If f is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of $f(A) \cap f(B)$. Then $x \in f(A)$ and $x \in f(B)$. That is, there exists $y \in A$ such that $f(y) = x$ and there exists $z \in B$ such that $f(z) = x$. Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$. We would like to find $u \in A \cap B$ s.t. $f(u) = x$. But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$. Since $y = z$, we have that $y \in B$. Therefore, setting $u = y$, we are done.

L1

H1. f is an injection

T1. $\boldsymbol{f(A) \cap f(B) \subset f(A \cap B)}$

1. Expand pre-universal target T1.

L1

H1. f is an injection

T2. $\boldsymbol{\forall x.(x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))}$

2. Apply ‘let’ trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(A) \cap f(B)$.

L1

x

H1. f is an injection

H2. $\boldsymbol{x \in f(A) \cap f(B)}$

T3. $x \in f(A \cap B)$

3. Quantifier-free expansion of hypothesis H2.

Since $x \in f(A) \cap f(B)$, $x \in f(A)$ and $x \in f(B)$.

L1

x

H1. f is an injection

H3. $\boldsymbol{x \in f(A)}$

H4. $x \in f(B)$

T3. $x \in f(A \cap B)$

4. Expand pre-existential hypothesis H3.

By definition, since $x \in f(A)$, there exists $y \in A$ such that $f(y) = x$.

L1

$x \ y$

H1. f is an injection

H5. $y \in A$

H6. $f(y) = x$

H4. $\boldsymbol{x \in f(B)}$

T3. $x \in f(A \cap B)$

5. Expand pre-existential hypothesis H4.

By definition, since $x \in f(B)$, there exists $z \in B$ such that $f(z) = x$.

L1

$x \ y \ z$

H1. $\boldsymbol{f}$ **is an injection**

H5. $y \in A$

H6. $\boldsymbol{f(y) = x}$

H7. $z \in B$

H8. $\boldsymbol{f(z) = x}$

T3. $x \in f(A \cap B)$

6. Forwards reasoning using H1 with (H6,H8).

Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

T3. $\boldsymbol{x \in f(A \cap B)}$

7. Expand pre-existential target T3.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

T4. $\boldsymbol{\exists u.(u \in A \cap B \wedge f(u) = x)}$

8. Unlock existential target T4.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

L2$^\blacklozenge$

$u^\blacklozenge$

T5. $\boldsymbol{u^\blacklozenge \in A \cap B}$
T6. $f(u^\blacklozenge) = x$

9. Quantifier-free expansion of target T5.

But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $\boldsymbol{y = z}$

L2$^\blacklozenge$

$u^\blacklozenge$

T7. $u^\blacklozenge \in A$
T8. $u^\blacklozenge \in B$
T6. $f(u^\blacklozenge) = x$

10. Rewrite z as y throughout the tableau using hypothesis H9.

Since $y = z$, we have that $y \in B$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H10. $y \in B$

L2$^\blacklozenge$

$u^\blacklozenge$

T7. $u^\blacklozenge \in A$
T8. $u^\blacklozenge \in B$
T6. $f(u^\blacklozenge) = x$

11. Moved H6 down, as x can only be utilised by T6.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H10. $y \in B$

L2$^\blacklozenge$

$u^\blacklozenge$

T7. $u^\blacklozenge \in A$
T8. $u^\blacklozenge \in B$

L3

H6. $f(y) = x$ [Vuln.; From L1.]

T6. $f(u^\blacklozenge) = x$

12. Choosing $u^\blacklozenge = y$ matches all targets inside L2$^\blacklozenge$ against hypotheses, so L2$^\blacklozenge$ is done.

Therefore, setting $u = y$, we are done.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H10. $y \in B$

L2$^\blacklozenge$ Done

13. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.

If g,f are injections then (g o f) is an injection.

Let x, y and z be such that $g(f(x)) = z$ and $g(f(y)) = z$. Then, since g is an injection, we have that $f(x) = f(y)$. Therefore, since f is an injection, $x = y$ if $f(y) = f(y)$. Since g is an injection and $g(f(y)) = z$, $f(y) = f(y)$ if $g(f(y)) = z$. But this is clearly the case, so we are done.

L1

H1. f is an injection
H2. g is an injection

T1. $g \circ f$ is an injection

1. Expand pre-universal target T1.

L1

H1. f is an injection
H2. g is an injection

T2. $\forall x, y, z.(g(f(x)) = z \wedge g(f(y)) = z \Rightarrow x = y)$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x, y and z be such that $g(f(x)) = z$ and $g(f(y)) = z$.

L1

x y z

H1. f is an injection
H2. g is an injection
H3. $g(f(x)) = z$
H4. $g(f(y)) = z$

T3. $x = y$

3. Forwards reasoning using H2 with (H3,H4).

Since g is an injection, $g(f(x)) = z$ and $g(f(y)) = z$, we have that $f(x) = f(y)$.

L1

x y z

H1. f is an injection	
H2. g is an injection	[Vuln.; Used with (H3,H4).]
H3. $g(f(x)) = z$	[Vuln.]
H4. $g(f(y)) = z$	[Vuln.]
H5. $f(x) = f(y)$	

T3. $x = y$

4. Backwards reasoning using H1 with (T3,H5).

Since f is an injection and $f(x) = f(y)$, $x = y$ if $f(y) = f(y)$.

L1

x y z

H1. f is an injection	[Vuln.]
H2. g is an injection	**[Vuln.; Used with (H3,H4).]**
H3. $g(f(x)) = z$	[Vuln.]
H4. $g(f(y)) = z$	**[Vuln.]**
H5. $f(x) = f(y)$	[Vuln.]

T4. $f(y) = f(y)$

5. Backwards reasoning using H2 with (T4,H4).

Since g is an injection and $g(f(y)) = z$, $f(y) = f(y)$ if $g(f(y)) = z$.

L1

	$x\ y\ z$
H1. f is an injection	[Vuln.]
H2. g is an injection	[Vuln.; Used with (H3,H4).]
H3. $g(f(x)) = z$	[Vuln.]
H4. $\boldsymbol{g(f(y)) = z}$	**[Vuln.]**
H5. $f(x) = f(y)$	[Vuln.]
T5. $\boldsymbol{g(f(y)) = z}$	

6. Hypothesis H4 matches target T5, so L1 is done.

But this is clearly the case, so we are done.

L1 Done

Problem solved.

Prove that $A \subseteq f^{-1}(f(A))$

Let x be an element of A. We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$. But this is clearly the case, so we are done.

L1

T1. $\boldsymbol{A \subset f^{-1}(f(A))}$

1. Expand pre-universal target T1.

L1

T2. $\boldsymbol{\forall x.(x \in A \Rightarrow x \in f^{-1}(f(A)))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of A.

L1

x

H1. $x \in A$

T3. $\boldsymbol{x \in f^{-1}(f(A))}$

3. Quantifier-free expansion of target T3.

We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$.

L1

x

H1. $\boldsymbol{x \in A}$

T4. $\boldsymbol{f(x) \in f(A)}$

4. All conjuncts of T4 (after expansion) can be simultaneously matched against H1 or rendered trivial by choosing $y = x$, so L1 is done.

We would like to show that $f(x) \in f(A)$. But this is clearly the case, so we are done.

L1 Done

Problem solved.

Prove that $f(f^{-1}(A)) \subset A$

Let x be an element of $f(f^{-1}(A))$. Then there exists $y \in f^{-1}(A)$ such that $f(y) = x$. Since $y \in f^{-1}(A)$, we have that $f(y) \in A$. Since $f(y) = x$, we have that $x \in A$. But this is clearly the case, so we are done.

L1

T1. $\boldsymbol{f(f^{-1}(A)) \subset A}$

1. Expand pre-universal target T1.

L1

T2. $\boldsymbol{\forall x.(x \in f(f^{-1}(A)) \Rightarrow x \in A)}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(f^{-1}(A))$.

L1

x

H1. $\boldsymbol{x \in f(f^{-1}(A))}$

T3. $x \in A$

3. Expand pre-existential hypothesis H1.

By definition, since $x \in f(f^{-1}(A))$, there exists $y \in f^{-1}(A)$ such that $f(y) = x$.

L1

$x \; y$

H2. $\boldsymbol{y \in f^{-1}(A)}$

H3. $f(y) = x$

T3. $x \in A$

4. Quantifier-free expansion of hypothesis H2.

Since $y \in f^{-1}(A)$, we have that $f(y) \in A$.

L1

$x \; y$

H4. $f(y) \in A$

H3. $\boldsymbol{f(y) = x}$

T3. $x \in A$

5. Rewrite $f(y)$ as x throughout the tableau using hypothesis H3.

Since $f(y) = x$, we have that $x \in A$.

L1

$x \; y$

H5. $\boldsymbol{x \in A}$

T3. $\boldsymbol{x \in A}$

6. Hypothesis H5 matches target T3, so L1 is done.

But this is clearly the case, so we are done.

L1 Done

Problem solved.

Prove that $f(A \cap B) \subset f(A) \cap f(B)$

By definition, since $y \in f(A \cap B)$, there exists $z \in A \cap B$ such that $f(z) = y$. Since $z \in A \cap B$, $z \in A$ and $z \in B$. We would like to show that $y \in f(A) \cap f(B)$, i.e. that $y \in f(A)$ and $y \in f(B)$. We would like to show that $y \in f(A)$. But this is clearly the case, so we are done. Thus $y \in f(B)$ and we are done.

L1

H1. $\boldsymbol{y \in f(A \cap B)}$

T1. $y \in f(A) \cap f(B)$

1. Expand pre-existential hypothesis H1.

By definition, since $y \in f(A \cap B)$, there exists $z \in A \cap B$ such that $f(z) = y$.

L1

z

H2. $\boldsymbol{z \in A \cap B}$

H3. $f(z) = y$

T1. $y \in f(A) \cap f(B)$

2. Quantifier-free expansion of hypothesis H2.

Since $z \in A \cap B$, $z \in A$ and $z \in B$.

L1

z

H4. $z \in A$

H5. $z \in B$

H3. $f(z) = y$

T1. $\boldsymbol{y \in f(A) \cap f(B)}$

3. Quantifier-free expansion of target T1.

We would like to show that $y \in f(A) \cap f(B)$, i.e. that $y \in f(A)$ and $y \in f(B)$.

L1

z

H4. $\boldsymbol{z \in A}$

H5. $z \in B$

H3. $\boldsymbol{f(z) = y}$

T2. $\boldsymbol{y \in f(A)}$

T3. $y \in f(B)$

4. All conjuncts of T2 (after expansion) can be simultaneously matched against H4 and H3 or rendered trivial by choosing $u = z$, so we can remove T2.

We would like to show that $y \in f(A)$. But this is clearly the case, so we are done.

L1

z

H4. $z \in A$

H5. $\boldsymbol{z \in B}$

H3. $\boldsymbol{f(z) = y}$

T3. $\boldsymbol{y \in f(B)}$

5. All conjuncts of T3 (after expansion) can be simultaneously matched against H5 and H3 or rendered trivial by choosing $u = z$, so L1 is done.

We would like to show that $y \in f(B)$. But this is clearly the case, so we are done.

