COMP319 Algorithms 1
Lecture 10
Sets and Hashing

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Disjoint set representation and operations

Hashing and collision resolution

Textbook Chapters 5 and 12

Slide credit: J. Lillis, UIC's CS 201 Data Structures and Discrete Mathematics I

Disjoint set representation

Union-Find operations

Collapsing rule

Textbook chapter 5

SETS

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Set Definition and Operations

- Set definition
  - A collection of data with **UNIQUE** elements
    - No DUPLICATE elements
- Notations
  - Ø: empty set
  - *Z*: integer set
  - *R*: real numbers
  - *N*: natural numbers

- Operations
  - Union
  - Intersection
  - Difference
  - Complement
  - Combinations of the above
- Membership: $\in$
  - Check if a specific element belongs to a set

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Disjoint Set Definition

- Pairwise disjoint set definition

  $S_i \cap S_j = \emptyset \leftrightarrow$ If S_i and S_j, for $i \neq j$, are disjoint sets, then there is no element that is in both S_i and S_j.

- Disjoint set operations
  - Cardinality: $|S|$ = the number of elements of set S
  - Disjoint set union
    - If S_i and S_j are two disjoint sets, then their union $S_i \cup S_j$ = {all elements x such that $x \in S_i$ or $x \in S_j$}.
    - $|S_i \cup S_j| = |S_i| + |S_j| - |S_i \cap S_j| = |S_i| + |S_j|$
  - Find(x)
    - Find the set containing element x
    - UNIQUE set is found, by the definition of disjoint set

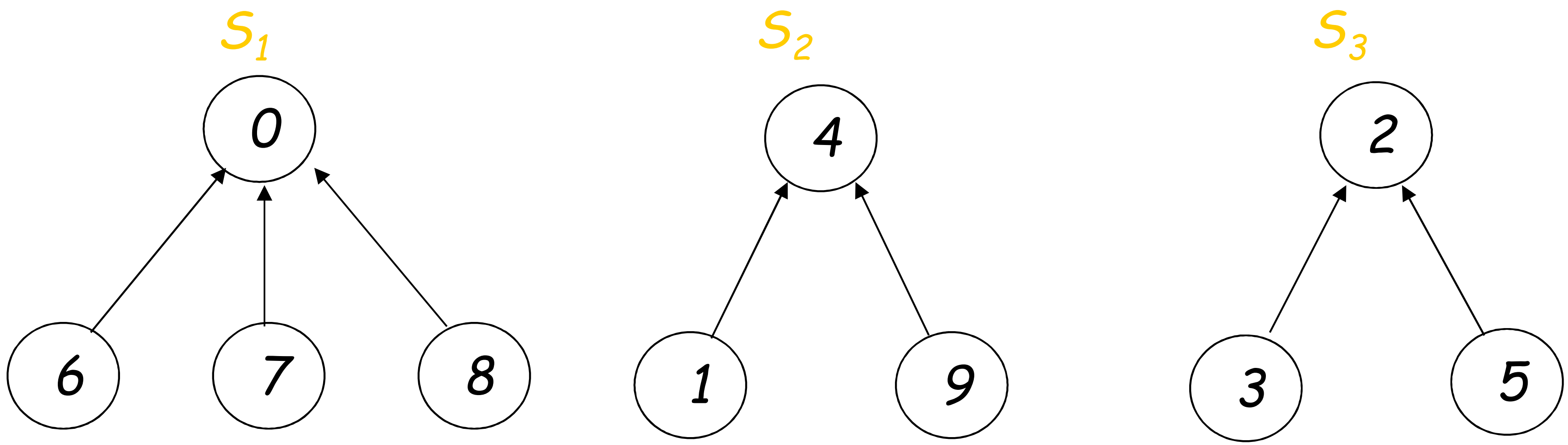
Disjoint Set ADT

- The universe consists of N elements, named 1, …, N .
- The disjoint set ADT is a collection of sets, such that
  - Sets are disjoint: each element is in exactly one set
- Operations
  - NewSet = **union** (Set1, Set2)
    - creates a **NEW** set, composed of all elements which are either in Set1 or Set2
  - Setname = **find** (element_name)
    - returns the name of the UNIQUE set that contains the given element

Tree Representation of Disjoint Sets

*The **has-a** relationship can be represented by TREEs*

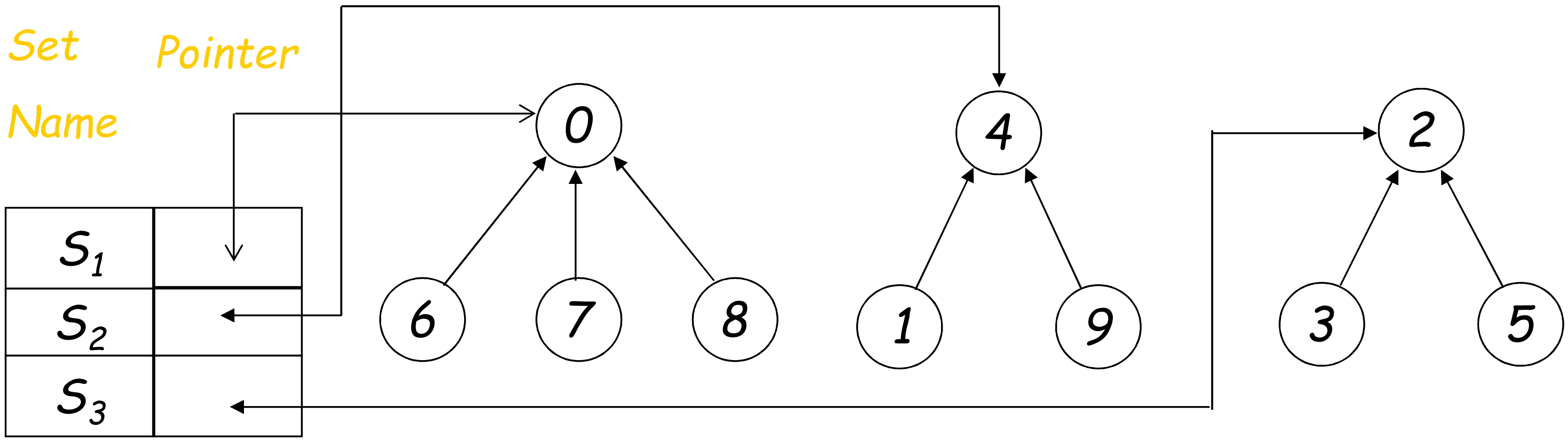
Set 2 = {4 , 1 ,9}



Set I = {0 , 6 ,7 ,8 }

Set 3 = {2 , 3 , 5}

Pointer Representation for S1, S2, S3



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Unions of Sets

- To obtain the union of two sets, just set the parent field of one of the roots to the other root.

- To figure out which set an element is belonged to, just follow its parent link to the root and then follow the pointer in the root to the set name.

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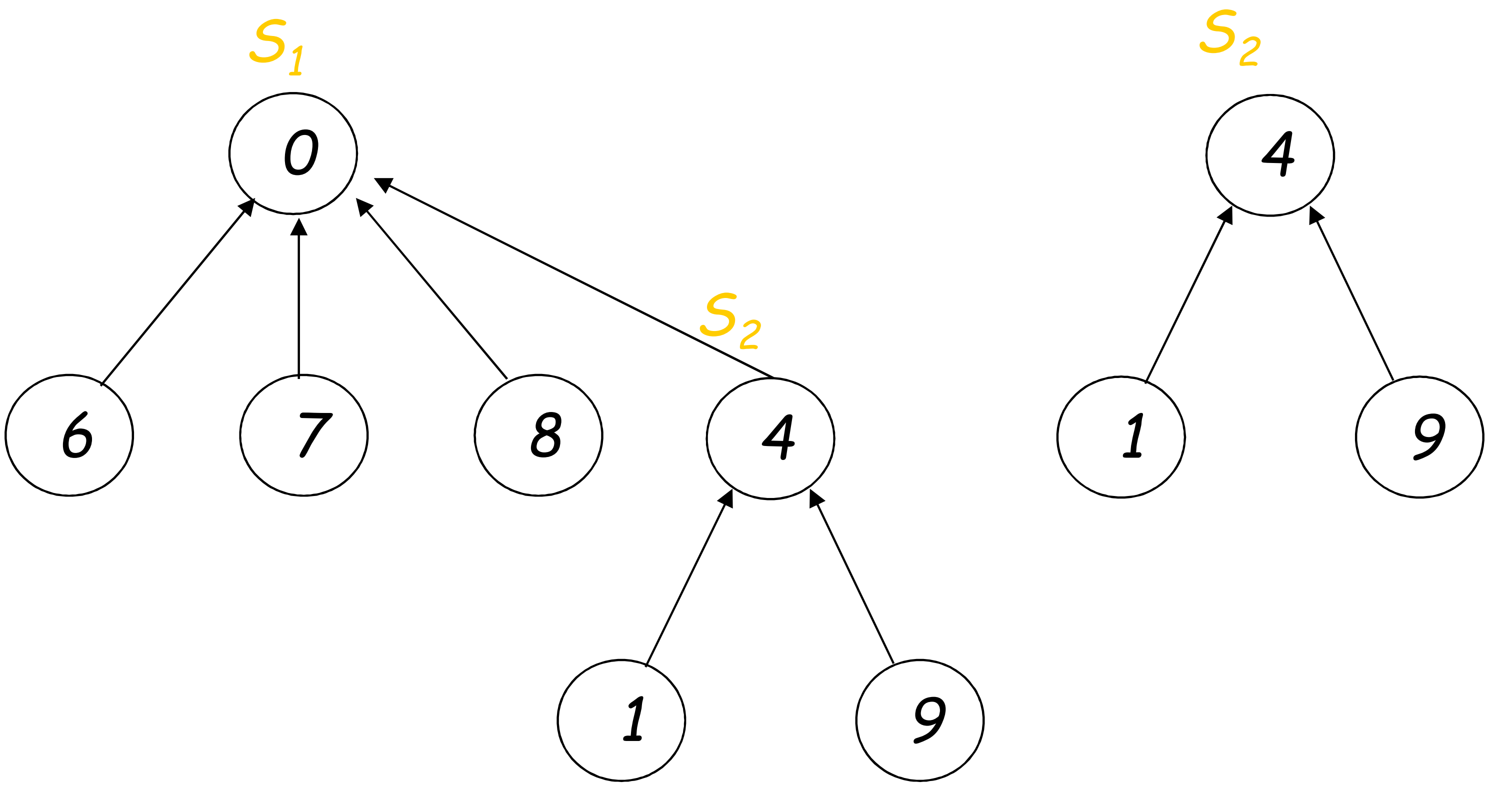
Representations of S1 ∪ S2 (1)



