March 29th, 2019

Remark. lim sup is the limit of sup. If sup is easy to calculate, find sup and take the limit.

Quiz 1 Solutions

#1. Given set A, $\text{int}(A)$, A', determine whether the set is open or closed.

1. $A = \mathbb{N} \subset \mathbb{R}$. $\text{int}(A) = \emptyset$, $A' = \emptyset$, A is closed.

2. $\mathbb{Q} \subset \mathbb{R}$. $\text{int}(\mathbb{Q}) = \emptyset$, $\mathbb{Q}' = \mathbb{R}$, $\mathbb{Q}$ is neither open nor closed.

3. $C = [0,1] \cup (2,3) \cap \{4\} \subset \mathbb{R}$. $\text{int}(C) = (0,1) \cup (2,3)$, $C' = [0,1] \cup [2,3]$, C is neither open nor closed.

4. $D = \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) : 0 \leq y \leq 1\} \subset \mathbb{R}^2$. $\text{int}(D) = \emptyset$, $D' = D \cup \{(0,y) : 0 \leq y \leq 1\}$, D is neither open nor closed. ($\because \text{int}D \neq D$, $\overline{D} \neq D$)

#2. Find a limit point of given set.

1. $A = \mathbb{Q} \subset \mathbb{R}$. 0 is a limit point. (Directly follows from Archimedes' principle)

2. $B = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. 0 is a limit point of B. (Also directly follows from Archimedes')

3. $C = \{2^{-n} + 3^{-m} : n, m \in \mathbb{N}\} \subset \mathbb{R}$. 0 is a limit point of C. Given $\epsilon > 0$, exists $N \in \mathbb{N}$ such that for $n, m \geq N$, $2^{-n} < \epsilon/2$, $3^{-m} < \epsilon/2$. Then $0 \neq 2^{-n} + 3^{-m} < \epsilon$.

#3. True or False? If false, find a counterexample.

1. $\overline{A \cup B} = \overline{A} \cup \overline{B}$ **True**

2. $\overline{A \cap B} = \overline{A} \cap \overline{B}$ **False**. Set $A = (0,1), B = (1,2)$.
   Correct Statement: $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

3. $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$ **False**. Set $A = [0,1], B = [1,2]$.
   Correct Statement: $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$

4. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$ **True**

Thm. $A \subset B \implies \overline{A} \subset \overline{B}, \mathrm{int}(A) \subset \mathrm{int}(B)$.

Proof.

- We need to show $A' \subset B'$. Let $x \in A'$.
  $\implies \forall \epsilon > 0,\ N(x,\epsilon) \cap (A \backslash \{x\}) \neq \emptyset$.
  $\implies \forall \epsilon > 0, N(x,\epsilon) \cap (B \backslash \{x\}) \neq \emptyset$
  $\implies x \in B'$.

- Let $x \in \mathrm{int}(A)$
  $\implies \exists \epsilon > 0, N(x,\epsilon) \subset A \implies N(x,\epsilon) \subset B \implies x \in \mathrm{int}(B)$.

Proof of (c). $A, B \subset A \cup B$
$\implies \mathrm{int}(A), \mathrm{int}(B) \subset \mathrm{int}(A \cup B)$. Thus $\mathrm{int}(A) \cup \mathrm{int}(B) \subset \mathrm{int}(A \cup B)$

