Computer Graphics: Review of Basic Math
Part 1

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Introduction

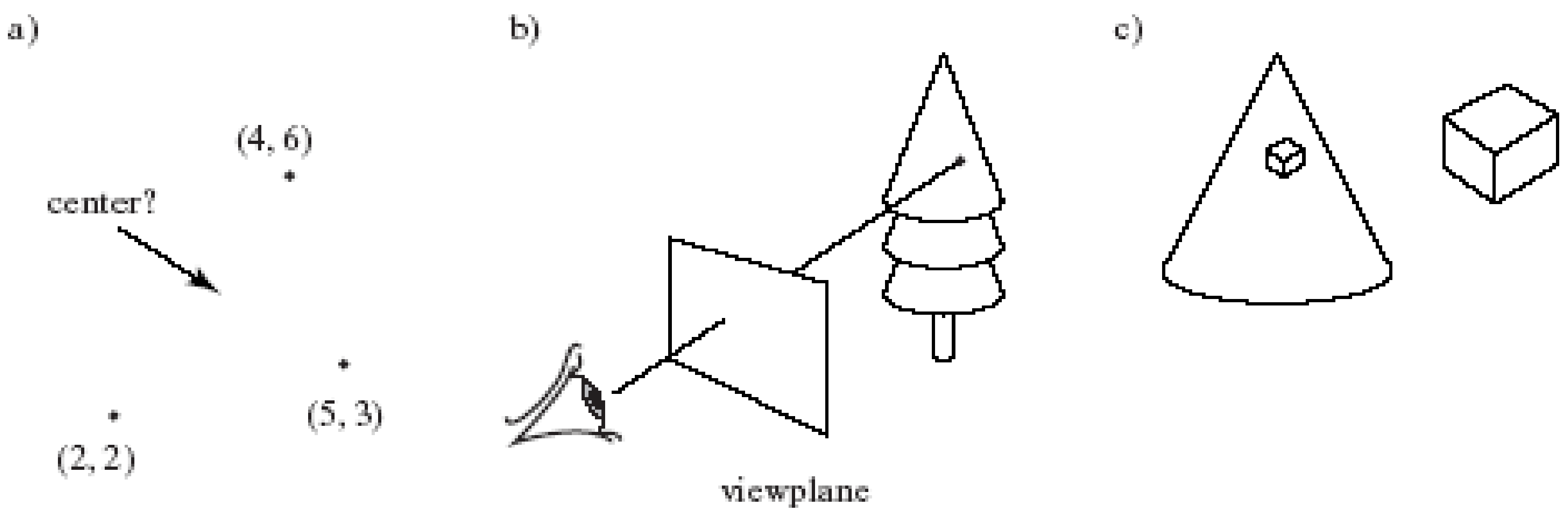
- In computer graphics, we work with objects defined in a three dimensional world (with 2D objects and worlds being just special cases).
- All objects to be drawn, and the cameras used to draw them, have shape, position, and orientation.
- We must write computer programs that somehow describe these objects, and describe how light bounces around illuminating them, so that the final pixel values on the display can be computed.

Introduction (2)

- The two fundamental sets of tools that come to our aid in graphics are *vector analysis* and *transformations.*
- We develop methods to describe various geometric objects, and we learn how to convert geometric ideas to numbers.
- This provides a collection of crucial algorithms that we can use in graphics programs.

Easy Problems for Vectors

- Where is the center of the circle through the 3 points? What image shape appears on the viewplane, and where? Where does the reflection of the cube appear on the shiny cone, and what is the exact shape of the reflection?