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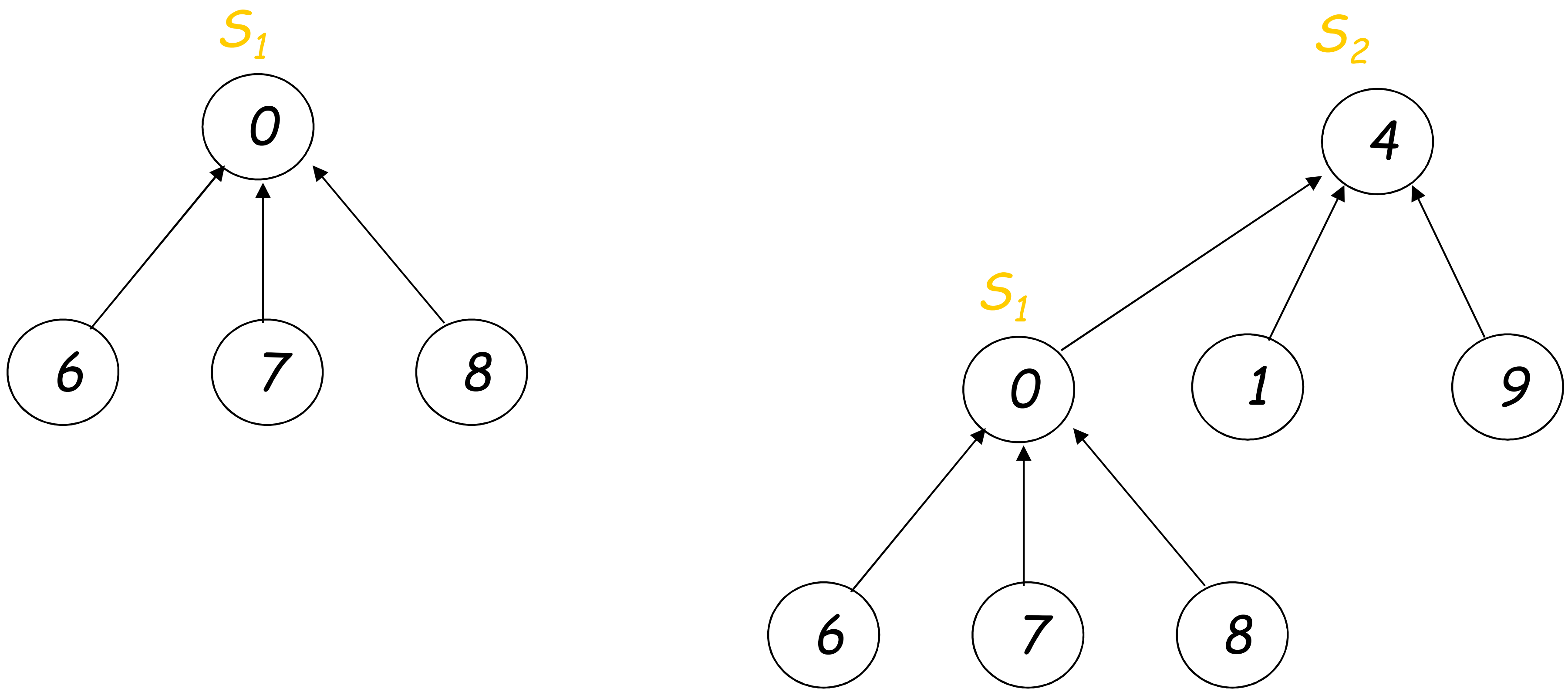
Representations of S1 ∪ S2 (2)



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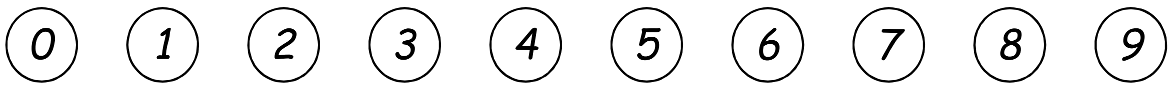
Array Representation

- We could use an array to represent each set.
  - Assume set elements are numbered 0 through n-1.
- To start, each set contains one element
- Array value: **-1**
  - It is a root of a set with itself only



i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

-1 means no parents ➔ root: a set of one element

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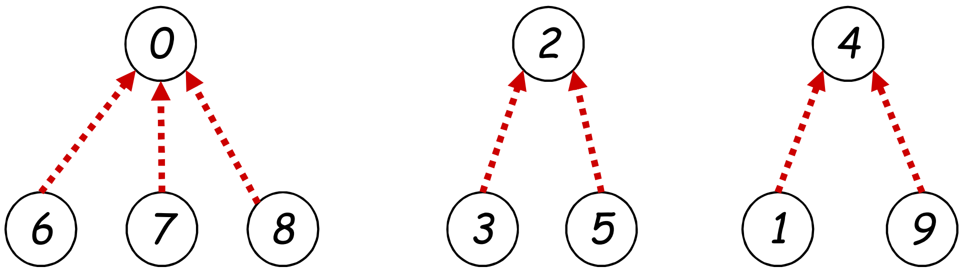
Array Representation

- Array value -1: root
- Any nonnegative integer: index of its parent
  - Parent can be either root or any other element

i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	4	-1	2	-1	2	0	0	0	4

Root

Try drawing this array into set graph



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Array Representation

```
void Union1( int i , int j )
{
    parent[i] = j ;
}
```

EX: S1 ∪ S2   Union1(0 , 2) ;

i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	4	-1	2	-1	2	0	0	0	4
	2									

EX: Find1(5) ;

i = 2

```
int Find1( int i )
{
        for(;parent[i] >= 0 ; i = parent[i]) ;
        return i ;
}
```

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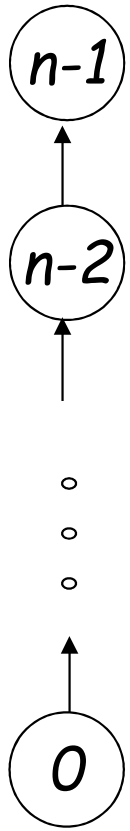
Worst-case total running time of a sequence of f finds and u unions

Collapsing rule for efficient find

DISJOINT SET ANALYSIS AND COLLAPSING RULE

Analysis Union-Find Operations

- For a set of n elements each in a set of its own, then the result of the union function is a degenerate tree (chained).
- The time complexity of the following sequence of $(n-1)$ union-find operation is $O(n^2)$.
- The complexity can be improved by using weighting rule for union.



union(0, 1), find(0)   *Union operation*

union(1, 2), find(0)   *O(n)*

⋮

Find operation

union(n-2, n-1), find(0) $O(n^2)$

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Weighting Rule

- The purpose of weighting rule is to prevent a degenerate tree as a result of successive union operations
- Definition [Weighting rule for union(i, j)]
  - If the number of nodes in the tree with root i is less than the number in the tree with root j, then make j the parent of i ; otherwise make i the parent of j.
  - root(union(S_i,S_j)) := root(S_i) if $|S_i| > |S_j|$; otherwise root(S_j)
- Modification of the value of root in the array
  - Instead of simple -1, assign $-|S|$ to store the number of elements in a set (cardinality)

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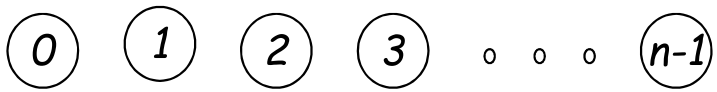
i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

```
void union2 (int i, int j)
{
    int temp = parent[i] + parent[j];
    if ( parent[i]>parent[j]) {
        parent[i]=j;
        parent[j]=temp;
    }
    else {
        parent[j]=i;
        parent[i]=temp;
    }
}
```



EX: unoin2 (0 , 1) , unoin2 (0 , 2) , unoin2 (0 , 3)

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i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

```
void union2 (int i, int j)
{
    int temp = parent[i] + parent[j];
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        parent[j]=temp;
    }
    else {
        parent[j]=i;
        parent[i]=temp;
    }
}
```
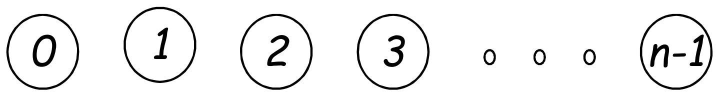

unoin2 (0 , 1)

temp = -2

The absolute value is the number of elements in the union set



EX: unoin2 (0 , 1) , unoin2 (0 , 2) , unoin2 (0 , 3)

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i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-2	0	-1	-1	-1	-1	-1	-1	-1	-1

```
void union2 (int i, int j)
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   }
   else {
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      parent[i]=temp;
   }
}
```

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temp = -2

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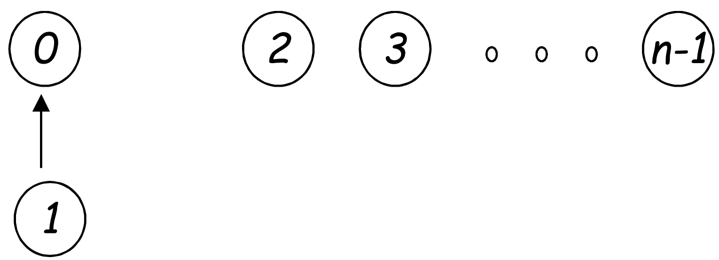

EX: unoin2 (0 , 1) , unoin2 (0 , 2) , unoin2 (0 , 3)

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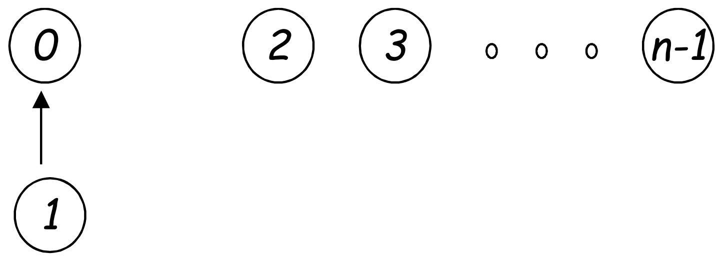
i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-2	0	-1	-1	-1	-1	-1	-1	-1	-1

```
void union2 (int i, int j)
{
    int temp = parent[i] + parent[j];
    if ( parent[i]>parent[j]) {
        parent[i]=j;
        parent[j]=temp;
    }
    else {
        parent[j]=i;
        parent[i]=temp;
    }
}
```

unoin2 (0 , 1)
unoin2 (0 , 2)

temp = -3

The absolute value is the number of elements in the union set



EX: unoin2 (0 , 1) , unoin2 (0 , 2) , unoin2 (0 , 3)

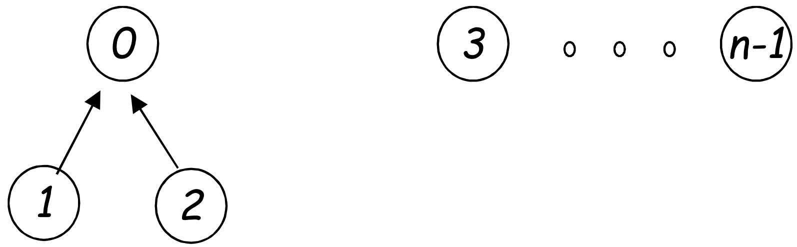
i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-3	0	0	-1	-1	-1	-1	-1	-1	-1

```
void union2 (int i, int j)
{
    int temp = parent[i] + parent[j];
    if ( parent[i]>parent[j]) {
        parent[i]=j;
        parent[j]=temp;
    }
    else {
        parent[j]=i;
        parent[i]=temp;
    }
}
```

unoin2 (0 , 1)
unoin2 (0 , 2)

temp = -3

The absolute value is the number of elements in the union set



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i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-3	0	0	-1	-1	-1	-1	-1	-1	-1

```
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{
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        parent[i]=j;
        parent[j]=temp;
    }
    else {
        parent[j]=i;
        parent[i]=temp;
    }
}
```
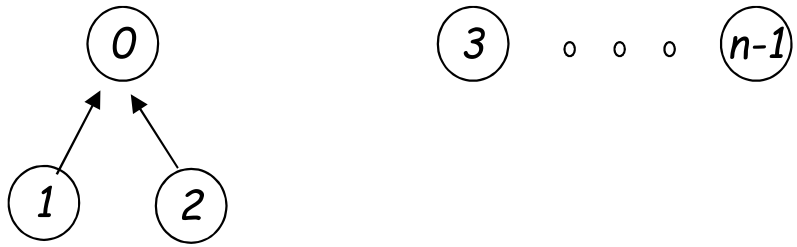

unoin2 (0 , 1)
unoin2 (0 , 2)
unoin2 (0 , 3)

temp = -4

The absolute value is the number of elements in the union set



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i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-4	0	0	0	-1	-1	-1	-1	-1	-1

```
void union2 (int i, int j)
{
    int temp = parent[i] + parent[j];
    if ( parent[i]>parent[j]) {
        parent[i]=j;
        parent[j]=temp;
    }
    else {
        parent[j]=i;
        parent[i]=temp;
    }
}
```

unoin2 (0 , 1)
unoin2 (0 , 2)
unoin2 (0 , 3)

temp = -4

The absolute value is the number of elements in the union set

EX: unoin2 (0 , 1) , unoin2 (0 , 2) , unoin2 (0 , 3)

