Formulation of Prospective Active Inference as Quadratic Programming

1 Problem Statement

Consider the discrete-time Linear Time-Invariant (LTI) dynamical system model described by:

$$x_{k+1} = Ax_k + Bu_k + Fw_k, \tag{1}$$
$$y_k = Cx_k + v_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$ represents the system state, $u_k \in \mathbb{R}^m$ is the control input, and $y_k \in \mathbb{R}^p$ is the observation output.

In the active inference framework, we introduce a *generative model* to encapsulate the dynamics of the internal state. Let z_k denote the **internal state**, which evolves according to the internal model:

$$z_{i+1|k} = (A + BK)z_{i|k} + w_{i|k}, \tag{3}$$

where i denotes the prediction step index relative to the current time step k.

2 Future Variational Free Energy (F-VFE) Objective

At time step k, considering a prediction horizon N, we define the cumulative Variational Free Energy, denoted as $\mathcal{F}_k$. This objective function is composed of two primary components: the *external state error* (deviation between the predicted system state x and the internal state z) and the *internal dynamic consistency error*:

$$\mathcal{F}_k = \underbrace{\frac{1}{2}\sum_{i=1}^{N} \left\| x_{i|k} - z_{i|k} \right\|^2_{\tilde{R}}}_{\text{State/Observation Error}} + \underbrace{\frac{1}{2}\sum_{i=1}^{N} \left\| z_{i|k} - (A + BK)z_{i-1|k} \right\|^2_{\tilde{Q}}}_{\text{Internal Dynamics Consistency}}. \tag{4}$$

3 Compact Matrix Formulation

To formulate the optimization problem as a standard Quadratic Program (QP), we define the *lifted vectors* over the horizon N:

$$\mathbf{X}_k = \begin{bmatrix} x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix}, \quad \mathbf{Z}_k = \begin{bmatrix} z_{1|k} \\ \vdots \\ z_{N|k} \end{bmatrix}, \quad \mathbf{U}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}. \tag{5}$$

Based on the system dynamics (1), the predicted state trajectory $\mathbf{X}_k$ can be expressed as an affine function of the initial state and future control inputs:

$$\mathbf{X}_k = \mathbf{F}x_k + \mathbf{\Phi}\mathbf{U}_k, \tag{6}$$

where $\mathbf{F}$ and $\mathbf{\Phi}$ are the standard prediction matrices derived from (A, B).

To capture the internal state dynamics (3), we define the shift matrix $\mathbf{S}$ and the block-diagonal dynamics matrix $\mathbf{M}$:

$$\mathbf{S} = I_{N\times N}, \quad \mathbf{M} = \text{blockdiag}(A + BK, \ldots, A + BK). \tag{7}$$

The cost function (4) can then be rewritten in a compact quadratic form:

$$\mathcal{F}_k = \frac{1}{2} \|\mathbf{X}_k - \mathbf{Z}_k\|^2_{\tilde{R}} + \frac{1}{2} \|(\mathbf{S} - \mathbf{M})\mathbf{Z}_k\|^2_{\tilde{Q}}. \tag{8}$$

4 Transformation to Standard QP Form

We define the joint optimization variable $\boldsymbol{\xi}(k)$ containing both control inputs and internal states:

$$\boldsymbol{\xi}(k) = \begin{bmatrix} \mathbf{U}_k \\ \mathbf{Z}_k \end{bmatrix}. \tag{9}$$

We introduce transformation matrices T_1 and T_2 to express the error terms with respect to $\boldsymbol{\xi}(k)$:

$$T_1 = \begin{bmatrix} \mathbf{\Phi} & -I \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & I \end{bmatrix}. \tag{10}$$

Substituting $\mathbf{X}_k = \mathbf{F}x_k + \mathbf{\Phi}\mathbf{U}_k$ into the cost function terms yields:

$$\mathbf{X}_k - \mathbf{Z}_k = \mathbf{F}x_k + \mathbf{\Phi}\mathbf{U}_k - \mathbf{Z}_k = \mathbf{F}x_k + T_1\boldsymbol{\xi}(k), \tag{11}$$

$$(\mathbf{S} - \mathbf{M})\mathbf{Z}_k = (\mathbf{S} - \mathbf{M})T_2\boldsymbol{\xi}(k). \tag{12}$$

Expanding the cost function $\mathcal{F}_k(\boldsymbol{\xi})$, we obtain the standard quadratic form used for optimization:

$$\min_{\boldsymbol{\xi}} \quad \frac{1}{2}\boldsymbol{\xi}^T\mathbf{H}\boldsymbol{\xi} + \mathbf{g}^T\boldsymbol{\xi}, \tag{13}$$

where the Hessian matrix $\mathbf{H}$ and gradient vector $\mathbf{g}$ are given by:

$$\mathbf{H} = T_1^T\tilde{R}T_1 + T_2^T(\mathbf{S} - \mathbf{M})^T\tilde{Q}(\mathbf{S} - \mathbf{M})T_2, \tag{14}$$

$$\mathbf{g} = T_1^T\tilde{R}\mathbf{F}x_k. \tag{15}$$

This formulation allows the Prospective Active Inference problem to be solved efficiently using standard QP solvers (e.g., qpOASES, OSQP), optimizing simultaneously for the control actions $\mathbf{U}_k$ and the internal state trajectory $\mathbf{Z}_k$.