해석개론 및 연습 1 과제 #2

2017-18570 컴퓨터공학부 이성찬

1. (1) Given $\forall M > 0$, set $N = \dfrac{M+3}{2}$. Then for all $n > N$, $2n - 3 > M$. Thus $-2n + 3 < -M$. ($-2n+3$ can be made an arbitrarily small negative number)
$\therefore \lim_{n\to\infty}(-2n+3) = -\infty$

(2) Given $\forall M > 0$, set $N = \dfrac{\tan^{-1} M}{\pi - 2\tan^{-1} M} > 0$.[1] Then for all $n > N$, $n > \dfrac{\tan^{-1} M}{\pi - 2\tan^{-1} M}$. Simplify with respect to M which leads to $n\pi > (2n+1)\tan^{-1} M$.

$$\frac{\pi}{2} > \frac{n\pi}{2n+1} > \tan^{-1} M$$

Since $\tan x$ is an increasing function on $(0, \pi/2)$,

$$M < \tan\frac{n\pi}{2n+1} \quad \text{for all } M > 0$$

$\therefore \lim_{n\to\infty} \tan\dfrac{n\pi}{2n+1} = \infty$

2. Since every convergent sequence is bounded, there exists $A \in \mathbb{R}$ such that $|a_n| < A$ for all $n \in \mathbb{N}$. $\therefore -A < a_n < A$.

(1) $\lim_{n\to\infty} b_n = \infty \implies \forall M > 0, \exists N \in \mathbb{N}$ such that $(n \geq N \Rightarrow b_n > M)$.
$\implies \forall M > A$, $\exists N \in \mathbb{N}$ such that $(n \geq N \Rightarrow b_n > M)$.
$\implies \forall M' = M - A > 0$, $\exists N \in \mathbb{N}$ such that $(n \geq N \Rightarrow a_n + b_n > M - A = M' > 0)$.
$\implies \lim_{n\to\infty}(a_n + b_n) = \infty$.

(2) (i) Suppose $a > 0$.
For any $0 < \epsilon < a$, there exists $N_1 \in \mathbb{N}$ such that $(n > N_1 \Rightarrow |a_n - a| < \epsilon)$.
$\therefore 0 < a - \epsilon < a_n < a + \epsilon$.
Also for any $M > 0$, there exists $N_2 \in \mathbb{N}$ such that $(n > N_2 \Rightarrow b_n > M)$. Take $N = \max\{N_1, N_2\}$. Then if $n > N$, we have $a_n b_n > M(a - \epsilon) > 0$, and $M(a-\epsilon)$ can be chosen arbitrarily large. $\therefore \lim_{n\to\infty} a_n b_n = +\infty = a \cdot \infty = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$. The desired result holds for this case.

(ii) Suppose $a < 0$.
For any $0 < \epsilon < -a$, there exists $N_1 \in \mathbb{N}$ such that $(n > N_1 \Rightarrow |a_n - a| < \epsilon)$.
$\therefore a - \epsilon < a_n < a + \epsilon < 0$.
Also for any $M > 0$, there exists $N_2 \in \mathbb{N}$ such that $(n > N_2 \Rightarrow b_n > M)$. Take $N = \max\{N_1, N_2\}$. Then if $n > N$, we have $a_n b_n < M a_n < M(a+\epsilon) < 0$, and $M(a+\epsilon)$ can be any negative real. $\therefore \lim_{n\to\infty} a_n b_n = -\infty = a \cdot \infty = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$. The desired result also holds for this case.

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[1] $0 < \tan^{-1} x < \pi/2$ for $x > 0$.

(3) Set $a_n = (-1)^n \dfrac{1}{n}$, and $b_n = n$. We see that $\lim_{n\to\infty} a_n = 0$ from the following inequality and taking limits on all sides. ($1/n$ can be made arbitrarily close to 0)

$$-\frac{1}{n} \le a_n \le \frac{1}{n} \implies 0 \le \lim_{n\to\infty} a_n \le 0$$

$\lim_{n\to\infty} b_n = \infty$ since n can be made arbitrarily large. But $a_n b_n = (-1)^n$, and we have shown in class that $(-1)^n$ diverges and oscillates between ± 1.

