March 29th, 2019

Remark. lim sup is the limit of sup. If sup is easy to calculate, find sup and take the limit.

Quiz 1 Solutions

#1. Given set A, $\mathrm{int}(A)$, A', determine whether the set is open or closed.

(1) $A = \mathbb{N} \subset \mathbb{R}$. $\mathrm{int}(A) = \emptyset$, $A' = \emptyset$, A is closed.

(2) $\mathbb{Q} \subset \mathbb{R}$. $\mathrm{int}(\mathbb{Q}) = \emptyset$, $\mathbb{Q}' = \mathbb{R}$, $\mathbb{Q}$ is neither open nor closed.

(3) $C = [0,1] \cup (2,3) \cap \{4\} \subset \mathbb{R}$. $\mathrm{int}(C) = (0,1) \cup (2,3)$, $C' = [0,1] \cup [2,3]$, C is neither open nor closed.

(4) $D = \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) : 0 \le y \le 1\} \subset \mathbb{R}^2$. $\mathrm{int}(D) = \emptyset$, $D' = D \cup \{(0,y) : 0 \le y \le 1\}$, D is neither open nor closed. ($\because$ $\mathrm{int}D \neq D$, $\overline{D} \neq D$)

#2. Find a limit point of given set.

(1) $A = \mathbb{Q} \subset \mathbb{R}$. 0 is a limit point. (Directly follows from Archimedes' principle)

(2) $B = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. 0 is a limit point of B. (Also directly follows from Archimedes')

(3) $C = \{2^{-n} + 3^{-m} : n, m \in \mathbb{N}\} \subset \mathbb{R}$. 0 is a limit point of C. Given $\epsilon > 0$, exists $N \in \mathbb{N}$ such that for $n, m \ge N$, $2^{-n} < \epsilon/2$, $3^{-m} < \epsilon/2$. Then $0 \neq 2^{-n} + 3^{-m} < \epsilon$.

#3. True or False? If false, find a counterexample.

(1) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ **True**

(2) $\overline{A \cap B} = \overline{A} \cap \overline{B}$ **False**. Set $A = (0,1), B = (1,2)$.
Correct Statement: $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

(3) $\mathrm{int}(A \cup B) = \mathrm{int}(A) \cup \mathrm{int}(B)$ **False**. Set $A = [0,1], B = [1,2]$.
Correct Statement: $\mathrm{int}(A) \cup \mathrm{int}(B) \subset \mathrm{int}(A \cup B)$

(4) $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$ **True**

Thm. $A \subset B \implies \overline{A} \subset \overline{B}, \text{int}(A) \subset \text{int}(B)$.

Proof.

- We need to show $A' \subset B'$. Let $x \in A'$.
  $\implies \forall \epsilon > 0,\ N(x, \epsilon) \cap (A \backslash \{x\}) \neq \emptyset$.
  $\implies \forall \epsilon > 0, N(x, \epsilon) \cap (B \backslash \{x\}) \neq \emptyset$
  $\implies x \in B'$.

- Let $x \in \text{int}(A)$
  $\implies \exists \epsilon > 0, N(x, \epsilon) \subset A \implies N(x, \epsilon) \subset B \implies x \in \text{int}(B)$.

Proof of (c). $A, B \subset A \cup B$
$\implies \text{int}(A), \text{int}(B) \subset \text{int}(A \cup B)$. Thus $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$

Proof of (d). $A \cap B \subset A, B \implies \text{int}(A, B) \subset \text{int}(A), \text{int}(B)$. Thus $\text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B)$
Suppose $x \in \text{int}(A) \cap \text{int}(B)$. Then $\exists \epsilon_A, \epsilon_B > 0$ s.t. $N(x, \epsilon_A) \subset A, N(x, \epsilon_B) \subset B$. Take $\epsilon = \min\{\epsilon_A, \epsilon_B\}/2$. Then $N(x, \epsilon) \subset A, B$. Therefore $N(x, \epsilon) \subset A \cap B$, $x \in \text{int}(A \cap B)$.
Example. $A = \{(x, y) : x^2 + 2y^2 < 1\}$. $\text{int}(A) = A, A' = \{(x, y) : x^2 + 2y^2 \leq 1\}$.
Suppose $(x_0, y_0) \in A$. $x_0^2 + 2y_0^2 = 1 - \delta < 1$ for some $\delta > 0$. By symmetry, let $x_0, y_0 > 0$. From

$$(x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 = x_0^2 + 2y_0^2 + \epsilon(2x_0 + 4y_0 + 3\epsilon) < 1$$

, we want $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \delta$. Set $\epsilon < 1/10$. Then $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \epsilon(2x_0 + 4y_0 + 3) < \delta$.
Now set $\epsilon = \min\left\{\dfrac{1}{2(2x_0 + 4y_0 + 3)}, \dfrac{1}{100}\right\} > 0$.
Then $|x - x_0| < \epsilon, |y - y_0| < \epsilon$. $x_0^2 + 2y_0^2 < (x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 < 1$. $N((x_0, y_0), \epsilon) \subset A$.
Interior points are limit points, and for the points (x_0, y_0) on the border, consider a sequence $(x_0 - 1/n, y_0 - 1/n)$. Then the elements are in A and they converge to (x_0, y_0). Thus the border is also included in A'.

April 1st, 2019

int$A$: $x \in A$ s.t. $N(x,\epsilon) \subset A$ for some $\epsilon > 0$.
A' : $x \in \mathbb{R}^d$ s.t. $N(x,\epsilon) \cap (A \backslash \{x\}) \neq \emptyset$ for $\forall \epsilon > 0$
$\overline{A}$: $x \in \mathbb{R}^d$ s.t. $N(x,\epsilon) \cap A \neq \emptyset$, $\forall \epsilon > 0$, $\overline{A} = A \cup A'$

Example. $A = [0,1) \cup \{2\}$. $1 \in A'$, $2 \notin A'$, $2 \in \overline{A}$

Prop 2.3.3 $x \in A' \implies N(x,\epsilon) \cap (A \backslash \{x\})$ 는 무한집합이다.
Proof. 유한집합이라고 가정하자. $N(x,\epsilon) \cap (A \backslash \{x\}) = \{x_1, \ldots, x_n\}$ 이라 할 수 있다. Set $\delta = \min\{\|x - x_i\| : \forall i\}$. Then $N(x,\delta) \cap (A \backslash \{x\}) = \emptyset$. 모순.
그래서 사실은 공집합이 아닌 것으로 정의했지만 **사실은** 무한집합이다.

Remark. $A' \neq \emptyset \implies A$는 무한집합.
(대우) A가 유한집합이면 극한점이 존재하지 않는다. (2.2 보기 4)
(역) **거짓**. $A = \{1, 2, \cdots\}$ 이면 $A' = \emptyset$.
그러면 역이 언제 성립하나요? 다음 단원 내용!

Definition. Convergence in $\mathbb{R}^d$
Let $\langle x_n \rangle$ be a sequence in $\mathbb{R}^d$.

$$\lim_{n\to\infty} x_n = x \iff \forall \epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies \|x_n - x\| < \epsilon)$$

Exercise. $x_n = (x_n^{(1)}, \ldots)$, $x = (x^{(1)}, \ldots)$ 일 때, $x_n \to x \iff \forall i, x_n^{(i)} \to x^{(i)}$

Notation. $A \subset \mathbb{R}^d$; $\langle x_n \rangle$ is a sequence in $A \iff \forall n, x_n \in A$

Theorem 2.2.2

(1) $x \in A' \iff \exists \langle x_n \rangle$ in $A \backslash \{x\}$ such that $x_n \to x$

(2) $x \in \overline{A} \iff \exists \langle x_n \rangle$ in A such that $x_n \to x$

Proof.

(1) ($\implies$) $x_n \in N\left(x, \frac{1}{n}\right) \cap (A \backslash \{x\})$ 이라 하자. (공집합이 아니므로 이러한 원소가 존재한다.) 그러면 $\|x_n - x\| < 1/n$ 이므로 x_n 은 x 로 수렴한다. 그리고 $x_n \in A \backslash \{x\}$ 이므로 수열이 $A \backslash \{x\}$ 에 있다.

(2) Left as exercise. Replace $A \backslash \{x\}$ with A.

Theorem 2.2.3. The following are equivalent.

(1) F is closed.

(2) $F' \subset F$.

(3) $F = \overline{F}$

(4) For a sequence $\langle x_n \rangle$ in F, $\lim_{n\to\infty} x_n = x \implies x \in F$.

Proof.
(1) $\iff$ (3) ($\overline{F}$: smallest closed set containing F.)
(2) $\iff$ (3) 은 자명.
(1) $\iff$ (4) by the above theorem. (Thm 2.2.2)

Applications.

(1) A' is closed.
Proof. We want to show that $(A')' \subset A'$.
We want to show: $x \in (A')' \implies x \in A'$.
(A' 이 공집합이면 자명. 공집합이 아니라고 가정하고...)
Given $\epsilon > 0$, $N(x,\epsilon) \cap (A' \backslash \{x\}) \neq \emptyset$. Take an element $y \in A'$ from this set. Now set $\delta = \min\{\|x-y\|, \epsilon - \|x-y\|\}$ then we have $N(y,\delta) \cap (A \backslash \{y\}) \neq \emptyset$. ($\because y \in A'$)
$z \in N(y,\delta) \cap (A \backslash \{y\})$ 라 하자.

(a) $z \in A \backslash \{y\} \subset A$.

(b) $\|x-z\| \leq \|x-y\| + \|y-z\| < \|x-y\| + \delta \leq \epsilon$ ($z \in N(y,\delta)$)

(c) $\|x-z\| \geq \|x-y\| - \|y-z\| > \|x-y\| - \delta \geq 0$ (By the choice of δ.) Thus $x \neq z$.

Therefore $z \in N(x,\epsilon)$ (by (b)), $z \in A \backslash \{x\}$ (by (a), (c)).
$x \in A'$ since $N(x,\epsilon) \cap (A \backslash \{x\})$ is not empty.

(2) $A \subset \mathbb{R}$: closed and bounded $\implies$ $\inf A = \min A$, $\sup A = \max A$. (Existence)
Proof. Let $\sup A = x \notin A$. ($\sup A \in A$ 이면 자명)
Claim. $x \in A'$.
Proof of Claim. $\forall \epsilon > 0$, $N(x,\epsilon) = (x-\epsilon, x+\epsilon)$
$x = \sup A$ 이므로 $x - \epsilon$ is not an upper bound.
$\exists y$ such that $y \in (x-\epsilon, x)$
$y \in N(x,\epsilon) \cap (A \backslash \{x\}) \neq \emptyset$ 이므로 x 는 극한점.
따라서 $x \in A' \subset A$ (closed set 이므로 Thm 2.2.3 (2)) 모순.
$\sup A \in A$ 이므로 이 값이 최댓값이다.

2.3 유계집합과 코시수열

핵심: Thm 2.3.4, Thm 2.3.7

Definition. $\langle x_n \rangle$: 유계수열(bounded sequence) $\iff$ $\exists M > 0$ s.t. $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Definition. $n_1 < n_2 < \cdots$: sequence in $\mathbb{N}$ 이라 하자. $\langle x_{n_k} \rangle_{k=1}^{\infty} = (x_{n_1}, x_{n_2}, \ldots)$ 를 $\langle x_n \rangle$의 부분수열(subsequence)이라 한다.

