CS 603: Minimum Spanning Trees: Prim and Kruskal's algorithms

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Outline

1. Minimum Spanning Trees
2. Kruskal's Algorithm
3. Prim's Algorithm

Minimum Spanning Trees

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  - Ethernet Cable
  - Electricity
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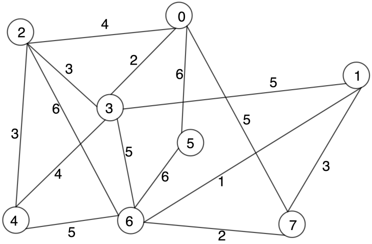
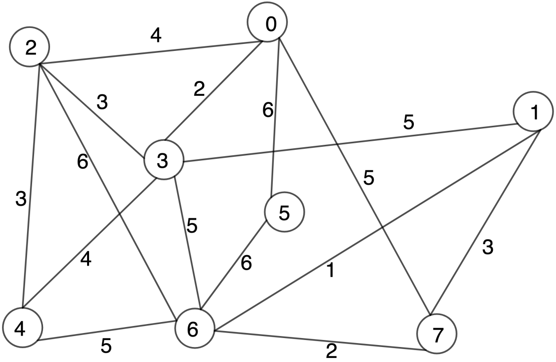
- There is a cost in connecting each location to each other.
- We want to find the minimum cost!



Claim: If the cost is non-negative, the min cost can always be achieved on a tree. (Why?)

Two Greedy Algorithms to find the Min Spanning Tree



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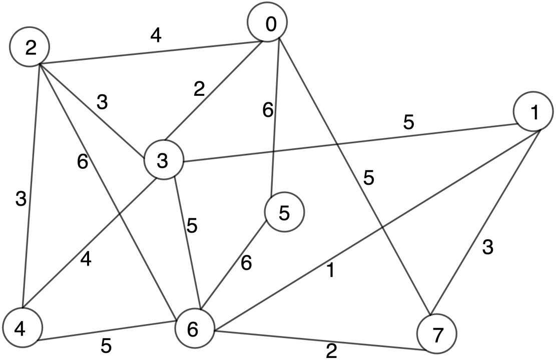

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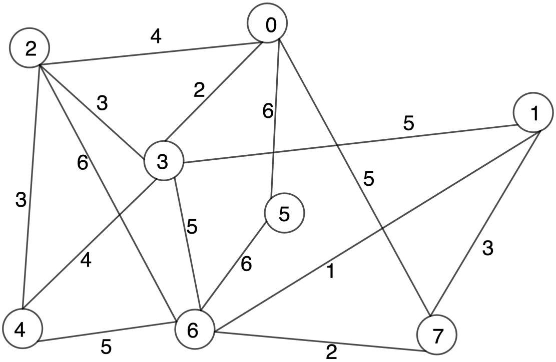

- Kruskal's Algorithm: add cheapest edges that don't create a cycle
- Prim's Algorithm: add cheapest edges from current sub-tree to new node

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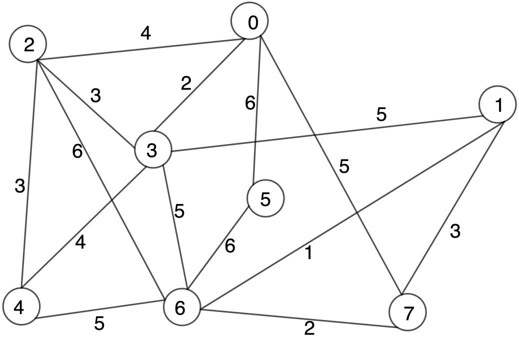

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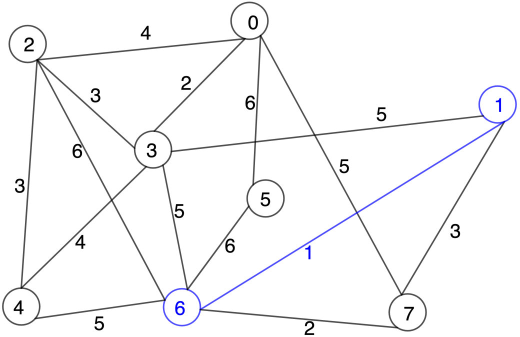

