해석개론 및 연습 1 과제 #6

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1. **(1)** Since x^2 is increasing, for the given partition,

$$M_i = \frac{i}{n} \quad m_i = \frac{i-1}{n}$$

thus

$$U(f, P_n) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} \left(\frac{i}{n}\right) \cdot \frac{1}{n} = \frac{n(n+1)(2n+1)}{6n^3}$$
$$L(f, P_n) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right) \cdot \frac{1}{n} = \frac{n(n-1)(2n-1)}{6n^3}$$

(2)

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{1 \cdot (1+1/n) \cdot (2+1/n)}{6} = \frac{1}{3}$$
$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} \frac{1 \cdot (1-1/n) \cdot (2-1/n)}{6} = \frac{1}{3}$$

2. f is not Riemann Integrable. For any partition $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$, $m_i = 0$ since there exists an irrational number in $[x_{i-1}, x_i]$. Also, $M_i = x_i$ since for any $\epsilon > 0$, there exists a rational number in $(x_i - \epsilon, x_i)$. Thus $L(f, P) = 0$, and

$$U(f, P) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}) \geq \sum_{i=1}^{n} \frac{x_i + x_{i-1}}{2}(x_i - x_{i-1}) = \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2) = \frac{1}{2}x_n^2 = \frac{1}{2}$$

Therefore, $U(f, P) - L(f, P) = \frac{1}{2}$, and cannot be made arbitrarily small.

3. **(1)** For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$, $M_i = m_i = c$.

$$U(f, P) = L(f, P) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c(b - a)$$

$U(f, P), L(f, P)$ are always constant, so the upper/lower integrals also have the value $c(b-a)$ and thus f is integrable.

(2) Let $Y = \{y_i : i = 1, \dots, n\}$ be the set of discontinuities of f. For any $\epsilon > 0$, set

$$\delta = \min\left\{\frac{\epsilon}{2\sum_{j=1}^{n}|c - f(y_j)|}, \min_{1<j\leq n}\left\{\frac{y_j - y_{j-1}}{4}\right\}, \frac{y_1 - a}{4}, \frac{b - y_n}{4}\right\}$$

Now consider $P = \{a < y_1 - \delta < y_1 + \delta < \cdots < y_n - \delta < y_n + \delta < b\}$, and let each element be $x_0, \dots, x_{2n+2}$ in ascending order.

$$U(f, P) - L(f, P) = \sum_{i=1}^{2n+2}(M_i - m_i)(x_i - x_{i-1}) = \sum_{j=1}^{n}|c - f(y_j)| \cdot 2\delta < \epsilon$$

the last equality holds because $M_i - m_i$ is $|c - f(y_j)|$ for $(y_j - \delta, y_j + \delta)$, 0 otherwise. Thus f is Riemann Integrable. Now we know that $\inf U(f,P)$ will equal $\int_a^b f$.

$$\begin{aligned}
U(f,P) &= \sum_{i=1}^{2n+2} M_i(x_i - x_{i-1}) \\
&= c(y_1 - \delta - a) + \sum_{j=1}^{n} \max\{c, f(y_i)\} \cdot 2\delta + c(b - y_n - \delta) + \sum_{j=2}^{n} c(y_i - y_{i-1} - 2\delta) \\
&= c(b-a) + c(y_1 - y_n - 2\delta) + c(y_n - y_1 - 2(n-1)\delta) + \sum_{j=1}^{n} \max\{c, f(y_i)\} \cdot 2\delta \\
&= c(b-a) - 2nc\delta + \sum_{j=1}^{n} \max\{c, f(y_i)\} \cdot 2\delta \\
&= c(b-a) + 2\delta\left(\sum_{j=1}^{n} \max\{c, f(y_i)\} - nc\right) \geq c(b-a)
\end{aligned}$$

The last inequality holds from $\max\{c, f(y_i)\} \geq c$. Now setting $\delta \to 0$ (by setting $\epsilon \to 0$) will give $\inf U(f,P) = c(b-a) = \int_a^b f$.

4. We know that if f, g are Riemann Integrable, $f - g$ is also Riemann Integrable. For any $P \in \mathcal{P}[a,b]$,

$$\begin{aligned}
\int_a^b f &= \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \qquad (x_i^* \in [x_{i-1}, x_i]) \\
&\geq \lim_{\|P\| \to 0} \sum_{i=1}^{n} g(y_i^*)(x_i - x_{i-1}) \qquad (y_i^* \in [x_{i-1}, x_i]) \\
&= \int_a^b g
\end{aligned}$$

This inequality works because f, g are integrable and the Riemann sums exist.

