Prove that $A \subseteq f^{-1}(f(A))$

Let x be an element of A. We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$. But this is clearly the case, so we are done.

L1

T1. $\boldsymbol{A \subset f^{-1}(f(A))}$

1. Expand pre-universal target T1.

L1

T2. $\boldsymbol{\forall x.(x \in A \Rightarrow x \in f^{-1}(f(A)))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of A.

L1

x

H1. $x \in A$

T3. $\boldsymbol{x \in f^{-1}(f(A))}$

3. Quantifier-free expansion of target T3.

We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$.

L1

x

H1. $\boldsymbol{x \in A}$

T4. $\boldsymbol{f(x) \in f(A)}$

4. All conjuncts of T4 (after expansion) can be simultaneously matched against H1 or rendered trivial by choosing $y = x$, so L1 is done.

We would like to show that $f(x) \in f(A)$. But this is clearly the case, so we are done.

L1 Done

Problem solved.

If g,f are surjections then (g o f) is a surjection.

Let y be an element of C. Then, since g from B to C is a surjection, we have that $y \in g(B)$. That is, there exists $z \in B$ such that $g(z) = y$. Since f from A to B is a surjection and $z \in B$, we have that $z \in f(A)$. That is, there exists $u \in A$ such that $f(u) = z$. We would like to find $v \in A$ s.t. $g(f(v)) = y$.

L1

H1. f from A to B is a surjection
H2. g from B to C is a surjection

T1. $g \circ f$ from A to C is a surjection

1. Expand pre-universal target T1.

L1

H1. f from A to B is a surjection
H2. g from B to C is a surjection

T2. $\forall y.(y \in C \Rightarrow y \in g \circ f(A))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let y be an element of C.

L1

y

H1. f from A to B is a surjection
H2. g from B to C is a surjection
H3. $y \in C$

T3. $y \in g \circ f(A)$

3. Forwards reasoning using H2 with H3.

Since g from B to C is a surjection and $y \in C$, we have that $y \in g(B)$.

L1

y

H1. f from A to B is a surjection
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $y \in g(B)$

T3. $y \in g \circ f(A)$

4. Deleted H3, as this unexpandable atomic statement is unmatchable.

L1

y

H1. f from A to B is a surjection
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $y \in g(B)$

T3. $y \in g \circ f(A)$

5. Delete H2 as no other statement mentions C.

L1

y

H1. f from A to B is a surjection
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $y \in g(B)$

T3. $y \in g \circ f(A)$

6. Expand pre-existential hypothesis H4.

By definition, since $y \in g(B)$, there exists $z \in B$ such that $g(z) = y$.

L1

$y\ z$

H1. f from A to B is a surjection
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H5. $z \in B$
H6. $g(z) = y$

T3. $y \in g \circ f(A)$

7. Forwards reasoning using H1 with H5.

Since f from A to B is a surjection and $z \in B$, we have that $z \in f(A)$.

L1

$y\ z$

H1. f from A to B is a surjection [Vuln.; Used with H5.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H5. $z \in B$ [Vuln.]
H6. $g(z) = y$
H7. $z \in f(A)$

T3. $y \in g \circ f(A)$

8. Deleted H5, as this unexpandable atomic statement is unmatchable.

L1

$y\ z$

H1. f from A to B is a surjection [Vuln.; Used with H5.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H5. $z \in B$ [Vuln.]
H6. $g(z) = y$
H7. $z \in f(A)$

T3. $y \in g \circ f(A)$

9. Delete H1 as no other statement mentions B.

L1

$y\ z$

H1. f from A to B is a surjection [Vuln.; Used with H5.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H5. $z \in B$ [Vuln.]
H6. $g(z) = y$
H7. $z \in f(A)$

T3. $y \in g \circ f(A)$

10. Expand pre-existential hypothesis H7.

By definition, since $z \in f(A)$, there exists $u \in A$ such that $f(u) = z$.