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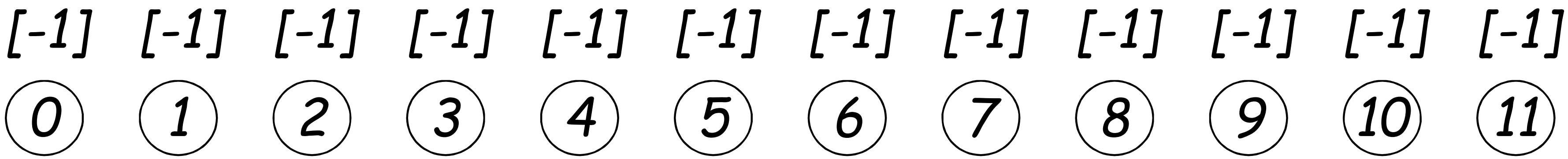
Weighted Union Time Complexity

- Lemma 5.5: Assume that we start with a forest of trees, each having one node.
  - Let T be a tree with m nodes created as a result of a sequence of unions each performed using function **WeightedUnion**.
  - The height of T is no greater than

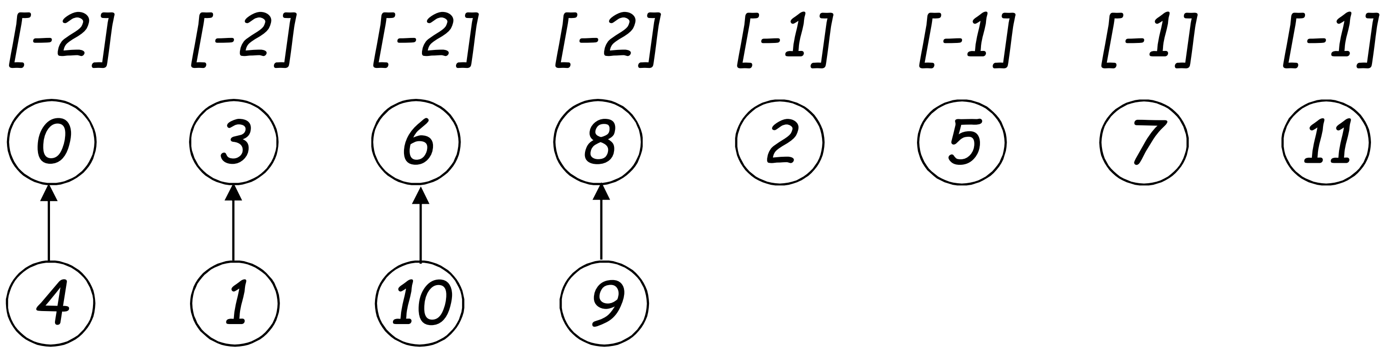
$$floor\ (\log_2 m) + 1 .$$

- For the processing of an intermixed sequence of $u - 1$ unions and f find operations, the time complexity is $O(u + f \log u)$.

Example 5.3

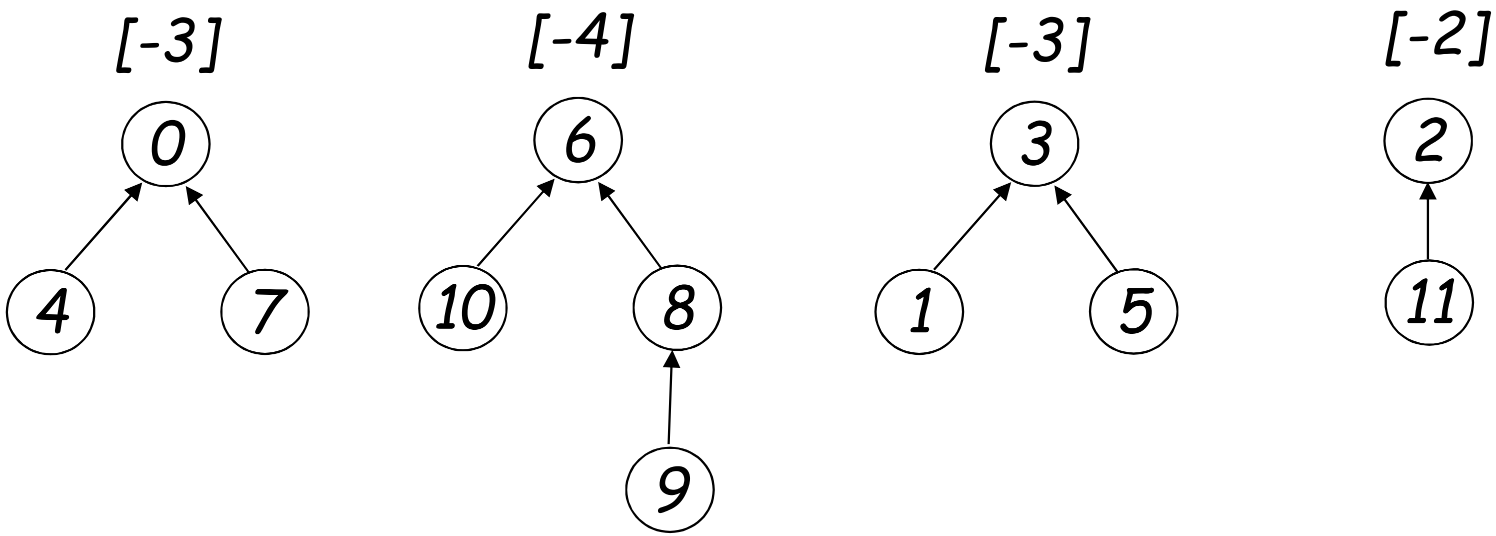


(a) Initial trees



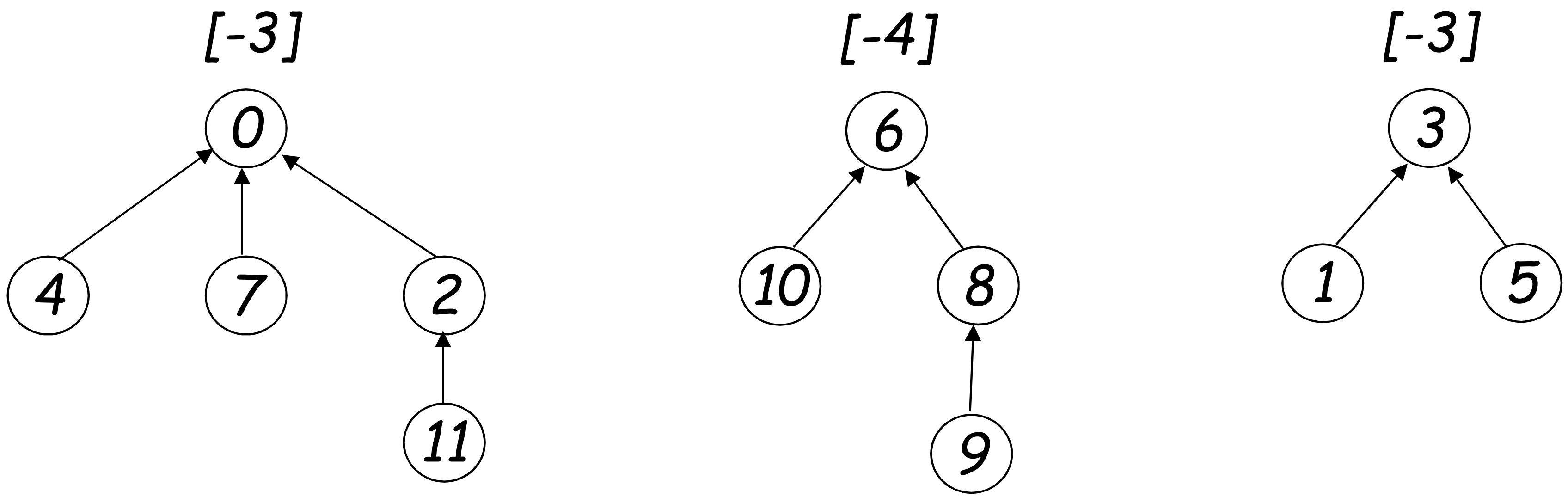
(b) Height-2 trees following $0 \cup 4$, $3 \cup 1$, $6 \cup 10$, and $8 \cup 9$

Example 5.3 (Cont.)



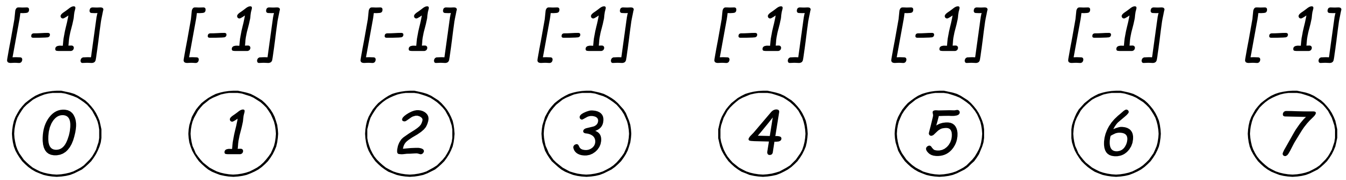
(c) Tree following $7 \cup 4$, $6 \cup 8$, $3 \cup 5$, and $2 \cup 11$

Example 5.3 (Cont.)

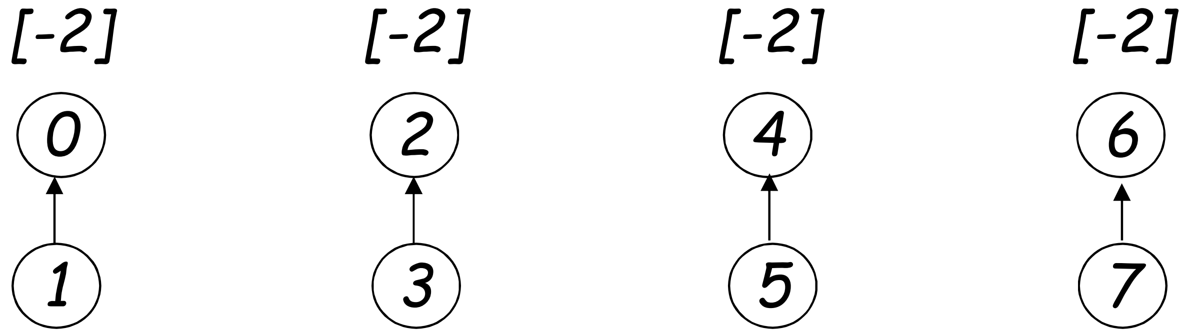


(d) Tree following 11 ∪ 0

Trees Achieving Worst-Case Bound

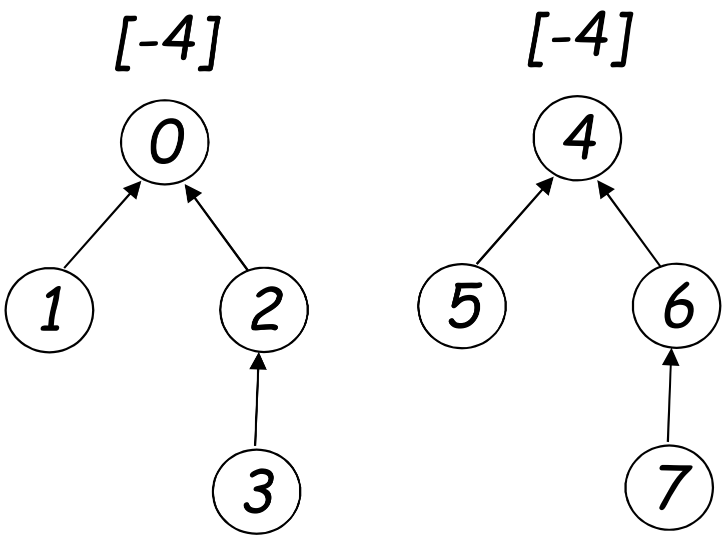


(a) Initial height trees

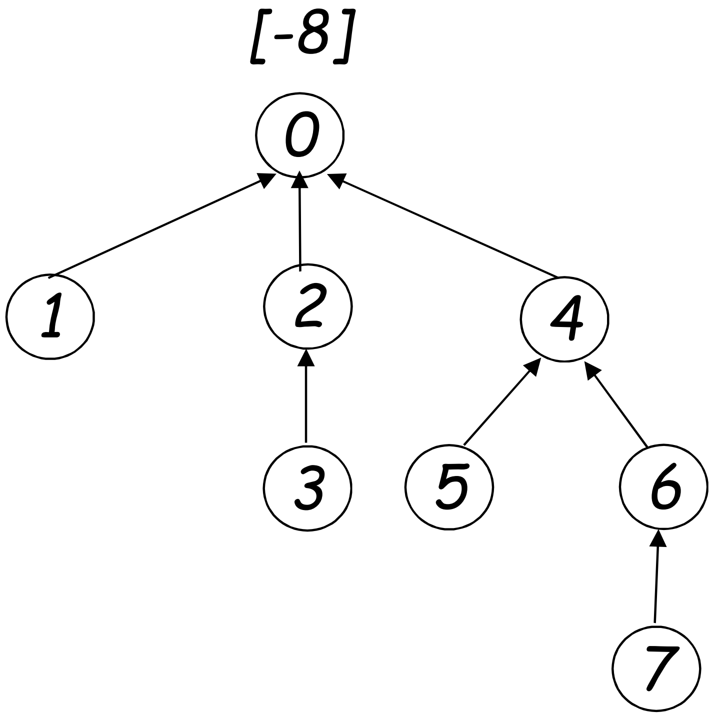


(b) Height-2 trees following union (0, 1), (2, 3), (4, 5), and (6, 7)

Worst-Case Bound (Cont.)



