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Sources: None

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1. (5 pts.) **Getting started.** Please read the course policies on the syllabus, especially the course policies on collaboration. If you have any questions, contact the instructors. Once you have done this, please write "I understand the course policies." on your homework to get credit for this problem.

I understand the course policies

2. (36 pts.) **Comparing growth rates.** In each of the following situations, indicate whether $f = O(g)$, or $f = \Omega(g)$, or both (in which case $f = \Theta(g)$). Give a one sentence justification for each of your answers.

	$f(n)$	$g(n)$
a)	$6n \cdot 2^n + n^{100}$	3^n
b)	$\log(2n)$	$\log(3n)$
c)	$\sqrt{n}$	$\sqrt[3]{n}$
d)	$\frac{n^2}{\log n}$	$n(\log n)^4$
e)	$n\log n + n^2$	$10n^2 + (\log n)^5$
f)	$(\log_2 n)^{\log_2 n}$	$2^{(\log_2 n)^2}$
g)	$n\log(n^{20})$	$\log(3n!)$
h)	$\log(n^9 + \log n)$	$\log(2n)$
i)	$8^n \cdot n^2$	$(\lfloor\sqrt{n}\rfloor)!$

a. $f(n) = 6n \cdot 2^n + n^{100}, g(n) = 3^n$

$$\lim_{n \to \infty} 6n \cdot 2^n + n^{100} = \infty$$

$\lim_{n \to \infty} 3^n = \infty \to$ g(n) grows at a much faster rate than f(n) $\to$ 3^n >> 2^n

As the limit approaches 0, it becomes evident that the growth rate of g(n) surpasses that of f(n), implying that f = O(3n).

b. f(n) = log(2n), g(n) = log(3n)

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{log(2n)}{log(3n)} = \frac{(\frac{2}{x})}{(\frac{3}{x})} = \frac{2}{3}$$

As the limit approaches 2/3, the growth rates of f(n) and g(n) become approximately equal, indicating that f(n) is proportional to log(n).

c.

f(n) = $\sqrt{n}$, g(n) = $\sqrt[3]{n}$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt[3]{n}} = \lim_{n \to \infty} \sqrt[6]{n} = \infty$$

As the limit tends to infinity, it indicates that the growth rate of f(n) surpasses that of g(n), signifying that f(n) is in the order of 3n.

d.

$f(n) = \frac{n^2}{log(n)}$, $g(n) = n(log(n))^4$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{\frac{n^2}{log(n)}}{n(log(n))^4} = \lim_{n \to \infty} \frac{n}{(log(n))^5} = \infty$$

As n approaches infinity, f(n) exhibits faster growth than g(n), indicating that f is proportional to log(n).

e.

$f(n) = n \cdot log(n) + n^2$

$g(n) = 10n^2 + (log(n))^5$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{n \cdot log(n) + n^2}{10n^2 + (log(n))^5} = \lim_{n \to \infty} \frac{\frac{log(n)}{n} + 1}{10 + \frac{(log(n))^5}{n^2}} = \frac{1}{10}$$

As the limit approaches ⅔, the growth rates of f(n) and g(n) become nearly identical, implying that f(n) behaves as n^2.

F.

$f(n) = (log_2 n)^{log_2 n}$, $g(n) = 2^{(log_2 n)^2}$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{(log_2 n)^{log_2 n}}{2^{(log_2 n)^2}} = 0$$

As the limit approaches 0, it indicates that g(n) grows faster than f(n), signifying that f(n) is in O(2n).

G.

$f(n) = n \cdot log(n^{20}), g(n) = log(3n!)$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{n \cdot log(n^{20})}{log(3n!)}$$

$$= \infty$$

As the limit tends to infinity, it indicates that f(n) grows faster than g(n), implying that f(n) is proportional to n log(n).

H.

$f(n) = log(n^{9} + log(n))$
$g(n) = log(2n)$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{log(n^{9}+log(n))}{log(2n)}$$

$$= 9$$

As the limit approaches 9, the growth rates of f(n) and g(n) become nearly identical, indicating that f(n) behaves as logarithmically proportional to n, denoted as f = log(n).

3. (15 pts.) **Geometric progressions growth.** Prove the following:

$$\sum_{i=0}^{k} c^i = \begin{cases} \Theta(c^k) & \text{if } c > 1, \\ \Theta(k) & \text{if } c = 1, \\ \Theta(1) & \text{if } 0 < c < 1. \end{cases}$$

Hint - when $c \neq 1$, the partial sum of a geometric series $S(k) = \sum_{i=0}^{k} c^i = \frac{1-c^{k+1}}{1-c}$.

a) C > 1:
- Sum of terms from i to KC: $c + c^2 + c^3 + ... + ck$
- Viewed as a Geometric Progression with first term C and common ratio r = c
- Sum expression: a x (r^n - 1) / (r - 1)
- Simplified sum expression: c x ck-1c - 1
- Special case: If all terms are 1, the sum is k (Θ(k)).
- Observation: When c > 1, 'c' is nearly equal to c - 1.

b)