L1 Done

Problem solved.

Prove that $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$

Since $x \in f^{-1}(A \cap B)$, we have that $f(x) \in A \cap B$. Then $f(x) \in A$ and $f(x) \in B$. We would like to show that $x \in f^{-1}(A) \cap f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. We would like to show that $x \in f^{-1}(B)$, i.e. that $f(x) \in B$. But this is clearly the case, so we are done.

L1
H1. $\boldsymbol{x \in f^{-1}(A \cap B)}$

T1. $x \in f^{-1}(A) \cap f^{-1}(B)$

1. Quantifier-free expansion of hypothesis H1.

Since $x \in f^{-1}(A \cap B)$, we have that $f(x) \in A \cap B$.

L1
H2. $\boldsymbol{f(x) \in A \cap B}$

T1. $x \in f^{-1}(A) \cap f^{-1}(B)$

2. Quantifier-free expansion of hypothesis H2.

Since $f(x) \in A \cap B$, $f(x) \in A$ and $f(x) \in B$.

L1
H3. $f(x) \in A$
H4. $f(x) \in B$

T1. $\boldsymbol{x \in f^{-1}(A) \cap f^{-1}(B)}$

3. Quantifier-free expansion of target T1.

We would like to show that $x \in f^{-1}(A) \cap f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$.

L1
H3. $f(x) \in A$
H4. $f(x) \in B$

T2. $\boldsymbol{x \in f^{-1}(A)}$
T3. $x \in f^{-1}(B)$

4. Quantifier-free expansion of target T2.

We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$.

L1
H3. $\boldsymbol{f(x) \in A}$
H4. $f(x) \in B$

T4. $\boldsymbol{f(x) \in A}$
T3. $x \in f^{-1}(B)$

5. Hypothesis H3 matches target T4, so we can remove T4.

L1
H3. $f(x) \in A$
H4. $f(x) \in B$

T3. $\boldsymbol{x \in f^{-1}(B)}$

6. Quantifier-free expansion of target T3.

We would like to show that $x \in f^{-1}(B)$, i.e. that $f(x) \in B$.

L1
H3. $f(x) \in A$
H4. $\boldsymbol{f(x) \in B}$

T5. $\boldsymbol{f(x) \in B}$

7. Hypothesis H4 matches target T5, so L1 is done.

But this is clearly the case, so we are done.

L1 Done

Problem solved.

Prove that $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$

Let x be an element of $f^{-1}(A) \cap f^{-1}(B)$. Then $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Then $f(x) \in A$ and $f(x) \in B$. We would like to show that $x \in f^{-1}(A \cap B)$, i.e. that $f(x) \in A \cap B$. We would like to show that $f(x) \in A \cap B$, i.e. that $f(x) \in A$ and $f(x) \in B$. But this is clearly the case, so we are done.

L1

T1. $\boldsymbol{f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)}$

1. Expand pre-universal target T1.

L1

T2. $\boldsymbol{\forall x.(x \in f^{-1}(A) \cap f^{-1}(B) \Rightarrow x \in f^{-1}(A \cap B))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f^{-1}(A) \cap f^{-1}(B)$.

L1

x

H1. $\boldsymbol{x \in f^{-1}(A) \cap f^{-1}(B)}$

T3. $x \in f^{-1}(A \cap B)$

3. Quantifier-free expansion of hypothesis H1.

Since $x \in f^{-1}(A) \cap f^{-1}(B)$, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$.

L1

x

H2. $\boldsymbol{x \in f^{-1}(A)}$

H3. $x \in f^{-1}(B)$

T3. $x \in f^{-1}(A \cap B)$

4. Quantifier-free expansion of hypothesis H2.

Since $x \in f^{-1}(A)$, we have that $f(x) \in A$.

L1

x

H4. $f(x) \in A$

H3. $\boldsymbol{x \in f^{-1}(B)}$

T3. $x \in f^{-1}(A \cap B)$

5. Quantifier-free expansion of hypothesis H3.

Since $x \in f^{-1}(B)$, we have that $f(x) \in B$.

L1

x

H4. $f(x) \in A$

H5. $f(x) \in B$

T3. $\boldsymbol{x \in f^{-1}(A \cap B)}$

6. Quantifier-free expansion of target T3.

We would like to show that $x \in f^{-1}(A \cap B)$, i.e. that $f(x) \in A \cap B$.

L1

x

H4. $f(x) \in A$

H5. $f(x) \in B$

T4. $\boldsymbol{f(x) \in A \cap B}$

7. Quantifier-free expansion of target T4.

We would like to show that $f(x) \in A \cap B$, i.e. that $f(x) \in A$ and $f(x) \in B$.

L1

x

H4. $\boldsymbol{f(x) \in A}$

H5. $f(x) \in B$

T5. $\boldsymbol{f(x) \in A}$

T6. $f(x) \in B$

8. Hypothesis H4 matches target T5, so we can remove T5.

L1

x

H4. $f(x) \in A$

H5. $\boldsymbol{f(x) \in B}$

T6. $\boldsymbol{f(x) \in B}$

9. Hypothesis H5 matches target T6, so L1 is done.

But this is clearly the case, so we are done.

L1 Done

Problem solved.

Prove that $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$

Let x be an element of $f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$. Then $f(x) \in A$ or $f(x) \in B$. We would like to show that $x \in f^{-1}(A) \cup f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. But this is clearly the case, so we are done. We would like to show that $x \in f^{-1}(A) \cup f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. We would like to show that $x \in f^{-1}(B)$, i.e. that $f(x) \in B$. But this is clearly the case, so we are done.

L1

T1. $\boldsymbol{f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)}$

1. Expand pre-universal target T1.

L1

T2. $\boldsymbol{\forall x.(x \in f^{-1}(A \cup B) \Rightarrow x \in f^{-1}(A) \cup f^{-1}(B))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f^{-1}(A \cup B)$.

L1

x

H1. $\boldsymbol{x \in f^{-1}(A \cup B)}$

T3. $x \in f^{-1}(A) \cup f^{-1}(B)$

3. Quantifier-free expansion of hypothesis H1.

Since $x \in f^{-1}(A \cup B)$, we have that $f(x) \in A \cup B$.

L1

x

H2. $\boldsymbol{f(x) \in A \cup B}$

T3. $x \in f^{-1}(A) \cup f^{-1}(B)$

4. Quantifier-free expansion of hypothesis H2.

Since $f(x) \in A \cup B$, $f(x) \in A$ or $f(x) \in B$.

L1

x

H3. $\boldsymbol{f(x) \in A \vee f(x) \in B}$

T3. $x \in f^{-1}(A) \cup f^{-1}(B)$

5. Split into cases to handle disjunctive hypothesis H3.

L1

x

L2

H4. $f(x) \in A$

T4. $\boldsymbol{x \in f^{-1}(A) \cup f^{-1}(B)}$

L3

H5. $f(x) \in B$

T5. $x \in f^{-1}(A) \cup f^{-1}(B)$

6. Quantifier-free expansion of target T4.

We would like to show that $x \in f^{-1}(A) \cup f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$.

L1

x

L2

H4. $f(x) \in A$

T6. $\boldsymbol{x \in f^{-1}(A) \vee x \in f^{-1}(B)}$

L3

H5. $f(x) \in B$

T5. $x \in f^{-1}(A) \cup f^{-1}(B)$

7. Split up disjunctive target T6.

L1

x

L2

H4. $f(x) \in A$

L6

T9. $\boldsymbol{x \in f^{-1}(A)}$

$\vee$

L7

T10. $x \in f^{-1}(B)$

L3

H5. $f(x) \in B$

T5. $x \in f^{-1}(A) \cup f^{-1}(B)$

8. Quantifier-free expansion of target T9.

We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$.

L1

x

L2

H4. $\boldsymbol{f(x) \in A}$

L6

T11. $\boldsymbol{f(x) \in A}$

$\vee$

L7

T10. $x \in f^{-1}(B)$

L3

H5. $f(x) \in B$

T5. $x \in f^{-1}(A) \cup f^{-1}(B)$

9. Hypothesis H4 matches target T11, so L6 is done.

But this is clearly the case, so we are done.

L1

x

L2

H4. $f(x) \in A$

L6 Done

$\vee$

L7

T10. $x \in f^{-1}(B)$

L3

H5. $f(x) \in B$

T5. $x \in f^{-1}(A) \cup f^{-1}(B)$

10. Some disjunct of the target of L2 is ‘Done’, so L2 is itself ‘Done’.

L1 x

L2 Done

L3 H5. $f(x) \in B$

T5. $x \in f^{-1}(A) \cup f^{-1}(B)$

11. Remove ‘Done’ targets of L1.

L1 x

L3 H5. $f(x) \in B$

T5. $\boldsymbol{x \in f^{-1}(A) \cup f^{-1}(B)}$

12. Quantifier-free expansion of target T5.

We would like to show that $x \in f^{-1}(A) \cup f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$.

L1 x

L3 H5. $f(x) \in B$

T12. $\boldsymbol{x \in f^{-1}(A) \lor x \in f^{-1}(B)}$

13. Split up disjunctive target T12.

L1 x

L3 H5. $f(x) \in B$

L10 **T15.** $\boldsymbol{x \in f^{-1}(A)}$ $\lor$ L11 T16. $x \in f^{-1}(B)$

14. Quantifier-free expansion of target T15.

We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$.

L1 x

L3 H5. $f(x) \in B$

L10 T17. $f(x) \in A$ $\lor$ L11 **T16.** $\boldsymbol{x \in f^{-1}(B)}$

15. Quantifier-free expansion of target T16.

We would like to show that $x \in f^{-1}(B)$, i.e. that $f(x) \in B$.

L1 x

L3 **H5.** $\boldsymbol{f(x) \in B}$

L10 T17. $f(x) \in A$ $\lor$ L11 **T18.** $\boldsymbol{f(x) \in B}$

16. Hypothesis H5 matches target T18, so L11 is done.

But this is clearly the case, so we are done.

L1

x

L3

H5. $f(x) \in B$

L10

T17. $f(x) \in A$ $\vee$ L11 Done

17. Some disjunct of the target of L3 is ‘Done’, so L3 is itself ‘Done’.

L1

x

L3 Done

18. All targets of L1 are ‘Done’, so L1 is itself done.

L1 Done

Problem solved.

Prove that $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$

Let x be an element of $f^{-1}(A) \cup f^{-1}(B)$. Then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. Since $x \in f^{-1}(A)$, we have that $f(x) \in A$. Since $x \in f^{-1}(B)$, we have that $f(x) \in B$. We would like to show that $x \in f^{-1}(A \cup B)$, i.e. that $f(x) \in A \cup B$. We would like to show that $f(x) \in A \cup B$, i.e. that $f(x) \in A$ or $f(x) \in B$. But this is clearly the case, so we are done. We would like to show that $x \in f^{-1}(A \cup B)$, i.e. that $f(x) \in A \cup B$. We would like to show that $f(x) \in A \cup B$, i.e. that $f(x) \in A$ or $f(x) \in B$. But this is clearly the case, so we are done.