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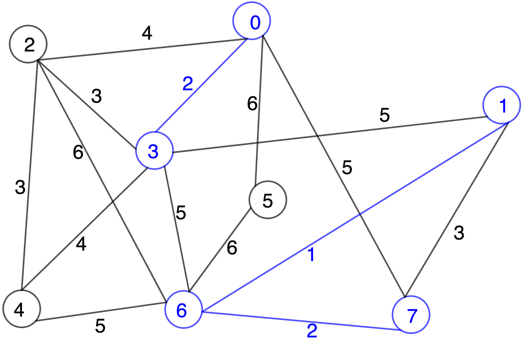

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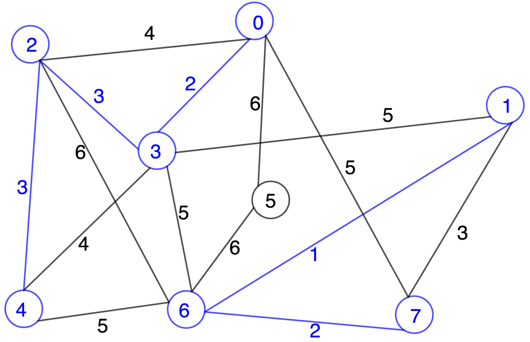

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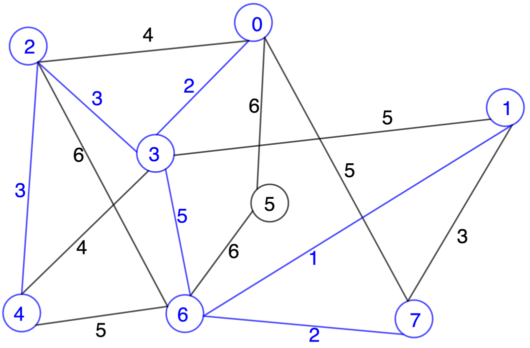

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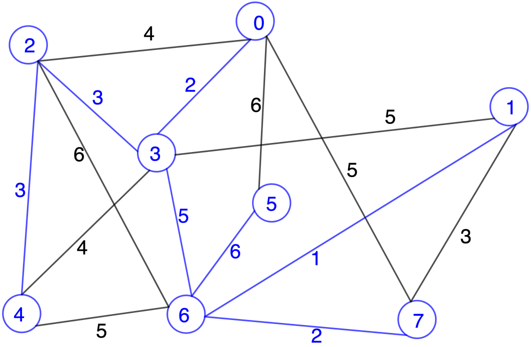

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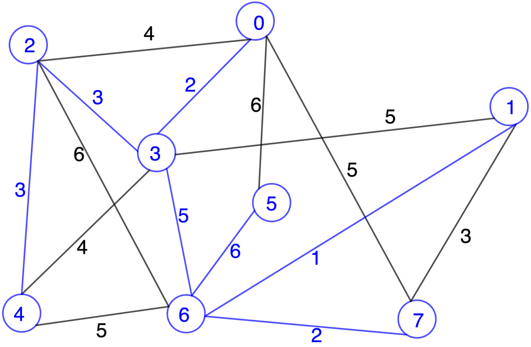

Thus, the weight of the minimum spanning tree is:

$$1+2+2+3+3+5+6=22$$

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In total: $O(m\log(m)) = O(m\log(n))$ (b/c $n - 1 \leq m < n^2$).

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- Maybe you already know the following well-known tree theorem (we'll prove next class):

Theorem

Suppose a graph on n vertices has any two of the following three properties:

1. *Connected*
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3. *Has $n - 1$ edges*

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- Can prove Kruskal's Algorithm works using the following:

Lemma

Suppose T and T' are both spanning trees of the same graph. Suppose, further, that e is in T but not in T'. Then we can find an e' in T' such that $T' + e - e'$ is a tree.

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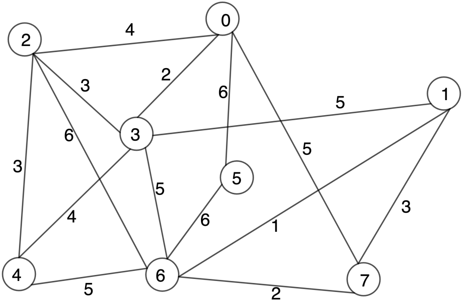

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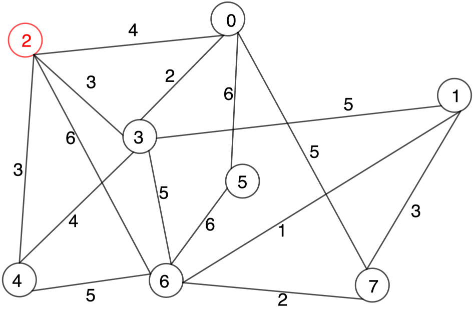

