If x_n is convergent then x_n is Cauchy.

Let $\epsilon > 0$. We would like to find N s.t. $\forall m, n.(m \geqslant N$ and $n \geqslant N \Rightarrow d(a_m, a_n) < \epsilon)$. Assume now that $m \geqslant N$ and $n \geqslant N$.

L1
H1. $\exists a.(a_n \to a)$

T1. (a_n) is Cauchy

1. Expand pre-universal target T1.

L1
H1. $\exists a.(a_n \to a)$

T2. $\boldsymbol{\forall \epsilon.(\exists N.(\forall m, n.(m \geqslant N \wedge n \geqslant N \Rightarrow d(a_m, a_n) < \epsilon)))}$

2. pply 'let' trick and move premise of universal target T2 above the line.

Let $\epsilon > 0$.

L1
ϵ
H1. $\exists a.(a_n \to a)$

T3. $\boldsymbol{\exists N.(\forall m, n.(m \geqslant N \wedge n \geqslant N \Rightarrow d(a_m, a_n) < \epsilon))}$

3. Unlock existential-universal-conditional target T3.

We would like to find N s.t. $\forall m, n.(m \geqslant N$ and $n \geqslant N \Rightarrow d(a_m, a_n) < \epsilon)$.

L1
ϵ
H1. $\exists a.(a_n \to a)$

L2$^{\blacklozenge}$
$N^{\blacklozenge}[\overline{m}, \overline{n}]$ m n
H2. $m \geqslant N^{\blacklozenge}[\overline{m}, \overline{n}]$ [From L1.]
H3. $n \geqslant N^{\blacklozenge}[\overline{m}, \overline{n}]$ [From L1.]

T4. $d(a_m, a_n) < \epsilon$

4. Replacing diamonds with bullets in L2$^{\blacklozenge}$.

Assume now that $m \geqslant N$ and $n \geqslant N$.

L1
ϵ
H1. $\exists a.(a_n \to a)$

L2
$N^{\bullet}[\overline{m}, \overline{n}]$ m n
H2. $m \geqslant N^{\bullet}[\overline{m}, \overline{n}]$ [From L1.]
H3. $n \geqslant N^{\bullet}[\overline{m}, \overline{n}]$ [From L1.]

T4. $d(a_m, a_n) < \epsilon$

No moves possible.

Prove that $A \subseteq f^{-1}(f(A))$

Let x be an element of A. We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$. But this is clearly the case, so we are done.

L1

T1. $\boldsymbol{A \subset f^{-1}(f(A))}$

1. Expand pre-universal target T1.

L1

T2. $\boldsymbol{\forall x.(x \in A \Rightarrow x \in f^{-1}(f(A)))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of A.

L1

x

H1. $x \in A$

T3. $\boldsymbol{x \in f^{-1}(f(A))}$

3. Quantifier-free expansion of target T3.

We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$.

L1

x

H1. $\boldsymbol{x \in A}$

T4. $\boldsymbol{f(x) \in f(A)}$

4. All conjuncts of T4 (after expansion) can be simultaneously matched against H1 or rendered trivial by choosing $y = x$, so L1 is done.

We would like to show that $f(x) \in f(A)$. But this is clearly the case, so we are done.

L1 Done

Problem solved.

Prove that $f(f^{-1}(A)) \subset A$

Let x be an element of $f(f^{-1}(A))$. Then there exists $y \in f^{-1}(A)$ such that $f(y) = x$. Since $y \in f^{-1}(A)$, we have that $f(y) \in A$. Since $f(y) = x$, we have that $x \in A$ and we are done.

L1

T1. $\boldsymbol{f(f^{-1}(A)) \subset A}$

1. Expand pre-universal target T1.

L1

T2. $\boldsymbol{\forall x.(x \in f(f^{-1}(A)) \Rightarrow x \in A)}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(f^{-1}(A))$.

L1

x

H1. $\boldsymbol{x \in f(f^{-1}(A))}$

T3. $x \in A$

3. Expand pre-existential hypothesis H1.

By definition, since $x \in f(f^{-1}(A))$, there exists $y \in f^{-1}(A)$ such that $f(y) = x$.

L1

x y

H2. $\boldsymbol{y \in f^{-1}(A)}$

H3. $f(y) = x$

T3. $x \in A$

4. Quantifier-free expansion of hypothesis H2.

Since $y \in f^{-1}(A)$, we have that $f(y) \in A$.

L1

x y

H4. $f(y) \in A$

H3. $\boldsymbol{f(y) = x}$

T3. $x \in A$

5. Rewrite $f(y)$ as x throughout the tableau using hypothesis H3.

Since $f(y) = x$, we have that $x \in A$.

L1

x y

H5. $\boldsymbol{x \in A}$

T3. $\boldsymbol{x \in A}$

6. Hypothesis H5 matches target T3, so L1 is done.

We are done.

L1 Done

Problem solved.

Prove that $f(A \cap B) \subset f(A) \cap f(B)$

By definition, since $y \in f(A \cap B)$, there exists $z \in A \cap B$ such that $f(z) = y$. Since $z \in A \cap B$, $z \in A$ and $z \in B$. We would like to show that $y \in f(A) \cap f(B)$, i.e. that $y \in f(A)$ and $y \in f(B)$. We would like to show that $y \in f(A)$. But this is clearly the case, so we are done. Thus $y \in f(B)$ and we are done.

L1

H1. $y \in f(A \cap B)$

T1. $y \in f(A) \cap f(B)$

1. Expand pre-existential hypothesis H1.

By definition, since $y \in f(A \cap B)$, there exists $z \in A \cap B$ such that $f(z) = y$.

L1

z

H2. $z \in A \cap B$

H3. $f(z) = y$

T1. $y \in f(A) \cap f(B)$

2. Quantifier-free expansion of hypothesis H2.

Since $z \in A \cap B$, $z \in A$ and $z \in B$.

L1

z

H4. $z \in A$

H5. $z \in B$

H3. $f(z) = y$

T1. $y \in f(A) \cap f(B)$

3. Quantifier-free expansion of target T1.

We would like to show that $y \in f(A) \cap f(B)$, i.e. that $y \in f(A)$ and $y \in f(B)$.

L1

z

H4. $z \in A$

H5. $z \in B$

H3. $f(z) = y$

T2. $y \in f(A)$

T3. $y \in f(B)$

4. All conjuncts of T2 (after expansion) can be simultaneously matched against H4 and H3 or rendered trivial by choosing $u = z$, so we can remove T2.

We would like to show that $y \in f(A)$. But this is clearly the case, so we are done.

L1

z

H4. $z \in A$

H5. $z \in B$

H3. $f(z) = y$

T3. $y \in f(B)$

5. All conjuncts of T3 (after expansion) can be simultaneously matched against H5 and H3 or rendered trivial by choosing $u = z$, so L1 is done.

We would like to show that $y \in f(B)$. But this is clearly the case, so we are done.

L1 Done

Problem solved.