3. (1) If a sequence $\langle a_n \rangle$ is not bounded below, we define $\liminf_{n\to\infty} a_n = -\infty$. Otherwise, let us define $z_n = \inf\{a_k : k \ge n\}$ for each $n \in \mathbb{N}$. If the increasing sequence $\langle z_n \rangle$ is not bounded above, we define $\liminf_{n\to\infty} a_n = \infty$. If $\langle z_n \rangle$ is bounded above, $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} z_n$.

(2) ($\Longrightarrow$) Since $\lim_{n\to\infty} a_n = \infty$, $\forall M > 0$, $\exists N \in \mathbb{N}$ such that ($a_n > M$ for all $n > N$) $\cdots (*)$

i. *$\langle a_n \rangle$ is not bounded above.*

Proof. Suppose a_n is bounded above. Then there exists $M_* > 0$ such that $a_n < M_*$ for all n. This contradicts $(*)$. Thus $\limsup_{n\to\infty} a_n = \infty$.

ii. *$\langle z_n \rangle$ is an increasing sequence.*

Proof. $z_n = \inf\{a_k : k \ge n\} = \inf\{a_n, \inf\{a_k : k \ge n+1\}\} = \inf\{a_n, z_{n+1}\} \le z_{n+1}$.

iii. *$\langle z_n \rangle$ is not bounded above.*

Proof. If z_n is bounded, there exists $M' > 0$ such that $z_n < M'$ for all n. But from $(*)$, one can find $N' \in \mathbb{N}$ for given M' so that $a_n > M'$ for all $n > N'$. Thus for any $n > N'$, $z_n = \inf\{a_k : k \ge n\}$ is at least M'. We have a contradiction, and $\liminf_{n\to\infty} a_n = \infty$.

($\Longleftarrow$) $\liminf_{n\to\infty} a_n = \infty \implies$ For all $M > 0$, there exists $N \in \mathbb{N}$ such that $z_n = \inf\{a_k : k \ge n\} > M$ if $n \ge N$. But by definition, $z_n = \inf\{a_k : k \ge n\} \le a_n$, so $a_n \ge M$ for $n \ge N$. $\lim_{n\to\infty} a_n = \infty$.

(3) $a_n = (-1)^n n$. a_n is neither bounded above nor bounded below. Suppose such bound M_1, M_2 existed so that $M_1 < a_n < M_2$. To violate the first inequality, choose an odd number from $n \ge \max\{M_1, M_2\}$, and choose an even number to violate the second inequality. Thus by definition, $\limsup_{n\to\infty} a_n = \infty$ and $\liminf_{n\to\infty} a_n = -\infty$.

4. (1) (0.1) If either the limit superior of $\langle a_n \rangle, \langle b_n \rangle$ is ∞, there is nothing to prove. And for the case where either one of them is $-\infty$, suppose $\limsup_{n\to\infty} a_n = -\infty$, without loss of generality. Then the sequence $y_n = \sup_{k\ge n} a_k$ is not bounded below. Thus $\sup_{k\ge n}(a_k + b_k)$ is also not bounded below.[2], which gives us $\limsup_{n\to\infty}(a_n + b_n) = -\infty$, satisfying the inequality.

Now suppose $\limsup_{n\to\infty} a_n = \alpha$, $\limsup_{n\to\infty} b_n = \beta$. ($\alpha, \beta \in \mathbb{R}$) $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$

[2]Except for the case where $\sup_{k\ge n} b_k$ diverges to ∞.

such that for all $n \geq N$, $a_n < \alpha + \epsilon/2$ and $b_n < \beta + \epsilon/2$.
Then we have $a_n + b_n < \alpha + \beta + \epsilon$ for $n \geq N$, thus $\limsup_{n\to\infty}(a_n + b_n)$ is at most $\alpha + \beta$.

(0.2) If either the limit inferior of $\langle a_n \rangle, \langle b_n \rangle$ is $-\infty$, there is nothing to prove. And for the case where either one of them is ∞, suppose $\liminf_{n\to\infty} a_n = \infty$, without loss of generality. Then the sequence $z_n = \inf_{k\geq n} a_k$ is not bounded above. Thus $\inf_{k\geq n}(a_k + b_k)$ is also not bounded above.[3], which gives us $\liminf_{n\to\infty}(a_n + b_n) = \infty$, satisfying the inequality.
Now suppose $\liminf_{n\to\infty} a_n = \alpha$, $\liminf_{n\to\infty} b_n = \beta$. ($\alpha, \beta \in \mathbb{R}$) $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > \alpha - \epsilon/2$ and $b_n > \beta - \epsilon/2$.
Then we have $a_n + b_n > \alpha + \beta - \epsilon$ for $n \geq N$, thus $\liminf_{n\to\infty}(a_n + b_n)$ is at least $\alpha + \beta$.