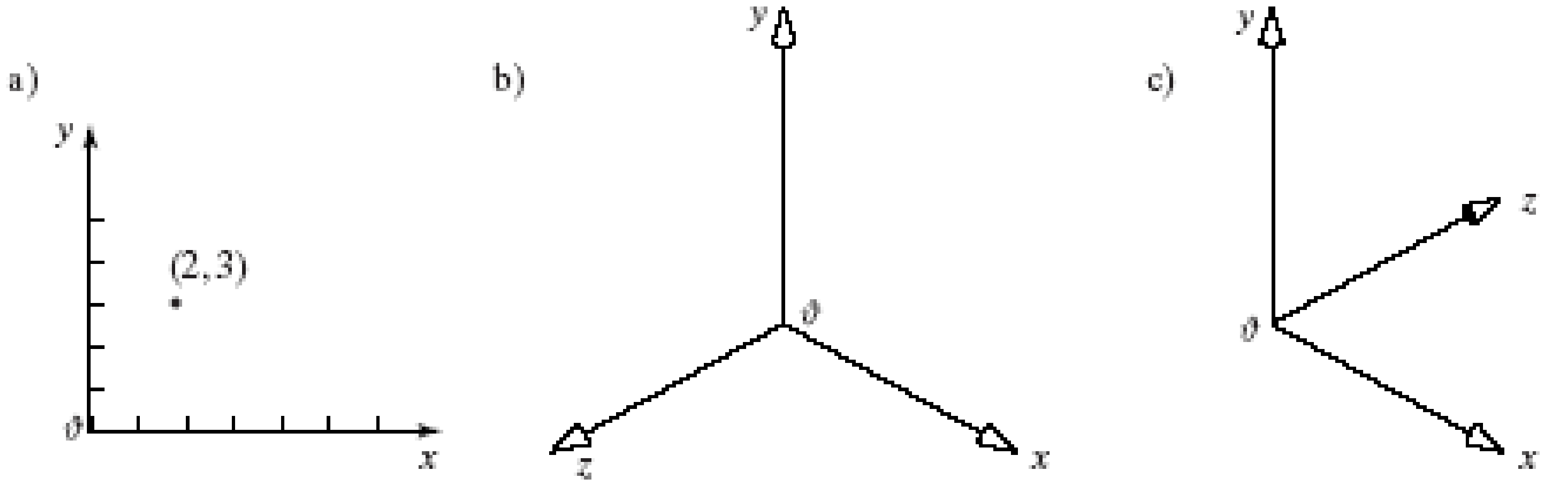


Vectors

- Vectors provide easy ways of solving some tough problems.
- A vector has length and direction, but not position (relative to a coordinate system). It can be moved anywhere.
- A point has position but not length and direction (relative to a coordinate system).
- A scalar has only size (a number).

Basics of Points and Vectors

- All points and vectors are defined relative to some coordinate system. Shown below are a 2D coordinate system and a right- and a left-handed 3-D coordinate system.



Left and Right Handedness

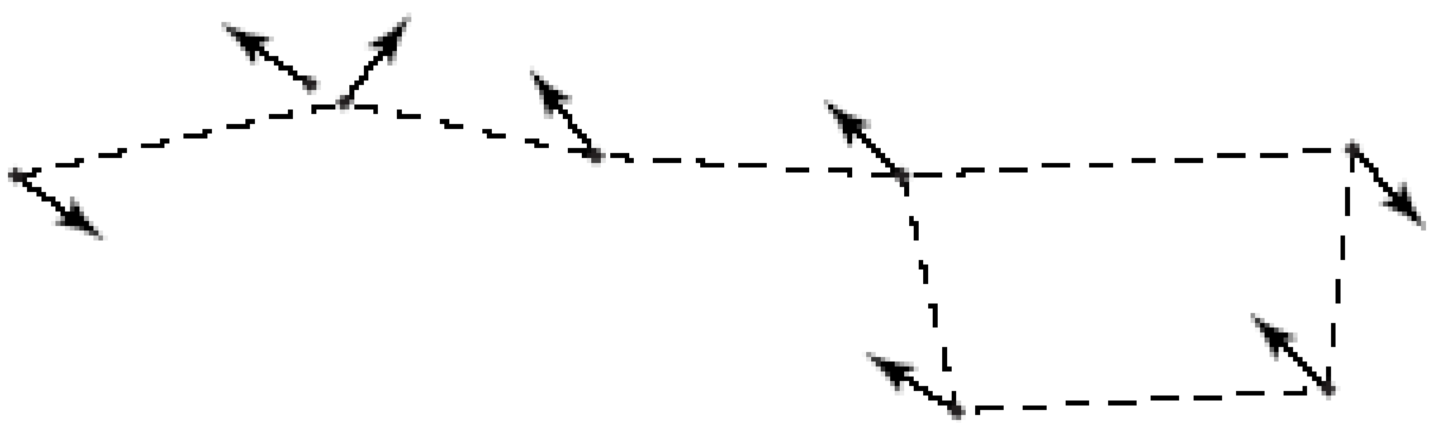
- In a 3D system, using your right hand, curl your fingers around going from the x-axis to the y-axis. Your thumb is at right angles to your fingers.
  - If your thumb points along the direction of the z-axis, the system is right handed.
  - If your thumb points opposite to the direction of the z-axis, the system is left handed.

