March 29th, 2019

Remark. $\limsup$ is the limit of sup. If sup is easy to calculate, find sup and take the limit.

Quiz 1 Solutions

#1. Given set A, $\text{int}(A)$, A', determine whether the set is open or closed.

(1) $A = \mathbb{N} \subset \mathbb{R}$. $\text{int}(A) = \emptyset$, $A' = \emptyset$, A is closed.

(2) $\mathbb{Q} \subset \mathbb{R}$. $\text{int}(\mathbb{Q}) = \emptyset$, $\mathbb{Q}' = \mathbb{R}$, $\mathbb{Q}$ is neither open nor closed.

(3) $C = [0,1] \cup (2,3) \cap \{4\} \subset \mathbb{R}$. $\text{int}(C) = (0,1) \cup (2,3)$, $C' = [0,1] \cup [2,3]$, C is neither open nor closed.

(4) $D = \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) : 0 \leq y \leq 1\} \subset \mathbb{R}^2$. $\text{int}(D) = \emptyset$, $D' = D \cup \{(0,y) : 0 \leq y \leq 1\}$, D is neither open nor closed. ($\because \text{int}D \neq D$, $\overline{D} \neq D$)

#2. Find a limit point of given set.

(1) $A = \mathbb{Q} \subset \mathbb{R}$. 0 is a limit point. (Directly follows from Archimedes' principle)

(2) $B = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. 0 is a limit point of B. (Also directly follows from Archimedes')

(3) $C = \{2^{-n} + 3^{-m} : n, m \in \mathbb{N}\} \subset \mathbb{R}$. 0 is a limit point of C. Given $\epsilon > 0$, exists $N \in \mathbb{N}$ such that for $n, m \geq N$, $2^{-n} < \epsilon/2$, $3^{-m} < \epsilon/2$. Then $0 \neq 2^{-n} + 3^{-m} < \epsilon$.

#3. True or False? If false, find a counterexample.

(1) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ **True**

(2) $\overline{A \cap B} = \overline{A} \cap \overline{B}$ **False**. Set $A = (0,1), B = (1,2)$.
Correct Statement: $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

(3) $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$ **False**. Set $A = [0,1], B = [1,2]$.
Correct Statement: $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$

(4) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$ **True**

Thm. $A \subset B \implies \overline{A} \subset \overline{B}, \mathrm{int}(A) \subset \mathrm{int}(B)$.
Proof.

- We need to show $A' \subset B'$. Let $x \in A'$.
  $\implies \forall \epsilon > 0,\ N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset$.
  $\implies \forall \epsilon > 0, N(x, \epsilon) \cap (B \setminus \{x\}) \neq \emptyset$
  $\implies x \in B'$.

- Let $x \in \mathrm{int}(A)$
  $\implies \exists \epsilon > 0, N(x, \epsilon) \subset A \implies N(x, \epsilon) \subset B \implies x \in \mathrm{int}(B)$.

Proof of (c). $A, B \subset A \cup B$
$\implies \mathrm{int}(A), \mathrm{int}(B) \subset \mathrm{int}(A \cup B)$. Thus $\mathrm{int}(A) \cup \mathrm{int}(B) \subset \mathrm{int}(A \cup B)$

Proof of (d). $A \cap B \subset A, B \implies \mathrm{int}(A, B) \subset \mathrm{int}(A), \mathrm{int}(B)$. Thus $\mathrm{int}(A \cap B) \subset \mathrm{int}(A) \cap \mathrm{int}(B)$
Suppose $x \in \mathrm{int}(A) \cap \mathrm{int}(B)$. Then $\exists \epsilon_A, \epsilon_B > 0$ s.t. $N(x, \epsilon_A) \subset A, N(x, \epsilon_B) \subset B$. Take $\epsilon = \min\{\epsilon_A, \epsilon_B\}/2$. Then $N(x, \epsilon) \subset A, B$. Therefore $N(x, \epsilon) \subset A \cap B$, $x \in \mathrm{int}(A \cap B)$.
Example. $A = \{(x, y) : x^2 + 2y^2 < 1\}$. $\mathrm{int}(A) = A, A' = \{(x, y) : x^2 + 2y^2 \leq 1\}$.
Suppose $(x_0, y_0) \in A$. $x_0^2 + 2y_0^2 = 1 - \delta < 1$ for some $\delta > 0$. By symmetry, let $x_0, y_0 > 0$. From

$$(x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 = x_0^2 + 2y_0^2 + \epsilon(2x_0 + 4y_0 + 3\epsilon) < 1$$

, we want $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \delta$. Set $\epsilon < 1/10$. Then $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \epsilon(2x_0 + 4y_0 + 3) < \delta$.
Now set $\epsilon = \min\left\{\frac{1}{2(2x_0 + 4y_0 + 3)}, \frac{1}{100}\right\} > 0$.
Then $|x - x_0| < \epsilon, |y - y_0| < \epsilon$. $x_0^2 + 2y_0^2 < (x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 < 1$. $N((x_0, y_0), \epsilon) \subset A$.
Interior points are limit points, and for the points (x_0, y_0) on the border, consider a sequence $(x_0 - 1/n, y_0 - 1/n)$. Then the elements are in A and they converge to (x_0, y_0). Thus the border is also included in A'.

April 1st, 2019

int$A$: $x \in A$ s.t. $N(x, \epsilon) \subset A$ for some $\epsilon > 0$.
A' : $x \in \mathbb{R}^d$ s.t. $N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset$ for $\forall \epsilon > 0$
$\overline{A}$: $x \in \mathbb{R}^d$ s.t. $N(x, \epsilon) \cap A \neq \emptyset$, $\forall \epsilon > 0$, $\overline{A} = A \cup A'$

Example. $A = [0, 1) \cup \{2\}$. $1 \in A'$, $2 \notin A'$, $2 \in \overline{A}$

Prop 2.3.3 $x \in A' \implies N(x, \epsilon) \cap (A \setminus \{x\})$ 는 무한집합이다.
Proof. 유한집합이라고 가정하자. $N(x, \epsilon) \cap (A \backslash \{x\}) = \{x_1, \dots, x_n\}$ 이라 할 수 있다. Set $\delta = \min\{\|x - x_i\| : \forall i\}$. Then $N(x, \delta) \cap (A \backslash \{x\}) = \emptyset$. 모순.
그래서 사실은 공집합이 아닌 것으로 정의했지만 **사실은** 무한집합이다.

Remark. $A' \neq \emptyset \implies A$는 무한집합.
(대우) A가 유한집합이면 극한점이 존재하지 않는다. (2.2 보기 4)
(역) **거짓**. $A = \{1, 2, \cdots\}$ 이면 $A' = \emptyset$.
그러면 역이 언제 성립하나요? 다음 단원 내용!

Definition. Convergence in $\mathbb{R}^d$
Let $\langle x_n \rangle$ be a sequence in $\mathbb{R}^d$.

$$\lim_{n \to \infty} x_n = x \iff \forall \epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies \|x_n - x\| < \epsilon)$$

Exercise. $x_n = (x_n^{(1)}, \dots)$, $x = (x^{(1)}, \dots)$ 일 때, $x_n \to x \iff \forall i, x_n^{(i)} \to x^{(i)}$

Notation. $A \subset \mathbb{R}^d$; $\langle x_n \rangle$ is a sequence in $A \iff \forall n, x_n \in A$

Theorem 2.2.2

(1) $x \in A' \iff \exists \langle x_n \rangle$ in $A \setminus \{x\}$ such that $x_n \to x$

(2) $x \in \overline{A} \iff \exists \langle x_n \rangle$ in A such that $x_n \to x$

Proof.

(1) ($\implies$) $x_n \in N\left(x, \frac{1}{n}\right) \cap (A \backslash \{x\})$ 이라 하자. (공집합이 아니므로 이러한 원소가 존재한다.) 그러면 $\|x_n - x\| < 1/n$ 이므로 x_n 은 x 로 수렴한다. 그리고 $x_n \in A \backslash \{x\}$ 이므로 수열이 $A \backslash \{x\}$ 에 있다.

(2) Left as exercise. Replace $A \setminus \{x\}$ with A.

Theorem 2.2.3. The following are equivalent.

(1) F is closed.

(2) $F' \subset F$.

(3) $F = \overline{F}$

(4) For a sequence $\langle x_n \rangle$ in F, $\lim_{n\to\infty} x_n = x \implies x \in F$.

Proof.
(1) $\iff$ (3) ($\overline{F}$: smallest closed set containing F.)
(2) $\iff$ (3) 은 자명.
(1) $\iff$ (4) by the above theorem. (Thm 2.2.2)

Applications.

(1) A' is closed.
Proof. We want to show that $(A')' \subset A'$.
We want to show: $x \in (A')' \implies x \in A'$.
(A' 이 공집합이면 자명. 공집합이 아니라고 가정하고...)
Given $\epsilon > 0$, $N(x,\epsilon) \cap (A' \setminus \{x\}) \neq \emptyset$. Take an element $y \in A'$ from this set. Now set $\delta = \min\{\|x-y\|, \epsilon - \|x-y\|\}$ then we have $N(y,\delta) \cap (A \setminus \{y\}) \neq \emptyset$. ($\because y \in A'$)
$z \in N(y,\delta) \cap (A \setminus \{y\})$ 라 하자.

(a) $z \in A \setminus \{y\} \subset A$.
(b) $\|x-z\| \le \|x-y\| + \|y-z\| < \|x-y\| + \delta \le \epsilon$ ($z \in N(y,\delta)$)
(c) $\|x-z\| \ge \|x-y\| - \|y-z\| > \|x-y\| - \delta \ge 0$ (By the choice of δ.) Thus $x \neq z$.

Therefore $z \in N(x,\epsilon)$ (by (b)), $z \in A \setminus \{x\}$ (by (a), (c)).
$x \in A'$ since $N(x,\epsilon) \cap (A \setminus \{x\})$ is not empty.

(2) $A \subset \mathbb{R}$: closed and bounded $\implies \inf A = \min A$, $\sup A = \max A$. (Existence)
Proof. Let $\sup A = x \notin A$. ($\sup A \in A$ 이면 자명)
Claim. $x \in A'$.
Proof of Claim. $\forall \epsilon > 0$, $N(x,\epsilon) = (x-\epsilon, x+\epsilon)$
$x = \sup A$ 이므로 $x - \epsilon$ is not an upper bound.
$\exists y$ such that $y \in (x-\epsilon, x)$
$y \in N(x,\epsilon) \cap (A \setminus \{x\}) \neq \emptyset$ 이므로 x 는 극한점.
따라서 $x \in A' \subset A$ (closed set 이므로 Thm 2.2.3 (2)) 모순.
$\sup A \in A$ 이므로 이 값이 최댓값이다.

2.3 유계집합과 코시수열

핵심: Thm 2.3.4, Thm 2.3.7

Definition. $\langle x_n \rangle$: 유계수열(bounded sequence) $\iff \exists M > 0$ s.t. $\|x_n\| \le M$ for all $n \in \mathbb{N}$.

Definition. $n_1 < n_2 < \cdots$: sequence in $\mathbb{N}$ 이라 하자. $\langle x_{n_k} \rangle_{k=1}^{\infty} = (x_{n_1}, x_{n_2}, \dots)$ 를 $\langle x_n \rangle$의 부분수열(subsequence)이라 한다.

Theorem 2.3.4 (Bolzano-Weierstrass Theorem)
If $\langle x_n \rangle$ is bounded, there exists a convergent subsequence of $\langle x_n \rangle$.

Idea of Proof. Equivalent formulation for sets.

Definition. Set A is bounded $\iff \exists M > 0$ such that $\|x\| < M$ for all $x \in A$.

Theorem 2.3.2 (Equivalent of 2.3.4) A가 유계이고 무한집합이면, $A' \neq \emptyset$.

Remark. $A' \neq \emptyset \implies A$: 무한집합.
역이 성립하기 위해서는 A가 유계라는 조건이 필요하다.

극한점이 중요한 이유는 계속 수열과 관련이 있기 때문이다.
Example. $A = \{1/n : n \in \mathbb{N}\}$ 을 고려하는 것은 수열 $x_n = 1/n$ 을 고려하는 것이나 마찬가지이다. 이 수열 x_n 이 x 로 수렴하는 것은 $A' = \{x\}$ 와 동치이다. (Hence the name “limit point”)
이로부터 $x \in A' \iff$ Exists a subsequence of $\langle x_n \rangle$ in $A \setminus \{x\}$ converging to x.

Proof of 2.3.2

(1) **Lemma 2.3.1** 축소구간정리 in $\mathbb{R}^d$.
B is a closed box in $\mathbb{R}^d \iff B = I_1 \times I_2 \times \cdots \times I_d$, where $I_i = [a_i, b_i]$ for $i = 1, \dots, d$.
(I_i is a closed and bounded interval.)

$$B_1 \supset B_2 \supset \cdots \implies \bigcap_{n=1}^{\infty} B_n \neq \emptyset$$

Proof. 각 ‘좌표’ I_i 별로 1차원 축소구간정리를 적용하면 된다.

(2) **Divide and Conquer Strategy**
B: Box 일 때, $\mathrm{diam}(B) = \sup\{\|x - y\| : x, y \in B\} = \sqrt{(a_1 - b_1)^2 + \cdots + (a_d - b_d)^2}$
Claim. There exists closed boxes $B_1, B_2, \dots$ s.t.

(a) $B_1 \supset B_2 \supset \cdots$
(b) $\mathrm{diam} B_n = \frac{1}{2^{n-1}} \mathrm{diam} B_1$

(c) $B_n \cap A$: 무한집합

Proof. (Induction) $n = 1$; B_1: 충분히 커서 $A \subset B_1$ 인 box 를 잡으면 된다.
Suppose we have $B_1, \ldots, B_n$; B_n을 2^d 등분하면 적어도 하나는 A의 원소를 무한개 포함하고 있다. 그 집합을 B_{n+1} 으로 잡는다. (비둘기집의 원리)
이제 $x \in \bigcap_{n=1}^{\infty} B_n$ 으로 잡으면 (축소구간정리에 의해 잡을 수 있다) $x \in A'$. ($A' \neq \emptyset$)
$\because \forall \epsilon > 0$, $\mathrm{diam} B_n < \epsilon$ 인 $N \in \mathbb{N}$ 을 찾아 $n \geq N$ 일 때 부등식이 성립하도록 할 수 있다.
이러한 n 들에 대하여 $B_n \subset N(x, \epsilon)$. 그러면 $N(x, \epsilon) \cap (A \setminus \{x\}) \supset B_n \cap (A \setminus \{x\})$.

April 3rd, 2019

우리가 지금 2.3 을 하고 있는데, 2 가지 중요한 결과가 있어요.

Theorem 2.3.4 $\langle x_n \rangle$ 이 bounded 이면 수렴하는 부분수열을 갖는다.[1]

Theorem 2.3.2 A가 유계인 집합이고 무한집합이면 극한점을 가진다. $A' \neq \emptyset$
증명은 축소구간정리를 박스로 확장해가지고 분할 정복하면 된다.

Recall 2.3.3 $x \in A' \implies N(x, \epsilon) \cap (A \setminus \{x\})$ 는 무한집합이다.

Proof of 2.3.4. $A = \{x_1, x_2, \dots, x_n\}$ 라고 하면 이 집합은 유계이다. (수열이 유계이므로)

(1) A가 유한집합: 자명.
$\exists x$ such that x appears infinitely many times in $\langle x_n \rangle$. (PHP) 이 경우에는 부분수열을 $x, x, \dots$ 로 잡으면 된다. 이는 수렴하는 부분수열이다.

(2) A가 무한집합[2]
$A' \neq \emptyset$ 이므로 $\alpha \in A'$ 이라 하자.
Claim. $\exists n_1 < n_2 < \dots$ such that $\|x_{n_k} - \alpha\| < 1/k$.
Proof. (첨자들이 증가하면서 가까워져야 한다는 것이 유일하게 tricky 한 부분이다. 귀납법을 사용하자.) $k = 1$: $x_{n_1} \in N(\alpha, 1) \cap (A \setminus \{\alpha\})$ 로 잡으면 된다.
$x_{n_1}, \cdots, x_{n_k}$를 잡았다고 가정: $N(\alpha, \frac{1}{k+1}) \cap (A \setminus \{\alpha\})$ 에서 $x_{n_{k+1}}$를 잡아야 하는데 이 집합은 무한집합이다. (Recall 2.3.3) 이 집합에서 첨자가 n_k보다 큰 항이 반드시 존재하므로 그 중 하나를 $x_{n_{k+1}}$ 이라 잡으면 된다.
따라서 $\lim_{k \to \infty} x_{n_k} = \alpha$ (Check as exercise)

Application. (Characterization of $\limsup$ and $\liminf$)
x_n 이 bounded 이면, $A = \{x : \exists \text{ subsequence of } x_n \text{ converging to } x\}$. 이 때 Theorem 2.3.4에 의해 $A \neq \emptyset$ 임을 증명하였다.

(1) A: closed and bounded $\implies \max(A), \min(A)$ 가 존재한다.
Proof. $B = \{x_1, x_2, \dots\}$, $C = \{\langle x_n \rangle$ 에 무한 번 나타나는 수$\}$ 로 잡자. $A = B' \cup C, C \subset B, C' \subset B'$ 임을 확인해보라! 이를 이용하면 $B' \cup C = (B' \cup C') \cup C = B' \cup (C' \cup C) = B' \cup \overline{C}$ 가 되어 닫힌집합의 합집합은 닫힌 집합이다. A는 closed and bounded 이다.

(2) $\limsup x_n = \max(A)$, $\liminf x_n = \min(A)$
(부분수열이 가질 수 있는 극한값들 중 가장 큰 값이 $\limsup$, 가장 작은 값이 $\liminf$)

[1] 증명이 가장 테크니컬 해요!
[2] 이제 Thm 2.3.2 를 사용할 수 있다. 사실 경우를 나눈 것은 예외적인 case 를 처리하기 위한 것이었다.

Proof. Recall

$$\limsup x_n = \alpha \iff \begin{cases} \text{(i) } \forall \epsilon > 0, \exists N \text{ s.t } (n \geq N \implies x_n < \alpha + \epsilon) \\ \text{(ii) } \forall \epsilon > 0, x_n > \alpha - \epsilon \text{ for infinitely many } n \end{cases}$$

(a) 부분수열 $\langle x_{n_k} \rangle \to \beta$ 이면 (i)에 의해 $k \geq N \implies x_{n_k} < \alpha + \epsilon$ 이 되어 $\beta \leq \alpha + \epsilon$. $\beta \leq \alpha$. 그러므로 $\max(A) \leq \alpha$ 이다.

(b) $\forall \epsilon > 0$, (i), (ii)에 의해 $x_n \in (\alpha - \epsilon, \alpha + \epsilon)$ 인 n 이 무한히 많다. 이 유계인 구간에 속하는 수열의 항들에 대해 부분수열을 잡아 (further subsequence) γ 로 수렴하도록 할 수 있다. (Theorem 2.3.4) 그러면 $\langle x_{m_k} \rangle \to \gamma \in [\alpha - \epsilon, \alpha + \epsilon]$. 따라서 $\alpha - \epsilon \leq \gamma \leq \max(A)$ 가 되어 $\alpha \leq \max(A)$.

따라서 $\max(A) = \alpha$.

Definition. $\langle x_n \rangle$: **Cauchy Sequence** $\iff \forall \epsilon > 0, \exists N \text{ s.t. } [m, n \geq N \implies \|x_m - x_n\| < \epsilon]$

Prop 2.3.6, Thm 2.3.8 $\langle x_n \rangle$: convergent $\iff \langle x_n \rangle$: Cauchy sequence[3]

Proof. ($\implies$) 자명. $\|x_m - x_n\| \leq \|x_m - \alpha\| + \|x_n - \alpha\| < \epsilon/2 + \epsilon/2 = \epsilon$ 인 $m, n \geq N$ 존재.

($\impliedby$) 수렴 값이 없는 상태에서 증명해야 한다. 먼저 수렴 값을 찾아보자.

(1) $\langle x_n \rangle$ is bounded.
Proof. $\exists N$ s.t. $\|x_m - x_n\| < 1$ for all $m, n \geq N$.
Set $M = \max\{\|x_1\|, \dots, \|x_{N-1}\|, \|x_N\| + 1\}$. ($\|x_m\| < \|x_N\| + 1$)
따라서 $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

(2) There exists a subsequence $\langle x_{n_k} \rangle$ converging to some α. (Thm 2.3.4)

(3) $\langle x_n \rangle$ converges to α.
Proof. $\epsilon > 0$ 에 대해,

(a) 코시 수열의 성질에 의해 $\exists N_1$ s.t. $\|x_m - x_n\| < \epsilon/2$ for all $m, n \geq N_1$.

(b) 부분수열이 α로 수렴하므로 $\exists N_2$ s.t. $\|x_{n_k} - \alpha\| < \epsilon/2$ for all $k \geq N_2$.

Let $N = \max\{N_1, N_2\}$. $n \geq N_1, n_N \geq n_{N_1} \geq N_1$ 이므로,

$$n > N \implies \|x_n - \alpha\| \leq \|x_n - x_{n_N}\| + \|x_{n_N} - \alpha\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

[3] 중간고사 전 까지 가장 중요한 정리.

Remark. 우리의 여정을 돌아보자.