(c) Height-3 trees following union (0, 2), (4, 6)



(d) Height-4 trees following union (0, 4)

Collapsing Rule

- Definition [Collapsing rule]
  - While processing operation find, if *j* is a node on the path from *i* to its root,
    and parent[*i*]≠ root(*i*), then set parent[*j*] to root(*i*).

- The first run of find operation will collapse the tree. Therefore, all following find operation of the same element only goes up one link to find the root.

- All the parents of node *i* become its siblings – flattened.

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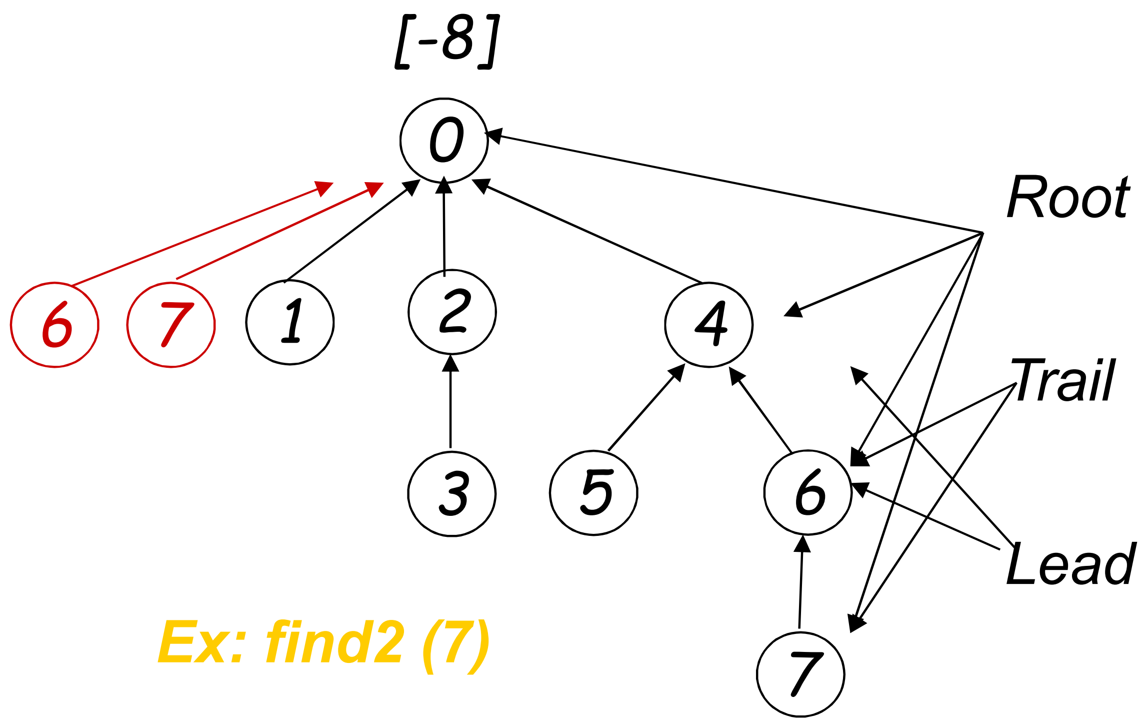
```
int find2(int i)
{
   int root, trail, lead;
   for (root=i; parent[root]>=0; root=parent[root]);
   for (trail=i; trail!=root;  trail=lead) {
      lead = parent[trail];
      parent[trail]= root;
   }
   return root:
}
```



```
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{
    int root, trail, lead;
    for (root=i; parent[root]>=0; root=parent[root]);
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        lead = parent[trail];
        parent[trail]= root;
    }
    return root:
}
```

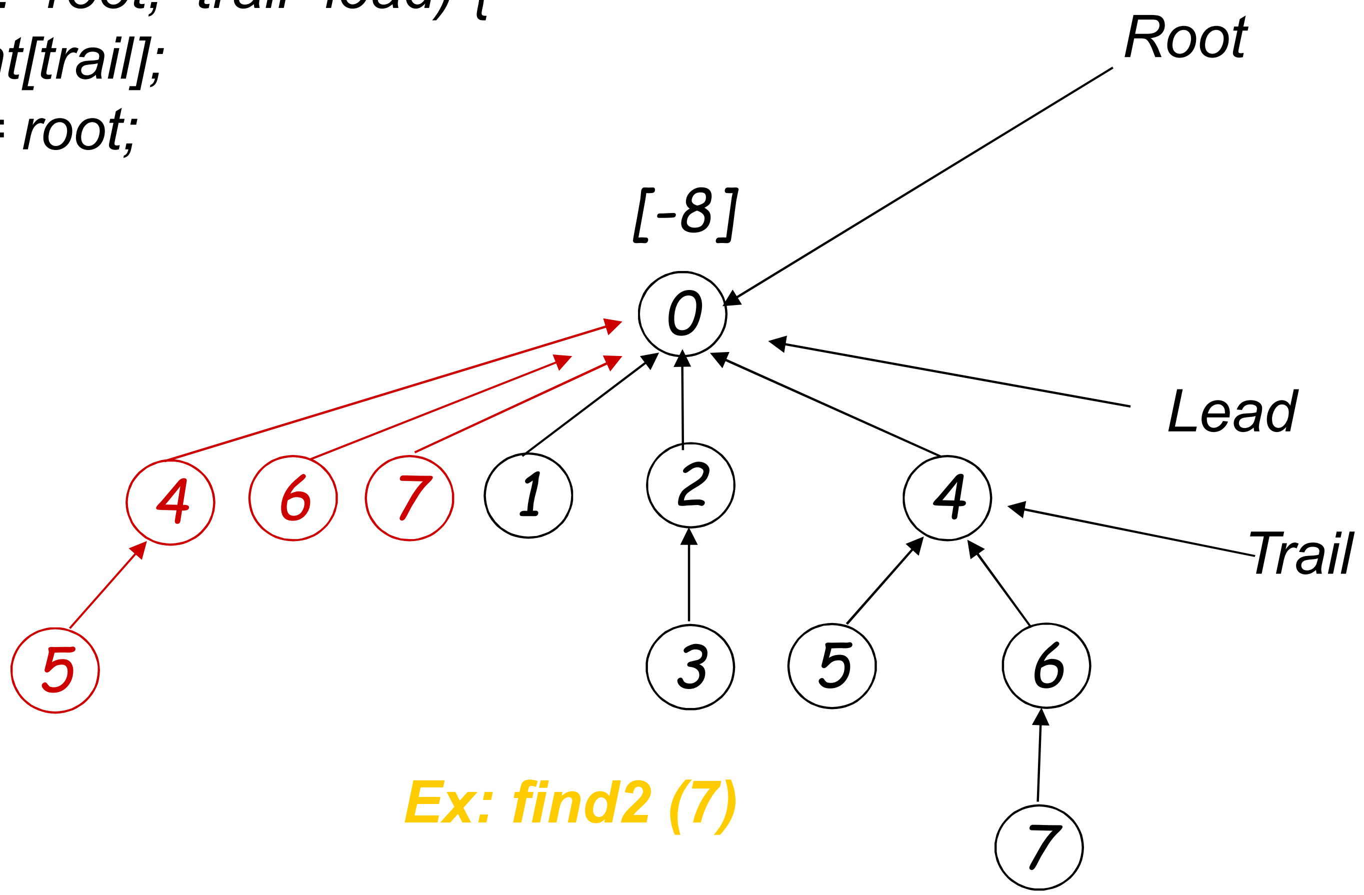

WeightedUnion and CollapsingFind

- Analysis of WeightedUnion and CollapsingFind
  - The use of collapsing rule roughly double the time for an individual find. However, it reduces the worst-case time over a sequence of finds.

```
int find2(int i)
{
    int root, trail, lead;
    for (root=i; parent[root]>=0; root=parent[root]);
    for (trail=i; trail!=root;  trail=lead) {
        lead = parent[trail];
        parent[trail]= root;
    }
    return root:
}
```

