If f is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of $f(A) \cap f(B)$. Then $x \in f(A)$ and $x \in f(B)$. That is, there exists $y \in A$ such that $f(y) = x$ and there exists $z \in B$ such that $f(z) = x$. Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$. We would like to find $u \in A \cap B$ s.t. $f(u) = x$. But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$. Since $y = z$, we have that $y \in B$. Therefore, setting $u = y$, we are done.

If g,f are injections then (g o f) is an injection.

Let x, y and z be such that $g(f(x)) = z$ and $g(f(y)) = z$. Then, since g is an injection, we have that $f(x) = f(y)$. Therefore, since f is an injection, $x = y$ if $f(y) = f(y)$. Since g is an injection and $g(f(y)) = z$, $f(y) = f(y)$ if $g(f(y)) = z$. But this is clearly the case, so we are done.

Prove that $A \subseteq f^{-1}(f(A))$

Let x be an element of A. We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$. But this is clearly the case, so we are done.

Prove that $f(f^{-1}(A)) \subset A$

Let x be an element of $f(f^{-1}(A))$. Then there exists $y \in f^{-1}(A)$ such that $f(y) = x$. Since $y \in f^{-1}(A)$, we have that $f(y) \in A$. Since $f(y) = x$, we have that $x \in A$. But this is clearly the case, so we are done.

Prove that $f(A \cap B) \subset f(A) \cap f(B)$

By definition, since $y \in f(A \cap B)$, there exists $z \in A \cap B$ such that $f(z) = y$. Since $z \in A \cap B$, $z \in A$ and $z \in B$. We would like to show that $y \in f(A) \cap f(B)$, i.e. that $y \in f(A)$ and $y \in f(B)$. We would like to show that $y \in f(A)$. But this is clearly the case, so we are done. Thus $y \in f(B)$ and we are done.

Prove that $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$

Since $x \in f^{-1}(A \cap B)$, we have that $f(x) \in A \cap B$. Then $f(x) \in A$ and $f(x) \in B$. We would like to show that $x \in f^{-1}(A) \cap f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. We would like to show that $x \in f^{-1}(B)$, i.e. that $f(x) \in B$. But this is clearly the case, so we are done.

Prove that $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$

Let x be an element of $f^{-1}(A) \cap f^{-1}(B)$. Then $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Then $f(x) \in A$ and $f(x) \in B$. We would like to show that $x \in f^{-1}(A \cap B)$, i.e. that $f(x) \in A \cap B$. We would like to show that $f(x) \in A \cap B$, i.e. that $f(x) \in A$ and $f(x) \in B$. But this is clearly the case, so we are done.

Prove that $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$

Let x be an element of $f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$. Then $f(x) \in A$ or $f(x) \in B$. We would like to show that $x \in f^{-1}(A) \cup f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. But this is clearly the case, so we are done. We would like to show that $x \in f^{-1}(A) \cup f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. We would like to show that $x \in f^{-1}(B)$, i.e. that $f(x) \in B$. But this is clearly the case, so we are done.

Prove that $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$

Let x be an element of $f^{-1}(A) \cup f^{-1}(B)$. Then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. Since $x \in f^{-1}(A)$, we have that $f(x) \in A$. Since $x \in f^{-1}(B)$, we have that $f(x) \in B$. We would like to show that $x \in f^{-1}(A \cup B)$, i.e. that $f(x) \in A \cup B$. We would like to show that $f(x) \in A \cup B$, i.e. that $f(x) \in A$ or $f(x) \in B$. But this is clearly the case, so we are done. We would like to show that $x \in f^{-1}(A \cup B)$, i.e. that $f(x) \in A \cup B$. We would like to show that $f(x) \in A \cup B$, i.e. that $f(x) \in A$ or $f(x) \in B$. But this is clearly the case, so we are done.

If A, B,and C are open sets, then $A \cup (B \cup C)$ is also open.

Let x be an element of $A \cup B \cup C$. Then $x \in A$ or $x \in B \cup C$. Since A is open and $x \in A$, there exists $\alpha > 0$ such that $w \in A$ whenever $d(x, w) < \alpha$. Since $x \in B \cup C$, $x \in B$ or $x \in C$. Since B is open and $x \in B$, there exists $\delta' > 0$ such that $r \in B$ whenever $d(x, r) < \delta'$. Since C is open and $x \in C$, there exists $\delta'' > 0$ such that $s \in C$ whenever $d(x, s) < \delta''$. We would like to find $\eta > 0$ s.t. $z \in A \cup B \cup C$ whenever $d(x, z) < \eta$. But $z \in A \cup B \cup C$ if and only if $z \in A$ or $z \in B \cup C$. We know that $z \in A$ if $d(x, z) < \alpha$. Therefore, setting $\eta = \alpha$, we are done. We would like to find $\beta > 0$ s.t. $v \in A \cup B \cup C$ whenever $d(x, v) < \beta$. But $v \in A \cup B \cup C$ if and only if $v \in A$ or $v \in B \cup C$. We would like to show that $v \in B \cup C$, i.e. that $v \in B$ or $v \in C$. We know that $v \in B$ if $d(x, v) < \delta'$. Therefore, setting $\beta = \delta'$, we are done. We would like to find $\gamma > 0$ s.t. $p \in A \cup B \cup C$ whenever $d(x, p) < \gamma$. But $p \in A \cup B \cup C$ if and only if $p \in A$ or $p \in B \cup C$. We would like to show that $p \in B \cup C$, i.e. that $p \in B$ or $p \in C$. We know that $p \in C$ if $d(x, p) < \delta''$. Therefore, setting $\gamma = \delta''$, we are done.

