

**If  $f$  is an injection then  $f(A) \cap f(B) \subset f(A \cap B)$**

Let  $x$  be an element of  $f(A) \cap f(B)$ . Then  $x \in f(A)$  and  $x \in f(B)$ . That is, there exists  $y \in A$  such that  $f(y) = x$  and there exists  $z \in B$  such that  $f(z) = x$ . Since  $f$  is an injection,  $f(y) = x$  and  $f(z) = x$ , we have that  $y = z$ . We would like to find  $u \in A \cap B$  s.t.  $f(u) = x$ . But  $u \in A \cap B$  if and only if  $u \in A$  and  $u \in B$ . Since  $y = z$ , we have that  $y \in B$ . Therefore, setting  $u = y$ , we are done.

**If  $g, f$  are injections then  $(g \circ f)$  is an injection.**

Let  $x, y$  and  $z$  be such that  $g(f(x)) = z$  and  $g(f(y)) = z$ . Then, since  $g$  is an injection, we have that  $f(x) = f(y)$ . Therefore, since  $f$  is an injection,  $x = y$  if  $f(x) = f(y)$ . Since  $g$  is an injection and  $g(f(y)) = z$ ,  $f(y) = f(y)$  if  $g(f(y)) = z$ . But this is clearly the case, so we are done.

**Prove that  $A \subseteq f^{-1}(f(A))$**

Let  $x$  be an element of  $A$ . We would like to show that  $x \in f^{-1}(f(A))$ , i.e. that  $f(x) \in f(A)$ . But this is clearly the case, so we are done.

**Prove that  $f(f^{-1}(A)) \subset A$**

Let  $x$  be an element of  $f(f^{-1}(A))$ . Then there exists  $y \in f^{-1}(A)$  such that  $f(y) = x$ . Since  $y \in f^{-1}(A)$ , we have that  $f(y) \in A$ . Since  $f(y) = x$ , we have that  $x \in A$ . But this is clearly the case, so we are done.

**Prove that  $f(A \cap B) \subseteq f(A) \cap f(B)$**

By definition, since  $y \in f(A \cap B)$ , there exists  $z \in A \cap B$  such that  $f(z) = y$ . Since  $z \in A \cap B$ ,  $z \in A$  and  $z \in B$ . We would like to show that  $y \in f(A) \cap f(B)$ , i.e. that  $y \in f(A)$  and  $y \in f(B)$ . We would like to show that  $y \in f(A)$ . But this is clearly the case, so we are done. Thus  $y \in f(B)$  and we are done.

**Prove that  $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$**

Since  $x \in f^{-1}(A \cap B)$ , we have that  $f(x) \in A \cap B$ . Then  $f(x) \in A$  and  $f(x) \in B$ . We would like to show that  $x \in f^{-1}(A) \cap f^{-1}(B)$ , i.e. that  $x \in f^{-1}(A)$  and  $x \in f^{-1}(B)$ . We would like to show that  $x \in f^{-1}(A)$ , i.e. that  $f(x) \in A$ . We would like to show that  $x \in f^{-1}(B)$ , i.e. that  $f(x) \in B$ . But this is clearly the case, so we are done.

**Prove that  $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$**

Let  $x$  be an element of  $f^{-1}(A) \cap f^{-1}(B)$ . Then  $x \in f^{-1}(A)$  and  $x \in f^{-1}(B)$ . Then  $f(x) \in A$  and  $f(x) \in B$ . We would like to show that  $x \in f^{-1}(A \cap B)$ , i.e. that  $f(x) \in A \cap B$ . We would like to show that  $f(x) \in A \cap B$ , i.e. that  $f(x) \in A$  and  $f(x) \in B$ . But this is clearly the case, so we are done.

