If f is a surjection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of $f(A) \cap f(B)$. Then $x \in f(A)$ and $x \in f(B)$. That is, there exists $y \in A$ such that $f(y) = x$ and there exists $z \in B$ such that $f(z) = x$. We would like to find $u \in A \cap B$ s.t. $f(u) = x$. But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$.

L1

H1. *surjection*(f)

T1. $\boldsymbol{f(A) \cap f(B) \subset f(A \cap B)}$

1. Expand pre-universal target T1.

L1

H1. *surjection*(f)

T2. $\boldsymbol{\forall x.(x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(A) \cap f(B)$.

L1

x

H1. *surjection*(f)

H2. $\boldsymbol{x \in f(A) \cap f(B)}$

T3. $x \in f(A \cap B)$

3. Quantifier-free expansion of hypothesis H2.

Since $x \in f(A) \cap f(B)$, $x \in f(A)$ and $x \in f(B)$.

L1

x

H1. *surjection*(f)

H3. $\boldsymbol{x \in f(A)}$

H4. $x \in f(B)$

T3. $x \in f(A \cap B)$

4. Expand pre-existential hypothesis H3.

By definition, since $x \in f(A)$, there exists $y \in A$ such that $f(y) = x$.

L1

$x \ y$

H1. *surjection*(f)

H5. $y \in A$

H6. $f(y) = x$

H4. $\boldsymbol{x \in f(B)}$

T3. $x \in f(A \cap B)$

5. Expand pre-existential hypothesis H4.

By definition, since $x \in f(B)$, there exists $z \in B$ such that $f(z) = x$.

L1

$x \ y \ z$

H1. *surjection*(f)

H5. $y \in A$

H6. $f(y) = x$

H7. $z \in B$

H8. $f(z) = x$

T3. $\boldsymbol{x \in f(A \cap B)}$

6. Expand pre-existential target T3.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1

$x\ y\ z$

H1. *surjection*(f)
H5. $y \in A$
H6. $f(y) = x$
H7. $z \in B$
H8. $f(z) = x$

T4. $\boldsymbol{\exists u.(u \in A \cap B \wedge f(u) = x)}$

7. Unlock existential target T4.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1

$x\ y\ z$

H1. *surjection*(f)
H5. $y \in A$
H6. $f(y) = x$
H7. $z \in B$
H8. $f(z) = x$

L2$^\blacklozenge$

$u^\blacklozenge$

T5. $\boldsymbol{u^\blacklozenge \in A \cap B}$
T6. $f(u^\blacklozenge) = x$

8. Quantifier-free expansion of target T5.

But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$.

L1

$x\ y\ z$

H1. *surjection*(f)
H5. $y \in A$
H6. $f(y) = x$
H7. $z \in B$
H8. $f(z) = x$

L2$^\blacklozenge$

$u^\blacklozenge$

T7. $u^\blacklozenge \in A$
T8. $u^\blacklozenge \in B$
T6. $f(u^\blacklozenge) = x$

No moves possible.

If f is an injection then $f(A) \cap f(B) \subset f(A \cap B)$

Let x be an element of $f(A) \cap f(B)$. Then $x \in f(A)$ and $x \in f(B)$. That is, there exists $y \in A$ such that $f(y) = x$ and there exists $z \in B$ such that $f(z) = x$. Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$. We would like to find $u \in A \cap B$ s.t. $f(u) = x$. But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$. Since $y = z$, we have that $y \in B$. Therefore, setting $u = y$, we are done.

L1

H1. f is an injection

T1. $f(A) \cap f(B) \subset f(A \cap B)$

1. Expand pre-universal target T1.

L1

H1. f is an injection

T2. $\forall x.(x \in f(A) \cap f(B) \Rightarrow x \in f(A \cap B))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(A) \cap f(B)$.

L1

x

H1. f is an injection

H2. $x \in f(A) \cap f(B)$

T3. $x \in f(A \cap B)$

3. Quantifier-free expansion of hypothesis H2.

Since $x \in f(A) \cap f(B)$, $x \in f(A)$ and $x \in f(B)$.

L1

x

H1. f is an injection

H3. $x \in f(A)$

H4. $x \in f(B)$

T3. $x \in f(A \cap B)$

4. Expand pre-existential hypothesis H3.

By definition, since $x \in f(A)$, there exists $y \in A$ such that $f(y) = x$.

L1

$x \; y$

H1. f is an injection

H5. $y \in A$

H6. $f(y) = x$

H4. $x \in f(B)$

T3. $x \in f(A \cap B)$

5. Expand pre-existential hypothesis H4.

By definition, since $x \in f(B)$, there exists $z \in B$ such that $f(z) = x$.

L1

$x \; y \; z$

H1. f is an injection

H5. $y \in A$

H6. $f(y) = x$

H7. $z \in B$

H8. $f(z) = x$

T3. $x \in f(A \cap B)$

6. Forwards reasoning using H1 with (H6,H8).

Since f is an injection, $f(y) = x$ and $f(z) = x$, we have that $y = z$.

L1

$x \; y \; z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

T3. $\boldsymbol{x \in f(A \cap B)}$

7. Expand pre-existential target T3.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1

$x \; y \; z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

T4. $\boldsymbol{\exists u.(u \in A \cap B \land f(u) = x)}$

8. Unlock existential target T4.

We would like to find $u \in A \cap B$ s.t. $f(u) = x$.