5. (1) Consider

$$f(x) = \begin{cases} 1 & (x = x_0 = \frac{a+b}{2}) \\ 0 & (x \neq x_0) \end{cases}$$

We know that $\int_a^b f = 0$ by **3 – (2)**.

(2) Since $f(x)$ is continuous, there exists $\delta > 0$ s.t.

$$x \in [a,b], |x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{f(x_0)}{2}$$

Then in the interval $X = [a,b] \cap (x_0 - \delta, x_0 + \delta)$, $\frac{f(x_0)}{2} < f(x)$.

$$\int_a^b f = \int_X f + \int_{[a,b]\setminus X} f \overset{(*)}{\geq} \int_X f > \frac{f(x_0)}{2} \min\{b-a, 2\delta\} > 0$$

$(*)$: $f(x) \geq 0$ implies that the integral is greater than or equal to 0 in $[a,b]\setminus X$.

6. (2) Let $M = \sup\{f(t) : t \in (a,b)\} - \inf\{f(t) : t \in (a,b)\}$, $D_f = \{d_1, d_2, \dots\}$. Let $\delta > 0$ and define $I \subset X = [a,b]$ as

$$I = \left\{x \in [a,b] : \exists \epsilon > 0 \text{ s.t. } \sup\{f(t) : t \in N_X(x,\epsilon)\} - \inf\{f(t) : t \in N_X(x,\epsilon)\} < \frac{\delta}{b-a}\right\}$$

Then I is open in X and contains all continuous points of f. In particular $[a,b] = I \cup D$. For $k \geq 1$ define D_k by

$$D_k = \left(d_k - \frac{\delta}{4M \cdot 2^k}, d_k + \frac{\delta}{4M \cdot 2^k}\right) \cap [a,b].$$

Then I and D_i will cover $[a,b]$, and because $[a,b]$ is compact it is already covered by a finite union

$$[a,b] = I \cup D_1 \cup D_2 \cup \cdots \cup D_n$$

for some $n \geq 1$. The complement

$$Y = [a,b] \setminus (D_1 \cup \cdots \cup D_n)$$

is compact (union of closed intervals) and contained in I. Therefore it can be covered by open intervals, such that on the closure of each interval, supremum $-$ infimum is less than $\delta/(b-a)$. By definition of I, every point has such a neighborhood.
Also by compactness of Y, a finite subcover of such intervals exists. Now all end points of this subcover can be considered as points in the partition. The partition will divide Y into finitely many closed intervals such that $\sup(f) - \inf(f) \leq \delta/(b-a)$ will hold on each interval. Now we have the closure of $D_1 \cup \cdots \cup D_n$ is itself a union of closed intervals with a total length less than δ/M.
The entire interval $[a,b]$ is now partitioned so that $U(f,P) - L(f,P) < 2\delta$. δ can be chosen arbitrarily small and f is Riemann Integrable.

(1) Finite sets are countable. The result follows directly from (2).

7. f, g should be bounded.

- $0 \leq \sup\{|f(x)| : a \leq x \leq b\} = M < \infty$. For given $\epsilon > 0$, $\exists P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ s.t.

$$\sum_{i=1}^{n}(x_i - x_{i-1})(M_i - m_i) < \frac{\epsilon}{2M+1}$$

Since

$$\left|f(x)^2 - f(y)^2\right| \leq |f(x) - f(y)| \left(|f(x)| + |f(y)|\right) \leq 2M\left|f(x) - f(y)\right|$$

, let $\tilde{M_i}, \tilde{m_i}$ be supremum and infimum of f^2 in $[x_{i-1}, x_i]$. Then

$$\tilde{M_i} - \tilde{m_i} \leq 2M(M_i - m_i)$$

Thus

$$\sum_{i=1}^{n}(x_i - x_{i-1})(\tilde{M_i} - \tilde{m_i}) \leq \sum_{i=1}^{n}(x_i - x_{i-1})2M(M_i - m_i) \leq 2M \cdot \frac{\epsilon}{2M+1} < \epsilon$$

and f^2 is integrable.

- Now write $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$ to observe that fg is integrable because $f+g, f-g, (f+g)^2, (f-g)^2$ are integrable.