If g,f are surjections then (g o f) is a surjection.

Let y be an element of C. Then, since g from B to C is a surjection, there exists $u \in B$ such that g(u) = y. Since f from A to B is a surjection and $u \in B$, there exists $v \in A$ such that f(v) = u. We would like to find $x \in A$ s.t. g(f(x)) = y.

 $H1. \ f \ \text{from } A \ \text{to } B \ \text{is a surjection}$ $H2. \ g \ \text{from } B \ \text{to } C \ \text{is a surjection}$

T1. $g \circ f$ from A to C is a surjection

1. Expand pre-universal target T1.

H1. f from A to B is a surjection

H2. g from B to C is a surjection

T2. $\forall y.(y \in C \Rightarrow \exists x.(x \in A \land g(f(x)) = y))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let y be an element of C.

H1. f from A to B is a surjection

H2. g from B to C is a surjection

H3. $y \in C$ T3. $\exists x. (x \in A \land g(f(x)) = y)$

3. Forwards reasoning using H2 with H3.

 $\begin{array}{c} \text{L1} & y \ u[y] \\ \text{H1. } f \text{ from } A \text{ to } B \text{ is a surjection} \\ \text{H2. } g \text{ from } B \text{ to } C \text{ is a surjection} \\ \text{H3. } y \in C \\ \text{H4. } u[y] \in B \\ \text{H5. } g(u[y]) = y \\ \hline \\ \text{T3. } \exists x. (x \in A \land g(f(x)) = y) \end{array}$

Since g from B to C is a surjection and $y \in C$, there exists $u \in B$ such that g(u) = y.

4. Deleted H3, as this unexpandable atomic statement is unmatchable.

L1) $y \ u[y]$ H1. f from A to B is a surjection
H2. g from B to C is a surjection
[Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]

H4. $u[y] \in B$ H5. g(u[y]) = yT3. $\exists x. (x \in A \land g(f(x)) = y)$

5. Delete H2 as no other statement mentions C.

 $\begin{array}{c} \textbf{L1}) & y \ u[y] \\ \textbf{H1.} \ \textbf{\textit{f} from } A \ \textbf{to } B \ \textbf{is a surjection} \\ \text{H2.} \ g \ \text{from } B \ \textbf{to } C \ \textbf{is a surjection} \\ \text{H3.} \ y \in C & [Vuln.; Used \ with \ H3.] \\ \textbf{H4.} \ u[y] \in B \\ \textbf{H5.} \ g(u[y]) = y \\ \hline \textbf{T3.} \ \exists x. (x \in A \land g(f(x)) = y) \end{array}$

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6. Forwards reasoning using H1 with H4.

L1 $y \ u[y] \ v[u[y]]$ H1. f from A to B is a surjection [Vuln.; Used with H4.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. $y \in C$ [Vuln.]
H4. $u[y] \in B$ [Vuln.]
H5. g(u[y]) = yH6. $v[u[y]] \in A$ H7. f(v[u[y]]) = u[y]T3. $\exists x. (x \in A \land g(f(x)) = y)$

Since f from A to B is a surjection and $u \in B$, there exists $v \in A$ such that f(v) = u.

7. Deleted H4, as this unexpandable atomic statement is unmatchable.

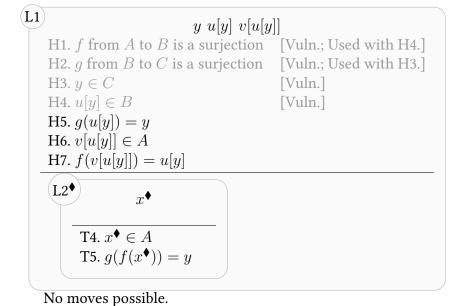
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L1) y \ u[y] \ v[u[y]]
H1. f from A to B is a surjection [Vuln.; Used with H4.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. y \in C [Vuln.]
H4. u[y] \in B [Vuln.]
H5. g(u[y]) = y
H6. v[u[y]] \in A
H7. f(v[u[y]]) = u[y]
T3. \exists x. (x \in A \land g(f(x)) = y)
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8. Delete H1 as no other statement mentions B.

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L1 y \ u[y] \ v[u[y]]
H1. f from A to B is a surjection [Vuln.; Used with H4.]
H2. g from B to C is a surjection [Vuln.; Used with H3.]
H3. y \in C [Vuln.]
H4. u[y] \in B [Vuln.]
H5. g(u[y]) = y
H6. v[u[y]] \in A
H7. f(v[u[y]]) = u[y]