L1

$x \; y \; z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $y = z$

L2$^\blacklozenge$

$u^\blacklozenge$

T5. $\boldsymbol{u^\blacklozenge \in A \cap B}$
T6. $f(u^\blacklozenge) = x$

9. Quantifier-free expansion of target T5.

But $u \in A \cap B$ if and only if $u \in A$ and $u \in B$.

L1

$x \; y \; z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H7. $z \in B$
H8. $f(z) = x$ [Vuln.]
H9. $\boldsymbol{y = z}$

L2$^\blacklozenge$

$u^\blacklozenge$

T7. $u^\blacklozenge \in A$
T8. $u^\blacklozenge \in B$
T6. $f(u^\blacklozenge) = x$

10. Rewrite z as y throughout the tableau using hypothesis H9.

Since $y = z$, we have that $y \in B$.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H6. $f(y) = x$ [Vuln.]
H10. $y \in B$

L2$^\blacklozenge$

$u^\blacklozenge$

T7. $u^\blacklozenge \in A$
T8. $u^\blacklozenge \in B$
T6. $f(u^\blacklozenge) = x$

11. Moved H6 down, as x can only be utilised by T6.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H10. $y \in B$

L2$^\blacklozenge$

$u^\blacklozenge$

T7. $u^\blacklozenge \in A$
T8. $u^\blacklozenge \in B$

L3
H6. $f(y) = x$ [Vuln.; From L1.]

T6. $f(u^\blacklozenge) = x$

12. Choosing $u^\blacklozenge = y$ matches all targets inside L2$^\blacklozenge$ against hypotheses, so L2$^\blacklozenge$ is done.

Therefore, setting $u = y$, we are done.

L1

$x\ y\ z$

H1. f is an injection [Vuln.; Used with (H6,H8).]
H5. $y \in A$
H10. $y \in B$

L2$^\blacklozenge$ Done

13. All targets of L1 are ‘Done’, so L1 is itself done.

L1 Done

Problem solved.

If A and B are open sets, then $A \cap B$ is also open.

Let x be an element of $A \cap B$. Then $x \in A$ and $x \in B$. Therefore, since A is open, there exists $\eta > 0$ such that $u \in A$ whenever $d(x, u) < \eta$ and since B is open, there exists $\theta > 0$ such that $v \in B$ whenever $d(x, v) < \theta$. We would like to find $\delta > 0$ s.t. $y \in A \cap B$ whenever $d(x, y) < \delta$. But $y \in A \cap B$ if and only if $y \in A$ and $y \in B$. We know that $y \in A$ whenever $d(x, y) < \eta$ and that $y \in B$ whenever $d(x, y) < \theta$. Assume now that $d(x, y) < \delta$. Then $d(x, y) < \eta$ if $\delta \leqslant \eta$ and $d(x, y) < \theta$ if $\delta \leqslant \theta$. We may therefore take $\delta = \min(\eta, \theta)$ and we are done.

L1

H1. A is open
H2. B is open

T1. $A \cap B$ is open

1. Expand pre-universal target T1.

L1

H1. A is open
H2. B is open

T2. $\boldsymbol{\forall x.(x \in A \cap B \Rightarrow \exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cap B)))}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $A \cap B$.

L1

x

H1. A is open
H2. B is open
H3. $\boldsymbol{x \in A \cap B}$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cap B))$

3. Quantifier-free expansion of hypothesis H3.

Since $x \in A \cap B$, $x \in A$ and $x \in B$.

L1

x

H1. A is open
H2. B is open
H4. $\boldsymbol{x \in A}$
H5. $x \in B$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cap B))$

4. Forwards reasoning using H1 with H4.

Since A is open and $x \in A$, there exists $\eta > 0$ such that $u \in A$ whenever $d(x, u) < \eta$.

L1

$x \ \eta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open
H4. $x \in A$ [Vuln.]
H5. $x \in B$
H6. $\forall u.(d(x, u) < \eta[x] \Rightarrow u \in A)$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in A \cap B))$

5. Deleted H4, as this unexpandable atomic statement is unmatchable.

L1

$x\ \eta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open
H4. $x \in A$ [Vuln.]
H5. $x \in B$
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

6. Deleted H1, as the conclusion of this implicative statement is unmatchable.

L1

$x\ \eta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. $\boldsymbol{B}$ is open
H4. $x \in A$ [Vuln.]
H5. $\boldsymbol{x \in B}$
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

7. Forwards reasoning using H2 with H5.

Since B is open and $x \in B$, there exists $\theta > 0$ such that $v \in B$ whenever $d(x,v) < \theta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

8. Deleted H5, as this unexpandable atomic statement is unmatchable.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

T3. $\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))$

9. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

T3. $\boldsymbol{\exists \delta.(\forall y.(d(x,y) < \delta \Rightarrow y \in A \cap B))}$

10. Unlock existential-universal-conditional target T3.

We would like to find $\delta > 0$ s.t. $y \in A \cap B$ whenever $d(x,y) < \delta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y$

H8. $d(x,y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T4. $\boldsymbol{y \in A \cap B}$

