

If g, f are surjections then $(g \circ f)$ is a surjection.

Let y be an element of C . Then, since g from B to C is a surjection, there exists $u \in B$ such that $g(u) = y$. Since f from A to B is a surjection and $u \in B$, there exists $v \in A$ such that $f(v) = u$. We would like to find $x \in A$ s.t. $g(f(x)) = y$.

L1

H1. f from A to B is a surjection
H2. g from B to C is a surjection
T1. $g \circ f$ from A to C is a surjection

1. Expand pre-universal target T1.

L1

H1. f from A to B is a surjection
H2. g from B to C is a surjection
T2. $\forall y. (y \in C \Rightarrow \exists x. (x \in A \wedge g(f(x)) = y))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let y be an element of C .

L1

y
H1. f from A to B is a surjection
H2. g from B to C is a surjection
H3. $y \in C$
T3. $\exists x. (x \in A \wedge g(f(x)) = y)$

3. Forwards reasoning using H2 with H3.

Since g from B to C is a surjection and $y \in C$, there exists $u \in B$ such that $g(u) = y$.

L1

$y \quad u[y]$
H1. f from A to B is a surjection
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$
H5. $g(u[y]) = y$
T3. $\exists x. (x \in A \wedge g(f(x)) = y)$

4. Deleted H3, as this unexpandable atomic statement is unmatchable.

L1

$y \quad u[y]$
H1. f from A to B is a surjection
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$
H5. $g(u[y]) = y$
T3. $\exists x. (x \in A \wedge g(f(x)) = y)$

5. Delete H2 as no other statement mentions C .

L1

$y \quad u[y]$
H1. f from A to B is a surjection
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$
H5. $g(u[y]) = y$
T3. $\exists x. (x \in A \wedge g(f(x)) = y)$

6. Forwards reasoning using H1 with H4.

L1	$y \ u[y] \ v[u[y]]$
H1.	f from A to B is a surjection [Vuln.; Used with H4.]
H2.	g from B to C is a surjection [Vuln.; Used with H3.]
H3.	$y \in C$ [Vuln.]
H4.	$u[y] \in B$ [Vuln.]
H5.	$g(u[y]) = y$
H6.	$v[u[y]] \in A$
H7.	$f(v[u[y]]) = u[y]$
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T3.	$\exists x.(x \in A \wedge g(f(x)) = y)$

Since f from A to B is a surjection and $u \in B$, there exists $v \in A$ such that $f(v) = u$.

7. Deleted H4, as this unexpandable atomic statement is unmatched.

L1	$y \ u[y] \ v[u[y]]$
H1.	f from A to B is a surjection [Vuln.; Used with H4.]
H2.	g from B to C is a surjection [Vuln.; Used with H3.]
H3.	$y \in C$ [Vuln.]
H4.	$u[y] \in B$ [Vuln.]
H5.	$g(u[y]) = y$
H6.	$v[u[y]] \in A$
H7.	$f(v[u[y]]) = u[y]$
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T3.	$\exists x.(x \in A \wedge g(f(x)) = y)$

8. Delete H1 as no other statement mentions B .

L1	$y \ u[y] \ v[u[y]]$
H1.	f from A to B is a surjection [Vuln.; Used with H4.]
H2.	g from B to C is a surjection [Vuln.; Used with H3.]
H3.	$y \in C$ [Vuln.]
H4.	$u[y] \in B$ [Vuln.]
H5.	$g(u[y]) = y$
H6.	$v[u[y]] \in A$
H7.	$f(v[u[y]]) = u[y]$
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T3.	$\exists x.(x \in A \wedge g(f(x)) = y)$

9. Unlock existential target T3.

We would like to find $x \in A$ s.t. $g(f(x)) = y$.

L1	$y \ u[y] \ v[u[y]]$
H1.	f from A to B is a surjection [Vuln.; Used with H4.]
H2.	g from B to C is a surjection [Vuln.; Used with H3.]
H3.	$y \in C$ [Vuln.]
H4.	$u[y] \in B$ [Vuln.]
H5.	$g(u[y]) = y$
H6.	$v[u[y]] \in A$
H7.	$f(v[u[y]]) = u[y]$
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L2♦	x^\diamond
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T4.	$x^\diamond \in A$
T5.	$g(f(x^\diamond)) = y$

No moves possible.

