Prove that $A \subseteq f^{-1}(f(A))$

Let x be an element of A. We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$. But this is clearly the case, so we are done.

$$oxed{ ext{T1. } A \subset f^{-1}(f(A))}$$

1. Expand pre-universal target T1.

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of A.

$$\begin{array}{c}
\text{L1} & x \\
 & \text{H1. } x \in A \\
\hline
 & \text{T3. } x \in f^{-1}(f(A))
\end{array}$$

3. Quantifier-free expansion of target T3.

$$\begin{array}{c} \text{L1} \\ \hline \textbf{H1.} \ x \in \overset{x}{A} \\ \hline \textbf{T4.} \ f(x) \in f(A) \end{array}$$

We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$.

4. All conjuncts of T4 (after expansion) can be simultaneously matched against H1 or rendered trivial by choosing y = x, so L1 is done.

L1 Done

Problem solved.

We would like to show that $f(x) \in f(A)$. But this is clearly the case, so we are done.

If g,f are surjections then (g o f) is a surjection.

Let y be an element of C. Then, since g from B to C is a surjection, we have that $y \in g(B)$. That is, there exists $z \in B$ such that g(z) = y. Since f from A to B is a surjection and $z \in B$, we have that $z \in f(A)$. That is, there exists $u \in A$ such that f(u) = z. We would like to find $v \in A$ s.t. g(f(v)) = y.

```
L1
H1. f from A to B is a surjection
H2. g from B to C is a surjection

T1. g \circ f from A to C is a surjection
```

1. Expand pre-universal target T1.

H1.
$$f$$
 from A to B is a surjection
H2. g from B to C is a surjection
T2. $\forall y.(y \in C \Rightarrow y \in g \circ f(A))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let y be an element of C.

```
L1) y
H1. f from A to B is a surjection
H2. g from B to C is a surjection
H3. y \in C
T3. <math>y \in g \circ f(A)
```

3. Forwards reasoning using H2 with H3.

```
\begin{array}{c} \text{L1} & y \\ \text{H1. } f \text{ from } A \text{ to } B \text{ is a surjection} \\ \text{H2. } g \text{ from } B \text{ to } C \text{ is a surjection} \\ \text{H3. } y \in C \\ \text{H4. } y \in g(B) \\ \hline \\ \text{T3. } y \in g \circ f(A) \end{array} \qquad [\text{Vuln.}]
```

Since g from B to C is a surjection and $y \in C$, we have that $y \in g(B)$.

4. Deleted H3, as this unexpandable atomic statement is unmatchable.

```
\begin{array}{c} \text{L1} & y \\ \text{H1. } f \text{ from } A \text{ to } B \text{ is a surjection} \\ \text{H2. } g \text{ from } B \text{ to } C \text{ is a surjection} \\ \text{H3. } y \in C \\ \text{H4. } y \in g(B) \\ \hline \\ \text{T3. } y \in g \circ f(A) \end{array} \qquad [\text{Vuln.}]
```

5. Delete H2 as no other statement mentions C.

```
\begin{array}{c} \textbf{L1} & y \\ \textbf{H1. } f \ \text{from } A \ \text{to } B \ \text{is a surjection} \\ \textbf{H2. } g \ \text{from } B \ \text{to } C \ \text{is a surjection} \\ \textbf{H3. } y \in C \\ \textbf{H4. } y \in g(B) \\ \hline \textbf{T3. } y \in g \circ f(A) \end{array}
```

6. Expand pre-existential hypothesis H4.

By definition, since $y \in g(B)$, there exists $z \in B$ such that g(z) = y

```
L1) y z

H1. f from A to B is a surjection

H2. g from B to C is a surjection [Vuln.; Used with H3.]

H3. y \in C [Vuln.]

H5. z \in B

H6. g(z) = y

T3. y \in g \circ f(A)
```

7. Forwards reasoning using H1 with H5.

```
\begin{array}{c} \text{L1} & y \ z \\ \text{H1. } f \ \text{from } A \ \text{to } B \ \text{is a surjection} & [\text{Vuln.; Used with H5.}] \\ \text{H2. } g \ \text{from } B \ \text{to } C \ \text{is a surjection} & [\text{Vuln.; Used with H3.}] \\ \text{H3. } y \in C & [\text{Vuln.}] \\ \text{H5. } z \in B & [\text{Vuln.}] \\ \text{H6. } g(z) = y \\ \text{H7. } z \in f(A) \\ \hline \hline \text{T3. } y \in g \circ f(A) \end{array}
```

Since f from A to B is a surjection and $z \in B$, we have that $z \in f(A)$.