Proof of (d). $A \cap B \subset A, B \implies \mathrm{int}(A, B) \subset \mathrm{int}(A), \mathrm{int}(B)$. Thus $\mathrm{int}(A \cap B) \subset \mathrm{int}(A) \cap \mathrm{int}(B)$
Suppose $x \in \mathrm{int}(A) \cap \mathrm{int}(B)$. Then $\exists \epsilon_A, \epsilon_B > 0$ s.t. $N(x, \epsilon_A) \subset A, N(x, \epsilon_B) \subset B$. Take $\epsilon = \min\{\epsilon_A, \epsilon_B\}/2$. Then $N(x,\epsilon) \subset A, B$. Therefore $N(x,\epsilon) \subset A \cap B$, $x \in \mathrm{int}(A \cap B)$.
Example. $A = \{(x,y) : x^2 + 2y^2 < 1\}$. $\mathrm{int}(A) = A, A' = \{(x,y) : x^2 + 2y^2 \leq 1\}$.
Suppose $(x_0, y_0) \in A$. $x_0^2 + 2y_0^2 = 1 - \delta < 1$ for some $\delta > 0$. By symmetry, let $x_0, y_0 > 0$. From

$$(x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 = x_0^2 + 2y_0^2 + \epsilon(2x_0 + 4y_0 + 3\epsilon) < 1$$

, we want $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \delta$. Set $\epsilon < 1/10$. Then $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \epsilon(2x_0 + 4y_0 + 3) < \delta$.
Now set $\epsilon = \min\left\{\frac{1}{2(2x_0 + 4y_0 + 3)}, \frac{1}{100}\right\} > 0$.
Then $|x - x_0| < \epsilon, |y - y_0| < \epsilon$. $x_0^2 + 2y_0^2 < (x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 < 1$. $N((x_0, y_0), \epsilon) \subset A$.
Interior points are limit points, and for the points (x_0, y_0) on the border, consider a sequence $(x_0 - 1/n, y_0 - 1/n)$. Then the elements are in A and they converge to (x_0, y_0). Thus the border is also included in A'.

April 1st, 2019

int$A$: $x \in A$ s.t. $N(x,\epsilon) \subset A$ for some $\epsilon > 0$.
A' : $x \in \mathbb{R}^d$ s.t. $N(x,\epsilon) \cap (A\backslash\{x\}) \neq \emptyset$ for $\forall\epsilon > 0$
$\overline{A}$: $x \in \mathbb{R}^d$ s.t. $N(x,\epsilon) \cap A \neq \emptyset$, $\forall\epsilon > 0$, $\overline{A} = A \cup A'$

Example. $A = [0,1) \cup \{2\}$. $1 \in A'$, $2 \notin A'$, $2 \in \overline{A}$

Prop 2.3.3 $x \in A' \implies N(x,\epsilon) \cap (A\backslash\{x\})$ 는 무한집합이다.
Proof. 유한집합이라고 가정하자. $N(x,\epsilon) \cap (A\backslash\{x\}) = \{x_1,\ldots,x_n\}$ 이라 할 수 있다. Set $\delta = \min\{\|x - x_i\| : \forall i\}$. Then $N(x,\delta) \cap (A\backslash\{x\}) = \emptyset$. 모순.
그래서 사실은 공집합이 아닌 것으로 정의했지만 **사실은** 무한집합이다.

Remark. $A' \neq \emptyset \implies A$는 무한집합.
(대우) A가 유한집합이면 극한점이 존재하지 않는다. (2.2 보기 4)
(역) **거짓**. $A = \{1,2,\cdots\}$ 이면 $A' = \emptyset$.
그러면 역이 언제 성립하나요? 다음 단원 내용!

Definition. Convergence in $\mathbb{R}^d$
Let $\langle x_n \rangle$ be a sequence in $\mathbb{R}^d$.

$$\lim_{n\to\infty} x_n = x \iff \forall\epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies \|x_n - x\| < \epsilon)$$

Exercise. $x_n = (x_n^{(1)},\ldots)$, $x = (x^{(1)},\ldots)$ 일 때, $x_n \to x \iff \forall i, x_n^{(i)} \to x^{(i)}$

Notation. $A \subset \mathbb{R}^d$; $\langle x_n \rangle$ is a sequence in $A \iff \forall n, x_n \in A$

Theorem 2.2.2

1. $x \in A' \iff \exists\langle x_n \rangle$ in $A\backslash\{x\}$ such that $x_n \to x$
2. $x \in \overline{A} \iff \exists\langle x_n \rangle$ in A such that $x_n \to x$

Proof.