"Not only Newton's laws, but also the other laws of physics, so far as we know, today, have the two properties which we call invariance under the translation of axes and rotation of axes. These properties are so important that a mathematical technique has been developed to take advantage of them in writing and using physical laws.. called vector analysis."

Richard Feynman

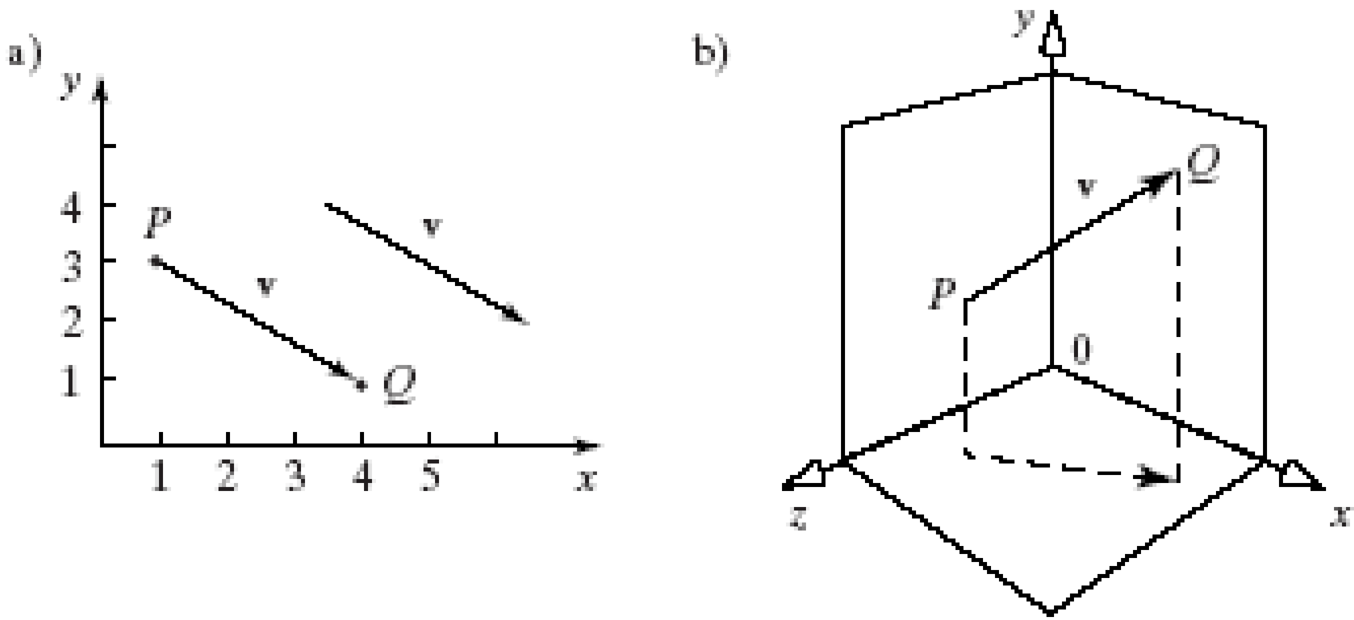
Review of Vectors

- Vectors are drawn as arrows of a certain length pointing in a certain direction.
- A vector is a *displacement* from one point to another. Shown below are displacements of the stars in the Big Dipper over the next 50,000 years.