8. Deleted H5, as this unexpandable atomic statement is unmatchable.

```
L1) y z
H1. f from A to B is a surjection [Vuln.; Used with H5.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. y \in C [Vuln.]
H5. z \in B [Vuln.]
H6. g(z) = y
H7. z \in f(A)

T3. y \in g \circ f(A)
```

9. Delete H1 as no other statement mentions B.

```
\begin{array}{c} \textbf{L1} & y \ z \\ \\ \textbf{H1.} \ f \ \text{from} \ A \ \text{to} \ B \ \text{is a surjection} & [\text{Vuln.; Used with H5.}] \\ \\ \textbf{H2.} \ g \ \text{from} \ B \ \text{to} \ C \ \text{is a surjection} & [\text{Vuln.; Used with H3.}] \\ \\ \textbf{H3.} \ y \in C & [\text{Vuln.}] \\ \\ \textbf{H5.} \ z \in B & [\text{Vuln.}] \\ \\ \textbf{H6.} \ g(z) = y \\ \\ \textbf{H7.} \ z \in \textbf{\textit{f}}(\textbf{\textit{A}}) \\ \hline \\ \textbf{T3.} \ y \in g \circ f(A) \end{array}
```

10. Expand pre-existential hypothesis H7.

```
L1 y z u

H1. f from A to B is a surjection [Vuln.; Used with H5.]

H2. g from B to C is a surjection [Vuln.; Used with H3.]

H3. y \in C [Vuln.]

H5. z \in B [Vuln.]

H6. g(z) = y

H8. u \in A

H9. f(u) = z

T3. y \in g \circ f(A)
```

11. Expand pre-existential target T3.

By definition, since $z \in f(A)$, there exists $u \in A$ such that f(u) = z.

We would like to find $v \in A$ s.t. g(f(v)) = y.

```
L1) y z u

H1. f from A to B is a surjection [Vuln.; Used with H5.]

H2. g from B to C is a surjection [Vuln.; Used with H3.]

H3. y \in C [Vuln.]

H5. z \in B [Vuln.]

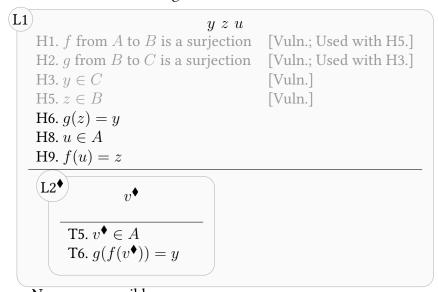
H6. g(z) = y

H8. u \in A

H9. f(u) = z

T4. \exists v \cdot (v \in A \land g(f(v)) = y)
```

12. Unlock existential target T4.



No moves possible.

We would like to find $v \in A$ s.t. g(f(v)) = y.

If f is a surjection then $f(A)^c \subset f(A^c)$

Let x be an element of $f(A)^c$. Then $x \notin f(A)$. Then it is not that case that $x \in f(A)$. We would like to find $y \in A^c$ s.t. f(y) = x. But $y \in A^c$ if and only if $y \notin A$. But $y \notin A$ if and only if it is not that case that $y \in A$.

H1.
$$f$$
 from A to C is a surjection
$$T1. f(A)^c \subset f(A^c)$$

1. Expand pre-universal target T1.

H1.
$$f$$
 from A to C is a surjection
$$T2. \forall x. (x \in f(A)^c \Rightarrow x \in f(A^c))$$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

L1 xH1. f from A to C is a surjection
H2. $x \in f(A)^c$ T3. $x \in f(A^c)$

Let x be an element of $f(A)^c$.

T3. $x \in f(A^c)$ 3. Quantifier-free expansion of hypothesis H2.

L1) xH1. f from A to C is a surjection
H3. $x \notin f(A)$

Since $x \in f(A)^c$, we have that $x \notin f(A)$.

H1. f from A to C is a surjection H3. $x \notin f(A)$ $T3. x \in f(A^c)$

4. Quantifier-free expansion of hypothesis H3.

L1 xH1. f from A to C is a surjection
H4. $\neg x \in f(A)$ T3. $x \in f(A^c)$

Since $x \notin f(A)$, it is not that case that $x \in f(A)$.

5. Expand pre-existential target T3.

L1

H1. f from A to C is a surjection

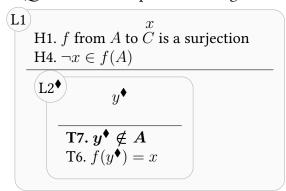
H4. $\neg x \in f(A)$ T4. $\exists y.(y \in A^c \land f(y) = x)$

We would like to find $y \in A^c$ s.t. f(y) = x.

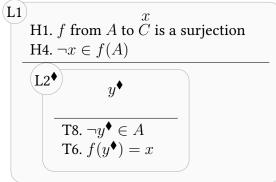
6. Unlock existential target T4.

We would like to find $y \in A^c$ s.t. f(y) = x.

7. Quantifier-free expansion of target T5.



8. Quantifier-free expansion of target T7.



No moves possible.

But $y \in A^c$ if and only if $y \notin A$.

But $y \notin A$ if and only if it is not that case that $y \in A$.

If f is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of $f(A)\cap f(B)$. Then $x\in f(A)$ and $x\in f(B)$. That is, there exists $y\in A$ such that f(y)=x and there exists $z\in B$ such that f(z)=x. Since f is an injection, f(y)=x and f(z)=x, we have that y=z. We would like to find $u\in A\cap B$ s.t. f(u)=x. But $u\in A\cap B$ if and only if $u\in A$ and $u\in B$. Since y=z, we have that $y\in B$. Therefore, setting u=y, we are done.