Theorem 2.3.4 (Bolzano-Weierstrass Theorem)

If $\langle x_n \rangle$ is bounded, there exists a convergent subsequence of $\langle x_n \rangle$.

Idea of Proof. Equivalent formulation for sets.

Definition. Set A is bounded $\iff$ $\exists M > 0$ such that $\|x\| < M$ for all $x \in A$.

Theorem 2.3.2 (Equivalent of 2.3.4) A가 유계이고 무한집합이면, $A' \neq \emptyset$.

Remark. $A' \neq \emptyset \implies A$: 무한집합.

역이 성립하기 위해서는 A가 유계라는 조건이 필요하다.

극한점이 중요한 이유는 계속 수열과 관련이 있기 때문이다.

Example. $A = \{1/n : n \in \mathbb{N}\}$ 을 고려하는 것은 수열 $x_n = 1/n$ 을 고려하는 것이나 마찬가지이다. 이 수열 x_n 이 x 로 수렴하는 것은 $A' = \{x\}$ 와 동치이다. (Hence the name “limit point”) 이로부터 $x \in A'$ $\iff$ Exists a subsequence of $\langle x_n \rangle$ in $A \backslash \{x\}$ converging to x.

Proof of 2.3.2

(1) **Lemma 2.3.1** 축소구간정리 in $\mathbb{R}^d$.

B is a closed box in $\mathbb{R}^d$ $\iff$ $B = I_1 \times I_2 \times \cdots \times I_d$, where $I_i = [a_i, b_i]$ for $i = 1, \ldots, d$. (I_i is a closed and bounded interval.)

$$B_1 \supset B_2 \supset \cdots \implies \bigcap_{n=1}^{\infty} B_n \neq \emptyset$$

Proof. 각 ‘좌표’ I_i 별로 1차원 축소구간정리를 적용하면 된다.

(2) **Divide and Conquer Strategy**

B: Box 일 때, $\text{diam}(B) = \sup\{\|x - y\| : x, y \in B\} = \sqrt{(a_1 - b_1)^2 + \cdots + (a_d - b_d)^2}$

Claim. There exists closed boxes $B_1, B_2, \ldots$ s.t.

(a) $B_1 \supset B_2 \supset \cdots$

(b) $\text{diam} B_n = \dfrac{1}{2^{n-1}} \text{diam} B_1$

(c) $B_n \cap A$: 무한집합

Proof. (Induction) $n = 1$; B_1: 충분히 커서 $A \subset B_1$ 인 box 를 잡으면 된다.
Suppose we have $B_1, \ldots, B_n$; B_n을 2^d 등분하면 적어도 하나는 A의 원소를 무한개 포함하고 있다. 그 집합을 B_{n+1} 으로 잡는다. (비둘기집의 원리)
이제 $x \in \bigcap_{n=1}^{\infty} B_n$ 으로 잡으면 (축소구간정리에 의해 잡을 수 있다) $x \in A'$. ($A' \neq \emptyset$)
$\because \forall \epsilon > 0$, $\mathrm{diam} B_n < \epsilon$ 인 $N \in \mathbb{N}$ 을 찾아 $n \geq N$ 일 때 부등식이 성립하도록 할 수 있다.
이러한 n 들에 대하여 $B_n \subset N(x, \epsilon)$. 그러면 $N(x, \epsilon) \cap (A \backslash \{x\}) \supset B_n \cap (A \backslash \{x\})$.

April 3rd, 2019

우리가 지금 2.3 을 하고 있는데, 2 가지 중요한 결과가 있어요.

Theorem 2.3.4 $\langle x_n \rangle$ 이 bounded 이면 수렴하는 부분수열을 갖는다.[1]

Theorem 2.3.2 A가 유계인 집합이고 무한집합이면 극한점을 가진다. $A' \neq \emptyset$
증명은 축소구간정리를 박스로 확장해가지고 분할 정복하면 된다.

Recall 2.3.3 $x \in A' \implies N(x, \epsilon) \cap (A \backslash \{x\})$ 는 무한집합이다.

Proof of 2.3.4. $A = \{x_1, x_2, \dots, x_n\}$ 라고 하면 이 집합은 유계이다. (수열이 유계이므로)

(1) A가 유한집합: 자명.
$\exists x$ such that x appears infinitely many times in $\langle x_n \rangle$. (PHP) 이 경우에는 부분수열을 $x, x, \dots$ 로 잡으면 된다. 이는 수렴하는 부분수열이다.

(2) A가 무한집합[2]
$A' \neq \emptyset$ 이므로 $\alpha \in A'$ 이라 하자.
Claim. $\exists n_1 < n_2 < \dots$ such that $\|x_{n_k} - \alpha\| < 1/k$.
Proof. (첨자들이 증가하면서 가까워져야 한다는 것이 유일하게 tricky 한 부분이다. 귀납법을 사용하자.) $k = 1$: $x_{n_1} \in N(\alpha, 1) \cap (A \backslash \{\alpha\})$ 로 잡으면 된다.
$x_{n_1}, \cdots, x_{n_k}$를 잡았다고 가정: $N(\alpha, \frac{1}{k+1}) \cap (A \backslash \{\alpha\})$ 에서 $x_{n_{k+1}}$를 잡아야 하는데 이 집합은 무한집합이다. (Recall 2.3.3) 이 집합에서 첨자가 n_k보다 큰 항이 반드시 존재하므로 그 중 하나를 $x_{n_{k+1}}$ 이라 잡으면 된다.
따라서 $\lim_{k \to \infty} x_{n_k} = \alpha$ (Check as exercise)

Application. (Characterization of $\limsup$ and $\liminf$)
x_n 이 bounded 이면, $A = \{x : \exists \text{ subsequence of } x_n \text{ converging to } x\}$. 이 때 Theorem 2.3.4에 의해 $A \neq \emptyset$ 임을 증명하였다.

(1) A: closed and bounded $\implies \max(A), \min(A)$ 가 존재한다.
Proof. $B = \{x_1, x_2, \dots\}$, $C = \{\langle x_n \rangle$ 에 무한 번 나타나는 수$\}$ 로 잡자. $A = B' \cup C, C \subset B, C' \subset B'$ 임을 확인해보라! 이를 이용하면 $B' \cup C = (B' \cup C') \cup C = B' \cup (C' \cup C) = B' \cup \overline{C}$ 가 되어 닫힌집합의 합집합은 닫힌 집합이다. A는 closed and bounded 이다.

(2) $\limsup x_n = \max(A)$, $\liminf x_n = \min(A)$
(부분수열이 가질 수 있는 극한값들 중 가장 큰 값이 $\limsup$, 가장 작은 값이 $\liminf$)

[1] 증명이 가장 테크니컬 해요!
[2] 이제 Thm 2.3.2 를 사용할 수 있다. 사실 경우를 나눈 것은 예외적인 case 를 처리하기 위한 것이었다.

Proof. Recall

$$\limsup x_n = \alpha \iff \begin{cases} \text{(i) } \forall \epsilon > 0, \exists N \text{ s.t } (n \geq N \implies x_n < \alpha + \epsilon) \\ \text{(ii) } \forall \epsilon > 0, x_n > \alpha - \epsilon \text{ for infinitely many } n \end{cases}$$

(a) 부분수열 $\langle x_{n_k} \rangle \to \beta$ 이면 (i)에 의해 $k \geq N \implies x_{n_k} < \alpha + \epsilon$ 이 되어 $\beta \leq \alpha + \epsilon$. $\beta \leq \alpha$. 그러므로 $\max(A) \leq \alpha$ 이다.

(b) $\forall \epsilon > 0$, (i), (ii)에 의해 $x_n \in (\alpha - \epsilon, \alpha + \epsilon)$ 인 n 이 무한히 많다. 이 유계인 구간에 속하는 수열의 항들에 대해 부분수열을 잡아 (further subsequence) γ 로 수렴하도록 할 수 있다. (Theorem 2.3.4) 그러면 $\langle x_{m_k} \rangle \to \gamma \in [\alpha - \epsilon, \alpha + \epsilon]$. 따라서 $\alpha - \epsilon \leq \gamma \leq \max(A)$ 가 되어 $\alpha \leq \max(A)$.

따라서 $\max(A) = \alpha$.

Definition. $\langle x_n \rangle$: **Cauchy Sequence** $\iff \forall \epsilon > 0, \exists N \text{ s.t. } [m, n \geq N \implies \|x_m - x_n\| < \epsilon]$

Prop 2.3.6, Thm 2.3.8 $\langle x_n \rangle$: convergent $\iff$ $\langle x_n \rangle$: Cauchy sequence[3]
Proof. ($\implies$) 자명. $\|x_m - x_n\| \leq \|x_m - \alpha\| + \|x_n - \alpha\| < \epsilon/2 + \epsilon/2 = \epsilon$ 인 $m, n \geq N$ 존재.
($\impliedby$) 수렴 값이 없는 상태에서 증명해야 한다. 먼저 수렴 값을 찾아보자.

(1) $\langle x_n \rangle$ is bounded.
Proof. $\exists N$ s.t. $\|x_m - x_n\| < 1$ for all $m, n \geq N$.
Set $M = \max\{\|x_1\|, \ldots, \|x_{N-1}\|, \|x_N\| + 1\}$. ($\|x_m\| < \|x_N\| + 1$)
따라서 $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

(2) There exists a subsequence $\langle x_{n_k} \rangle$ converging to some α. (Thm 2.3.4)

(3) $\langle x_n \rangle$ converges to α.
Proof. $\epsilon > 0$ 에 대해,

(a) 코시 수열의 성질에 의해 $\exists N_1$ s.t. $\|x_m - x_n\| < \epsilon/2$ for all $m, n \geq N_1$.

(b) 부분수열이 α로 수렴하므로 $\exists N_2$ s.t. $\|x_{n_k} - \alpha\| < \epsilon/2$ for all $k \geq N_2$.

Let $N = \max\{N_1, N_2\}$. $n \geq N_1, n_N \geq n_{N_1} \geq N_1$ 이므로,

$$n > N \implies \|x_n - \alpha\| \leq \|x_n - x_{n_N}\| + \|x_{n_N} - \alpha\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

[3]중간고사 전 까지 가장 중요한 정리.

Remark. 우리의 여정을 돌아보자.

(1) Archimedes' Principle 을 가정하면
Completeness Axiom $\implies$ Monotone Convergence Theorem $\implies$ 축소구간정리 $\implies$ Bolzano-Weierstrass Theorem $\implies$ **Cauchy Convergent Theorem**[4]
(Exercise) $\implies$ Completeness Axiom

(2) **Example**. $X = C([0,1])$. (Set of functions that are continuous in $[0,1]$) How would we define $\|f-g\|$? $\int_0^1 |f(x)-g(x)|\,dx$? $\max\{|f(x)-g(x)| : x \in [0,1]\}$? Only the second choice gives completeness for X.

(3) **Convergence Test** without limit value. (**Theorem 2.3.9**)
$\sum_{n=1}^{\infty} a_n$ is convergent $\iff \forall\epsilon > 0, \exists N$ s.t. $(n > m \geq N \implies |a_{m+1} + \cdots + a_n| < \epsilon)$
Proof. Trivial.