Vectors and Coordinate Systems

- A vector v between points P = (1, 3) and Q = (4, 1), with components of (3, -2), calculated by subtracting the coordinates individually (Q – P).
- To "go" from P to Q, we move down by 2 and right by 3. Since v has no position, the two arrows labeled v are the same vector. The 3D case is also shown.



Vector Operations

- The difference between 2 points is a vector: **v** = Q – P.
- The sum of a point and a vector is a point: P + **v** = Q.
- We represent an n-dimensional vector by an n-tuple of its components, e.g. **v** = (v_x, v_y, v_z). (We will usually use 2- or 3-dimensional vectors: e.g., **v** = (3, -2)).

Vector Representations

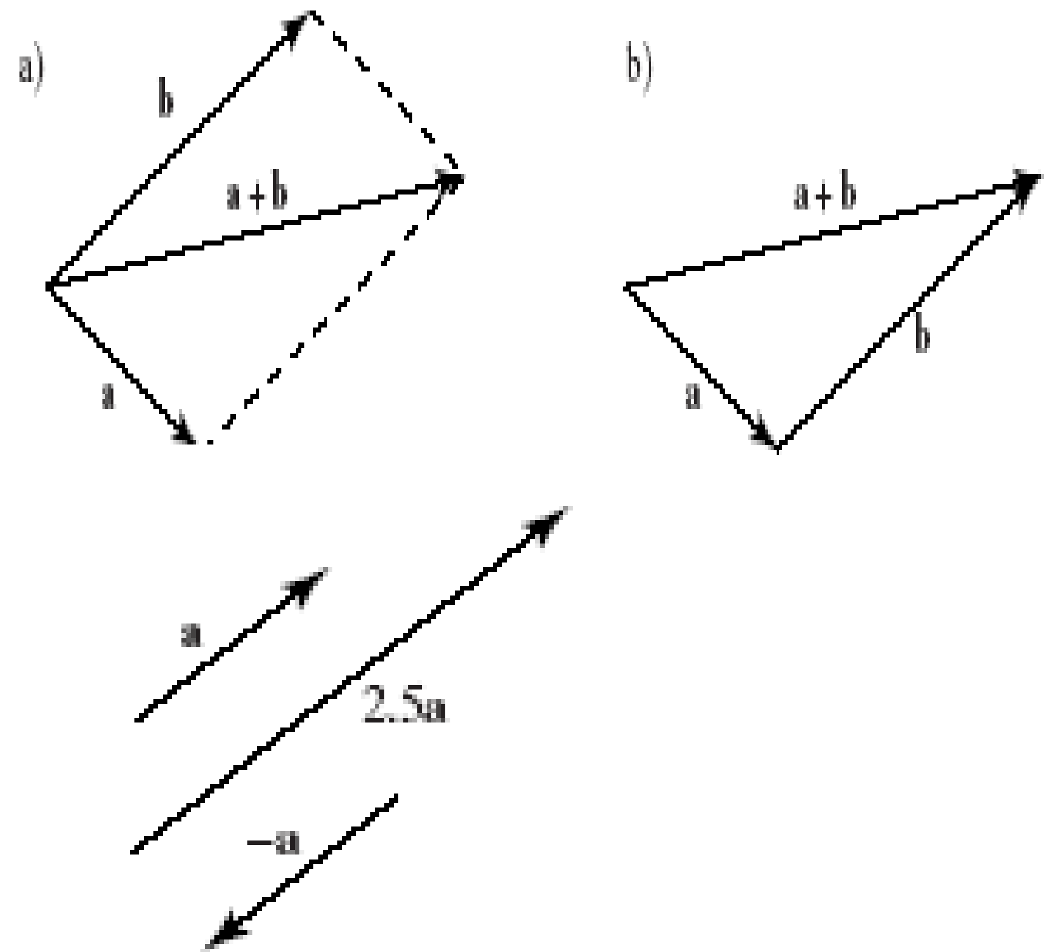
- A vector **v** = (33, 142.7, 89.1) is a row vector.
- A vector **v** = $(33, 142.7, 89.1)^T$ is a column vector.
  - It is the same as