(2) Take $a_n = (-1)^n, b_n = -(-1)^n$ for both inequalities. We have $\limsup_{n\to\infty}(a_n + b_n) = \liminf_{n\to\infty}(a_n + b_n) = 0$, $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$, $\liminf_{n\to\infty} a_n = \liminf_{n\to\infty} b_n = -1$. The equality parts do not hold for this case.

(3) Yes. It is enough to show (0.1), (0.2) with each of their inequality signs reversed. Since $\langle b_n \rangle$ is convergent, $\limsup_{n\to\infty} b_n = \liminf_{n\to\infty} b_n = \beta \in \mathbb{R}$. For all $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $|b_n - \beta| < \epsilon$. Then we have

$$a_n + \beta - \epsilon < a_n + b_n < a_n + \beta + \epsilon$$

From the first inequality, $\sup_{k\geq n}(a_k + b_k) \geq \sup_{k\geq n}(a_k) + \beta - \epsilon$.

$$\therefore \limsup_{n\to\infty}(a_n + b_n) \geq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

From the second inequality, $\inf_{k\geq n}(a_k + b_k) < \liminf_{n\to\infty}(a_k) + \beta + \epsilon$.

$$\therefore \liminf_{n\to\infty}(a_n + b_n) \leq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

Hence the equality holds.

(4) The statement is false. Let $a_n = -1$, $b_n = (-1)^n$. $\limsup_{n\to\infty} a_n b_n = 1$ while $\limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n = -1 \cdot 1 = -1$.

5. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$. Since $||a_n| - |a|| < |a_n - a|$, we have

$$-\epsilon < |a_n| - |a| < \epsilon$$

and obviously, $-|a_n| \leq b_n \leq |a_n|$. So the following holds.

$$-|a| - \epsilon < -|a_n| \leq b_n \leq |a_n| < |a| + \epsilon \qquad \cdots (*)$$

We can also see that b_n is a bounded sequence, since a_n is bounded.

[3] Except for the case where $\inf_{k\geq n} b_k$ diverges to $-\infty$.

(a) $\limsup_{n\to\infty} b_n = |a|$.
Given $\forall\epsilon > 0$, $b_n < |a| + \epsilon$ for large enough n (from $(*)$)
Now we must show that there are infinitely many n that satisfy $|a| - \epsilon < b_n$.

i. $a \geq 0$: Since $a - \epsilon < a_n$ and $a_n = b_n$ whenever n is even, there are infinitely many n.

ii. $a < 0$: Since $a_n < a + \epsilon$ and $a_n = -b_n$ whenever n is odd, we have $-b_n < a + \epsilon$, $b_n > -a - \epsilon$. There are infinitely many n.

By Prop 1.4.3, $\limsup_{n\to\infty} b_n = |a|$.

(b) $\liminf_{n\to\infty} b_n = -|a|$.
Given $\forall\epsilon > 0$, $-|a| - \epsilon < b_n$ for large enough n (from $(*)$)
Now we must show that there are infinitely many n that satisfy $b_n < -|a| + \epsilon$.

i. $a \geq 0$: Since $a - \epsilon < a_n$ and $a_n = -b_n$ whenever n is odd, we have $a - \epsilon < -b_n$, $b_n < -a + \epsilon$. There are infinitely many n.

ii. $a < 0$: Since $a_n < a + \epsilon$ and $a_n = b_n$ whenever n is even. There are infinitely many n.

By Prop 1.4.4, $\liminf_{n\to\infty} b_n = -|a|$.

