Assignment 2

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Definition. For a sequence $\langle a_n \rangle$, we write

$$\lim_{n\to\infty} a_n = \infty$$

if for any $M > 0$, there exists N such that $a_n > M$ for all $n \geq N$. In this case, we say that the sequence $\langle a_n \rangle$ diverges to infinity. Similarly, we write

$$\lim_{n\to\infty} a_n = -\infty$$

if for any $M > 0$, there exists N such that $a_n < -M$ for all $n \geq N$, and we say that $\langle a_n \rangle$ diverges to minus infinity.

We also use the following notations:

- $\infty + \infty = \infty$, $(-\infty) + (-\infty) = -\infty$
- For any $a \in \mathbb{R}$,

$$a + \infty = \infty + a = \infty \text{ and } a + (-\infty) = (-\infty) + a = -\infty.$$

- For any $a > 0$ and $b < 0$,

$$a \cdot \infty = \infty \cdot a = \infty \ , \ a \cdot (-\infty) = (-\infty) \cdot a = -\infty \ ,$$
$$b \cdot \infty = \infty \cdot b = -\infty \text{ and } b \cdot (-\infty) = (-\infty) \cdot b = \infty \ .$$

We shall not define $0 \cdot \infty$ and $\infty + (-\infty)$.

1. Prove the following limits:

(1) $\lim_{n\to\infty}(-2n+3) = -\infty$

(2) $\lim_{n\to\infty} \tan \frac{n}{2n+1}\pi = \infty$

2. Suppose that two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ satisfy $\lim a_n = a$ and $\lim b_n = \infty$.

(1) Prove that $\lim (a_n + b_n) = \infty$.

(2) Prove that $\lim a_n b_n = \lim a_n \lim b_n$ if $a \neq 0$.

(3) For $a = 0$, find $\langle a_n \rangle$ and $\langle b_n \rangle$ such that the sequence $\langle a_n b_n \rangle$ is neither convergent nor divergent to $\pm\infty$.

Definition. If a sequence $\langle a_n \rangle$ is not bounded above, we define $\limsup_{n\to\infty} a_n = +\infty$. Otherwise, let us define $y_n = \sup\{a_k : k \geq n\}$ for each $n \in \mathbb{N}$. If the decreasing sequence $\langle y_n \rangle$ is not bounded below, we define $\limsup y_n = -\infty$. Check that $\limsup_{n\to\infty} a_n$ is now defined for **any** sequence $\langle a_n \rangle$.

3. (1) Define $\liminf_{n\to\infty} a_n$ for any sequence $\langle a_n \rangle$ by a similar manner as above.

(2) Prove that a sequence $\langle a_n \rangle$ satisfies $\lim_{n\to\infty} a_n = \infty$ if and only if

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \infty.$$

(The same result holds for $-\infty$ as well; check if you want)

(3) Find a sequence $\langle a_n \rangle$ such that $\limsup a_n = \infty$ and $\liminf a_n = -\infty$.

4. (1) For any two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \ , \tag{0.1}$$

and that

$$\liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \ . \tag{0.2}$$

(2) Verify that the equality part of (0.1) and (0.2) may not hold.

(3) Assume in addition that $\langle b_n \rangle$ is convergent; $\lim b_n \in \mathbb{R}$. Can we replace the inequality at (0.1) or (0.2) with equality? Demonstrate your answer.

(4) Prove or disprove that

$$\limsup_{n\to\infty} a_n b_n \leq \limsup_{n\to\infty} a_n \limsup_{n\to\infty} b_n \ .$$

5. Suppose that $\lim a_n = a$, and define $b_n = (-1)^n a_n$. Compute $\limsup b_n$ and $\liminf b_n$ and prove your answer.

6. Compute $\limsup a_n$ and $\liminf a_n$ for the following sequences:
(1) $a_n = \frac{(n+2)(-1)^n + n}{n+1}$

(2) $a_n = \cos\sqrt{2019 + n^2\pi^2}$

7. Prove that the following sets are open.
(1) $A = \{(x,\, y) : 1 < x + y < 2\} \subset \mathbb{R}^2$

(2) $B = \{(x,\, y,\, z) : 4 < x^2 + y^2 + z^2 < 9\} \subset \mathbb{R}^3$

(3) $C = \{(x_1,\ \ldots, x_n) : \sum_{i=1}^n x_i^4 < 1\} \subset \mathbb{R}^n$

(4) $D = \{(x,\, y) : xy > 1\} \subset \mathbb{R}^2$

8. Prove that the following sets are closed.
(1) $A = \mathbb{N} \subset \mathbb{R}$

(2) $B = \{(x,\, y) : xy = 0\} \subset \mathbb{R}^2$

(3) $C = \{(x,\, y) : 3x + 2y = 1\} \subset \mathbb{R}^2$

(4) $D = \{(x,\, y,\, z) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^3$