$$v = \begin{pmatrix} 33 \\ 142.7 \\ 89.1 \end{pmatrix}$$

Vector Operations (2)

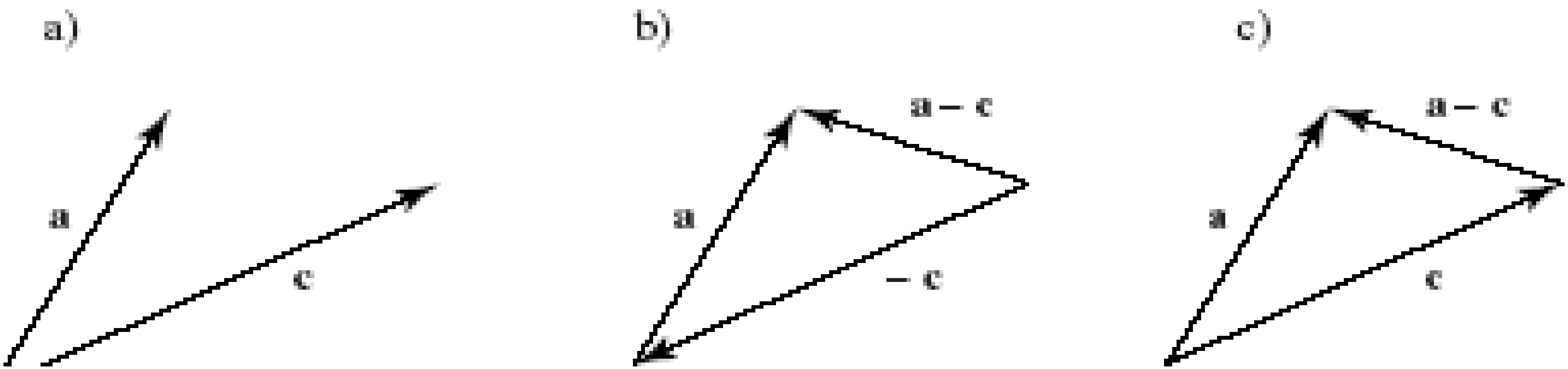
- Vectors have 2 fundamental operations: addition

  of 2 vectors and multiplication by a scalar.
- If **a** and **b** are vectors, so is **a** + **b**, and so is s**a**, where s is a scalar.



Vector Operations (3)

- Subtracting **c** from **a** is equivalent to adding **a** and (**-c**), where **–c** = (-1)**c**.



Linear Combinations of Vectors

Definition:
A **linear combination** of the *m* vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a vector of the form

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_m\mathbf{v}_m$$

where $a_1, a_2, \ldots, a_m$ are scalars.