11. Quantifier-free expansion of target T4.

But $y \in A \cap B$ if and only if $y \in A$ and $y \in B$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\boldsymbol{\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)}$
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y$

H8. $d(x,y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T5. $\boldsymbol{y \in A}$
T6. $y \in B$

12. Backwards reasoning using H6 with T5.

We know that $y \in A$ whenever $d(x,y) < \eta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y$

H8. $d(x,y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T7. $d(x,y) < \eta[x]$
T6. $y \in B$

13. Delete H6 as no other statement mentions A.

L1

$x \ \eta[x] \ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\boldsymbol{\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)}$

L2$^{\blacklozenge}$

$\delta^{\blacklozenge}[\overline{y}] \ y$

H8. $d(x,y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T7. $d(x,y) < \eta[x]$
T6. $\boldsymbol{y \in B}$

14. Backwards reasoning using H7 with T6.

We know that $y \in B$ whenever $d(x,y) < \theta$.

L1

$x \ \eta[x] \ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2$^{\blacklozenge}$

$\delta^{\blacklozenge}[\overline{y}] \ y$

H8. $d(x,y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T7. $d(x,y) < \eta[x]$
T8. $d(x,y) < \theta[x]$

15. Delete H7 as no other statement mentions B.

L1

$x \ \eta[x] \ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2$^{\blacklozenge}$

$\delta^{\blacklozenge}[\overline{y}] \ y$

H8. $d(x,y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T7. $d(x,y) < \eta[x]$
T8. $d(x,y) < \theta[x]$

16. Replacing diamonds with bullets in L2$^{\blacklozenge}$.

Assume now that $d(x,y) < \delta$.

L1

$x \ \eta[x] \ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}] \ y$

H8. $\boldsymbol{d(x,y) < \delta^\bullet[\overline{y}]}$ **[From L1.]**

T7. $\boldsymbol{d(x,y) < \eta[x]}$
T8. $d(x,y) < \theta[x]$

17. Backwards reasoning using library result "transitivity" with (T7,H8). Since $d(x,y) < \delta$, $d(x,y) < \eta$ if $\delta \leqslant \eta$.

L1

$x \ \eta[x] \ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}] \ y$

H8. $d(x,y) < \delta^\bullet[\overline{y}]$ [From L1.; Vuln.]

T9. $\delta^\bullet[\overline{y}] \leqslant \eta[x]$
T8. $d(x,y) < \theta[x]$

18. Moved H8 down, as x can only be utilised by T8.

L1

$x \ \eta[x] \ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}] \ y$

T9. $\delta^\bullet[\overline{y}] \leqslant \eta[x]$

L3

H8. $\boldsymbol{d(x,y) < \delta^\bullet[\overline{y}]}$ **[From L1.; Vuln.; From L2.]**

T8. $\boldsymbol{d(x,y) < \theta[x]}$

19. Backwards reasoning using library result "transitivity" with (T8,H8). Since $d(x,y) < \delta$, $d(x,y) < \theta$ if $\delta \leqslant \theta$.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}]\ y$

T9. $\delta^\bullet[\overline{y}] \leqslant \eta[x]$

L3

H8. $d(x,y) < \delta^\bullet[\overline{y}]$ [From L1.; Vuln.; From L2.]

T10. $\delta^\bullet[\overline{y}] \leqslant \theta[x]$

20. Delete H8 as no other statement mentions x.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}]\ y$

T9. $\delta^\bullet[\overline{y}] \leqslant \eta[x]$

L3

H8. $d(x,y) < \delta^\bullet[\overline{y}]$ [From L1.; Vuln.; From L2.]

T10. $\delta^\bullet[\overline{y}] \leqslant \theta[x]$

21. Collapsed subtableau L3 as it has no undeleted hypotheses.

L1

$x\ \eta[x]\ \theta[x]$

H1. A is open [Vuln.; Used with H4.]
H2. B is open [Vuln.; Used with H5.]
H4. $x \in A$ [Vuln.]
H5. $x \in B$ [Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$ [Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$ [Vuln.]

L2

$\delta^\bullet[\overline{y}]\ y$

T9. $\delta^\bullet[\overline{y}] \leqslant \eta[x]$
T10. $\delta^\bullet[\overline{y}] \leqslant \theta[x]$

22. Taking $\delta^\bullet[\overline{y}] = \min(\eta[x], \theta[x])$ matches all targets against a library solution, so L2 is done.

We may therefore take $\delta = \min(\eta, \theta)$. We are done.

L1

$x \; \eta[x] \; \theta[x]$

H1. A is open	[Vuln.; Used with H4.]
H2. B is open	[Vuln.; Used with H5.]
H4. $x \in A$	[Vuln.]
H5. $x \in B$	[Vuln.]
H6. $\forall u.(d(x,u) < \eta[x] \Rightarrow u \in A)$	[Vuln.]
H7. $\forall v.(d(x,v) < \theta[x] \Rightarrow v \in B)$	[Vuln.]

L2 Done

23. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.

The pre-image of an open set U under a continuous function f is open.