Definition. $\sum a_n$ is **absolutely convergent** $\iff$ $\sum |a_n|$ is convergent

Theorem. An absolutely convergent series converges.
Proof. Suppose $\sum |a_n|$ converges. For $\forall\epsilon > 0$, there exists N such that $||a_{m+1}| + \cdots + |a_n|| < \epsilon$ for all $m, n \geq N$. Therefore, for $m, n \geq N$,

$$|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \epsilon$$

and $\sum a_n$ converges.

[4] In any metric spaces, this is the condition for completeness.

April 5th, 2019

Theorem. $\overline{A} \cup \overline{B} = \overline{A \cup B}$

Proof. ($\subset$) Trivial.

($\supset$) $A \subset \overline{A}$, $B \subset \overline{B} \implies A \cup B \subset \overline{A} \cup \overline{B} \implies \overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. The closure of a closed set is itself.

6. (2) $a_n = \cos\sqrt{2019 + n^2\pi^2}$

Consider $\delta > 0$, such that

$$(n\pi - \delta)^2 < 2019 + n^2\pi^2 < (n\pi + \delta)^2$$

$$-2n\pi < \frac{2019}{\delta} \pm \delta < 2n\pi$$

We can find large enough N such that the above inequality holds for $n \geq N$.

Now we want $b_n = \sqrt{2019 + n^2\pi^2}$ bounded by $n\pi \pm \delta$.

$n \geq N$, n even $\implies n\pi - \delta < b_n < n\pi + \delta$

$\implies 1 \geq a_n > 1 - \epsilon$

$n \geq N$, n odd $\implies -1 \leq a_n < -1 + \epsilon$

Problem 2.3.5

(1) $x_{n+2} = \dfrac{x_n + x_{n+1}}{2}$

(2) $x_{n+1} = x_n + x(-1)^n \dfrac{1}{3n+1}$

Solution.

(1) Write $x_{n+2} - x_{n+1} = a(x_{n+1} - x_n)$ and observe that $a = -1/2$. Write as

$$x_n = x_{n-1} + \left(-\frac{1}{2}\right)^{n-2} (x_2 - x_1)$$

Then we have

$$x_n = x_2 + \sum_{k=1}^{n-2} \left(-\frac{1}{2}\right)^k (x_2 - x_1)$$

This series converges to $\dfrac{2x_2 + x_1}{3}$.

(2) This is an alternating series. Write as

$$x_n = x_1 + \sum_{k=1}^{n-1} (-1)^k \frac{x}{3n+1}$$

By alternating series test, the second summation term converges, and the series converges to x_1.

Since a converging sequence is a Cauchy sequence, x_1, x_2 can be any real number.

April 8th, 2019

Section 2.3: Bolzano-Weierstrass Theorem, Cauchy Convergent Theorem
In section 2.4, we will be studying about Convergence Tests.
정
2.4 급수의 수렴판정
Cor 2.3.9. $\sum_{n=1}^{\infty} a_n$ is convergent $\iff$ $s_n = \sum_{k=1}^{n} a_k$, $\langle s_n \rangle$ is convergent $\iff$ $\langle s_n \rangle$ is Cauchy.

(1) $\sum_{n=1}^{\infty} a_n$ is convergent $\implies$ $\lim_{n\to\infty} a_n = 0$.

(2) $\sum_{n=1}^{\infty} |a_n|$ is convergent $\implies$ $\sum_{n=1}^{\infty} a_n$ convergent.

Theorem 2.4.3 (Comparison Test) Suppose $\sum b_n$ converges. If $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$, $\sum a_n$ converges.
ProofLet $M = \sum b_n$, $s_n = \sum_{k=1}^{n} a_k$. s_n is increasing and s_n is bounded by M. s_n is convergent by Monotone Convergence Theorem.

Theorem. Suppose sequences a_n, b_n satisfy $0 \le |a_n| \le b_n$[5] and $\sum b_n$ converges. Then $\sum a_n$ is convergent.[6]
Proof. By comparison test and absolute convergence.

Prop 2.4.4 (Root Test) Suppose $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$.
If $\alpha < 1$, $\sum a_n$ converges. If $\alpha > 1$, $\sum a_n$ diverges.
Proof.

(1) $\alpha < 1$. Take $\epsilon > 0$ such that $\alpha < \alpha + \epsilon < 1$. Then there exists N such that $|a_n|^{1/n} < \alpha + \epsilon$ for all $n \ge N$. Therefore $|a_n| < (\alpha + \epsilon)^n$. Since $\alpha + \epsilon < 1$, $\sum (\alpha+\epsilon)^n$ converges. Apply the comparison test to see that $\sum a_n < \infty$.

(2) $\alpha > 1$. Take $\epsilon > 0$ such that $\alpha > \alpha - \epsilon > 1$. Then $|a_n|^{1/n} > \alpha - \epsilon$ for infinitely many n. Then $|a_n| > (\alpha - \epsilon)^n > 1$. Therefore $\lim a_n \ne 0$. $\sum a_n$ diverges.

Prop 2.4.5 (Ratio Test) Suppose $a_n \ne 0$. Let $\beta = \limsup |a_{n+1}/a_n|$, $\gamma = \liminf |a_{n+1}/a_n|$.
If $\beta < 1$, $\sum a_n$ converges. If $\gamma > 1$, $\sum a_n$ diverges.
Proof.

(1) $\beta < 1$. Take $\epsilon > 0$ such that $\beta < \beta + \epsilon < 1$. Then $\exists N$ s.t. $|a_{n+1}/a_n| < \beta + \epsilon$ for $n \ge N$.
$\implies$ $|a_n| = |a_N| \, |a_{N+1}/a_N| \cdots |a_n/a_{n-1}| < |a_N| \, (\beta+\epsilon)^{n-N}$.
Set $b_n = |a_N| \, (\beta+\epsilon)^{n-N}$ and apply comparison test to see that $\sum a_n < \infty$.

[5] Note that this condition can fail for finitely many n.
[6] a_n may be a very complex expression, but we want b_n to be simple, an expression we know that it is convergent.

(2) $\gamma > 1$. Take $\epsilon > 0$ such that $\gamma > \gamma - \epsilon > 1$. Then $\exists N$ s.t. $|a_{n+1}/a_n| > \gamma - \epsilon$ for $n \geq N$. Then we see that $|a_n|$ is increasing for $n \geq N$. Thus a_n cannot converge to 0. $\sum a_n$ is divergent.

Remark. If the above limits (ratio, root) exist, elementary tests can be applied. But if the limits turn out to be 1, the test fails. (ND: Non-Deterministic) Check it for $\sum 1/n, \sum 1/n^2$. Also, these are *weak tests*. For most of the series, the limit is 1. Moreover...

Theorem 2.4.6 Suppose $a_n \neq 0$.

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

Thus if the root test works, ratio test also works.[7]

Proof. We only need to prove the last inequality.

Let $\beta = \limsup |a_{n+1}/a_n|$, $\forall \epsilon > 0$. $\implies \exists N$ s.t. $|a_{n+1}/a_n| \leq \beta + \epsilon$ for $n \geq N$. Then if $n \geq N$, $|a_n| \leq |a_N| (\beta + \epsilon)^{n-N}$. (Similar to proof of 2.4.5) Then

$$|a_n|^{1/n} \leq (\beta + \epsilon) \left(\frac{|a_n|}{(\beta + \epsilon)^N}\right)^{1/n}$$

and take lim sup on both sides, then $\limsup |a_n|^{1/n} \leq \beta + \epsilon$.

Example. $\langle a_n \rangle = \begin{cases} 1/2^n & n \text{ odd} \\ 1/2^{n-2} & n \text{ even} \end{cases}$

Check that $\limsup |a_n|^{1/n} = 1/2 < 1$, and the series $\sum a_n$ converges by the root test.

But if we use the ratio test here, lim sup value is 2 and lim inf value is 1/8.[8] The ratio test does not tell us anything about the convergence. Also note that the series converges to 2.

Prop 2.4.1 (Rearrangement) $a_n \geq 0$.[9] Suppose a bijection $r : \mathbb{N} \to \mathbb{N}$ exists.

(1) $\displaystyle\sum_{n=1}^{\infty} a_n = s \iff \sum_{n=1}^{\infty} a_{r(n)} = s$

(2) $\displaystyle\sum_{n=1}^{\infty} = \infty \iff \sum_{n=1}^{\infty} a_{r(n)} = s$

Proof.

(1) ($\implies$) Let $t_n = \sum_{k=1}^{n} a_{r(k)}$. Then t_n is increasing and bounded by s. Thus t_n converges by MCT, and $\lim t_n \leq s$.
$s_n = \sum_{k=1}^{n} a_k \leq \sum_{n=1}^{\infty} a_{r(n)} = t = \lim t_n$. ($a_n \geq 0$ was used here.)
($\impliedby$) Use $r^{-1}(n)$.

[7]The limit for the ratio test is much easier to calculate than the root test. That's why we use the ratio test.
[8]The ratios are: 2, 1/8, 2, 1/8 ...
[9]This is the important condition.

(2) Contraposition of (1).

Prop 2.4.2 (Alternating Series Test) For a given sequence x_n, suppose the following holds.

- x_n is decreasing.
- $\lim x_n = 0$.

Then the series $\sum_{k=1}^{\infty}(-1)^{n-1}x_n$ is convergent.
Proof. Let $s_n = \sum_{k=1}^{n}(-1)^{k-1}x_k$. For $m < n$,

$$|s_n - s_m| = \left|(-1)^m x_{m+1} + \cdots + (-1)^{n-1}x_n\right| = |x_{m+1} - x_{m+2} + \cdots \pm x_n| \overset{(*)}{\in} [0, x_{m+1}]$$

$$\begin{aligned}(*) : x_{m+1} - x_{m+2} + \cdots + x_n &= (x_{m+1} - x_{m+2}) + \cdots + (x_{n-2} - x_{n-1}) + x_n \geq 0 \\ &= x_{m+1} - (x_{m+2} - x_{m+3}) - \cdots - (x_{n-1} - x_n) \leq x_{m+1}\end{aligned}$$

Check for the case with last term $-$.
Now, $\forall \epsilon > 0$, find N such that $|x_n| < \epsilon$ for $n \geq N$. Then for $n > m \geq N$, $|s_n - s_m| \leq x_{m+1} < \epsilon$.
Thus $\langle s_n \rangle$ is a Cauchy sequence and the given series converges.

Example. $a_n = (-1)^{n-1}/n$. $\sum a_n$ converges by alternating series test and converges to $\log 2$.

Remark. The rearrangement of the above example may not converge, or converge to a different value than $\log 2$.

Exam: 1.1 - 2.6

After the midterms we will be covering functions and continuity.
Chapter 1 has been about $\mathbb{R}$, and in Chapter 2, we have talked about subsets of $\mathbb{R}^n$.
2.1: What is $\mathbb{R}^n$? Vector Space, IPS, Metric Space, Normed Space...
2.2: Open, closed sets
2.3: Bounded sets and Cauchy sequences
(2.4: Convergence Tests)
2.5: Compact Sets
2.6: Connect Sets

April 10th, 2019

2.5 Compact Set

Definition. $\{U_i : i \in I\}$ (I is the index set, $U_i \subset \mathbb{R}^d$) is called "family of sets".