L1

T1. $\boldsymbol{f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)}$

1. Expand pre-universal target T1.

L1

T2. $\boldsymbol{\forall x.(x \in f^{-1}(A) \cup f^{-1}(B) \Rightarrow x \in f^{-1}(A \cup B))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f^{-1}(A) \cup f^{-1}(B)$.

L1

x

H1. $\boldsymbol{x \in f^{-1}(A) \cup f^{-1}(B)}$

T3. $x \in f^{-1}(A \cup B)$

3. Quantifier-free expansion of hypothesis H1.

Since $x \in f^{-1}(A) \cup f^{-1}(B)$, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$.

L1

x

H2. $\boldsymbol{x \in f^{-1}(A) \vee x \in f^{-1}(B)}$

T3. $x \in f^{-1}(A \cup B)$

4. Split into cases to handle disjunctive hypothesis H2.

L1

x

L2

H3. $\boldsymbol{x \in f^{-1}(A)}$

T4. $x \in f^{-1}(A \cup B)$

L3

H4. $x \in f^{-1}(B)$

T5. $x \in f^{-1}(A \cup B)$

5. Quantifier-free expansion of hypothesis H3.

Since $x \in f^{-1}(A)$, we have that $f(x) \in A$.

L1

x

L2

H5. $f(x) \in A$

T4. $x \in f^{-1}(A \cup B)$

L3

H4. $\boldsymbol{x \in f^{-1}(B)}$

T5. $x \in f^{-1}(A \cup B)$

6. Quantifier-free expansion of hypothesis H4.

L1

x

L2

H5. $f(x) \in A$

T4. $\boldsymbol{x \in f^{-1}(A \cup B)}$

L3

H6. $f(x) \in B$

T5. $x \in f^{-1}(A \cup B)$

Since $x \in f^{-1}(B)$, we have that $f(x) \in B$.

7. Quantifier-free expansion of target T4.

L1

x

L2

H5. $f(x) \in A$

T6. $\boldsymbol{f(x) \in A \cup B}$

L3

H6. $f(x) \in B$

T5. $x \in f^{-1}(A \cup B)$

We would like to show that $x \in f^{-1}(A \cup B)$, i.e. that $f(x) \in A \cup B$.

8. Quantifier-free expansion of target T6.

L1

x

L2

H5. $f(x) \in A$

T7. $\boldsymbol{f(x) \in A \vee f(x) \in B}$

L3

H6. $f(x) \in B$

T5. $x \in f^{-1}(A \cup B)$

We would like to show that $f(x) \in A \cup B$, i.e. that $f(x) \in A$ or $f(x) \in B$.

9. Split up disjunctive target T7.

L1

x

L2

H5. $\boldsymbol{f(x) \in A}$

L6

T10. $\boldsymbol{f(x) \in A}$

$\vee$

L7

T11. $f(x) \in B$

L3

H6. $f(x) \in B$

T5. $x \in f^{-1}(A \cup B)$

10. Hypothesis H5 matches target T10, so L6 is done.

But this is clearly the case, so we are done.

L1 x

L2 H5. $f(x) \in A$

L6 Done $\vee$ L7

T11. $f(x) \in B$

L3 H6. $f(x) \in B$

T5. $x \in f^{-1}(A \cup B)$

11. Some disjunct of the target of L2 is 'Done', so L2 is itself 'Done'.

L1 x

L2 Done

L3 H6. $f(x) \in B$

T5. $x \in f^{-1}(A \cup B)$

12. Remove 'Done' targets of L1.

L1 x

L3 H6. $f(x) \in B$

T5. $\boldsymbol{x \in f^{-1}(A \cup B)}$

13. Quantifier-free expansion of target T5.

We would like to show that $x \in f^{-1}(A \cup B)$, i.e. that $f(x) \in A \cup B$.

L1 x

L3 H6. $f(x) \in B$

T12. $\boldsymbol{f(x) \in A \cup B}$

14. Quantifier-free expansion of target T12.

We would like to show that $f(x) \in A \cup B$, i.e. that $f(x) \in A$ or $f(x) \in B$.

L1 x

L3 H6. $f(x) \in B$

T13. $\boldsymbol{f(x) \in A \vee f(x) \in B}$

15. Split up disjunctive target T13.

L1

x

L3

H6. $\boldsymbol{f(x) \in B}$

L10

T16. $f(x) \in A$

$\vee$

L11

T17. $\boldsymbol{f(x) \in B}$

16. Hypothesis H6 matches target T17, so L11 is done.

But this is clearly the case, so we are done.

L1

x

L3

H6. $f(x) \in B$

L10

T16. $f(x) \in A$

$\vee$

L11 Done

17. Some disjunct of the target of L3 is ‘Done’, so L3 is itself ‘Done’.

L1

x

L3 Done

18. All targets of L1 are ‘Done’, so L1 is itself done.

L1 Done

Problem solved.

If A, B,and C are open sets, then $A \cup (B \cup C)$ is also open.

Let x be an element of $A \cup B \cup C$. Then $x \in A$ or $x \in B \cup C$. Since A is open and $x \in A$, there exists $\alpha > 0$ such that $w \in A$ whenever $d(x, w) < \alpha$. Since $x \in B \cup C$, $x \in B$ or $x \in C$. Since B is open and $x \in B$, there exists $\delta' > 0$ such that $r \in B$ whenever $d(x, r) < \delta'$. Since C is open and $x \in C$, there exists $\delta'' > 0$ such that $s \in C$ whenever $d(x, s) < \delta''$. We would like to find $\eta > 0$ s.t. $z \in A \cup B \cup C$ whenever $d(x, z) < \eta$. But $z \in A \cup B \cup C$ if and only if $z \in A$ or $z \in B \cup C$. We know that $z \in A$ if $d(x, z) < \alpha$. Therefore, setting $\eta = \alpha$, we are done. We would like to find $\beta > 0$ s.t. $v \in A \cup B \cup C$ whenever $d(x, v) < \beta$. But $v \in A \cup B \cup C$ if and only if $v \in A$ or $v \in B \cup C$. We would like to show that $v \in B \cup C$, i.e. that $v \in B$ or $v \in C$. We know that $v \in B$ if $d(x, v) < \delta'$. Therefore, setting $\beta = \delta'$, we are done. We would like to find $\gamma > 0$ s.t. $p \in A \cup B \cup C$ whenever $d(x, p) < \gamma$. But $p \in A \cup B \cup C$ if and only if $p \in A$ or $p \in B \cup C$. We would like to show that $p \in B \cup C$, i.e. that $p \in B$ or $p \in C$. We know that $p \in C$ if $d(x, p) < \delta''$. Therefore, setting $\gamma = \delta''$, we are done.

L1

H1. A is open
H2. B is open
H3. C is open

T1. $\boldsymbol{A \cup B \cup C}$ is open

1. Expand pre-universal target T1.

L1

H1. A is open
H2. B is open
H3. C is open

T2. $\boldsymbol{\forall x.(x \in A \cup B \cup C \Rightarrow \exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cup B \cup C)))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $A \cup B \cup C$.

L1

x

H1. A is open
H2. B is open
H3. C is open
H4. $\boldsymbol{x \in A \cup B \cup C}$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cup B \cup C))$

3. Quantifier-free expansion of hypothesis H4.

Since $x \in A \cup B \cup C$, $x \in A$ or $x \in B \cup C$.

L1

x

H1. A is open
H2. B is open
H3. C is open
H5. $\boldsymbol{x \in A \vee x \in B \cup C}$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cup B \cup C))$

4. Split into cases to handle disjunctive hypothesis H5.

L1

x

H1. A is open

H2. B is open

H3. C is open

L2

H6. $x \in A$

T4. $\exists\eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))$

L3

H7. $x \in B \cup C$

T5. $\exists\theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B \cup C))$

5. Forwards reasoning using H1 with H6.

Since A is open and $x \in A$, there exists $\alpha > 0$ such that $w \in A$ whenever $d(x,w) < \alpha$.

L1

x

H1. A is open [Vuln.; Used with H6.]

H2. B is open

H3. C is open

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]

H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists\eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))$

L3

H7. $x \in B \cup C$

T5. $\exists\theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B \cup C))$

6. Deleted H6, as this unexpandable atomic statement is unmatchable.

L1

x

H1. A is open [Vuln.; Used with H6.]

H2. B is open

H3. C is open

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]

H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists\eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))$

L3

H7. $x \in B \cup C$

T5. $\exists\theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B \cup C))$

7. Deleted H1, as the conclusion of this implicative statement is unmatchable.

L1

x

H1. A is open [Vuln.; Used with H6.]

H2. B is open

H3. C is open

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]

H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists\eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))$

L3

H7. $x \in B \cup C$

T5. $\exists\theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B \cup C))$

8. Quantifier-free expansion of hypothesis H7.

Since $x \in B \cup C$, $x \in B$ or $x \in C$.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open
H3. C is open

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))$

L3

H9. $\boldsymbol{x \in B \vee x \in C}$

T5. $\exists \theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B \cup C))$

9. Split into cases to handle disjunctive hypothesis H9.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open
H3. C is open

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))$

L3

L4

H10. $x \in B$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5

H11. $x \in C$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

10. Collapsed subtableau L3 as it has no undeleted hypotheses.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. $\boldsymbol{B}$ is open
H3. C is open

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))$

L4

H10. $\boldsymbol{x \in B}$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5

H11. $x \in C$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

11. Forwards reasoning using H2 with H10.

Since B is open and $x \in B$, there exists $\delta' > 0$ such that $r \in B$ whenever $d(x,r) < \delta'$.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))$

L4

$\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5

H11. $x \in C$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

12. Deleted H10, as this unexpandable atomic statement is unmatchable.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))$

L4

$\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5

H11. $x \in C$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

13. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))$

L4

$\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5

H11. $x \in C$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

14. Forwards reasoning using H3 with H11.

Since C is open and $x \in C$, there exists $\delta'' > 0$ such that $s \in C$ whenever

L1

x

H1. A is open [Vuln.; Used with H6.]

H2. B is open [Vuln.; Used with H10.]

H3. C is open [Vuln.; Used with H11.]

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]

H8. $\forall w.(d(x, w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$

L4

$\delta'[x]$

H10. $x \in B$ [Vuln.]

H12. $\forall r.(d(x, r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5

$\delta''[x]$

H11. $x \in C$ [Vuln.]

H13. $\forall s.(d(x, s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$

15. Deleted H11, as this unexpandable atomic statement is unmatchable.

L1

x

H1. A is open [Vuln.; Used with H6.]

H2. B is open [Vuln.; Used with H10.]

H3. C is open [Vuln.; Used with H11.]

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]

H8. $\forall w.(d(x, w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x, z) < \eta \Rightarrow z \in A \cup B \cup C))$

L4

$\delta'[x]$

H10. $x \in B$ [Vuln.]

H12. $\forall r.(d(x, r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5

$\delta''[x]$

H11. $x \in C$ [Vuln.]

H13. $\forall s.(d(x, s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$

16. Deleted H3, as the conclusion of this implicative statement is unmatchable.

L1
x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L2
$\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\boldsymbol{\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B \cup C))}$

L4
$\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5
$\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

17. Unlock existential-universal-conditional target T4.

We would like to find $\eta > 0$ s.t. $z \in A \cup B \cup C$ whenever $d(x,z) < \eta$.

