# If f is a surjection then $f(A)\cap f(B)\subset f(A\cap B)$

Let x be an element of  $f(A) \cap f(B)$ . Then  $x \in f(A)$  and  $x \in f(B)$ . That is, there exists  $y \in A$  such that f(y) = x and there exists  $z \in B$  such that f(z) = x. We would like to find  $u \in A \cap B$  s.t. f(u) = x. But  $u \in A \cap B$  if and only if  $u \in A$  and  $u \in B$ .

```
11 H1. surjection(f) 11. f(A) \cap f(B) \subset f(A \cap B)
```

1. Expand pre-universal target T1.

$$\begin{array}{c} \text{L1} \\ \hline \text{H1. } \textit{surjection}(f) \\ \hline \text{T2. } \forall x. (x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B)) \end{array}$$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

L1) xH1. surjection(f)H2.  $x \in f(A) \cap f(B)$ T3.  $x \in f(A \cap B)$ 

Let x be an element of  $f(A) \cap f(B)$ .

3. Quantifier-free expansion of hypothesis H2.

L1) xH1. surjection(f)H3.  $x \in f(A)$ H4.  $x \in f(B)$   $T3. <math>x \in f(A \cap B)$ 

Since  $x \in f(A) \cap f(B)$ ,  $x \in f(A)$  and  $x \in f(B)$ .

4. Expand pre-existential hypothesis H3.

L1)  $x \ y$ H1. surjection(f)H5.  $y \in A$ H6. f(y) = xH4.  $x \in f(B)$ T3.  $x \in f(A \cap B)$ 

By definition, since  $x \in f(A)$ , there exists  $y \in A$  such that f(y) = x.

5. Expand pre-existential hypothesis H4.

L1  $x \ y \ z$ H1. surjection(f)H5.  $y \in A$ H6. f(y) = xH7.  $z \in B$ H8. f(z) = xT3.  $x \in f(A \cap B)$ 

By definition, since  $x \in f(B)$ , there exists  $z \in B$  such that f(z) = x.

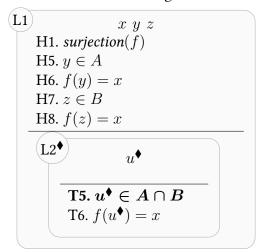
6. Expand pre-existential target T3.

We would like to find  $u \in A \cap B$  s.t. f(u) = x.

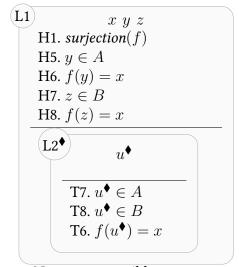
L1) 
$$x y z$$
H1.  $surjection(f)$ 
H5.  $y \in A$ 
H6.  $f(y) = x$ 
H7.  $z \in B$ 
H8.  $f(z) = x$ 

T4.  $\exists u.(u \in A \cap B \land f(u) = x)$ 

7. Unlock existential target T4.



8. Quantifier-free expansion of target T5.



No moves possible.

We would like to find  $u \in A \cap B$  s.t. f(u) = x.

But  $u \in A \cap B$  if and only if  $u \in A$  and  $u \in B$ .

# If f is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of  $f(A)\cap f(B)$ . Then  $x\in f(A)$  and  $x\in f(B)$ . That is, there exists  $y\in A$  such that f(y)=x and there exists  $z\in B$  such that f(z)=x. Since f is an injection, f(y)=x and f(z)=x, we have that y=z. We would like to find  $u\in A\cap B$  s.t. f(u)=x. But  $u\in A\cap B$  if and only if  $u\in A$  and  $u\in B$ . Since y=z, we have that  $y\in B$ . Therefore, setting u=y, we are done.

$$H1. \ f$$
 is an injection  $T1. \ f(A) \cap f(B) \subset f(A \cap B)$ 

1. Expand pre-universal target T1.

1. H1. 
$$f$$
 is an injection T2.  $\forall x. (x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))$ 

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

L1 xH1. f is an injection
H2.  $x \in f(A) \cap f(B)$ T3.  $x \in f(A \cap B)$ 

Let x be an element of  $f(A) \cap f(B)$ .

3. Quantifier-free expansion of hypothesis H2.

L1 xH1. f is an injection
H3.  $x \in f(A)$ H4.  $x \in f(B)$   $T3. <math>x \in f(A \cap B)$ 

Since  $x \in f(A) \cap f(B)$ ,  $x \in f(A)$  and  $x \in f(B)$ .

4. Expand pre-existential hypothesis H3.

L1) x yH1. f is an injection
H5.  $y \in A$ H6. f(y) = xH4.  $x \in f(B)$ T3.  $x \in f(A \cap B)$ 

By definition, since  $x \in f(A)$ , there exists  $y \in A$  such that f(y) = x.

5. Expand pre-existential hypothesis H4.

L1)  $x \ y \ z$ H1. f is an injection

H5.  $y \in A$ H6. f(y) = xH7.  $z \in B$ H8. f(z) = xT3.  $x \in f(A \cap B)$ 

By definition, since  $x \in f(B)$ , there exists  $z \in B$  such that f(z) = x.

6. Forwards reasoning using H1 with (H6,H8).

Since f is an injection, f(y) = x and f(z) = x, we have that y = z.

$$\begin{array}{c|cccc} \textbf{L1} & x & y & z \\ & \textbf{H1.} & f \text{ is an injection} & [\text{Vuln.; Used with (H6,H8).}] \\ & \textbf{H5.} & y \in A \\ & \textbf{H6.} & f(y) = x & [\text{Vuln.}] \\ & \textbf{H7.} & z \in B \\ & \textbf{H8.} & f(z) = x & [\text{Vuln.}] \\ & \textbf{H9.} & y = z \\ \hline & \textbf{T3.} & x \in f(A \cap B) \end{array}$$

7. Expand pre-existential target T3.

L1  $x \ y \ z$ H1. f is an injection [Vuln.; Used with (H6,H8).]
H5.  $y \in A$ H6. f(y) = x [Vuln.]
H7.  $z \in B$ H8. f(z) = x [Vuln.]
H9. y = z  $T4. \ \exists u. (u \in A \cap B \land f(u) = x)$ 

We would like to find  $u \in A \cap B$  s.t. f(u) = x.

8. Unlock existential target T4.

L1 x y zH1. f is an injection [Vuln.; Used with (H6,H8).]

H5.  $y \in A$ H6. f(y) = x [Vuln.]

H7.  $z \in B$ H8. f(z) = x [Vuln.]

H9. y = zL2• u• T5. u•  $\in A \cap B$ T6. f(u•) = x

We would like to find  $u \in A \cap B$  s.t. f(u) = x.

9. Quantifier-free expansion of target T5.

L1 x y zH1. f is an injection [Vuln.; Used with (H6,H8).]