**Prove that  $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$**

Let  $x$  be an element of  $f^{-1}(A \cup B)$ . Then  $f(x) \in A \cup B$ . Then  $f(x) \in A$  or  $f(x) \in B$ . We would like to show that  $x \in f^{-1}(A) \cup f^{-1}(B)$ , i.e. that  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$ . We would like to show that  $x \in f^{-1}(A)$ , i.e. that  $f(x) \in A$ . But this is clearly the case, so we are done. We would like to show that  $x \in f^{-1}(A) \cup f^{-1}(B)$ , i.e. that  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$ . We would like to show that  $x \in f^{-1}(A)$ , i.e. that  $f(x) \in A$ . We would like to show that  $x \in f^{-1}(B)$ , i.e. that  $f(x) \in B$ . But this is clearly the case, so we are done.

**Prove that  $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$**

Let  $x$  be an element of  $f^{-1}(A) \cup f^{-1}(B)$ . Then  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$ . Since  $x \in f^{-1}(A)$ , we have that  $f(x) \in A$ . Since  $x \in f^{-1}(B)$ , we have that  $f(x) \in B$ . We would like to show that  $x \in f^{-1}(A \cup B)$ , i.e. that  $f(x) \in A \cup B$ . We would like to show that  $f(x) \in A \cup B$ , i.e. that  $f(x) \in A$  or  $f(x) \in B$ . But this is clearly the case, so we are done. We would like to show that  $x \in f^{-1}(A \cup B)$ , i.e. that  $f(x) \in A \cup B$ . We would like to show that  $f(x) \in A \cup B$ , i.e. that  $f(x) \in A$  or  $f(x) \in B$ . But this is clearly the case, so we are done.

**If  $A$  and  $B$  are open sets, then  $A \cup B$  is also open.**

Let  $x$  be an element of  $A \cup B$ . Then  $x \in A$  or  $x \in B$ . Since  $A$  is open and  $x \in A$ , there exists  $\alpha > 0$  such that  $w \in A$  whenever  $d(x, w) < \alpha$ . Since  $B$  is open and  $x \in B$ , there exists  $\beta > 0$  such that  $p \in B$  whenever  $d(x, p) < \beta$ . We would like to find  $\eta > 0$  s.t.  $z \in A \cup B$  whenever  $d(x, z) < \eta$ . But  $z \in A \cup B$  if and only if  $z \in A$  or  $z \in B$ . We know that  $z \in A$  if  $d(x, z) < \alpha$ . Therefore, setting  $\eta = \alpha$ , we are done. We would like to find  $\theta > 0$  s.t.  $u \in A \cup B$  whenever  $d(x, u) < \theta$ . But  $u \in A \cup B$  if and only if  $u \in A$  or  $u \in B$ . We know that  $u \in B$  if  $d(x, u) < \beta$ . Therefore, setting  $\theta = \beta$ , we are done.

**If  $A, B,$  and  $C$  are open sets, then  $A \cup (B \cup C)$  is also open.**

Let  $x$  be an element of  $A \cup B \cup C$ . Then  $x \in A$  or  $x \in B \cup C$ . Since  $A$  is open and  $x \in A$ , there exists  $\alpha > 0$  such that  $w \in A$  whenever  $d(x, w) < \alpha$ . Since  $x \in B \cup C$ ,  $x \in B$  or  $x \in C$ . Since  $B$  is open and  $x \in B$ , there exists  $\delta' > 0$  such that  $r \in B$  whenever  $d(x, r) < \delta'$ . Since  $C$  is open and  $x \in C$ , there exists  $\delta'' > 0$  such that  $s \in C$  whenever  $d(x, s) < \delta''$ . We would like to find  $\eta > 0$  s.t.  $z \in A \cup B \cup C$  whenever  $d(x, z) < \eta$ . But  $z \in A \cup B \cup C$  if and only if  $z \in A$  or  $z \in B \cup C$ . We know that  $z \in A$  if  $d(x, z) < \alpha$ . Therefore, setting  $\eta = \alpha$ , we are done. We would like to find  $\beta > 0$  s.t.  $v \in A \cup B \cup C$  whenever  $d(x, v) < \beta$ . But  $v \in A \cup B \cup C$  if and only if  $v \in A$  or  $v \in B \cup C$ . We would like to show that  $v \in B \cup C$ , i.e. that  $v \in B$  or  $v \in C$ . We know that  $v \in B$  if  $d(x, v) < \delta'$ . Therefore, setting  $\beta = \delta'$ , we are done. We would like to find  $\gamma > 0$  s.t.  $p \in A \cup B \cup C$  whenever  $d(x, p) < \gamma$ . But  $p \in A \cup B \cup C$  if and only if  $p \in A$  or  $p \in B \cup C$ . We would like to show that  $p \in B \cup C$ , i.e. that  $p \in B$  or  $p \in C$ . We know that  $p \in C$  if  $d(x, p) < \delta''$ . Therefore, setting  $\gamma = \delta''$ , we are done.