(1) $\{U_i : i \in I\}$ is a **cover** of $K \subset \mathbb{R}^d \iff K \subset \bigcup_{i\in I} U_i$.

(2) $\{U_i : i \in I\}$ is a **open cover** $\iff U_i$ are open for $\forall i$.

(3) $J \subset I$, $\{U_i : i \in J\}$ is called a **subcover** of $\{U_i : i \in I\} \iff K \subset \bigcup_{i\in J} U_i$.

Definition. $K \subset \mathbb{R}^d$ is **compact** $\iff$ Any open cover of K has finite subcover.

Example.

(1) $\mathbb{N}$ is not compact. Set $U_k = (k - 1/2, k + 1/2)$, then $\{U_k : k \in \mathbb{N}\}$ is a (open) cover of $\mathbb{N}$. But there are no finite subcover.

(2) $A = (0, 1)$ is not compact. Set $U_k = (1/k, 1)$, then because $\bigcup_{k=1}^{\infty} U_k = (0, 1)$, $\{U_k : k \in \mathbb{N}\}$ is a (open) cover of A. But there are no finite subcover. $\bigcup_{i=1}^{m} U_{k_i} = U_{k_m} = (1/k_m, 1)$, which cannot contain $(0, 1)$.

(3) $A = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{R}^d$ is compact. $\{U_i : i \in I\}$ be a cover of A. There exists $i_1, \ldots, i_m \in I$ such that $a_k \in U_{i_k}$ for $k = 1, \ldots, m$. Then $\{U_{i_1}, U_{i_2}, \ldots, U_{i_m}\}$ is a finite subcover of A.

Main Theorem: **Heine-Borel Theorem**

$$K \text{ is compact} \iff K \text{ is bounded and closed.}$$

Remark.

(1) This is a part of Thm 2.5.4

(2) Proof: Prop 2.5.1, Thm 2.5.2, Prop 2.5.3

(3) **Characterization of compact sets in** $\mathbb{R}^d$.[10]

---

[10] Compact Set 을 이 단순한 공간 안에서는 characterize 할 수 있다!

Proof.

($\Longrightarrow$) (Prop 2.5.1)

(1) *Is K bounded?*
Set $U_k = N(0,k)$. Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d$. Thus $\{U_k : k \in \mathbb{N}\}$ is an open cover of K. There exists a finite subcover $U_{k_1}, \ldots, U_{k_m}$ $(k_1 < \cdots < k_m)$ of K. Then we have $K \subset \bigcup_{i=1}^{m} U_{k_i} = U_{k_m} = N(0, k_m)$. Therefore K is bounded.

(2) *Is K closed?*
Suppose $x \in K^C$. Set $U_k = \{y : \|y - x\| > 1/k\}$. Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d \backslash \{x\} \supset K$. (Open cover) There exists a finite subcover $U_{k_1}, \ldots, U_{k_m}$ of K. $K \subset \bigcup_{i=1}^{m} U_{k_i} = U_{k_m}$. Therefore $K^C \supset U_{k_m}^C = \{y : \|y - x\| \leq 1/k_m\} \supset N(x, 1/k_m)$. Thus K^C is open, K is closed.

($\Longleftarrow$)

(1) (Theorem 2.5.2) *Closed box is compact.*
$B = I_1 \times \cdots \times I_d$, $I_i = [a_i, b_i]$. Let $\{U_i : i \in I\}$ is an open cover of B.
(Contradiction) Suppose there is no finite subcover of B.
Claim. There exists $B = B_1 \supset B_2 \supset \cdots$ (closed boxes) such that

- $\operatorname{diam}(B_n) = \frac{1}{2^{n-1}} \operatorname{diam}(B_1)$
- There is no finite subcover of $\{U_i : i \in I\}$ covering B_n.

By Lemma 2.3.1, there exists $x \in \bigcap_{n=1}^{\infty} B_n$. Since $x \in B$, $\exists U_i$ such that $x \in U_i$. Then $\exists \epsilon > 0$ such that $N(x, \epsilon) \subset U_i$.[11] Set $\frac{1}{2^{n-1}} \operatorname{diam}(B_1) < \epsilon$.
If $y \in B_n \implies \|x - y\| \leq \operatorname{diam}(B_n) < \epsilon \implies y \in N(x, \epsilon)$. Then $B_n \subset N(x, \epsilon) \subset U_i$, contradiction.

(2) *K: compact, $F \subset K$, F is closed $\Longrightarrow$ F: compact.*
Let $\{U_i : i \in I\}$ be an open cover of F. Then $\{U_i : i \in I\} \cup \{F^C\}$ is an open cover of K. Because K is compact, there exists a finite subcover of K. There are two cases.

(a) $\{U_{i_1}, U_{i_2}, \ldots, U_{i_m}\}$: This is already a finite subcover of F.

(b) $\{U_{i_1}, U_{i_2}, \ldots, U_{i_m}, F^C\}$: Since F^C does not cover F, U_{i_k} must cover F.

(3) *Closed and bounded set is compact.*
Suppose K is bounded and closed. There exists a closed box B that contains K. Thus B is compact by (1), K is a closed subset of B. Then by (2), K is compact.

Theorem 2.5.2 가 가장 non-trivial 한 부분이다.

[11] n이 충분히 크면 ball 안에 box 가 들어가고 box 는 U_i 안에 있다? Claim 의 2번째에 모순.

Theorem 2.5.4 The following are equivalent.

(1) K is compact.

(2) K is bounded and closed.

(3) If A is an infinite subset of K, $\emptyset \neq A' \subset K$.

(4) For a sequence $\langle x_n \rangle$ in K, there exists a convergent subsequence whose limit is in K.

Proof.
(1) $\iff$ (2) by Heine-Borel Theorem.
(2) $\implies$ (3) Suppose A is infinite and bounded. ($A \subset K$) By Bolzano-Weierstrass, $A' \neq \emptyset$. $A' \subset A' \cup A = \overline{A} \subset K$. ($\overline{A}$ is the smallest closed set containing A, $A \subset K$.)
(3) $\implies$ (4) Let $A = \{x_1, x_2, \dots\}$

(1) If A is finite, trivial. (Take a constant subsequence, which constant $\in K$.)

(2) If A is infinite, $x \in A' \subset K$ by (3). ($x \in A'$ by Thm 2.3.4)

(4) $\implies$ (2)

(1) K is bounded.
(Contradiction) Suppose K is not bounded. Then $\forall n \in \mathbb{N}$, there exists $x_n \in K$, $\|x_n\| \geq n$. There are no convergent subsequences, contradiction.

(2) K is closed.
(Contradiction) Suppose K is not closed.

(a) K: finite $\to$ K: closed $\to$ Contradiction.

(b) K: infinite $\to$ K: infinite and bounded $\overset{\text{B-W}}{\to}$ $K' \neq \emptyset$

Note. $K' \subset K \iff K$: closed.
Then if K' is not a subset of K[12], there exists $x \in K' \backslash K$. Since $x \in K'$, there exists a sequence $\langle x_n \rangle$ in $K \backslash \{x\}$ $(= K)$[13] converging to x. Thus for a subsequence of $\langle x_n \rangle$, its limit must be in K. But x is the only possible limit value. $x \in K$. Contradiction.

[12] Contraposition
[13] $x \notin K$

April 12th, 2019

Problem 2.4.7 (바) $\sum \dfrac{1}{n^p - n^q}$ $(0 < q < p)$
$0 < n^p - n^q \le n^p$ 이므로 $1/n^p \le 1/(n^p - n^q)$ 가 되어 $p \le 1$ 이면 발산한다.
충분히 큰 N에 대하여 $n \ge N$ 일 때마다 $n^p - n^q \ge n^p/2$ 가 되게 할수 있다. (이 때 $n^p/2 \ge n^q$ 이므로 $n^{p-q} \ge 2$ 가 되어 N 을 잡을 수 있다) 비교판정법에 의해 수렴한다.

Problem 2.7.12 Given $\langle a_n \rangle$ such that $\lim a_n = a$, show that $\sigma_n = \dfrac{a_1 + \cdots + a_n}{n}$ also converges to a.

Problem 2.7.13 $r < 1$, $\|x_{n+2} - x_{n+1}\| \le r \|x_{n+1} - x_n\|$. Show that $\langle x_n \rangle$ is a Cauchy sequence.
Proof. $\|x_{n+1} - x_n\| \le r^{n-1} \|x_2 - x_1\| = r^{n-1}A$, for $A \in \mathbb{R}$. Given $\epsilon > 0$, exists N such that for all $n \ge N$, $\|x_{n+1} - x_n\| < Ar^{n-1} < \epsilon$. Then we have

$$\begin{aligned} m > n \ge N \Rightarrow \|x_n - x_m\| &\le \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\| \\ &\le \|x_{n+1} - x_n\| (1 + r + r^2 + \cdots) < \frac{\epsilon}{1-r} \end{aligned}$$

Remark. Counterexample for $\|x_{n+2} - x_{n+1}\| < \|x_{n+1} - x_n\|$. $x_n = \sum_{k=1}^{n} \frac{1}{k}$

Problem 2.7.14 $x_n \to x$, $A_k = \{x_i : i \ge k\}$. Show that $\bigcap_{k=1}^{\infty} \overline{A_k} = \{x\}$.
Proof. Given $\epsilon > 0$, there exists N such that $n \ge N \Rightarrow x_n \in (x - \epsilon, x + \epsilon)$. Either $x_n = x$, or $x_n \in (x - \epsilon, x + \epsilon) \backslash \{x\}$. Thus $x \in \overline{A_k}$ for all k. $\{x\} \subset \bigcap_{k=1}^{\infty} \overline{A_k}$.
For $y \in \mathbb{R} \backslash \{x\}$, we want to show that $y \notin \bigcap_{k=1}^{\infty} \overline{A_k}$. Then we want to find N such that $y \notin \overline{A_N}$. Since $\|x - y\| > 0$, set $\epsilon = \frac{1}{3} \|x - y\|$. There exists N such that $\|x_n - x\| < \epsilon$. Then $\forall x_n \notin N(y, \epsilon)$. $\overline{A_N} = \{x_N, x_{N+1}, \dots\}$, and y cannot be in $\overline{A_N}$. $\{x\}^C \subset \left(\bigcap_{k=1}^{\infty} \overline{A_k}\right)^C \Rightarrow \bigcap_{k=1}^{\infty} \overline{A_k} \subset \{x\}$.

Problem 2.7.15 $\sum a_n$ converges absolutely.

(1) $\sum a_n^2$
Proof. $a_n^2 < |a_n|$ for large n. Converges by comparison test.

(2) $\sum \dfrac{a_n}{1 + a_n}$
Proof. Since $a_n \to 0$, exists N such that $n \ge N \Rightarrow |a_n| < 1/3$. Then for $n \ge N$, $|1 + a_n| \ge 1 - |a_n| > 2/3 > 1/3$, $1/|1 + a_n| < 3$. We have $\left|\frac{a_n}{1+a_n}\right| < 3|a_n|$. Converges by comparison test.

(3) $\sum \dfrac{a_n^2}{1 + a_n^2}$
Proof. Trivial from 1, 2.