L1
x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L2
$\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

L6$^\blacklozenge$
$\eta^\blacklozenge[\overline{z}]$ z

H14. $d(x,z) < \eta^\blacklozenge[\overline{z}]$ [From L2.]

T8. $\boldsymbol{z \in A \cup B \cup C}$

L4
$\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5
$\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

18. Quantifier-free expansion of target T8.

But $z \in A \cup B \cup C$ if and only if $z \in A$ or $z \in B \cup C$.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

L6$^\blacklozenge$

$\eta^\blacklozenge[\overline{z}]$ z

H14. $d(x,z) < \eta^\blacklozenge[\overline{z}]$ [From L2.]

T9. $\boldsymbol{z \in A \vee z \in B \cup C}$

L4

$\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5

$\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

19. Split up disjunctive target T9.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L2

$\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\boldsymbol{\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)}$

L6$^\blacklozenge$

$\eta^\blacklozenge[\overline{z}]$ z

H14. $d(x,z) < \eta^\blacklozenge[\overline{z}]$ [From L2.]

L9

T12. $\boldsymbol{z \in A}$

$\vee$

L10

T13. $z \in B \cup C$

L4

$\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5

$\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

20. Backwards reasoning using H8 with T12.

We know that $z \in A$ if $d(x,z) < \alpha$.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L2 $\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x, w) < \alpha[x] \Rightarrow w \in A)$ **[Vuln.]**

L6$^\blacklozenge$ $\eta^\blacklozenge[\overline{z}]\ z$

H14. $d(x, z) < \eta^\blacklozenge[\overline{z}]$ [From L2.]

L9

T14. $d(x, z) < \alpha[x]$

$\vee$

L10

T13. $z \in B \cup C$

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x, r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x, s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$

21. Delete H8 as no other statement mentions A.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L2 $\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x, w) < \alpha[x] \Rightarrow w \in A)$ [Vuln.]

L6$^\blacklozenge$ $\eta^\blacklozenge[\overline{z}]\ z$

H14. $\boldsymbol{d(x, z) < \eta^\blacklozenge[\overline{z}]}$ **[From L2.]**

L9

T14. $\boldsymbol{d(x, z) < \alpha[x]}$

$\vee$

L10

T13. $z \in B \cup C$

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x, r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x, v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x, s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x, p) < \gamma \Rightarrow p \in A \cup B \cup C))$

22. Hypothesis H14 matches target T14 after choosing $\eta^\blacklozenge[\overline{z}] = \alpha[x]$, so L6$^\blacklozenge$ is done.

Therefore, setting $\eta = \alpha$, we are done.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L2 $\alpha[x]$

H6. $x \in A$ [Vuln.]
H8. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$ [Vuln.]

L6♦ Done

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

23. All targets of L2 are 'Done', so L2 is itself done.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L2 Done

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

T6. $\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))$

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

24. Remove 'Done' targets of L1.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

T6. $\boldsymbol{\exists \beta.(\forall v.(d(x,v) < \beta \Rightarrow v \in A \cup B \cup C))}$

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

25. Unlock existential-universal-conditional target T6.

We would like to find $\beta > 0$ s.t. $v \in A \cup B \cup C$ whenever $d(x,v) < \beta$.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

L11$^\blacklozenge$ $\beta^\blacklozenge[\overline{v}]$ v

H15. $d(x,v) < \beta^\blacklozenge[\overline{v}]$ [From L4.]

T15. $\boldsymbol{v \in A \cup B \cup C}$

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

26. Quantifier-free expansion of target T15.

But $v \in A \cup B \cup C$ if and only if $v \in A$ or $v \in B \cup C$.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

L11$^\blacklozenge$ $\beta^\blacklozenge[\overline{v}]$ v

H15. $d(x,v) < \beta^\blacklozenge[\overline{v}]$ [From L4.]

T16. $\boldsymbol{v \in A \vee v \in B \cup C}$

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

27. Split up disjunctive target T16.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

L11$^\blacklozenge$ $\beta^\blacklozenge[\overline{v}]$ v

H15. $d(x,v) < \beta^\blacklozenge[\overline{v}]$ [From L4.]

L14 T19. $v \in A$ $\vee$ **L15** **T20.** $\boldsymbol{v \in B \cup C}$

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

28. Quantifier-free expansion of target T20.

We would like to show that $v \in B \cup C$, i.e. that $v \in B$ or $v \in C$.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$

L11♦ $\beta^{\blacklozenge}[\overline{v}]\ v$

H15. $d(x,v) < \beta^{\blacklozenge}[\overline{v}]$ [From L4.]

L14: T19. $v \in A$ $\vee$ L15: **T21. $\boldsymbol{v \in B \vee v \in C}$**

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

29. Split up disjunctive target T21.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\boldsymbol{\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)}$

L11♦ $\beta^{\blacklozenge}[\overline{v}]\ v$

H15. $d(x,v) < \beta^{\blacklozenge}[\overline{v}]$ [From L4.]

L14: T19. $v \in A$ $\vee$ L15: (L18: **T24. $\boldsymbol{v \in B}$** $\vee$ L19: T25. $v \in C$)

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

30. Backwards reasoning using H12 with T24.

We know that $v \in B$ if $d(x,v) < \delta'$.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L4 $\delta'[x]$

H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$ [Vuln.]

L11♦ $\beta^{\blacklozenge}[\overline{v}]\ v$

H15. $d(x,v) < \beta^{\blacklozenge}[\overline{v}]$ [From L4.]

L14: T19. $v \in A$ $\vee$ L15: (L18: T26. $d(x,v) < \delta'[x]$ $\vee$ L19: T25. $v \in C$)

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

31. Delete H12 as no other statement mentions B.

L1
x
H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L4
$\delta'[x]$
H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$ [Vuln.]

L11$^\blacklozenge$
$\beta^\blacklozenge[\overline{v}]$ v
H15. $\boldsymbol{d(x,v) < \beta^\blacklozenge[\overline{v}]}$ **[From L4.]**

L14
T19. $v \in A$

$\vee$

L15

L18
T26. $\boldsymbol{d(x,v) < \delta'[x]}$

$\vee$

L19
T25. $v \in C$

L5
$\delta''[x]$
H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

32. Hypothesis H15 matches target T26 after choosing $\beta^\blacklozenge[\overline{v}] = \delta'[x]$, so L11$^\blacklozenge$ is done.

Therefore, setting $\beta = \delta'$, we are done.

L1
x
H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L4
$\delta'[x]$
H10. $x \in B$ [Vuln.]
H12. $\forall r.(d(x,r) < \delta'[x] \Rightarrow r \in B)$ [Vuln.]

L11$^\blacklozenge$ Done

L5
$\delta''[x]$
H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

33. All targets of L4 are 'Done', so L4 is itself done.

L1
x
H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L4 Done

L5
$\delta''[x]$
H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))$

34. Remove 'Done' targets of L1.

L1
x
H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L5
$\delta''[x]$
H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

T7. $\boldsymbol{\exists \gamma.(\forall p.(d(x,p) < \gamma \Rightarrow p \in A \cup B \cup C))}$

35. Unlock existential-universal-conditional target T7.

We would like to find $\gamma > 0$ s.t. $p \in A \cup B \cup C$ whenever $d(x,p) < \gamma$.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L5

$\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x, s) < \delta''[x] \Rightarrow s \in C)$

L20$^\blacklozenge$

$\gamma^\blacklozenge[\overline{p}]$ p

H16. $d(x, p) < \gamma^\blacklozenge[\overline{p}]$ [From L5.]

T27. $\boldsymbol{p \in A \cup B \cup C}$

36. Quantifier-free expansion of target T27.

But $p \in A \cup B \cup C$ if and only if $p \in A$ or $p \in B \cup C$.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L5

$\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x, s) < \delta''[x] \Rightarrow s \in C)$

L20$^\blacklozenge$

$\gamma^\blacklozenge[\overline{p}]$ p

H16. $d(x, p) < \gamma^\blacklozenge[\overline{p}]$ [From L5.]

T28. $\boldsymbol{p \in A \vee p \in B \cup C}$

37. Split up disjunctive target T28.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L5

$\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x, s) < \delta''[x] \Rightarrow s \in C)$

L20$^\blacklozenge$

$\gamma^\blacklozenge[\overline{p}]$ p

H16. $d(x, p) < \gamma^\blacklozenge[\overline{p}]$ [From L5.]

L23: T31. $p \in A$ $\vee$ L24: **T32. $\boldsymbol{p \in B \cup C}$**

38. Quantifier-free expansion of target T32.

We would like to show that $p \in B \cup C$, i.e. that $p \in B$ or $p \in C$.

L1

x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L5

$\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x, s) < \delta''[x] \Rightarrow s \in C)$

L20$^\blacklozenge$

$\gamma^\blacklozenge[\overline{p}]$ p

H16. $d(x, p) < \gamma^\blacklozenge[\overline{p}]$ [From L5.]

L23: T31. $p \in A$ $\vee$ L24: **T33. $\boldsymbol{p \in B \vee p \in C}$**

39. Split up disjunctive target T33.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$

L20$^\blacklozenge$ $\gamma^\blacklozenge[\overline{p}]$ p

H16. $d(x,p) < \gamma^\blacklozenge[\overline{p}]$ [From L5.]

L23 T31. $p \in A$ $\vee$ L24 (L27 T36. $p \in B$ $\vee$ L28 **T37.** $p \in C$)

40. Backwards reasoning using H13 with T37.

We know that $p \in C$ if $d(x,p) < \delta''$.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$ [Vuln.]

L20$^\blacklozenge$ $\gamma^\blacklozenge[\overline{p}]$ p

H16. $d(x,p) < \gamma^\blacklozenge[\overline{p}]$ [From L5.]

L23 T31. $p \in A$ $\vee$ L24 (L27 T36. $p \in B$ $\vee$ L28 T38. $d(x,p) < \delta''[x]$)

41. Delete H13 as no other statement mentions C.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$ [Vuln.]

L20$^\blacklozenge$ $\gamma^\blacklozenge[\overline{p}]$ p

H16. $d(x,p) < \gamma^\blacklozenge[\overline{p}]$ **[From L5.]**

L23 T31. $p \in A$ $\vee$ L24 (L27 T36. $p \in B$ $\vee$ L28 **T38.** $d(x,p) < \delta''[x]$)

42. Hypothesis H16 matches target T38 after choosing $\gamma^\blacklozenge[\overline{p}] = \delta''[x]$, so L20$^\blacklozenge$ is done.

Therefore, setting $\gamma = \delta''$, we are done.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L5 $\delta''[x]$

H11. $x \in C$ [Vuln.]
H13. $\forall s.(d(x,s) < \delta''[x] \Rightarrow s \in C)$ [Vuln.]

L20$^\blacklozenge$ Done

43. All targets of L5 are 'Done', so L5 is itself done.

L1 x

H1. A is open [Vuln.; Used with H6.]
H2. B is open [Vuln.; Used with H10.]
H3. C is open [Vuln.; Used with H11.]