6. (a) $|a_n| \leq \left|\frac{(n+2)(-1)^n+n}{n+1}\right| \leq \frac{|n+2|+|n|}{n+1} \leq 2$. a_n is bounded. And observe that

$$a_n = \begin{cases} 2 & (n \text{ even}) \\ -\dfrac{2}{n+1} & (n \text{ odd}) \end{cases}$$

i. Given $\forall\epsilon > 0$, $a_n < 2 + \epsilon$ for all n. For all even n, $2 - \epsilon < a_n$.
By Prop 1.4.3, $\limsup a_n = 2$.

ii. Given $\forall\epsilon > 0$, $a_n > 0 - \epsilon$ for $n \geq \max\{2/\epsilon - 1, 0\}$. For all odd n, $a_n < 0 + \epsilon$.
By Prop 1.4.4, $\liminf a_n = 0$.

(b) a_n is obviously bounded since cos is bounded.

i. Given $\forall\epsilon > 0$, $a_n < 1 + \epsilon$ for all n. Now we show that $a_n > 1 - \epsilon$.
We want to show that for any given $\delta > 0$, there exists $n, k \in \mathbb{N}$ such that $\left|\sqrt{2019 + \pi^2 n^2} - (2k\pi)\right| \leq \delta$. Set $n' = \sqrt{2019 + \pi^2 n^2} = 2k\pi + r$, $0 \leq r < 2\pi$. Such $k \in \mathbb{N}, r \in \mathbb{R}$ are unique. Define

$$f(n) = r = n' - \left\lfloor \frac{n'}{2\pi} \right\rfloor n'$$

For any large enough N, we can find $N < n_1, n_2 \in \mathbb{N}$ such that $f(n_1), f(n_2)$ are arbitrarily close. This is because $2\pi/m$ can be made arbitrarily smaller than δ for some $m \in \mathbb{N}$. Partition the interval $[0, 2\pi)$ into $\left[\frac{2i}{m}\pi, \frac{2i+2}{m}\pi\right)$ $(i = 0, \cdots, m-1)$ and by pigeonhole principle there exists n_1, n_2 such that $f(n_1), f(n_2)$ belong to the same interval.

Without loss of generality, assume $f(n_1) \le f(n_2)$, $n_i = 2k_i\pi + r_i$ $(i = 1, 2)$, $r_1 \le r_2$. Then $n_2 - n_1 = 2(k_2 - k_1)\pi + r_2 - r_1$. Since $f(n_2) - f(n_1) = r_2 - r_1 < 2\pi/m \le 2\pi$, $f(n_2 - n_1) = r_2 - r_1 < 2\pi/m < \delta \cdots (*)$.

Now, for $l \in \mathbb{N}$, if $lf(n) < 2\pi$, $f(ln) = lf(n)$. ($\because n = 2k\pi + r$, then $ln = 2kl\pi + lr$, $f(ln) = lr = lf(n) < 2\pi$.)

Claim. There exists $l \in \mathbb{N}$ such that $lf(n_2 - n_1) \in [-\delta, \delta]$.

Proof. (by contradiction) Suppose that such l does not exist. Then there exists $l_1 \in \mathbb{N}$ such that

$$\overbrace{l_1 f(n_2 - n_1) < -\delta}^{(1)} < \overbrace{\delta < (l_1 + 1)f(n_2 - n_1)}^{(2)}$$

Add (1), (2) to get $2\delta < f(n_2 - n_1)$, contradicting $(*)$.

Thus we have $f(l(n_2 - n_1)) = lf(n_2 - n_1) \in [-\delta, \delta]$, and there exists infinitely many n that satisfy $a_n \ge 1 - \epsilon$.[4]

By Prop 1.4.3, $\limsup a_n = 1$.

ii. $\liminf a_n = -\limsup(-a_n) = -\limsup a_n = -1$. (cos is an even function.)

7. (1) For any $z = (a, b) \in A$, set $\epsilon = \min\left\{\frac{a+b-1}{\sqrt{2}}, \frac{2-a-b}{\sqrt{2}}\right\}$. These two expressions were obtained from the distance formula. Each expression is the distance from point z to $x + y = 1$ and $x + y = 2$, respectively. Now we must show $N(z, \epsilon) \subset A$. Since $N(z, \epsilon) = \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < \epsilon^2\}$, we must show that $1 < x + y < 2$. From Cauchy-Schwarz inequality,

$$(1^2 + 1^2)\left((x - a)^2 + (y - b)^2\right) \ge (x + y - a - b)^2$$

Now we have

$$\frac{1}{2}(x + y - a - b)^2 \le (x - a)^2 + (y - b)^2 < \epsilon^2$$

By the choice of ϵ, the following holds.

$$\frac{1}{2}(x + y - a - b)^2 \le \left(\frac{a+b-1}{\sqrt{2}}\right)^2 \qquad \frac{1}{2}(x + y - a - b)^2 \le \left(\frac{2-a-b}{\sqrt{2}}\right)^2$$

Solving for $x + y$ gives $1 < x + y < 2$. Therefore A is open in $\mathbb{R}^2$.