H5.  $y \in A$ H6. f(y) = x [Vuln.]

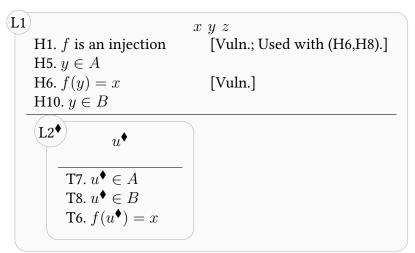
H7.  $z \in B$ H8. f(z) = x [Vuln.]

H9. y = zL2 $\bullet$   $u \bullet$   $T7. u \bullet \in A$   $T8. u \bullet \in B$   $T6. f(u \bullet) = x$ 

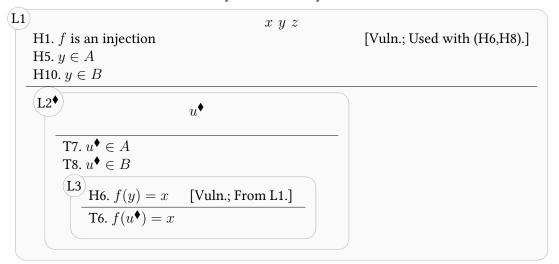
But  $u \in A \cap B$  if and only if  $u \in A$  and  $u \in B$ .

10. Rewrite z as y throughout the tableau using hypothesis H9.

Since y = z, we have that  $y \in B$ .



11. Moved H6 down, as x can only be utilised by T6.



12. Choosing  $u^{\blacklozenge}=y$  matches all targets inside  $\mathrm{L2}^{\spadesuit}$  against hypotheses, so  $\mathrm{L2}^{\spadesuit}$  is done.

Therefore, setting u = y, we are done.

- L1 x y zH1. f is an injection [Vuln.; Used with (H6,H8).]
  H5.  $y \in A$ H10.  $y \in B$ L2 Done
- 13. All targets of L1 are 'Done', so L1 is itself done.
- L1 Done

Problem solved.

## If A and B are open sets, then $A \cap B$ is also open.

Let x be an element of  $A\cap B$ . Then  $x\in A$  and  $x\in B$ . Therefore, since A is open, there exists  $\eta>0$  such that  $u\in A$  whenever  $d(x,u)<\eta$  and since B is open, there exists  $\theta>0$  such that  $v\in B$  whenever  $d(x,v)<\theta$ . We would like to find  $\delta>0$  s.t.  $y\in A\cap B$  whenever  $d(x,y)<\delta$ . But  $y\in A\cap B$  if and only if  $y\in A$  and  $y\in B$ . We know that  $y\in A$  whenever  $d(x,y)<\eta$  and that  $y\in B$  whenever  $d(x,y)<\theta$ . Assume now that  $d(x,y)<\delta$ . Then  $d(x,y)<\eta$  if  $\delta\leqslant\eta$  and  $d(x,y)<\theta$  if  $\delta\leqslant\theta$ . We may therefore take  $\delta=\min(\eta,\theta)$  and we are done.

```
A = \begin{bmatrix} L1 \\ H1. \ A \text{ is open} \\ H2. \ B \text{ is open} \\ \hline \textbf{T1. } A \cap B \text{ is open} \end{bmatrix}
```

1. Expand pre-universal target T1.

```
L1 H1. A is open H2. B is open T2. \forall x. (x \in A \cap B \Rightarrow \exists \delta. (\forall y. (d(x,y) < \delta \Rightarrow y \in A \cap B)))
```

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of  $A \cap B$ .

```
L1) x
H1. A is open
H2. B is open
H3. x \in A \cap B
T3. \exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cap B))
```

3. Quantifier-free expansion of hypothesis H3.

```
L1) x
H1. A is open
H2. B is open
H4. x \in A
H5. x \in B

T3. \exists \delta. (\forall y. (d(x,y) < \delta \Rightarrow y \in A \cap B))
```

4. Forwards reasoning using H1 with H4.

```
\begin{array}{c|c} \textbf{L1} & x \ \eta[x] \\ \textbf{H1.} \ A \ \text{is open} & [\text{Vuln.; Used with H4.}] \\ \textbf{H2.} \ B \ \text{is open} & \\ \textbf{H4.} \ x \in A & [\text{Vuln.}] \\ \textbf{H5.} \ x \in B & \\ \textbf{H6.} \ \forall u.(d(x,u) < \eta[x] \Rightarrow u \in A) \\ \hline \textbf{T3.} \ \exists \delta. (\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B)) \end{array}
```

5. Deleted H4, as this unexpandable atomic statement is unmatchable.

Since  $x \in A \cap B$ ,  $x \in A$  and  $x \in B$ .

Since A is open and  $x \in A$ , there exists  $\eta > 0$  such that  $u \in A$  whenever  $d(x, u) < \eta$ .

```
\begin{array}{c} \text{L1} & x \ \eta[x] \\ \text{H1. } A \text{ is open} & [\text{Vuln.; Used with H4.}] \\ \text{H2. } B \text{ is open} & \\ \text{H4. } x \in A & [\text{Vuln.}] \\ \text{H5. } x \in B & \\ \text{H6. } \forall u.(d(x,u) < \eta[x] \Rightarrow u \in A) \\ \hline \text{T3. } \exists \delta. (\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B)) \end{array}
```

6. Deleted H1, as the conclusion of this implicative statement is unmatchable.

```
L1 x \eta[x]
H1. A is open [Vuln.; Used with H4.]

H2. B is open

H4. x \in A [Vuln.]

H5. x \in B

H6. \forall u.(d(x, u) < \eta[x] \Rightarrow u \in A)

T3. \exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cap B))
```

7. Forwards reasoning using H2 with H5.

```
\begin{array}{c} \text{L1} & x \ \eta[x] \ \theta[x] \\ \text{H1. } A \text{ is open} & [\text{Vuln.; Used with H4.}] \\ \text{H2. } B \text{ is open} & [\text{Vuln.; Used with H5.}] \\ \text{H4. } x \in A & [\text{Vuln.}] \\ \text{H5. } x \in B & [\text{Vuln.}] \\ \text{H6. } \forall u.(d(x,u) < \eta[x] \Rightarrow u \in A) \\ \text{H7. } \forall v.(d(x,v) < \theta[x] \Rightarrow v \in B) \\ \hline \\ \text{T3. } \exists \delta. (\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B)) \end{array}
```

8. Deleted H5, as this unexpandable atomic statement is unmatchable.

9. Deleted H2, as the conclusion of this implicative statement is unmatchable.

```
L1) x \ \eta[x] \ \theta[x]
H1. \ A \text{ is open}
H2. \ B \text{ is open}
H4. \ x \in A
H5. \ x \in B
H6. \ \forall u. (d(x, u) < \eta[x] \Rightarrow u \in A)
H7. \ \forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)
T3. \ \exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in A \cap B))
[Vuln.; Used with H4.]
[Vuln.; Used with H5.]
[Vuln.]
[Vuln.]
[Vuln.]
```

10. Unlock existential-universal-conditional target T3.

Since B is open and  $x \in B$ , there exists  $\theta > 0$  such that  $v \in B$  whenever  $d(x, v) < \theta$ .