L5 Done

44. All targets of L1 are ‘Done’, so L1 is itself done.

L1 Done

Problem solved.

If A and B are open sets, then $A \cup B$ is also open.

Let x be an element of $A \cup B$. Then $x \in A$ or $x \in B$. Since A is open and $x \in A$, there exists $\alpha > 0$ such that $w \in A$ whenever $d(x, w) < \alpha$. Since B is open and $x \in B$, there exists $\beta > 0$ such that $p \in B$ whenever $d(x, p) < \beta$. We would like to find $\eta > 0$ s.t. $z \in A \cup B$ whenever $d(x, z) < \eta$. But $z \in A \cup B$ if and only if $z \in A$ or $z \in B$. We know that $z \in A$ if $d(x, z) < \alpha$. Therefore, setting $\eta = \alpha$, we are done. We would like to find $\theta > 0$ s.t. $u \in A \cup B$ whenever $d(x, u) < \theta$. But $u \in A \cup B$ if and only if $u \in A$ or $u \in B$. We know that $u \in B$ if $d(x, u) < \beta$. Therefore, setting $\theta = \beta$, we are done.

L1

H1. A is open
H2. B is open

T1. $A \cup B$ is open

1. Expand pre-universal target T1.

L1

H1. A is open
H2. B is open

T2. $\forall x.(x \in A \cup B \Rightarrow \exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cup B)))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $A \cup B$.

L1

x

H1. A is open
H2. B is open
H3. $x \in A \cup B$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cup B))$

3. Quantifier-free expansion of hypothesis H3.

Since $x \in A \cup B$, $x \in A$ or $x \in B$.

L1

x

H1. A is open
H2. B is open
H4. $x \in A \vee x \in B$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cup B))$

4. Split into cases to handle disjunctive hypothesis H4.

L1

x

H1. A is open
H2. B is open

L2

H5. $x \in A$

T4. $\exists \eta.(\forall z.(d(x, z) < \eta \Rightarrow z \in A \cup B))$

L3

H6. $x \in B$

T5. $\exists \theta.(\forall u.(d(x, u) < \theta \Rightarrow u \in A \cup B))$

5. Forwards reasoning using H1 with H5.

Since A is open and $x \in A$, there exists $\alpha > 0$ such that $w \in A$ whenever $d(x, w) < \alpha$.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open

L2

$\alpha[x]$

H5. $x \in A$ [Vuln.]

H7. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B))$

L3

H6. $x \in B$

T5. $\exists \theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))$

6. Deleted H5, as this unexpandable atomic statement is unmatchable.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open

L2

$\alpha[x]$

H5. $x \in A$ [Vuln.]

H7. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B))$

L3

H6. $x \in B$

T5. $\exists \theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))$

7. Deleted H1, as the conclusion of this implicative statement is unmatchable.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open

L2

$\alpha[x]$

H5. $x \in A$ [Vuln.]

H7. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B))$

L3

H6. $x \in B$

T5. $\exists \theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))$

8. Forwards reasoning using H2 with H6.

Since B is open and $x \in B$, there exists $\beta > 0$ such that $p \in B$ whenever $d(x,p) < \beta$.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

L2

$\alpha[x]$

H5. $x \in A$ [Vuln.]

H7. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

T4. $\exists \eta.(\forall z.(d(x,z) < \eta \Rightarrow z \in A \cup B))$

L3

$\beta[x]$

H6. $x \in B$ [Vuln.]

H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$

T5. $\exists \theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))$

9. Deleted H6, as this unexpandable atomic statement is unmatchable.

L1 x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

> L2 $\alpha[x]$
>
> H5. $x \in A$ [Vuln.]
>
> H7. $\forall w.(d(x, w) < \alpha[x] \Rightarrow w \in A)$
>
> T4. $\exists \eta.(\forall z.(d(x, z) < \eta \Rightarrow z \in A \cup B))$

> L3 $\beta[x]$
>
> H6. $x \in B$ [Vuln.]
>
> H8. $\forall p.(d(x, p) < \beta[x] \Rightarrow p \in B)$
>
> T5. $\exists \theta.(\forall u.(d(x, u) < \theta \Rightarrow u \in A \cup B))$

10. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1 x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

> L2 $\alpha[x]$
>
> H5. $x \in A$ [Vuln.]
>
> H7. $\forall w.(d(x, w) < \alpha[x] \Rightarrow w \in A)$
>
> **T4.** $\boldsymbol{\exists \eta.(\forall z.(d(x, z) < \eta \Rightarrow z \in A \cup B))}$

> L3 $\beta[x]$
>
> H6. $x \in B$ [Vuln.]
>
> H8. $\forall p.(d(x, p) < \beta[x] \Rightarrow p \in B)$
>
> T5. $\exists \theta.(\forall u.(d(x, u) < \theta \Rightarrow u \in A \cup B))$

11. Unlock existential-universal-conditional target T4.

We would like to find $\eta > 0$ s.t. $z \in A \cup B$ whenever $d(x, z) < \eta$.

L1 x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

> L2 $\alpha[x]$
>
> H5. $x \in A$ [Vuln.]
>
> H7. $\forall w.(d(x, w) < \alpha[x] \Rightarrow w \in A)$
>
> > L4♦ $\eta^{\blacklozenge}[\overline{z}]$ z
> >
> > H9. $d(x, z) < \eta^{\blacklozenge}[\overline{z}]$ [From L2.]
> >
> > **T6.** $\boldsymbol{z \in A \cup B}$

> L3 $\beta[x]$
>
> H6. $x \in B$ [Vuln.]
>
> H8. $\forall p.(d(x, p) < \beta[x] \Rightarrow p \in B)$
>
> T5. $\exists \theta.(\forall u.(d(x, u) < \theta \Rightarrow u \in A \cup B))$

12. Quantifier-free expansion of target T6.

But $z \in A \cup B$ if and only if $z \in A$ or $z \in B$.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

L2

$\alpha[x]$

H5. $x \in A$ [Vuln.]

H7. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$

L4$^\blacklozenge$

$\eta^\blacklozenge[\overline{z}]$ z

H9. $d(x,z) < \eta^\blacklozenge[\overline{z}]$ [From L2.]

T7. $\boldsymbol{z \in A \vee z \in B}$

L3

$\beta[x]$

H6. $x \in B$ [Vuln.]

H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$

T5. $\exists\theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))$

13. Split up disjunctive target T7.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

L2

$\alpha[x]$

H5. $x \in A$ [Vuln.]

H7. $\boldsymbol{\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)}$

L4$^\blacklozenge$

$\eta^\blacklozenge[\overline{z}]$ z

H9. $d(x,z) < \eta^\blacklozenge[\overline{z}]$ [From L2.]

L7

T10. $\boldsymbol{z \in A}$

$\vee$

L8

T11. $z \in B$

L3

$\beta[x]$

H6. $x \in B$ [Vuln.]

H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$

T5. $\exists\theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))$

14. Backwards reasoning using H7 with T10.

We know that $z \in A$ if $d(x,z) < \alpha$.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

L2

$\alpha[x]$

H5. $x \in A$ [Vuln.]

H7. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$ [Vuln.]

L4$^\blacklozenge$

$\eta^\blacklozenge[\overline{z}]$ z

H9. $d(x,z) < \eta^\blacklozenge[\overline{z}]$ [From L2.]

L7

T12. $d(x,z) < \alpha[x]$

$\vee$

L8

T11. $z \in B$

L3

$\beta[x]$

H6. $x \in B$ [Vuln.]

H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$

T5. $\exists\theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))$

15. Delete H7 as no other statement mentions A.

L1

x

H1. A is open [Vuln.; Used with H5.]
H2. B is open [Vuln.; Used with H6.]

L2

$\alpha[x]$

H5. $x \in A$ [Vuln.]
H7. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$ [Vuln.]

L4$^\blacklozenge$

$\eta^\blacklozenge[\overline{z}]$ z

H9. $\boldsymbol{d(x,z) < \eta^\blacklozenge[\overline{z}]}$ **[From L2.]**

L7

T12. $\boldsymbol{d(x,z) < \alpha[x]}$

$\vee$

L8

T11. $z \in B$

L3

$\beta[x]$

H6. $x \in B$ [Vuln.]
H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$

T5. $\exists \theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))$

16. Hypothesis H9 matches target T12 after choosing $\eta^\blacklozenge[\overline{z}] = \alpha[x]$, so L4$^\blacklozenge$ is done.

Therefore, setting $\eta = \alpha$, we are done.

L1

x

H1. A is open [Vuln.; Used with H5.]
H2. B is open [Vuln.; Used with H6.]

L2

$\alpha[x]$

H5. $x \in A$ [Vuln.]
H7. $\forall w.(d(x,w) < \alpha[x] \Rightarrow w \in A)$ [Vuln.]

L4$^\blacklozenge$ Done

L3

$\beta[x]$

H6. $x \in B$ [Vuln.]
H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$

T5. $\exists \theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))$

17. All targets of L2 are ‘Done’, so L2 is itself done.

L1

x

H1. A is open [Vuln.; Used with H5.]
H2. B is open [Vuln.; Used with H6.]

L2 Done

L3

$\beta[x]$

H6. $x \in B$ [Vuln.]
H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$

T5. $\exists \theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))$

18. Remove ‘Done’ targets of L1.

L1

x

H1. A is open [Vuln.; Used with H5.]
H2. B is open [Vuln.; Used with H6.]

L3

$\beta[x]$

H6. $x \in B$ [Vuln.]
H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$

T5. $\boldsymbol{\exists \theta.(\forall u.(d(x,u) < \theta \Rightarrow u \in A \cup B))}$

19. Unlock existential-universal-conditional target T5.

We would like to find $\theta > 0$ s.t. $u \in A \cup B$ whenever $d(x,u) < \theta$.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

> L3
>
> $\beta[x]$
>
> H6. $x \in B$ [Vuln.]
>
> H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$
>
> > L9$^\blacklozenge$
> >
> > $\theta^\blacklozenge[\overline{u}]\ u$
> >
> > H10. $d(x,u) < \theta^\blacklozenge[\overline{u}]$ [From L3.]
> >
> > **T13.** $\boldsymbol{u \in A \cup B}$

20. Quantifier-free expansion of target T13.

But $u \in A \cup B$ if and only if $u \in A$ or $u \in B$.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

> L3
>
> $\beta[x]$
>
> H6. $x \in B$ [Vuln.]
>
> H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$
>
> > L9$^\blacklozenge$
> >
> > $\theta^\blacklozenge[\overline{u}]\ u$
> >
> > H10. $d(x,u) < \theta^\blacklozenge[\overline{u}]$ [From L3.]
> >
> > **T14.** $\boldsymbol{u \in A \vee u \in B}$

21. Split up disjunctive target T14.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

> L3
>
> $\beta[x]$
>
> H6. $x \in B$ [Vuln.]
>
> **H8.** $\boldsymbol{\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)}$
>
> > L9$^\blacklozenge$
> >
> > $\theta^\blacklozenge[\overline{u}]\ u$
> >
> > H10. $d(x,u) < \theta^\blacklozenge[\overline{u}]$ [From L3.]
> >
> > L12: T17. $u \in A$ $\vee$ L13: **T18.** $\boldsymbol{u \in B}$

22. Backwards reasoning using H8 with T18.

We know that $u \in B$ if $d(x,u) < \beta$.