(1) Archimedes' Principle 을 가정하면
Completeness Axiom $\Longrightarrow$ Monotone Convergence Theorem $\Longrightarrow$ 축소구간정리 $\Longrightarrow$ Bolzano-Weierstrass Theorem $\Longrightarrow$ **Cauchy Convergent Theorem**[4]
(Exercise) $\Longrightarrow$ Completeness Axiom

(2) **Example**. $X = C([0,1])$. (Set of functions that are continuous in $[0,1]$) How would we define $\|f-g\|$? $\int_0^1 |f(x)-g(x)|\,dx$? $\max\{|f(x)-g(x)| : x \in [0,1]\}$? Only the second choice gives completeness for X.

(3) **Convergence Test** without limit value. (**Theorem 2.3.9**)
$\sum_{n=1}^{\infty} a_n$ is convergent $\iff \forall\epsilon > 0, \exists N$ s.t. $(n > m \geq N \implies |a_{m+1} + \cdots + a_n| < \epsilon)$
Proof. Trivial.

Definition. $\sum a_n$ is **absolutely convergent** $\iff \sum |a_n|$ is convergent

Theorem. An absolutely convergent series converges.
Proof. Suppose $\sum |a_n|$ converges. For $\forall\epsilon > 0$, there exists N such that $||a_{m+1}| + \cdots + |a_n|| < \epsilon$ for all $m, n \geq N$. Therefore, for $m, n \geq N$,

$$|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \epsilon$$

and $\sum a_n$ converges.

[4] In any metric spaces, this is the condition for completeness.

April 5th, 2019

Theorem. $\overline{A} \cup \overline{B} = \overline{A \cup B}$

Proof. ($\subset$) Trivial.

($\supset$) $A \subset \overline{A}$, $B \subset \overline{B} \implies A \cup B \subset \overline{A} \cup \overline{B} \implies \overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. The closure of a closed set is itself.

6. (2) $a_n = \cos\sqrt{2019 + n^2\pi^2}$

Consider $\delta > 0$, such that

$$(n\pi - \delta)^2 < 2019 + n^2\pi^2 < (n\pi + \delta)^2$$
$$-2n\pi < \frac{2019}{\delta} \pm \delta < 2n\pi$$

We can find large enough N such that the above inequality holds for $n \geq N$.

Now we want $b_n = \sqrt{2019 + n^2\pi^2}$ bounded by $n\pi \pm \delta$.

$n \geq N$, n even $\implies n\pi - \delta < b_n < n\pi + \delta$

$\implies 1 \geq a_n > 1 - \epsilon$

$n \geq N$, n odd $\implies -1 \leq a_n < -1 + \epsilon$

Problem 2.3.5

(1) $x_{n+2} = \dfrac{x_n + x_{n+1}}{2}$

(2) $x_{n+1} = x_n + x(-1)^n \dfrac{1}{3n+1}$

Solution.

(1) Write $x_{n+2} - x_{n+1} = a(x_{n+1} - x_n)$ and observe that $a = -1/2$. Write as

$$x_n = x_{n-1} + \left(-\frac{1}{2}\right)^{n-2} (x_2 - x_1)$$

Then we have

$$x_n = x_2 + \sum_{k=1}^{n-2} \left(-\frac{1}{2}\right)^k (x_2 - x_1)$$

This series converges to $\dfrac{2x_2 + x_1}{3}$.

(2) This is an alternating series. Write as

$$x_n = x_1 + \sum_{k=1}^{n-1} (-1)^k \frac{x}{3n+1}$$

By alternating series test, the second summation term converges, and the series converges to x_1.

Since a converging sequence is a Cauchy sequence, x_1, x_2 can be any real number.

April 8th, 2019

Section 2.3: Bolzano-Weierstrass Theorem, Cauchy Convergent Theorem
In section 2.4, we will be studying about Convergence Tests.
정
2.4 급수의 수렴판정
Cor 2.3.9. $\sum_{n=1}^{\infty} a_n$ is convergent $\iff$ $s_n = \sum_{k=1}^{n} a_k$, $\langle s_n \rangle$ is convergent $\iff$ $\langle s_n \rangle$ is Cauchy.

(1) $\sum_{n=1}^{\infty} a_n$ is convergent $\implies$ $\lim_{n\to\infty} a_n = 0$.

(2) $\sum_{n=1}^{\infty} |a_n|$ is convergent $\implies$ $\sum_{n=1}^{\infty} a_n$ convergent.

Theorem 2.4.3 (Comparison Test) Suppose $\sum b_n$ converges. If $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$, $\sum a_n$ converges.
ProofLet $M = \sum b_n$, $s_n = \sum_{k=1}^{n} a_k$. s_n is increasing and s_n is bounded by M. s_n is convergent by Monotone Convergence Theorem.

Theorem. Suppose sequences a_n, b_n satisfy $0 \le |a_n| \le b_n$[5] and $\sum b_n$ converges. Then $\sum a_n$ is convergent.[6]
Proof. By comparison test and absolute convergence.

Prop 2.4.4 (Root Test) Suppose $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$.
If $\alpha < 1$, $\sum a_n$ converges. If $\alpha > 1$, $\sum a_n$ diverges.
Proof.

(1) $\alpha < 1$. Take $\epsilon > 0$ such that $\alpha < \alpha + \epsilon < 1$. Then there exists N such that $|a_n|^{1/n} < \alpha + \epsilon$ for all $n \ge N$. Therefore $|a_n| < (\alpha + \epsilon)^n$. Since $\alpha + \epsilon < 1$, $\sum (\alpha + \epsilon)^n$ converges. Apply the comparison test to see that $\sum a_n < \infty$.

(2) $\alpha > 1$. Take $\epsilon > 0$ such that $\alpha > \alpha - \epsilon > 1$. Then $|a_n|^{1/n} > \alpha - \epsilon$ for infinitely many n. Then $|a_n| > (\alpha - \epsilon)^n > 1$. Therefore $\lim a_n \ne 0$. $\sum a_n$ diverges.

Prop 2.4.5 (Ratio Test) Suppose $a_n \ne 0$. Let $\beta = \limsup |a_{n+1}/a_n|$, $\gamma = \liminf |a_{n+1}/a_n|$.
If $\beta < 1$, $\sum a_n$ converges. If $\gamma > 1$, $\sum a_n$ diverges.
Proof.

(1) $\beta < 1$. Take $\epsilon > 0$ such that $\beta < \beta + \epsilon < 1$. Then $\exists N$ s.t. $|a_{n+1}/a_n| < \beta + \epsilon$ for $n \ge N$.
$\implies |a_n| = |a_N| \, |a_{N+1}/a_N| \cdots |a_n/a_{n-1}| < |a_N| \, (\beta + \epsilon)^{n-N}$.
Set $b_n = |a_N| \, (\beta + \epsilon)^{n-N}$ and apply comparison test to see that $\sum a_n < \infty$.

[5]Note that this condition can fail for finitely many n.
[6]a_n may be a very complex expression, but we want b_n to be simple, an expression we know that it is convergent.

(2) $\gamma > 1$. Take $\epsilon > 0$ such that $\gamma > \gamma - \epsilon > 1$. Then $\exists N$ s.t. $|a_{n+1}/a_n| > \gamma - \epsilon$ for $n \geq N$. Then we see that $|a_n|$ is increasing for $n \geq N$. Thus a_n cannot converge to 0. $\sum a_n$ is divergent.

Remark. If the above limits (ratio, root) exist, elementary tests can be applied. But if the limits turn out to be 1, the test fails. (ND: Non-Deterministic) Check it for $\sum 1/n, \sum 1/n^2$. Also, these are *weak tests*. For most of the series, the limit is 1. Moreover...

Theorem 2.4.6 Suppose $a_n \neq 0$.

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

Thus if the root test works, ratio test also works.[7]

Proof. We only need to prove the last inequality.
Let $\beta = \limsup |a_{n+1}/a_n|$, $\forall \epsilon > 0$. $\implies$ $\exists N$ s.t. $|a_{n+1}/a_n| \leq \beta + \epsilon$ for $n \geq N$. Then if $n \geq N$, $|a_n| \leq |a_N| (\beta + \epsilon)^{n-N}$. (Similar to proof of 2.4.5) Then

$$|a_n|^{1/n} \leq (\beta + \epsilon) \left(\frac{|a_n|}{(\beta + \epsilon)^N}\right)^{1/n}$$

and take lim sup on both sides, then $\limsup |a_n|^{1/n} \leq \beta + \epsilon$.

Example. $\langle a_n \rangle = \begin{cases} 1/2^n & n \text{ odd} \\ 1/2^{n-2} & n \text{ even} \end{cases}$

Check that $\limsup |a_n|^{1/n} = 1/2 < 1$, and the series $\sum a_n$ converges by the root test.
But if we use the ratio test here, lim sup value is 2 and lim inf value is 1/8.[8] The ratio test does not tell us anything about the convergence. Also note that the series converges to 2.

Prop 2.4.1 (Rearrangement) $a_n \geq 0$.[9] Suppose a bijection $r : \mathbb{N} \to \mathbb{N}$ exists.

(1) $\displaystyle\sum_{n=1}^{\infty} a_n = s \iff \sum_{n=1}^{\infty} a_{r(n)} = s$

(2) $\displaystyle\sum_{n=1}^{\infty} = \infty \iff \sum_{n=1}^{\infty} a_{r(n)} = s$

Proof.

(1) ($\implies$) Let $t_n = \sum_{k=1}^{n} a_{r(k)}$. Then t_n is increasing and bounded by s. Thus t_n converges by MCT, and $\lim t_n \leq s$.
$s_n = \sum_{k=1}^{n} a_k \leq \sum_{n=1}^{\infty} a_{r(n)} = t = \lim t_n$. ($a_n \geq 0$ was used here.)
($\impliedby$) Use $r^{-1}(n)$.

[7] The limit for the ratio test is much easier to calculate than the root test. That's why we use the ratio test.
[8] The ratios are: 2, 1/8, 2, 1/8 ...
[9] This is the important condition.

(2) Contraposition of (1).

Prop 2.4.2 (Alternating Series Test) For a given sequence x_n, suppose the following holds.

- x_n is decreasing.
- $\lim x_n = 0$.

Then the series $\sum_{k=1}^{\infty}(-1)^{n-1}x_n$ is convergent.
Proof. Let $s_n = \sum_{k=1}^{n}(-1)^{k-1}x_k$. For $m < n$,

$$|s_n - s_m| = \left|(-1)^m x_{m+1} + \cdots + (-1)^{n-1}x_n\right| = |x_{m+1} - x_{m+2} + \cdots \pm x_n| \overset{(*)}{\in} [0, x_{m+1}]$$

$$\begin{aligned}(*) : x_{m+1} - x_{m+2} + \cdots + x_n &= (x_{m+1} - x_{m+2}) + \cdots + (x_{n-2} - x_{n-1}) + x_n \geq 0 \\ &= x_{m+1} - (x_{m+2} - x_{m+3}) - \cdots - (x_{n-1} - x_n) \leq x_{m+1}\end{aligned}$$

Check for the case with last term $-$.
Now, $\forall \epsilon > 0$, find N such that $|x_n| < \epsilon$ for $n \geq N$. Then for $n > m \geq N$, $|s_n - s_m| \leq x_{m+1} < \epsilon$.
Thus $\langle s_n \rangle$ is a Cauchy sequence and the given series converges.

Example. $a_n = (-1)^{n-1}/n$. $\sum a_n$ converges by alternating series test and converges to $\log 2$.

Remark. The rearrangement of the above example may not converge, or converge to a different value than $\log 2$.

Exam: 1.1 - 2.6

After the midterms we will be covering functions and continuity.
Chapter 1 has been about $\mathbb{R}$, and in Chapter 2, we have talked about subsets of $\mathbb{R}^n$.
2.1: What is $\mathbb{R}^n$? Vector Space, IPS, Metric Space, Normed Space...
2.2: Open, closed sets
2.3: Bounded sets and Cauchy sequences
(2.4: Convergence Tests)
2.5: Compact Sets
2.6: Connect Sets

April 10th, 2019

2.5 Compact Set

Definition. $\{U_i : i \in I\}$ (I is the index set, $U_i \subset \mathbb{R}^d$) is called "family of sets".

(1) $\{U_i : i \in I\}$ is a **cover** of $K \subset \mathbb{R}^d \iff K \subset \bigcup_{i \in I} U_i$.

(2) $\{U_i : i \in I\}$ is a **open cover** $\iff U_i$ are open for $\forall i$.

(3) $J \subset I$, $\{U_i : i \in J\}$ is called a **subcover** of $\{U_i : i \in I\} \iff K \subset \bigcup_{i \in J} U_i$.

Definition. $K \subset \mathbb{R}^d$ is **compact** $\iff$ Any open cover of K has finite subcover.

Example.

(1) $\mathbb{N}$ is not compact. Set $U_k = (k - 1/2, k + 1/2)$, then $\{U_k : k \in \mathbb{N}\}$ is a (open) cover of $\mathbb{N}$. But there are no finite subcover.

(2) $A = (0, 1)$ is not compact. Set $U_k = (1/k, 1)$, then because $\bigcup_{k=1}^{\infty} U_k = (0, 1)$, $\{U_k : k \in \mathbb{N}\}$ is a (open) cover of A. But there are no finite subcover. $\bigcup_{i=1}^{m} U_{k_i} = U_{k_m} = (1/k_m, 1)$, which cannot contain $(0, 1)$.

(3) $A = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{R}^d$ is compact. $\{U_i : i \in I\}$ be a cover of A. There exists $i_1, \ldots, i_m \in I$ such that $a_k \in U_{i_k}$ for $k = 1, \ldots, m$. Then $\{U_{i_1}, U_{i_2}, \ldots, U_{i_m}\}$ is a finite subcover of A.

Main Theorem: **Heine-Borel Theorem**

$$K \text{ is compact} \iff K \text{ is bounded and closed.}$$

Remark.

(1) This is a part of Thm 2.5.4

(2) Proof: Prop 2.5.1, Thm 2.5.2, Prop 2.5.3

(3) **Characterization of compact sets in** $\mathbb{R}^d$.[10]

---

[10] Compact Set 을 이 단순한 공간 안에서는 characterize 할 수 있다!

Proof.

($\Longrightarrow$) (Prop 2.5.1)

(1) *Is K bounded?*
Set $U_k = N(0,k)$. Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d$. Thus $\{U_k : k \in \mathbb{N}\}$ is an open cover of K. There exists a finite subcover $U_{k_1}, \ldots, U_{k_m}$ $(k_1 < \cdots < k_m)$ of K. Then we have $K \subset \bigcup_{i=1}^{m} U_{k_i} = U_{k_m} = N(0, k_m)$. Therefore K is bounded.

(2) *Is K closed?*
Suppose $x \in K^C$. Set $U_k = \{y : \|y - x\| > 1/k\}$. Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d \setminus \{x\} \supset K$. (Open cover) There exists a finite subcover $U_{k_1}, \ldots, U_{k_m}$ of K. $K \subset \bigcup_{i=1}^{m} U_{k_i} = U_{k_m}$. Therefore $K^C \supset U_{k_m}^C = \{y : \|y - x\| \le 1/k_m\} \supset N(x, 1/k_m)$. Thus K^C is open, K is closed.

($\Longleftarrow$)

(1) (Theorem 2.5.2) *Closed box is compact.*
$B = I_1 \times \cdots \times I_d$, $I_i = [a_i, b_i]$. Let $\{U_i : i \in I\}$ is an open cover of B.
(Contradiction) Suppose there is no finite subcover of B.
Claim. There exists $B = B_1 \supset B_2 \supset \cdots$ (closed boxes) such that

- $\mathrm{diam}(B_n) = \frac{1}{2^{n-1}}\mathrm{diam}(B_1)$
- There is no finite subcover of $\{U_i : i \in I\}$ covering B_n.

By Lemma 2.3.1, there exists $x \in \bigcap_{n=1}^{\infty} B_n$. Since $x \in B$, $\exists U_i$ such that $x \in U_i$. Then $\exists \epsilon > 0$ such that $N(x, \epsilon) \subset U_i$.[11] Set $\frac{1}{2^{n-1}}\mathrm{diam}(B_1) < \epsilon$.
If $y \in B_n \implies \|x - y\| \le \mathrm{diam}(B_n) < \epsilon \implies y \in N(x, \epsilon)$. Then $B_n \subset N(x, \epsilon) \subset U_i$, contradiction.

(2) *K: compact, $F \subset K$, F is closed $\implies$ F: compact.*
Let $\{U_i : i \in I\}$ be an open cover of F. Then $\{U_i : i \in I\} \cup \{F^C\}$ is an open cover of K. Because K is compact, there exists a finite subcover of K. There are two cases.

  (a) $\{U_{i_1}, U_{i_2}, \ldots, U_{i_m}\}$: This is already a finite subcover of F.

  (b) $\{U_{i_1}, U_{i_2}, \ldots, U_{i_m}, F^C\}$: Since F^C does not cover F, U_{i_k} must cover F.

(3) *Closed and bounded set is compact.*
Suppose K is bounded and closed. There exists a closed box B that contains K. Thus B is compact by (1), K is a closed subset of B. Then by (2), K is compact.

Theorem 2.5.2 가 가장 non-trivial 한 부분이다.

[11] n이 충분히 크면 ball 안에 box 가 들어가고 box 는 U_i 안에 있다? Claim 의 2번째에 모순.

Theorem 2.5.4 The following are equivalent.

(1) K is compact.

(2) K is bounded and closed.

(3) If A is an infinite subset of K, $\emptyset \neq A' \subset K$.

(4) For a sequence $\langle x_n \rangle$ in K, there exists a convergent subsequence whose limit is in K.

Proof.
(1) $\iff$ (2) by Heine-Borel Theorem.
(2) $\implies$ (3) Suppose A is infinite and bounded. ($A \subset K$) By Bolzano-Weierstrass, $A' \neq \emptyset$. $A' \subset A' \cup A = \overline{A} \subset K$. ($\overline{A}$ is the smallest closed set containing A, $A \subset K$.)
(3) $\implies$ (4) Let $A = \{x_1, x_2, \dots\}$

(1) If A is finite, trivial. (Take a constant subsequence, which constant $\in K$.)

(2) If A is infinite, $x \in A' \subset K$ by (3). ($x \in A'$ by Thm 2.3.4)

(4) $\implies$ (2)

(1) K is bounded.
(Contradiction) Suppose K is not bounded. Then $\forall n \in \mathbb{N}$, there exists $x_n \in K$, $\|x_n\| \geq n$. There are no convergent subsequences, contradiction.

(2) K is closed.
(Contradiction) Suppose K is not closed.

(a) K: finite $\to$ K: closed $\to$ Contradiction.

(b) K: infinite $\to$ K: infinite and bounded $\overset{\text{B-W}}{\to}$ $K' \neq \emptyset$

Note. $K' \subset K \iff K$: closed.
Then if K' is not a subset of K[12], there exists $x \in K' \setminus K$. Since $x \in K'$, there exists a sequence $\langle x_n \rangle$ in $K \setminus \{x\}$ $(= K)$[13] converging to x. Thus for a subsequence of $\langle x_n \rangle$, its limit must be in K. But x is the only possible limit value. $x \in K$. Contradiction.

[12] Contraposition
[13] $x \notin K$

April 12th, 2019

Problem 2.4.7 (바) $\sum \dfrac{1}{n^p - n^q}$ $(0 < q < p)$
$0 < n^p - n^q \le n^p$ 이므로 $1/n^p \le 1/(n^p - n^q)$ 가 되어 $p \le 1$ 이면 발산한다.
충분히 큰 N에 대하여 $n \ge N$ 일 때마다 $n^p - n^q \ge n^p/2$ 가 되게 할수 있다. (이 때 $n^p/2 \ge n^q$ 이므로 $n^{p-q} \ge 2$ 가 되어 N 을 잡을 수 있다) 비교판정법에 의해 수렴한다.

Problem 2.7.12 Given $\langle a_n \rangle$ such that $\lim a_n = a$, show that $\sigma_n = \dfrac{a_1 + \cdots + a_n}{n}$ also converges to a.

Problem 2.7.13 $r < 1$, $\|x_{n+2} - x_{n+1}\| \le r \|x_{n+1} - x_n\|$. Show that $\langle x_n \rangle$ is a Cauchy sequence.
Proof. $\|x_{n+1} - x_n\| \le r^{n-1} \|x_2 - x_1\| = r^{n-1}A$, for $A \in \mathbb{R}$. Given $\epsilon > 0$, exists N such that for all $n \ge N$, $\|x_{n+1} - x_n\| < Ar^{n-1} < \epsilon$. Then we have

$$\begin{aligned} m > n \ge N \Rightarrow \|x_n - x_m\| &\le \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\| \\ &\le \|x_{n+1} - x_n\| (1 + r + r^2 + \cdots) < \frac{\epsilon}{1-r} \end{aligned}$$

Remark. Counterexample for $\|x_{n+2} - x_{n+1}\| < \|x_{n+1} - x_n\|$. $x_n = \sum_{k=1}^{n} \frac{1}{k}$

Problem 2.7.14 $x_n \to x$, $A_k = \{x_i : i \ge k\}$. Show that $\bigcap_{k=1}^{\infty} \overline{A_k} = \{x\}$.
Proof. Given $\epsilon > 0$, there exists N such that $n \ge N \Rightarrow x_n \in (x - \epsilon, x + \epsilon)$. Either $x_n = x$, or $x_n \in (x - \epsilon, x + \epsilon) \setminus \{x\}$. Thus $x \in \overline{A_k}$ for all k. $\{x\} \subset \bigcap_{k=1}^{\infty} \overline{A_k}$.
For $y \in \mathbb{R} \setminus \{x\}$, we want to show that $y \notin \bigcap_{k=1}^{\infty} \overline{A_k}$. Then we want to find N such that $y \notin \overline{A_N}$. Since $\|x - y\| > 0$, set $\epsilon = \frac{1}{3} \|x - y\|$. There exists N such that $\|x_n - x\| < \epsilon$. Then $\forall x_n \notin N(y, \epsilon)$. $\overline{A_N} = \{x_N, x_{N+1}, \dots\}$, and y cannot be in $\overline{A_N}$. $\{x\}^C \subset \left(\bigcap_{k=1}^{\infty} \overline{A_k}\right)^C \Rightarrow \bigcap_{k=1}^{\infty} \overline{A_k} \subset \{x\}$.