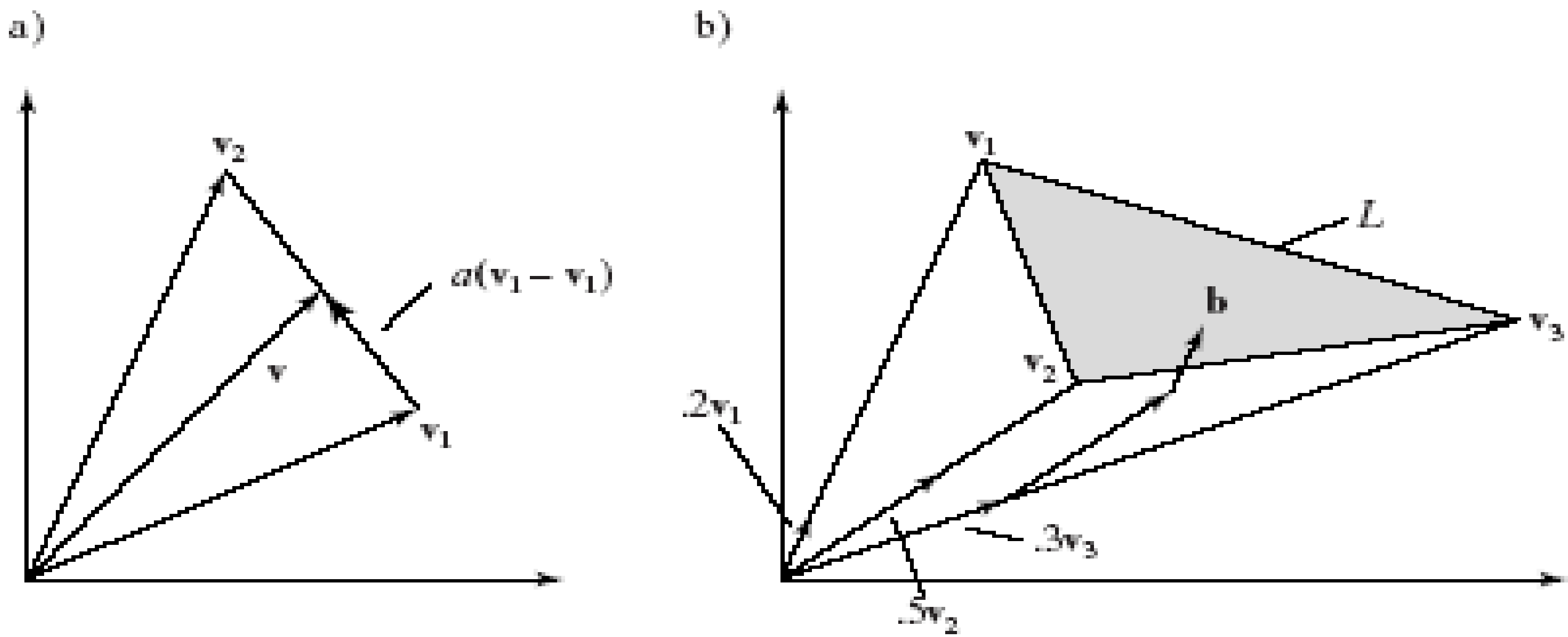
- $\mathbf{v}_1 \pm \mathbf{v}_2 = (v_{1x} \pm v_{2x}, v_{1y} \pm v_{2y}, v_{1z} \pm v_{2z})$
- $s\mathbf{v} = (sv_x, sv_y, sv_z)$
- A linear combination of the m vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$ is $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_m\mathbf{v}_m$.
  - Example: 2(3, 4,-1) + 6(-1, 0, 2) forms the vector (0, 8, 10).

Linear Combinations of Vectors

- The linear combination becomes an affine combination if $a_1 + a_2 + \ldots + a_m = 1$.
  - Example: 3**a** + 2 **b** - 4 **c** is an affine combination of **a**, **b**, and **c**, but 3 **a** + **b** - 4 **c** is not.
  - (1-*t*) **a** + (*t*) **b** is an affine combination of **a** and **b**.
- The affine combination becomes a convex combination if $a_i \geq 0$ for $1 \leq i \leq m$.
  - Example: .3**a**+.7**b** is a convex combination of **a** and **b**, but 1.8**a** -.8**b** is not.

The Set of All Convex Combinations of 2 or 3 Vectors

- $\mathbf{v} = (1 - a)\mathbf{v}_1 + a\mathbf{v}_2$, as a varies from 0 to 1, gives the set of all convex combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$. An example is shown below.