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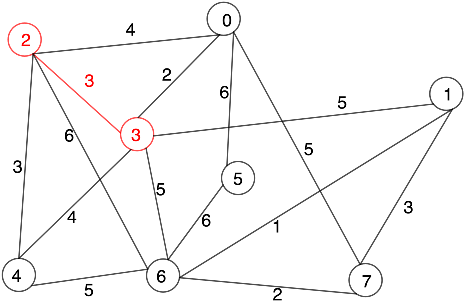

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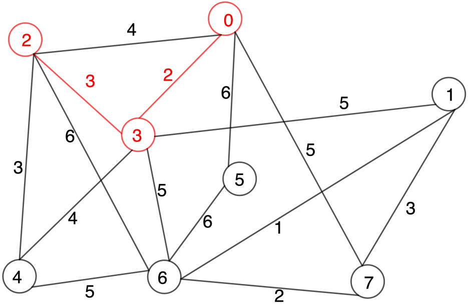

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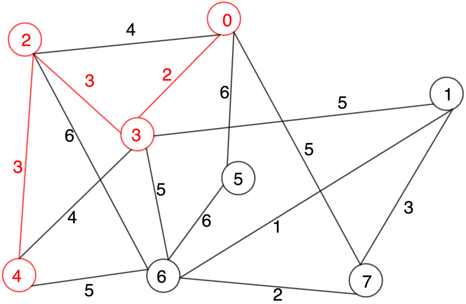

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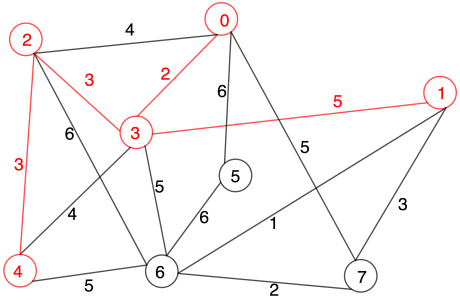

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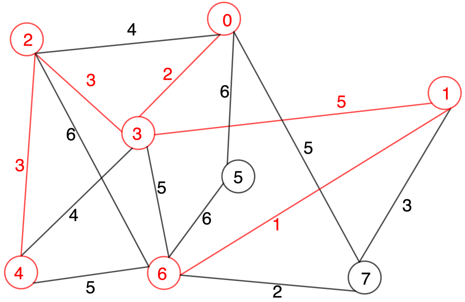

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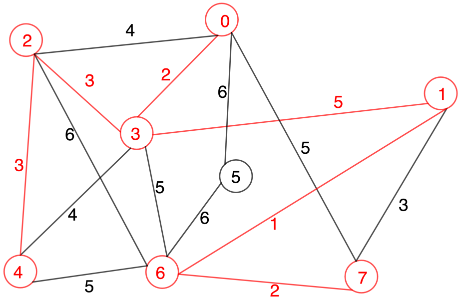

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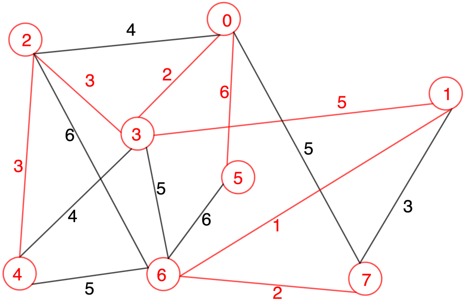

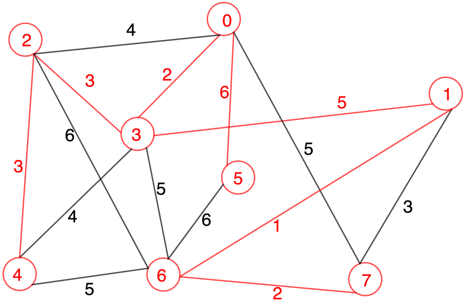

Thus, the weight of the minimum spanning tree is:

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  - In each step, we add a vertex u to the sub-tree, by removing the vertex of min distance from the heap.
    - When we add vertex u to our sub-tree, we consider all edges from u to a vertex v not in our sub-tree
    - If v's current distance to the tree is larger than weight of that edge, update the distance to be the weight of that edge
    - Otherwise, don't update v's distance to the tree.
  - Continue until each vertex is pulled.

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In total: $O(m\log(n))$.

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"Inverse Kruskal:". Remove the most expensive edges which do not disconnect the graph, until you have a tree.

All of these greedy algorithms imply a kind of robustness for the MST problem.