We would like to find  $\delta > 0$  s.t.  $y \in A \cap B$  whenever  $d(x, y) < \delta$ .

11. Quantifier-free expansion of target T4.

L1) 
$$x \eta[x] \theta[x]$$
H1.  $A$  is open 
$$[Vuln.; Used with H4.]$$
H2.  $B$  is open 
$$[Vuln.; Used with H5.]$$
H4.  $x \in A$  
$$[Vuln.]$$
H5.  $x \in B$  
$$[Vuln.]$$
H6.  $\forall u.(d(x, u) < \eta[x] \Rightarrow u \in A)$ 
H7.  $\forall v.(d(x, v) < \theta[x] \Rightarrow v \in B)$ 

$$\boxed{L2^{\bullet}} \qquad \delta^{\bullet}[\overline{y}] y$$

$$\boxed{H8. \ d(x, y) < \delta^{\bullet}[\overline{y}] \ [From L1.]}$$

$$\boxed{T5. \ y \in A}$$

$$\boxed{T6. \ y \in B}$$

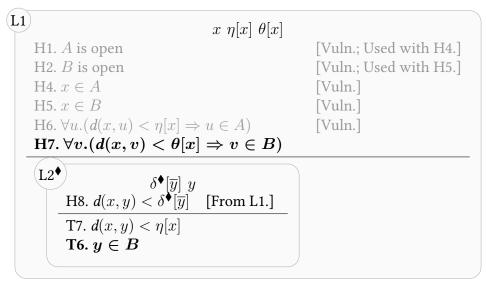
But  $y \in A \cap B$  if and only if  $y \in A$  and  $y \in B$ .

12. Backwards reasoning using H6 with T5.

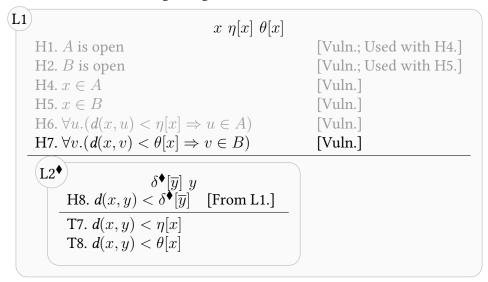
```
(L1)
                                            x \eta[x] \theta[x]
      H1. A is open
                                                                  [Vuln.; Used with H4.]
     H2. B is open
                                                                  [Vuln.; Used with H5.]
     H4. x \in A
                                                                  [Vuln.]
     H5. x \in B
                                                                  [Vuln.]
     H6. \forall u.(d(x, u) < \eta[x] \Rightarrow u \in A)
                                                                  [Vuln.]
     H7. \forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)
      (L2♦)
                              \delta^{lack}[\overline{y}] y
            H8. d(x,y) < \delta^{\bullet}[\overline{y}]
                                          [From L1.]
            T7. d(x, y) < \eta[x]
            T6. y \in B
```

We know that  $y \in A$  whenever  $d(x, y) < \eta$ .

13. Delete H6 as no other statement mentions A.



14. Backwards reasoning using H7 with T6.



15. Delete H7 as no other statement mentions B.

```
(L1)
                                          x \eta[x] \theta[x]
                                                               [Vuln.; Used with H4.]
     H1. A is open
                                                               [Vuln.; Used with H5.]
     H2. B is open
     H4. x \in A
                                                               [Vuln.]
     H5. x \in B
                                                               [Vuln.]
     H6. \forall u.(d(x, u) < \eta[x] \Rightarrow u \in A)
                                                               [Vuln.]
     H7. \forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)
                                                               [Vuln.]
     (L2*)
            H8. d(x,y) < \delta^{\bullet}[\overline{y}]
                                        [From L1.]
            T7. d(x, y) < \eta[x]
            T8. d(x, y) < \theta[x]
```

16. Replacing diamonds with bullets in  $L2^{\blacklozenge}$ .

We know that  $y \in B$  whenever  $d(x, y) < \theta$ .

Assume now that  $d(x, y) < \delta$ .

```
(L1)
                                             x \eta[x] \theta[x]
     H1. A is open
                                                                      [Vuln.; Used with H4.]
                                                                      [Vuln.; Used with H5.]
     H2. B is open
     H4. x \in A
                                                                      [Vuln.]
     H5. x \in B
                                                                      [Vuln.]
     H6. \forall u.(d(x, u) < \eta[x] \Rightarrow u \in A)
                                                                      [Vuln.]
     H7. \forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)
                                                                      [Vuln.]
     (L2)
                                \delta^{\bullet}[\overline{y}] y
           H8. d(x,y) < \delta^{\bullet}[\overline{y}]
                                            [From L1.]
           T7. d(x, y) < \eta[x]
           T8. d(x,y) < \theta[x]
```

17. Backwards reasoning using library result "transitivity" with (T7,H8).

(L1) $x \eta[x] \theta[x]$ H1. A is open [Vuln.; Used with H4.] H2. B is open [Vuln.; Used with H5.] H4.  $x \in A$ [Vuln.] H5.  $x \in B$ [Vuln.] H6.  $\forall u.(d(x, u) < \eta[x] \Rightarrow u \in A)$ [Vuln.] H7.  $\forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)$ [Vuln.] (L2) [From L1.; Vuln.] H8.  $d(x,y) < \delta^{\bullet}[\overline{y}]$ T9.  $\delta^{\bullet}[\overline{y}] \leqslant \eta[x]$ T8.  $d(x,y) < \theta[x]$ 

Since  $d(x, y) < \delta$ ,  $d(x, y) < \eta$  if  $\delta \leqslant \eta$ .