If f is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of $f(A) \cap f(B)$. Then $x \in f(A)$ and $x \in f(B)$. That is, there exists $y \in A$ such that $f(y) = x$ and there exists $z \in B$ such that $f(z) = x$. Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$. We would like to find $u \in A \cap B$ s.t. $f(u) = x$. But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$. Since $y = z$, we have that $y \in B$. Therefore, setting $u = y$, we are done.

L1
H1. f is an injection

T1. $f(A) \cap f(B) \subset f(A \cap B)$

1. Expand pre-universal target T1.

L1
H1. f is an injection

T2. $\forall x. (x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(A) \cap f(B)$.

L1
 x
H1. f is an injection
H2. $x \in f(A) \cap f(B)$

T3. $x \in f(A \cap B)$

3. Quantifier-free expansion of hypothesis H2.

Since $x \in f(A) \cap f(B)$,
 $x \in f(A)$ and $x \in f(B)$.

L1
 x
H1. f is an injection
H3. $x \in f(A)$
H4. $x \in f(B)$

T3. $x \in f(A \cap B)$

4. Expand pre-existential hypothesis H3.

By definition, since $x \in f(A)$, there exists $y \in A$ such that $f(y) = x$.

L1
 $x \ y$
H1. f is an injection
H5. $y \in A$
H6. $f(y) = x$
H4. $x \in f(B)$

T3. $x \in f(A \cap B)$

5. Expand pre-existential hypothesis H4.

By definition, since $x \in f(B)$, there exists $z \in B$ such that $f(z) = x$.

L1
 $x \ y \ z$
H1. f is an injection
H5. $y \in A$
H6. $f(y) = x$
H7. $z \in B$
H8. $f(z) = x$

T3. $x \in f(A \cap B)$

6. Forwards reasoning using H1 with (H6,H8).

Since f is an injection,
 $f(y) = x$ and $f(z) = x$,
we have that $y = z$.

L1	$x \ y \ z$
H1. f is an injection	[Vuln.; Used with (H6,H8).]
H5. $y \in A$	
H6. $f(y) = x$	[Vuln.]
H7. $z \in B$	
H8. $f(z) = x$	[Vuln.]
H9. $y = z$	
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T3. $x \in f(A \cap B)$	

7. Expand pre-existential target T3.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1	$x \ y \ z$
H1. f is an injection	[Vuln.; Used with (H6,H8).]
H5. $y \in A$	
H6. $f(y) = x$	[Vuln.]
H7. $z \in B$	
H8. $f(z) = x$	[Vuln.]
H9. $y = z$	
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T4. $\exists u. (u \in A \cap B \wedge f(u) = x)$	

8. Unlock existential target T4.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1	$x \ y \ z$
H1. f is an injection	[Vuln.; Used with (H6,H8).]
H5. $y \in A$	
H6. $f(y) = x$	[Vuln.]
H7. $z \in B$	
H8. $f(z) = x$	[Vuln.]
H9. $y = z$	
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L2♦	u^\diamond
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T5. $u^\diamond \in A \cap B$	
T6. $f(u^\diamond) = x$	

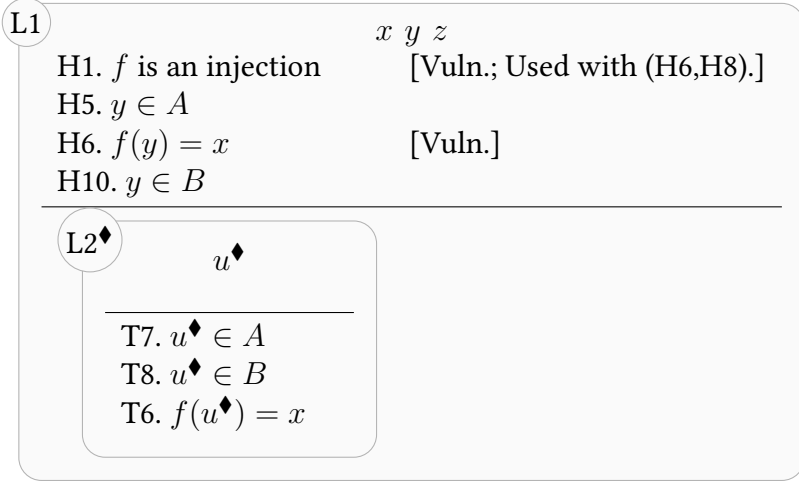
9. Quantifier-free expansion of target T5.