(2) For any $\alpha = (a, b, c) \in B$, set $\epsilon = \min\{r - 2, 3 - r\}$ where $r^2 = a^2 + b^2 + c^2$. Then for any $\beta = (x, y, z) \in N(\alpha, \epsilon)$, $\|\beta - \alpha\| < \epsilon$. And we have $2 < \|\alpha\| = r < 3$.

$$\|\beta\| = \|\beta - \alpha + \alpha\| \le \|\beta - \alpha\| + \|\alpha\| < \epsilon + r \le 3 - r + r = 3$$

$$2 = r + 2 - r \le \|\alpha\| - \epsilon < \|\alpha\| - \|\beta - \alpha\| \le \|\alpha + \beta - \alpha\| = \|\beta\|$$

Thus $\beta \in B$, $N(\alpha, \epsilon) \subset B$. Therefore B is open in $\mathbb{R}^3$.

[4] cos is a continuous function ...

(3) For $\alpha \in \mathbb{R}^n, r > 0$, define

$$N_4(\alpha, r) = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : \|\mathbf{x} - \alpha\|_4 < r\}$$

. The given set C is equal to $N_4(\mathbf{0}, 1)$. For any $\alpha = (x_1, \ldots, x_n) \in C$, consider $X = N_4(\alpha, 1 - \|\alpha\|_4)$.

i. $X \subset N_4(\mathbf{0}, 1)$.
For any $x \in X$, $\|x - \alpha\|_4 < 1 - \|\alpha\|_4$. By Minkowski's inequality, $\|x\|_4 \leq \|x - \alpha\|_4 + \|\alpha\|_4 < 1$. $\therefore x \in N_4(\mathbf{0}, 1)$.

ii. $N(\alpha, \epsilon) \subset X$, where $\epsilon = 1 - \|\alpha\|_4$.
Lemma. Since $x_i^2 \geq 0$,

$$\sum_{i=1}^{n} x_i^4 \leq \left(\sum_{i=1}^{n} x_i^2\right)^2$$

Proof. RHS $-$ LHS $= 2\sum_{i \neq j} x_i^2 x_j^2 \geq 0$.
Now suppose $x \in N(\alpha, \epsilon)$. Then $\|x - \alpha\| < \epsilon$, and $(\|x - \alpha\|_4)^4 \leq (\|x - \alpha\|)^4$ by the lemma.

$$\|x - \alpha\|_4 \leq \|x - \alpha\| < 1 - \|\alpha\|_4$$

Therefore x is also in X.

Now we see that for all $\alpha \in C$ there exists $\epsilon > 0$ such that $N(\alpha, \epsilon) \subset X \subset C$. C is open in $\mathbb{R}^n$.

(4) For $z = (x, y) \in D$, we can write

$$N(z, \epsilon) = \{(x + r\cos\theta, y + r\sin\theta) : 0 \leq r < \epsilon, 0 \leq \theta < 2\pi\}$$

Now we want to show that there exists some $\epsilon > 0$ such that $N(z, \epsilon) \subset D$. To show this, it is enough to show that there exists some r_0 such that for all $r < r_0$, $0 \leq \theta < 2\pi$, $(x + r\cos\theta)(y + r\sin\theta) > 1$. We will show the following.

$$xy + r(x\cos\theta + y\sin\theta) + r^2\cos\theta\sin\theta - 1 > 0$$

Set $r_0 = \min\left\{\dfrac{xy - 1}{|x| + |y| + 1}, 1\right\}$. For $r < r_0$,

$$\begin{aligned} \text{LHS} &= xy - 1 - \left|r(x\cos\theta + y\sin\theta) + r^2\cos\theta\sin\theta\right| \\ &\overset{(*)}{\geq} xy - 1 - r(|x| + |y| + 1) > 0 \qquad \text{(by the choice of } r_0\text{)} \end{aligned}$$

Proof of $(*)$. $|r(x\cos\theta + y\sin\theta) + r^2\cos\theta\sin\theta| \leq r(|x| + |y| + r) < r(|x| + |y| + 1)$.
The last inequality is from $r < 1$.
$\therefore N(z, \epsilon) \subset D$. D is open in $\mathbb{R}^2$.