1. $(\implies)$ $x_n \in N\left(x, \frac{1}{n}\right) \cap (A\backslash\{x\})$ 이라 하자. (공집합이 아니므로 이러한 원소가 존재한다.) 그러면 $\|x_n - x\| < 1/n$ 이므로 x_n 은 x 로 수렴한다. 그리고 $x_n \in A\backslash\{x\}$ 이므로 수열이 $A\backslash\{x\}$ 에 있다.
2. Left as exercise. Replace $A\backslash\{x\}$ with A.

Theorem 2.2.3. The following are equivalent.

1. F is closed.
2. $F' \subset F$.
3. $F = \overline{F}$
4. For a sequence $\langle x_n \rangle$ in F, $\lim_{n\to\infty} x_n = x \implies x \in F$.

Proof.
(1) $\iff$ (3) ($\overline{F}$: smallest closed set containing F.)
(2) $\iff$ (3) 은 자명.
(1) $\iff$ (4) by the above theorem. (Thm 2.2.2)

Applications.

1. A' is closed.
   Proof. We want to show that $(A')' \subset A'$.
   We want to show: $x \in (A')' \implies x \in A'$.
   (A' 이 공집합이면 자명. 공집합이 아니라고 가정하고...)
   Given $\epsilon > 0$, $N(x,\epsilon) \cap (A' \backslash \{x\}) \neq \emptyset$. Take an element $y \in A'$ from this set. Now set $\delta = \min\{\|x-y\|, \epsilon - \|x-y\|\}$ then we have $N(y,\delta) \cap (A \backslash \{y\}) \neq \emptyset$. ($\because y \in A'$)
   $z \in N(y,\delta) \cap (A \backslash \{y\})$ 라 하자.

   (a) $z \in A \backslash \{y\} \subset A$.
   (b) $\|x-z\| \leq \|x-y\| + \|y-z\| < \|x-y\| + \delta \leq \epsilon$ ($z \in N(y,\delta)$)
   (c) $\|x-z\| \geq \|x-y\| - \|y-z\| > \|x-y\| - \delta \geq 0$ (By the choice of δ.) Thus $x \neq z$.

   Therefore $z \in N(x,\epsilon)$ (by (b)), $z \in A \backslash \{x\}$ (by (a), (c)).
   $x \in A'$ since $N(x,\epsilon) \cap (A \backslash \{x\})$ is not empty.

2. $A \subset \mathbb{R}$: closed and bounded $\implies$ $\inf A = \min A$, $\sup A = \max A$. (Existence)
   Proof. Let $\sup A = x \notin A$. ($\sup A \in A$ 이면 자명)
   Claim. $x \in A'$.
   Proof of Claim. $\forall \epsilon > 0$, $N(x,\epsilon) = (x-\epsilon, x+\epsilon)$
   $x = \sup A$ 이므로 $x - \epsilon$ is not an upper bound.
   $\exists y$ such that $y \in (x-\epsilon, x)$
   $y \in N(x,\epsilon) \cap (A \backslash \{x\}) \neq \emptyset$ 이므로 x 는 극한점.
   따라서 $x \in A' \subset A$ (closed set 이므로 Thm 2.2.3 (2)) 모순.
   $\sup A \in A$ 이므로 이 값이 최댓값이다.

2.3 유계집합과 코시수열

핵심: Thm 2.3.4, Thm 2.3.7

Definition. $\langle x_n \rangle$: 유계수열(bounded sequence) $\iff \exists M > 0$ s.t. $\|x_n\| \le M$ for all $n \in \mathbb{N}$.

Definition. $n_1 < n_2 < \cdots$: sequence in $\mathbb{N}$ 이라 하자. $\langle x_{n_k} \rangle_{k=1}^{\infty} = (x_{n_1}, x_{n_2}, \ldots)$ 를 $\langle x_n \rangle$의 부분수열(subsequence)이라 한다.

