Prove that $A \subseteq f^{-1}(f(A))$

Let x be an element of A. We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$. But this is clearly the case, so we are done.

Prove that $f(f^{-1}(A)) \subset A$

Let x be an element of $f(f^{-1}(A))$. Then there exists $y \in f^{-1}(A)$ such that $f(y) = x$. Since $y \in f^{-1}(A)$, we have that $f(y) \in A$. Since $f(y) = x$, we have that $x \in A$. But this is clearly the case, so we are done.

Prove that $f(A \cap B) \subset f(A) \cap f(B)$

By definition, since $y \in f(A \cap B)$, there exists $z \in A \cap B$ such that $f(z) = y$. Since $z \in A \cap B$, $z \in A$ and $z \in B$. We would like to show that $y \in f(A) \cap f(B)$, i.e. that $y \in f(A)$ and $y \in f(B)$. We would like to show that $y \in f(A)$. But this is clearly the case, so we are done. Thus $y \in f(B)$ and we are done.

Prove that $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$

Since $x \in f^{-1}(A \cap B)$, we have that $f(x) \in A \cap B$. Then $f(x) \in A$ and $f(x) \in B$. We would like to show that $x \in f^{-1}(A) \cap f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. We would like to show that $x \in f^{-1}(B)$, i.e. that $f(x) \in B$. But this is clearly the case, so we are done.

Prove that $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$

Let x be an element of $f^{-1}(A) \cap f^{-1}(B)$. Then $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Then $f(x) \in A$ and $f(x) \in B$. We would like to show that $x \in f^{-1}(A \cap B)$, i.e. that $f(x) \in A \cap B$. We would like to show that $f(x) \in A \cap B$, i.e. that $f(x) \in A$ and $f(x) \in B$. But this is clearly the case, so we are done.

Prove that $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$

Let x be an element of $f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$. Then $f(x) \in A$ or $f(x) \in B$. We would like to show that $x \in f^{-1}(A) \cup f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. But this is clearly the case, so we are done. We would like to show that $x \in f^{-1}(A) \cup f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. We would like to show that $x \in f^{-1}(B)$, i.e. that $f(x) \in B$. But this is clearly the case, so we are done.

Prove that $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$

Let x be an element of $f^{-1}(A) \cup f^{-1}(B)$. Then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. Since $x \in f^{-1}(A)$, we have that $f(x) \in A$. Since $x \in f^{-1}(B)$, we have that $f(x) \in B$. We would like to show that $x \in f^{-1}(A \cup B)$, i.e. that $f(x) \in A \cup B$. We would like to show that $f(x) \in A \cup B$, i.e. that $f(x) \in A$ or $f(x) \in B$. But this is clearly the case, so we are done. We would like to show that $x \in f^{-1}(A \cup B)$, i.e. that $f(x) \in A \cup B$. We would like to show that $f(x) \in A \cup B$, i.e. that $f(x) \in A$ or $f(x) \in B$. But this is clearly the case, so we are done.