Let x be an element of $f^{-1}(U)$. Then $f(x) \in U$. Therefore, since U is open, there exists $\eta > 0$ such that $u \in U$ whenever $d(f(x), u) < \eta$. We would like to find $\delta > 0$ s.t. $y \in f^{-1}(U)$ whenever $d(x, y) < \delta$. But $y \in f^{-1}(U)$ if and only if $f(y) \in U$. We know that $f(y) \in U$ whenever $d(f(x), f(y)) < \eta$. Since f is continuous, there exists $\theta > 0$ such that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \theta$. Therefore, setting $\delta = \theta$, we are done.

L1

H1. f is continuous
H2. U is open

T1. $f^{-1}(U)$ is open

1. Expand pre-universal target T1.

L1

H1. f is continuous
H2. U is open

T2. $\forall x.(x \in f^{-1}(U) \Rightarrow \exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U))))$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f^{-1}(U)$.

L1

x

H1. f is continuous
H2. U is open
H3. $x \in f^{-1}(U)$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$

3. Quantifier-free expansion of hypothesis H3.

Since $x \in f^{-1}(U)$, we have that $f(x) \in U$.

L1

x

H1. f is continuous
H2. U is open
H4. $f(x) \in U$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$

4. Forwards reasoning using H2 with H4.

Since U is open and $f(x) \in U$, there exists $\eta > 0$ such that $u \in U$ whenever $d(f(x), u) < \eta$.

L1

$x \ \eta[f(x)]$

H1. f is continuous
H2. U is open [Vuln.; Used with H4.]
H4. $f(x) \in U$ [Vuln.]
H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$

5. Deleted H4, as this unexpandable atomic statement is unmatchable.

L1

$x \ \eta[f(x)]$

H1. f is continuous
H2. U is open [Vuln.; Used with H4.]
H4. $f(x) \in U$ [Vuln.]
H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))$

6. Deleted H2, as the conclusion of this implicative statement is unmatchable.

L1

$x\ \eta[f(x)]$

H1. f is continuous

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$

T3. $\boldsymbol{\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow y \in f^{-1}(U)))}$

7. Unlock existential-universal-conditional target T3.

We would like to find $\delta > 0$ s.t. $y \in f^{-1}(U)$ whenever $d(x, y) < \delta$.

L1

$x\ \eta[f(x)]$

H1. f is continuous

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$

L2♦

$\delta^{\blacklozenge}[\overline{y}]\ y$

H6. $d(x, y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T4. $\boldsymbol{y \in f^{-1}(U)}$

8. Quantifier-free expansion of target T4.

But $y \in f^{-1}(U)$ if and only if $f(y) \in U$.

L1

$x\ \eta[f(x)]$

H1. f is continuous

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\boldsymbol{\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)}$

L2♦

$\delta^{\blacklozenge}[\overline{y}]\ y$

H6. $d(x, y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T5. $\boldsymbol{f(y) \in U}$

9. Backwards reasoning using H5 with T5.

We know that $f(y) \in U$ whenever $d(f(x), f(y)) < \eta$.

L1

$x\ \eta[f(x)]$

H1. f is continuous

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$ [Vuln.]

L2♦

$\delta^{\blacklozenge}[\overline{y}]\ y$

H6. $d(x, y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T6. $d(f(x), f(y)) < \eta[f(x)]$

10. Delete H5 as no other statement mentions U.

L1

$x\ \eta[f(x)]$

H1. f is continuous

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$ [Vuln.]

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y$

H6. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T6. $d(f(x), f(y)) < \eta[f(x)]$

11. Backwards reasoning using H1 with T6.

Since f is continuous, there exists $\theta > 0$ such that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \theta$.

L1

$x\ \eta[f(x)]$

H1. f is continuous [Vuln.]

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$ [Vuln.]

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y\ \theta[z, \epsilon]$

H6. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T7. $d(x, y) < \theta[x, \eta[f(x)]]$

12. Delete H1 as no other statement mentions f.

L1

$x\ \eta[f(x)]$

H1. f is continuous [Vuln.]

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$ [Vuln.]

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}]\ y\ \theta[z, \epsilon]$

H6. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T7. $d(x, y) < \theta[x, \eta[f(x)]]$

13. Hypothesis H6 matches target T7 after choosing $\delta^\blacklozenge[\overline{y}] = \theta[x, \eta[f(x)]]$, so L2$^\blacklozenge$ is done.

Therefore, setting $\delta = \theta$, we are done.

L1

$x\ \eta[f(x)]$

H1. f is continuous [Vuln.]

H2. U is open [Vuln.; Used with H4.]

H4. $f(x) \in U$ [Vuln.]

H5. $\forall u.(d(f(x), u) < \eta[f(x)] \Rightarrow u \in U)$ [Vuln.]

L2$^\blacklozenge$ Done

14. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.

If f and g are continuous functions, then $g \circ f$ is continuous.

Take x and $\epsilon > 0$. We would like to find $\delta > 0$ s.t. $d(g(f(x)), g(f(y))) < \epsilon$ whenever $d(x, y) < \delta$. Since g is continuous, there exists $\eta > 0$ such that $d(g(f(x)), g(f(y))) < \epsilon$ whenever $d(f(x), f(y)) < \eta$. Since f is continuous, there exists $\theta > 0$ such that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \theta$. Therefore, setting $\delta = \theta$, we are done.

