $y \in B$. Therefore, setting $u = y$, we are done.

L1
 H1. f is an injection

 T1. $f(A) \cap f(B) \subset f(A \cap B)$

1. Expand pre-universal target T1.

L1
 H1. f is an injection

 T2. $\forall x. (x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(A) \cap f(B)$.

L1
 x
 H1. f is an injection
 H2. $x \in f(A) \cap f(B)$

 T3. $x \in f(A \cap B)$

3. Quantifier-free expansion of hypothesis H2.

Since $x \in f(A) \cap f(B)$,
 $x \in f(A)$ and $x \in f(B)$.

L1
 x
 H1. f is an injection
 H3. $x \in f(A)$
 H4. $x \in f(B)$

 T3. $x \in f(A \cap B)$

4. Expand pre-existential hypothesis H3.

By definition, since $x \in f(A)$, there exists $y \in A$ such that $f(y) = x$.

L1
 $x \ y$
 H1. f is an injection
 H5. $y \in A$
 H6. $f(y) = x$
 H4. $x \in f(B)$

 T3. $x \in f(A \cap B)$

5. Expand pre-existential hypothesis H4.

By definition, since $x \in f(B)$, there exists $z \in B$ such that $f(z) = x$.

L1
 $x \ y \ z$
 H1. f is an injection
 H5. $y \in A$
 H6. $f(y) = x$
 H7. $z \in B$
 H8. $f(z) = x$

 T3. $x \in f(A \cap B)$

6. Forwards reasoning using H1 with (H6,H8).

Since f is an injection,
 $f(y) = x$ and $f(z) = x$,
 we have that $y = z$.

| | |
|---|-----------------------------|
| L1 | $x \ y \ z$ |
| H1. f is an injection | [Vuln.; Used with (H6,H8).] |
| H5. $y \in A$ | |
| H6. $f(y) = x$ | [Vuln.] |
| H7. $z \in B$ | |
| H8. $f(z) = x$ | [Vuln.] |
| H9. $y = z$ | |
| <hr/> | |
| T3. $x \in f(A \cap B)$ | |

7. Expand pre-existential target T3.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

| | |
|---|-----------------------------|
| L1 | $x \ y \ z$ |
| H1. f is an injection | [Vuln.; Used with (H6,H8).] |
| H5. $y \in A$ | |
| H6. $f(y) = x$ | [Vuln.] |
| H7. $z \in B$ | |
| H8. $f(z) = x$ | [Vuln.] |
| H9. $y = z$ | |
| <hr/> | |
| T4. $\exists u. (u \in A \cap B \wedge f(u) = x)$ | |

8. Unlock existential target T4.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

| | |
|---|-----------------------------|
| L1 | $x \ y \ z$ |
| H1. f is an injection | [Vuln.; Used with (H6,H8).] |
| H5. $y \in A$ | |
| H6. $f(y) = x$ | [Vuln.] |
| H7. $z \in B$ | |
| H8. $f(z) = x$ | [Vuln.] |
| H9. $y = z$ | |
| <hr/> | |
| L2♦ | u^\diamond |
| <hr/> | |
| T5. $u^\diamond \in A \cap B$ | |
| T6. $f(u^\diamond) = x$ | |

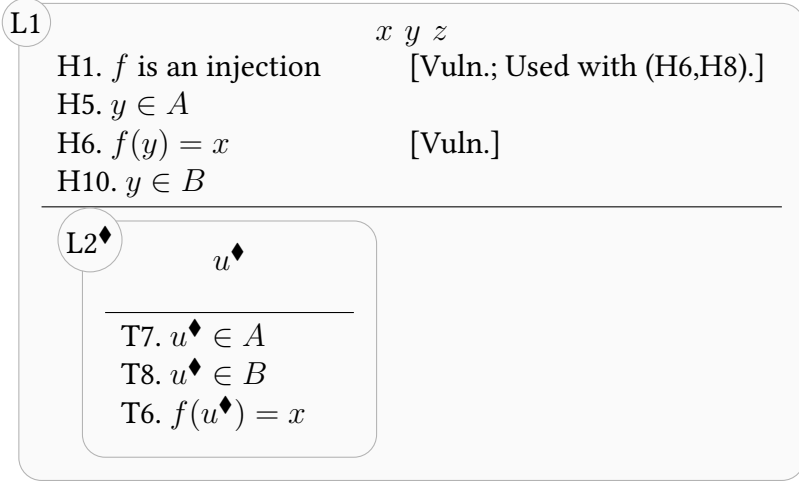
9. Quantifier-free expansion of target T5.