Theorem 2.3.4 (Bolzano-Weierstrass Theorem)
If $\langle x_n \rangle$ is bounded, there exists a convergent subsequence of $\langle x_n \rangle$.

Idea of Proof. Equivalent formulation for sets.

Definition. Set A is bounded $\iff \exists M > 0$ such that $\|x\| < M$ for all $x \in A$.

Theorem 2.3.2 (Equivalent of 2.3.4) A가 유계이고 무한집합이면, $A' \neq \emptyset$.

Remark. $A' \neq \emptyset \implies A$: 무한집합.
역이 성립하기 위해서는 A가 유계라는 조건이 필요하다.

극한점이 중요한 이유는 계속 수열과 관련이 있기 때문이다.
Example. $A = \{1/n : n \in \mathbb{N}\}$ 을 고려하는 것은 수열 $x_n = 1/n$ 을 고려하는 것이나 마찬가지이다. 이 수열 x_n 이 x 로 수렴하는 것은 $A' = \{x\}$ 와 동치이다. (Hence the name “limit point”)
이로부터 $x \in A' \iff$ Exists a subsequence of $\langle x_n \rangle$ in $A \backslash \{x\}$ converging to x.

Proof of 2.3.2

1. **Lemma 2.3.1** 축소구간정리 in $\mathbb{R}^d$.
   B is a closed box in $\mathbb{R}^d \iff B = I_1 \times I_2 \times \cdots \times I_d$, where $I_i = [a_i, b_i]$ for $i = 1, \ldots, d$.
   (I_i is a closed and bounded interval.)

   $$B_1 \supset B_2 \supset \cdots \implies \bigcap_{n=1}^{\infty} B_n \neq \emptyset$$

   Proof. 각 ‘좌표’ I_i 별로 1차원 축소구간정리를 적용하면 된다.

2. **Divide and Conquer Strategy**
   B: Box 일 때, $\mathrm{diam}(B) = \sup\{\|x - y\| : x, y \in B\} = \sqrt{(a_1 - b_1)^2 + \cdots + (a_d - b_d)^2}$
   Claim. There exists closed boxes $B_1, B_2, \ldots$ s.t.

   (a) $B_1 \supset B_2 \supset \cdots$
   (b) $\mathrm{diam} B_n = \dfrac{1}{2^{n-1}} \mathrm{diam} B_1$

(c) $B_n \cap A$: 무한집합

Proof. (Induction) $n = 1$; B_1: 충분히 커서 $A \subset B_1$ 인 box 를 잡으면 된다.
Suppose we have $B_1, \ldots, B_n$; B_n을 2^d 등분하면 적어도 하나는 A의 원소를 무한개 포함하고 있다. 그 집합을 B_{n+1} 으로 잡는다. (비둘기집의 원리)
이제 $x \in \bigcap_{n=1}^{\infty} B_n$ 으로 잡으면 (축소구간정리에 의해 잡을 수 있다) $x \in A'$. ($A' \neq \emptyset$)
$\because \forall \epsilon > 0$, $\mathrm{diam} B_n < \epsilon$ 인 $N \in \mathbb{N}$ 을 찾아 $n \geq N$ 일 때 부등식이 성립하도록 할 수 있다.
이러한 n 들에 대하여 $B_n \subset N(x, \epsilon)$. 그러면 $N(x, \epsilon) \cap (A \backslash \{x\}) \supset B_n \cap (A \backslash \{x\})$.

April 3rd, 2019

우리가 지금 2.3 을 하고 있는데, 2 가지 중요한 결과가 있어요.

Theorem 2.3.4 $\langle x_n \rangle$ 이 bounded 이면 수렴하는 부분수열을 갖는다.[1]

Theorem 2.3.2 A가 유계인 집합이고 무한집합이면 극한점을 가진다. $A' \neq \emptyset$
증명은 축소구간정리를 박스로 확장해가지고 분할 정복하면 된다.