If A and B are open sets, then $A \cup B$ is also open.

Let x be an element of $A \cup B$. Then $x \in A$ or $x \in B$. Since A is open and $x \in A$, there exists $\alpha > 0$ such that $w \in A$ whenever $d(x, w) < \alpha$. Since B is open and $x \in B$, there exists $\beta > 0$ such that $p \in B$ whenever $d(x, p) < \beta$. We would like to find $\eta > 0$ s.t. $z \in A \cup B$ whenever $d(x, z) < \eta$. But $z \in A \cup B$ if and only if $z \in A$ or $z \in B$. We know that $z \in A$ if $d(x, z) < \alpha$. Therefore, setting $\eta = \alpha$, we are done. We would like to find $\theta > 0$ s.t. $u \in A \cup B$ whenever $d(x, u) < \theta$. But $u \in A \cup B$ if and only if $u \in A$ or $u \in B$. We know that $u \in B$ if $d(x, u) < \beta$. Therefore, setting $\theta = \beta$, we are done.

If A and B are open sets, then $A \cap B$ is also open.

Let x be an element of $A \cap B$. Then $x \in A$ and $x \in B$. Therefore, since A is open, there exists $\eta > 0$ such that $u \in A$ whenever $d(x, u) < \eta$ and since B is open, there exists $\theta > 0$ such that $v \in B$ whenever $d(x, v) < \theta$. We would like to find $\delta > 0$ s.t. $y \in A \cap B$ whenever $d(x, y) < \delta$. But $y \in A \cap B$ if and only if $y \in A$ and $y \in B$. We know that $y \in A$ whenever $d(x, y) < \eta$ and that $y \in B$ whenever $d(x, y) < \theta$. Assume now that $d(x, y) < \delta$. Then $d(x, y) < \eta$ if $\delta \leqslant \eta$ and $d(x, y) < \theta$ if $\delta \leqslant \theta$. We may therefore take $\delta = \min(\eta, \theta)$ and we are done.

If A and B are closed sets, then $A \cap B$ is also closed.

Let (a_n) and a be such that (a_n) is a sequence in $A \cap B$ and $a_n \to a$. Then (a_n) is a sequence in A and (a_n) is a sequence in B. Therefore, since A is closed and $a_n \to a$, we have that $a \in A$ and since B is closed and $a_n \to a$, we have that $a \in B$. We would like to show that $a \in A \cap B$, i.e. that $a \in A$ and $a \in B$. But this is clearly the case, so we are done.

The pre-image of a closed set U under a continuous function f is closed.

Let (a_n) and a be such that (a_n) is a sequence in $f^{-1}(U)$ and $a_n \to a$. Then $f(a_n)$ is a sequence in U. We would like to show that $a \in f^{-1}(U)$, i.e. that $f(a) \in U$ and since U is closed, $f(a) \in U$ if $f(a_n) \to f(a)$. Let $\epsilon > 0$. We would like to find N s.t. $d(f(a), f(a_n)) < \epsilon$ whenever $n \geqslant N$. Since f is continuous, there exists $\delta > 0$ such that $d(f(a), f(a_n)) < \epsilon$ whenever $d(a, a_n) < \delta$. Since $a_n \to a$, there exists N' such that $d(a, a_n) < \delta$ whenever $n \geqslant N'$. Therefore, setting $N = N'$, we are done.

The pre-image of an open set U under a continuous function f is open.

Let x be an element of $f^{-1}(U)$. Then $f(x) \in U$. Therefore, since U is open, there exists $\eta > 0$ such that $u \in U$ whenever $d(f(x), u) < \eta$. We would like to find $\delta > 0$ s.t. $y \in f^{-1}(U)$ whenever $d(x, y) < \delta$. But $y \in f^{-1}(U)$ if and only if $f(y) \in U$. We know that $f(y) \in U$ whenever $d(f(x), f(y)) < \eta$. Since f is continuous, there exists $\theta > 0$ such that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \theta$. Therefore, setting $\delta = \theta$, we are done.

If f and g are continuous functions, then $g \circ f$ is continuous.

Take x and $\epsilon > 0$. We would like to find $\delta > 0$ s.t. $d(g(f(x)), g(f(y))) < \epsilon$ whenever $d(x, y) < \delta$. Since g is continuous, there exists $\eta > 0$ such that $d(g(f(x)), g(f(y))) < \epsilon$ whenever $d(f(x), f(y)) < \eta$. Since f is continuous, there exists $\theta > 0$ such that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \theta$. Therefore, setting $\delta = \theta$, we are done.

If f is a continuous function and $(a_n) \to a$, then $(f(a_n)) \to f(a)$

Let $\epsilon > 0$. We would like to find N s.t. $d(f(a), f(a_n)) < \epsilon$ whenever $n \geqslant N$. Since f is continuous, there exists $\delta > 0$ such that $d(f(a), f(a_n)) < \epsilon$ whenever $d(a, a_n) < \delta$. Since $a_n \to a$, there exists N' such that $d(a, a_n) < \delta$ whenever $n \geqslant N'$. Therefore, setting $N = N'$, we are done.