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Textbook chapter 12

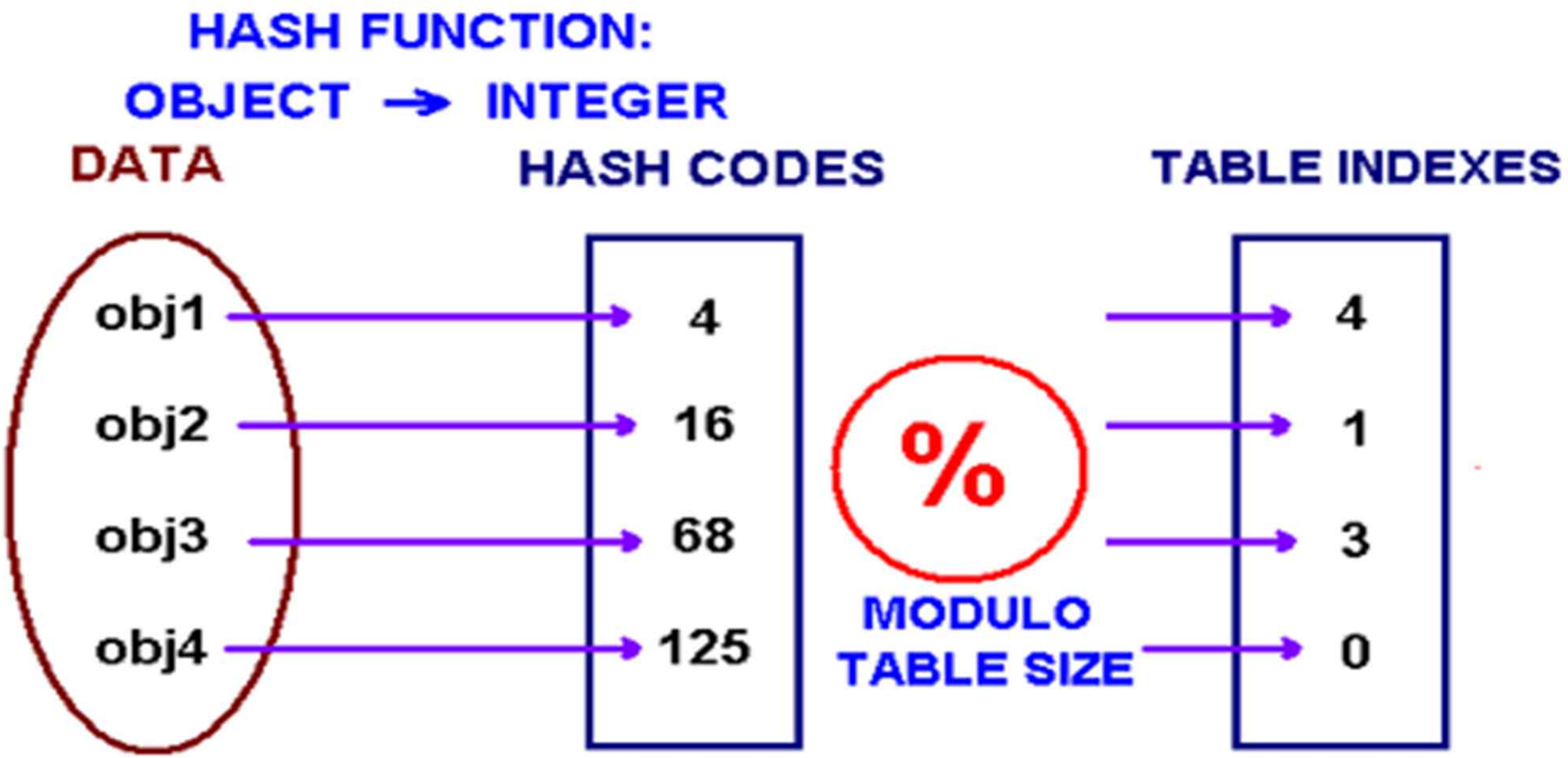
HASHING

Searching in General

- Consider the problem of searching an array for a given value
  - If the array is ***not sorted***, the search requires O(*n*) time
    - If the value isn't there, we need to search all *n* elements
    - If the value is there, we search *n*/2 elements on average
  - If the array is ***sorted***, we can do in O(log *n*) time
    - A binary search requires O(log *n*) time
    - About equally fast whether the element is found or not
  - It doesn't seem like we could do much better
    - How about an **O(1)**, that is, constant time search?
    - We can do it *if the array is organized* in a particular way

Direct Mapping

- Suppose we were to come up with a MAGIC FUNCTION that, given a value to search for, would tell us exactly where in the array to look
  - If it is in that location, it is in the array
  - If it is NOT in that location, it is NOT in the array
- If we look at the function's inputs and outputs, their relationships WON'T MAKE SENSE
  - This function would have no other purpose but searching
- This function is called a HASH FUNCTION because it MAKES HASH of its inputs



HASH TABLE

OBJ4	OBJ2		OBJ3	OBJ1
0	1	2	3	4

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Hash Function Definition

- A hash function is a function that:
  - When applied to an Object, returns a number
  - When applied to *equal* Objects, returns the *same* number for each
  - When applied to ***unequal*** Objects, is ***very unlikely*** to return the same number for each
- Hash function is often called hash map
- Hash functions turn out to be very important for searching, that is, looking things up fast
  - Required for associative memory

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Hash Tables

- Motivation: symbol tables
  - A compiler uses a *symbol table* to relate symbols to associated data
    - Symbols: variable names, procedure names, etc.
    - Associated data: memory location, call graph, etc.
  - For a symbol table (also called a *dictionary*), we care about search, insertion, and deletion
- Hash table:
  - Given a table T and a record x, with key (= symbol) and satellite data, support:
    - Insert (T, x)　　Delete (T, x)　　Search(T, x)
  - We want these to be fast, but don't care about sorting the records
  - Supports all the above in O(1) expected time!

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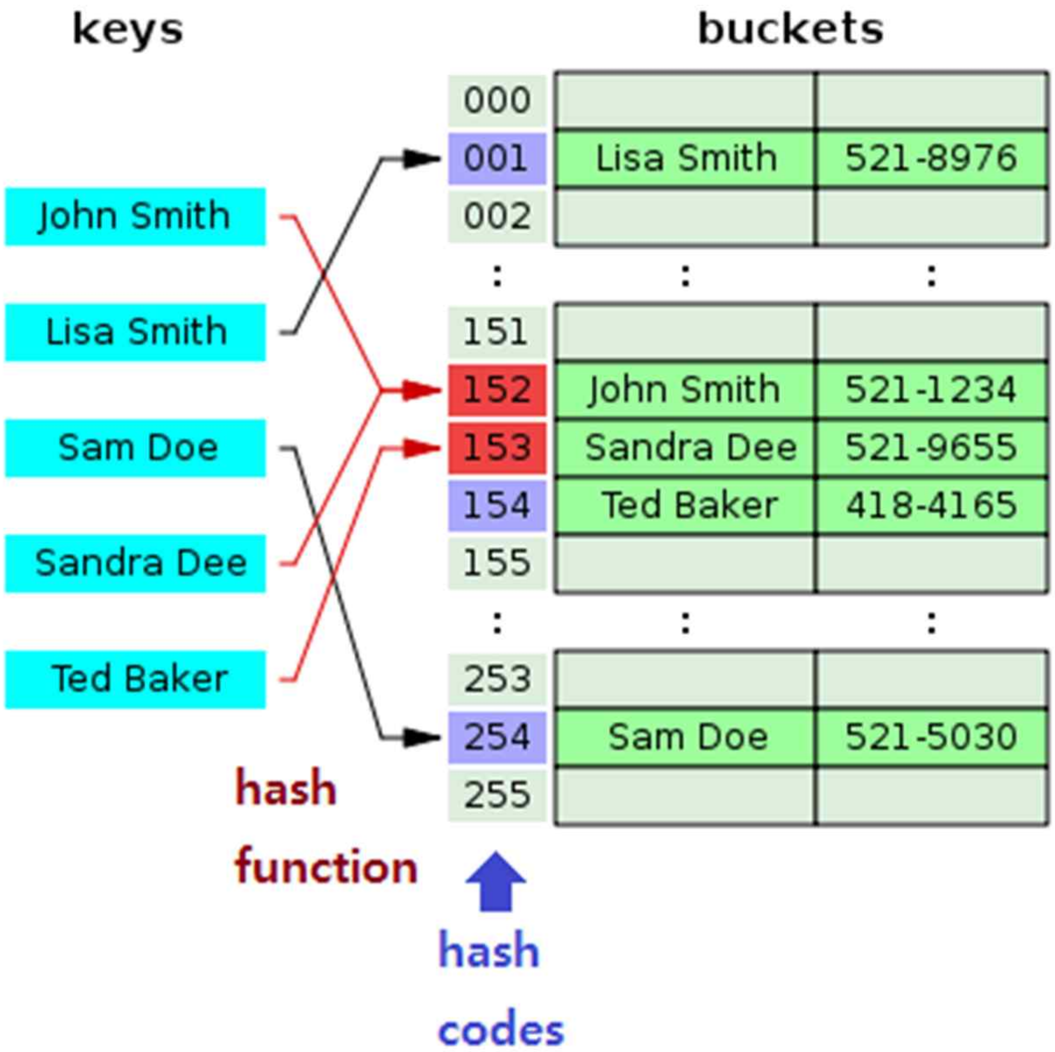
Hash Table Constituents

- Keys
  - Unique values to identify objects
- Hash function
  - Mapping from keys to hash codes
- Hash codes (values)
  - Output of the hash function
- Buckets
  - Actual data storage



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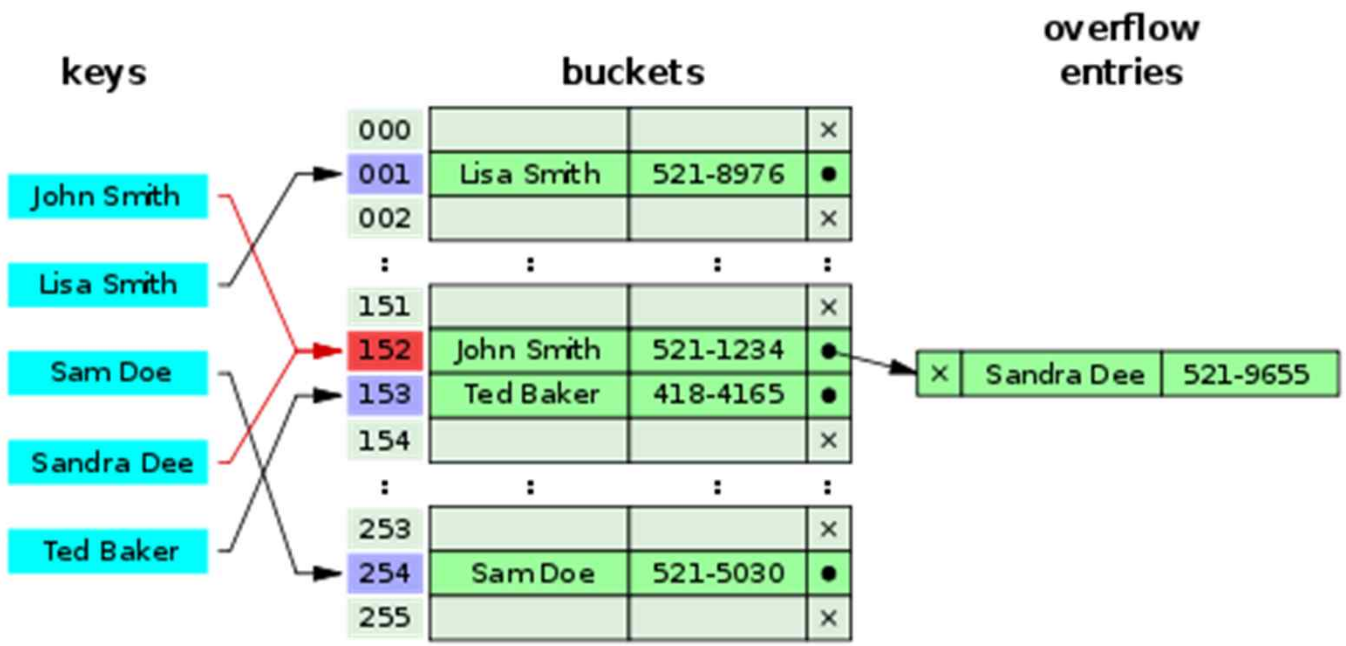
Issues in Hashing

- Keys assignment
  - How can we convert floats to natural numbers for hashing purposes?
  - How can we convert ASCII strings to natural numbers for hashing purposes?
  - How to ensure ***uniqueness***?
- Collision in hash codes
  - What if the output of hash functions are the same for distinct key values?
  - May results in bucket entry ***overflow***



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HASH FUNCTION DESIGN

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Finding the hash function

- Clearly choosing the hash function well is crucial
  - *What will a worst-case hash function do?*
  - *What will be the time to search in this case?*
- *What are desirable features of the hash function?*
  - Should distribute keys uniformly into slots
    - Collision may be unavoidable, but try being as less as possible
  - Should not depend on patterns in the data
    - Fast only for specific data patterns ➔ not desirable
- *How can we come up with MAGIC hash function?*
  - In a few specific cases, where all the possible values are known in advance, it has been possible to compute a perfect hash function

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Simple Example: Direct Addressing

- Suppose:
  - The range of keys is $0..m-1$
  - Keys are all distinct
- The idea:
  - Set up an array $T[0..m-1]$ in such a way that
    - $T[i] = x$ if $x \in T$ and $key[x] = i$
    - $T[i]$ = NULL otherwise
  - This is called a *direct-address table*
    - Operations take O(1) time!
    - *So what's the problem?*

"NULL" in this context does not mean "NULL pointer" only, but a special value to indicate that the array location is empty. For example, -1 can be used in this case because the valid key value ranges from 0 to (m-1)

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The Problem With Direct Addressing

- Direct addressing works well when the range of *m* keys is relatively small
- But what if the keys are 32-bit integers?
  - Problem 1: direct-address table will have $2^{32} = 4.295*10^9$ entries > 4 billion
  - Problem 2: even if memory is not an issue, the time to initialize the elements to NULL may be time-consuming
- Solution: map keys to smaller range

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Example (ideal) Hash Function

