Assignment 3

Due Date: 2019/04/17, 1:30 PM

1. Find $\operatorname{int} A$, A' and $\overline{A}$ for the following sets and prove your answer:

(1) $A = \{(x,\, y) \in \mathbb{R}^2 : xy \geq 1\}$

(2) $A = \left\{(-1)^n \frac{n}{n+1} : n \in \mathbb{N}\right\}$

(3) $A = \{(x,\, y,\, z) \in \mathbb{R}^3 : x^2 + y^2 < 1,\ z = 0\}$

(4) $A = \left\{(\frac{m}{n},\, \frac{1}{n}) : m,\, n \in \mathbb{N}\right\}$

(5) $A = \{(x,\, y) \in \mathbb{R}^2 : |x| \leq 1,\ |x+y| < 1\}$.

2. Prove or disprove (by a counter example) the following properties for $A \subset \mathbb{R}^d$:

(1) $\operatorname{int} \overline{A} = \operatorname{int} A$

(2) $\overline{\operatorname{int} A} = A$ if A is a closed set.

(3) $\operatorname{int} A \cap \overline{A^c} = \emptyset$

(4) $\operatorname{int} A \cup \overline{A^c} = \mathbb{R}^n$

3. **(1)** Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely in $\mathbb{R}$, and $\langle b_n \rangle$ is a Cauchy sequence in $\mathbb{R}$. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.
(2) Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or not (and prove your answer).

4. Suppose that A is a bounded set in $\mathbb{R}^d$. Prove that $\overline{A}$ is a compact set.

5. Find an open cover of the following set $A \subset \mathbb{R}^2$ that does not have finite subcover, and demonstrate your answer.

$$A = \{|x| + 2|y| < 1\}$$

6. Let $\langle a_n \rangle$ be a sequence in $\mathbb{R}^d$ and let $a \in \mathbb{R}^d$. Suppose that every subsequence of $\langle a_n \rangle$ has a further subsequence that converges to a. Prove that $\lim a_n = a$.

Select only one problem out of the following two problems.

7. The following is an alternative proof of Bolzano-Weierstrass Theorem (Theorem 2.3.4 of the textbook): Suppose that $\langle a_n \rangle$ is a bounded sequence in $\mathbb{R}$. We say that $\ell \in \mathbb{N}$ is a **peak** of the sequence $\langle a_n \rangle$ if we have $a_\ell > a_m$ for all $m > \ell$.

(1) If $\langle a_n \rangle$ has infinitely many peaks, then prove that $\langle a_n \rangle$ has a decreasing subsequence.

(2) If $\langle a_n \rangle$ has finitely many peaks, then prove that $\langle a_n \rangle$ has an increasing subsequence.

(3) Prove Theorem 2.3.4 for sequences in $\mathbb{R}$.

(4) Prove Theorem 2.3.4 for sequences in $\mathbb{R}^d$.

8. Suppose that $\langle a_n \rangle$ is a positive sequence in $\mathbb{R}$, and that there exist $c > 1$ and $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{c}{n} \text{ for all } n \geq N .$$

Prove that $\sum_{n=1}^{\infty} a_n$ converges.