April 15th, 2019

K: compact $\iff$ Exists an open cover of K that has *finite* subcover.

Theorem 2.5.4 (Heine-Borel) For $\mathbb{R}^d$, K: compact $\iff$ K is bounded and closed.

Theorem 2.5.5 (Cantor's Intersection Theorem)[14]
Given family of **compact** sets $\{K_i : i \in I\}$, for all **finite** $J \subset I$, $\bigcap_{i\in J} K_i \neq \emptyset$. Then

$$\bigcap_{i\in I} K_i \neq \emptyset$$

Proof. (Contradiction) $\bigcap_{i\in I} K_i = \emptyset \implies \bigcup_{i\in I} K^C = \mathbb{R}^d$. (Complement)
Take any K_a ($a \in I$), then $K_a \subset \bigcup_{i\in I} K_i^C (= \mathbb{R}^d) \implies \{K_i^C : i \in I\}$ is an open cover of K_a. Then there exists a finite subcover, $\{K_i^C : i \in J\}$ (K_a is compact) Now we can write $K_a \subset \bigcup_{i\in J} K_i^C$. Take complement on both sides to get $K_a^C \supset \bigcap_{i\in J} K_i$. Then $K_a \cap \bigcap_{i\in J} K_i = \emptyset$, contradiction.

Remark. Let $K_i = [a_i, b_i]$ (Compact in $\mathbb{R}$) and set $K_1 \supset K_2 \supset \cdots$
$\implies$ For $J = \{j_1, \dots, j_m\}$ ($j_1 < \cdots < j_m$), $\bigcap_{i\in J} K_i = K_{j_m} \neq \emptyset$
$\implies \bigcap_{i=1}^{\infty} K_i \neq \emptyset$ (축소구간정리)

2.6 Connected Set
p46-p47 (Section 2.2)

Definition. $X \subset \mathbb{R}^d$, $x \in X$. Define

$$N_X(x, r) = \{y \in X : \|y - x\| < r\} = N(x, \epsilon) \cap X$$

Definition. $U \subset X$ is open in $X \iff x \in U, \exists\, \epsilon > 0$ such that $N_X(x, \epsilon) \subset U$.

Example.

- $U = \{3\}$. U is open in $X = \mathbb{N}$. $N_{\mathbb{N}}(3, 1/10) = 3 \subset U$. (But not open in $\mathbb{R}$)
- For $X = [0, 10]$, $U = [0, 1)$. $x \in U$, $N(x, 1-x) = (2x-1, 1)$, and this might not be subset of U. But

$$N_X(x, 1-x) = \begin{cases} (2x-1, 1) & (x > 1/2) \\ [0, 1) & (x \leq 1/2) \end{cases}$$

  For both cases $N_X(x, 1-x) \subset U$.

[14] 축소구간정리의 가장 일반적인 형태

Prop 2.2.5 U is open in $X \iff U = X \cap V$ for some open set V in $\mathbb{R}^d$.

Remark. First example: $\{3\} = \mathbb{N} \cap (2.9, 3.1)$, Second example: $[0,1) = [0,10] \cap (-1,1)$. Some references may write this definition as "*relatively*" open in X.

Proof of 2.2.5
$(\implies)$ $x \in U$, $\exists \epsilon_x > 0$ such that $N_X(x, \epsilon_x) \subset U$. Select $V = \bigcup_{x \in U} N(x, \epsilon_x)$, which is open.[15] Then we have $X \cap V = \bigcup_{x \in U} X \cap N(x, \epsilon_x) = \bigcup_{x \in U} N_X(x, \epsilon_x)$, which is exactly equal to U.

$(\impliedby)$ $x \in U = X \cap V \implies x \in V$. Thus $\exists\, \epsilon > 0$ such that $N(x, \epsilon) \subset V$. Then

$$N_X(x, \epsilon) = X \cap N(x, \epsilon) \subset X \cap V = U$$

Thus U is open in X.

Cor. U: open in X, $Y \subset X$. $\implies U \cap Y$: open in Y.
Proof. $U = X \cap V$ (V: open) $\implies U \cap Y = X \cap V \cap Y = V \cap (X \cap Y) = V \cap Y$.

Definition. $S \subset \mathbb{R}^d$: **disconnected** $\iff$ There exists **non-empty** sets U, V such that

(1) $U \cap V = \emptyset$

(2) $U \cup V = S$

(3) U and V are open in S

$S \subset \mathbb{R}^d$: **connected** $\iff$ S is not disconnected.

Question. Find all $A \subset \mathbb{R}^d$ such that A is open and closed.
Proof. The only possible sets are $A = \emptyset, \mathbb{R}^d$.
If A is open and closed $\implies A$: open, A^C: open. Then $\mathbb{R}^d = A \cup A^C$, and $\mathbb{R}^d$ is disconnected. But $\mathbb{R}^d$ is connected. Contradiction if either A or A^C is empty.

Theorem. The following are equivalent for $S \subset \mathbb{R}$.

(1) S is connected.

(2) $\forall a, b \in S$ s.t. $a < b$, and $c \in (a, b) \implies c \in S$.

(3) $S = [a, b]$ or $[a, b)$ or $(a, b]$ or (a, b) (a, b can be $\pm\infty$)

[15] $N(x, \epsilon)$ is open and union of open sets are always open.

Remark. Prop 2.5.1 (1' $\iff$ 2') + Disscussion above (2 $\iff$ 3)

Proof.

(1 $\implies$ 2) (Contradiction) Assume $a, b \in S, c \notin S$ for some $a < c < b$. Set $U = (-\infty, c) \cap S$, $V = (c, \infty) \cap S$. U, V are non-empty.[16] $U \cap V = \emptyset$ and $U \cup V = S$. (Note that $c \notin S$) And U, V are open in S. (Prop 2.2.5) Then S is disconnected.

(2 $\implies$ 1) (Contradiction) Assume S is disconnected. There exists U, V that satisfy the definition of disconnected set. For $a \in U, b \in V$, (WLOG $a < b$). By (2), $[a, b] \subset S$.
Let $c = \sup([a, b] \cap U)$.
Case I) $c \in U$. Then $c \neq b \implies c \in [a, b) = Y \implies c \in U \cap Y$.
Since U is open in S and $Y \subset S \implies U \cap Y$ is open in Y. (Cor of 2.2.5)
$\implies \exists\, \epsilon > 0$ such that $N_Y(c, \epsilon) \subset U \cap Y \subset U \cap [a, b]$.

$$Y \cap N(c, \epsilon) = [a, b) \cap (c - \epsilon, c + \epsilon) \supset [c, b) \cap [c, c + \epsilon) = [c, \min\{b, c + \epsilon\})$$

Therefore, we have

$$[c, \min\{b, c + \epsilon\}) \subset N_Y(c, \epsilon) \subset U \cap [a, b]$$

and since c was the supremum, contradiction.
Case II) $c \in V$. Similarly, contradiction.

(2 $\implies$ 3) $\inf S = u$, $\sup S = v$. (If S is not bounded below, $u = -\infty$, if S is not bounded above, $v = \infty$). Then if $c \in (u, v) \implies c \in S$. There exists $a, b \in S$ such that $u \leq a < c < b \leq v$, meaning that S must be one of $[u, v], [u, v), (u, v], (u, v)$.

(3 $\implies$ 2) Trivial.

[16] Always check! $a \in U, b \in V$.

April 17th, 2019

Definition. $S \subset \mathbb{R}^d$: **disconnected** $\iff$ There exists **non-empty** sets U, V such that

(1) $U \cap V = \emptyset$

(2) $U \cup V = S$

(3) U and V are open in S

Last time we characterized all connected sets of $\mathbb{R}$.

Theorem 2.6.2 Suppose $\{C_i : i \in I\}$ is a family of connected sets.[17]

$$\bigcap_{i \in I} C_i \neq \emptyset \implies \bigcup_{i \in I} C_i \text{ is connected}$$

Proof. (Routine) Assume $C = \bigcup_{i \in I} C_i$ is disconnected. C can be decomposed into 2 sets U, V (that satisfy condition (1), (2), (3) from the definition). Let

$$U_i = C_i \cap U, \quad V_i = C_i \cap V \quad (\forall i)$$

then U_i, V_i are open in C_i.[18] Now U_i, V_i satisfy (2) and (3) for C_i. Since C_i is connected, (1) should not hold, in other words, either U_i or V_i must be $\emptyset$.
Define: $I_1 = \{i \in I : U_i = \emptyset, V_i = C_i\}$, $I_2 = \{i \in I : U_i = C_i, V_i = \emptyset\}$. If $I_1 = \emptyset \implies I_2 = I \implies V_i = \emptyset\ (\forall i) \implies V = \bigcup_{i \in I} V_i = \emptyset$[19], contradiction. Similarly if $I_2 = \emptyset$, contradiction.
Select $i_1 \in I_1, i_2 \in I_2$. Then $C_{i_1} = V_{i_1} \subset V$, $C_{i_2} = U_{i_2} \subset U$. Therefore $C_{i_1} \cap C_{i_2} = \emptyset$. Contradiction.

Example.

(1) $x, y \in \mathbb{R}^d$, $[x, y] = \{tx + (1-t)y : t \in [0, 1]\}$ is connected. (Proof similar to Prop 2.6.1)

(2) $N(x, r) = \bigcup_{y \in N(x,r)} [x, y]$ is connected by the theorem above. ($\bigcap_{y \in N(x,r)} [x, y] = \{x\} \neq \emptyset$)

(3) $\mathbb{R}^d = \bigcup_{y \in \mathbb{R}^d} [0, y]$ is connected.

(4) Convex sets are connected. $A = \bigcup_{y \in A} [x, y]$.

---

[17] 활용 보다도 증명이 중요하니 꼭 기억해 두자.
[18] U: open in X and $Y \subset X \implies U \cap Y$: open in Y.
[19] Check!

Definition. Set A is **convex** $\iff x, y \in A \implies [x, y] \subset A$.

Comment. Homework problem: Show that $S = \{(x, y) : xy > 1\}$ is open.
Proof. 1. Show that $N(z, \epsilon) \subset S$ for all $z \in S$.
2. Instead show that $F = \{(x, y) : xy \leq 1\}$ is closed.
Use Thm 2.2.3 (4). Let (x_n, y_n) be a sequence in F that converges to (x, y).

$$xy = \lim x_n \lim y_n = \lim x_n y_n \leq 1 \implies (x, y) \in F$$

Example. $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, define $A \times B \subset \mathbb{R}^{n+m}$ as

$$A \times B = \{(x, y) : a \in A, b \in B\}$$

If $m = n = 1$, $A \times B$ is a rectangular box in $\mathbb{R}^2$.
If A, B is open/closed/compact/connected, $A \times B$ is open/closed/compact/connected.
Proof.

(1) (Open) $(a, b) \in A \times B$. There exists $\epsilon_1, \epsilon_2 > 0$ such that $N(a, \epsilon_1) \subset A$, $N(b, \epsilon_2) \subset B$. Choose $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. If $(x, y) \in N((a, b), \epsilon) \subset \mathbb{R}^{n+m}$,[20] we have

$$\epsilon^2 > \|(x, y) - (a, b)\|^2 = \|x - a\|^2 + \|y - b\|^2$$

$\|x - a\| < \epsilon < \epsilon_1$ and $\|y - b\| < \epsilon < \epsilon_2$. $x \in A$, $y \in B$.
Therefore $(x, y) \in A \times B$, and $N((a, b), \epsilon) \subset A \times B$.