L1

H1. f is continuous

H2. g is continuous

T1. $g \circ f$ is continuous

1. Expand pre-universal target T1.

L1

H1. f is continuous

H2. g is continuous

T2. $\forall x, \epsilon.(\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow d(g(f(x)), g(f(y))) < \epsilon)))$

2. pply 'let' trick and move premise of universal target T2 above the line.

Take x and $\epsilon > 0$.

L1

$x \ \epsilon$

H1. f is continuous

H2. g is continuous

T3. $\exists \delta.(\forall y.(d(x, y) < \delta \Rightarrow d(g(f(x)), g(f(y))) < \epsilon))$

3. Unlock existential-universal-conditional target T3.

We would like to find $\delta > 0$ s.t. $d(g(f(x)), g(f(y))) < \epsilon$ whenever $d(x, y) < \delta$.

L1

$x \ \epsilon$

H1. f is continuous

H2. g is continuous

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}] \ y$

H3. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T4. $d(g(f(x)), g(f(y))) < \epsilon$

4. Backwards reasoning using H2 with T4.

Since g is continuous, there exists $\eta > 0$ such that $d(g(f(x)), g(f(y))) < \epsilon$ whenever $d(f(x), f(y)) < \eta$.

L1

$x \ \epsilon$

H1. f is continuous

H2. g is continuous [Vuln.]

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}] \ y \ \eta[z, \theta]$

H3. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T5. $d(f(x), f(y)) < \eta[f(x), \epsilon]$

5. Delete H2 as no other statement mentions g.

L1

$x \ \epsilon$

H1. f is continuous

H2. g is continuous [Vuln.]

L2$^\blacklozenge$

$\delta^\blacklozenge[\overline{y}] \ y \ \eta[z, \theta]$

H3. $d(x, y) < \delta^\blacklozenge[\overline{y}]$ [From L1.]

T5. $d(f(x), f(y)) < \eta[f(x), \epsilon]$

6. Backwards reasoning using H1 with T5.

Since f is continuous, there exists $\theta > 0$ such that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \theta$.

L1

$x\ \epsilon$

H1. f is continuous [Vuln.]

H2. g is continuous [Vuln.]

L2$^{\blacklozenge}$

$\delta^{\blacklozenge}[\overline{y}]\ y\ \eta[z, \theta]\ \theta[z, \alpha]$

H3. $d(x, y) < \delta^{\blacklozenge}[\overline{y}]$ [From L1.]

T6. $d(x, y) < \theta[x, \eta[f(x), \epsilon]]$

7. Delete H1 as no other statement mentions f.

L1

$x\ \epsilon$

H1. f is continuous [Vuln.]

H2. g is continuous [Vuln.]

L2$^{\blacklozenge}$

$\delta^{\blacklozenge}[\overline{y}]\ y\ \eta[z, \theta]\ \theta[z, \alpha]$

H3. $\boldsymbol{d(x, y) < \delta^{\blacklozenge}[\overline{y}]}$ **[From L1.]**

T6. $\boldsymbol{d(x, y) < \theta[x, \eta[f(x), \epsilon]]}$

8. Hypothesis H3 matches target T6 after choosing $\delta^{\blacklozenge}[\overline{y}] = \theta[x, \eta[f(x), \epsilon]]$, so L2$^{\blacklozenge}$ is done.

Therefore, setting $\delta = \theta$, we are done.

L1

$x\ \epsilon$

H1. f is continuous [Vuln.]

H2. g is continuous [Vuln.]

L2$^{\blacklozenge}$ Done

9. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.

If f is a continuous function and $(a_n) \to a$, then $(f(a_n)) \to f(a)$

Let $\epsilon > 0$. We would like to find N s.t. $d(f(a), f(a_n)) < \epsilon$ whenever $n \geqslant N$. Since f is continuous, there exists $\delta > 0$ such that $d(f(a), f(a_n)) < \epsilon$ whenever $d(a, a_n) < \delta$. Since $a_n \to a$, there exists N' such that $d(a, a_n) < \delta$ whenever $n \geqslant N'$. Therefore, setting $N = N'$, we are done.

L1
H1. f is continuous
H2. $a_n \to a$

T1. $f(a_n) \to f(a)$

1. Expand pre-universal target T1.

L1
H1. f is continuous
H2. $a_n \to a$

T2. $\forall \epsilon.(\exists N.(\forall n.(n \geqslant N \Rightarrow d(f(a), f(a_n)) < \epsilon)))$

2. pply 'let' trick and move premise of universal target T2 above the line.

Let $\epsilon > 0$.

L1
ϵ
H1. f is continuous
H2. $a_n \to a$

T3. $\exists N.(\forall n.(n \geqslant N \Rightarrow d(f(a), f(a_n)) < \epsilon))$

3. Unlock existential-universal-conditional target T3.

We would like to find N s.t. $d(f(a), f(a_n)) < \epsilon$ whenever $n \geqslant N$.

L1
ϵ
H1. f is continuous
H2. $a_n \to a$

L2$^\blacklozenge$
$N^\blacklozenge[\overline{n}]$ n
H3. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T4. $d(f(a), f(a_n)) < \epsilon$

4. Backwards reasoning using H1 with T4.

Since f is continuous, there exists $\delta > 0$ such that $d(f(a), f(a_n)) < \epsilon$ whenever $d(a, a_n) < \delta$.