Problem 2.7.15 $\sum a_n$ converges absolutely.

(1) $\sum a_n^2$
Proof. $a_n^2 < |a_n|$ for large n. Converges by comparison test.

(2) $\sum \dfrac{a_n}{1 + a_n}$
Proof. Since $a_n \to 0$, exists N such that $n \ge N \Rightarrow |a_n| < 1/3$. Then for $n \ge N$, $|1 + a_n| \ge 1 - |a_n| > 2/3 > 1/3$, $1/|1 + a_n| < 3$. We have $\left|\frac{a_n}{1+a_n}\right| < 3|a_n|$. Converges by comparison test.

(3) $\sum \dfrac{a_n^2}{1 + a_n^2}$
Proof. Trivial from 1, 2.

April 15th, 2019

K: compact $\iff$ Exists an open cover of K that has *finite* subcover.

Theorem 2.5.4 (Heine-Borel) For $\mathbb{R}^d$, K: compact $\iff$ K is bounded and closed.

Theorem 2.5.5 (Cantor's Intersection Theorem)[14]
Given family of **compact** sets $\{K_i : i \in I\}$, for all **finite** $J \subset I$, $\bigcap_{i \in J} K_i \neq \emptyset$. Then

$$\bigcap_{i \in I} K_i \neq \emptyset$$

Proof. (Contradiction) $\bigcap_{i \in I} K_i = \emptyset \implies \bigcup_{i \in I} K^C = \mathbb{R}^d$. (Complement)
Take any K_a ($a \in I$), then $K_a \subset \bigcup_{i \in I} K_i^C (= \mathbb{R}^d) \implies \{K_i^C : i \in I\}$ is an open cover of K_a. Then there exists a finite subcover, $\{K_i^C : i \in J\}$ (K_a is compact) Now we can write $K_a \subset \bigcup_{i \in J} K_i^C$. Take complement on both sides to get $K_a^C \supset \bigcap_{i \in J} K_i$. Then $K_a \cap \bigcap_{i \in J} K_i = \emptyset$, contradiction.

Remark. Let $K_i = [a_i, b_i]$ (Compact in $\mathbb{R}$) and set $K_1 \supset K_2 \supset \cdots$
$\implies$ For $J = \{j_1, \ldots, j_m\}$ ($j_1 < \cdots < j_m$), $\bigcap_{i \in J} K_i = K_{j_m} \neq \emptyset$
$\implies \bigcap_{i=1}^{\infty} K_i \neq \emptyset$ (축소구간정리)

2.6 Connected Set

p46-p47 (Section 2.2)

Definition. $X \subset \mathbb{R}^d$, $x \in X$. Define

$$N_X(x, r) = \{y \in X : \|y - x\| < r\} = N(x, \epsilon) \cap X$$

Definition. $U \subset X$ is open in X $\iff$ $x \in U, \exists\, \epsilon > 0$ such that $N_X(x, \epsilon) \subset U$.

Example.

- $U = \{3\}$. U is open in $X = \mathbb{N}$. $N_{\mathbb{N}}(3, 1/10) = 3 \subset U$. (But not open in $\mathbb{R}$)
- For $X = [0, 10]$, $U = [0, 1)$. $x \in U$, $N(x, 1-x) = (2x-1, 1)$, and this might not be subset of U. But

$$N_X(x, 1-x) = \begin{cases} (2x-1, 1) & (x > 1/2) \\ [0, 1) & (x \leq 1/2) \end{cases}$$

  For both cases $N_X(x, 1-x) \subset U$.

[14] 축소구간정리의 가장 일반적인 형태

Prop 2.2.5 U is open in $X \iff U = X \cap V$ for some open set V in $\mathbb{R}^d$.

Remark. First example: $\{3\} = \mathbb{N} \cap (2.9, 3.1)$, Second example: $[0,1) = [0,10] \cap (-1,1)$.
Some references may write this definition as "*relatively*" open in X.

Proof of 2.2.5
$(\implies)$ $x \in U$, $\exists \epsilon_x > 0$ such that $N_X(x, \epsilon_x) \subset U$. Select $V = \bigcup_{x\in U} N(x, \epsilon_x)$, which is open.[15]
Then we have $X \cap V = \bigcup_{x\in U} X \cap N(x, \epsilon_x) = \bigcup_{x\in U} N_X(x, \epsilon_x)$, which is exactly equal to U.

$(\impliedby)$ $x \in U = X \cap V \implies x \in V$. Thus $\exists\, \epsilon > 0$ such that $N(x, \epsilon) \subset V$. Then

$$N_X(x, \epsilon) = X \cap N(x, \epsilon) \subset X \cap V = U$$

Thus U is open in X.

Cor. U: open in X, $Y \subset X$. $\implies U \cap Y$: open in Y.
Proof. $U = X \cap V$ (V: open) $\implies U \cap Y = X \cap V \cap Y = V \cap (X \cap Y) = V \cap Y$.

Definition. $S \subset \mathbb{R}^d$: **disconnected** $\iff$ There exists **non-empty** sets U, V such that

(1) $U \cap V = \emptyset$

(2) $U \cup V = S$

(3) U and V are open in S

$S \subset \mathbb{R}^d$: **connected** $\iff$ S is not disconnected.

Question. Find all $A \subset \mathbb{R}^d$ such that A is open and closed.
Proof. The only possible sets are $A = \emptyset, \mathbb{R}^d$.
If A is open and closed $\implies A$: open, A^C: open. Then $\mathbb{R}^d = A \cup A^C$, and $\mathbb{R}^d$ is disconnected.
But $\mathbb{R}^d$ is connected. Contradiction if either A or A^C is empty.

Theorem. The following are equivalent for $S \subset \mathbb{R}$.

(1) S is connected.

(2) $\forall a, b \in S$ s.t. $a < b$, and $c \in (a, b) \implies c \in S$.

(3) $S = [a, b]$ or $[a, b)$ or $(a, b]$ or (a, b) (a, b can be $\pm\infty$)

[15] $N(x, \epsilon)$ is open and union of open sets are always open.

Remark. Prop 2.5.1 (1' $\iff$ 2') + Disscussion above (2 $\iff$ 3)

Proof.

(1 $\implies$ 2) (Contradiction) Assume $a, b \in S, c \notin S$ for some $a < c < b$. Set $U = (-\infty, c) \cap S$, $V = (c, \infty) \cap S$. U, V are non-empty.[16] $U \cap V = \emptyset$ and $U \cup V = S$. (Note that $c \notin S$) And U, V are open in S. (Prop 2.2.5) Then S is disconnected.

(2 $\implies$ 1) (Contradiction) Assume S is disconnected. There exists U, V that satisfy the definition of disconnected set. For $a \in U, b \in V$, (WLOG $a < b$). By (2), $[a, b] \subset S$.

Let $c = \sup([a, b] \cap U)$.

Case I) $c \in U$. Then $c \neq b \implies c \in [a, b) = Y \implies c \in U \cap Y$.

Since U is open in S and $Y \subset S \implies U \cap Y$ is open in Y. (Cor of 2.2.5)

$\implies \exists\, \epsilon > 0$ such that $N_Y(c, \epsilon) \subset U \cap Y \subset U \cap [a, b]$.

$$Y \cap N(c, \epsilon) = [a, b) \cap (c - \epsilon, c + \epsilon) \supset [c, b) \cap [c, c + \epsilon) = [c, \min\{b, c + \epsilon\})$$

Therefore, we have

$$[c, \min\{b, c + \epsilon\}) \subset N_Y(c, \epsilon) \subset U \cap [a, b]$$

and since c was the supremum, contradiction.

Case II) $c \in V$. Similarly, contradiction.

(2 $\implies$ 3) $\inf S = u$, $\sup S = v$. (If S is not bounded below, $u = -\infty$, if S is not bounded above, $v = \infty$). Then if $c \in (u, v) \implies c \in S$. There exists $a, b \in S$ such that $u \leq a < c < b \leq v$, meaning that S must be one of $[u, v], [u, v), (u, v], (u, v)$.

(3 $\implies$ 2) Trivial.

[16] Always check! $a \in U, b \in V$.

April 17th, 2019

Definition. $S \subset \mathbb{R}^d$: **disconnected** $\iff$ There exists **non-empty** sets U, V such that

(1) $U \cap V = \emptyset$

(2) $U \cup V = S$

(3) U and V are open in S

Last time we characterized all connected sets of $\mathbb{R}$.

Theorem 2.6.2 Suppose $\{C_i : i \in I\}$ is a family of connected sets.[17]

$$\bigcap_{i \in I} C_i \neq \emptyset \implies \bigcup_{i \in I} C_i \text{ is connected}$$

Proof. (Routine) Assume $C = \bigcup_{i \in I} C_i$ is disconnected. C can be decomposed into 2 sets U, V (that satisfy condition (1), (2), (3) from the definition). Let

$$U_i = C_i \cap U, \quad V_i = C_i \cap V \quad (\forall i)$$

then U_i, V_i are open in C_i.[18] Now U_i, V_i satisfy (2) and (3) for C_i. Since C_i is connected, (1) should not hold, in other words, either U_i or V_i must be $\emptyset$.
Define: $I_1 = \{i \in I : U_i = \emptyset, V_i = C_i\}$, $I_2 = \{i \in I : U_i = C_i, V_i = \emptyset\}$. If $I_1 = \emptyset \implies I_2 = I \implies V_i = \emptyset \ (\forall i) \implies V = \bigcup_{i \in I} V_i = \emptyset$[19], contradiction. Similarly if $I_2 = \emptyset$, contradiction.
Select $i_1 \in I_1, i_2 \in I_2$. Then $C_{i_1} = V_{i_1} \subset V$, $C_{i_2} = U_{i_2} \subset U$. Therefore $C_{i_1} \cap C_{i_2} = \emptyset$. Contradiction.

Example.

(1) $x, y \in \mathbb{R}^d$, $[x, y] = \{tx + (1-t)y : t \in [0,1]\}$ is connected. (Proof similar to Prop 2.6.1)

(2) $N(x, r) = \displaystyle\bigcup_{y \in N(x,r)} [x, y]$ is connected by the theorem above. ($\bigcap_{y \in N(x,r)} [x, y] = \{x\} \neq \emptyset$)

(3) $\mathbb{R}^d = \displaystyle\bigcup_{y \in \mathbb{R}^d} [0, y]$ is connected.

(4) Convex sets are connected. $A = \displaystyle\bigcup_{y \in A} [x, y]$.

[17] 활용 보다도 증명이 중요하니 꼭 기억해 두자.
[18] U: open in X and $Y \subset X \implies U \cap Y$: open in Y.
[19] Check!

Definition. Set A is **convex** $\iff x, y \in A \implies [x, y] \subset A$.

Comment. Homework problem: Show that $S = \{(x, y) : xy > 1\}$ is open.
Proof. 1. Show that $N(z, \epsilon) \subset S$ for all $z \in S$.
2. Instead show that $F = \{(x, y) : xy \leq 1\}$ is closed.
Use Thm 2.2.3 (4). Let (x_n, y_n) be a sequence in F that converges to (x, y).

$$xy = \lim x_n \lim y_n = \lim x_n y_n \leq 1 \implies (x, y) \in F$$

Example. $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, define $A \times B \subset \mathbb{R}^{n+m}$ as

$$A \times B = \{(x, y) : a \in A, b \in B\}$$

If $m = n = 1$, $A \times B$ is a rectangular box in $\mathbb{R}^2$.
If A, B is open/closed/compact/connected, $A \times B$ is open/closed/compact/connected.
Proof.

(1) (Open) $(a, b) \in A \times B$. There exists $\epsilon_1, \epsilon_2 > 0$ such that $N(a, \epsilon_1) \subset A$, $N(b, \epsilon_2) \subset B$. Choose $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. If $(x, y) \in N((a, b), \epsilon) \subset \mathbb{R}^{n+m}$,[20] we have

$$\epsilon^2 > \|(x, y) - (a, b)\|^2 = \|x - a\|^2 + \|y - b\|^2$$

$\|x - a\| < \epsilon < \epsilon_1$ and $\|y - b\| < \epsilon < \epsilon_2$. $x \in A$, $y \in B$.
Therefore $(x, y) \in A \times B$, and $N((a, b), \epsilon) \subset A \times B$.

(2) (Closed) (x_k, y_k): sequence in $A \times B$. $(x_k \in A, y_k \in B)$
Suppose $(x_k, y_k) \to (x, y)$ $(x_k \to x, y_k \to y)$. Since A is closed and x_k is a sequence in A, $x \in A$. Similarly, $y \in B$. Thus $(x, y) \in A \times B$, and $A \times B$ is closed.

(3) (Compact) A, B are closed and bounded. Closed is proven by (2).
Since A, B are bounded, $\exists\, M_1, M_2$ such that $\|a\| \leq M_1$, $\|b\| \leq M_2$ for all $a \in A$, $b \in B$. For all $(a, b) \in A \times B$,

$$\|(a, b)\| = \sqrt{\|a\|^2 + \|b\|^2} \leq \sqrt{M_1^2 + M_2^2}$$

Therefore $A \times B$ is bounded. Thus compact.

(4) (Connected) $a \in A \implies \{a\} \times B$ is connected. $b \in B \implies A \times \{b\}$ is connected.
Proof. If the set is disconnected, exists $\{a\} \times U$, $\{a\} \times V$ such that splits B.
Since $(A \times \{b\}) \cap (\{a\} \times B) = \{(a, b)\} \neq \emptyset$, $(A \times \{b\}) \cup (\{a\} \times B)$ is connected by Thm 2.6.2. Now fix $a \in A$, and define $C_b = (A \times \{b\}) \cup (\{a\} \times B)$.
Then $\{C_b : b \in B\}$ is a family of connected sets, and $\bigcap_{b \in B} = \{a\} \times B \neq \emptyset$. $A \times B = \bigcup_{b \in B} C_b$ is connected by Thm 2.6.2.

[20] Do not write as $\mathbb{R}^{m+n}$. Fist coordinate is n-dimension, second is m-dimension.

April 22nd, 2019

3. Continuous Functions

3.1 Limit of a Function & Continuous Functions

특별한 언급이 없으면 다음과 같은 가정을 한다.[21]

$$f : X \to Y \quad (X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n)$$

Definition. For $x_0 \in X'$, $\lim\limits_{x \to x_0} f(x) = y_0 \iff$

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ s.t. } \big(\mathbf{0} < \|x - x_0\| < \delta \Rightarrow \|f(x) - y_0\| < \epsilon\big)$$

Remark. Why X'? $X = [0,1] \cup \{2\}$, consider $f(x) = 2x$ on X. $\lim\limits_{x \to 2} f(x)$ is nonsense.

Example.

(1) $f(x) = \begin{cases} x^2 & (x \neq 0) \\ 1 & (x = 0) \end{cases}$, $\lim\limits_{x \to 0} f(x) = 0$.[22]
For $\epsilon > 0$, set $\delta = \sqrt{\epsilon}$. Then $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| = |x^2| < \delta^2 = \epsilon$.

(2) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$. $(X = \mathbb{R} \setminus \{2\}, Y = \mathbb{R},\ 2 \in X')$
For $\epsilon > 0$, set $\delta = \epsilon$. Then $0 < |x - 2| < \delta \Rightarrow |f(x) - 4| = |x - 2| < \delta = \epsilon$.

Prop 3.1.1 $f, g : X \to Y$, $x_0 \in X'$[23]. If $\lim\limits_{x \to x_0} f(x) = y_0$, $\lim\limits_{x \to x_0} g(x) = z_0$, then

(1) $\lim\limits_{x \to x_0} af(x) + bg(x) = ay_0 + bz_0$

(2) $\lim\limits_{x \to x_0} f(x)g(x) = y_0 z_0$

(3) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{y_0}{z_0}$ $(z_0 \neq 0)$

연속을 3가지로 정의한다. 세 정의들이 서로 **동치**임을 이해하는 것이 중요하다.
Definition. Let $f : X \to Y$, $x_0 \in X$. f is **continuous** at x_0 $\iff$

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ s.t. } \big(\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon\big)$$

Remark. $\|x - x_0\| < \delta$ should be satisfied for $x \in X$. The $0 <$ condition is omitted here since the inequality holds trivially for x_0.

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[21] 치역이 중요하지 공역은 뭐...
[22] 특별한 언급이 없으면 $X = f$ 가 정의되는 곳, $Y = \mathbb{R}^n$ 으로 생각한다.
[23] 책에 X로 되어있는데 이는 오타.

(1) $x_0 \in X'$: f is continuous at $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$.

(2) $x_0 \in X \setminus X'$ (isolated point): f is continuous at x_0.

Definition.

(1) $A \subset X, f : X \to Y$. If f is continuous at x_0 for all $x \in A \implies f$ is continuous on A.

(2) If f is continuous on $X \implies f$ is continuous.

Prop 3.1.3 The following are equivalent for $f : X \to Y$.

(1) f: continuous at $x_0 \in X$.

(2) If there exists a sequence $\langle x_n \rangle$ in X converging to $x_0 \implies \lim_{n \to \infty} f(x_n) = f(x_0)$.

Proof.
(1 $\implies$ 2) Given $\epsilon > 0$,

(i) $\exists\, \delta > 0$ s.t. $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$

(ii) Since $x_n \to x_0$, $\exists\, N$ s.t. for $n \geq N \implies \|x_n - x_0\| < \delta$.

Therefore, $n \geq N \implies \|x_n - x_0\| < \delta \implies \|f(x_n) - f(x_0)\| < \epsilon$.

(2 $\implies$ 1) (Contradiction) Suppose there exists $\epsilon_0 > 0$ such that no δ statisfies $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon_0$. (i.e. For all $\delta > 0$, $\exists\, x \in X$ s.t. $\|x - x_0\| < \delta$ and $\|f(x) - f(x_0)\| \geq \epsilon_0$)

Thus for all $n \in \mathbb{N}$, there exists $x_n \in X$ s.t. $\|x_n - x_0\| < 1/n$ and $\|f(x_n) - f(x_0)\| \geq \epsilon_0$. ($\delta = 1/n$) Then we have $\lim_{n \to \infty} x_n = x_0$, but $\lim_{n \to \infty} f(x_n) \neq f(x_0)$. Contradiction.

Definition. $f : X \to Y$, $A \subset X$, $B \subset Y$. Define

$$f(A) = \{f(x) : x \in A\} \quad f^{-1}(B) = \{x \in X : f(x) \in B\}$$

Remark.

(1) $A \subseteq f^{-1}(f(A))$
$x \in A$ and let $y = f(x)$. Then $y \in f(A)$, thus $x \in f^{-1}(f(A))$.

(2) $f(f^{-1}(B)) \subseteq B$
$y \in f(f^{-1}(B))$ then $y = f(x)$ for some $x \in f^{-1}(B)$. Thus we have $x \in f^{-1}(B) \iff f(x) \in B$. $\therefore y = f(x) \in B$.

Also remember the counterexamples where the equality does not hold. (1) doesn't hold if f is not injective, (2) doesn't hold if f is not surjective.

Theorem 3.1.5 The following are equivalent for $f : X \to Y$.

(1) f is continuous on X.

(2) B: open set in $Y \implies f^{-1}(B)$: open in X.

(3) B: closed in $Y \implies f^{-1}(B)$: closed in X.

Proof. **(2 $\iff$ 3)** Trivial. Check $f^{-1}(B^C)$.
(1 $\implies$ 2) Observation. f is continuous at $x_0 \iff \forall\epsilon > 0,\ \delta > 0$ s.t. $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$. Re-write the last two inequality as $x \in N_X(x, \delta)$ and $f(x) \in N_Y(f(x_0), \epsilon)$. Then continuity condition is equivalent to

$$\forall\epsilon > 0, \exists\, \delta > 0 \text{ s.t. } f(N_X(x, \delta)) \subset N_Y(f(x_0), \epsilon)$$

Now suppose $x_0 \in f^{-1}(B) \iff f(x_0) \in B$. Since B is open, there exists $\epsilon > 0$ s.t. $N_Y(f(x_0), \epsilon) \subset B$. Then there exists $\delta > 0$ s.t. $f(N_X(x_0, \delta)) \subset N_Y(f(x_0), \epsilon) \subset B$. Take f^{-1} on both sides. $N_X(x_0, \delta) \subset f^{-1}(f(N_X(x_0, \delta))) \subset f^{-1}(B)$. Thus $f^{-1}(B)$ is open in X.