18. Moved H8 down, as x can only be utilised by T8.

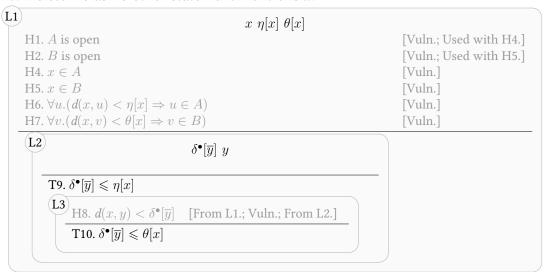
```
(L1)
                                                                      x \eta[x] \theta[x]
      H1. A is open
                                                                                                                     [Vuln.; Used with H4.]
                                                                                                                     [Vuln.; Used with H5.]
      H2. B is open
      H4. x \in A
                                                                                                                     [Vuln.]
      H5. x \in B
                                                                                                                     [Vuln.]
      H6. \forall u.(d(x, u) < \eta[x] \Rightarrow u \in A)
                                                                                                                     [Vuln.]
      H7. \forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)
                                                                                                                     [Vuln.]
      (L2)
                                                       \delta^{\bullet}[\overline{y}] y
            T9. \delta^{\bullet}[\overline{y}] \leqslant \eta[x]
            \overset{	ext{(L3)}}{	o} H8. d(x,y)<\delta^ullet[\overline{y}]
                                                      [From L1.; Vuln.; From L2.]
                   T8. d(x,y) < \theta[x]
```

19. Backwards reasoning using library result "transitivity" with (T8,H8).

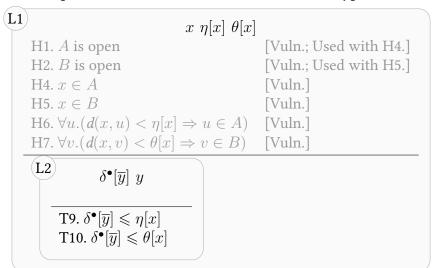
Since  $d(x,y) < \delta$ ,  $d(x,y) < \theta$  if  $\delta \leqslant \theta$ .

```
(L1)
                                                                       x \eta[x] \theta[x]
      H1. A is open
                                                                                                                      [Vuln.; Used with H4.]
      H2. B is open
                                                                                                                      [Vuln.; Used with H5.]
      H4. x \in A
                                                                                                                      [Vuln.]
      H5. x \in B
                                                                                                                      [Vuln.]
      H6. \forall u.(d(x, u) < \eta[x] \Rightarrow u \in A)
                                                                                                                      [Vuln.]
      H7. \forall v. (d(x, v) < \theta[x] \Rightarrow v \in B)
                                                                                                                      [Vuln.]
       (L2)
                                                        \delta^{\bullet}[\overline{y}] y
            T9. \delta^{\bullet}[\overline{y}] \leqslant \eta[x]
              \text{H8. } d(x,y) < \delta^{\bullet}[\overline{y}] 
                                                      [From L1.; Vuln.; From L2.]
                    T10. \delta^{\bullet}[\overline{y}] \leqslant \theta[x]
```

#### 20. Delete H8 as no other statement mentions x.

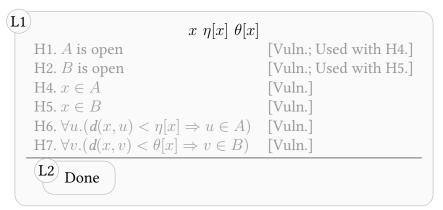


### 21. Collapsed subtableau L3 as it has no undeleted hypotheses.



22. Taking  $\delta^{\bullet}[\overline{y}] = \min(\eta[x], \theta[x])$  matches all targets against a library solution, so L2 is done.

We may therefore take  $\delta = \min(\eta, \theta)$ . We are done.



23. All targets of L1 are 'Done', so L1 is itself done.



Problem solved.

# The pre-image of an open set U under a continuous function f is open.

Let x be an element of  $f^{-1}(U)$ . Then  $f(x) \in U$ . Therefore, since U is open, there exists  $\eta > 0$  such that  $u \in U$  whenever  $d(f(x), u) < \eta$ . We would like to find  $\delta > 0$  s.t.  $y \in f^{-1}(U)$  whenever  $d(x,y) < \delta$ . But  $y \in f^{-1}(U)$  if and only if  $f(y) \in U$ . We know that  $f(y) \in U$  whenever  $d(f(x), f(y)) < \eta$ . Since f is continuous, there exists  $\theta > 0$  such that  $d(f(x), f(y)) < \eta$  whenever  $d(x, y) < \theta$ . Therefore, setting  $\delta = \theta$ , we are done.

- H1. f is continuous

  H2. U is open

  T1.  $f^{-1}(U)$  is open
- 1. Expand pre-universal target T1.

H1. 
$$f$$
 is continuous

H2.  $U$  is open

T2.  $\forall x. (x \in f^{-1}(U) \Rightarrow \exists \delta. (\forall y. (d(x,y) < \delta \Rightarrow y \in f^{-1}(U))))$ 

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of  $f^{-1}(U)$ .

L1 
$$\begin{array}{c} x \\ \text{H1. } f \text{ is continuous} \\ \text{H2. } U \text{ is open} \\ \hline \textbf{H3. } x \in f^{-1}(U) \\ \hline \hline \text{T3. } \exists \delta. (\forall y. (d(x,y) < \delta \Rightarrow y \in f^{-1}(U))) \end{array}$$

- 3. Quantifier-free expansion of hypothesis H3.
- L1 xH1. f is continuous
  H2. U is open
  H4.  $f(x) \in U$ T3.  $\exists \delta. (\forall y. (d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$
- 4. Forwards reasoning using H2 with H4.
- $\begin{array}{c} \text{L1} & x \ \eta[f(x)] \\ \text{H1. } f \text{ is continuous} \\ \text{H2. } U \text{ is open} & [\text{Vuln.; Used with H4.}] \\ \text{H4. } f(x) \in U & [\text{Vuln.}] \\ \hline \text{H5. } \forall u.(d(f(x),u) < \eta[f(x)] \Rightarrow u \in U) \\ \hline \text{T3. } \exists \delta. (\forall y.(d(x,y) < \delta \Rightarrow y \in f^{-1}(U))) \end{array}$
- 5. Deleted H4, as this unexpandable atomic statement is unmatchable.
- $\begin{array}{c} \text{L1} & x \ \eta[f(x)] \\ \text{H1. } f \text{ is continuous} \\ \text{H2. } U \text{ is open} & \text{[Vuln.; Used with H4.]} \\ \text{H4. } f(x) \in U & \text{[Vuln.]} \\ \hline \text{H5. } \forall u.(d(f(x),u) < \eta[f(x)] \Rightarrow u \in U) \\ \hline \hline \text{T3. } \exists \delta. (\forall y.(d(x,y) < \delta \Rightarrow y \in f^{-1}(U))) \end{array}$

15

Since  $x \in f^{-1}(U)$ , we have that  $f(x) \in U$ .

Since U is open and  $f(x) \in U$ , there exists  $\eta > 0$  such that  $u \in U$  whenever  $d(f(x), u) < \eta$ .