Recall 2.3.3 $x \in A' \implies N(x,\epsilon) \cap (A \backslash \{x\})$ 는 무한집합이다.

Proof of 2.3.4. $A = \{x_1, x_2, \ldots, x_n\}$ 라고 하면 이 집합은 유계이다. (수열이 유계이므로)

1. A가 유한집합: 자명.
   $\exists x$ such that x appears infinitely many times in $\langle x_n \rangle$. (PHP) 이 경우에는 부분수열을 $x, x, \ldots$ 로 잡으면 된다. 이는 수렴하는 부분수열이다.

2. A가 무한집합[2]
   $A' \neq \emptyset$ 이므로 $\alpha \in A'$ 이라 하자.
   Claim. $\exists n_1 < n_2 < \ldots$ such that $\|x_{n_k} - \alpha\| < 1/k$.
   Proof. (첨자들이 증가하면서 가까워져야 한다는 것이 유일하게 tricky 한 부분이다. 귀납법을 사용하자.) $k = 1$: $x_{n_1} \in N(\alpha, 1) \cap (A \backslash \{\alpha\})$ 로 잡으면 된다.
   $x_{n_1}, \cdots, x_{n_k}$를 잡았다고 가정: $N(\alpha, \frac{1}{k+1}) \cap (A \backslash \{\alpha\})$ 에서 $x_{n_{k+1}}$를 잡아야 하는데 이 집합은 무한집합이다. (Recall 2.3.3) 이 집합에서 첨자가 n_k보다 큰 항이 반드시 존재하므로 그 중 하나를 $x_{n_{k+1}}$ 이라 잡으면 된다.
   따라서 $\lim\limits_{k\to\infty} x_{n_k} = \alpha$ (Check as exercise)

Application. (Characterization of $\limsup$ and $\liminf$)
x_n 이 bounded 이면, $A = \{x : \exists \text{ subsequence of } x_n \text{ converging to } x\}$. 이 때 Theorem 2.3.4에 의해 $A \neq \emptyset$ 임을 증명하였다.

1. A: closed and bounded $\implies \max(A), \min(A)$ 가 존재한다.
   Proof. $B = \{x_1, x_2, \ldots\}$, $C = \{\langle x_n \rangle$ 에 무한 번 나타나는 수$\}$ 로 잡자. $A = B' \cup C, C \subset B, C' \subset B'$ 임을 확인해보라! 이를 이용하면 $B' \cup C = (B' \cup C') \cup C = B' \cup (C' \cup C) = B' \cup \overline{C}$ 가 되어 닫힌집합의 합집합은 닫힌 집합이다. A는 closed and bounded 이다.

2. $\limsup x_n = \max(A)$, $\liminf x_n = \min(A)$
   (부분수열이 가질 수 있는 극한값들 중 가장 큰 값이 $\limsup$, 가장 작은 값이 $\liminf$)

[1]증명이 가장 테크니컬 해요!
[2]이제 Thm 2.3.2 를 사용할 수 있다. 사실 경우를 나눈 것은 예외적인 case 를 처리하기 위한 것이었다.

Proof. Recall

$$\limsup x_n = \alpha \iff \begin{cases} \text{(i) } \forall \epsilon > 0, \exists N \text{ s.t } (n \geq N \implies x_n < \alpha + \epsilon) \\ \text{(ii) } \forall \epsilon > 0, x_n > \alpha - \epsilon \text{ for infinitely many } n \end{cases}$$

(a) 부분수열 $\langle x_{n_k} \rangle \to \beta$ 이면 (i)에 의해 $k \geq N \implies x_{n_k} < \alpha + \epsilon$ 이 되어 $\beta \leq \alpha + \epsilon$. $\beta \leq \alpha$. 그러므로 $\max(A) \leq \alpha$ 이다.