- Suppose our hash function gave us the following values:

```
hashCode("apple") = 5
hashCode("watermelon") = 3
hashCode("grapes") = 8
hashCode("cantaloupe") = 7
hashCode("kiwi") = 0
hashCode("strawberry") = 9
hashCode("mango") = 6
hashCode("banana") = 2
```

- Note that there is no same hash values so no collision in the buckets

0	*kiwi*
1	
2	*banana*
3	*watermelon*
4	
5	*apple*
6	*mango*
7	*cantaloupe*
8	*grapes*
9	*strawberry*

Sets and Tables

- Hashing for set:
  - Sometimes we just want a set of things—objects are either in it, or they are not in it
- Hashing for map:
  - Sometimes we want a map—a way of looking up one thing based on the value of another
  - We use a *key* to find a place in the map
  - The associated *value* is the information we are trying to look up
- Hashing works the same for both sets and maps
  - Most of our examples will be sets

	key	value
. . .		
141		
142	robin	robin info
143	sparrow	sparrow info
144	hawk	hawk info
145	seagull	seagull info
146		
147	bluejay	bluejay info
148	owl	owl info

Choosing A Hash Function

- Choosing the hash function well is crucial
  - Bad hash function puts all elements in same slot
  - A good hash function:
    - Should distribute keys uniformly into slots
    - Should not depend on patterns in the data
- Three methods:
  - Division method
  - Multiplication method
  - Universal hashing

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Hash Functions: The Division Method

- $h(k) = k \bmod m$
  - In words: hash k into a table with m slots using the slot given by the remainder of k divided by m
- *What happens to elements with adjacent values of k?*
  - Elements with adjacent keys hashed to different slots: good
- *What happens if m is a power of 2 (say 2^P)?*
  - If keys bear relation to m: bad
- *What if m is a power of 10?*
- Upshot: pick table size m = prime number not too close to a power of 2 (or 10)

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The Multiplication Method

- For a constant A, $0 < A < 1$:
- $h(k) = \lfloor m\ \underbrace{(kA - \lfloor kA \rfloor)}_{\textit{Fractional part of } kA} \rfloor$

- Choose $m = 2^P$
- Choose A not too close to 0 or 1
- Knuth: Good choice for $A = (\sqrt{5} - 1)/2$

Hash Functions: Universal Hashing

- *Universal hashing*: pick a hash function randomly when the algorithm begins (*not* upon every insert!)
  - Guarantees good performance on average, no matter what keys adversary chooses
  - Need a family of hash functions to choose from

- Let ς be a (finite) collection of hash functions
  - that map a given universe U of keys into the range $\{0, 1, \ldots, m - 1\}$.
- ς is said to be *universal* if:
  - for each pair of distinct keys $x, y \in U$, the number of hash functions h $\in \varsigma$ for which $h(x) = h(y)$ is $|\varsigma|/m$
  - With a random hash function from ς, the chance of a collision between x and y is exactly $1/m$ $(x \neq y)$

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Universal Hashing: Theorem 12.3

- Choose *h* from a universal family of hash functions
- Hash *n* keys into a table of *m* slots, $n \leq m$
- Then the expected number of collisions involving a particular key *x* is less than 1
- Proof:
  - For each pair of keys *y*, *z*, let c_{yx} = 1 if *y* and *z* collide, 0 otherwise
  - $E[c_{yz}] = 1/m$ (by definition)
  - Let C_x be total number of collisions involving key *x*

$$E[C_x] = \sum_{\substack{y \in T \\ y \neq x}} E[c_{xy}] = \frac{n-1}{m}$$

  - Since $n \leq m$, we have $E[C_x] < 1$

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A Universal Hash Function

- Choose table size *m* to be prime
- Decompose key *x* into *r*+1 bytes, so that $x = \{x_0, x_1, \ldots, x_r\}$
  - Only requirement is that max value of byte < *m*
  - Let $a = \{a_0, a_1, \ldots, a_r\}$ denote a sequence of *r*+1 elements chosen randomly from $\{0, 1, \ldots, m - 1\}$
  - Define corresponding hash function $h_a \in \varsigma$:

$$h_a(x) = \sum_{i=0}^{r} a_i x_i \bmod m$$

  - With this definition, ς has m^{r+1} members

A Universal Hash Function

- ς is a universal collection of hash functions (Theorem 12.4)
- How to use:
  - Pick *r* based on *m* and the range of keys in *U*
  - Pick a hash function by (randomly) picking the *a*'s
  - Use that hash function on all keys

Imperfect Hash Function

- Suppose our hash function gave us the following values:

```
hash("apple") = 5
hash("watermelon") = 3
hash("grapes") = 8
hash("cantaloupe") = 7
hash("kiwi") = 0
hash("strawberry") = 9
hash("mango") = 6
hash("banana") = 2
```

- Now the new item's bucket is already filled:

```
hash("honeydew") = 6
```

- *Now what?*

0	*kiwi*
1	
2	*banana*
3	*watermelon*
4	
5	*apple*
6	*mango*
7	*cantaloupe*
8	*grapes*
9	*strawberry*

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Collisions

- A perfect hash function would tell us exactly *where to look*.
- Imperfect hash function gives
  - Same hash values for different key values – called a ***collision***.
  - Collisions are normally treated as "first come, first served"—the first value that hashes to the location gets it
- What is the next best thing?
  - The best we can do is a function that tells us *where to start looking*!
  - Find something to do with the second and subsequent values that hash to this same location

RESOLVING COLLISIONS

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Handling collisions

- What can we do when two different values attempt to occupy the same place in an array?
  - **Solution #1 (open addressing):** Search from there for an empty location
    - Can stop searching when we find the value *or* an empty location
    - Search must be end-around
  - **Solution #2 (rehashing):** Use a 2^{nd} hash function
    - ...and a third, and a fourth, and a fifth, ...
  - **Solution #3 (chaining):** Use the array location as the header of a linked list of values that hash to this location
- All these solutions work, provided:
  - We use the same technique to *add* things to the array as we use to *search* for things in the array

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Open Addressing

- Basic idea (details in Section 12.4):
  - To insert: if slot is full, try another slot, ..., until an open slot is found (probing)
  - To search, follow same sequence of probes as would be used when inserting the element
    - If reach element with correct key, return it
    - If reach a NULL pointer, element is not in table
- Good for fixed sets (adding but no deletion)
  - Example: spell checking
- Table needn't be much bigger than *n*

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Open Addressing: Insertion, I

- Suppose you want to add seagull to this hash table
- Also suppose:
  - hashCode(seagull) = 143
  - table[143] is not empty
  - table[143] != seagull
  - table[144] is not empty
  - table[144] != seagull
  - table[145] is empty
- Therefore, put seagull at location 145

Open Addressing: Searching, I

- Suppose you want to look up seagull in this hash table
- Also suppose:
  - hashCode(seagull) = 143
  - table[143] is not empty
  - table[143] != seagull
  - table[144] is not empty
  - table[144] != seagull
  - table[145] is not empty
  - table[145] == seagull !
- We found seagull at location 145

. . .	
141	
142	*robin*
143	*sparrow*
144	*hawk*
145	*seagull*
146	
147	*bluejay*
148	*owl*
. . .	

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Open Addressing: Searching, II

- Suppose you want to look up cow in this hash table
- Also suppose:
  - hashCode(cow) = 144
  - table[144] is not empty
  - table[144] != cow
  - table[145] is not empty
  - table[145] != cow
  - table[146] is empty
- If cow were in the table, we should have found it by now
- Therefore, it isn't here

...	
141	
142	robin
143	sparrow
144	hawk
145	seagull
146	
147	bluejay
148	owl
...	

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Open Addressing: Insertion, II

- Suppose you want to add hawk to this hash table
- Also suppose
  - hashCode(hawk) = 143
  - table[143] is not empty
  - table[143] != hawk
  - table[144] is not empty
  - table[144] == hawk
- hawk is already in the table, so do nothing

Open Addressing: Insertion, III

- Suppose:
  - You want to add cardinal to this hash table
  - hashCode(cardinal) = 147
  - The last location is 148
  - 147 and 148 are occupied
- Solution:
  - Treat the table as circular; after 148 comes 0
  - Hence, cardinal goes in location 0 (or 1, or 2, or ...)

. . .	
141	
142	*robin*
143	*sparrow*
144	*hawk*
145	*seagull*
146	
147	*bluejay*
148	*owl*

Efficiency

- Hash tables are actually surprisingly efficient
- Until the table is about 70% full, the number of probes (places looked at in the table) is typically only 2 or 3
- Sophisticated mathematical analysis is required to prove that the expected cost of inserting into a hash table, or looking something up in the hash table, is O(1)
- Even if the table is nearly full (leading to occasional long searches), efficiency is usually still quite high

Analysis Of Hash Tables

- *Simple uniform hashing*: each key in table is equally likely to be hashed to any slot
- *Load factor* $\alpha = n/m$ = average # keys per slot
  - Average cost of unsuccessful search = $O(1+\alpha)$
  - Successful search: $O(1+ \alpha/2) = O(1+ \alpha)$
  - If n is proportional to m, $\alpha = O(1)$
- So the cost of searching = O(1) if we size our table appropriately

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Solution #2: Rehashing

- In the event of a collision, another approach is to rehash: compute another hash function
  - Since we may need to rehash many times, we need an easily computable sequence of functions
- Simple example: in the case of hashing Strings, we might take the previous hash code and add the length of the String to it
  - Probably better if the length of the string was not a component in computing the original hash function
- Possibly better yet: add the length of the String plus the number of probes made so far
  - Problem: are we sure we will look at every location in the array?
- Rehashing is a fairly uncommon approach, and we won't pursue it any further here

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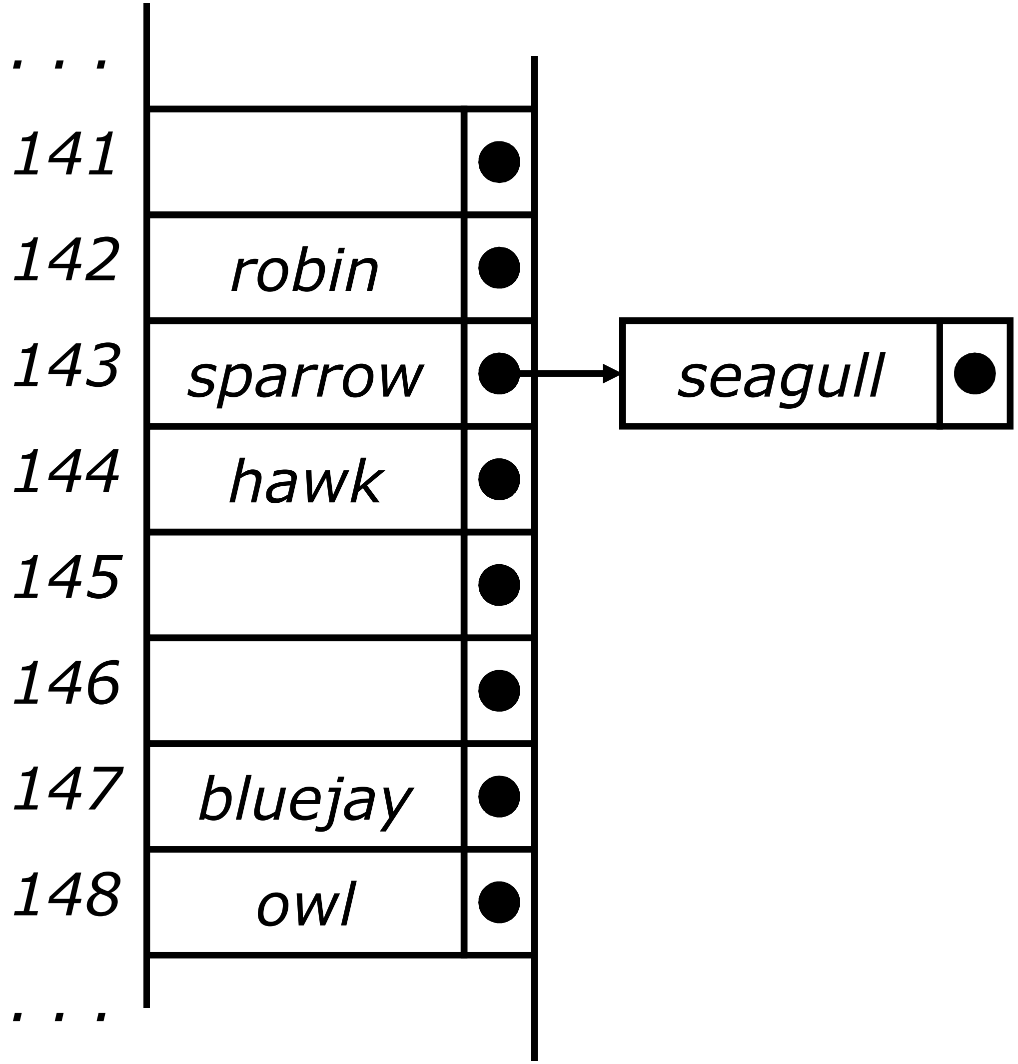
Solution #3: Chaining / Bucket hashing