(2 $\implies$ 1) $x_0 \in X$, $f(x_0) \in Y$. Given $\epsilon > 0$, $N_Y(f(x_0), \epsilon)$ is open in Y. By (2), $f^{-1}(N_Y(f(x_0), \epsilon))$ is open in X. Observe that this set always contains x_0. Then $\exists\, \delta$ s.t. $N_X(x_0, \delta) \subset f^{-1}(N_Y(f(x_0), \epsilon))$. Now take f on both sides. $f(N_X(x_0, \delta)) \subset f(f^{-1}(N_Y(f(x_0), \epsilon))) \subset N_Y(f(x_0), \epsilon)$. Thus f is continuous at x_0.

April 24th, 2019

연속함수의 기본적 성질

Prop 3.1.2 Suppose $f, g : X \to \mathbb{R}^n$ are continuous on X.

(1) $af + bg$: continuous

(2) $(n = 1)$ fg: continuous

(3) $\dfrac{f}{g}$: continuous ($g \neq 0$ on X)

Proof. (2) Given $\epsilon > 0$, $\exists\, \delta_1$ s.t. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\epsilon}{2|g(x_0)|+1}$, $\exists\, \delta_2$ s.t. $|x - x_0| < \delta_2 \implies |g(x) - g(x_0)| < \dfrac{\epsilon}{2(|f(x_0)| + \frac{\epsilon}{2|g(x_0)|+1})}$. Then we have

$$\begin{aligned}|f(x)g(x) - f(x_0)g(x_0)| &= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\ &\leq |f(x)|\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon\end{aligned}$$

Thus we have continuity.

Proof 2. By sequential definition, exists $\langle x_n \rangle \to x_0$ in X such that $f(x_n) \to f(x_0), g(x_n) \to g(x_0)$. Then we have $f(x_n)g(x_n) \to f(x_0)g(x_0)$.

Prop 3.1.4 Suppose we have two continuous functions $f : X \to Y$, $g : Y \to Z$. If f is continuous at $x_0 \in X$, and if g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof. Given $\epsilon > 0$, $\exists\, \delta_1 > 0$ s.t. $\|y - f(x_0)\| < \delta_1 \implies \|g(y) - g(f(x_0))\| < \epsilon$. Also, $\exists\, \delta_2 > 0$ s.t. $\|x - x_0\| < \delta_2 \implies \|f(x) - f(x_0)\| < \delta_1$. Now we automatically have $\|g(f(x)) - g(f(x_0))\| = \|(g \circ f)(x) - (g \circ f)(x_0)\| < \epsilon$.

Remark. Suppose f: continuous X, g: continuous on Y (or on $f(X)$). Then $g \circ f$ is continuous on X.

Example.

(1) Polynomials are continuous. Use continuity of $f(x) = x$.

(2) $f(x) = \sqrt{x}$.

(3) $f(x) = \sqrt{x^4 + 1}$ is continuous.

(4) $f(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$ is not continuous.

Proof. $x_0 \in \mathbb{R}$. Suppose there exists a sequence $\langle x_n \rangle$ in $\mathbb{Q}$ converging to x_0. Then $\langle f(x_n) \rangle \to 1$. ($x_n = \lfloor n x_0 \rfloor / n$) But there also exists a sequence $\langle x_n \rangle$ in $\mathbb{R} \setminus \mathbb{Q}$ converging to x_0. Then $\langle f(x_n) \rangle \to 0$. ($x_n = \lfloor \sqrt{2} n x_0 \rfloor / \sqrt{2} n$) $f(x)$ cannot be continuous anywhere.

3.2 Extreme Value Theorem & Intermediate Value Theorem

Theorem 3.2.1 If $f : X \to Y$ is continuous, surjective and X: compact, then Y: compact.

Proof. Suppose $\{U_i : i \in I\}$ is an open cover of Y. $V_i = U_i \cap Y$ is an open set in Y, and $\{V_i : i \in I\}$ is also an open cover of Y. Consider $\{f^{-1}(V_i) : i \in I\}$, which is an open cover of X. Since X is compact, there exists a finite subcover $\{f^{-1}(V_i) : i \in J\}$ $(J \subset I)$ of X. Then $\{V_i : i \in J\}$ is a finite subcover of Y.

$$Y = f(X) = f\left(\bigcup_{i\in J} f^{-1}(V_i)\right) = \bigcup_{i\in J} f(f^{-1}(V_i)) \subset \bigcup_{i\in J} V_i$$

We have a finite subcover of Y. Thus Y is compact.

Check. $\forall A \subset X$. f: surjective $\implies$, $f(f^{-1}(A)) = A$. f: injective $\implies$ $f^{-1}(f(A)) = A$.

Remark.

(1) $f : \mathbb{R}^m \to \mathbb{R}^n$, f: continuous. If $K \subset \mathbb{R}^m$ is compact, $f(K)$ is compact.
Set $f : K \to f(K)$.

(2) Image of compact set is compact.

Cor 3.2.2 Suppose X is compact. $f : X \to \mathbb{R} \implies f$ has maximum and minimum.

Proof. Set $f : X \to f(X)$, then f is surjective and $f(X)$ is compact. Check that if $K \subset \mathbb{R}$, K: compact, then $\inf K, \sup K \in K$ and $\inf K = \min K$, $\sup K = \max K$.

Cor 3.2.4 (Extreme Value Theorem) If f is a continuous function defined on $[a, b]$, f has a maximum and minimum.

Proof. $[a, b]$ is compact.

Cor 3.2.3 Suppose X is compact and $f : X \to \mathbb{R}$ is continuous. If $f(x) > 0$ for all $x \in X$, then $\exists\, \delta > 0$ s.t. $f(x) \geq \delta > 0$ for all $x \in X$.

Proof. Let $\delta = \min f(X) = f(u) > 0$ for some u.

Remark. $X = [1, \infty)$, $f(x) = 1/x$. (X is not compact.)

Cor 3.2.5 Suppose X is compact and $f : X \to Y$ is bijective and continuous. Then f^{-1} is continuous.

Check. $f : X \to Y$. $A \subset X, B \subset Y$. Image: $f(A)$, pre-image: $f^{-1}(B)$. We must check if image of B on f^{-1} is equal to the pre-image of B. (Well-definedness!)

April 26th, 2019

Assignment 3.5 #3: Check and remember.

(2) $f\left(\bigcap_{i\in\mathcal{I}} A_i\right) \subset \bigcap_{i\in\mathcal{I}} f(A_i)$

Problem 3.1.2 $f : X \to \mathbb{R}^n$, $f(x) = \big(f_1(x), \dots, f_n(x)\big)$ $(x \in X)$. The following are equivalent.

(1) f is continuous at x.

(2) For all i, $f_i : X \to \mathbb{R}$ is continuous at x.

Proof. **(1 $\implies$ 2)** $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|y - x\| < \delta \implies \|f(y) - f(x)\| < \epsilon$. Then we have $\|f_i(y) - f_i(x)\| \le \|f(y) - f(x)\| < \epsilon$, for any i.
(2 $\implies$ 1) $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|x - y\| < \delta \implies \|f_i(x) - f_i(y))\| < \epsilon/\sqrt{n}$. Then

$$\|x - y\| < \delta \implies \|f(x) - f(y)\| = \sqrt{\sum_{i=1}^{n} \|f_i(x) - f_i(y)\|^2} < \sqrt{n \cdot \frac{\epsilon^2}{n}} = \epsilon$$

Prop 3.1.2 (3) f, g: continuous $\implies$ f/g: continuous ($g \ne 0$ on X)
Proof. $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t. for all $x_0 \in X$,
$|x - x_0| < \delta \implies |g(x) - g(x_0)| < \min\{\frac{1}{2}|g(x_0)|, \frac{1}{4}\frac{|g(x_0)|^2\epsilon}{|f(x_0)|+1}\}$, $|f(x) - f(x_0)| < \frac{1}{4}|g(x_0)|\,\epsilon$.

$$\begin{aligned}\left|\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}\right| &\le \frac{|g(x_0)|\,|f(x) - f(x_0)| + |f(x_0)|\,|g(x) - g(x_0)|}{|g(x)|\,|g(x_0)|} \\ &\le \frac{|g(x_0)|\,\frac{1}{4}|g(x_0)|\,\epsilon + |f(x_0)|\,\frac{1}{4}\frac{|g(x_0)|^2\epsilon}{|f(x_0)|+1}}{\frac{1}{2}|g(x_0)|^2} < \frac{\frac{1}{4}|g(x_0)|^2\epsilon + \frac{1}{4}|g(x_0)|^2\epsilon}{\frac{1}{2}|g(x_0)|^2} = \epsilon\end{aligned}$$

Example. $g(x) = \begin{cases} 0 & (x = 0, 1 \text{ or } x \in \mathbb{R}\setminus\mathbb{Q}) \\ 1/q & (x = p/q, \text{irreducible fraction}) \end{cases}$

(i) $x_0 \in \mathbb{Q} \cap (0, 1)$ then $g(x_0) > 0$. Set $\epsilon = \frac{1}{2}g(x_0) > 0$. For all $\delta > 0$, $\exists\, y \in \mathbb{Q}^C \cap [0, 1]$ s.t. $|y - x_0| < \delta$, but $|g(y) - g(x_0)| = g(x_0) \ge \epsilon$. Thus f is not continuous at x_0.

(ii) $x_0 \in \mathbb{Q}^C \cup \{0, 1\}$. $g(x_0) = 0$. $\forall \epsilon > 0$, $\exists\, N \ge 1$ s.t. $1/N < \epsilon$. Then there are finitely many y such that $g(y) \ge 1/N$. ($\frac{1}{N}, \frac{1}{N-1}, \dots, \frac{1}{2}$ is finite) Let them be $y_1, \dots, y_k$ and set $\delta = \min_{1\le i\le k} |y_i - x_0| > 0$. If $\|y - x_0\| < \delta$, then $0 \le g(y) < 1/N < \epsilon$. $|g(y) - g(x_0)| = g(y) < \epsilon$.

Problem 3.5.1

(1) $f(x) = 0$, $f(\mathbb{R}) = \{0\}$ (closed)

(3) $f(x) = e^x$, $f(\mathbb{R}) = (0, \infty)$ (open)

April 29th, 2019

3.2 EVT & IVT

Theorem 3.2.1 Suppose $f : X \to Y$ is continuous and surjective.[24] If X is compact, Y is also compact.

Remark. $f : X \to Y$ continuous, $K \subset X$: compact $\implies f(K)$: compact. Inverse does not hold. Consider $f(x) = \sin x$. Image is $[0, 1]$ (compact), but pre-image is $\mathbb{R}$ (not bounded).

Definition. Function $f : X \to \mathbb{R}$ has **maximum** M if there exists $u \in X$ s.t. $f(u) = M$, and $\forall x \in X$, $f(x) \leq M$.

Cor 3.2.5 Suppose $f : X \to Y$ is continuous and bijective. If X is compact, $f^{-1} : Y \to X$ is continuous.[25]

Proof. Let $f^{-1} = g : Y \to X$. For any open set U in X, it is enough to show that $g^{-1}(U)$ is open in Y. But $g^{-1}(U) = (f^{-1})^{-1}(U) = f(U)$. Check that $Y \setminus f(U) = f(X \setminus U)$. Since a closed subset of a compact set is compact, $Y \setminus f(U) = f(X \setminus U)$ is compact, and hence closed in $\mathbb{R}^d$. Then $f(U) = (Y \setminus f(U))^C \cap Y$ is open in Y.

Example. $f : X = \{0\} \cup (1, 2) \to Y = [0, 1)$. $f(0) = 0$, $f(x) = x - 1$ on $(1, 2)$. By definition, f is continuous on X. Consider f^{-1}. $f^{-1}(0) = 0$, $f^{-1}(x) = x + 1$ on $(0, 1)$. f^{-1} is not continuous.[26]

Application. (Distance between sets) Define dist as follows.

$$A, B \subset \mathbb{R}^d, \quad \mathrm{dist}(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}$$

Example. $A = \{(x, y) : x \leq 0\}$, $B = \{(x, y) : xy \geq 1, x, y > 0\}$. $\mathrm{dist}(A, B) \leq \left\|(0, n) - (\frac{1}{n}, n)\right\| = 1/n$ for all n. Thus $\mathrm{dist}(A, B) = 0$.

Theorem. A : compact, B : closed. $A \cap B = \emptyset \implies \mathrm{dist}(A, B) > 0$.

Proof. $f : A \to \mathbb{R}$, $f(x) = \mathrm{dist}(\{x\}, B)$ $(x \in A)$.

(i) $f(x) > 0$ for all $x \in A$.
$\because N(x, \epsilon) \subset B^C$ (open) $\implies \mathrm{dist}(\{x\}, B) \geq \epsilon > 0$.

(ii) f: continuous, $b \in B$. For $x, y \in A$, $\|x - b\| \leq \|x - y\| + \|y - b\|$. Take infimum over $b \in B$. Then we have $f(x) \leq \|x - y\| + f(y)$. Similarly we have $f(y) \leq \|x - y\| + f(x)$. Hence $\|f(x) - f(y)\| \leq \|x - y\|$. (Continuity follows easily by setting $\delta = \epsilon$)

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[24] Not necessarily. Adjust Y to be $f(X)$.

[25] Thm 3.1.5 was about the pre-image of an open set. In this corollary, we must show that the image of an open set is also open.

[26] 수학적으로 장난질 치는게 아니라 본질적인 의미가 있는 예시입니다.

Lipschitz Continuous: $\|f(x)-f(y)\| \le k\|x-y\|$ for some $k \ge 0$ (Set $\delta = \epsilon/k$ to show continuity)
Contraction: Lipschitz continuous and $k = 1$.

By Cor 3.2.3, $\exists\,\delta > 0$ s.t. $f(x) \ge \delta > 0$ for all $x \in A$. Then $\mathrm{dist}(A, B) \ge \delta > 0$.

Theorem 3.2.8 Suppose $f : X \to Y$ is continuous and surjective. If X is connected, Y is also connected.
Proof.[27] (Contradiction) Assume Y is disconnected. Then there exists non-empty sets U, V that are open in Y, and $U \cap V = \emptyset$, $U \cup V = Y$. Consider $f^{-1}(U), f^{-1}(V)$. We will show that X is disconnected. Since f is surjective, $f^{-1}(U), f^{-1}(V)$ are non-empty. Decomposition conditions can be checked easily, (use theorems from assignment) and openness holds by continuity.

Remark. Suppose $f : X \to Y$ is continuous. If $C \subset X$ is connected, $f(C)$ is also connected.

Cor 3.2.9 Suppose $f : I \to \mathbb{R}$ is continuous where I is any interval of $\mathbb{R}$. Then $f(I)$ is also an interval and hence connected.[28]

Cor 3.2.10 (Intermediate Value Theorem) Suppose $f : [a, b] \to \mathbb{R}$ is continuous. If α is in between $f(a)$ and $f(b)$,[29] then $\exists\, c \in [a, b]$ s.t. $f(c) = \alpha$.[30]
Proof. $f([a, b])$ is an **interval** (Cor 3.2.9) which includes $f(a), f(b)$. Then it must include α.[31]

Cor 3.2.11 Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then $f([a, b])$ is a closed interval.
Proof. $f([a, b])$ is an interval (Cor 3.2.9) and compact (Thm 3.2.1).

Cor 3.2.12 Suppose $f : [a, b] \to [a, b]$ is continuous. Then $\exists\, c \in [a, b]$ s.t. $f(c) = c$. We call such c a fixed point.
Proof. Apply IVT on $g(x) = x - f(x)$, set $\alpha = 0$. Then we have

$$g(a) = a - f(a) \le 0 = \alpha = 0 \le b - f(b) = g(b)$$

and the result follows directly.

Application. (**Path-Connected Set**)

Remark. $x, y \in \mathbb{R}^d \implies [x, y] = \{tx + (1-t)y : 0 \le t \le 1\}$ (convex combination)

[27] 책과 약간 다릅니다. 책의 증명도 읽어보세요.
[28] 이런 집합을 구간으로만 이해를 하면 우리가 아무것도 못 해요. 그런데 얘를 연결집합으로 이해하면 뭔가 할 것들이 생기고 여기서 중간값 정리가 바로 나오죠.
[29] $(f(a)-\alpha)(f(b)-\alpha) < 0$
[30] 이 정리를 위해 달려온 것...
[31] 구간은 볼록집합임을 이용해도 α 를 포함함을 보일 수 있다.

Set $f : [0,1] \to [x,y]$ as $f(t) = tx + (1-t)y$. Then f is continuous. (Lipschitz continuity can be easily checked and f is surjective)

Definition. Let $a, b \in \mathbb{R}$, $a < b$. Suppose $f : [a,b] \to \mathbb{R}^d$ is continuous. Then $f([a,b])$ is called a **path**.

Remark. Define $f : [a,b] \to \mathbb{R}^3$ as $f(t) = (\sin t, \cos t, \frac{1}{1+t^2})$ (Parameterized curve)
Also note that a path is compact and connected. ($[a,b]$ is compact and connected)

Definition. $C \subset \mathbb{R}^d$ is called **path-connected** if for any $x, y \in C$, there exists a path **in** C connecting x and y.

Theorem. Path-connected $\implies$ Connected

Proof. (Contradiction) Assume X is path-connected but disconnected. Then there exists sets U, V such that satisfy disconnectedness for X. Let $x \in U$, $y \in V$. From path-connected condition, there exists $f : [a,b] \to X$ s.t. f is continuous, $f(a) = x$, and $f(b) = y$. Let $Y = f([a,b]) \subset X$. Then Y can be decomposed into $Y \cap U$ and $Y \cap V$. These two sets satisfy the disconnectedness condition, (check) hence Y is disconnected. But since paths are always connected, contradiction.

Remark. The converse of the above theorem is **false**. Consider $f(x) = \sin \frac{1}{x}$ $(x > 0)$. Set $A = \{(x, \sin \frac{1}{x}) : x \in (0,1)\} \subset \mathbb{R}^2$. A is a path and therefore connected.
But the problem arises when we consider $\overline{A}$. We can easily check that the closure of a connected set is connected. We can also check that $\overline{A} = A \cup \{(0,t) : t \in [-1,1]\}$, which is not path-connected.[32]

[32] We need a jump from $x = 0$ to $x > 0$...

May 1st, 2019

3.3 Uniform Continuity

Definition. $f : X \to Y$ is **uniformly continuous** $\iff$ $\forall \epsilon > 0, \exists\, \delta > 0$ s.t. $x, y \in X$, $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$.

Remark. "$f : X \to Y$ is continuous at $x_0 \in X$" meant that $\forall \epsilon > 0, \exists\, \delta > 0$ s.t. $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$. In this definition, δ was a function of x_0. But in the definition of uniform continuity, δ is only dependent of ϵ.

Example.

(1) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ (Not uniformly continuous)
For $\epsilon = 1$, suppose we have $\delta > 0$. Set $x = 1/\delta + \delta/2$, $y = 1/\delta$. Then $|x - y| = \delta/2 < \delta$, but $|f(x) - f(y)| = |x^2 - y^2| = 1 + \delta^2/4 > \epsilon$.

(2) $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$ (Uniformly continuous & Lipschitz continuous)[33]
Given $\epsilon > 0$, $\delta = \epsilon/2$. If $|x - y| < \delta$ then $|f(x) - f(y)| = |x + y|\,|x - y| < 2\delta = \epsilon$.

(3) Lipschitz Continuity $\implies$ Uniform Continuity
Suppose $\forall x, y \in X$, $\exists\, k > 0$ s.t. $\|f(x) - f(y)\| \leq k\,\|x - y\|$. Then set $\delta = \epsilon/k$ to show uniform continuity.

(4) **Lipschitz $\implies$ Uniform $\implies$ Continuous**
$f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$.

  (a) Not Lipschitz continuous.
  $|f(x) - f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq k\,|x - y|$ for all $x, y \in X$? Impossible.

  (b) Uniform continuous.
  Set $\delta = \epsilon^2$. $|f(x) - f(y)| = \left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|} < \sqrt{\delta} = \epsilon$

Theorem 3.3.1 (Heine's Theorem) Suppose $f : X \to Y$ is continuous. If X is compact, f is uniformly continuous.

Proof. Given $\epsilon > 0$, $x \in X$, $\exists\, \delta(x) > 0$ s.t. $\|y - x\| < \delta(x) \implies \|f(y) - f(x)\| < \epsilon/2$.
Define $U_x = N(x, \delta(x)/2)$. Then $\{U_x : x \in X\}$ is a open cover of X. By compactness, there exists a finite subcover $\{U_{x_i}\}_{i=1}^n$. Set $\delta = \frac{1}{2}\min\{\delta(x_1), \ldots, \delta(x_n)\}$.
Suppose $\|x - y\| < \delta$. For some k, $x \in U_{x_k}$, and then $y \in N(x_k, \delta(x_k))$. This is because

$$\|x - x_k\| < \delta(x_k)/2, \quad \|y - x_k\| \leq \|y - x\| + \|x - x_k\| < \delta + \delta(x_k)/2 < \delta(x_k)$$

[33] 함수의 성질일 뿐만 아니라 domain 의 성질이기도 하다? Domain 도 중요한 역할을 한다.

Then we have

$$\|f(x) - f(y)\| \leq \|f(x) - f(x_k)\| + \|f(x_k) - f(y)\| < \epsilon/2 + \epsilon/2 = \epsilon$$

by continuity of f. Thus f is uniformly continuous.