8. (1) $A = \mathbb{N}$. Check if $\mathbb{N}^C$ is open. We will show that for all $x \in \mathbb{N}^C$, $N(x, \epsilon) \subset \mathbb{N}^C$ where $\epsilon = \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}$. Because $N(x, \epsilon) = (x - \epsilon, x + \epsilon)$,

$$x + \epsilon < x + \lceil x \rceil - x < \lceil x \rceil \qquad x - \epsilon > x - x + \lfloor x \rfloor > \lfloor x \rfloor$$

Thus for any $n \in \mathbb{N}$, $n \notin N(x, \epsilon)$. $\therefore N(x, \epsilon) \subset \mathbb{N}^C$. $\mathbb{N}^C$ is open thus $\mathbb{N}$ is closed.

(2) $B^C = \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$. We will show that for all $z = (x_0, y_0) \in B^C$, $N(z, \epsilon) \subset B^C$ where $\epsilon = \min\{|x_0|, |y_0|\}$. For all $z' = (x, y) \in N(z, \epsilon)$, suppose $x - \epsilon < 0 < x + \epsilon$. Then $x < \epsilon$, $-\epsilon < x$ which gives us $|x| < \epsilon$. Contradiction; Thus either $x < x + \epsilon \leq 0$ or $x > x - \epsilon \geq 0$. Similarily, $y < y + \epsilon \leq 0$ or $y > y - \epsilon \geq 0$. For all cases, $xy \neq 0$.
Thus $z' \in B^C$ and B^C is open. Hence B is closed.

(3) $C^C = \{(x, y) \in \mathbb{R}^2 : 3x + 2y \neq 1\}$. For $z = (x_0, y_0) \in C^C$, set $\epsilon = \dfrac{|3x_0 + 2y_0 - 1|}{\sqrt{13}}$.
Then by Cauchy-Schwarz, if $z' = (x, y) \in N(z, \epsilon)$,

$$(3x - 3x_0 + 2y - 2y_0)^2 \leq 13\left((x - x_0)^2 + (y - y_0)^2\right) < 13\epsilon^2 = (3x_0 + 2y_0 - 1)^2$$

Solving this gives,

$$(3x + 2y - 6x_0 - 4y_0 + 1)(3x + 2y - 1) \leq 0$$

i. $3x + 2y - 1 > 0$
$\implies 0 \geq 3x + 2y - 6x_0 - 4y_0 + 1 > 2 - 6x_0 - 4y_0$
$\implies 3x_0 + 2y_0 > 1$

ii. $3x + 2y - 1 < 0$
$\implies 0 \leq 3x + 2y - 6x_0 - 4y_0 + 1 < 2 - 6x_0 - 4y_0$
$\implies 3x_0 + 2y_0 < 1$

Therefore $z' \in C^C$. $N(z, \epsilon) \subset C^C$, C is closed.

(4) Take $\alpha = (x_0, y_0, z_0) \in D^C$. Then $x_0^2 + y_0^2 > 1$. Consider $\beta = (x, y, z) \in N(\alpha, \epsilon)$ where $\epsilon = \sqrt{x_0^2 + y_0^2} - 1$. Now take $\gamma = (0, 0, z_0), \gamma' = (0, 0, z) \in D$. Then we have $\|\beta - \alpha\| < \epsilon$, $\|\alpha - \gamma\| = 1 + \epsilon$. Using the triangle inequality twice gives

$$\|\alpha - \beta\| + \|\beta - \gamma'\| \geq \|\alpha - \gamma'\| \geq \|\gamma - \gamma'\| + \|\gamma - \alpha\| \geq \|\gamma - \alpha\| = 1 + \epsilon$$

Now we know that $\sqrt{x^2 + y^2} = \|\beta - \gamma'\| \geq 1 + \epsilon - \|\alpha - \beta\| > 1$, which implies that $\beta \in D^C$. Thus $N(\alpha, \epsilon) \subset D^C$, and D^C is open. Thus D is closed.