L1

x

H1. A is open [Vuln.; Used with H5.]

H2. B is open [Vuln.; Used with H6.]

> L3
>
> $\beta[x]$
>
> H6. $x \in B$ [Vuln.]
>
> H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$ [Vuln.]
>
> > L9$^\blacklozenge$
> >
> > $\theta^\blacklozenge[\overline{u}]\ u$
> >
> > H10. $d(x,u) < \theta^\blacklozenge[\overline{u}]$ [From L3.]
> >
> > L12: T17. $u \in A$ $\vee$ L13: T19. $d(x,u) < \beta[x]$

23. Delete H8 as no other statement mentions B.

L1

x

H1. A is open [Vuln.; Used with H5.]
H2. B is open [Vuln.; Used with H6.]

L3

$\beta[x]$

H6. $x \in B$ [Vuln.]
H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$ [Vuln.]

L9$^\blacklozenge$

$\theta^\blacklozenge[\overline{u}]$ u

H10. $\boldsymbol{d(x,u) < \theta^\blacklozenge[\overline{u}]}$ **[From L3.]**

L12

T17. $u \in A$

$\vee$

L13

T19. $\boldsymbol{d(x,u) < \beta[x]}$

24. Hypothesis H10 matches target T19 after choosing $\theta^\blacklozenge[\overline{u}] = \beta[x]$, so L9$^\blacklozenge$ is done.

Therefore, setting $\theta = \beta$, we are done.

L1

x

H1. A is open [Vuln.; Used with H5.]
H2. B is open [Vuln.; Used with H6.]

L3

$\beta[x]$

H6. $x \in B$ [Vuln.]
H8. $\forall p.(d(x,p) < \beta[x] \Rightarrow p \in B)$ [Vuln.]

L9$^\blacklozenge$ Done

25. All targets of L3 are ‘Done’, so L3 is itself done.

L1

x

H1. A is open [Vuln.; Used with H5.]
H2. B is open [Vuln.; Used with H6.]

L3 Done

26. All targets of L1 are ‘Done’, so L1 is itself done.

L1 Done

Problem solved.

If A and B are open sets, then $A \cap B$ is also open.

Let x be an element of $A \cap B$. Then $x \in A$ and $x \in B$. Therefore, since A is open, there exists $\eta > 0$ such that $u \in A$ whenever $d(x,u) < \eta$ and since B is open, there exists $\theta > 0$ such that $v \in B$ whenever $d(x,v) < \theta$. We would like to find $\delta > 0$ s.t. $y \in A \cap B$ whenever $d(x,y) < \delta$. But $y \in A \cap B$ if and only if $y \in A$ and $y \in B$. We know that $y \in A$ whenever $d(x,y) < \eta$ and that $y \in B$ whenever $d(x,y) < \theta$. Assume now that $d(x,y) < \delta$. Then $d(x,y) < \eta$ if $\delta \leqslant \eta$ and $d(x,y) < \theta$ if $\delta \leqslant \theta$. We may therefore take $\delta = \min(\eta, \theta)$ and we are done.

L1

H1. A is open
H2. B is open

T1. $\boldsymbol{A \cap B}$ is open

1. Expand pre-universal target T1.

L1

H1. A is open
H2. B is open

T2. $\boldsymbol{\forall x.(x \in A \cap B \Rightarrow \exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B)))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $A \cap B$.

L1

x

H1. A is open
H2. B is open
H3. $\boldsymbol{x \in A \cap B}$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

3. Quantifier-free expansion of hypothesis H3.

Since $x \in A \cap B$, $x \in A$ and $x \in B$.

L1

x

H1. $\boldsymbol{A}$ is open
H2. B is open
H4. $\boldsymbol{x \in A}$
H5. $x \in B$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

4. Forwards reasoning using H1 with H4.

Since A is open and $x \in A$, there exists $\eta > 0$ such that $u \in A$ whenever $d(x,u) < \eta$.

L1

$x \ \eta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open
H4. $x \in A$ [Vuln.]
H5. $x \in B$
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

5. Deleted H4, as this unexpandable atomic statement is unmatchable.

L1

$x\ \eta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open
H4. $x \in A$ [Vuln.]
H5. $x \in B$
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

6. Deleted H1, as the conclusion of this implicative statement is unmatchable.

L1

$x\ \eta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open
H4. $x \in A$ [Vuln.]
H5. $x \in B$
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

7. Forwards reasoning using H2 with H5.

Since B is open and $x \in B$, there exists $\theta > 0$ such that $v \in B$ whenever $d(x,v) < \theta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

8. Deleted H5, as this unexpandable atomic statement is unmatchable.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

9. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

T3. $\boldsymbol{\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))}$

10. Unlock existential-universal-conditional target T3.

We would like to find $\delta > 0$ s.t. $y \in A \cap B$ whenever $d(x,y) < \delta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y$

H8. $d(x,y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T4. $\boldsymbol{y \in A \cap B}$

11. Quantifier-free expansion of target T4.

But $y \in A \cap B$ if and only if $y \in A$ and $y \in B$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\boldsymbol{\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)}$
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y$

H8. $d(x,y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T5. $\boldsymbol{y \in A}$
T6. $y \in B$

12. Backwards reasoning using H6 with T5.

We know that $y \in A$ whenever $d(x,y) < \eta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y$

H8. $d(x,y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T7. $d(x,y) < \eta[x]$
T6. $y \in B$

13. Delete H6 as no other statement mentions A.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\boldsymbol{\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)}$

L2$^{\blacklozenge}$

$\delta^{\blacklozenge}[\overline{y}]\ y$

H8. $d(x,y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T7. $d(x,y) < \eta[x]$
T6. $\boldsymbol{y \in B}$

14. Backwards reasoning using H7 with T6.

We know that $y \in B$ whenever $d(x,y) < \theta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2$^{\blacklozenge}$

$\delta^{\blacklozenge}[\overline{y}]\ y$

H8. $d(x,y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T7. $d(x,y) < \eta[x]$
T8. $d(x,y) < \theta[x]$

15. Delete H7 as no other statement mentions B.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2$^{\blacklozenge}$

$\delta^{\blacklozenge}[\overline{y}]\ y$

H8. $d(x,y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T7. $d(x,y) < \eta[x]$
T8. $d(x,y) < \theta[x]$

16. Replacing diamonds with bullets in L2$^{\blacklozenge}$.

Assume now that $d(x,y) < \delta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}]\ y$

H8. $d(x,y) < \delta^\bullet[\overline{y}]$ [From L1.]

T7. $d(x,y) < \eta[x]$
T8. $d(x,y) < \theta[x]$

17. Backwards reasoning using library result "transitivity" with (T7,H8). Since $d(x,y) < \delta$, $d(x,y) < \eta$ if $\delta \leqslant \eta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}]\ y$

H8. $d(x,y) < \delta^\bullet[\overline{y}]$ [From L1.; Vuln.]

T9. $\delta^\bullet[\overline{y}] \leqslant \eta[x]$
T8. $d(x,y) < \theta[x]$

18. Moved H8 down, as x can only be utilised by T8.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}]\ y$

T9. $\delta^\bullet[\overline{y}] \leqslant \eta[x]$

L3

H8. $d(x,y) < \delta^\bullet[\overline{y}]$ [From L1.; Vuln.; From L2.]

T8. $d(x,y) < \theta[x]$

19. Backwards reasoning using library result "transitivity" with (T8,H8). Since $d(x,y) < \delta$, $d(x,y) < \theta$ if $\delta \leqslant \theta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}]\ y$

T9. $\delta^\bullet[\overline{y}] \leqslant \eta[x]$

L3

H8. $d(x,y) < \delta^\bullet[\overline{y}]$ [From L1.; Vuln.; From L2.]

T10. $\delta^\bullet[\overline{y}] \leqslant \theta[x]$

20. Delete H8 as no other statement mentions x.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}]\ y$

T9. $\delta^\bullet[\overline{y}] \leqslant \eta[x]$

L3

H8. $d(x,y) < \delta^\bullet[\overline{y}]$ [From L1.; Vuln.; From L2.]

T10. $\delta^\bullet[\overline{y}] \leqslant \theta[x]$

21. Collapsed subtableau L3 as it has no undeleted hypotheses.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}]\ y$

T9. $\delta^\bullet[\overline{y}] \leqslant \eta[x]$
T10. $\delta^\bullet[\overline{y}] \leqslant \theta[x]$

22. Taking $\delta^\bullet[\overline{y}] = \min(\eta[x], \theta[x])$ matches all targets against a library solution, so L2 is done.

We may therefore take $\delta = \min(\eta, \theta)$. We are done.

L1

$x \ \eta[x] \ \theta[x]$

H1. A is open	[Vuln.; Used with H4.]
H2. B is open	[Vuln.; Used with H5.]
H4. $x \in A$	[Vuln.]
H5. $x \in B$	[Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$	[Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$	[Vuln.]

L2 Done

23. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.

If A and B are closed sets, then $A \cap B$ is also closed.

Let (a_n) and a be such that (a_n) is a sequence in $A \cap B$ and $a_n \to a$. Then (a_n) is a sequence in A and (a_n) is a sequence in B. Therefore, since A is closed and $a_n \to a$, we have that $a \in A$ and since B is closed and $a_n \to a$, we have that $a \in B$. We would like to show that $a \in A \cap B$, i.e. that $a \in A$ and $a \in B$. But this is clearly the case, so we are done.

L1

H1. A is closed
H2. B is closed

T1. $\boldsymbol{A \cap B}$ is closed

1. Expand pre-universal target T1.

L1

H1. A is closed
H2. B is closed

T2. $\boldsymbol{\forall (a_n), a.((a_n) \text{ is a sequence in } A \cap B \wedge a_n \to a \Rightarrow a \in A \cap B)}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let (a_n) and a be such that (a_n) is a sequence in $A \cap B$ and $a_n \to a$.

L1

(a_n) a

H1. A is closed
H2. B is closed
H3. $\boldsymbol{(a_n)}$ is a sequence in $\boldsymbol{A \cap B}$
H4. $a_n \to a$

T3. $a \in A \cap B$

3. Quantifier-free expansion of hypothesis H3.

Since (a_n) is a sequence in $A \cap B$, (a_n) is a sequence in A and (a_n) is a sequence in B.

L1

(a_n) a

H1. $\boldsymbol{A}$ is closed
H2. B is closed
H5. $\boldsymbol{(a_n)}$ is a sequence in $\boldsymbol{A}$
H6. (a_n) is a sequence in B
H4. $\boldsymbol{a_n \to a}$

T3. $a \in A \cap B$

4. Forwards reasoning using H1 with (H5,H4).

Since A is closed, (a_n) is a sequence in A and $a_n \to a$, we have that $a \in A$.