**If  $A$  and  $B$  are open sets, then  $A \cap B$  is also open.**

Let  $x$  be an element of  $A \cap B$ . Then  $x \in A$  and  $x \in B$ . Therefore, since  $A$  is open, there exists  $\eta > 0$  such that  $u \in A$  whenever  $d(x, u) < \eta$  and since  $B$  is open, there exists  $\theta > 0$  such that  $v \in B$  whenever  $d(x, v) < \theta$ . We would like to find  $\delta > 0$  s.t.  $y \in A \cap B$  whenever  $d(x, y) < \delta$ . But  $y \in A \cap B$  if and only if  $y \in A$  and  $y \in B$ . We know that  $y \in A$  whenever  $d(x, y) < \eta$  and that  $y \in B$  whenever  $d(x, y) < \theta$ . Assume now that  $d(x, y) < \delta$ . Then  $d(x, y) < \eta$  if  $\delta \leq \eta$  and  $d(x, y) < \theta$  if  $\delta \leq \theta$ . We may therefore take  $\delta = \min(\eta, \theta)$  and we are done.

**If  $A$  and  $B$  are closed sets, then  $A \cap B$  is also closed.**

Let  $(a_n)$  and  $a$  be such that  $(a_n)$  is a sequence in  $A \cap B$  and  $a_n \rightarrow a$ . Then  $(a_n)$  is a sequence in  $A$  and  $(a_n)$  is a sequence in  $B$ . Therefore, since  $A$  is closed and  $a_n \rightarrow a$ , we have that  $a \in A$  and since  $B$  is closed and  $a_n \rightarrow a$ , we have that  $a \in B$ . We would like to show that  $a \in A \cap B$ , i.e. that  $a \in A$  and  $a \in B$ . But this is clearly the case, so we are done.

**The pre-image of a closed set  $U$  under a continuous function  $f$  is closed.**

Let  $(a_n)$  and  $a$  be such that  $(a_n)$  is a sequence in  $f^{-1}(U)$  and  $a_n \rightarrow a$ . Then  $f(a_n)$  is a sequence in  $U$ . We would like to show that  $a \in f^{-1}(U)$ , i.e. that  $f(a) \in U$  and since  $U$  is closed,  $f(a) \in U$  if  $f(a_n) \rightarrow f(a)$ . Let  $\epsilon > 0$ . We would like to find  $N$  s.t.  $d(f(a), f(a_n)) < \epsilon$  whenever  $n \geq N$ . Since  $f$  is continuous, there exists  $\delta > 0$  such that  $d(f(a), f(a_n)) < \epsilon$  whenever  $d(a, a_n) < \delta$ . Since  $a_n \rightarrow a$ , there exists  $N'$  such that  $d(a, a_n) < \delta$  whenever  $n \geq N'$ . Therefore, setting  $N = N'$ , we are done.