(b) $\forall \epsilon > 0$, (i), (ii)에 의해 $x_n \in (\alpha - \epsilon, \alpha + \epsilon)$ 인 n 이 무한히 많다. 이 유계인 구간에 속하는 수열의 항들에 대해 부분수열을 잡아 (further subsequence) γ 로 수렴하도록 할 수 있다. (Theorem 2.3.4) 그러면 $\langle x_{m_k} \rangle \to \gamma \in [\alpha - \epsilon, \alpha + \epsilon]$. 따라서 $\alpha - \epsilon \leq \gamma \leq \max(A)$ 가 되어 $\alpha \leq \max(A)$.

따라서 $\max(A) = \alpha$.

Definition. $\langle x_n \rangle$: **Cauchy Sequence** $\iff \forall \epsilon > 0, \exists N \text{ s.t. } [m, n \geq N \implies \|x_m - x_n\| < \epsilon]$

Prop 2.3.6, Thm 2.3.8 $\langle x_n \rangle$: convergent $\iff$ $\langle x_n \rangle$: Cauchy sequence[3]
Proof. ($\implies$) 자명. $\|x_m - x_n\| \leq \|x_m - \alpha\| + \|x_n - \alpha\| < \epsilon/2 + \epsilon/2 = \epsilon$ 인 $m, n \geq N$ 존재.
($\impliedby$) 수렴 값이 없는 상태에서 증명해야 한다. 먼저 수렴 값을 찾아보자.

1. $\langle x_n \rangle$ is bounded.
   Proof. $\exists N$ s.t. $\|x_m - x_n\| < 1$ for all $m, n \geq N$.
   Set $M = \max\{\|x_1\|, \ldots, \|x_{N-1}\|, \|x_N\| + 1\}$. ($\|x_m\| < \|x_N\| + 1$)
   따라서 $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

2. There exists a subsequence $\langle x_{n_k} \rangle$ converging to some α. (Thm 2.3.4)

3. $\langle x_n \rangle$ converges to α.
   Proof. $\epsilon > 0$ 에 대해,

   (a) 코시 수열의 성질에 의해 $\exists N_1$ s.t. $\|x_m - x_n\| < \epsilon/2$ for all $m, n \geq N_1$.

   (b) 부분수열이 α로 수렴하므로 $\exists N_2$ s.t. $\|x_{n_k} - \alpha\| < \epsilon/2$ for all $k \geq N_2$.

   Let $N = \max\{N_1, N_2\}$. $n \geq N_1, n_N \geq n_{N_1} \geq N_1$ 이므로,

$$n > N \implies \|x_n - \alpha\| \leq \|x_n - x_{n_N}\| + \|x_{n_N} - \alpha\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

[3]중간고사 전 까지 가장 중요한 정리.

Remark. 우리의 여정을 돌아보자.

1. Archimedes' Principle 을 가정하면
   Completeness Axiom $\implies$ Monotone Convergence Theorem $\implies$ 축소구간정리 $\implies$ Bolzano-Weierstrass Theorem $\implies$ **Cauchy Convergent Theorem**[4]
   (Exercise) $\implies$ Completeness Axiom

2. **Example**. $X = C([0,1])$. (Set of functions that are continuous in $[0,1]$) How would we define $\|f-g\|$? $\int_0^1 |f(x)-g(x)|\,dx$? $\max\{|f(x)-g(x)| : x \in [0,1]\}$? Only the second choice gives completeness for X.

3. **Convergence Test** without limit value. (**Theorem 2.3.9**)
   $\sum_{n=1}^{\infty} a_n$ is convergent $\iff \forall\epsilon > 0, \exists N$ s.t. $(n > m \geq N \implies |a_{m+1} + \cdots + a_n| < \epsilon)$
   Proof. Trivial.