- The previous solutions used open hashing: all entries went into a “flat” (unstructured) array
- Another solution is to make each array location the header of a linked list of values that hash to that location



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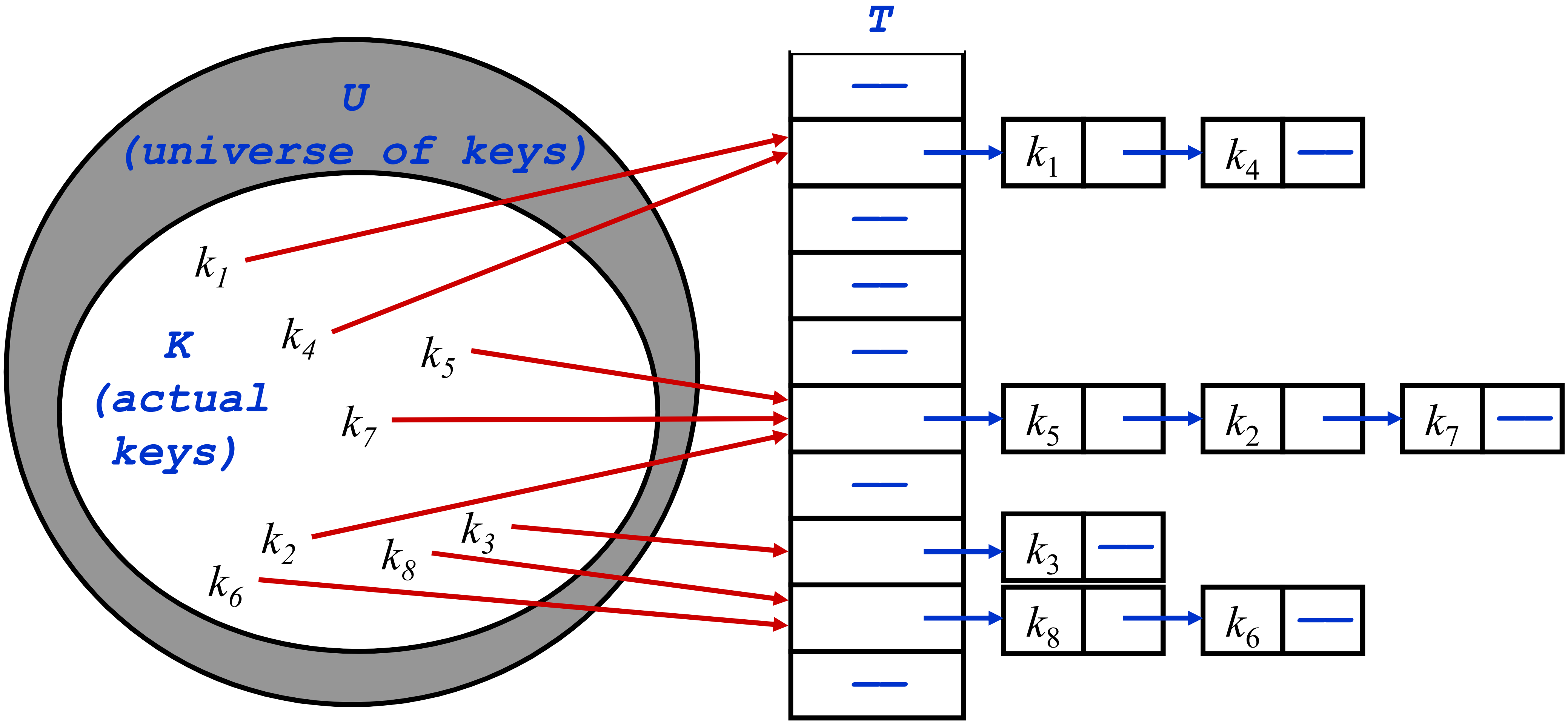
Chaining

- Chaining puts elements that hash to the same slot in a linked list:



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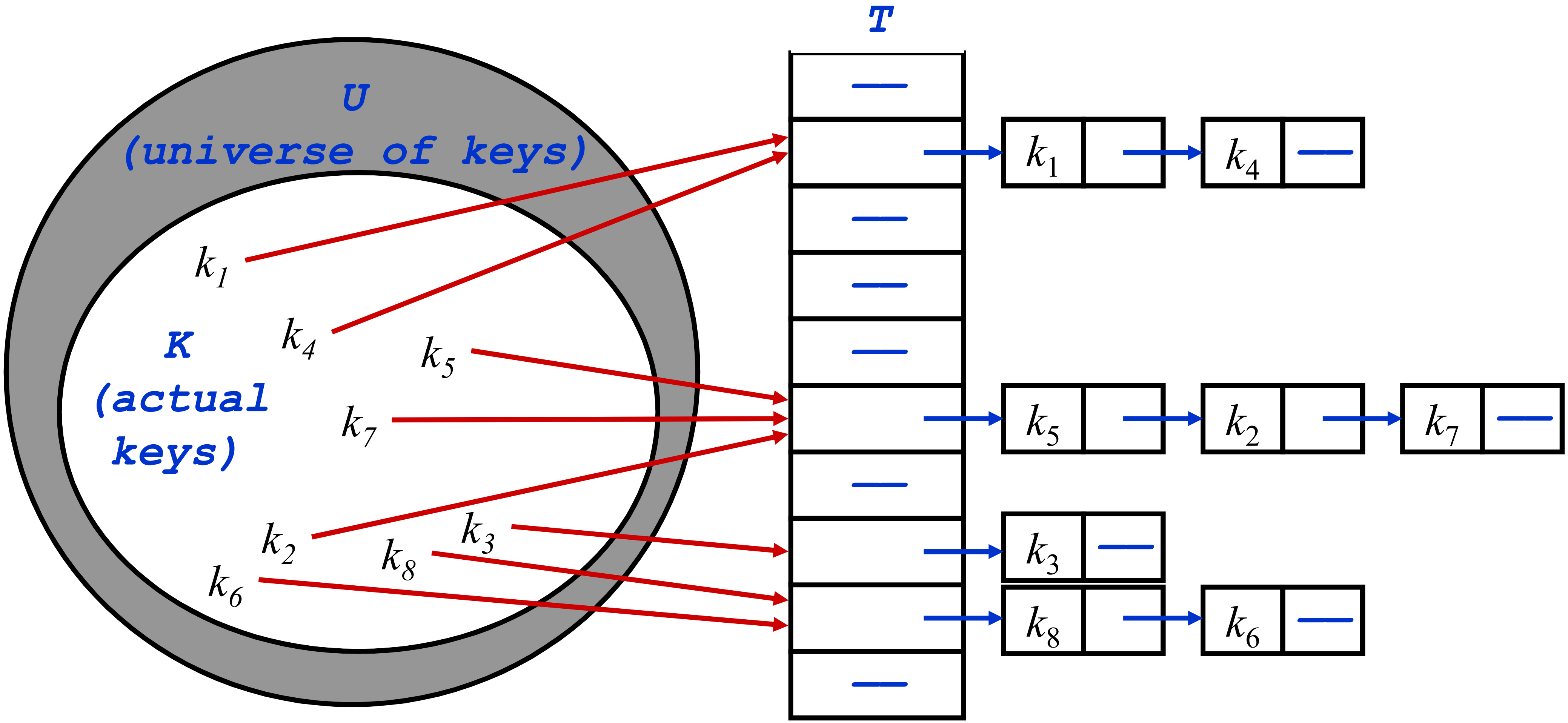
Chaining

- *How do we insert an element?*



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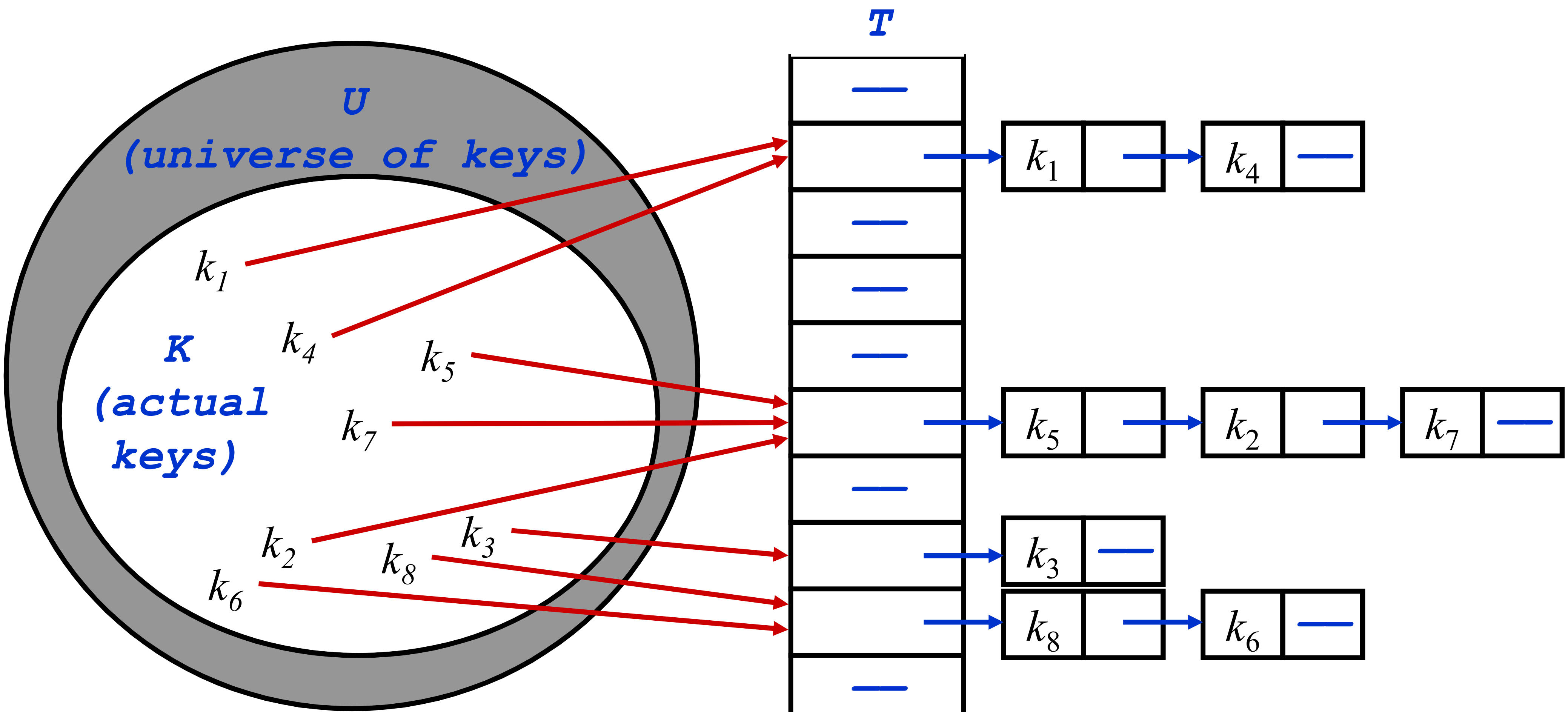
Chaining