L1
ϵ
H1. f is continuous [Vuln.]
H2. $a_n \to a$

L2$^\blacklozenge$
$N^\blacklozenge[\overline{n}]$ n $\delta[x, \eta]$
H3. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T5. $d(a, a_n) < \delta[a, \epsilon]$

5. Delete H1 as no other statement mentions f.

L1
ϵ
H1. f is continuous [Vuln.]
H2. $a_n \to a$

L2$^\blacklozenge$
$N^\blacklozenge[\overline{n}]$ n $\delta[x, \eta]$
H3. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T5. $d(a, a_n) < \delta[a, \epsilon]$

6. Backwards reasoning using H2 with T5.

L1

ϵ

H1. f is continuous [Vuln.]

H2. $a_n \to a$ [Vuln.]

L2♦

$N^\blacklozenge[\overline{n}]$ n $\delta[x,\eta]$ $N'[\eta]$

H3. $n \geqslant N^\blacklozenge[\overline{n}]$ [From L1.]

T6. $n \geqslant N'[\delta[a,\epsilon]]$

Since $a_n \to a$, there exists N' such that $d(a, a_n) < \delta$ whenever $n \geqslant N'$.

7. Delete H2 as no other statement mentions a.

L1

ϵ

H1. f is continuous [Vuln.]

H2. $a_n \to a$ [Vuln.]

L2♦

$N^\blacklozenge[\overline{n}]$ n $\delta[x,\eta]$ $N'[\eta]$

H3. $\boldsymbol{n \geqslant N^\blacklozenge[\overline{n}]}$ **[From L1.]**

T6. $\boldsymbol{n \geqslant N'[\delta[a,\epsilon]]}$

8. Hypothesis H3 matches target T6 after choosing $N^\blacklozenge[\overline{n}] = N'[\delta[a,\epsilon]]$, so L2♦ is done.

L1

ϵ

H1. f is continuous [Vuln.]

H2. $a_n \to a$ [Vuln.]

L2♦ Done

Therefore, setting $N = N'$, we are done.

9. All targets of L1 are 'Done', so L1 is itself done.

L1 Done

Problem solved.

A closed subset A of a complete metric space X is complete.

Let (a_n) be a Cauchy sequence in A. Then, since X is complete, we have that (a_n) converges. That is, there exists a such that $a_n \to a$. Since A is closed in X, (a_n) is a sequence in A and $a_n \to a$, we have that $a \in A$. Thus (a_n) converges in A and we are done.

L1

H1. X is complete
H2. A is closed in X

T1. A is a complete space

1. Expand pre-universal target T1.

L1

H1. X is complete
H2. A is closed in X

T2. $\forall(a_n).((a_n)$ is Cauchy $\wedge$ (a_n) is a sequence in $A \Rightarrow (a_n)$ converges in $A)$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let (a_n) be a Cauchy sequence in A.

L1

(a_n)

H1. X is complete
H2. A is closed in X
H3. (a_n) is Cauchy
H4. (a_n) is a sequence in A

T3. (a_n) converges in A

3. Forwards reasoning using H1 with H3.

Since X is complete and (a_n) is Cauchy, we have that (a_n) converges.

L1

(a_n)

H1. X is complete [Vuln.; Used with H3.]
H2. A is closed in X
H3. (a_n) is Cauchy [Vuln.]
H4. (a_n) is a sequence in A
H5. (a_n) converges

T3. (a_n) converges in A

4. Deleted H1, as the conclusion of this implicative statement is unmatchable.

L1

(a_n)

H1. X is complete [Vuln.; Used with H3.]
H2. A is closed in X
H3. (a_n) is Cauchy [Vuln.]
H4. (a_n) is a sequence in A
H5. (a_n) converges

T3. (a_n) converges in A

5. Expand pre-existential hypothesis H5.

By definition, since (a_n) converges, there exists a such that $a_n \to a$.

L1

(a_n) a

H1. X is complete	[Vuln.; Used with H3.]
H2. A is closed in X	
H3. (a_n) is Cauchy	[Vuln.]
H4. (a_n) is a sequence in A	
H6. $a_n \to a$	

T3. (a_n) converges in A

6. Forwards reasoning using library result "a closed set contains its limit points" with (H2,H4,H6).

Since A is closed in X, (a_n) is a sequence in A and $a_n \to a$, we have that $a \in A$.

L1

(a_n) a

H1. X is complete	[Vuln.; Used with H3.]
H2. A is closed in X	[Vuln.]
H3. (a_n) is Cauchy	[Vuln.]
H4. (a_n) is a sequence in A	[Vuln.]
H6. $a_n \to a$	[Vuln.]
H7. $a \in A$	

T3. (a_n) converges in A

7. Delete H2 as no other statement mentions X.

L1

(a_n) a

H1. X is complete	[Vuln.; Used with H3.]
H2. A is closed in X	[Vuln.]
H3. (a_n) is Cauchy	[Vuln.]
H4. (a_n) is a sequence in A	[Vuln.]
H6. $a_n \to a$	**[Vuln.]**
H7. $a \in A$	

T3. (a_n) converges in A

8. All conjuncts of T3 (after expansion) can be simultaneously matched against H7 and H6 or rendered trivial by choosing $z = a$, so L1 is done.