L1

(a_n) a

H1. A is closed [Vuln.; Used with (H4,H5).]
H2. B is closed
H5. (a_n) is a sequence in A [Vuln.]
H6. (a_n) is a sequence in B
H4. $a_n \to a$ [Vuln.]
H7. $a \in A$

T3. $a \in A \cap B$

5. Deleted H1, as the conclusion of this implicative statement is unmatchable.

L1

(a_n) a

H1. A is closed	[Vuln.; Used with (H4,H5).]
H2. B is closed	
H5. (a_n) is a sequence in A	[Vuln.]
H6. (a_n) is a sequence in B	
H4. $a_n \to a$	**[Vuln.]**
H7. $a \in A$	

T3. $a \in A \cap B$

6. Forwards reasoning using H2 with (H6,H4).

Since B is closed, (a_n) is a sequence in B and $a_n \to a$, we have that $a \in B$.

L1

(a_n) a

H1. A is closed	[Vuln.; Used with (H4,H5).]
H2. B is closed	[Vuln.; Used with (H4,H6).]
H5. (a_n) is a sequence in A	[Vuln.]
H6. (a_n) is a sequence in B	[Vuln.]
H4. $a_n \to a$	[Vuln.]
H7. $a \in A$	
H8. $a \in B$	

T3. $a \in A \cap B$

7. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1

(a_n) a

H1. A is closed	[Vuln.; Used with (H4,H5).]
H2. B is closed	[Vuln.; Used with (H4,H6).]
H5. (a_n) is a sequence in A	[Vuln.]
H6. (a_n) is a sequence in B	[Vuln.]
H4. $a_n \to a$	[Vuln.]
H7. $a \in A$	
H8. $a \in B$	

T3. $a \in A \cap B$

8. Quantifier-free expansion of target T3.

We would like to show that $a \in A \cap B$, i.e. that $a \in A$ and $a \in B$.

L1

(a_n) a

H1. A is closed	[Vuln.; Used with (H4,H5).]
H2. B is closed	[Vuln.; Used with (H4,H6).]
H5. (a_n) is a sequence in A	[Vuln.]
H6. (a_n) is a sequence in B	[Vuln.]
H4. $a_n \to a$	[Vuln.]
H7. $a \in A$	
H8. $a \in B$	

T4. $a \in A$
T5. $a \in B$

9. Hypothesis H7 matches target T4, so we can remove T4.

L1

(a_n) a

H1. A is closed	[Vuln.; Used with (H4,H5).]
H2. B is closed	[Vuln.; Used with (H4,H6).]
H5. (a_n) is a sequence in A	[Vuln.]
H6. (a_n) is a sequence in B	[Vuln.]
H4. $a_n \to a$	[Vuln.]
H7. $a \in A$	
H8. $\boldsymbol{a \in B}$	

T5. $\boldsymbol{a \in B}$

10. Hypothesis H8 matches target T5, so L1 is done.

But this is clearly the case, so we are done.

L1 Done

Problem solved.

The pre-image of a closed set U under a continuous function f is closed.

Let (a_n) and a be such that (a_n) is a sequence in $f^{-1}(U)$ and $a_n \to a$. Then $f(a_n)$ is a sequence in U. We would like to show that $a \in f^{-1}(U)$, i.e. that $f(a) \in U$ and since U is closed, $f(a) \in U$ if $f(a_n) \to f(a)$. Let $\epsilon > 0$. We would like to find N s.t. $d(f(a), f(a_n)) < \epsilon$ whenever $n \geqslant N$. Since f is continuous, there exists $\delta > 0$ such that $d(f(a), f(a_n)) < \epsilon$ whenever $d(a, a_n) < \delta$. Since $a_n \to a$, there exists N' such that $d(a, a_n) < \delta$ whenever $n \geqslant N'$. Therefore, setting $N = N'$, we are done.

L1

H1. f is continuous
H2. U is closed

T1. $f^{-1}(U)$ is closed

1. Expand pre-universal target T1.

L1

H1. f is continuous
H2. U is closed

T2. $\forall (a_n), a.((a_n)$ is a sequence in $f^{-1}(U) \wedge a_n \to a \Rightarrow a \in f^{-1}(U))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let (a_n) and a be such that (a_n) is a sequence in $f^{-1}(U)$ and $a_n \to a$.

L1

(a_n) a

H1. f is continuous
H2. U is closed
H3. (a_n) is a sequence in $f^{-1}(U)$
H4. $a_n \to a$

T3. $a \in f^{-1}(U)$

3. Quantifier-free expansion of hypothesis H3.

Since (a_n) is a sequence in $f^{-1}(U)$, we have that $f(a_n)$ is a sequence in U.

L1

(a_n) a

H1. f is continuous
H2. U is closed
H5. $f(a_n)$ is a sequence in U
H4. $a_n \to a$

T3. $a \in f^{-1}(U)$

4. Quantifier-free expansion of target T3.

We would like to show that $a \in f^{-1}(U)$, i.e. that $f(a) \in U$.

L1

(a_n) a

H1. f is continuous
H2. U is closed
H5. $f(a_n)$ is a sequence in U
H4. $a_n \to a$

T4. $f(a) \in U$

5. Backwards reasoning using H2 with (T4,H5).

Since U is closed and $f(a_n)$ is a sequence in U, $f(a) \in U$ if $f(a_n) \to f(a)$.

L1

(a_n) a

H1. f is continuous
H2. U is closed [Vuln.]
H5. $f(a_n)$ is a sequence in U [Vuln.]
H4. $a_n \to a$

T5. $f(a_n) \to f(a)$

6. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1

(a_n) a

H1. f is continuous
H2. U is closed [Vuln.]
H5. $f(a_n)$ is a sequence in U [Vuln.]
H4. $a_n \to a$

T5. $f(a_n) \to f(a)$

7. Delete H5 as no other statement mentions U.

L1

(a_n) a

H1. f is continuous
H2. U is closed [Vuln.]
H5. $f(a_n)$ is a sequence in U [Vuln.]
H4. $a_n \to a$

T5. $\boldsymbol{f(a_n) \to f(a)}$

8. Expand pre-universal target T5.

L1

(a_n) a

H1. f is continuous
H2. U is closed [Vuln.]
H5. $f(a_n)$ is a sequence in U [Vuln.]
H4. $a_n \to a$

T6. $\boldsymbol{\forall \epsilon.(\exists N.(\forall n.(n \geqslant N \Rightarrow d(f(a), f(a_n)) < \epsilon)))}$

9. Apply 'let' trick and move premise of universal target T6 above the line.

Let $\epsilon > 0$.

L1

(a_n) a ϵ

H1. f is continuous
H2. U is closed [Vuln.]
H5. $f(a_n)$ is a sequence in U [Vuln.]
H4. $a_n \to a$

T7. $\boldsymbol{\exists N.(\forall n.(n \geqslant N \Rightarrow d(f(a), f(a_n)) < \epsilon))}$

10. Unlock existential-universal-conditional target T7.

We would like to find N s.t. $d(f(a), f(a_n)) < \epsilon$ whenever $n \geqslant N$.

L1

(a_n) a ϵ

H1. $\boldsymbol{f}$ is continuous
H2. U is closed [Vuln.]
H5. $f(a_n)$ is a sequence in U [Vuln.]
H4. $a_n \to a$

L2$^{\blacklozenge}$

$N^{\blacklozenge}[\overline{n}]$ n

H6. $n \geqslant N^{\blacklozenge}[\overline{n}]$ [From L1.]

T8. $\boldsymbol{d(f(a), f(a_n)) < \epsilon}$

11. Backwards reasoning using H1 with T8.

Since f is continuous, there exists $\delta > 0$ such that $d(f(a), f(a_n)) < \epsilon$ whenever $d(a, a_n) < \delta$.

L1

$(a_n)\ a\ \epsilon$

H1. f is continuous [Vuln.]
H2. U is closed [Vuln.]
H5. $f(a_n)$ is a sequence in U [Vuln.]
H4. $a_n \to a$

L2♦

$N^\blacklozenge[\overline{n}]\ n\ \delta[x, \eta]$

H6. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T9. $d(a, a_n) < \delta[a, \epsilon]$

12. Delete H1 as no other statement mentions f.

L1

$(a_n)\ a\ \epsilon$

H1. f is continuous [Vuln.]
H2. U is closed [Vuln.]
H5. $f(a_n)$ is a sequence in U [Vuln.]
H4. $\boldsymbol{a_n \to a}$

L2♦

$N^\blacklozenge[\overline{n}]\ n\ \delta[x, \eta]$

H6. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T9. $\boldsymbol{d(a, a_n) < \delta[a, \epsilon]}$

13. Backwards reasoning using H4 with T9.

Since $a_n \to a$, there exists N' such that $d(a, a_n) < \delta$ whenever $n \geqslant N'$.

L1

$(a_n)\ a\ \epsilon$

H1. f is continuous [Vuln.]
H2. U is closed [Vuln.]
H5. $f(a_n)$ is a sequence in U [Vuln.]
H4. $a_n \to a$ [Vuln.]

L2♦

$N^\blacklozenge[\overline{n}]\ n\ \delta[x, \eta]\ N'[\eta]$

H6. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T10. $n \geqslant N'[\delta[a, \epsilon]]$

14. Delete H4 as no other statement mentions a.

L1

$(a_n)\ a\ \epsilon$

H1. f is continuous [Vuln.]
H2. U is closed [Vuln.]
H5. $f(a_n)$ is a sequence in U [Vuln.]
H4. $a_n \to a$ [Vuln.]

L2♦

$N^\blacklozenge[\overline{n}]\ n\ \delta[x, \eta]\ N'[\eta]$

H6. $\boldsymbol{n \geqslant N^\blacklozenge[\overline{n}]}$ **[From L1.]**

T10. $\boldsymbol{n \geqslant N'[\delta[a, \epsilon]]}$

15. Hypothesis H6 matches target T10 after choosing $N^\blacklozenge[\overline{n}] = N'[\delta[a, \epsilon]]$, so L2♦ is done.

Therefore, setting $N = N'$, we are done.

L1

(a_n) a ϵ

H1. f is continuous [Vuln.]

H2. U is closed [Vuln.]

H5. $f(a_n)$ is a sequence in U [Vuln.]

H4. $a_n \to a$ [Vuln.]

L2♦ Done

16. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.

The pre-image of an open set U under a continuous function f is open.

Let x be an element of $f^{-1}(U)$. Then $f(x) \in U$. Therefore, since U is open, there exists $\eta > 0$ such that $u \in U$ whenever $d(f(x), u) < \eta$. We would like to find $\delta > 0$ s.t. $y \in f^{-1}(U)$ whenever $d(x, y) < \delta$. But $y \in f^{-1}(U)$ if and only if $f(y) \in U$. We know that $f(y) \in U$ whenever $d(f(x), f(y)) < \eta$. Since f is continuous, there exists $\theta > 0$ such that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \theta$. Therefore, setting $\delta = \theta$, we are done.

L1

H1. f is continuous
H2. U is open

T1. $f^{-1}(U)$ is open

1. Expand pre-universal target T1.

L1

H1. f is continuous
H2. U is open

T2. $\forall x.(x \in f^{-1}(U) \Rightarrow \exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U))))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f^{-1}(U)$.