(2) (Closed) (x_k, y_k): sequence in $A \times B$. $(x_k \in A, y_k \in B)$
Suppose $(x_k, y_k) \to (x, y)$ $(x_k \to x, y_k \to y)$. Since A is closed and x_k is a sequence in A, $x \in A$. Similarly, $y \in B$. Thus $(x, y) \in A \times B$, and $A \times B$ is closed.

(3) (Compact) A, B are closed and bounded. Closed is proven by (2).
Since A, B are bounded, $\exists\, M_1, M_2$ such that $\|a\| \leq M_1$, $\|b\| \leq M_2$ for all $a \in A$, $b \in B$. For all $(a, b) \in A \times B$,

$$\|(a, b)\| = \sqrt{\|a\|^2 + \|b\|^2} \leq \sqrt{M_1^2 + M_2^2}$$

Therefore $A \times B$ is bounded. Thus compact.

(4) (Connected) $a \in A \implies \{a\} \times B$ is connected. $b \in B \implies A \times \{b\}$ is connected.
Proof. If the set is disconnected, exists $\{a\} \times U$, $\{a\} \times V$ such that splits B.
Since $(A \times \{b\}) \cap (\{a\} \times B) = \{(a, b)\} \neq \emptyset$, $(A \times \{b\}) \cup (\{a\} \times B)$ is connected by Thm 2.6.2. Now fix $a \in A$, and define $C_b = (A \times \{b\}) \cup (\{a\} \times B)$.
Then $\{C_b : b \in B\}$ is a family of connected sets, and $\bigcap_{b \in B} = \{a\} \times B \neq \emptyset$. $A \times B = \bigcup_{b \in B} C_b$ is connected by Thm 2.6.2.

[20]Do not write as $\mathbb{R}^{m+n}$. Fist coordinate is n-dimension, second is m-dimension.

April 22nd, 2019

3. Continuous Functions

3.1 함수의 극한과 연속함수의 정의

특별한 언급이 없으면 다음과 같은 가정을 한다.[21]

$$f : X \to Y \quad (X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n)$$

Definition. For $x_0 \in X'$, $\lim\limits_{x \to x_0} f(x) = y_0 \iff$

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ s.t. } \big(\mathbf{0} < \|x - x_0\| < \delta \Rightarrow \|f(x) - y_0\| < \epsilon\big)$$

Remark. Why X'? $X = [0, 1] \cup \{2\}$, consider $f(x) = 2x$ on X. $\lim\limits_{x \to 2} f(x)$ is nonsense.

Example.

(1) $f(x) = \begin{cases} x^2 & (x \neq 0) \\ 1 & (x = 0) \end{cases}$, $\lim\limits_{x \to 0} f(x) = 0.$[22]
For $\epsilon > 0$, set $\delta = \sqrt{\epsilon}$. Then $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| = |x^2| < \delta^2 = \epsilon$.

(2) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$. ($X = \mathbb{R} \backslash \{2\}, Y = \mathbb{R}$, $2 \in X'$)
For $\epsilon > 0$, set $\delta = \epsilon$. Then $0 < |x - 2| < \delta \Rightarrow |f(x) - 4| = |x - 2| < \delta = \epsilon$.

Prop 3.1.1 $f, g : X \to Y$, $x_0 \in X'$[23]. If $\lim\limits_{x \to x_0} f(x) = y_0$, $\lim\limits_{x \to x_0} g(x) = z_0$, then

(1) $\lim\limits_{x \to x_0} af(x) + bg(x) = ay_0 + bz_0$

(2) $\lim\limits_{x \to x_0} f(x)g(x) = y_0 z_0$

(3) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{y_0}{z_0}$ $(z_0 \neq 0)$

연속을 3가지로 정의한다. 세 정의들이 서로 **동치**임을 이해하는 것이 중요하다.

Definition. Let $f : X \to Y$, $x_0 \in X$. f is **continuous** at $x_0 \iff$

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ s.t. } \big(\, \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon\big)$$

Remark. $|x - x_0| < \delta$ should be satisfied for $x \in X$. The $0 <$ condition is omitted here since the inequality holds trivially for x_0.

---

[21] 치역이 중요하지 공역은 뭐...

[22] 특별한 언급이 없으면 $X = f$ 가 정의되는 곳, $Y = \mathbb{R}^n$ 으로 생각한다.

[23] 책에 X로 되어있는데 이는 오타.

(1) $x_0 \in X'$: f is continuous at $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$.

(2) $x_0 \in X \backslash X'$ (isolated point): f is continuous at x_0.

Definition.

(1) $A \subset X, f : X \to Y$. If f is continuous at x_0 for all $x \in A \implies f$ is continuous on A.

(2) If f is continuous on $X \implies f$ is continuous.

Prop 3.1.3 The following are equivalent for $f : X \to Y$.

(1) f: continuous at $x_0 \in X$.

(2) If there exists a sequence $\langle x_n \rangle$ in X converging to $x_0 \implies \lim_{n\to\infty} f(x_n) = f(x_0)$.

Proof.
(1 $\implies$ 2) Given $\epsilon > 0$,

(i) $\exists\, \delta > 0$ s.t. $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$

(ii) Since $x_n \to x_0$, $\exists\, N$ s.t. for $n \geq N \implies \|x_n - x_0\| < \delta$.

Therefore, $n \geq N \implies \|x_n - x_0\| < \delta \implies \|f(x_n) - f(x_0)\| < \epsilon$.

(2 $\implies$ 1) (Contradiction) Suppose there exists $\epsilon_0 > 0$ such that no δ statisfies $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon_0$. (i.e. For all $\delta > 0$, $\exists\, x \in X$ s.t. $\|x - x_0\| < \delta$ and $\|f(x) - f(x_0)\| \geq \epsilon_0$)

Thus for all $n \in \mathbb{N}$, there exists $x_n \in X$ s.t. $\|x_n - x_0\| < 1/n$ and $\|f(x_n) - f(x_0)\| \geq \epsilon_0$. ($\delta = 1/n$) Then we have $\lim_{n\to\infty} x_n = x_0$, but $\lim_{n\to\infty} f(x_n) \neq f(x_0)$. Contradiction.

Definition. $f : X \to Y$, $A \subset X$, $B \subset Y$. Define

$$f(A) = \{f(x) : x \in A\} \quad f^{-1}(B) = \{x \in X : f(x) \in B\}$$

Remark.

(1) $A \subseteq f^{-1}(f(A))$
$x \in A$ and let $y = f(x)$. Then $y \in f(A)$, thus $x \in f^{-1}(f(A))$.

(2) $f(f^{-1}(B)) \subseteq B$
$y \in f(f^{-1}(B))$ then $y = f(x)$ for some $x \in f^{-1}(B)$. Thus we have $x \in f^{-1}(B) \iff f(x) \in B$. $\therefore y = f(x) \in B$.

Also remember the counterexamples where the equality does not hold. (1) doesn't hold if f is not injective, (2) doesn't hold if f is not surjective.

Theorem 3.1.5 The following are equivalent for $f : X \to Y$.

(1) f is continuous on X.

(2) B: open set in $Y \implies f^{-1}(B)$: open in X.

(3) B: closed in $Y \implies f^{-1}(B)$: closed in X.

Proof. **(2 $\iff$ 3)** Trivial. Check $f^{-1}(B^C)$.
(1 $\implies$ 2) Observation. f is continuous at $x_0 \iff \forall\epsilon > 0$, $\delta > 0$ s.t. $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$. Re-write the last two inequality as $x \in N_X(x, \delta)$ and $f(x) \in N_Y(f(x_0), \epsilon)$. Then continuity condition is equivalent to

$$\forall\epsilon > 0, \exists\,\delta > 0 \text{ s.t. } f(N_X(x, \delta)) \subset N_Y(f(x_0), \epsilon)$$

Now suppose $x_0 \in f^{-1}(B) \iff f(x_0) \in B$. Since B is open, there exists $\epsilon > 0$ s.t. $N_Y(f(x_0), \epsilon) \subset B$. Then there exists $\delta > 0$ s.t. $f(N_X(x_0, \delta)) \subset N_Y(f(x_0), \epsilon) \subset B$. Take f^{-1} on both sides. $N_X(x_0, \delta) \subset f^{-1}(f(N_X(x_0, \delta))) \subset f^{-1}(B)$. Thus $f^{-1}(B)$ is open in X.

(2 $\implies$ 1) $x_0 \in X$, $f(x_0) \in Y$. Given $\epsilon > 0$, $N_Y(f(x_0), \epsilon)$ is open in Y. By (2), $f^{-1}(N_Y(f(x_0), \epsilon))$ is open in X. Observe that this set always contains x_0. Then $\exists\,\delta$ s.t. $N_X(x_0, \delta) \subset f^{-1}(N_Y(f(x_0), \epsilon))$. Now take f on both sides. $f(N_X(x_0, \delta)) \subset f(f^{-1}(N_Y(f(x_0), \epsilon))) \subset N_Y(f(x_0), \epsilon)$. Thus f is continuous at x_0.

April 24th, 2019

연속함수의 기본적 성질

Prop 3.1.2 Suppose $f, g : X \to \mathbb{R}^n$ are continuous on X.

(1) $af + bg$: continuous

(2) $(n = 1)$ fg: continuous

(3) $\dfrac{f}{g}$: continuous ($g \neq 0$ on X)

Proof. (2) Given $\epsilon > 0$, $\exists\, \delta_1$ s.t. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\epsilon}{2|g(x_0)|+1}$, $\exists\, \delta_2$ s.t. $|x - x_0| < \delta_2 \implies |g(x) - g(x_0)| < \dfrac{\epsilon}{2(|f(x_0)| + \frac{\epsilon}{2|g(x_0)|+1})}$. Then we have

$$\begin{aligned} |f(x)g(x) - f(x_0)g(x_0)| &= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\ &\leq |f(x)|\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Thus we have continuity.

Proof 2. By sequential definition, exists $\langle x_n \rangle \to x_0$ in X such that $f(x_n) \to f(x_0), g(x_n) \to g(x_0)$. Then we have $f(x_n)g(x_n) \to f(x_0)g(x_0)$.

Prop 3.1.4 Suppose we have two continuous functions $f : X \to Y$, $g : Y \to Z$. If f is continuous at $x_0 \in X$, and if g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof. Given $\epsilon > 0$, $\exists\, \delta_1 > 0$ s.t. $\|y - f(x_0)\| < \delta_1 \implies \|g(y) - g(f(x_0))\| < \epsilon$. Also, $\exists\, \delta_2 > 0$ s.t. $\|x - x_0\| < \delta_2 \implies \|f(x) - f(x_0)\| < \delta_1$. Now we automatically have $\|g(f(x)) - g(f(x_0))\| = \|(g \circ f)(x) - (g \circ f)(x_0)\| < \epsilon$.

Remark. Suppose f: continuous X, g: continuous on Y (or on $f(X)$). Then $g \circ f$ is continuous on X.

Example.

(1) Polynomials are continuous. Use continuity of $f(x) = x$.