But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$.

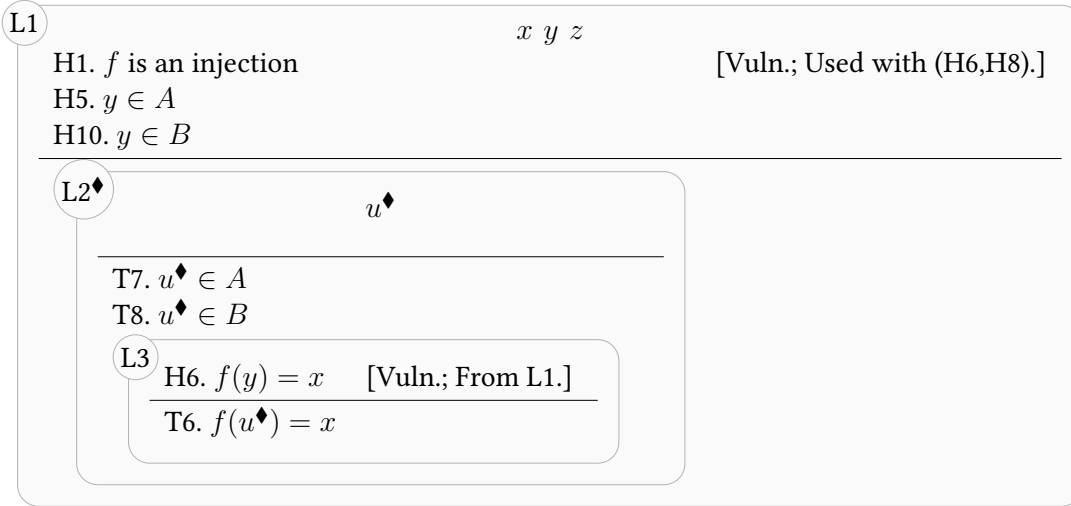
| | |
|---|-----------------------------|
| L1 | $x \ y \ z$ |
| H1. f is an injection | [Vuln.; Used with (H6,H8).] |
| H5. $y \in A$ | |
| H6. $f(y) = x$ | [Vuln.] |
| H7. $z \in B$ | |
| H8. $f(z) = x$ | [Vuln.] |
| H9. $y = z$ | |
| <hr/> | |
| L2♦ | u^\diamond |
| <hr/> | |
| T7. $u^\diamond \in A$ | |
| T8. $u^\diamond \in B$ | |
| T6. $f(u^\diamond) = x$ | |

10. Rewrite z as y throughout the tableau using hypothesis H9.

Since $y = z$, we have that $y \in B$.

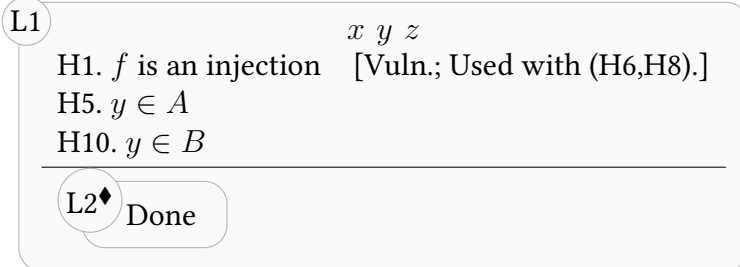


11. Moved H6 down, as x can only be utilised by T6.



12. Choosing $u^\blacklozenge = y$ matches all targets inside L2 \blacklozenge against hypotheses, so L2 \blacklozenge is done.

Therefore, setting $u = y$, we are done.



13. All targets of L1 are 'Done', so L1 is itself done.



Problem solved.

