If g,f are injections then (g o f) is an injection.

Let x, y and z be such that $g(f(x)) = z$ and $g(f(y)) = z$. Then, since g is an injection, we have that $f(x) = f(y)$. Therefore, since f is an injection, $x = y$ if $f(y) = f(y)$. Since g is an injection and $g(f(y)) = z$, $f(y) = f(y)$ if $g(f(y)) = z$. But this is clearly the case, so we are done.

Prove that $f(A \cap B) \subset f(A) \cap f(B)$

By definition, since $y \in f(A \cap B)$, there exists $z \in A \cap B$ such that $f(z) = y$. Since $z \in A \cap B$, $z \in A$ and $z \in B$. We would like to show that $y \in f(A) \cap f(B)$, i.e. that $y \in f(A)$ and $y \in f(B)$. We would like to show that $y \in f(A)$. But this is clearly the case, so we are done. Thus $y \in f(B)$ and we are done.

Prove that $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$

Since $x \in f^{-1}(A \cap B)$, we have that $f(x) \in A \cap B$. Then $f(x) \in A$ and $f(x) \in B$. We would like to show that $x \in f^{-1}(A) \cap f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. We would like to show that $x \in f^{-1}(B)$, i.e. that $f(x) \in B$. But this is clearly the case, so we are done.

Prove that $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$

Let x be an element of $f^{-1}(A) \cap f^{-1}(B)$. Then $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Then $f(x) \in A$ and $f(x) \in B$. We would like to show that $x \in f^{-1}(A \cap B)$, i.e. that $f(x) \in A \cap B$. We would like to show that $f(x) \in A \cap B$, i.e. that $f(x) \in A$ and $f(x) \in B$. But this is clearly the case, so we are done.

Prove that $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$

Let x be an element of $f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$. Then $f(x) \in A$ or $f(x) \in B$. We would like to show that $x \in f^{-1}(A) \cup f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. But this is clearly the case, so we are done. We would like to show that $x \in f^{-1}(A) \cup f^{-1}(B)$, i.e. that $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We would like to show that $x \in f^{-1}(A)$, i.e. that $f(x) \in A$. We would like to show that $x \in f^{-1}(B)$, i.e. that $f(x) \in B$. But this is clearly the case, so we are done.

Prove that $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$

Let x be an element of $f^{-1}(A) \cup f^{-1}(B)$. Then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. Since $x \in f^{-1}(A)$, we have that $f(x) \in A$. Since $x \in f^{-1}(B)$, we have that $f(x) \in B$. We would like to show that $x \in f^{-1}(A \cup B)$, i.e. that $f(x) \in A \cup B$. We would like to show that $f(x) \in A \cup B$, i.e. that $f(x) \in A$ or $f(x) \in B$. But this is clearly the case, so we are done. We would like to show that $x \in f^{-1}(A \cup B)$, i.e. that $f(x) \in A \cup B$. We would like to show that $f(x) \in A \cup B$, i.e. that $f(x) \in A$ or $f(x) \in B$. But this is clearly the case, so we are done.