We would like to show that (a_n) converges in A. But this is clearly the case, so we are done.

L1 Done

Problem solved.

$$f(f^{-1}(A)) \subset A$$

Let x be an element of $f(f^{-1}(A))$. Then there exists $y \in f^{-1}(A)$ such that $f(y) = x$. Since $y \in f^{-1}(A)$, we have that $f(y) \in A$. Since $f(y) = x$, we have that $x \in A$ and we are done.

L1

T1. $\boldsymbol{f(f^{-1}(A)) \subset A}$

1. Expand pre-universal target T1.

L1

T2. $\boldsymbol{\forall x.(x \in f(f^{-1}(A)) \Rightarrow x \in A)}$

2. Apply 'let' trick and move premise of universal-conditional target T2 above the line.

Let x be an element of $f(f^{-1}(A))$.

L1

x

H1. $\boldsymbol{x \in f(f^{-1}(A))}$

T3. $x \in A$

3. Expand pre-existential hypothesis H1.

By definition, since $x \in f(f^{-1}(A))$, there exists $y \in f^{-1}(A)$ such that $f(y) = x$.

L1

$x \; y$

H2. $\boldsymbol{y \in f^{-1}(A)}$

H3. $f(y) = x$

T3. $x \in A$

4. Quantifier-free expansion of hypothesis H2.

Since $y \in f^{-1}(A)$, we have that $f(y) \in A$.

L1

$x \; y$

H4. $f(y) \in A$

H3. $\boldsymbol{f(y) = x}$

T3. $x \in A$

5. Rewrite $f(y)$ as x throughout the tableau using hypothesis H3.

Since $f(y) = x$, we have that $x \in A$.

L1

$x \; y$

H5. $\boldsymbol{x \in A}$

T3. $\boldsymbol{x \in A}$

6. Hypothesis H5 matches target T3, so L1 is done.

We are done.

L1 Done

Problem solved.

$$A \subset f^{-1}(f(A))$$

Let x be an element of A. We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$. But this is clearly the case, so we are done.

L1

T1. $\boldsymbol{A \subset f^{-1}(f(A))}$

1. Expand pre-universal target T1.

L1

T2. $\boldsymbol{\forall x.(x \in A \Rightarrow x \in f^{-1}(f(A)))}$

2. Apply ‘let’ trick and move premise of universal-conditional target T2 above the line.

Let x be an element of A.

L1

x

H1. $x \in A$

T3. $\boldsymbol{x \in f^{-1}(f(A))}$

3. Quantifier-free expansion of target T3.

We would like to show that $x \in f^{-1}(f(A))$, i.e. that $f(x) \in f(A)$.

L1

x

H1. $\boldsymbol{x \in A}$

T4. $\boldsymbol{f(x) \in f(A)}$

4. All conjuncts of T4 (after expansion) can be simultaneously matched against H1 or rendered trivial by choosing $y = x$, so L1 is done.

We would like to show that $f(x) \in f(A)$. But this is clearly the case, so we are done.

L1 Done

Problem solved.

The intersection of two subgroups is a subgroup

Let x and y be such that $x \in H \cap K$ and $y \in H \cap K$. Since H is a subgroup, H is closed under taking inverses, $e \in H$ and H is closed under multiplication. Since K is a subgroup, K is closed under taking inverses, $e \in K$ and K is closed under multiplication. Since $x \in H \cap K$, $x \in H$ and $x \in K$. Since $y \in H \cap K$, $y \in H$ and $y \in K$. Therefore, since H is closed under multiplication, we have that $xy \in H$ and since K is closed under multiplication and $x \in K$, we have that $xy \in K$. We would like to show that $xy \in H \cap K$, i.e. that $xy \in H$ and $xy \in K$ and we are done.

L1

H1. H is a subgroup
H2. K is a subgroup

T1. $\boldsymbol{\forall x, y.(x \in H \cap K \wedge y \in H \cap K \Rightarrow xy \in H \cap K)}$

1. Apply 'let' trick and move premise of universal-conditional target T1 above the line.

Let x and y be such that $x \in H \cap K$ and $y \in H \cap K$.

L1

x y

H1. $\boldsymbol{H}$ is a subgroup
H2. K is a subgroup
H3. $x \in H \cap K$
H4. $y \in H \cap K$

T2. $xy \in H \cap K$

2. Quantifier-free expansion of hypothesis H1.

Since H is a subgroup, H is closed under taking inverses, $e \in H$ and H is closed under multiplication.

L1

x y

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication
H2. $\boldsymbol{K}$ is a subgroup
H3. $x \in H \cap K$
H4. $y \in H \cap K$

T2. $xy \in H \cap K$

3. Quantifier-free expansion of hypothesis H2.

Since K is a subgroup, K is closed under taking inverses, $e \in K$ and K is closed under multiplication.

L1

x y

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication
H3. $\boldsymbol{x \in H \cap K}$
H4. $y \in H \cap K$

T2. $xy \in H \cap K$

4. Quantifier-free expansion of hypothesis H3.

Since $x \in H \cap K$, $x \in H$ and $x \in K$.

