Assignment 5

Due Date: 2019/05/29, 1:30 PM

A function $f : A \to \mathbb{R}$ is called C^n-function on A if f is n times differentiable and $f^{(n)}$ is a continuous on A.

1. Suppose that $f : [a,\, b] \to \mathbb{R}$ is a C^1-function on $[a,\, b]$, and that $f'(x) > 0$ for all $x \in [a,\, b]$. Prove that f is a strictly increasing function on $[a,\, b]$. In other words, $f(x) > f(y)$ for all $x > y$.

2. Suppose that $f,\, g : (a,\, b) \to \mathbb{R}$ are differentiable functions on $(a,\, b)$, and that $g(x) \neq 0$ for all $x \in (a,\, b)$. Prove that f/g is differentiable on $(a,\, b)$, and moreover

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \text{ for all } x \in (a,\, b)\ .$$

3. Find the Taylor expansion of the following functions around 0 (in other words, in equation (8) at the page 112 of the textbook, $a = 0$).

(1) $f(x) = \sum_{k=0}^{n} c_k x^k$ (a polynomial of degree n; here $c_0,\, \ldots,\, c_n$ are real numbers)

(2) $f(x) = e^{2x+1}$

(3) $f(x) = \cos(x^2)$

4. Let $f : [a,\, b] \to \mathbb{R}$ be a C^2-function on $[a,\, b]$. In the plane $\mathbb{R}^2$, the line segement connecting two points $(a,\, f(a))$ and $(b,\, f(b))$ intersects with the graph of f at some point $(c,\, f(c))$. Prove that there exists $d \in [a,\, b]$ such that $f''(d) = 0$. (**Hint.** Lemma 4.2.1)

5. **(1)** Prove the following inequality for all $x \geq 0$ and $n \in \mathbb{N}$

$$e^x \geq \frac{x^n}{n!}$$

(2) Define $f : \mathbb{R} \to [0, \infty)$ as following:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Prove that f is a differentiable function on $\mathbb{R}$.

(3) Prove that, for each $n \in \mathbb{N}$, there exists a polynomial $Q_n(t)$ of degree $3n$ so that

$$f^{(n)}(x) = Q_n\left(\frac{1}{x}\right) e^{-1/x^2} \text{ for all } x > 0 \text{ .}$$

(4) Prove that f is a C^∞-function on $\mathbb{R}$.

(5) Prove that there exists $g : \mathbb{R} \to [0, \infty)$ satisfying all the conditions below:

- g is a C^∞-function
- $g(x) = 0$ for $x \notin (-1, 1)$
- $g(x) > 0$ for $x \in (-1, 1)$

Such a function g is called *smooth mollifier*, and plays extremely important role in the study of analysis. (**Hint.** Use the function of the form $f(x-a)$ or $f(a-x)$ in a creative way.)

6. **(1)** Let $f : (a, b) \to \mathbb{R}$ be a C^2-function on (a, b). For $x \in (a, b)$, prove the following limit.

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

(2) Let $f : (a, b) \to \mathbb{R}$ be a C^3-function on (a, b). For $x \in (a, b)$, prove the following limit.

$$\lim_{h\to 0} \frac{f(x+2h) - 3f(x+h) - f(x-h) + 3f(x)}{h^3} = f'''(x)$$

Note. Since f is not a C^∞-function, you cannot use the Taylor expansion (although it provides a good intuition on this problem). There might be a better and simple way.