6. Deleted H2, as the conclusion of this implicative statement is unmatchable.

```
L1 x \eta[f(x)]
H1. f is continuous
H2. U is open
H4. f(x) \in U
[Vuln.; Used with H4.]
H5. \forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)

T3. \exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))
```

7. Unlock existential-universal-conditional target T3.

```
L1 x \eta[f(x)]
H1. f is continuous
H2. U is open
H4. f(x) \in U
[Vuln.; Used with H4.]
H5. \forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)

L2^{\blacklozenge}
\delta^{\blacklozenge}[\overline{y}] y
H6. <math>d(x, y) < \delta^{\blacklozenge}[\overline{y}] [From L1.]
T4. y \in f^{-1}(U)
```

We would like to find  $\delta > 0$  s.t.  $y \in f^{-1}(U)$  whenever  $d(x,y) < \delta$ .

8. Quantifier-free expansion of target T4.

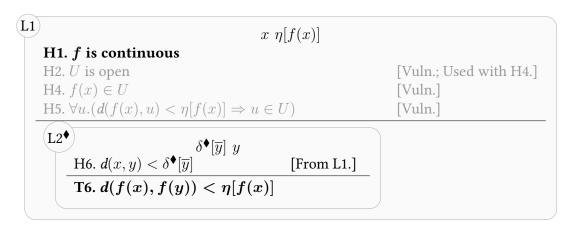
But  $y \in f^{-1}(U)$  if and only if  $f(y) \in U$ .

9. Backwards reasoning using H5 with T5.

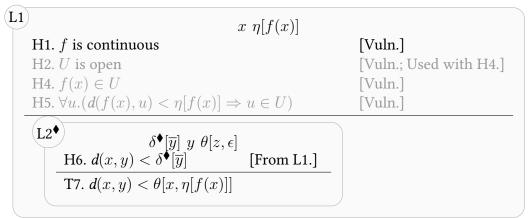
```
\begin{array}{c} \text{L1} & x \ \eta[f(x)] \\ \text{H1. } f \text{ is continuous} \\ \text{H2. } U \text{ is open} \\ \text{H4. } f(x) \in U \\ \text{H5. } \forall u.(d(f(x),u) < \eta[f(x)] \Rightarrow u \in U) \\ \hline \\ \text{L2}^{\blacklozenge} & \delta^{\blacklozenge}[\overline{y}] \ y \\ \hline \\ \text{L2}^{\blacklozenge} & \delta^{\blacklozenge}[\overline{y}] \ \text{[From L1.]} \\ \hline \\ \text{T6. } d(f(x),f(y)) < \eta[f(x)] \\ \hline \end{array}
```

We know that  $f(y) \in U$  whenever  $d(f(x), f(y)) < \eta$ .

10. Delete H5 as no other statement mentions U.



11. Backwards reasoning using H1 with T6.



Since f is continuous, there exists  $\theta > 0$  such that  $d(f(x), f(y)) < \eta$  whenever  $d(x, y) < \theta$ .

12. Delete H1 as no other statement mentions f.

13. Hypothesis H6 matches target T7 after choosing  $\delta^{\blacklozenge}[\overline{y}] = \theta[x, \eta[f(x)]]$ , so L2 $^{\blacklozenge}$  is done.

 $\begin{array}{c|c} \textbf{L1} & x \ \eta[f(x)] \\ & \text{H1. } f \text{ is continuous} & [\text{Vuln.}] \\ & \text{H2. } U \text{ is open} & [\text{Vuln.; Used with H4.}] \\ & \text{H4. } f(x) \in U & [\text{Vuln.}] \\ & \text{H5. } \forall u.(d(f(x),u) < \eta[f(x)] \Rightarrow u \in U) & [\text{Vuln.}] \\ \hline \hline & \textbf{L2} \bullet \\ \hline \textbf{Done} & \\ \end{array}$ 

Therefore, setting  $\delta = \theta$ , we are done.

14. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.

## If f and g are continuous functions, then $g \circ f$ is continuous.

Take x and  $\epsilon > 0$ . We would like to find  $\delta > 0$  s.t.  $d(g(f(x)),g(f(y))) < \epsilon$  whenever  $d(x,y) < \delta$ . Since g is continuous, there exists  $\eta > 0$  such that  $d(g(f(x)),g(f(y))) < \epsilon$  whenever  $d(f(x),f(y)) < \eta$ . Since f is continuous, there exists  $\theta > 0$  such that  $d(f(x),f(y)) < \eta$  whenever  $d(x,y) < \theta$ . Therefore, setting  $\delta = \theta$ , we are done.

H1. f is continuous

H2. g is continuous

T1.  $g \circ f$  is continuous

1. Expand pre-universal target T1.

H1. 
$$f$$
 is continuous

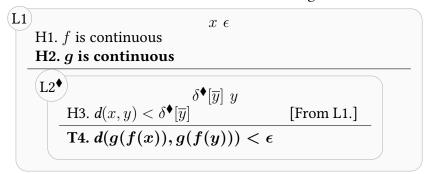
H2.  $g$  is continuous

T2.  $\forall x, \epsilon. (\exists \delta. (\forall y. (d(x,y) < \delta \Rightarrow d(g(f(x)), g(f(y))) < \epsilon)))$ 

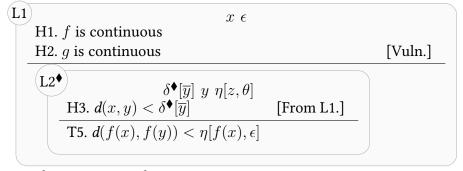
2. pply 'let' trick and move premise of universal target T2 above the line.

L1  $x \in H1. \ f$  is continuous  $H2. \ g$  is continuous  $T3. \ \exists \delta. (\forall y. (d(x,y) < \delta \Rightarrow d(g(f(x)),g(f(y))) < \epsilon))$ 

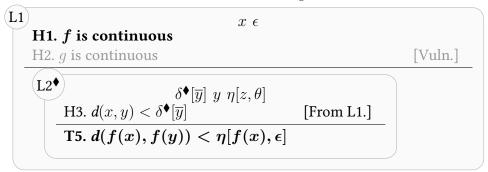
3. Unlock existential-universal-conditional target T3.



4. Backwards reasoning using H2 with T4.



5. Delete H2 as no other statement mentions g.

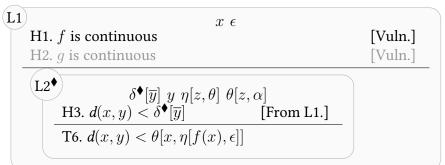


Take x and  $\epsilon > 0$ .

We would like to find  $\delta > 0$  s.t.  $d(g(f(x)), g(f(y))) < \epsilon$  whenever  $d(x, y) < \delta$ .