(2) $f(x) = \sqrt{x}$.[24]

(3) $f(x) = \sqrt{x^4 + 1}$ is continuous.

(4) $f(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \backslash \mathbb{Q}) \end{cases}$ is not continuous.

Proof. $x_0 \in \mathbb{R}$. Suppose there exists a sequence $\langle x_n \rangle$ in $\mathbb{Q}$ converging to x_0. Then $\langle f(x_n) \rangle \to 1$. ($x_n = \lfloor n x_0 \rfloor / n$) But there also exists a sequence $\langle x_n \rangle$ in $\mathbb{R} \backslash \mathbb{Q}$ converging to x_0. Then $\langle f(x_n) \rangle \to 0$. ($x_n = \lfloor \sqrt{2} n x_0 \rfloor / \sqrt{2} n$) $f(x)$ cannot be continuous anywhere.

[24] 연속이지만 고른연속은 아닌 함수

3.2 최대최소정리와 중간값정리

Theorem 3.2.1 Suppose $f : X \to Y$ is surjective and X is compact. Then Y is compact.[25]

Proof. Suppose $\{U_i : i \in I\}$ is an open cover of Y. $V_i = U_i \cap Y$ is an open set in Y, and $\{V_i : i \in I\}$ is also an open cover of Y. Consider $\{f^{-1}(V_i) : i \in I\}$, which is an open cover of X.[26] Since X is compact, there exists a finite subcover $\{f^{-1}(V_i) : i \in J\}$ $(J \subset I)$ of X. Then $\{V_i : i \in J\}$ is a finite subcover of Y.

$$Y = f(X) = f\left(\bigcup_{i \in J} f^{-1}(V_i)\right) = \bigcup_{i \in J} f(f^{-1}(V_i)) \subset \bigcup_{i \in J} V_i$$

We have a finite subcover of Y. Thus Y is compact.

Check. $\forall A \subset X$. f: surjective $\implies$, $f(f^{-1}(A)) = A$. f: injective $\implies$ $f^{-1}(f(A)) = A$.

Remark.

(1) $f : \mathbb{R}^m \to \mathbb{R}^n$, f: continuous. If $K \subset \mathbb{R}^m$ is compact, $f(K)$ is compact.
Set $f : K \to f(K)$.

(2) Image of compact set is compact.

Cor 3.2.2 Suppose X is compact. $f : X \to \mathbb{R} \implies f$ has maximum and minimum.

Proof. Set $f : X \to f(X)$, then f is surjective and $f(X)$ is compact. Check that if $K \subset \mathbb{R}$, K: compact, then $\inf K, \sup K \in K$ and $\inf K = \min K$, $\sup K = \max K$.

Cor 3.2.4 (Extreme Value Theorem) If f is a continuous function defined on $[a, b]$, f has a maximum and minimum.

Proof. $[a, b]$ is compact.

Cor 3.2.3 Suppose X is compact and $f : X \to \mathbb{R}$ is continuous. If $f(x) > 0$ for all $x \in X$, then $\exists\, \delta > 0$ s.t. $f(x) \geq \delta > 0$ for all $x \in X$.

Proof. Let $\delta = \min f(X) = f(u) > 0$ for some u.

Remark. $X = [1, \infty)$, $f(x) = 1/x$. (X is not compact.)

Cor 3.2.5 Suppose X is compact and $f : X \to Y$ is bijective and continuous. Then f^{-1} is continuous.

Check. $f : X \to Y$. $A \subset X, B \subset Y$. Image: $f(A)$, pre-image: $f^{-1}(B)$. We must check if image of B on f^{-1} is equal to the pre-image of B. (Well-definedness!)

[25] 연속성이 필요없나?

[26] Check at assignment 3.5.

April 26th, 2019

Assignment 3.5 #3: Check and remember.

(2) $f\left(\bigcap_{i\in\mathcal{I}} A_i\right) \subset \bigcap_{i\in\mathcal{I}} f(A_i)$

Problem 3.1.2 $f: X \to \mathbb{R}^n$, $f(x) = \big(f_1(x), \ldots, f_n(x)\big)$ $(x \in X)$. The following are equivalent.

(1) f is continuous at x.

(2) For all i, $f_i : X \to \mathbb{R}$ is continuous at x.

Proof. **(1 $\implies$ 2)** $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|y - x\| < \delta \implies \|f(y) - f(x)\| < \epsilon$. Then we have $\|f_i(y) - f_i(x)\| \le \|f(y) - f(x)\| < \epsilon$, for any i.
(2 $\implies$ 1) $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|x - y\| < \delta \implies \|f_i(x) - f_i(y))\| < \epsilon/\sqrt{n}$. Then

$$\|x - y\| < \delta \implies \|f(x) - f(y)\| = \sqrt{\sum_{i=1}^{n} \|f_i(x) - f_i(y)\|^2} < \sqrt{n \cdot \frac{\epsilon^2}{n}} = \epsilon$$

Prop 3.1.2 (3) f, g: continuous $\implies$ f/g: continuous ($g \neq 0$ on X)
Proof. $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t. for all $x_0 \in X$,
$|x - x_0| < \delta \implies |g(x) - g(x_0)| < \min\{\frac{1}{2}|g(x_0)|, \frac{1}{4}\frac{|g(x_0)|^2\epsilon}{|f(x_0)|+1}\}$, $|f(x) - f(x_0)| < \frac{1}{4}|g(x_0)|\,\epsilon$.

$$\begin{aligned}\left|\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}\right| &\le \frac{|g(x_0)|\,|f(x) - f(x_0)| + |f(x_0)|\,|g(x) - g(x_0)|}{|g(x)|\,|g(x_0)|} \\ &\le \frac{|g(x_0)|\,\frac{1}{4}|g(x_0)|\,\epsilon + |f(x_0)|\,\frac{1}{4}\frac{|g(x_0)|^2\epsilon}{|f(x_0)|+1}}{\frac{1}{2}|g(x_0)|^2} < \frac{\frac{1}{4}|g(x_0)|^2\epsilon + \frac{1}{4}|g(x_0)|^2\epsilon}{\frac{1}{2}|g(x_0)|^2} = \epsilon\end{aligned}$$

Example. $g(x) = \begin{cases} 0 & (x = 0, 1 \text{ or } x \in \mathbb{R}\backslash\mathbb{Q}) \\ 1/q & (x = p/q, \text{irreducible fraction}) \end{cases}$

(i) $x_0 \in \mathbb{Q} \cap (0, 1)$ then $g(x_0) > 0$. Set $\epsilon = \frac{1}{2}g(x_0) > 0$. For all $\delta > 0$, $\exists\, y \in \mathbb{Q}^C \cap [0, 1]$ s.t. $|y - x_0| < \delta$, but $|g(y) - g(x_0)| = g(x_0) \ge \epsilon$. Thus f is not continuous at x_0.

(ii) $x_0 \in \mathbb{Q}^C \cup \{0, 1\}$. $g(x_0) = 0$. $\forall \epsilon > 0$, $\exists\, N \ge 1$ s.t. $1/N < \epsilon$. Then there are finitely many y such that $g(y) \ge 1/N$. ($\frac{1}{N}, \frac{1}{N-1}, \ldots, \frac{1}{2}$ is finite) Let them be $y_1, \ldots, y_k$ and set $\delta = \min_{1\le i\le k} |y_i - x_0| > 0$. If $\|y - x_0\| < \delta$, then $0 \le g(y) < 1/N < \epsilon$. $|g(y) - g(x_0)| = g(y) < \epsilon$.

Problem 3.5.1

(1) $f(x) = 0$, $f(\mathbb{R}) = \{0\}$ (closed)

(3) $f(x) = e^x$, $f(\mathbb{R}) = (0, \infty)$ (open)

April 29th, 2019

3.2 EVT & IVT

Theorem 3.2.1 Suppose $f : X \to Y$ is continuous and surjective.[27] If X is compact, Y is also compact.

Remark. $f : X \to Y$ continuous, $K \subset X$: compact $\implies$ $f(K)$: compact. Inverse does not hold. Consider $f(x) = \sin x$. Image is $[0, 1]$ (compact), but pre-image is $\mathbb{R}$ (not bounded).

Definition. Function $f : X \to \mathbb{R}$ has **maximum** M if there exists $u \in X$ s.t. $f(u) = M$, and $\forall x \in X$, $f(x) \le M$.

Cor 3.2.5 Suppose $f : X \to Y$ is continuous and bijective. If X is compact, $f^{-1} : Y \to X$ is continuous.[28]

Proof. Let $f^{-1} = g : Y \to X$. For any open set U in X, it is enough to show that $g^{-1}(U)$ is open in Y. But $g^{-1}(U) = (f^{-1})^{-1}(U) = f(U)$. Check that $Y\backslash f(U) = f(X\backslash U)$. Since a closed subset of a compact set is compact, $Y\backslash f(U) = f(X\backslash U)$ is compact, and hence closed in $\mathbb{R}^d$. Then $f(U) = (Y\backslash f(U))^C \cap Y$ is open in Y.

Example. $f : X = \{0\} \cup (1, 2) \to Y = [0, 1)$. $f(0) = 0$, $f(x) = x - 1$ on $(1, 2)$. By definition, f is continuous on X. Consider f^{-1}. $f^{-1}(0) = 0$, $f^{-1}(x) = x + 1$ on $(0, 1)$. f^{-1} is not continuous.[29]

Application. (Distance between sets) Define dist as follows.

$$A, B \subset \mathbb{R}^d, \quad \mathrm{dist}(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}$$

Example. $A = \{(x, y) : x \le 0\}$, $B = \{(x, y) : xy \ge 1, x, y > 0\}$. $\mathrm{dist}(A, B) \le \left\|(0, n) - (\frac{1}{n}, n)\right\| = 1/n$ for all n. Thus $\mathrm{dist}(A, B) = 0$.

Theorem. A : compact, B : closed. $A \cap B = \emptyset \implies \mathrm{dist}(A, B) > 0$.

Proof. $f : A \to \mathbb{R}$, $f(x) = \mathrm{dist}(\{x\}, B)$ $(x \in A)$.

(i) $f(x) > 0$ for all $x \in A$.
$\because N(x, \epsilon) \subset B^C$ (open) $\implies$ $\mathrm{dist}(\{x\}, B) \ge \epsilon > 0$.

(ii) f: continuous, $b \in B$. For $x, y \in A$, $\|x - b\| \le \|x - y\| + \|y - b\|$. Take infimum over $b \in B$. Then we have $f(x) \le \|x - y\| + f(y)$. Similarly we have $f(y) \le \|x - y\| + f(x)$. Hence $\|f(x) - f(y)\| \le \|x - y\|$. (Continuity follows easily by setting $\delta = \epsilon$)

---

[27] Not necessarily. Adjust Y to be $f(X)$.

[28] Thm 3.1.5 was about the pre-image of an open set. In this corollary, we must show that the image of an open set is also open.

[29] 수학적으로 장난질 치는게 아니라 본질적인 의미가 있는 예시입니다.

Lipschitz Continuous: $\|f(x)-f(y)\| \leq k\|x-y\|$ for some $k \geq 0$ (Set $\delta = \epsilon/k$ to show continuity)
Contraction: Lipschitz continuous and $k = 1$.