T3. \exists x.(x \in A \land g(f(x)) = y)
```

9. Unlock existential target T3.



We would like to find $x \in A$ s.t. g(f(x)) = y.

If f is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of $f(A)\cap f(B)$. Then $x\in f(A)$ and $x\in f(B)$. That is, there exists $y\in A$ such that f(y)=x and there exists $z\in B$ such that f(z)=x. Since f is an injection, f(y)=x and f(z)=x, we have that y=z. We would like to find $u\in A\cap B$ s.t. f(u)=x. But $u\in A\cap B$ if and only if $u\in A$ and $u\in B$. Since y=z, we have that $y\in B$. Therefore, setting u=y, we are done.

$$H1. \ f$$
 is an injection $T1. \ f(A) \cap f(B) \subset f(A \cap B)$

1. Expand pre-universal target T1.

1. H1.
$$f$$
 is an injection T2. $\forall x. (x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

L1 xH1. f is an injection
H2. $x \in f(A) \cap f(B)$ T3. $x \in f(A \cap B)$

Let x be an element of $f(A) \cap f(B)$.

3. Quantifier-free expansion of hypothesis H2.

L1 xH1. f is an injection
H3. $x \in f(A)$ H4. $x \in f(B)$ $T3. <math>x \in f(A \cap B)$

Since $x \in f(A) \cap f(B)$, $x \in f(A)$ and $x \in f(B)$.

4. Expand pre-existential hypothesis H3.

L1) $x \ y$ H1. f is an injection
H5. $y \in A$ H6. f(y) = xH4. $x \in f(B)$ T3. $x \in f(A \cap B)$

By definition, since $x \in f(A)$, there exists $y \in A$ such that f(y) = x.

5. Expand pre-existential hypothesis H4.

L1) x y zH1. f is an injection

H5. $y \in A$ H6. f(y) = xH7. $z \in B$ H8. f(z) = xT3. $x \in f(A \cap B)$

By definition, since $x \in f(B)$, there exists $z \in B$ such that f(z) = x.

6. Forwards reasoning using H1 with (H6,H8).

Since f is an injection, f(y) = x and f(z) = x, we have that y = z.

$$\begin{array}{c|cccc} \textbf{L1} & x & y & z \\ & \textbf{H1.} & f \text{ is an injection} & [\text{Vuln.; Used with (H6,H8).}] \\ & \textbf{H5.} & y \in A \\ & \textbf{H6.} & f(y) = x & [\text{Vuln.}] \\ & \textbf{H7.} & z \in B \\ & \textbf{H8.} & f(z) = x & [\text{Vuln.}] \\ & \textbf{H9.} & y = z \\ \hline & \textbf{T3.} & x \in f(A \cap B) \end{array}$$

7. Expand pre-existential target T3.

L1 x y zH1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$ H6. f(y) = x [Vuln.]
H7. $z \in B$ H8. f(z) = x [Vuln.]
H9. y = z $T4. \exists u. (u \in A \cap B \land f(u) = x)$

We would like to find $u \in A \cap B$ s.t. f(u) = x.

8. Unlock existential target T4.

L1 x y zH1. f is an injection [Vuln.; Used with (H6,H8).]

H5. $y \in A$ H6. f(y) = x [Vuln.]

H7. $z \in B$ H8. f(z) = x [Vuln.]

H9. y = zL2• u• T5. u• $\in A \cap B$ T6. f(u•) = x

We would like to find $u \in A \cap B$ s.t. f(u) = x.

9. Quantifier-free expansion of target T5.

L1

H1. f is an injection

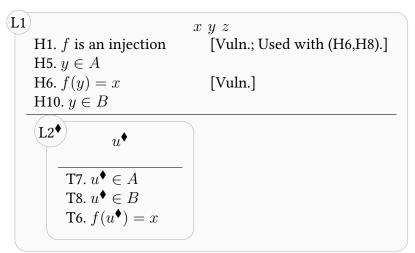
H5. $y \in A$ H6. f(y) = xH7. $z \in B$ H8. f(z) = xH9. y = z[Vuln.]

T7. $u^{\bullet} \in A$ T8. $u^{\bullet} \in B$ T6. $f(u^{\bullet}) = x$

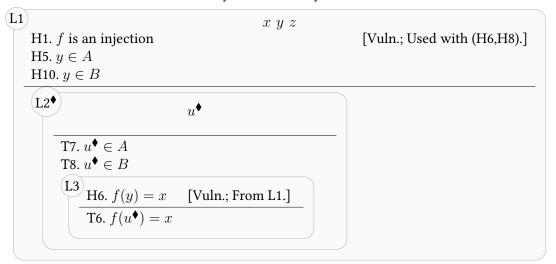
But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$.

10. Rewrite *z* as *y* throughout the tableau using hypothesis H9.

Since y = z, we have that $y \in B$.

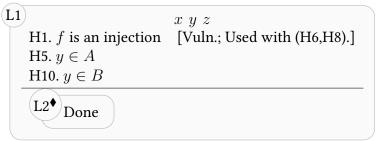


11. Moved H6 down, as x can only be utilised by T6.



12. Choosing $u^{\blacklozenge}=y$ matches all targets inside $\mathrm{L2}^{\spadesuit}$ against hypotheses, so $\mathrm{L2}^{\spadesuit}$ is done.

Therefore, setting u = y, we are done.



13. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.