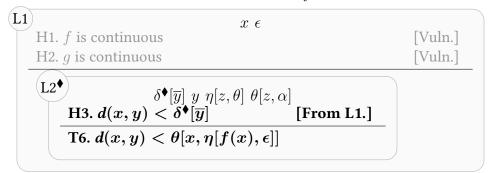
Since g is continuous, there exists  $\eta > 0$  such that  $d(g(f(x)), g(f(y))) < \epsilon$  whenever  $d(f(x), f(y)) < \eta$ .

6. Backwards reasoning using H1 with T5.



Since f is continuous, there exists  $\theta > 0$  such that  $d(f(x), f(y)) < \eta$  whenever  $d(x, y) < \theta$ .

7. Delete H1 as no other statement mentions f.



8. Hypothesis H3 matches target T6 after choosing  $\delta^{\blacklozenge}[\overline{y}] = \theta[x, \eta[f(x), \epsilon]]$ , so L2<sup> $\blacklozenge$ </sup> is done.

Therefore, setting  $\delta = \theta$ , we are done.

9. All targets of L1 are 'Done', so L1 is itself done.

Problem solved.

# If f is a continuous function and $(a_n) \to a$ , then $(f(a_n)) \to f(a)$

Let  $\epsilon > 0$ . We would like to find N s.t.  $d(f(a), f(a_n)) < \epsilon$  whenever  $n \geqslant N$ . Since f is continuous, there exists  $\delta > 0$  such that  $d(f(a), f(a_n)) < \epsilon$  whenever  $d(a, a_n) < \delta$ . Since  $a_n \to a$ , there exists N' such that  $d(a, a_n) < \delta$  whenever  $n \geqslant N'$ . Therefore, setting N = N', we are done.

H1. 
$$f$$
 is continuous
$$\frac{\text{H2. } a_n \to a}{\text{T1. } f(a_n) \to f(a)}$$

1. Expand pre-universal target T1.

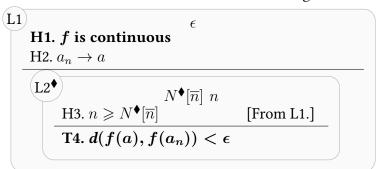
$$egin{aligned} egin{aligned} \operatorname{L1} & \operatorname{H1.}\ f \ ext{is continuous} \ & \operatorname{H2.}\ a_n o a \ & \operatorname{T2.}\ orall \epsilon. (\exists N. (orall n. (n \geqslant N \Rightarrow d(f(a), f(a_n)) < \epsilon))) \end{aligned}$$

2. pply 'let' trick and move premise of universal target T2 above the line.

Let  $\epsilon > 0$ .

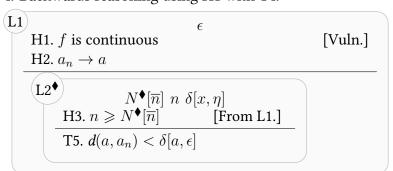
```
egin{aligned} \mathsf{L1} & \epsilon \ \mathsf{H1.} \ f \ \mathsf{is} \ \mathsf{continuous} \ \mathsf{H2.} \ a_n 	o a \ \hline  \mathbf{T3.} \ \exists N. ( \forall n. (n \geqslant N \Rightarrow d(f(a), f(a_n)) < \epsilon)) \end{aligned}
```

3. Unlock existential-universal-conditional target T3.



We would like to find N s.t.  $d(f(a), f(a_n)) < \epsilon$  whenever  $n \ge N$ .

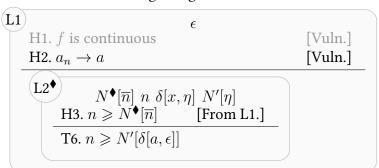
4. Backwards reasoning using H1 with T4.



Since f is continuous, there exists  $\delta > 0$  such that  $d(f(a), f(a_n)) < \epsilon$  whenever  $d(a, a_n) < \delta$ .

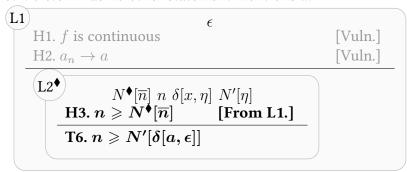
5. Delete H1 as no other statement mentions f.

6. Backwards reasoning using H2 with T5.



Since  $a_n \to a$ , there exists N' such that  $d(a, a_n) < \delta$  whenever  $n \ge N'$ .

7. Delete H2 as no other statement mentions a.



8. Hypothesis H3 matches target T6 after choosing  $N^{\blacklozenge}[\overline{n}] = N'[\delta[a,\epsilon]]$ , so  $L2^{\blacklozenge}$  is done.

Therefore, setting N=N', we are done.

```
\begin{array}{c|c} \textbf{L1} & \boldsymbol{\epsilon} \\ & \textbf{H1.} \ f \ \text{is continuous} & [\textbf{Vuln.}] \\ & \textbf{H2.} \ a_n \rightarrow a & [\textbf{Vuln.}] \\ \hline & \textbf{L2}^{\blacklozenge} \textbf{Done} \end{array}
```

9. All targets of L1 are 'Done', so L1 is itself done.

Problem solved.

## A closed subset A of a complete metric space X is complete.

Let  $(a_n)$  be a Cauchy sequence in A. Then, since X is complete, we have that  $(a_n)$  converges. That is, there exists a such that  $a_n \to a$ . Since A is closed in X,  $(a_n)$  is a sequence in A and  $a_n \to a$ , we have that  $a \in A$ . Thus  $(a_n)$  converges in A and we are done.

H1. X is complete H2. A is closed in XT1. A is a complete space

1. Expand pre-universal target T1.

L1
H1. X is complete
H2. A is closed in XT2.  $\forall (a_n).((a_n) \text{ is Cauchy } \land (a_n) \text{ is a sequence in } A \Rightarrow (a_n) \text{ converges in } A)$ 

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let  $(a_n)$  be a Cauchy sequence in A.

L1  $(a_n)$ H1. X is complete

H2. A is closed in XH3.  $(a_n)$  is Cauchy

H4.  $(a_n)$  is a sequence in AT3.  $(a_n)$  converges in A

3. Forwards reasoning using H1 with H3.

Since X is complete and  $(a_n)$  is Cauchy, we have that  $(a_n)$  converges.

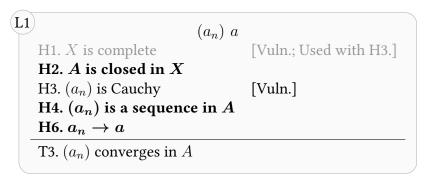
4. Deleted H1, as the conclusion of this implicative statement is unmatchable.