Theorem 3.3.2 Suppose $f : X \to Y$ is uniformly continuous. If $\langle x_n \rangle$ is a Cauchy sequence in X, $\langle f(x_n) \rangle$ is also a Cauchy sequence.
Proof. Given $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$. For this δ, $\exists\, N$ s.t. $m, n \geq N \implies \|x_m - x_n\| < \delta$. Then we have

$$m, n \geq N \implies \|x_m - x_n\| < \delta \implies \|f(x_m) - f(x_n)\| < \epsilon$$

Remark. If $f : X \to Y$ is continuous, $\langle x_n \rangle \to x$ then $\langle f(x_n) \rangle \to f(x)$. In this case, $\langle x_n \rangle, x$ must be in X, $\langle f(x_n) \rangle, f(x)$ must be in Y.
Consider $f : (0, 1) \to \mathbb{R}$, $f(x) = 1/x$. $x_n = 1/n$ converges, and is a Cauchy sequence. But $f(x_n) = n$ is not Cauchy. The limit value of $\langle x_n \rangle$ does not have to be in X for a uniform continuous function.

Definition. Suppose $f : X \to Y$ is continuous, $X \subset A, Y \subset B$. If $g : A \to B$ satisfies $g(x) = f(x)$ for $x \in X$, and if g is continuous on A, we say that g is a **continuous extension** of f to A.

Example.

(1) $f : (0, 1) \to \mathbb{R}$, $f(x) = x$.
Consider $A = (0, 2)$. $g(x) = x$ on $(0, 2)$ is a continuous extension, $h(x) = x$ on $(0, 1)$, $h(x) = 1$ on $[1, 2)$ is also a continuous extension.
Consider $A = [0, 1]$. Then $g(0) = 0, g(1) = 1$, $g(x) = x$ on $(0, 1)$ is a unique continuous extension of f.

(2) $f : (0, 1) \to \mathbb{R}$, $f(x) = 1/x$.
Consider $A = [0, 1)$. It is impossible to find a continuous extension.

Cor 3.3.3 Suppose $f : X \to Y$ is uniformly continuous. Then there exists a unique continuous extension of f to $\overline{X}$.[34]
Proof. Take $x_0 \in \overline{X} \setminus X$. Set $g(x) = f(x)$ for $x \in X$. Now for $g(x_0)$, recall that $x_0 \in \overline{X}$, so there exists a sequence $\langle x_n \rangle$ in X s.t. $x_n \to x_0$. Since $\langle x_n \rangle$ is convergent, $\langle x_n \rangle$ is Cauchy sequence and by Thm 3.3.2, $\langle f(x_n) \rangle$ is also a Cauchy sequence. Thus $\langle f(x_n) \rangle$ converges. Define $g(x_0)$ as the limit of $f(x_n)$.

[34] Y is assumed to be extended to $\mathbb{R}^d$.

Now we must check if $g(x_0)$ is well-defined. In other words: For any two sequence $\langle x_n \rangle, \langle y_n \rangle$ that converge to x_0, does $f(x_n), f(y_n)$ converge to the same value?
Consider $\langle z_n \rangle = x_1, y_1, x_2, y_2, \ldots$. It is trivial that $z_n \to x_0$. Since $\langle z_n \rangle$ is Cauchy, $\langle f(z_n) \rangle$ is also Cauchy by uniform continuity. Let its limit be γ. Then $\langle f(x_n) \rangle, \langle f(y_n) \rangle$ is a subsequence of $\langle f(z_n) \rangle$, thus they both must converge to γ. Uniqueness directly follows from this proof, and we can easily check that g is continuous.

May 8th, 2019

3.4 Monotone Function

For this section, $f : X \to \mathbb{R}$, $X \subset \mathbb{R}$, X is an interval.

Definition. f is **monotonically increasing** if $x < y$ then $f(x) \leq f(y)$.[35] f is **monotonically decreasing** if $x < y$ then $f(x) \geq f(y)$.

Definition. f is **increasing** if $x < y$ then $f(x) < f(y)$, **decreasing** if $x < y$ then $f(x) > f(y)$.

Remark. Monotonically increasing = Weakly increasing. Increasing = Strongly increasing.

Example. $f(x) = \begin{cases} \sin \dfrac{1}{|x|} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$ has no left/right limits at $x = 0$.

Definition. $f : X \to \mathbb{R}$, $x_0 \in X$, $\alpha \in \mathbb{R}$.[36]

(1) (Right Limit) $\lim\limits_{x \to x_0+} f(x) = \alpha$, $f(x_0+) = \alpha \iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } (x_0, x_0 + \delta) \subset X \text{ and } x \in (x_0, x_0 + \delta) \implies |f(x) - \alpha| < \epsilon$$

(2) (Left Limit) $\lim\limits_{x \to x_0-} f(x) = \alpha$, $f(x_0-) = \alpha \iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } (x_0 - \delta, x_0) \subset X \text{ and } x \in (x_0 - \delta, x_0) \implies |f(x) - \alpha| < \epsilon$$

Exercise. $\lim\limits_{x \to x_0} f(x) = \alpha \iff f(x_0+) = f(x_0-) = \alpha$.

Definition. (Infinite Limits)

(1) $f(x_0+) = \infty \iff$

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } (x_0, x_0 + \delta) \subset X \text{ and } x \in (x_0, x_0 + \delta) \implies f(x) > M$$

(2) $f(x_0+) = -\infty \iff$

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } (x_0, x_0 + \delta) \subset X \text{ and } x \in (x_0, x_0 + \delta) \implies f(x) < -M$$

Remark. $x_0 \in \text{int}X$, we define

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[35] Watch out for the "$\leq$".

[36] $(x_0, x_0 + \delta) \subset X$ condition is necessary. Consider $X = [0, 1]$, the right limit of $x = 1$ can be any real number...

$$\lim_{x \to x_0} f(x) = \pm\infty \iff f(x_0+) = f(x_0-) = \pm\infty$$

Theorem 3.4.1 Suppose $f : X \to \mathbb{R}$ is monotone on $X = (a, b)$.

(1) $\forall x_0 \in (a, b) \implies$ Both $f(x_0+), f(x_0-)$ exist.

(2) $f(a+), f(b-)$ exist.

(3) For $a < x < y < b$, if f is monotonically increasing,

$$f(a+) \le f(x-) \le f(x) \le f(x+) \le f(y-) \le f(y) \le f(y+) \le f(b-)$$

Proof. WLOG, suppose f is monotonically increasing.

(1) Define $\alpha = \inf\{f(t) : t \in (x_0, b)\}$. (the set is bounded below by $f(x_0)$)

   Claim. $f(x_0+) = \alpha$.
   Proof. $\forall \epsilon > 0, \exists\, x_1 \in (x_0, b)$ s.t. $f(x_1) < \alpha + \epsilon$. ($\alpha$ is infimum) Now set $\delta = x_1 - x_0$. Then $(x_0, x_0 + \delta) \subset X$. For the second condition, if $x \in (x_0, x_0 + \delta) = (x_0, x_1) \implies \alpha \le f(x) \le f(x_1) < \alpha + \epsilon$. Thus $|f(x) - \alpha| < \epsilon$.

   From the claim we have $f(x_0+) = \inf\{f(t) : t \in (x_0, b)\}$, $f(x_0-) = \sup\{f(t) : t \in (a, x_0)\}$

(2) Define $\alpha = \inf\{f(t) : t \in (a, b)\}$ if the set is bounded below, $-\infty$ otherwise. Then we have $f(a+) = \alpha$. (Left as exercise)
   Also define $\beta = \sup\{f(t) : t \in (a, b)\}$ if the set is bounded above, ∞ otherwise. Then we have $f(b-) = \beta$.[37]

(3) Trivial. Check $f(x+) \le f(y-)$. ($\frac{x+y}{2}$ is in both $(x, b), (a, y)$)

$$f(x+) = \inf\{f(t) : t \in (x, b)\} \le f\left(\frac{x + y}{2}\right) \le \sup\{f(t) : t \in (a, y)\} = f(y-)$$

Cor 3.4.2 Suppose $f : X \to \mathbb{R}$ is monotone and X is an interval. Define

$$D = \{x_0 \in X : f \text{ is discontinuous at } x_0\}$$

then D is finite or countable.
Proof. WLOG, suppose f is monotonically increasing.
Suppose $x_0 \in D' = D \setminus \{\text{two endpoints of } X\}$. By Thm 3.4.1, left, right limits at x_0 exist, and $f(x_0+) > f(x_0-)$. (If equality holds, f is continuous at x_0)
Define $g : D' \to \mathbb{Q}$ by $g(x_0) = q_{x_0} \in (f(x_0-), f(x_0+))$ (any rational) Then $g : D' \to g(D') \subset \mathbb{Q}$

[37] 극한값이 ∞ 인 경우도 존재한다고 표현하는가?

is bijective. Since $g(D')$ is finite or countable (subset of $\mathbb{Q}$), D' is also finite or countable.

Theorem 3.4.3 Suppose $f : X \to \mathbb{R}$ is continuous and X is an interval.[38] The following are equivalent.

(1) f is injective.

(2) f is strongly increasing or decreasing.

Proof. (책과 다름) **(2 $\Longrightarrow$ 1)** Trivial.
(1 $\Longrightarrow$ 2) Define $D \subset \mathbb{R}^2$, $D = \{(x, y) : x, y \in X, x < y\}$. $g : D \to \mathbb{R}$, $g(x, y) = f(x) - f(y)$.

(1) D is connected. (Convex) (Check!)

(2) g is continuous. (Trivial by sequence definition)

Thus $g(D)$ is connected, and since it is a subset of $\mathbb{R}$, $g(D)$ is an interval. Also, $0 \notin g(D)$ since $x < y$ in the definition of D and $f(x) - f(y)$ is never 0 by injectivity.
Hence $g(D)$ is a subset of $(0, \infty)$ or $(-\infty, 0)$. If $g(D) \subset (0, \infty)$, f is decreasing. f is increasing for the second case.

Remark. Suppose $f : X \to \mathbb{R}$ is continuous and X is an interval. If f is increasing (or decreasing), $f : X \to f(X)$ is bijective, (injective by Thm 3.4.3) and $f^{-1} : f(X) \to X$ is continuous.
Proof. $\delta = \min\{f(x_0) - f(x_0 - \epsilon), f(x_0 + \epsilon) - f(x_0)\}$

[38] Note that this is the first time supposing continuity.

May 13th, 2019

4. 미분가능함수의 성질

4.1 Differentiability

For this section, suppose $f : I \to \mathbb{R}$, $I = (a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$.

Definition. f is **differentiable** at $x_0 \in I \iff$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \alpha \in \mathbb{R}$$

Remark.

(1) Denote $\alpha = f'(x_0)$. (**Derivative** of f at x_0)

(2) Differentiability is defined point-wise.

(3) f is differentiable on $I \iff f$ is differentiable at all $x_0 \in I$

Prop 4.1.1 The following are equivalent for $f : I \to \mathbb{R}$, $x_0 \in I$.

(1) f is differentiable at x_0.

(2) $\exists\, \alpha \in \mathbb{R}, \exists\, \eta : N(0, \delta) \setminus \{0\} \to \mathbb{R}$ s.t.

  (a) $f(x_0 + h) - f(x_0) = \alpha h + |h| \cdot \eta(h)$.[39]

  (b) $\lim_{h \to 0} \eta(h) = 0$.

Proof. ($\mathbf{1} \implies \mathbf{2}$) Define

$$\eta(h) := \frac{f(x_0 + h) - f(x_0) - f'(x_0)h}{|h|} \quad (h \neq 0)$$

Now check if (b) is satisfied. Then

$$f(x_0 + h) - f(x_0) = f'(x_0)h + |h| \cdot \eta(h)$$

($\mathbf{2} \implies \mathbf{1}$)

$$\frac{f(x_0 + h) - f(x_0)}{h} = \alpha + \frac{|h|}{h}\eta(h) \to \alpha = f'(x_0)$$

since $||h|\, \eta(h)/h| \to 0$ as $h \to 0$.

Example. Define

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

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[39] $|h|$ 로 정의한 이유는 벡터 함수를 다루기 위함!

f is differentiable at $x = 0$.[40]

Proof. $f(h) - f(0) = h^2 \sin \frac{1}{h} - 0 = 0 \cdot h + |h|\,|h| \sin \frac{1}{h}$, and set $\eta(h) = |h| \sin \frac{1}{h}$.
Note that

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

and f' is not continuous at 0.

Definition. Suppose $n \in \mathbb{N}$, $f : I \to \mathbb{R}$.[41]

$$f \in C^n \iff f \text{ is differentiable } n \text{ times, } f^{(n)} \text{ is continuous on } I$$

Remark. Differentiable at $x = x_0 \implies$ Continuous at $x = x_0$.

Remark. f is **nowhere differentiable** if $f : I \to \mathbb{R}$ is continuous, and f is not differentiable at all $x_0 \in I$. f exists, and it describes Brownian motion.

Prop 4.1.3 Suppose $f, g : I \to \mathbb{R}$ are differentiable at $x_0 \in I$. Then $f + g$, fg, f/g are also differentiable at x_0, and

(1) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

(2) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

(3) $(f/g)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$ $(g(x_0) \neq 0)$

Prop 4.1.4 (Chain Rule) Suppose $f : I \to J$, $g : J \to \mathbb{R}$, $x_0 \in I$, $y_0 = f(x_0) \in J$.
f is differentiable at x_0 and g is differentiable at $y_0 \implies g \circ f$ is differentiable at x_0, and

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Proof. By Prop 4.1.1, there exists $\alpha(h), \beta(h)$ s.t.

$$g(y_0 + h) - g(y_0) = g'(y_0) \cdot h + |h|\,\alpha(h)$$

$$f(x_0 + h) - f(x_0) = f'(x_0) \cdot h + |h|\,\beta(h)$$

Then we have

$$\begin{aligned} g(f(x_0+h)) - g(f(x_0)) =& g(y_0 + [f(x_0+h) - f(x_0)]) - g(y_0) \\ =& g'(y_0)(f(x_0+h) - f(x_0)) + |f(x_0+h) - f(x_0)|\,\alpha(f(x_0+h) - f(x_0)) \\ =& g'(f(x_0))(f'(x_0)h + |h|\,\beta(h)) \\ & \quad + |f(x_0+h) - f(x_0)|\,\alpha(f(x_0+h) - f(x_0)) \end{aligned}$$

---

[40] 미분가능성의 장점을 거의 사용할 수 없는 (쓸데 없는) 함수...

[41] $f^{(n)}$: 다들 아실테니까 정의 안하고 쓸게요!

Therefore we set

$$\eta(h) = \beta(h)g'(f(x_0)) + \left|\frac{f(x_0+h) - f(x_0)}{h}\right| \alpha(f(x_0+h) - f(x_0))$$

and check if $\eta(h) \to 0$ as $h \to 0$. Use $\lim_{h\to 0} \alpha(h) = \lim_{h \to 0} \beta(h) = 0$.

Remark.

(1) In $g(y_0 + h) - g(y_0) = g'(y_0) \cdot h + |h|\,\alpha(h)$, 0 was not in the domain of α. But defining $\alpha(0) = 0$ will solve the problem.

(2) If $f : [a, b] \to \mathbb{R}$ define right and left derivative at $x = a, b$ as

$$f'(a) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

if they exist.

May 15th, 2019

4.2 Mean Value Theorem

Lemma 4.2.1 (Rolle's Theorem) Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, there exists $c \in (a, b)$ s.t. $f'(c) = 0$.

Proof.

(1) Maximum of f = Minimum of $f = f(a) = f(b)$
f is constant. Trivial.

(2) Maximum of f is not $f(a), f(b)$
Suppose f attains maximum at $x = c \in (a, b)$ Then $f'(c) = \lim_{h\to 0} \dfrac{f(c + h) - f(c)}{h}$ must be 0. ($\because f'_+(c) \leq 0$ and $f'_-(c) \geq 0$)

(3) Minimum of f is not $f(a), f(b)$
(Proof is identical to that of (2))

Theorem 4.2.2 (Cauchy's Mean Value Theorem) Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ s.t.

$$(g(a) - g(b))f'(c) = (f(a) - f(b))g'(c)$$

Proof. Set $h(x) = (g(a) - g(b))f(x) - (f(a) - f(b))g(x)$ and apply Rolle's Thm.

Theorem 4.2.3 (Mean Value Theorem) Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Set $g(x) = x$ in Cauchy's MVT.

Theorem 4.2.5 (L'Hopital's Rule) Suppose $f, g : (a, b) \to \mathbb{R}$ are differentiable on (a, b). For $x_0 \in (a, b)$, if $f(x_0) = g(x_0) = 0$ and $\lim_{x\to x_0} \dfrac{f'(x)}{g'(x)} = \alpha$, then $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \alpha$.

Proof. Given $\epsilon > 0$, there exists $\delta > 0$ s.t. if $|x - x_0| < \delta$ then $|f'(x)/g'(x) - \alpha| < \epsilon$.
By Cauchy's MVT, there exists c_x in between x_0 and x s.t.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c_x)}{g'(c_x)}$$

If $|x - x_0| < \delta$,

$$\left|\frac{f(x)}{g(x)} - \alpha\right| = \left|\frac{f'(c_x)}{g'(c_x)} - \alpha\right| < \epsilon$$

since $|c_x - x_0| < |x - x_0| < \delta$.

4.3 Taylor Expansion

Suppose I is a closed interval, and $a \in I$.

Theorem 4.3.1 Suppose $f, g : I \to \mathbb{R} \in C^\infty(I)$. If $x \in \mathrm{int}(I)$, there exists c_x between a and x s.t.

$$\left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right) g^{(n+1)}(c_x) = \left(g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(x-a)^k\right) f^{(n+1)}(c_x)$$

Proof. Fix x. Define

$$F(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

Then[42]

$$F'(t) = \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(-1)^k(x-t)^{k-1} = \frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

Similarly define $G(t)$ and calculate $G'(t) = g^{(n+1)}(t)/n! \cdot (x-t)^n$.
By Cauchy's MVT, there exists c_x between a and x s.t.

$$(F(x) - F(a))G'(c_x) = (G(x) - G(a))F'(c_x)$$

which simplifies to

$$(f(x) - F(a))g^{(n+1)}(c_x)\frac{(x-c_x)^n}{n!} = (g(x) - G(a))f^{(n+1)}(c_x)\frac{(x-c_x)^n}{n!}$$

and now the result directly follows.

Remark.

(1) Taylor Expansion (around a)

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

(2) (In the book) $f, g \in C^n(I)$, and $f^{(n)}, g^{(n)}$ should be differentiable on $\mathrm{int}(I)$.

(3) **(Taylor's Theorem)** Set $g(x) = (x-a)^{n+1}$. $g^{(0)}(a) = \cdots = g^{(n)}(a) = 0$, but $g^{(n+1)}(x) = (n+1)!$ (constant). Then we have

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f^{(n+1)}(c_x)\frac{(x-a)^{n+1}}{(n+1)!}$$

[42] Note the $k = 1$ in the second term.

Prop 4.3.3 Suppose $f : I \to \mathbb{R} \in C^{\infty}(I)$.[43] For $a, x \in I$, define J as a interval with a, x as two endpoints. If there exists $M > 0$ s.t. $\left|f^{(n)}(y)\right| \leq M$ for $\forall n \in \mathbb{N}, \forall y \in J$,[44] then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Proof. Define

$$S_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

then we want to show that $\lim_{n\to\infty} |S_n(x) - f(x)| = 0$.
By Taylor's Theorem, $\exists\, c_x \in J$ s.t.

$$|f(x) - S_n(x)| \leq \left|f^{(n+1)}(c_x)\right| \frac{|x-a|^{n+1}}{(n+1)!} \leq M\frac{|x-a|^{n+1}}{(n+1)!} \to 0$$

The last term converges to 0 since factorials increase faster than exponents.

Example. $f(x) = \sin x$ satisfies the conditions of Prop 4.3.3, and calculating $f^{(k)}(0)$ gives

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Example. $f(x) = e^x$, at $a = 0$. $x \in \mathbb{R}_{\geq 0}$, $\{f^{(n)}(t) : t \in [0, x], n \in \mathbb{N}\}$ is bounded by e^x. Thus $f(x) = \sum_{k=0}^{\infty} x^k/k!$ $(x \geq 0)$

[43] Such functions are called **smooth**.

[44] 이 조건은 매우 **과한** 조건이다.

May 20th, 2019

Example. $f(x) = \log(1+x)$, $I = [0, \infty) \overset{?}{\implies} f(x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k}$

This cannot be done *yet*. (Chap 6)

Definition. Suppose $f : X \to \mathbb{R}$ $(X \subset \mathbb{R}^d)$.

(1) f has a **local maximum** $f(x_0)$ at x_0
$\iff$ Exists $\delta > 0$ s.t. $f(x_0) \geq f(x)$ for all $x \in N(x_0, \delta) \cap X$

(2) f has a **local minimum** $f(x_0)$ at x_0
$\iff$ Exists $\delta > 0$ s.t. $f(x_0) \leq f(x)$ for all $x \in N(x_0, \delta) \cap X$

Theorem. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable and has local maximum (minimum) at $c \in [a, b]$.[45]

(1) If $c \in (a, b)$ then $f'(c) = 0$.