**The pre-image of an open set  $U$  under a continuous function  $f$  is open.**

Let  $x$  be an element of  $f^{-1}(U)$ . Then  $f(x) \in U$ . Therefore, since  $U$  is open, there exists  $\eta > 0$  such that  $u \in U$  whenever  $d(f(x), u) < \eta$ . We would like to find  $\delta > 0$  s.t.  $y \in f^{-1}(U)$  whenever  $d(x, y) < \delta$ . But  $y \in f^{-1}(U)$  if and only if  $f(y) \in U$ . We know that  $f(y) \in U$  whenever  $d(f(x), f(y)) < \eta$ . Since  $f$  is continuous, there exists  $\theta > 0$  such that  $d(f(x), f(y)) < \eta$  whenever  $d(x, y) < \theta$ . Therefore, setting  $\delta = \theta$ , we are done.

**If  $f$  and  $g$  are continuous functions, then  $g \circ f$  is continuous.**

Take  $x$  and  $\epsilon > 0$ . We would like to find  $\delta > 0$  s.t.  $d(g(f(x)), g(f(y))) < \epsilon$  whenever  $d(x, y) < \delta$ . Since  $g$  is continuous, there exists  $\eta > 0$  such that  $d(g(f(x)), g(f(y))) < \epsilon$  whenever  $d(f(x), f(y)) < \eta$ . Since  $f$  is continuous, there exists  $\theta > 0$  such that  $d(f(x), f(y)) < \eta$  whenever  $d(x, y) < \theta$ . Therefore, setting  $\delta = \theta$ , we are done.

**If  $f$  is a continuous function and  $(a_n) \rightarrow a$ , then  $(f(a_n)) \rightarrow f(a)$**

Let  $\epsilon > 0$ . We would like to find  $N$  s.t.  $d(f(a), f(a_n)) < \epsilon$  whenever  $n \geq N$ . Since  $f$  is continuous, there exists  $\delta > 0$  such that  $d(f(a), f(a_n)) < \epsilon$  whenever  $d(a, a_n) < \delta$ . Since  $a_n \rightarrow a$ , there exists  $N'$  such that  $d(a, a_n) < \delta$  whenever  $n \geq N'$ . Therefore, setting  $N = N'$ , we are done.

**Prove that  $A^c \cap B^c \subseteq (A \cup B)^c$ .**

Let  $x$  be an element of  $(A \cup B)^c$ . Then  $x \notin A \cup B$  and  $x \notin A$ . Then it is not that case that  $x \in A$ . Since  $x \notin A \cup B$ , we have that  $x \notin B$ . Then it is not that case that  $x \in B$ . Since  $x \notin A \cup B$ , it is not that case that  $x \in A \cup B$ . We would like to show that  $x \in (A \cup B)^c$ , i.e. that  $x \notin A \cup B$ . We would like to show that  $x \notin A \cup B$ , i.e. that it is not that case that  $x \in A \cup B$ . But this is clearly the case, so we are done.

**If  $g, f$  are surjections then  $(g \circ f)$  is a surjection.**

Let  $y$  be an element of  $C$ . Then, since  $g$  from  $B$  to  $C$  is a surjection, there exists  $u \in B$  such that  $g(u) = y$  and  $g(u) \in C$ . Since  $f$  from  $A$  to  $B$  is a surjection and  $u \in B$ , there exists  $v \in A$  such that  $f(v) = u$  and  $f(v) \in B$ . We would like to find  $x \in A$  s.t.  $g(f(x)) = y$  and  $g(f(x)) \in C$ .

**If  $f$  is a surjection then  $f(A)^c \subset f(A^c)$**

Let  $x$  be an element of  $(f(A))^c$ . Then  $x \notin f(A)$ . Then it is not that case that  $x \in f(A)$ . We would like to find  $y \in (A)^c$  s.t.  $f(y) = x$ . But  $y \in (A)^c$  if and only if  $y \notin A$ . But  $y \notin A$  if and only if it is not that case that  $y \in A$ .