L1  $(a_n)$ H1. X is complete [Vuln.; Used with H3.]
H2. A is closed in XH3.  $(a_n)$  is Cauchy [Vuln.]
H4.  $(a_n)$  is a sequence in AH5.  $(a_n)$  converges

T3.  $(a_n)$  converges in A

5. Expand pre-existential hypothesis H5.

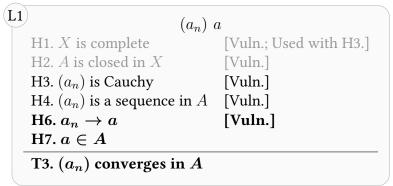
By definition, since  $(a_n)$  converges, there exists a such that  $a_n \to a$ .



6. Forwards reasoning using library result "a closed set contains its limit points" with (H2,H4,H6).

Since A is closed in X,  $(a_n)$  is a sequence in A and  $a_n \to a$ , we have that  $a \in A$ .

7. Delete H2 as no other statement mentions X.



8. All conjuncts of T3 (after expansion) can be simultaneously matched against H7 and H6 or rendered trivial by choosing z=a, so L1 is done.

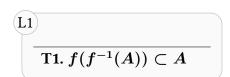
L1 Done

Problem solved.

We would like to show that  $(a_n)$  converges in A. But this is clearly the case, so we are done.

$$f(f^{-1}(A)) \subset A$$

Let x be an element of  $f(f^{-1}(A))$ . Then there exists  $y \in f^{-1}(A)$  such that f(y) = x. Since  $y \in f^{-1}(A)$ , we have that  $f(y) \in A$ . Since f(y) = x, we have that  $x \in A$  and we are done.



1. Expand pre-universal target T1.

L1 
$$T2. \, orall x. (x \in f(f^{-1}(A)) \Rightarrow x \in A)$$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

3. Expand pre-existential hypothesis H1.

L1) 
$$x \ y$$
H2.  $y \in f^{-1}(A)$ 
H3.  $f(y) = x$ 
T3.  $x \in A$ 

4. Quantifier-free expansion of hypothesis H2.

$$\begin{array}{c} \text{L1} & x \ y \\ \text{H4.} \ f(y) \in A \\ \hline \text{H3.} \ f(y) = x \\ \hline \hline \text{T3.} \ x \in A \end{array}$$

5. Rewrite f(y) as x throughout the tableau using hypothesis H3.

6. Hypothesis H5 matches target T3, so L1 is done.

Problem solved.

Let x be an element of  $f(f^{-1}(A))$ .

By definition, since  $x \in f(f^{-1}(A))$ , there exists  $y \in f^{-1}(A)$  such that f(y) = x.

Since  $y \in f^{-1}(A)$ , we have that  $f(y) \in A$ .

Since f(y) = x, we have that  $x \in A$ .

We are done.

$$A \subset f^{-1}(f(A))$$

Let x be an element of A. We would like to show that  $x \in f^{-1}(f(A))$ , i.e. that  $f(x) \in f(A)$ . But this is clearly the case, so we are done.

$$T1.\ A\subset f^{-1}(f(A))$$

1. Expand pre-universal target T1.

$$T2.\, orall x.(x\in A\Rightarrow x\in f^{-1}(f(A)))$$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of A.

$$\begin{array}{c}
\text{L1} & x \\
 & \text{H1. } x \in A \\
\hline
 & \text{T3. } x \in f^{-1}(f(A))
\end{array}$$

3. Quantifier-free expansion of target T3.

$$\begin{array}{c|c} \text{L1} & x \\ \hline \textbf{H1.} \ x \in \overset{x}{A} \\ \hline \textbf{T4.} \ f(x) \in f(A) \end{array}$$

We would like to show that  $x \in f^{-1}(f(A))$ , i.e. that  $f(x) \in f(A)$ .

4. All conjuncts of T4 (after expansion) can be simultaneously matched against H1 or rendered trivial by choosing y = x, so L1 is done.

L1 Done

Problem solved.

We would like to show that  $f(x) \in f(A)$ . But this is clearly the case, so we are done.

### The intersection of two subgroups is a subgroup

Let x and y be such that  $x \in H \cap K$  and  $y \in H \cap K$ . Since H is a subgroup, H is closed under taking inverses,  $e \in H$  and H is closed under multiplication. Since K is a subgroup, K is closed under taking inverses,  $E \in K$  and E is closed under multiplication. Since E is closed under multiplication, we have that E is closed under multiplication, we have that E is closed under multiplication and E is that E is and E is an analysis of E in the E is an analysis of E in the E in the E is an analysis of E in the E is an analysis of E in the E in the E is an analysis of E in the E in the E is an analysis of E in the E in the E is an analysis of E in the E in the E in the E is an analysis of E in the E in the E in the E is an analysis of E in the E in t

- L1 H1. H is a subgroup H2. K is a subgroup T1.  $\forall x, y. (x \in H \cap K \land y \in H \cap K \Rightarrow xy \in H \cap K)$
- 1. Apply 'let' trick and move premise of universal-conditional target T1 above the line.

Let x and y be such that  $x \in H \cap K$  and  $y \in H \cap K$ .

L1) 
$$x y$$

H1.  $H$  is a subgroup

H2.  $K$  is a subgroup

H3.  $x \in H \cap K$ 

H4.  $y \in H \cap K$ 

T2.  $xy \in H \cap K$ 

2. Quantifier-free expansion of hypothesis H1.

 $\begin{array}{c} \textbf{L1} & x \ y \\ \textbf{H5.} \ H \ \text{is closed under taking inverses} \\ \textbf{H6.} \ e \in H \\ \textbf{H7.} \ H \ \text{is closed under multiplication} \\ \textbf{\textbf{H2.}} \ \textbf{\textbf{\textbf{K}}} \ \textbf{is a subgroup} \\ \textbf{H3.} \ x \in H \cap K \\ \textbf{H4.} \ y \in H \cap K \\ \hline \textbf{\textbf{T2.}} \ xy \in H \cap K \\ \end{array}$ 

Since H is a subgroup, H is closed under taking inverses,  $e \in H$  and H is closed under multiplication.

3. Quantifier-free expansion of hypothesis H2.

H5. H is closed under taking inverses H6.  $e \in H$ H7. H is closed under multiplication H8. K is closed under taking inverses H9.  $e \in K$ H10. K is closed under multiplication H3.  $\mathbf{x} \in \mathbf{H} \cap \mathbf{K}$ H4.  $y \in H \cap K$ T2.  $xy \in H \cap K$ 

Since K is a subgroup, K is closed under taking inverses,  $e \in K$  and K is closed under multiplication.

4. Quantifier-free expansion of hypothesis H3.

Since  $x \in H \cap K$ ,  $x \in H$  and  $x \in K$ .