(2) If $c = a$, $f'(a) \leq 0$ (≥ 0)

(3) If $c = b$, $f'(b) \geq 0$ (≤ 0)

Proof. (1) : Compare left/right-hand limits. Since they must be the same, $f'(c) = 0$.
(2), (3) : Inspect right-hand and left-hand limits, respectively. Right-hand limit should be negative, left-hand limit should be positive.

Remark. Maximum (Minimum) $\implies$ Local Maximum (Minimum)

Recall.

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

Definition. Suppose $F : I \to \mathbb{R}$ is differentiable. If $F' = f$, F is an **antiderivative** of f.

Theorem 4.2.6 (Darboux's Theorem) Suppose $F : I \to \mathbb{R}$ is a differentiable function defined on a closed interval, and let $F' = f$. If a, b are points in I with $a < b$ and $f(a) < \alpha < f(b)$, then there exists $c \in (a, b)$ s.t. $f(c) = \alpha$.
Proof. Define $G(x) = F(x) - \alpha x$. $G(x)$ is continuous and differentiable on I and has a minimum $G(c)$. $G'(a) = F'(a) - \alpha = f(a) - \alpha < 0$, $G'(b) = F'(b) - \alpha = f(b) - \alpha > 0$. Since c is minimum, it must be a local minimum. If $c = a$, $G'(c) \geq 0$, if $c = b$, $G'(c) \leq 0$. Thus $c \neq a, b$

[45] Statements for local minimum in brackets.

and $c \in (a, b)$, therefore we have $G'(c) = f(c) - \alpha = 0$.

Cor 4.2.7 Suppose $F : I \to \mathbb{R}$ is a differentiable function and $F' = f$. For any interval $J \subset I$, $f(J)$ is also an interval.[46]

Example. Does $f(x) = \begin{cases} x & (x < 0) \\ x + 1 & (x \geq 0) \end{cases}$ have an antiderivative ?
No. $f([-1, 1]) = [-1, 0) \cup [1, 2]$, which is not an interval.

[46] Intermediate value property 를 이용하여 구간의 상이 **연결집합**임을 보일 수 있었다!

$$\int_a^b f(x)dx$$

We learned about Riemann integrals, when f was continuous. There are two generalizations.

- Riemann-Stieltjes Integrals $\displaystyle\int_a^b f(x)dg(x)$
- Lebesgue Integrals: $\displaystyle\int_a^b f d\mu$ (μ: measure) (Most general)

미분은 하면 할수록 함수가 안좋아져요, 그런데 적분은 하면 할수록 함수가 좋아져요!

5. 적분 가능 함수의 성질

5.1 Riemann Integrals [47]

Definition.

(1) P is a **partition** of $[a,b]$ if $P \subset [a,b]$ is a finite subset and $a, b \in P$.

(2) $\mathcal{P}[a,b]$ is the **collection** of all partitions of $[a,b]$.

Example. Consider $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Then we divided $[a,b]$ into $[x_0, x_1]$, $\ldots$, $[x_{n-1}, x_n]$.

Definition. Suppose $f : [a,b] \to \mathbb{R}$ is bounded.[48] Given $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a,b]$, define

$$m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\} \qquad M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\}$$

then we define **lower/upper Riemann sum**s as[49]

(1) (Lower) $L(f,P) = \displaystyle\sum_{i=1}^{n} (x_i - x_{i-1}) m_i$

(2) (Upper) $U(f,P) = \displaystyle\sum_{i=1}^{n} (x_i - x_{i-1}) M_i$

Prop 5.1.1 Suppose $f : [a,b] \to \mathbb{R}$ is bounded.

(1) $P, Q \in \mathcal{P}[a,b]$, if $P \subset Q$ (Q is a finer partition than P)

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

[47] If we define integration only with Riemann integrals, there aren't so many integrable functions.
[48] $\exists M \ge 0$ s.t. $|f(x)| \le M$ for all $x \in [a,b]$.
[49] We define it this way so that Riemann integrals can be defined also for non-continuous functions.

(2) $P, P' \in \mathcal{P}[a,b] \implies L(f,P) \leq U(f,P')$

Proof. (1) : For partition P, consider an interval $[x_i, x_{i+1}]$. This interval adds $M_{i+1}(x_{i+1} - x_i)$ to the upper sum $U(f,P)$. Meanwhile, in partition Q, $[x_i, x_{i+1}]$ can be considered as $[y_a, y_b]$ for some a, b and this interval adds $\sum_{j=a+1}^{b} M_j^Q (y_j - y_{j-1})$ to the upper sum $U(f,Q)$.

$$M_{i+1} = \sup\{f(t) : t \in [x_i, x_{i+1}]\} \qquad M_j^Q = \sup\{f(t) : t \in [y_{j-1}, y_j]\}$$

If $j = a+1, \ldots, b$, $M_j^Q \leq M_{i+1}$, and thus

$$\sum_{j=a+1}^{b} M_j^Q (y_j - y_{j-1}) \leq \sum_{j=a+1}^{b} M_{i+1}(y_j - y_{j-1}) = M_{i+1}(y_b - y_a) = M_{i+1}(x_{i+1} - x_i)$$

(2) : $L(f,P) \leq L(f, P \cup P') \leq U(f, P \cup P') \leq U(f, P')$

Definition. We define the following.

- Upper Integral $\overline{\int_a^b} f = \inf\{U(f,P) : P \in \mathcal{P}[a,b]\}$
- Lower Integral $\underline{\int_a^b} f = \sup\{L(f,P) : P \in \mathcal{P}[a,b]\}$

By Prop 5.1.1 (2), $\underline{\int_a^b} f \leq \overline{\int_a^b} f$, and if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

we say that f is **Riemann integrable**.

May 22nd, 2019

Review

$f : [a, b] \to \mathbb{R}$ is bounded.

$P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$

$$m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\} \qquad M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\}$$

(1) (Lower) $L(f, P) = \displaystyle\sum_{i=1}^{n}(x_i - x_{i-1})m_i$

(2) (Upper) $U(f, P) = \displaystyle\sum_{i=1}^{n}(x_i - x_{i-1})M_i$

Prop 5.1.1 Suppose $f : [a, b] \to \mathbb{R}$ is bounded.

(1) $P, Q \in \mathcal{P}[a, b]$, if $P \subset Q$ (Q is a finer partition than P)

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

(2) $P, P' \in \mathcal{P}[a, b] \implies L(f, P) \leq U(f, P')$

Define

- Upper Integral $\overline{\int_a^b} f = \inf\{U(f, P) : P \in \mathcal{P}[a, b]\}$
- Lower Integral $\underline{\int_a^b} f = \sup\{L(f, P) : P \in \mathcal{P}[a, b]\}$

By Prop 5.1.1 (2), $\underline{\int_a^b} f \leq \overline{\int_a^b} f$, and if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

we say that f is **Riemann integrable**.

Example. $f : [0, 1] \to \mathbb{R}$, $f(x) = \begin{cases} 2 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$

For any partition P, $M_i = 2$, $m_i = 0$ for all i. Then $U(f, P) = 2$, $L(f, P) = 0$, thus not Riemann Integrable.[50]

[50] 리만 적분의 약함을 보여주는 상징적인 예입니다.

Remark. $\displaystyle\int_0^1 f(x)dx$ should be 0. Cardinality of $\mathbb{R}\setminus\mathbb{Q}$ is larger than $\mathbb{Q}$. f is Lebesgue Integrable and the value is 0.

Prop 5.1.2 The following are equivalent for bounded $f:[a,b]\to\mathbb{R}$.

(1) f is Riemann Integrable.

(2) $\forall\epsilon>0$, $\exists\, P\in\mathcal{P}[a,b]$ s.t. $U(f,P)-L(f,P)<\epsilon$.

Proof. **(1 $\Longrightarrow$ 2)** Suppose there exists partitions $P_1,P_2\in\mathcal{P}[a,b]$ s.t.

$$\overline{\int_a^b} f+\frac{\epsilon}{2}>U(f,P_1)\qquad \underline{\int_a^b} f-\frac{\epsilon}{2}<L(f,P_2)$$

Since upper/lower integrals are equal, we have

$$L(f,P_2)\le L(f,P_1\cup P_2)\le U(f,P_1\cup P_2)\le U(f,P_1)$$

and then $U(f,P_1\cup P_2)-L(f,P_1\cup P_2)<\epsilon$.
(2 $\Longrightarrow$ 1) For all $\epsilon>0$,

$$\epsilon>U(f,P)-L(f,P)\ge\overline{\int_a^b} f-\underline{\int_a^b} f\ge 0$$

Thus upper/lower integrals must be same, and f is Riemann Integrable.

Example. Riemann Integrable Functions

(1) f: Continuous

(2) f: Monotone

(3) $f(x)=\begin{cases}0 & (0\le 0<1, 2<x\le 3)\\ 1 & (1\le x\le 2)\end{cases}$
Consider the partition

$$P=\left\{0,1-\frac{\epsilon}{5},1+\frac{\epsilon}{5},2-\frac{\epsilon}{5},2+\frac{\epsilon}{5},3\right\}$$

Then $U(f,P)-L(f,P)=\displaystyle\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})=\frac{4}{5}\epsilon<\epsilon.$

Theorem 5.1.3 Suppose $f,g:[a,b]\to\mathbb{R}$ is bounded and Riemann Integrable.

(1) $f+g$ is Riemann Integrable, and $\displaystyle\int_a^b(f+g)=\int_a^b f+\int_a^b g$

(2) $\alpha\in\mathbb{R}$, αf is Riemann Integrable, and $\displaystyle\int_a^b\alpha f=\alpha\int_a^b f$

Proof.

(1) It is enough to show the following inequality.

$$\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b} (f+g) \leq \overline{\int_a^b} (f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$$

(a) For $P = \{a = x_0 < \cdots < x_n = b\}$, define the following

$$m_i^f = \inf\{f(t) : t \in [x_{i-1}, x_i]\}$$

$$m_i^g = \inf\{g(t) : t \in [x_{i-1}, x_i]\}$$

$$m_i^{f+g} = \inf\{(f+g)(t) : t \in [x_{i-1}, x_i]\}$$

Then we have[51]

$$m_i^{f+g} \geq m_i^f + m_i^g$$

(b) From the definition of lower Riemann sum, we have[52]

$$L(f+g, P) \geq L(f, P) + L(g, P)$$

(c) $\forall \epsilon > 0$, there exists $P_1, P_2 \in \mathcal{P}[a, b]$ s.t.

$$L(f, P_1) > \underline{\int_a^b} f - \frac{\epsilon}{2} \qquad L(g, P_2) > \underline{\int_a^b} g - \frac{\epsilon}{2}$$

(d)

$$\begin{aligned} \underline{\int_a^b} (f+g) &\geq L(f+g, P_1 \cup P_2) \underset{(b)}{\geq} L(f, P_1 \cup P_2) + L(g, P_1 \cup P_2) \\ &\geq L(f, P_1) + L(g, P_2) \underset{(c)}{\geq} \underline{\int_a^b} f + \underline{\int_a^b} g - \epsilon \end{aligned}$$

Take $\epsilon \to 0$ to prove the first inequality. (Last inequality can be proved similarly.)

(2) (a) $\alpha > 0$, then

$$U(\alpha f, P) = \alpha \cdot U(f, P) \qquad L(\alpha f, P) = \alpha \cdot L(f, P)$$

thus

$$\overline{\int_a^b} \alpha f = \alpha \overline{\int_a^b} f \qquad \underline{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f$$

(b) $\alpha < 0$, then

$$U(\alpha f, P) = \alpha \cdot L(f, P) \qquad L(\alpha f, P) = \alpha \cdot U(f, P)$$

thus

$$\overline{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f \qquad \underline{\int_a^b} \alpha f = \alpha \overline{\int_a^b} f$$

Thus Riemann Integrable in both cases.

[51] 각각을 최적화 한 것이 합쳐서 최적화 한 것보다 좋다.

[52] sup 을 양변에 취하는 시도는 실패한다.

Theorem 5.1.4 Suppose $f : [a, b] \to I$ is bounded and Riemann Integrable. Then for $c \in (a, b)$

(1) f is Riemann Integrable on $[a, c], [c, b]$.

(2) $\displaystyle\int_a^b f = \int_a^c f + \int_c^b f$

Proof.

(1) $\forall \epsilon > 0$, $\exists\, P \in \mathcal{P}[a, b]$ s.t. $U(f, P) - L(f, P) < \epsilon$. Suppose the partition is $P = \{a = x_0 < x_1 < \cdots < x_{l-1} \le c \le x_l < \cdots < x_n = b\}$. Define a partition $Q = \{x_0 < x_1 < \cdots < x_{l-1} \le c\}$. Then we have

$$U(f, Q) - L(f, Q) = \sum_{i=1}^{l-1} (M_i - m_i)(x_i - x_{i-1}) + (M'_l - m'_l)(c - x_{l-1})$$

$$U(f, P) - L(f, P) = \sum_{i=1}^{l-1} (M_i - m_i)(x_i - x_{i-1}) + (M_l - m_l)(x_l - x_{l-1}) + \sum_{i=l+1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

Thus

$$U(f, Q) - L(f, Q) \le U(f, P) - L(f, P) < \epsilon$$

and since $Q \in \mathcal{P}[a, c]$, f is Riemann Integrable on $[a, c]$ by Prop 5.1.2.

(2) It is enough to show that

$$\overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f \qquad \underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f$$

We show the first equation.

($\ge$) $\forall \epsilon > 0$, exists $Q \in \mathcal{P}[a, c]$, $R \in \mathcal{P}[c, b]$ s.t.

$$\overline{\int_a^c} f + \frac{\epsilon}{2} > U(f, Q) \qquad \overline{\int_c^b} f + \frac{\epsilon}{2} > U(f, R)$$

Then we have

$$\overline{\int_a^c} f + \overline{\int_c^b} f + \epsilon > U(f, Q) + U(f, R) = U(f, Q \cup R) \ge \overline{\int_a^b} f$$

($\le$) Define $P = \{a = x_0 < x_1 < \cdots < x_{l-1} \le c \le x_l < \cdots < x_n = b\}$. Define a partition $Q = \{x_0 < x_1 < \cdots < x_{l-1} \le c\}$, $R = \{c \le x_l < \cdots < x_n = b\}$.
$\forall \epsilon > 0$,

$$\overline{\int_a^c} f + \overline{\int_c^b} f \le U(f, Q) + U(f, R) = U(f, P \cup \{c\}) \le U(f, P) \le \overline{\int_a^b} f + \epsilon$$

(There exists P s.t. satisfy the last inequality)

May 27th, 2019

Currently: We are given bounded $f : [a,b] \to \mathbb{R}$. For $P \in \mathcal{P}[a,b]$, we defined $U(f,P)$ and $L(f,P)$. Then we defined $\overline{\int_a^b} f$ and $\underline{\int_a^b}$, and f was Riemann Integrable when these two values were the same.

Theorem 5.1.5 If $f : [a,b] \to \mathbb{R}$ is Riemann Integrable, then $|f|$ is also Riemann Integrable. Also, the following holds.

$$\int_a^b |f| \leq \left| \int_a^b f \right|$$

Proof. From $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$, and for $\epsilon > 0$,

$$U(|f|, P) - L(|f|, P) \leq U(f,P) - L(f,P) < \epsilon$$

Thus $|f|$ is integrable, and $-|f| \leq f \leq |f|$ gives the inequality.

5.2 Riemann Integrable Functions

Theorem 5.2.1 Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Then f is Riemann Integrable.

Proof. Given $\epsilon > 0$, our objective is finding a partition P s.t. $U(f,P) - L(f,P) < \epsilon$.

(1) Our first observation is that f is uniformly continuous, since the domain is compact. Thus there exists $\delta > 0$ s.t.

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b-a}$$

(2) Now we set a partition as $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ s.t. $x_i - x_{i-1} < \delta$ for all i.

(3) From EVT, for each closed interval $[x_{i-1}, x_i]$, there exists maximum and minimum $f(u_i), f(v_i)$. Thus $M_i = f(u_i)$, $m_i = f(v_i)$.

(4) Now we have

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^n (f(u_i) - f(v_i))(x_i - x_{i-1}) \\ &< \sum_{i=1}^n \frac{\epsilon}{b-a}(x_i - x_{i-1}) = \epsilon \end{aligned}$$

Theorem 5.2.2 Suppose $f : [a,b] \to \mathbb{R}$ is monotone. Then f is Riemann Integrable.

Proof. WLOG, suppose f is increasing.

Given $\epsilon > 0$, we want to find a partition P. Take $n \in \mathbb{N}$ s.t.

$$n > \frac{(b-a)(f(b) - f(a))}{\epsilon}$$

Consider a partition as

$$x_i = a + \frac{b-a}{n} i \implies P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

Now

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \frac{b-a}{n} \\ &= \frac{b-a}{n}(f(x_n) - f(x_0)) = \frac{(b-a)(f(b) - f(a))}{n} < \epsilon \end{aligned}$$

Definition. For $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a,b]$, define the **norm** of P as[53]

$$\|P\| = \max_{1 \le i \le n} \{x_i - x_{i-1}\}$$

And we say that P is finer than Q if $\|P\| \le \|Q\|$. Also, if $P \subset Q$, $\|Q\| \le \|P\|$.

Definition. **Riemann Sum** $R(f,P)$ is defined as

$$R(f,P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \quad (t_i \in [x_{i-1}, x_i])$$

Remark.

(1) $R(f,P) = R(f,P,t_1,t_2,\ldots,t_n)$

(2)

$$U(f,P) = \sup_{t_1,\ldots,t_n} R(f,P) \qquad L(f,P) = \inf_{t_1,\ldots,t_n} R(f,P)$$

(3)

$$L(f,P) \le R(f,P) \le U(f,P)$$

Theorem 5.2.3 Characterization of Riemann Integral via Riemann sums.
The following are equivalent for bounded $f : [a,b] \to \mathbb{R}$.

(1) f is Riemann Integrable and $\displaystyle\int_a^b f = A$.

(2) $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$\|P\| < \delta \implies |R(f,P) - A| < \epsilon \quad (\forall t_1, \ldots, t_n)$$

This is also written as $\displaystyle\lim_{\|P\| \to 0} R(f,P) = A$.

[53] 기존에 알고있던 norm 의 성질을 만족하지는 않는다. 좋은 이름은 아니다.

(3) $\forall \epsilon > 0$, $\exists\, P_0 \in \mathcal{P}[a,b]$ s.t.

$$P \supset P_0 \implies |R(f,P) - A| < \epsilon$$

Proof. **(1 $\implies$ 2)**
Claim.

(i) $\exists\, \delta_1 > 0$ s.t. $\|P\| < \delta_1 \implies U(f,P) < A + \epsilon$

(ii) $\exists\, \delta_2 > 0$ s.t. $\|P\| < \delta_2 \implies L(f,P) > A - \epsilon$

Setting $\delta = \min\{\delta_1, \delta_2\}$ will prove (2) since

$$A - \epsilon < L(f,P) \leq R(f,P) \leq U(f,P) < A + \epsilon$$

Proof of (i). ((ii) is similar)

(1) $f > 0$
$\exists\, P_0 \in \mathcal{P}[a,b]$ s.t. $U(f,P_0) < A + \epsilon/2$ (By Riemann Integrability of f)
Set $P_0 = \{a = x_0 < x_1 < \cdots < x_n = b\}$, M as the upper bound of f. Now set

$$\delta_1 = \frac{\epsilon}{2Mn}$$

Now $P = \{a = y_0 < y_1 < \cdots < y_m = b\}$, with $\|P\| < \delta_1$. Define

$$I = \{i : x_j \in (y_{i-1}, y_i) \text{ for some } j\} \qquad J = \{i : [y_{i-1}, y_i] \subset [x_{j-1}, x_j] \text{ for some } j\}$$

Then

$$U(f,P) = \overbrace{\sum_{i\in I} M_i(y_i - y_{i-1})}^{\leq M\cdot\delta_1\cdot n} + \overbrace{\sum_{i\in J} M_i(y_i - y_{i-1})}^{\leq U(f,P_0)} \leq U(f,P_0) + \delta_1 \cdot nM < A + \epsilon$$

(2) For general f: Set $g = f + c$ where c is a positive constant large enough that $g > 0$.
Then $\exists\, \delta_1$ s.t.

$$\|P\| < \delta_1 \implies U(g,P) < \int_a^b g + \epsilon \quad (*)$$

Note that

$$U(g,P) = \sum_{i=1}^{n} M_i^g(x_i - x_{i-1}) = \sum_{i=1}^{n}(M_i^f + c)(x_i - x_{i-1}) = U(f,P) + c(b-a)$$

Also

$$\int_a^b g = \int_a^b (f+c) = \int_a^b f + \int_a^b c = A + c(b-a)$$

Thus inequality $(*)$ is equivalent to

$$U(f,P) + c(b-a) < A + c(b-a) + \epsilon$$

and canceling $c(b-a)$ gives the desired inequality.

(2 $\implies$ 3) Let P_0 be any partition s.t. $\|P_0\| < \delta$. If $P_0 \subset P$, $\|P\| \leq \|P_0\| < \delta$. Therefore we have $|R(f,P) - A| < \epsilon$.