Definition. $\sum a_n$ is **absolutely convergent** $\iff \sum |a_n|$ is convergent

Theorem. An absolutely convergent series converges.
Proof. Suppose $\sum |a_n|$ converges. For $\forall\epsilon > 0$, there exists N such that $||a_{m+1}| + \cdots + |a_n|| < \epsilon$ for all $m, n \geq N$. Therefore, for $m, n \geq N$,

$$|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \epsilon$$

and $\sum a_n$ converges.

[4] In any metric spaces, this is the condition for completeness.

April 5th, 2019

Theorem. $\overline{A} \cup \overline{B} = \overline{A \cup B}$

Proof. ($\subset$) Trivial.

($\supset$) $A \subset \overline{A}$, $B \subset \overline{B} \implies A \cup B \subset \overline{A} \cup \overline{B} \implies \overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. The closure of a closed set is itself.

6. (2) $a_n = \cos\sqrt{2019 + n^2\pi^2}$

Consider $\delta > 0$, such that

$$(n\pi - \delta)^2 < 2019 + n^2\pi^2 < (n\pi + \delta)^2$$

$$-2n\pi < \frac{2019}{\delta} \pm \delta < 2n\pi$$

We can find large enough N such that the above inequality holds for $n \geq N$.

Now we want $b_n = \sqrt{2019 + n^2\pi^2}$ bounded by $n\pi \pm \delta$.

$n \geq N$, n even $\implies n\pi - \delta < b_n < n\pi + \delta$

$\implies 1 \geq a_n > 1 - \epsilon$

$n \geq N$, n odd $\implies -1 \leq a_n < -1 + \epsilon$

Problem 2.3.5

(1) $x_{n+2} = \dfrac{x_n + x_{n+1}}{2}$

(2) $x_{n+1} = x_n + x(-1)^n \dfrac{1}{3n+1}$

Solution.

(1) Write $x_{n+2} - x_{n+1} = a(x_{n+1} - x_n)$ and observe that $a = -1/2$. Write as

$$x_n = x_{n-1} + \left(-\frac{1}{2}\right)^{n-2}(x_2 - x_1)$$

Then we have

$$x_n = x_2 + \sum_{k=1}^{n-2}\left(-\frac{1}{2}\right)^k (x_2 - x_1)$$

This series converges to $\dfrac{2x_2 + x_1}{3}$.

(2) This is an alternating series. Write as

$$x_n = x_1 + \sum_{k=1}^{n-1}(-1)^k \frac{x}{3n+1}$$

By alternating series test, the second summation term converges, and the series converges to x_1.

Since a converging sequence is a Cauchy sequence, x_1, x_2 can be any real number.

April 8th, 2019

Section 2.3: Bolzano-Weierstrass Theorem, Cauchy Convergent Theorem
In section 2.4, we will be studying about Convergence Tests.
정
2.4 급수의 수렴판정
Cor 2.3.9. $\sum_{n=1}^{\infty} a_n$ is convergent $\iff$ $s_n = \sum_{k=1}^{n} a_k$, $\langle s_n \rangle$ is convergent $\iff$ $\langle s_n \rangle$ is Cauchy.

1. $\sum_{n=1}^{\infty} a_n$ is convergent $\implies$ $\lim_{n\to\infty} a_n = 0$.
2. $\sum_{n=1}^{\infty} |a_n|$ is convergent $\implies$ $\sum_{n=1}^{\infty} a_n$ convergent.

Theorem 2.4.3 (Comparison Test) Suppose $\sum b_n$ converges. If $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$, $\sum a_n$ converges.
ProofLet $M = \sum b_n$, $s_n = \sum_{k=1}^{n} a_k$. s_n is increasing and s_n is bounded by M. s_n is convergent by Monotone Convergence Theorem.

Theorem. Suppose sequences a_n, b_n satisfy $0 \le |a_n| \le b_n$[5] and $\sum b_n$ converges. Then $\sum a_n$ is convergent.[6]
Proof. By comparison test and absolute convergence.