L1

$x\ y$

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication
H11. $x \in H$
H12. $x \in K$
H4. $\boldsymbol{y \in H \cap K}$

T2. $xy \in H \cap K$

5. Quantifier-free expansion of hypothesis H4.

Since $y \in H \cap K$, $y \in H$ and $y \in K$.

L1

$x\ y$

H5. H is closed under taking inverses
H6. $e \in H$
H7. $\boldsymbol{H}$ is closed under multiplication
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication
H11. $\boldsymbol{x \in H}$
H12. $x \in K$
H13. $\boldsymbol{y \in H}$
H14. $y \in K$

T2. $xy \in H \cap K$

6. Forwards reasoning using library result “” with (H7,H11,H13).

Since H is closed under multiplication, $x \in H$ and $y \in H$, we have that $xy \in H$.

L1

$x\ y$

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication [Vuln.]
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication
H11. $x \in H$ [Vuln.]
H12. $x \in K$
H13. $y \in H$ [Vuln.]
H14. $y \in K$
H15. $xy \in H$

T2. $xy \in H \cap K$

7. Deleted H7, as this unexpandable atomic statement is unmatchable.

L1

x y

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication [Vuln.]
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication
H11. $x \in H$ [Vuln.]
H12. $x \in K$
H13. $y \in H$ [Vuln.]
H14. $y \in K$
H15. $xy \in H$

T2. $xy \in H \cap K$

8. Deleted H11, as this unexpandable atomic statement is unmatchable.

L1

x y

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication [Vuln.]
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication
H11. $x \in H$ [Vuln.]
H12. $x \in K$
H13. $y \in H$ [Vuln.]
H14. $y \in K$
H15. $xy \in H$

T2. $xy \in H \cap K$

9. Deleted H13, as this unexpandable atomic statement is unmatchable.

L1

x y

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication [Vuln.]
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication
H11. $x \in H$ [Vuln.]
H12. $x \in K$
H13. $y \in H$ [Vuln.]
H14. $y \in K$
H15. $xy \in H$

T2. $xy \in H \cap K$

10. Forwards reasoning using library result “” with (H10,H12,H14).

Since K is closed under multiplication, $x \in K$ and $y \in K$, we have that $xy \in K$.

L1

$x \ y$

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication [Vuln.]
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication [Vuln.]
H11. $x \in H$ [Vuln.]
H12. $x \in K$ [Vuln.]
H13. $y \in H$ [Vuln.]
H14. $y \in K$ [Vuln.]
H15. $xy \in H$
H16. $xy \in K$

T2. $xy \in H \cap K$

11. Deleted H10, as this unexpandable atomic statement is unmatchable.

L1

$x \ y$

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication [Vuln.]
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication [Vuln.]
H11. $x \in H$ [Vuln.]
H12. $x \in K$ [Vuln.]
H13. $y \in H$ [Vuln.]
H14. $y \in K$ [Vuln.]
H15. $xy \in H$
H16. $xy \in K$

T2. $xy \in H \cap K$

12. Deleted H12, as this unexpandable atomic statement is unmatchable.

L1

$x \ y$

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication [Vuln.]
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication [Vuln.]
H11. $x \in H$ [Vuln.]
H12. $x \in K$ [Vuln.]
H13. $y \in H$ [Vuln.]
H14. $y \in K$ [Vuln.]
H15. $xy \in H$
H16. $xy \in K$

T2. $xy \in H \cap K$

13. Deleted H14, as this unexpandable atomic statement is unmatchable.

L1

$x\ y$

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication [Vuln.]
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication [Vuln.]
H11. $x \in H$ [Vuln.]
H12. $x \in K$ [Vuln.]
H13. $y \in H$ [Vuln.]
H14. $y \in K$ [Vuln.]
H15. $xy \in H$
H16. $xy \in K$

T2. $\boldsymbol{xy \in H \cap K}$

14. Quantifier-free expansion of target T2.

We would like to show that $xy \in H \cap K$, i.e. that $xy \in H$ and $xy \in K$.

L1

$x\ y$

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication [Vuln.]
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication [Vuln.]
H11. $x \in H$ [Vuln.]
H12. $x \in K$ [Vuln.]
H13. $y \in H$ [Vuln.]
H14. $y \in K$ [Vuln.]
H15. $\boldsymbol{xy \in H}$
H16. $xy \in K$

T3. $\boldsymbol{xy \in H}$
T4. $xy \in K$

15. Hypothesis H15 matches target T3, so we can remove T3.

L1

$x\ y$

H5. H is closed under taking inverses
H6. $e \in H$
H7. H is closed under multiplication [Vuln.]
H8. K is closed under taking inverses
H9. $e \in K$
H10. K is closed under multiplication [Vuln.]
H11. $x \in H$ [Vuln.]
H12. $x \in K$ [Vuln.]
H13. $y \in H$ [Vuln.]
H14. $y \in K$ [Vuln.]
H15. $xy \in H$
H16. $\boldsymbol{xy \in K}$

T4. $\boldsymbol{xy \in K}$

16. Hypothesis H16 matches target T4, so L1 is done.

We are done.

L1 Done

Problem solved.