(3 $\implies$ 1) $\forall \epsilon > 0$, $\exists\, P_0$ s.t. $P_0 \subset P$ s.t. $|R(f,P) - A| < \epsilon/3$. Then

$$A - \frac{\epsilon}{3} < R(f,P) < A + \frac{\epsilon}{3}$$

Taking $\inf_{t_1,\ldots,t_n}$ and $\sup_{t_1,\ldots,t_n}$ on left/right inequalities respectively gives

$$U(f,P) \leq A + \frac{\epsilon}{3} \quad L(f,P) \geq A - \frac{\epsilon}{3}$$

Therefore

$$U(f,P) - L(f,P) \leq \frac{2\epsilon}{3} < \epsilon$$

and f is Riemann Integrable. Also,

$$A - \frac{\epsilon}{3} \leq L(f,P) \leq U(f,P) \leq A + \frac{\epsilon}{3}$$

We can infer that

$$A - \frac{\epsilon}{3} \leq \underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f \leq A + \frac{\epsilon}{3}$$

and taking $\epsilon \to 0$ gives $\displaystyle\int_a^b f = A$.

May 29th, 2019

Theorem 5.3.1 + 5.3.3 (Fundamental Theorem of Calculus) Suppose $f : [a, b] \to \mathbb{R}$ is bounded and Riemann Integrable.

(1) Suppose $F(x) = \displaystyle\int_a^x f(t)dt$, and f is continuous at x_0. Then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

(2) If $F' = f$ on $[a, b]$, $\displaystyle\int_a^b f(t)dt = F(b) - F(a)$.

Remark.

(1) (For 1) If f is continuous on $[a, b]$, $F' = f$ on $[a, b]$, and thus continuous functions have an antiderivative.

(2) Consider $f(x) = \begin{cases} 0 & (0 \le x < 1) \\ 1 & (1 \le x \le 2) \end{cases}$ then F is not differentiable at $x = 1$.

(3) (For 1) F is Lipschitz continuous.
$\because |f(x)| \le M$. For $x > y$,

$$|F(x) - F(y)| = \left| \int_a^x f - \int_a^y f \right| = \left| \int_y^x f \right| \le \int_y^x |f| \le M(x - y)$$

Proof.

(1) $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.
If $x > x_0$, we want to show that $\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| \to 0$.

$$\begin{aligned}
\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| &= \left|\frac{1}{x - x_0}\int_{x_0}^x f(t)dt - \frac{1}{x - x_0}\int_{x_0}^x f(x_0)dt\right| \\
&= \frac{1}{|x - x_0|}\left|\int_{x_0}^x \big(f(t) - f(x_0)\big)dt\right| \\
&\le \frac{1}{|x - x_0|}\int_{x_0}^x |f(t) - f(x_0)|\, dt \\
&< \frac{1}{|x - x_0|}\int_{x_0}^x \epsilon\, dt = \epsilon \quad (\because |t - x_0| < \delta \implies |f(t) - f(x_0)| < \epsilon)
\end{aligned}$$

Therefore the right derivative of F at x_0 is $f(x_0)$. The proof is similar for the left derivative.

(2) Take any $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$.

$$\begin{aligned}
F(b) - F(a) = F(x_n) - F(x_0) &= \sum_{k=1}^n \big(F(x_k) - F(x_{k-1})\big) \\
&\overset{\text{MVT}}{=} \sum_{k=1}^n (x_k - x_{k-1})f(t_k) \quad (\exists\, t_k \in (x_{k-1}, x_k)) \\
&= R(f, P)
\end{aligned}$$

Now since f is Riemann Integrable, $\displaystyle\int_a^b f(t)dt = F(b) - F(a)$

Cor 5.3.2 (Mean Value Theorem for Integrals) Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then there exists $c \in (a, b)$ such that

$$\frac{1}{b-a}\int_a^b f(t)dt = f(c)$$

Proof. Consider $F(x) = \displaystyle\int_a^x f(t)dt$. F is differentiable and apply MVT.

Prop 5.3.4 (Substitution Rule) Suppose $g : [a, b] \to [c, d]$ is a C^1-function and $f : [a, b] \to \mathbb{R}$ is continuous. Then

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))\, g'(t)\, dt$$

Proof. $H(y) = \displaystyle\int_{g(a)}^{y} f(t)dt$. Then H is differentiable and $H' = f$. Set

$$F_1(x) = \int_{g(a)}^{g(x)} f(t)dt = H(g(x)) \quad F_2(x) = \int_a^x f(g(t))g'(t)dt$$

Then $F_1'(x) = H'(g(x))g'(x) = f(g(x))g'(x) = F_2'(x)$. Thus $F_1(x) - F_2(x) = c$ (constant), and evaluating this at $x = 0$ gives $c = 0$.

Prop 5.3.5 (Integration by Parts) Suppose $f, g : [a, b] \to \mathbb{R}$ are C^1-functions. Then[54]

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

Proof. Use $(fg)' = fg' + f'g$.

5.4 Function of Bounded Variation (BV function)

Given $\alpha : [a, b] \to \mathbb{R}$,

$$\sum_{i=1}^{n} f(t_i)(\alpha(x_i) - \alpha(x_{i-1})) \xrightarrow[\|P\|\to 0]{} \int_a^b f d\alpha$$

If this limit exists, f is Stieltjes Integrable w.r.t α. Here, α must be at least of bounded variation.

Definition. For $f : [a, b] \to \mathbb{R}$, $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$. Define

$$V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

[54]모든 미분가능한 함수 f 에 대해 부분적분 식을 만족하면 g' 을 g 의 도함수로 정의하기도 한다. '미분 가능'의 범위를 넓히는 개념. 극한으로 정의하면 넓힐 방법이 없다...

and the **total variation** of f over $[a, b]$ by

$$V_a^b(f) = \sup\{V(f, P) : P \in \mathcal{P}[a, b]\}$$

And f is said to be of **bounded variation** if the total variation is finite. $V_a^b(f) < \infty$.

Example. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$ is not BV. Consider

$$P_n = \left\{0 = x_0 < \frac{2}{(2n+1)\pi} < \frac{2}{(2n-1)\pi} < \cdots < \frac{2}{3\pi} < \frac{2}{\pi} < 1\right\}$$

Then $f(\frac{2}{(2k+1)\pi}) = \frac{2}{(2k+1)\pi}(-1)^k$ and

$$\left| f\left(\frac{2}{(2k+1)\pi}\right) - f\left(\frac{2}{(2k-1)\pi}\right) \right| = \frac{2}{(2k+1)\pi} + \frac{2}{(2k-1)\pi} > \frac{2}{(2k-1)\pi}$$

Then the total variation diverges.

$$V(f, P_n) > \frac{2}{(2n+1)\pi} + \frac{2}{(2n-1)\pi} + \cdots + \frac{2}{\pi} = \frac{2}{\pi}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}\right) \to \infty$$

.

Example. $f : [a, b] \to \mathbb{R}$.

(1) f: monotone $\implies$ f is of bounded variation.
Proof. WLOG suppose f is increasing. Then $V(f, P) = f(b) - f(a)$.

(2) f: Lipschitz continuous $\implies$ f is of bounded variation.
Proof. $\exists\, M$ s.t. $|f(x) - f(y)| \leq M\,|x - y|$. Then $V(f, P) \leq M(b - a)$.

(3) $f \in C^1$, f' is bounded $\implies$ f: Lipschitz continuous $\implies$ f: Bounded variation.

(4) f: continuous does not imply that f is of bounded variation. (Counterexample above)

Lemma. If $f : [a, b] \to \mathbb{R}$ is of bounded variation, f is bounded.
Proof. Let $x \in [a, b]$. $P = \{a, x, b\}$.

$$|f(x)| \leq |f(a)| + |f(x) - f(a)| \leq |f(a)| + \overbrace{|f(x) - f(a)| + |f(b) - f(x)|}^{V(f,P)} \leq |f(a)| + |V(f)|$$

May 31st, 2019

Theorem 5.3.1 Suppose $f : [a, b] \to \mathbb{R}$ is bounded and Riemann integrable.

$$F(x) = \int_a^x f(t)dt \quad (a \leq x \leq b)$$

is uniformly continuous. If f is continuous then F is differentiable.

Problem 5.3.1 F: differentiable does not imply that f is continuous.

Problem 5.3.2 $f(x) = \displaystyle\int_{x^2}^x \sqrt{1+t^2}dt \implies f'(x) = \sqrt{1+x^2} - 2x\sqrt{1+x^4}$

Problem 5.6.2 If f, g are integrable, $\max\{f, g\}, \min\{f, g\}$ are also integrable.
Proof. Use $\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$.

Problem 5.6.3 If f, g are integrable, fg is integrable.
Proof.

(1) $0 \leq \sup\{|f(x)| : a \leq x \leq b\} = M < \infty$. For given $\epsilon > 0$, $\exists\, P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ s.t.

$$\sum_{i=1}^n (x_i - x_{i-1})(M_i - m_i) < \frac{\epsilon}{2M+1}$$

Since

$$\left|f(x)^2 - f(y)^2\right| \leq |f(x) - f(y)|\,(|f(x)| + |f(y)|) \leq 2M\,|f(x) - f(y)|$$

, let $\widetilde{M_i}, \widetilde{m_i}$ be supremum and infimum of f^2 in $[x_{i-1}, x_i]$. Then

$$\widetilde{M_i} - \widetilde{m_i} \leq 2M(M_i - m_i)$$

Thus

$$\sum_{i=1}^n (x_i - x_{i-1})(\widetilde{M_i} - \widetilde{m_i}) \leq \sum_{i=1}^n (x_i - x_{i-1})2M(M_i - m_i) \leq 2M \cdot \frac{\epsilon}{2M+1} < \epsilon$$

and f^2 is integrable.

(2) Now write $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$ to observe that fg is integrable.

Problem 5.6.4 $f : [a, b] \to \mathbb{R}$ is integrable. Prove that

$$\lim_{n\to\infty} \sum_{k=1}^n \frac{b-a}{n} f\left(a + \frac{b-a}{n}k\right) = \int_a^b f(x)dx$$

Proof. f: integrable. $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|P\| < \delta \implies \left| R(f,P) - \int_a^b f \right| < \epsilon$.

Take N so that $\dfrac{b-a}{N} < \delta$. Then for $n \geq N$,

$$\left| \sum_{k=1}^{n} \frac{b-a}{n} f\left(a + \frac{b-a}{n}k\right) - \int_a^b f \right| < \epsilon$$

Converse: False. $f : [0,1] \to \mathbb{R}$. $f(x) = 1$ if $x \in \mathbb{Q}$, 0 otherwise. f is not integrable, but the Riemann sum above equals 1.

Problem

(1) $\displaystyle \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \int_0^1 \frac{1}{1+t^2}\, dt = \frac{\pi}{4}$

(2) $\displaystyle \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{k^2+n^2}} = \int_0^1 \frac{1}{\sqrt{1+t^2}}\, dt = \sinh^{-1}(1)$

Problem 5.6.5 $f : [0,1] \to \mathbb{R}$, continuous and $f \geq 0$. If $\int_0^1 f(x)dx = 0$, show that $f \equiv 0$.

Proof. (Contradiction) Suppose $\exists\, a \in [0,1]$ s.t. $f(a) > 0$.

For $a \in (0,1)$, $\exists\, \delta > 0$ s.t. $[a-\delta, a+\delta] \subset [0,1]$ and $|f(x) - f(a)| < \frac{f(a)}{2}$ if $x \in [a-\delta, a+\delta]$.

$$0 = \int_0^1 f = \int_0^{a-\delta} f + \int_{a-\delta}^{a+\delta} f + \int_{a+\delta}^{1} f \geq 0 + \int_{a-\delta}^{a+\delta} \frac{f(a)}{2} + 0 = 2\delta \cdot \frac{f(a)}{2} > 0$$

Problem 5.6.9 $f : \mathbb{R} \to \mathbb{R}$, continuous and bounded. If for all $[a,b]$, $\int_a^b f = 0$ then $f \equiv 0$.

Proof. Similar to 5.6.5. (Contradiction) WLOG $f(a) > 0$ ...

Problem 5.6.6 $f : [a,b] \to \mathbb{R}$ is continuous. Show that

$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^n\, dx \right)^{1/n} = \max\{|f(x)| : x \in [a,b]\}$$

Proof. WLOG $f \geq 0$. Let $M = \max\{|f(x)| : x \in [a,b]\}$

($\leq$) For all n, $\displaystyle \left(\int_a^b |f(x)|^n\, dx \right)^{1/n} \leq (M^n(b-a))^{1/n} = M(b-a)^{1/n}$.
Take $\limsup$ on both sides to get (LHS) $\leq M$.

($\geq$) $\exists\, c \in [a,b]$ s.t. $f(c) = M$.
$\forall \epsilon > 0$, we want to show that

$$\liminf_{n\to\infty} \left(\int_a^b |f(x)|^n\, dx \right)^{1/n} \geq M - \epsilon$$

$\exists\, \delta > 0$ s.t. $[c-\delta, c+\delta] \subset [a,b]$, and $x \in [c-\delta, c+\delta] \implies M - \epsilon \leq |f(x)| \leq M$.

$$\left(\int_a^b |f(x)|^n\, dx \right)^{1/n} \geq \left(\int_{c-\delta}^{c+\delta} |f(x)|^n\, dx \right)^{1/n} \geq (2\delta)^{1/n}(M - \epsilon)$$

Take $\liminf$ on both sides to show the desired inequality.

June 3rd, 2019

$f : [a, b] \to \mathbb{R}$, $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$
$V(f, P) = V_a^b(f, P) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$
If $\{V(f, P) : P \in \mathcal{P}[a, b]\}$ is bounded above, f is a function of bounded variation. And we write the total variation of f over $[a, b]$ as $V(f) = V_a^b(f) = \sup\{V(f, P) : P \in \mathcal{P}[a, b]\}$

Remark.

(1) For two partitions P, Q s.t. $P \subset Q$, then $V(f, P) \le V(f, Q)$.

(2) $f(x) = \begin{cases} x \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$ is not BV.

(3) $f \in C^1[a, b] \implies f$: differentiable, f': bounded $\implies f$: Lipschitz continuous $\implies f$: BV

(4) f: BV $\implies f$: bounded.

Prop 5.4.1 Suppose $f, g : [a, b] \to \mathbb{R}$ is BV. Also, $\exists\, M_f, M_g$ s.t. $|f| \le M_f, |g| \le M_g$.[55]

(1) $f + g$ is BV, $V(f + g) \le V(f) + V(g)$.

(2) fg is BV, $V(fg) \le M_f \cdot V(g) + M_g \cdot V(f)$.

(3) αf is BV, $V(\alpha f) = |\alpha|\, V(f)$.

Proof.

(1) We know that $f + g$ is BV by

$$\begin{aligned} V(f+g, P) &= \sum_{i=1}^n |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\ &\le V(f, P) + V(g, P) \le V(f) + V(g) \end{aligned}$$

and taking sup over all $P \in \mathcal{P}[a, b]$ gives $V(f + g) \le V(f) + V(g)$.

(2) Sum the following inequality from $i = 1$ to n.

$$\begin{aligned} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| &= |f(x_i)(g(x_{i-1}) - g(x_{i-1})) + g(x_{i-1})(f(x_i) - f(x_{i-1}))| \\ &\le M_f\, |g(x_i) - g(x_{i-1})| + M_g\, |f(x_i) - f(x_{i-1})| \end{aligned}$$

Thus $V(fg, P) \le M_f \cdot V(g, P) + M_g \cdot V(f, P) \le M_f \cdot V(g) + M_g \cdot V(f)$ and fg is BV. Taking sup over all $P \in \mathcal{P}[a, b]$ gives $V(fg) \le M_f \cdot V(g) + M_g \cdot V(f)$.

(3) Exercise.

[55] Now we see that any linear combination of BV functions are BV.

Prop 5.4.2 Suppose $f : [a,b] \to \mathbb{R}$, $c \in (a,b)$. The following are equivalent.

(1) f is of bounded variation on $[a,b]$.

(2) f is of bounded variation on $[a,c]$ and $[c,b]$.

Moreover, if (1), (2) both hold, then

$$V_a^b(f) = V_a^c(f) + V_c^b(f)$$

Proof.

- Show that (1) $\implies [(2), V_a^c(f) + V_c^b(f) \le V_a^b(f)]$
  For $Q \in \mathcal{P}[a,c]$, $R \in \mathcal{P}[c,b]$ define $P = Q \cup R \in \mathcal{P}[a,b]$. By definition and (1),

  $$V_a^c(f,Q) + V_c^b(f,R) = V_a^b(f,P) \le V_a^b(f)$$

  Since $V(*)$ is positive, (2) holds by

  $$V_a^c(f,Q) \le V_a^b(f) \quad V_c^b(f,R) \le V_a^b(f)$$

  and taking sup over partitions of $[a,c]$, $[c,b]$ will give the desired inequality.

- Show that (2) $\implies [(1), V_a^c(f) + V_c^b(f) \ge V_a^b(f)]$
  Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a,b]$. set $c \in [x_{l-1}, x_l]$. Define

  $$Q = \{a = x_0 < x_1 < \cdots < x_{l-1} \le c\} \in \mathcal{P}[a,c] \quad R = \{c \le x_l < \cdots < x_n = b\} \in \mathcal{P}[c,b]$$

  Then

  $$\begin{aligned} &V_a^c(f,Q) + V_c^b(f,R) \\ &= \sum_{i=1}^{l-1} |f(x_{i-1}) - f(x_i)| + |f(x_{l-1}) - f(c)| + |f(c) - f(x_l)| + \sum_{i=l+1}^{n} |f(x_{i-1}) - f(x_i)| \\ &\ge \sum_{1 \le i \le n, i \ne l} |f(x_{i-1}) - f(x_i)| + |f(x_{l-1}) - f(x_l)| \ge \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)| = V_a^b(f,P) \end{aligned}$$

  $$V_a^b(f,P) \le V_a^c(f,Q) + V_c^b(f,R) \le V_a^c(f) + V_c^b(f)$$

  Thus f is BV on $[a,b]$ and $V_a^b(f) \le V_a^c(f) + V_c^b(f)$.

Theorem 5.4.2 The following are equivalent for $f : [a,b] \to \mathbb{R}$.

(1) f is of bounded variation.

(2) There exists monotonically increasing functions $g, h : [a,b] \to \mathbb{R}$ s.t. $f = g - h$.

Proof. **(2 $\implies$ 1)** Monotonic $\implies$ BV. Thus $g - f$ is BV.
(1 $\implies$ 2) Consider $g(x) = V_a^x(f)$ and $h(x) = g(x) - f(x)$. Then g is obviously monotonically increasing and $f = g - h$. Now we show that h is monotonically increasing.

$$\begin{aligned} h(y) - h(x) = g(y) - g(x) - [f(y) - f(x)] &= V_x^y(f) - [(f(y) - f(x)] \\ &\geq V_x^y(f, P) - [f(y) - f(x)] \geq |f(y) - f(x)| - [f(y) - f(x)] \geq 0 \end{aligned}$$

Remark.

(1) In (2), g, h are not unique, and setting $G(x) = g(x) + x$, $H(x) = h(x) + x$ gives strictly increasing functions that satisfy $f = G - H$.

(2) However, $f = \widehat{g} - \widehat{h}$ and if $\widehat{g}, \widehat{h}$ are monotonically increasing, $\widehat{g}(a) = 0$.
Then $\widehat{g}(x) \geq V_a^x(f)$ for all $x \in [a, b]$.

Why is BV important? 1. Length of Curve. 2. Stieltjes Integral.

Definition. **(Length of Curve)** For curve $\alpha : [a, b] \to \mathbb{R}^m$. For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$, define

$$\Lambda(\alpha, P) = \sum_{i=1}^{n} \|\alpha(x_i) - \alpha(x_{i-1})\|$$

. If $\{\Lambda(\alpha, P) : P \in \mathcal{P}[a, b]\}$ is bounded above, we define the supremum of this set as the **length of curve** α and denote it as $\Lambda(\alpha)$.

Theorem 5.4.4 + 5.4.5 Suppose $\alpha : [a, b] \to \mathbb{R}^m$, $\alpha(t) = \big(\alpha_1(t), \ldots, \alpha_m(t)\big)$.

(1) $\Lambda(\alpha) < \infty \iff \alpha_i$ is BV for all i.

(2) For all i, if $\alpha_i \in C^1([a, b]) \implies \Lambda(\alpha) = \displaystyle\int_a^b \sqrt{\alpha_1'(t)^2 + \cdots + \alpha_m'(t)^2}\, dt$

Proof.

(1) We use that fact that

$$V(\alpha_i, P) \leq \Lambda(\alpha, P) = \sum_{i=1}^{n} \|\alpha(x_i) - \alpha(x_{i-1})\| \leq \sum_{j=1}^{m} \sum_{i=1}^{n} |\alpha_j(x_i) - \alpha_j(x_{i-1})| = \sum_{j=1}^{m} V(\alpha_j, P)$$

Thus if $\Lambda(\alpha) < \infty$, $V(\alpha_i, P) \leq \Lambda(\alpha)$ and α_i is BV.
Also, if α_i are BV, $\Lambda(\alpha, P)$ is upper bounded by $V(\alpha_j, P) \leq V(\alpha_j)$. Thus $\Lambda(\alpha)$ is finite.

(2) Apply MVT for each component of $\alpha(x_i) - \alpha(x_{i-1})$.

$$\Lambda(\alpha, P) = \sum_{i=1}^{n} \|\alpha(x_i) - \alpha(x_{i-1})\| = \sum_{i=1}^{n} (x_i - x_{i-1}) \sqrt{\sum_{j=1}^{m} \alpha_j'(s_j)^2}$$

where $s_j \in (x_{i-1}, x_i)$ for each j. Use uniform continuity to bound... (omitted here)

June 5th, 2019

5.5 Stieltjes Integral

$f, \alpha : [a,b] \to \mathbb{R}$, we want to define $\int f\, d\alpha$. We define this for cases where α is monotonically increasing, and of bounded variation.