By Cor 3.2.3, $\exists\, \delta > 0$ s.t. $f(x) \geq \delta > 0$ for all $x \in A$. Then $\text{dist}(A, B) \geq \delta > 0$.

Theorem 3.2.8 Suppose $f : X \to Y$ is continuous and surjective. If X is connected, Y is also connected.
Proof.[30] (Contradiction) Assume Y is disconnected. Then there exists non-empty sets U, V that are open in Y, and $U \cap V = \emptyset$, $U \cup V = Y$. Consider $f^{-1}(U), f^{-1}(V)$. We will show that X is disconnected. Since f is surjective, $f^{-1}(U), f^{-1}(V)$ are non-empty. Decomposition conditions can be checked easily, (use theorems from assignment) and openness holds by continuity.

Remark. Suppose $f : X \to Y$ is continuous. If $C \subset X$ is connected, $f(C)$ is also connected.

Cor 3.2.9 Suppose $f : I \to \mathbb{R}$ is continuous where I is any interval of $\mathbb{R}$. Then $f(I)$ is also an interval and hence connected.[31]

Cor 3.2.10 (Intermediate Value Theorem) Suppose $f : [a, b] \to \mathbb{R}$ is continuous. If α is in between $f(a)$ and $f(b)$,[32] then $\exists\, c \in [a, b]$ s.t. $f(c) = \alpha$.[33]
Proof. $f([a, b])$ is an **interval** (Cor 3.2.9) which includes $f(a), f(b)$. Then it must include α.[34]

Cor 3.2.11 Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then $f([a, b])$ is a closed interval.
Proof. $f([a, b])$ is an interval (Cor 3.2.9) and compact (Thm 3.2.1).

Cor 3.2.12 Suppose $f : [a, b] \to [a, b]$ is continuous. Then $\exists\, c \in [a, b]$ s.t. $f(c) = c$. We call such c a fixed point.
Proof. Apply IVT on $g(x) = x - f(x)$, set $\alpha = 0$. Then we have

$$g(a) = a - f(a) \leq 0 = \alpha = 0 \leq b - f(b) = g(b)$$

and the result follows directly.

Application. (**Path-Connected Set**)

Remark. $x, y \in \mathbb{R}^d \implies [x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$ (convex combination)

---

[30] 책과 약간 다릅니다. 책의 증명도 읽어보세요.
[31] 이런 집합을 구간으로만 이해를 하면 우리가 아무것도 못 해요. 그런데 얘를 연결집합으로 이해하면 뭔가 할 것들이 생기고 여기서 중간값 정리가 바로 나오죠.
[32] $(f(a) - \alpha)(f(b) - \alpha) < 0$
[33] 이 정리를 위해 달려온 것...
[34] 구간은 볼록집합임을 이용해도 α 를 포함함을 보일 수 있다.

Set $f : [0,1] \to [x,y]$ as $f(t) = tx + (1-t)y$. Then f is continuous. (Lipschitz continuity can be easily checked and f is surjective)

Definition. Let $a, b \in \mathbb{R}$, $a < b$. Suppose $f : [a,b] \to \mathbb{R}^d$ is continuous. Then $f([a,b])$ is called a **path**.

Remark. Define $f : [a,b] \to \mathbb{R}^3$ as $f(t) = (\sin t, \cos t, \frac{1}{1+t^2})$ (Parameterized curve)
Also note that a path is compact and connected. ($[a,b]$ is compact and connected)

Definition. $C \subset \mathbb{R}^d$ is called **path-connected** if for any $x, y \in C$, there exists a path **in** C connecting x and y.

Theorem. Path-connected $\implies$ Connected
Proof. (Contradiction) Assume X is path-connected but disconnected. Then there exists sets U, V such that satisfy disconnectedness for X. Let $x \in U$, $y \in V$. From path-connected condition, there exists $f : [a,b] \to X$ s.t. f is continuous, $f(a) = x$, and $f(b) = y$. Let $Y = f([a,b]) \subset X$. Then Y can be decomposed into $Y \cap U$ and $Y \cap V$. These two sets satisfy the disconnectedness condition, (check) hence Y is disconnected. But since paths are always connected, contradiction.

Remark. The converse of the above theorem is **false**. Consider $f(x) = \sin \frac{1}{x}$ $(x > 0)$. Set $A = \{(x, \sin \frac{1}{x}) : x \in (0,1)\} \subset \mathbb{R}^2$. A is a path and therefore connected.
But the problem arises when we consider $\overline{A}$. We can easily check that the closure of a connected set is connected. We can also check that $\overline{A} = A \cup \{(0,t) : t \in [-1,1]\}$, which is not path-connected.[35]

[35] We need a jump from $x = 0$ to $x > 0$...

May 1st, 2019

3.3 Uniform Continuity

Definition. $f : X \to Y$ is **uniformly continuous** $\iff$ $\forall \epsilon > 0, \exists \delta > 0$ s.t. $x, y \in X$, $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$.

Remark. "$f : X \to Y$ is continuous at $x_0 \in X$" meant that $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$. In this definition, δ was a function of x_0. But in the definition of uniform continuity, δ is only dependent of ϵ.

Example.

(1) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ (Not uniformly continuous)
For $\epsilon = 1$, suppose we have $\delta > 0$. Set $x = 1/\delta + \delta/2$, $y = 1/\delta$. Then $|x - y| = \delta/2 < \delta$, but $|f(x) - f(y)| = |x^2 - y^2| = 1 + \delta^2/4 > \epsilon$.

(2) $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$ (Uniformly continuous & Lipschitz continuous)[36]
Given $\epsilon > 0$, $\delta = \epsilon/2$. If $|x - y| < \delta$ then $|f(x) - f(y)| = |x + y|\,|x - y| < 2\delta = \epsilon$.

(3) Lipschitz Continuity $\implies$ Uniform Continuity
Suppose $\forall x, y \in X$, $\exists k > 0$ s.t. $\|f(x) - f(y)\| \leq k\,\|x - y\|$. Then set $\delta = \epsilon/k$ to show uniform continuity.

(4) **Lipschitz** $\implies$ **Uniform** $\implies$ **Continuous**
$f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$.

(a) Not Lipschitz continuous.
$|f(x) - f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq k\,|x - y|$ for all $x, y \in X$? Impossible.

(b) Uniform continuous.
Set $\delta = \epsilon^2$. $|f(x) - f(y)| = \left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|} < \sqrt{\delta} = \epsilon$

Theorem 3.3.1 (Heine's Theorem) Suppose $f : X \to Y$ is continuous. If X is compact, f is uniformly continuous.

Proof. Given $\epsilon > 0$, $x \in X$, $\exists \delta(x) > 0$ s.t. $\|y - x\| < \delta(x) \implies \|f(y) - f(x)\| < \epsilon/2$.
Define $U_x = N(x, \delta(x)/2)$. Then $\{U_x : x \in X\}$ is a open cover of X. By compactness, there exists a finite subcover $\{U_{x_i}\}_{i=1}^n$. Set $\delta = \frac{1}{2}\min\{\delta(x_1), \ldots, \delta(x_n)\}$.
Suppose $\|x - y\| < \delta$. For some k, $x \in U_{x_k}$, and then $y \in N(x_k, \delta(x_k))$. This is because

$$\|x - x_k\| < \delta(x_k)/2, \quad \|y - x_k\| \leq \|y - x\| + \|x - x_k\| < \delta + \delta(x_k)/2 < \delta(x_k)$$

[36] 함수의 성질일 뿐만 아니라 domain 의 성질이기도 하다? Domain 도 중요한 역할을 한다.

Then we have

$$\|f(x) - f(y)\| \le \|f(x) - f(x_k)\| + \|f(x_k) - f(y)\| < \epsilon/2 + \epsilon/2 = \epsilon$$

by continuity of f. Thus f is uniformly continuous.

Theorem 3.3.2 Suppose $f : X \to Y$ is uniformly continuous. If $\langle x_n \rangle$ is a Cauchy sequence in X, $\langle f(x_n) \rangle$ is also a Cauchy sequence.
Proof. Given $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$. For this δ, $\exists\, N$ s.t. $m, n \ge N \implies \|x_m - x_n\| < \delta$. Then we have

$$m, n \ge N \implies \|x_m - x_n\| < \delta \implies \|f(x_m) - f(x_n)\| < \epsilon$$

Remark. If $f : X \to Y$ is continuous, $\langle x_n \rangle \to x$ then $\langle f(x_n) \to f(x) \rangle$. In this case, $\langle x_n \rangle, x$ must be in X, $\langle f(x_n) \rangle, f(x)$ must be in Y.
Consider $f : (0, 1) \to \mathbb{R}$, $f(x) = 1/x$. $x_n = 1/n$ converges, and is a Cauchy sequence. But $f(x_n) = n$ is not Cauchy. The limit value of $\langle x_n \rangle$ does not have to be in X for a uniform continuous function.

Definition. Suppose $f : X \to Y$ is continuous, $X \subset A, Y \subset B$. If $g : A \to B$ satisfies $g(x) = f(x)$ for $x \in X$, and if g is continuous on A, we say that g is a **continuous extension** of f to A.

Example.

(1) $f : (0, 1) \to \mathbb{R}$, $f(x) = x$.
Consider $A = (0, 2)$. $g(x) = x$ on $(0, 2)$ is a continuous extension, $h(x) = x$ on $(0, 1)$, $h(x) = 1$ on $[1, 2)$ is also a continuous extension.
Consider $A = [0, 1]$. Then $g(0) = 0, g(1) = 1$, $g(x) = x$ on $(0, 1)$ is a unique continuous extension of f.

(2) $f : (0, 1) \to \mathbb{R}$, $f(x) = 1/x$.
Consider $A = [0, 1)$. It is impossible to find a continuous extension.

Cor 3.3.3 Suppose $f : X \to Y$ is uniformly continuous. Then there exists a unique continuous extension of f to $\overline{A}$.[37]
Proof. Take $x_0 \in \overline{X} \backslash X$. Set $g(x) = f(x)$ for $x \in X$. Now for $g(x_0)$, recall that $x_0 \in \overline{X}$, so there exists a sequence $\langle x_n \rangle$ in X s.t. $x_n \to x_0$. Since $\langle x_n \rangle$ is convergent, $\langle x_n \rangle$ is Cauchy sequence and by Thm 3.3.2, $\langle f(x_n) \rangle$ is also a Cauchy sequence. Thus $\langle f(x_n) \rangle$ converges. Define $g(x_0)$ as the limit of $f(x_n)$.

[37] Y is assumed to be extended to $\mathbb{R}^d$.

Now we must check if $g(x_0)$ is well-defined. In other words: For any two sequence $\langle x_n \rangle, \langle y_n \rangle$ that converge to x_0, does $f(x_n), f(y_n)$ converge to the same value?
Consider $\langle z_n \rangle = x_1, y_1, x_2, y_2, \ldots$. It is trivial that $z_n \to x_0$. Since $\langle z_n \rangle$ is Cauchy, $\langle f(z_n) \rangle$ is also Cauchy by uniform continuity. Let its limit be γ. Then $\langle f(x_n) \rangle, \langle f(y_n) \rangle$ is a subsequence of $\langle f(z_n) \rangle$, thus they both must converge to γ. Uniqueness directly follows from this proof, and we can easily check that g is continuous.

May 3rd, 2019