L1

x

H1. f is continuous
H2. U is open
H3. $x \in f^{-1}(U)$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$

3. Quantifier-free expansion of hypothesis H3.

Since $x \in f^{-1}(U)$, we have that $f(x) \in U$.

L1

x

H1. f is continuous
H2. U is open
H4. $f(x) \in U$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$

4. Forwards reasoning using H2 with H4.

Since U is open and $f(x) \in U$, there exists $\eta > 0$ such that $u \in U$ whenever $d(f(x), u) < \eta$.

L1

$x \; \eta[f(x)]$

H1. f is continuous
H2. U is open [Vuln.; Used with H4.]
H4. $f(x) \in U$ [Vuln.]
H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$

5. Deleted H4, as this unexpandable atomic statement is unmatchable.

L1

$x \; \eta[f(x)]$

H1. f is continuous
H2. U is open [Vuln.; Used with H4.]
H4. $f(x) \in U$ [Vuln.]
H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$

6. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1

$x\ \eta[f(x)]$

H1. f is continuous

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$

T3. $\boldsymbol{\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))}$

7. Unlock existential-universal-conditional target T3.

We would like to find $\delta > 0$ s.t. $y \in f^{-1}(U)$ whenever $d(x, y) < \delta$.

L1

$x\ \eta[f(x)]$

H1. f is continuous

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y$

H6. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T4. $\boldsymbol{y \in f^{-1}(U)}$

8. Quantifier-free expansion of target T4.

But $y \in f^{-1}(U)$ if and only if $f(y) \in U$.

L1

$x\ \eta[f(x)]$

H1. f is continuous

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\boldsymbol{\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)}$

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y$

H6. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T5. $\boldsymbol{f(y) \in U}$

9. Backwards reasoning using H5 with T5.

We know that $f(y) \in U$ whenever $d(f(x), f(y)) < \eta$.

L1

$x\ \eta[f(x)]$

H1. f is continuous

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$ [Vuln.]

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y$

H6. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T6. $d(f(x), f(y)) < \eta[f(x)]$

10. Delete H5 as no other statement mentions U.

L1

$x \ \eta[f(x)]$

H1. f is continuous

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$ [Vuln.]

L2♦

$\delta^{\blacklozenge}[\overline{y}] \ y$

H6. $d(x, y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T6. $d(f(x), f(y)) < \eta[f(x)]$

11. Backwards reasoning using H1 with T6.

Since f is continuous, there exists $\theta > 0$ such that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \theta$.

L1

$x \ \eta[f(x)]$

H1. f is continuous [Vuln.]

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$ [Vuln.]

L2♦

$\delta^{\blacklozenge}[\overline{y}] \ y \ \theta[z, \epsilon]$

H6. $d(x, y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T7. $d(x, y) < \theta[x, \eta[f(x)]]$

12. Delete H1 as no other statement mentions f.

L1

$x \ \eta[f(x)]$

H1. f is continuous [Vuln.]

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$ [Vuln.]

L2♦

$\delta^{\blacklozenge}[\overline{y}] \ y \ \theta[z, \epsilon]$

H6. $d(x, y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T7. $d(x, y) < \theta[x, \eta[f(x)]]$

13. Hypothesis H6 matches target T7 after choosing $\delta^{\blacklozenge}[\overline{y}] = \theta[x, \eta[f(x)]]$, so L2♦ is done.

Therefore, setting $\delta = \theta$, we are done.

L1

$x \ \eta[f(x)]$

H1. f is continuous [Vuln.]

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$ [Vuln.]

L2♦ Done

14. All targets of L1 are ‘Done’, so L1 is itself done.

L1 Done

Problem solved.

If f and g are continuous functions, then $g \circ f$ is continuous.

> Take x and $\epsilon > 0$. We would like to find $\delta > 0$ s.t. $d(g(f(x)), g(f(y))) < \epsilon$ whenever $d(x, y) < \delta$. Since g is continuous, there exists $\eta > 0$ such that $d(g(f(x)), g(f(y))) < \epsilon$ whenever $d(f(x), f(y)) < \eta$. Since f is continuous, there exists $\theta > 0$ such that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \theta$. Therefore, setting $\delta = \theta$, we are done.

L1

H1. f is continuous
H2. g is continuous

T1. $g \circ f$ is continuous

1. Expand pre-universal target T1.

L1

H1. f is continuous
H2. g is continuous

T2. $\forall x, \epsilon.(\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow d(g(f(x)), g(f(y))) < \epsilon)))$

2. Apply ‘let’ trick and move premise of universal target T2 above the line.

Take x and $\epsilon > 0$.

L1

$x \ \epsilon$

H1. f is continuous
H2. g is continuous

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow d(g(f(x)), g(f(y))) < \epsilon))$

3. Unlock existential-universal-conditional target T3.

We would like to find $\delta > 0$ s.t. $d(g(f(x)), g(f(y))) < \epsilon$ whenever $d(x, y) < \delta$.

L1

$x \ \epsilon$

H1. f is continuous
H2. g is continuous

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}] \ y$

H3. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T4. $d(g(f(x)), g(f(y))) < \epsilon$

4. Backwards reasoning using H2 with T4.

Since g is continuous, there exists $\eta > 0$ such that $d(g(f(x)), g(f(y))) < \epsilon$ whenever $d(f(x), f(y)) < \eta$.

L1

$x \ \epsilon$

H1. f is continuous
H2. g is continuous [Vuln.]

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}] \ y \ \eta[z, \theta]$

H3. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T5. $d(f(x), f(y)) < \eta[f(x), \epsilon]$

5. Delete H2 as no other statement mentions g.

L1

$x \ \epsilon$

H1. f is continuous
H2. g is continuous [Vuln.]

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}] \ y \ \eta[z, \theta]$

H3. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T5. $d(f(x), f(y)) < \eta[f(x), \epsilon]$

6. Backwards reasoning using H1 with T5.

Since f is continuous, there exists $\theta > 0$ such that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \theta$.

L1

$x \ \epsilon$

H1. f is continuous [Vuln.]

H2. g is continuous [Vuln.]

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}] \ y \ \eta[z, \theta] \ \theta[z, \alpha]$

H3. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T6. $d(x, y) < \theta[x, \eta[f(x), \epsilon]]$

7. Delete H1 as no other statement mentions f.

L1

$x \ \epsilon$

H1. f is continuous [Vuln.]

H2. g is continuous [Vuln.]

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}] \ y \ \eta[z, \theta] \ \theta[z, \alpha]$

H3. $\boldsymbol{d(x, y) < \delta^\blacklozenge[\overline{y}]}$ **[From L1.]**

T6. $\boldsymbol{d(x, y) < \theta[x, \eta[f(x), \epsilon]]}$

8. Hypothesis H3 matches target T6 after choosing $\delta^\blacklozenge[\overline{y}] = \theta[x, \eta[f(x), \epsilon]]$, so L2$^\blacklozenge$ is done.

Therefore, setting $\delta = \theta$, we are done.

L1

$x \ \epsilon$

H1. f is continuous [Vuln.]

H2. g is continuous [Vuln.]

L2$^\blacklozenge$ Done

9. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.

If f is a continuous function and $(a_n) \to a$, then $(f(a_n)) \to f(a)$

Let $\epsilon > 0$. We would like to find N s.t. $d(f(a), f(a_n)) < \epsilon$ whenever $n \geqslant N$. Since f is continuous, there exists $\delta > 0$ such that $d(f(a), f(a_n)) < \epsilon$ whenever $d(a, a_n) < \delta$. Since $a_n \to a$, there exists N' such that $d(a, a_n) < \delta$ whenever $n \geqslant N'$. Therefore, setting $N = N'$, we are done.

L1
H1. f is continuous
H2. $a_n \to a$

T1. $\boldsymbol{f(a_n) \to f(a)}$

1. Expand pre-universal target T1.

L1
H1. f is continuous
H2. $a_n \to a$

T2. $\boldsymbol{\forall \epsilon.(\exists N.(\forall n.(n \geqslant N \Rightarrow d(f(a), f(a_n)) < \epsilon)))}$

2. Apply 'let' trick and move premise of universal target T2 above the line.

Let $\epsilon > 0$.

L1
ϵ
H1. f is continuous
H2. $a_n \to a$

T3. $\boldsymbol{\exists N.(\forall n.(n \geqslant N \Rightarrow d(f(a), f(a_n)) < \epsilon))}$

3. Unlock existential-universal-conditional target T3.

We would like to find N s.t. $d(f(a), f(a_n)) < \epsilon$ whenever $n \geqslant N$.

L1
ϵ
H1. f is continuous
H2. $a_n \to a$

L2$^\blacklozenge$
$N^\blacklozenge[\overline{n}]$ n
H3. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T4. $\boldsymbol{d(f(a), f(a_n)) < \epsilon}$

4. Backwards reasoning using H1 with T4.

Since f is continuous, there exists $\delta > 0$ such that $d(f(a), f(a_n)) < \epsilon$ whenever $d(a, a_n) < \delta$.

L1
ϵ
H1. f is continuous [Vuln.]
H2. $a_n \to a$

L2$^\blacklozenge$
$N^\blacklozenge[\overline{n}]$ n $\delta[x, \eta]$
H3. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T5. $d(a, a_n) < \delta[a, \epsilon]$

5. Delete H1 as no other statement mentions f.

L1
ϵ
~~H1. f is continuous [Vuln.]~~
H2. $\boldsymbol{a_n \to a}$

L2$^\blacklozenge$
$N^\blacklozenge[\overline{n}]$ n $\delta[x, \eta]$
H3. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T5. $\boldsymbol{d(a, a_n) < \delta[a, \epsilon]}$

6. Backwards reasoning using H2 with T5.

L1

ϵ

H1. f is continuous [Vuln.]

H2. $a_n \to a$ [Vuln.]

L2$^\blacklozenge$

$N^\blacklozenge[\overline{n}]$ n $\delta[x,\eta]$ $N'[\eta]$

H3. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T6. $n \geqslant N'[\delta[a,\epsilon]]$

Since $a_n \to a$, there exists N' such that $d(a, a_n) < \delta$ whenever $n \geqslant N'$.

7. Delete H2 as no other statement mentions a.

L1

ϵ

H1. f is continuous [Vuln.]

H2. $a_n \to a$ [Vuln.]

L2$^\blacklozenge$

$N^\blacklozenge[\overline{n}]$ n $\delta[x,\eta]$ $N'[\eta]$

H3. $\boldsymbol{n \geqslant N^\blacklozenge[\overline{n}]}$ **[From L1.]**

T6. $\boldsymbol{n \geqslant N'[\delta[a,\epsilon]]}$

8. Hypothesis H3 matches target T6 after choosing $N^\blacklozenge[\overline{n}] = N'[\delta[a,\epsilon]]$, so L2$^\blacklozenge$ is done.

Therefore, setting $N = N'$, we are done.

L1

ϵ

H1. f is continuous [Vuln.]

H2. $a_n \to a$ [Vuln.]

L2$^\blacklozenge$ Done

9. All targets of L1 are ‘Done’, so L1 is itself done.

L1 Done

Problem solved.