But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$.

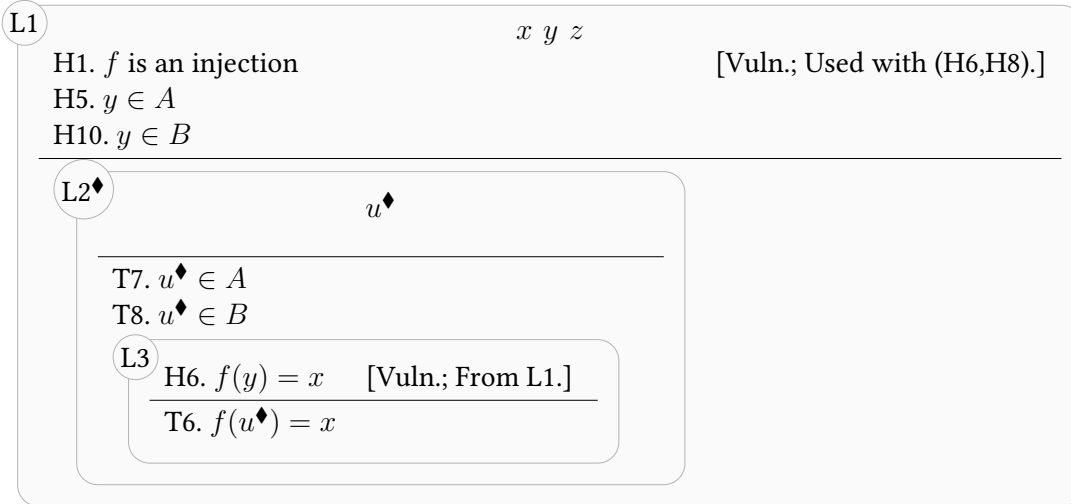
L1	$x \ y \ z$
H1. f is an injection	[Vuln.; Used with (H6,H8).]
H5. $y \in A$	
H6. $f(y) = x$	[Vuln.]
H7. $z \in B$	
H8. $f(z) = x$	[Vuln.]
H9. $y = z$	
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L2♦	u^\diamond
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T7. $u^\diamond \in A$	
T8. $u^\diamond \in B$	
T6. $f(u^\diamond) = x$	

10. Rewrite z as y throughout the tableau using hypothesis H9.

Since $y = z$, we have that $y \in B$.

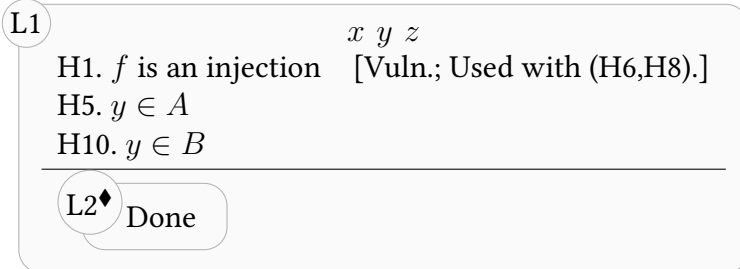


11. Moved H6 down, as x can only be utilised by T6.



12. Choosing $u^\blacklozenge = y$ matches all targets inside L2 \blacklozenge against hypotheses, so L2 \blacklozenge is done.

Therefore, setting $u = y$, we are done.



13. All targets of L1 are 'Done', so L1 is itself done.



Problem solved.