- *How do we delete an element?*
  - *Do we need a doubly-linked list for efficient delete?*



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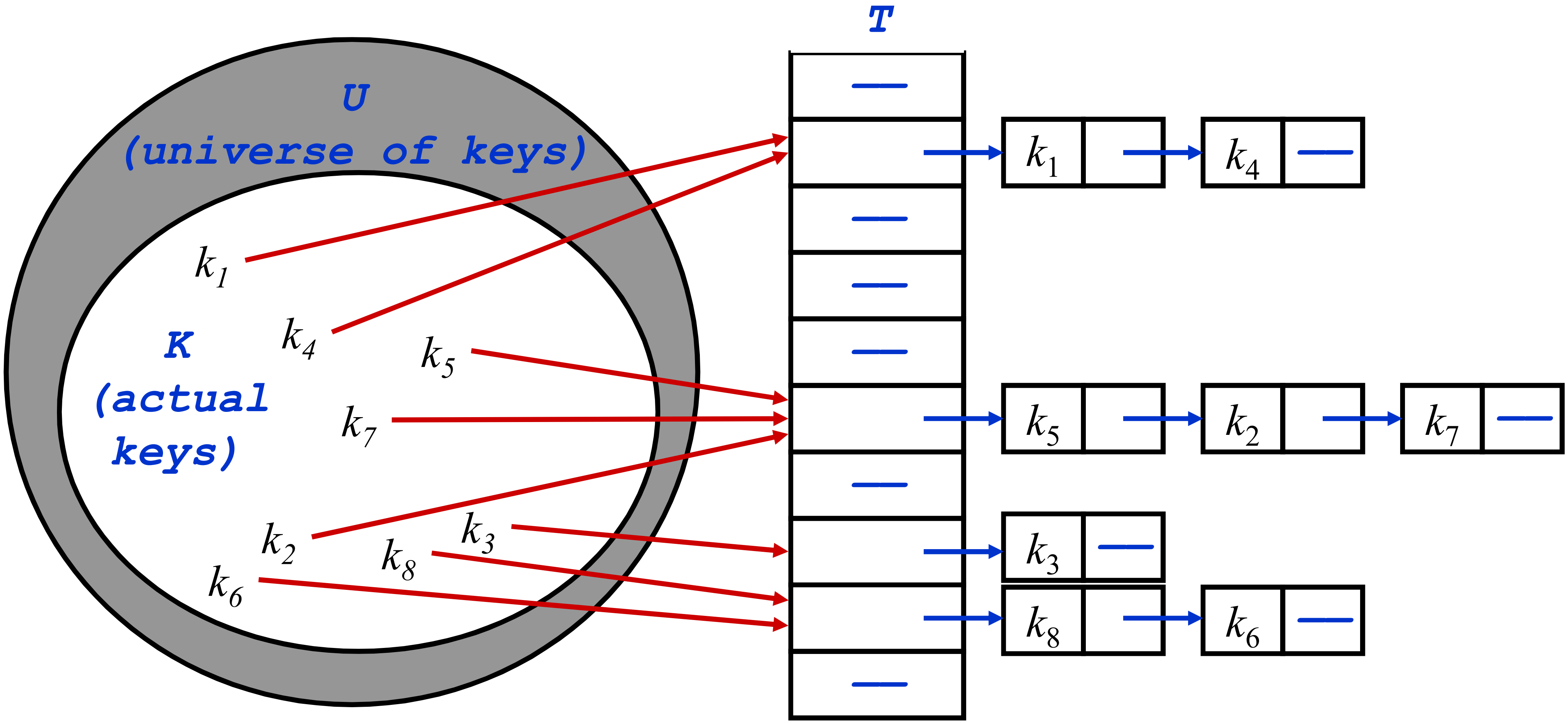
Chaining

- *How do we search for a element with a given key?*



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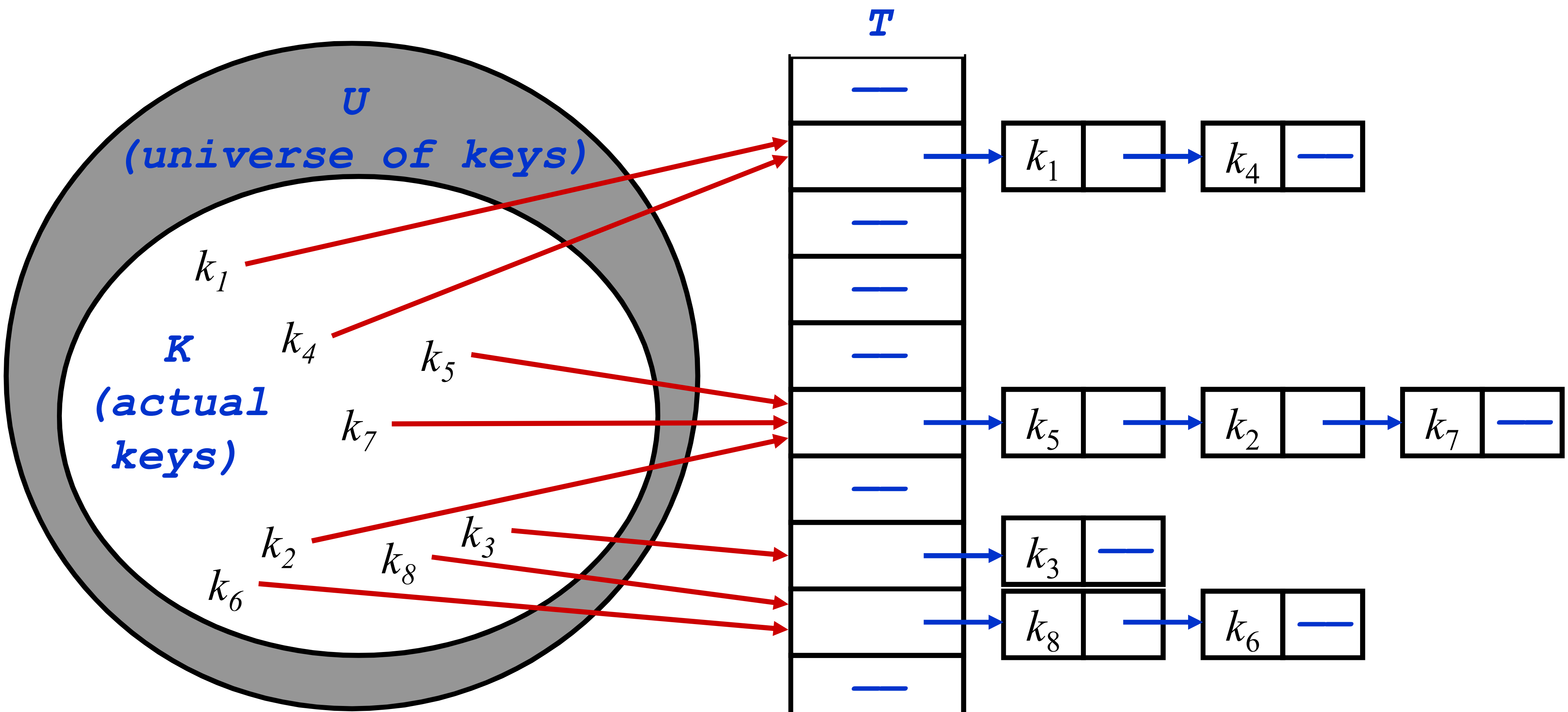
Review: Chaining

- Chaining puts elements that hash to the same slot in a linked list:



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Analysis of Chaining

- Assume *simple uniform hashing*: each key in table is equally likely to be hashed to any slot
- Given n keys and m slots, the *load factor* $\alpha = n/m$ = average # keys per slot
- *What will be the average cost of an unsuccessful search for a key?*
  - A: O(1+α)
- *What will be the average cost of a successful search?*
  - A: O(1 + α/2) = O(1 + α)
- *If the number of keys n is proportional to the number of slots in the table, what is α?*
  - A: α = O(1)
  - We can make the expected cost of searching constant if we make α constant

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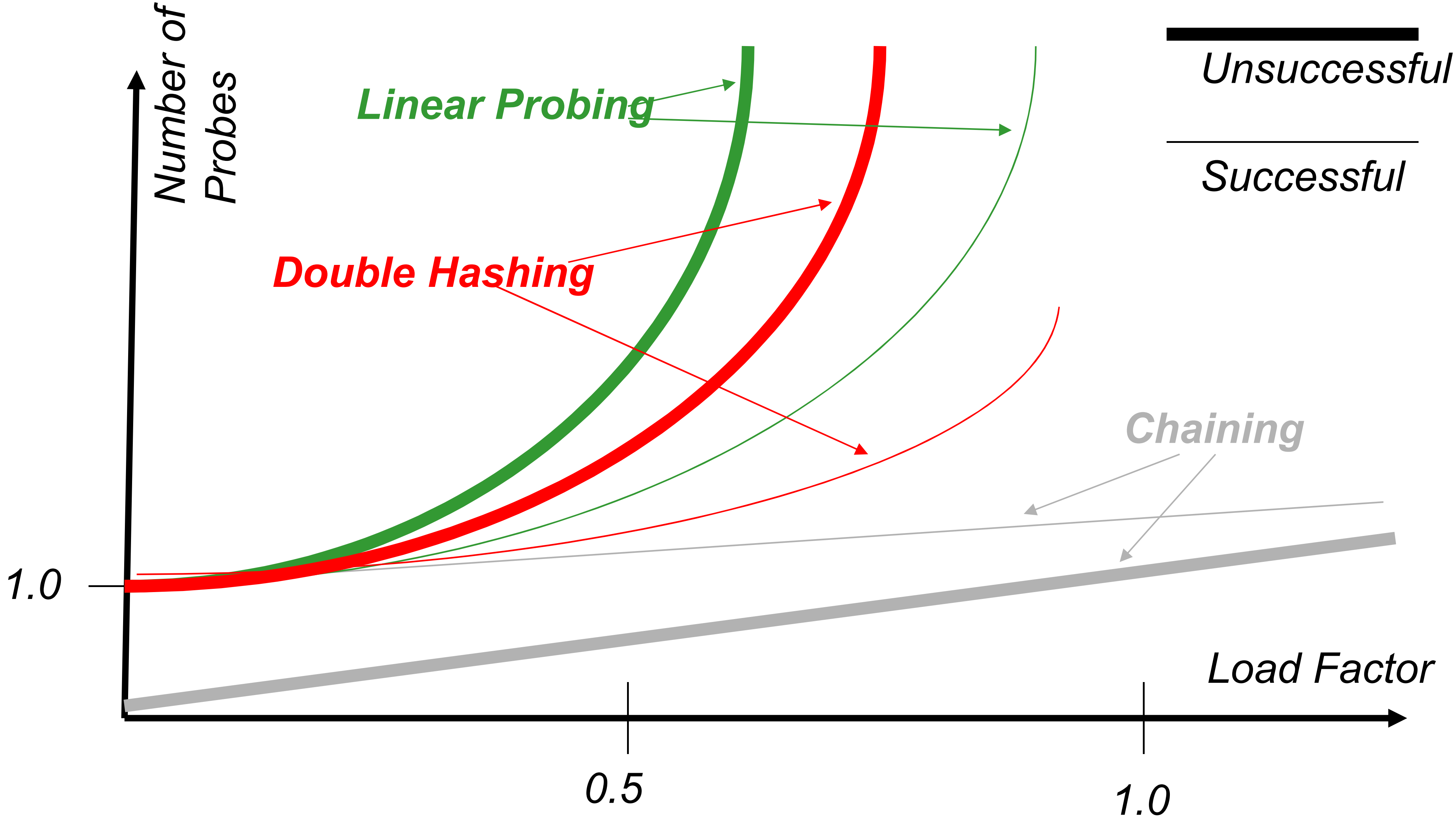
Expected Number of Probes vs. Load Factor



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END OF LECTURE 10