H1.
$$f$$
 is an injection
$$T1. f(A) \cap f(B) \subset f(A \cap B)$$

1. Expand pre-universal target T1.

H1.
$$f$$
 is an injection
$$T2. \forall x. (x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))$$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

L1 xH1. f is an injection
H2. $x \in f(A) \cap f(B)$ T3. $x \in f(A \cap B)$

Let x be an element of $f(A) \cap f(B)$.

3. Quantifier-free expansion of hypothesis H2.

L1 xH1. f is an injection
H3. $x \in f(A)$ H4. $x \in f(B)$ $T3. <math>x \in f(A \cap B)$

Since $x \in f(A) \cap f(B)$, $x \in f(A)$ and $x \in f(B)$.

4. Expand pre-existential hypothesis H3.

L1) x yH1. f is an injection
H5. $y \in A$ H6. f(y) = xH4. $x \in f(B)$ T3. $x \in f(A \cap B)$

By definition, since $x \in f(A)$, there exists $y \in A$ such that f(y) = x.

5. Expand pre-existential hypothesis H4.

L1) x y zH1. f is an injection

H5. $y \in A$ H6. f(y) = xH7. $z \in B$ H8. f(z) = xT3. $x \in f(A \cap B)$

By definition, since $x \in f(B)$, there exists $z \in B$ such that f(z) = x.

6. Forwards reasoning using H1 with (H6,H8).

Since f is an injection, f(y) = x and f(z) = x, we have that y = z.

$$\begin{array}{c|cccc} \textbf{L1} & x & y & z \\ & \textbf{H1.} & f \text{ is an injection} & [\text{Vuln.; Used with (H6,H8).}] \\ & \textbf{H5.} & y \in A \\ & \textbf{H6.} & f(y) = x & [\text{Vuln.}] \\ & \textbf{H7.} & z \in B \\ & \textbf{H8.} & f(z) = x & [\text{Vuln.}] \\ & \textbf{H9.} & y = z \\ \hline & \textbf{T3.} & x \in f(A \cap B) \end{array}$$

7. Expand pre-existential target T3.

L1 x y zH1. f is an injection [Vuln.; Used with (H6,H8).]

H5. $y \in A$ H6. f(y) = x [Vuln.]

H7. $z \in B$ H8. f(z) = x [Vuln.]

H9. y = zT4. $\exists u.(u \in A \cap B \land f(u) = x)$

We would like to find $u \in A \cap B$ s.t. f(u) = x.

8. Unlock existential target T4.

L1 x y zH1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$ H6. f(y) = x [Vuln.]
H7. $z \in B$ H8. f(z) = x [Vuln.]
H9. y = zL2• uT5. u• $\in A \cap B$ T6. f(u•) = x

We would like to find $u \in A \cap B$ s.t. f(u) = x.

9. Quantifier-free expansion of target T5.

L1 x y zH1. f is an injection [Vuln.; Used with (H6,H8).]

H5. $y \in A$ H6. f(y) = x [Vuln.]

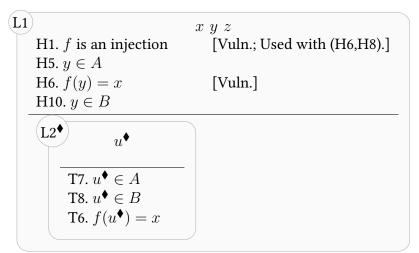
H7. $z \in B$ H8. f(z) = x [Vuln.]

H9. y = zL2 \bullet $u \bullet$ $T7. u \bullet \in A$ $T8. u \bullet \in B$ $T6. f(u \bullet) = x$

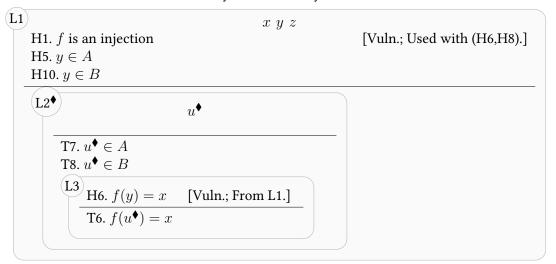
But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$.

10. Rewrite *z* as *y* throughout the tableau using hypothesis H9.

Since y = z, we have that $y \in B$.



11. Moved H6 down, as x can only be utilised by T6.



12. Choosing $u^{\blacklozenge}=y$ matches all targets inside $L2^{\blacklozenge}$ against hypotheses, so $L2^{\blacklozenge}$ is done.

Therefore, setting u = y, we are done.

- L1 x y zH1. f is an injection [Vuln.; Used with (H6,H8).]
 H5. $y \in A$ H10. $y \in B$ L2 Done
- 13. All targets of L1 are 'Done', so L1 is itself done.
- L1 Done

Problem solved.