H5. 
$$H$$
 is closed under taking inverses H6.  $e \in H$ 
H7.  $H$  is closed under multiplication H8.  $K$  is closed under taking inverses H9.  $e \in K$ 
H10.  $K$  is closed under multiplication H11.  $x \in H$ 
H12.  $x \in K$ 
H4.  $y \in H \cap K$ 

T2.  $xy \in H \cap K$ 

5. Quantifier-free expansion of hypothesis H4.

H5. H is closed under taking inverses H6.  $e \in H$ H7. H is closed under multiplication H8. K is closed under taking inverses H9.  $e \in K$ H10. K is closed under multiplication H11. K is closed under multiplication H12. K is closed under multiplication H12. K is closed under multiplication H11. K is closed u

Since  $y \in H \cap K$ ,  $y \in H$  and  $y \in K$ .

6. Forwards reasoning using library result "" with (H7,H11,H13).

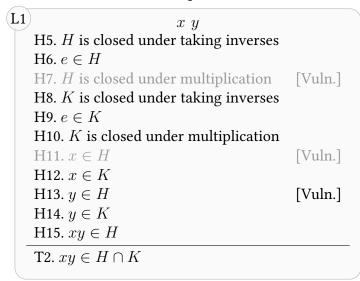
(L1)H5. H is closed under taking inverses H6.  $e \in H$ H7. H is closed under multiplication [Vuln.] H8. K is closed under taking inverses H9.  $e \in K$ H10. K is closed under multiplication H11.  $x \in H$ [Vuln.] H12.  $x \in K$ H13.  $y \in H$ [Vuln.] H14.  $y \in K$ H15.  $xy \in H$ T2.  $xy \in H \cap K$ 

Since H is closed under multiplication,  $x \in H$  and  $y \in H$ , we have that  $xy \in H$ .

7. Deleted H7, as this unexpandable atomic statement is unmatchable.

```
(L1)
                          x y
    H5. H is closed under taking inverses
    H6. e \in H
    H7. H is closed under multiplication
                                             [Vuln.]
    H8. K is closed under taking inverses
    H9. e \in K
    H10. K is closed under multiplication
    H11. x \in H
                                             [Vuln.]
    H12. x \in K
    H13. y \in H
                                             [Vuln.]
    H14. y \in K
    H15. xy \in H
    T2. xy \in H \cap K
```

8. Deleted H11, as this unexpandable atomic statement is unmatchable.



9. Deleted H13, as this unexpandable atomic statement is unmatchable.

L1	x y	
	H5. $H$ is closed under taking inverses	
	H6. $e \in H$	
	H7. $H$ is closed under multiplication	[Vuln.]
	H8. $K$ is closed under taking inverses	
	H9. $e \in K$	
	H10. $K$ is closed under multiplication	
	H11. $x \in H$	[Vuln.]
	H12. $x \in K$	
	H13. $y \in H$	[Vuln.]
	H14. $y \in K$	
	H15. $xy \in H$	
	T2. $xy \in H \cap K$	

10. Forwards reasoning using library result "" with (H10,H12,H14).

Since K is closed under multiplication,  $x \in K$  and  $y \in K$ , we have that  $xy \in K$ .

```
(L1)
                           x y
    H5. H is closed under taking inverses
    H6. e \in H
    H7. H is closed under multiplication
                                             [Vuln.]
    H8. K is closed under taking inverses
    H9. e \in K
    H10. K is closed under multiplication
                                             [Vuln.]
    H11. x \in H
                                              [Vuln.]
    H12. x \in K
                                             [Vuln.]
    H13. y \in H
                                              [Vuln.]
    H14. y \in K
                                             [Vuln.]
    H15. xy \in H
    H16. xy \in K
    T2. xy \in H \cap K
```

11. Deleted H10, as this unexpandable atomic statement is unmatchable.

(L1)	x y	
	H5. $H$ is closed under taking inverses	
	H6. $e \in H$	
	H7. $H$ is closed under multiplication	[Vuln.]
	H8. $K$ is closed under taking inverses	
	H9. $e \in K$	
	H10. $K$ is closed under multiplication	[Vuln.]
	H11. $x \in H$	[Vuln.]
	H12. $x \in K$	[Vuln.]
	H13. $y \in H$	[Vuln.]
	H14. $y \in K$	[Vuln.]
	H15. $xy \in H$	
	H16. $xy \in K$	
	T2. $xy \in H \cap K$	

12. Deleted H12, as this unexpandable atomic statement is unmatchable.

```
(L1)
                          x y
    H5. H is closed under taking inverses
    H6. e \in H
    H7. H is closed under multiplication
                                             [Vuln.]
    H8. K is closed under taking inverses
    H9. e \in K
    H10. K is closed under multiplication
                                             [Vuln.]
    H11. x \in H
                                              [Vuln.]
    H12. x \in K
                                              [Vuln.]
    H13. y \in H
                                              [Vuln.]
    H14. y \in K
                                             [Vuln.]
    H15. xy \in H
    H16. xy \in K
    T2. xy \in H \cap K
```

13. Deleted H14, as this unexpandable atomic statement is unmatchable.

```
(L1)
                          x y
    H5. H is closed under taking inverses
    H6. e \in H
    H7. H is closed under multiplication
                                             [Vuln.]
    H8. K is closed under taking inverses
    H9. e \in K
    H10. K is closed under multiplication
                                             [Vuln.]
    H11. x \in H
                                             [Vuln.]
    H12. x \in K
                                             [Vuln.]
    H13. y \in H
                                             [Vuln.]
    H14. y \in K
                                             [Vuln.]
    H15. xy \in H
    H16. xy \in K
    T2. xy \in H \cap K
```

14. Quantifier-free expansion of target T2.

(L1)H5. H is closed under taking inverses H6.  $e \in H$ H7. H is closed under multiplication [Vuln.] H8. *K* is closed under taking inverses H9.  $e \in K$ H10. K is closed under multiplication [Vuln.] H11.  $x \in H$ [Vuln.] H12.  $x \in K$ [Vuln.] H13.  $y \in H$ [Vuln.] H14.  $y \in K$ [Vuln.] H15.  $xy \in H$ H16.  $xy \in K$ T3.  $xy \in H$ T4.  $xy \in K$ 

15. Hypothesis H15 matches target T3, so we can remove T3.

x y	
H5. $H$ is closed under taking inverses	
H6. $e \in H$	
H7. $H$ is closed under multiplication	[Vuln.]
H8. $K$ is closed under taking inverses	
H9. $e \in K$	
H10. $K$ is closed under multiplication	[Vuln.]
H11. $x \in H$	[Vuln.]
H12. $x \in K$	[Vuln.]
H13. $y \in H$	[Vuln.]
H14. $y \in K$	[Vuln.]
H15. $xy \in H$	
H16. $xy \in K$	
$\boxed{T4. xy \in K}$	
· ·	

16. Hypothesis H16 matches target T4, so L1 is done.

L1 Done

Problem solved.

We would like to show that  $xy \in H \cap K$ , i.e. that  $xy \in H$  and  $xy \in K$ .

We are done.