Vector Magnitude and Unit Vectors

- The magnitude (length, size) of *n*-vector **w** is written |**w**|.

$$|\mathbf{w}| = \sqrt{w_1^{\ 2} + w_2^{\ 2} + \ldots + w_n^{\ 2}}$$

- Example: the magnitude of **w** = (4, -2) is $\sqrt{20}$ and that of **w** = (1, -3, 2) is $\sqrt{14}$.
- A unit vector has magnitude |**v**| = 1.
- The unit vector pointing in the same direction as vector **a** is $\hat{a} = \frac{a}{|a|}$ (if |a| ≠0).

- Converting **a** to $\hat{a}$ is called *normalizing* vector **a**.

Vector Magnitude and Unit Vectors (2)

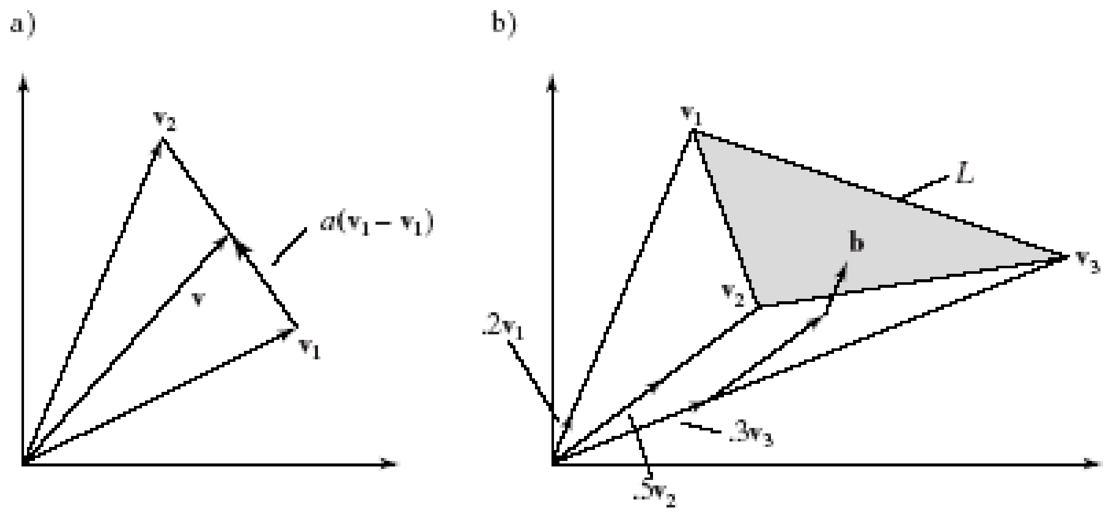
- At times we refer to a unit vector as a **direction**.
- Any vector can be written as its magnitude times its direction: **a** = |a|

$$\hat{a}$$

Exercise

what values, or range of values, for a1 and a2 create the following sets?

a.v1.
b.The line joining v1 and v2.
c.The vector midway between v2 and v3.
d.The centroid of the triangle.



Normalize each of the following vectors: (1, -2, .5); b). (8, 6); c). (4, 3)

To represent w vector as Linear Combinations

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + (1 - a_1 - a_2)\,\mathbf{v}_3$$

a) $a_1 = 1,\ a_2 = 0$

b) $1 - a_1 - a_2 = 0,\ a_1 = 1 - a_2,\ 0 \le a_2 \le 1$

c) $a_1 = 0,\ a_2 = 0.5$

d) $a_1 = a_2 = 1/3$

2. Solution: Normalizing Vectors.

a) $|\hat{a}| = |(1, -2, 0.5)| = \sqrt{1^2 + (-2)^2 + 0.5^2} = 2.29129$

So the normalized version is:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = (0.436436, -0.872872, 0.218218)$$

b) $|\hat{a}| = |(8,6)| = \sqrt{8^2 + 6^2} = 10$

So the normalized version is:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = (0.8, 0.6)$$

c) $|\hat{a}| = |(4,3)| = \sqrt{4^2 + 3^2} = 5$

So the normalized version is:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = (0.8, 0.6)$$