**Prove that  $(A \cap B)^c \subset A^c \cup B^c$**

Let  $x$  be an element of  $(A \cap B)^c$ . Then  $x \notin A \cap B$ . Then it is not that case that  $x \in A \cap B$ . We would like to show that  $x \in (A)^c \cup (B)^c$ , i.e. that  $x \in (A)^c$  or  $x \in (B)^c$ . We would like to show that  $x \in (A)^c$ , i.e. that  $x \notin A$ . We would like to show that  $x \notin A$ , i.e. that it is not that case that  $x \in A$ . We would like to show that  $x \in (B)^c$ , i.e. that  $x \notin B$ . We would like to show that  $x \notin B$ , i.e. that it is not that case that  $x \in B$ .

**Prove that  $A^c \cup B^c \subset (A \cap B)^c$**

Let  $x$  be an element of  $(A)^c \cup (B)^c$ . Then  $x \in (A)^c$  or  $x \in (B)^c$ . Since  $x \in (A)^c$ , we have that  $x \notin A$ . Then it is not that case that  $x \in A$ . Since  $x \in (B)^c$ , we have that  $x \notin B$ . Then it is not that case that  $x \in B$ . We would like to show that  $x \in (A \cap B)^c$ , i.e. that  $x \notin A \cap B$ . We would like to show that  $x \notin A \cap B$ , i.e. that it is not that case that  $x \in A \cap B$ . We would like to show that  $x \in (A \cap B)^c$ , i.e. that  $x \notin A \cap B$ . We would like to show that  $x \notin A \cap B$ , i.e. that it is not that case that  $x \in A \cap B$ .

**Prove that  $(A \cup B)^c = A^c \cap B^c$**

We would like to show that  $(A \cup B)^c \subset (A)^c \cap (B)^c$ , i.e. that  $(A \cup B)^c \subset (A)^c$  and  $(A \cup B)^c \subset (B)^c$ .

**If  $A, B,$  and  $C$  are open sets, then  $A \cap (B \cap C)$  is also open.**

Let  $x$  be an element of  $A \cap B \cap C$ . Then  $x \in A$  and  $x \in B \cap C$ . Therefore, since  $A$  is open, there exists  $\eta > 0$  such that  $u \in A$  whenever  $d(x, u) < \eta$  and  $x \in B$  and  $x \in C$ . Therefore, since  $B$  is open, there exists  $\theta > 0$  such that  $v \in B$  whenever  $d(x, v) < \theta$  and since  $C$  is open, there exists  $\alpha > 0$  such that  $w \in C$  whenever  $d(x, w) < \alpha$ . We would like to find  $\delta > 0$  s.t.  $y \in A \cap B \cap C$  whenever  $d(x, y) < \delta$ . But  $y \in A \cap B \cap C$  if and only if  $y \in A$  and  $y \in B \cap C$ . We know that  $y \in A$  whenever  $d(x, y) < \eta$ . But  $y \in B \cap C$  if and only if  $y \in B$  and  $y \in C$ . We know that  $y \in B$  whenever  $d(x, y) < \theta$  and that  $y \in C$  whenever  $d(x, y) < \alpha$ . Assume now that  $d(x, y) < \delta$ . Then  $d(x, y) < \eta$  if  $\delta \leq \eta$ ,  $d(x, y) < \theta$  if  $\delta \leq \theta$  and  $d(x, y) < \alpha$  if  $\delta \leq \alpha$ .

**If  $A$  and  $B$  are closed sets, then  $A \cup B$  is also closed.**

Let  $(a_n)$  and  $a$  be such that  $(a_n)$  is a sequence in  $A \cup B$  and  $a_n \rightarrow a$ . We would like to show that  $a \in A \cup B$ , i.e. that  $a \in A$  or  $a \in B$ . Since  $A$  is closed and  $a_n \rightarrow a$ ,  $a \in A$  if  $(a_n)$  is a sequence in  $A$ . Since  $B$  is closed and  $a_n \rightarrow a$ ,  $a \in B$  if  $(a_n)$  is a sequence in  $B$ . Take  $n$ . Take  $n'$ .