L1

$y\ z\ u$

H1. f from A to B is a surjection [Vuln.; Used with H5.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H5. $z \in B$ [Vuln.]
H6. $g(z) = y$
H8. $u \in A$
H9. $f(u) = z$

T3. $y \in g \circ f(A)$

11. Expand pre-existential target T3.

We would like to find $v \in A$ s.t. $g(f(v)) = y$.

L1

$y \; z \; u$

H1. f from A to B is a surjection [Vuln.; Used with H5.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H5. $z \in B$ [Vuln.]
H6. $g(z) = y$
H8. $u \in A$
H9. $f(u) = z$

T4. $\boldsymbol{\exists v.(v \in A \wedge g(f(v)) = y)}$

12. Unlock existential target T4.

We would like to find $v \in A$ s.t. $g(f(v)) = y$.

L1

$y \; z \; u$

H1. f from A to B is a surjection [Vuln.; Used with H5.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H5. $z \in B$ [Vuln.]
H6. $g(z) = y$
H8. $u \in A$
H9. $f(u) = z$

L2$^{\blacklozenge}$

$v^{\blacklozenge}$

T5. $v^{\blacklozenge} \in A$
T6. $g(f(v^{\blacklozenge})) = y$

No moves possible.

If f is a surjection then $f(A)^c \subset f(A^c)$

Let x be an element of $f(A)^c$. Then $x \notin f(A)$. Then it is not that case that $x \in f(A)$. We would like to find $y \in A^c$ s.t. $f(y) = x$. But $y \in A^c$ if and only if $y \notin A$. But $y \notin A$ if and only if it is not that case that $y \in A$.

L1

H1. f from A to C is a surjection

T1. $\boldsymbol{f(A)^c \subset f(A^c)}$

1. Expand pre-universal target T1.

L1

H1. f from A to C is a surjection

T2. $\boldsymbol{\forall x.(x \in f(A)^c \Rightarrow x \in f(A^c))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(A)^c$.

L1

x

H1. f from A to C is a surjection

H2. $\boldsymbol{x \in f(A)^c}$

T3. $x \in f(A^c)$

3. Quantifier-free expansion of hypothesis H2.

Since $x \in f(A)^c$, we have that $x \notin f(A)$.

L1

x

H1. f from A to C is a surjection

H3. $\boldsymbol{x \notin f(A)}$

T3. $x \in f(A^c)$

4. Quantifier-free expansion of hypothesis H3.

Since $x \notin f(A)$, it is not that case that $x \in f(A)$.

L1

x

H1. f from A to C is a surjection

H4. $\neg x \in f(A)$

T3. $\boldsymbol{x \in f(A^c)}$

5. Expand pre-existential target T3.

We would like to find $y \in A^c$ s.t. $f(y) = x$.

L1

x

H1. f from A to C is a surjection

H4. $\neg x \in f(A)$

T4. $\boldsymbol{\exists y.(y \in A^c \wedge f(y) = x)}$

6. Unlock existential target T4.

We would like to find $y \in A^c$ s.t. $f(y) = x$.

L1

x

H1. f from A to C is a surjection

H4. $\neg x \in f(A)$

L2$^\blacklozenge$

$y^\blacklozenge$

T5. $\boldsymbol{y^\blacklozenge \in A^c}$

T6. $f(y^\blacklozenge) = x$

7. Quantifier-free expansion of target T5.

But $y \in A^c$ if and only if $y \notin A$.

L1

x

H1. f from A to C is a surjection

H4. $\neg x \in f(A)$

L2$^\blacklozenge$

$y^\blacklozenge$

T7. $\boldsymbol{y^\blacklozenge \notin A}$

T6. $f(y^\blacklozenge) = x$

8. Quantifier-free expansion of target T7.

But $y \notin A$ if and only if it is not that case that $y \in A$.