α: Monotonically Increasing Case

Definition. Given bounded function $f : [a,b] \to \mathbb{R}$, monotonically increasing $\alpha : [a,b] \to \mathbb{R}$, and a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, define

$$U(f,P,\alpha) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1})) \quad L(f,P,\alpha) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1}))$$

Also, Prop 5.1.1 holds.[56]

(1) For $P, Q \in \mathcal{P}[a,b]$, if $P \subset Q$,

$$U(f,P,\alpha) \geq U(f,Q,\alpha) \geq L(f,Q,\alpha) \geq L(f,P,\alpha)$$

(2) $P, Q \in \mathcal{P}[a,b] \implies U(f,P,\alpha) \geq L(f,Q,\alpha)$.

Proof. (1): For $t \in [x_{i-1}, x_i]$, define $X = [x_{i-1}, t]$ and $Y = [t, x_i]$. We only need to check

$$M_i(\alpha(x_i) - \alpha(x_{i-1})) \geq M_i^X(\alpha(t) - \alpha(x_{i-1})) + M_i^Y(\alpha(x_i) - \alpha(t))$$

This inequality holds because α is monotonically increasing.

We can also define

$$\overline{\int_a^b} f\, d\alpha = \inf\{U(f,P,\alpha) : P \in \mathcal{P}[a,b]\} \quad \underline{\int_a^b} f\, d\alpha = \sup\{L(f,P,\alpha) : P \in \mathcal{P}[a,b]\}$$

and if these two values are the same, f is **Stieltjes Integrable** w.r.t. α. We write $f \in \mathcal{R}(\alpha)$, and

$$\int_a^b f\, d\alpha = \int_a^b f(x)\, d\alpha(x) = \overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$$

Theorem 5.5.1 Suppose $f, g : [a,b] \to \mathbb{R}$ is bounded and given monotonically increasing $\alpha, \beta : [a,b] \to \mathbb{R}$, $c \in \mathbb{R}$.

(1) If $f, g \in \mathcal{R}(\alpha)$, $f+g, cf \in \mathcal{R}(\alpha)$ and

$$\int_a^b (f+g)\, d\alpha = \int_a^b f\, d\alpha + \int_a^b g\, d\alpha \quad \int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$$

[56] Setting $\alpha(x) = x$ will give the definition of Riemann Integrals.

(2) $a < p < b$. If $f \in \mathcal{R}(\alpha)$ on $[a, b] \iff f \in \mathcal{R}(\alpha)$ on $[a, p]$ and $[p, b]$, and

$$\int_a^b f\, d\alpha = \int_a^p f\, d\alpha + \int_p^b f\, d\alpha$$

(3) If $f \in \mathcal{R}(\alpha), \mathcal{R}(\beta)$, then $f \in \mathcal{R}(\alpha + \beta), \mathcal{R}(c\alpha)$ for $c \geq 0$. And

$$\int_a^b f\, d(\alpha + \beta) = \int_a^b f\, d\alpha + \int_a^b f\, d\beta \quad \int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha$$

Theorem 5.5.2 (1/2) The following are euqivalent for bounded f and monotonically increasing α.

(1) $f \in \mathcal{R}(\alpha)$

(2) $\forall \epsilon > 0, \exists\, P \in \mathcal{P}[a, b]$ s.t. $U(f, P, \alpha) - L(f, P, \alpha) < \epsilon$

Note that the above theorem could only be used for testing integrability. So to calculate the value of the integral, we define a Stieltjes Sum by

$$S(f, P, \alpha) = \sum_{i=1}^{n} f(t_i)(\alpha(x_i) - \alpha(x_{i-1})) \qquad (t_i \in [x_{i-1}, x_i])$$

Also note that on the proof of 5.2.3 (2) $\alpha(x) = x$ was heavily used.

Theorem 5.5.2 (2/2)

(1) $\displaystyle\int_a^b f\, d\alpha = A$

(2) $\forall \epsilon > 0, \exists\, P_0 \in \mathcal{P}[a, b]$ s.t. $P_0 \subset P \implies |S(f, P, \alpha) - A| < \epsilon$

Why is Stieltjes integral important?

(1) Intermediate object between Riemann Integral and Lebesgue Integral.

(2) Ex. $E(f(X)) = \int_a^b f\, dF$ (F: cumulative distribution function)

Remark. 5.5.2 (2/2) $\implies \alpha \in C^1$ then $\displaystyle\int_a^b f\, d\alpha = \int_a^b f(x)\, \alpha'(x)\, dx$

Example. $\alpha(x) = \begin{cases} 0 & (x \leq 0) \\ 1 & (x > 0) \end{cases}$. Calculate $\displaystyle\int_{-1}^1 f\, d\alpha$ for bounded $f : [-1, 1] \to \mathbb{R}$, $\displaystyle\lim_{x\to 0^+} f(x) = f(0)$.

$\forall \epsilon > 0, \exists\, \delta > 0$ s.t. $x \in [0, \delta] \implies |f(x) - f(0)| < \epsilon$ since $\lim_{x\to 0^+} f(x) = f(0)$.

Set $P = \{-1, 0, \delta, 1\}$. Check that

$$U(f, P, \alpha) = M_2 \quad L(f, P, \alpha) = m_2$$

and $M_2 = \sup\{f(x) : x \in [0, \delta]\} < f(0) + \epsilon$, $m_2 = \inf\{f(x) : x \in [0, \delta]\} > f(0) - \epsilon$. Therefore $U(f, P, \alpha) - L(f, P, \alpha) < 2\epsilon$ and f is Stieltjes Integrable. Take $\epsilon \to 0$. The answer is $f(0)$.

This is counter-intuitive... *Dirac delta function...*

$$\therefore \lim_{x \to 0^+} f(x) = f(0) \implies f \in \mathcal{R}(\alpha), \int_{-1}^{1} f\, d\alpha = f(0)$$

Remark. Consider this as $\int_{-1}^{1} f(x)\alpha'(x)dx$. For our α, $\alpha'(x) = \infty$ at $x = 0$, 0 otherwise. Also consider $f(x) = 2$ for $x \geq 0$, 0 otherwise. Then setting $x_{i-1} < 0 < x_i$, with $\|P\| < \delta$ will give $S(f, P, \alpha) = f(t_i)$, which might be either 0 or 2 depending on t_i's sign. Thus we cannot say that $\lim_{\|P\| \to 0} S(f, P, \alpha) = \int_a^b f\, d\alpha$.

α: Bounded Variation Case

Definition. Suppose $f : [a, b] \to \mathbb{R}$ is bounded and $\alpha : [a, b] \to \mathbb{R}$ is of bounded variation. If $\alpha = \alpha_1 - \alpha_2$ for some monotonically increasing functions α_1, α_2, and $f \in \mathcal{R}(\alpha_1)$, $f \in \mathcal{R}(\alpha_2)$

$$\implies \int_a^b f\, d\alpha = \int_a^b f\, d\alpha_1 - \int_a^b f\, d\alpha_2$$

Well-Definedness!

If $\alpha = \alpha_1 - \alpha_2 = \beta_1 - \beta_2$,

$$\int_a^b f\, d\alpha_1 - \int_a^b f\, d\alpha_2 = \int_a^b f\, d\beta_1 - \int_a^b f\, d\beta_2$$

holds because of Thm 5.5.1 (1)

Theorem 5.5.3 If $f : [a, b] \to \mathbb{R}$ is **continuous** and $\alpha : [a, b] \to \mathbb{R}$ is BV, $f \in \mathcal{R}(\alpha)$.

Proof. Enough to show for monotonically increasing α.

For $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{M(\alpha(b) - \alpha(a))}$, where M is an upper bound of $|f|$. For $P \in \mathcal{P}[a, b]$ s.t. $\|P\| < \delta$,

$$U(f, P, \alpha) - L(f, P, \alpha) = \sum_{i=1}^{n} (M_i - m_i)(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^{n} (f(u_i) - f(v_i))(\alpha(x_i) - \alpha(x_{i-1}))$$

for some u_i, v_i, since f is continuous. (EVT) But setting $u_i, v_i \in [x_{i-1}, x_i] \implies |u_i - v_i| < \delta$. Thus

$$\leq \frac{\epsilon}{M(\alpha(b) - \alpha(a))} \sum_{i=1}^{n} (\alpha(x_i) - \alpha(x_{i-1})) < \frac{\epsilon}{M}$$

Now if α is BV, $\alpha = \alpha_1 - \alpha_2$ for monotonically increasing α_1, α_2. Then $f \in \mathcal{R}(\alpha_1)$, $f \in \mathcal{R}(\alpha_2)$. Then $f \in \mathcal{R}(\alpha)$ and $\int_a^b f\, d\alpha = \int_a^b f\, d\alpha_1 - \int_a^b f\, d\alpha_2$ by definition.

June 7th, 2019

Remark. If $\alpha = \alpha_1 - \alpha_2$, (BV)

$$S(f, P, \alpha) = \sum_{t_i \in [x_{i-1}, x_i]} f(t_i)(\alpha(x_i) - \alpha(x_{i-1})) = S(f, P, \alpha_1) - S(f, P, \alpha_2)$$

Theorem 5.5.4 Thm 5.5.2 holds for BV function α.[57]
Proof. Define $V(x) = V_a^x(\alpha)$ as **variation of** α **on** $[a, x]$. Let $\alpha = V - (V - \alpha)$ where V is monotonically increasing. Note that α, $V - \alpha$ are both monotonically increasing.

(1 $\implies$ 2) Exists monotonically increasing α_1, α_2 s.t. $f \in \mathcal{R}(\alpha_1)$, $f \in \mathcal{R}(\alpha_2)$. Let $\int_a^b f\,d\alpha_1 = A_1$, $\int_a^b f\,d\alpha_2 = A_2$. Then $A = A_1 - A_2$. By Thm 5.5.2, there exists P_1 s.t. $|S(f, P, \alpha_1) - A_1| < \frac{\epsilon}{2}$ if $P_1 \subset P$. Also, there exists P_2 s.t. $|S(f, P, \alpha_2) - A_2| < \frac{\epsilon}{2}$ if $P_2 \subset P$. Now set $P_0 = P_1 \cup P_2$, and for $P \supset P_0$,

$$|S(f, P, \alpha) - A| = |S(f, P, \alpha_1) - A_1 - (S(f, P, \alpha_2) - A_2)| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(2 $\implies$ 1) Claim. (2) implies

(i) $f \in \mathcal{R}(V)$, $\int_a^b f\,dV = A_1$.

(ii) $f \in \mathcal{R}(V - \alpha)$, $\int_a^b f\,d(V - \alpha) = A_1 - A$.

(i) $\implies$ (ii): $S(f, P, V - \alpha) = S(f, P, V) - S(f, P, \alpha)$. $\exists\, P_0$ s.t. $|S(f, P, \alpha) - A| < \frac{\epsilon}{2}$ if $P \supset P_0$ (By (2)). And $\exists\, P_1$ s.t. $|S(f, P, V) - A_1| < \frac{\epsilon}{2}$ if $P \supset P_1$ (By (i), Thm 5.5.2). Set $P_2 = P_0 \cup P_1$, and for $P \supset P_2$,

$$|S(f, P, V - \alpha) - (A_1 - A)| \le |S(f, P, V) - A_1| + |S(f, P, \alpha) - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

Since $V - \alpha$ is increasing, by Thm 5.5.2 (ii) holds.

(i), (ii) $\implies$ (1): $f \in \mathcal{R}(V)$, $f \in \mathcal{R}(V - \alpha)$, $V, V - \alpha$ are monotonically increasing. Thus $f \in \mathcal{R}(V - (V - \alpha)) = \mathcal{R}(\alpha)$, and

$$\int_a^b f\,d\alpha = \int_a^b f\,dV - \int_a^b f\,d(V - \alpha) = A_1 - (A_1 - A) = A$$

Proof of (i). $\forall \epsilon > 0$, we try to find a P s.t. $U(f, P, V) - L(f, P, V) < \epsilon$. Note that $V(b) = V_a^b(\alpha)$.

[57] 리만 적분에서는 norm 이 작기만 하면 되었는데, 스틸체스는 그렇지 않아요. 예를 들어 α 가 불연속점을 가질 때 불연속을 포함하게 자르면 안 됐죠. 그니까 어떤 잘 써는 partition P_0 에 대해 그거 보다 저 잘 썰면 스틸체스합이 수렴한다는 뜻입니다.

There exists P_1 s.t. $P \supset P_1 \implies |V(\alpha, P) - V(\alpha)| < \epsilon'$.[58]
By (2), $\forall \epsilon > 0$, $\exists P_0$ s.t. $|S(f, P, \alpha) - A| < \epsilon'$ if $P \supset P_0$. Set $P = P_0 \cup P_1$, then $P \supset P_0$ and $P \supset P_1$.

$$\begin{aligned}U(f,P,V) - L(f,P,V) =& \sum (M_i - m_i)(V(x_i) - V(x_{i-1})) \\ =& \sum (M_i - m_i)(V(x_i) - V(x_{i-1}) - |\alpha(x_i) - \alpha(x_{i-1})|) \quad \cdots \quad c_1 \\ &+ \sum (M_i - m_i)\,|\alpha(x_i) - \alpha(x_{i-1})| \quad \cdots \quad c_2\end{aligned}$$

Let M be an upper bound of $|f|$ on $[a, b]$.

$$0 \le c_1 \le 2M \sum_{i=1}^{n} \{V(x_i) - V(x_{i-1}) - |\alpha(x_i) - \alpha(x_{i-1})|\} = 2M\{V_a^b(\alpha) - V(\alpha, P)\} < 2M\epsilon'$$

, since $P \supset P_1$. Now for c_2,

$$M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\} \implies \exists\, u_i \text{ s.t. } f(u_i) > M_i - \epsilon'$$

$$m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\} \implies \exists\, v_i \text{ s.t. } f(v_i) < m_i + \epsilon'$$

therefore $f(u_i) - f(v_i) > M_i - m_i - 2\epsilon'$ and $M_i - m_i < f(u_i) - f(v_i) + 2\epsilon'$. Define

$$(t_i, s_i) = \begin{cases} (u_i, v_i) & \text{if } \alpha(x_i) - \alpha(x_{i-1}) \ge 0 \\ (v_i, u_i) & \text{if } \alpha(x_i) - \alpha(x_{i-1}) < 0 \end{cases}$$

Then

$$\begin{aligned}c_2 \le& \sum_{i=1}^{n} \big(f(u_i) - f(v_i) + 2\epsilon'\big)\,|\alpha(x_i) - \alpha(x_{i-1})| \\ =& \sum_{i=1}^{n} \big(f(u_i) - f(v_i)\big)\,|\alpha(x_i) - \alpha(x_{i-1})| + 2\epsilon' V(\alpha, P) \\ \le& \sum_{i=1}^{n} \big(f(t_i) - f(s_i)\big)\,|\alpha(x_i) - \alpha(x_{i-1})| + 2\epsilon' V(\alpha) \\ =& S_1(f, P, \alpha, t_1, \dots, t_n) - S(f, P, \alpha, s_1, \dots, s_n) + 2\epsilon' V(\alpha)\end{aligned}$$

Now we use (2), for $P \supset P_0$,[59]

$$|S_1(f, P, \alpha) - S_2(f, P, \alpha)| \le |S_1(f, P, \alpha) - A| + |S_2(f, P, \alpha) - A| < \epsilon' + \epsilon' = 2\epsilon'$$

Overall,

$$U(f,P,V) - L(f,P,V) = c_1 + c_2 \le 2M\epsilon' + 2\epsilon' + 2\epsilon' V(\alpha) = \epsilon'(2M + 2 + 2V(\alpha))$$

$\epsilon = \epsilon'(2M + 2V(\alpha) + 2)$ will show what we wanted.

Theorem 5.5.4 The following are equivalent for BV α.

[58] Check as assignment.
[59] 이 P_0 는 어디서 왔을까?

(1) $f \in \mathcal{R}(\alpha)$, $\displaystyle\int_a^b f\, d\alpha = A$.

(2) $\forall \epsilon > 0$, $\exists\, P_0 \in \mathcal{P}[a,b]$ s.t. $|S(f,P,\alpha) - A| < \epsilon$ for all $P \supset P_0$.

Remark. $f \in \mathcal{R}(\alpha) \implies f \in \mathcal{R}(V)$!!

Theorem 5.5.5 Suppose $\alpha \in C_1 : [a,b] \to \mathbb{R}$. If $f \in \mathcal{R}(\alpha)$, $f\alpha'$ is Riemann Integrable, and

$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx = \int_a^b f\alpha'$$

Proof. $S(f,P,\alpha) = \sum_{i=1}^n f(t_i)(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^n f(t_i)\alpha'(s_i)(x_i - x_{i-1})$ for $s_i \in (x_{i-1}, x_i)$ by MVT, and $R(f\alpha', P) = \sum_{i=1}^n f(t_i)\alpha'(t_i)(x_i - x_{i-1})$

(1) $\exists\, P_0$ s.t. $P \supset P_0 \implies |S(f,P,\alpha) - A| < \frac{\epsilon}{2}$ (Thm 5.5.4)

(2) $\exists\, \delta > 0$ s.t. $|x - y| < \delta \implies |\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{2M(b-a)}$ (Uniform continuity of α on $[a,b]$)

Set P_1 as any superset of P_0 s.t. $\|P_1\| < \delta$. If $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \supset P_1$,

$$\begin{aligned}|R(f\alpha', P) - A| &\le |R(f\alpha', P) - S(f,P,\alpha)| + |S(f,P,\alpha) - A| \\ &\le \sum_{i=1}^n |f(t_i)|\, |\alpha'(s_i) - \alpha'(t_i)|\, (x_i - x_{i-1}) + \frac{\epsilon}{2} \\ &< M\frac{\epsilon}{2M(b-a)}\sum_{i=1}^n (x_i - x_{i-1}) + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon\end{aligned}$$

now $f\alpha'$ is Riemann Integrable and the integral is equal to A.

Example.

(1) $\int_{-1}^1 x^3\, d(x^2) = \int_{-1}^1 x^3 \cdot 2x dx = \frac{4}{5}$, since x^2 is C^1 ($\implies$ Lipschitz $\implies$ BV)

(2) $\int_{-1}^1 x^3\, d\,|x|$.[60]
$|x| = \alpha_1 - \alpha_2$ where

$$\alpha_1(x) = \begin{cases} 0 & (x < 0) \\ x & (x \ge 0) \end{cases} \qquad \alpha_2(x) = \begin{cases} x & (x < 0) \\ 0 & (x \ge 0) \end{cases}$$

These are both increasing, then splitting the integral and a simple calculation yields $\frac{1}{2}$.

(3) $\int_{-1}^1 x^3\, d\alpha$ for $\alpha(x) = \begin{cases} -x & (-1 \le x \le 0) \\ x+1 & (0 < x \le 1) \end{cases}$.
Let $\alpha = \beta_1 + \beta_2$ s.t. $\beta_1(x) = |x|$ and $\beta_2(x) = 1$ for $x > 0$, 0 otherwise. β_1, β_2 are both BV,

[60] You can show that $|x|$ is Lipschitz continuous then it is BV.

and their sum α is BV.

$$\int_{-1}^{1} f\,d\alpha = \int_{-1}^{1} f\,d\beta_1 + \int_{-1}^{1} f\,d\beta_2 = \frac{1}{2} + f(0) = \frac{1}{2}$$

(Check the first equality for BV functions)

Theorem 5.5.6 Suppose $f, \alpha : [a, b] \to \mathbb{R}$ and f, α is BV. If $f \in \mathcal{R}(\alpha)$, then $\alpha \in \mathcal{R}(f)$ and

$$\int_a^b \alpha\,df = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f\,d\alpha$$

Proof. Let $\int_a^b f\,d\alpha = A$. Then by Thm 5.5.4 $\forall \epsilon > 0, \exists\, P_0$ s.t. $P \supset P_0 \implies |S(f, P, \alpha) - A| < \epsilon$.

$$S(\alpha, P, f) = \sum_{i=1}^{n} \alpha(t_i)(f(x_i) - f(x_{i-1}))$$

Rewrite

$$f(b)\alpha(b) - f(a)\alpha(b) = \sum_{i=1}^{n} \big(\alpha(x_i)f(x_i) - \alpha(x_{i-1})f(x_{i-1})\big)$$

And for $P \supset P_0$,

$$\begin{aligned}
&\big|S(\alpha, P, f) - \big(f(b)\alpha(b) - f(a)\alpha(a) - A\big)\big| \\
&= \left|\sum_{i=1}^{n} f(x_i)[\alpha(x_i) - \alpha(t_i)] + \sum_{i=1}^{n} f(x_{i-1})[\alpha(t_i) - \alpha(x_{i-1})] - A\right| \\
&= |S(f, Q, \alpha) - A| < \epsilon
\end{aligned}$$

where $Q = \{a = x_0 \le t_0 \le x_1 \le t_1 \le x_2 \le \cdots \le t_n \le x_n = b\} = P \cup \{t_1, \cdots, t_n\} \supset P \supset P_0$.