L1

x

H1. f from A to C is a surjection

H4. $\neg x \in f(A)$

L2$^\blacklozenge$

$y^\blacklozenge$

T8. $\neg y^\blacklozenge \in A$

T6. $f(y^\blacklozenge) = x$

No moves possible.

If f is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of $f(A) \cap f(B)$. Then $x \in f(A)$ and $x \in f(B)$. That is, there exists $y \in A$ such that $f(y) = x$ and there exists $z \in B$ such that $f(z) = x$. Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$. We would like to find $u \in A \cap B$ s.t. $f(u) = x$. But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$. Since $y = z$, we have that $y \in B$. Therefore, setting $u = y$, we are done.

L1

H1. f is an injection

T1. $\boldsymbol{f(A) \cap f(B) \subset f(A \cap B)}$

1. Expand pre-universal target T1.

L1

H1. f is an injection

T2. $\boldsymbol{\forall x.(x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(A) \cap f(B)$.

L1

x

H1. f is an injection

H2. $\boldsymbol{x \in f(A) \cap f(B)}$

T3. $x \in f(A \cap B)$

3. Quantifier-free expansion of hypothesis H2.

Since $x \in f(A) \cap f(B)$, $x \in f(A)$ and $x \in f(B)$.

L1

x

H1. f is an injection

H3. $\boldsymbol{x \in f(A)}$

H4. $x \in f(B)$

T3. $x \in f(A \cap B)$

4. Expand pre-existential hypothesis H3.

By definition, since $x \in f(A)$, there exists $y \in A$ such that $f(y) = x$.

L1

$x \; y$

H1. f is an injection

H5. $y \in A$

H6. $f(y) = x$

H4. $\boldsymbol{x \in f(B)}$

T3. $x \in f(A \cap B)$

5. Expand pre-existential hypothesis H4.

By definition, since $x \in f(B)$, there exists $z \in B$ such that $f(z) = x$.

L1

$x \; y \; z$

H1. $\boldsymbol{f}$ is an injection

H5. $y \in A$

H6. $\boldsymbol{f(y) = x}$

H7. $z \in B$

H8. $\boldsymbol{f(z) = x}$

T3. $x \in f(A \cap B)$

6. Forwards reasoning using H1 with (H6,H8).

Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

T3. $\boldsymbol{x \in f(A \cap B)}$

7. Expand pre-existential target T3.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

T4. $\boldsymbol{\exists u.(u \in A \cap B \wedge f(u) = x)}$

8. Unlock existential target T4.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

L2$^\blacklozenge$

$u^\blacklozenge$

T5. $\boldsymbol{u^\blacklozenge \in A \cap B}$
T6. $f(u^\blacklozenge) = x$

9. Quantifier-free expansion of target T5.

But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $\boldsymbol{y = z}$

L2$^\blacklozenge$

$u^\blacklozenge$

T7. $u^\blacklozenge \in A$
T8. $u^\blacklozenge \in B$
T6. $f(u^\blacklozenge) = x$

10. Rewrite z as y throughout the tableau using hypothesis H9.

Since $y = z$, we have that $y \in B$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H10. $y \in B$

L2$^\blacklozenge$

$u^\blacklozenge$

T7. $u^\blacklozenge \in A$
T8. $u^\blacklozenge \in B$
T6. $f(u^\blacklozenge) = x$

11. Moved H6 down, as x can only be utilised by T6.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H10. $y \in B$

L2$^\blacklozenge$

$u^\blacklozenge$

T7. $u^\blacklozenge \in A$
T8. $u^\blacklozenge \in B$

L3

H6. $f(y) = x$ [Vuln.; From L1.]

T6. $f(u^\blacklozenge) = x$

12. Choosing $u^\blacklozenge = y$ matches all targets inside L2$^\blacklozenge$ against hypotheses, so L2$^\blacklozenge$ is done.

Therefore, setting $u = y$, we are done.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H10. $y \in B$

L2$^\blacklozenge$ Done

13. All targets of L1 are ‘Done’, so L1 is itself done.

L1 Done

Problem solved.