Prop 2.4.4 (Root Test) Suppose $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$.
If $\alpha < 1$, $\sum a_n$ converges. If $\alpha > 1$, $\sum a_n$ diverges.
Proof.

1. $\alpha < 1$. Take $\epsilon > 0$ such that $\alpha < \alpha + \epsilon < 1$. Then there exists N such that $|a_n|^{1/n} < \alpha + \epsilon$ for all $n \ge N$. Therefore $|a_n| < (\alpha+\epsilon)^n$. Since $\alpha + \epsilon < 1$, $\sum(\alpha+\epsilon)^n$ converges. Apply the comparison test to see that $\sum a_n < \infty$.
2. $\alpha > 1$. Take $\epsilon > 0$ such that $\alpha > \alpha - \epsilon > 1$. Then $|a_n|^{1/n} > \alpha - \epsilon$ for infinitely many n. Then $|a_n| > (\alpha - \epsilon)^n > 1$. Therefore $\lim a_n \ne 0$. $\sum a_n$ diverges.

Prop 2.4.5 (Ratio Test) Suppose $a_n \ne 0$. Let $\beta = \limsup |a_{n+1}/a_n|$, $\gamma = \liminf |a_{n+1}/a_n|$.
If $\beta < 1$, $\sum a_n$ converges. If $\beta > 1$, $\sum a_n$ diverges.
Proof.

1. $\beta < 1$. Take $\epsilon > 0$ such that $\beta < \beta + \epsilon < 1$. Then $\exists N$ s.t. $|a_{n+1}/a_n| < \beta + \epsilon$ for $n \ge N$.
   $\implies |a_n| = |a_N|\,|a_{N+1}/a_N| \cdots |a_n/a_{n-1}| < |a_N|\,(\beta+\epsilon)^{n-N}$.
   Set $b_n = |a_N|\,(\beta+\epsilon)^{n-N}$ and apply comparison test to see that $\sum a_n < \infty$.

---
[5]Note that this condition can fail for finitely many n.
[6]a_n may be a very complex expression, but we want b_n to be simple, an expression we know that it is convergent.

2. $\gamma > 1$. Take $\epsilon > 0$ such that $\gamma > \gamma - \epsilon > 1$. Then $\exists N$ s.t. $|a_{n+1}/a_n| > \gamma - \epsilon$ for $n \geq N$. Then we see that $|a_n|$ is increasing for $n \geq N$. Thus a_n cannot converge to 0. $\sum a_n$ is divergent.

Remark. If the above limits (ratio, root) exist, elementary tests can be applied. But if the limits turn out to be 1, the test fails. (ND: Non-Deterministic) Check it for $\sum 1/n, \sum 1/n^2$. Also, these are *weak tests*. For most of the series, the limit is 1. Moreover...

Theorem 2.4.6 Suppose $a_n \neq 0$.

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

Thus if the root test works, ratio test also works.[7]

Proof. We only need to prove the last inequality.

Let $\beta = \limsup |a_{n+1}/a_n|$, $\forall \epsilon > 0$. $\implies \exists N$ s.t. $|a_{n+1}/a_n| \leq \beta + \epsilon$ for $n \geq N$. Then if $n \geq N$, $|a_n| \leq |a_N| (\beta + \epsilon)^{n-N}$. (Similar to proof of 2.4.5) Then

$$|a_n|^{1/n} \leq (\beta + \epsilon) \left(\frac{|a_n|}{(\beta + \epsilon)^N}\right)^{1/n}$$

and take lim sup on both sides, then $\limsup |a_n|^{1/n} \leq \beta + \epsilon$.

Example. $\langle a_n \rangle = \begin{cases} 1/2^n & n \text{ odd} \\ 1/2^{n-2} & n \text{ even} \end{cases}$

Check that $\limsup |a_n|^{1/n} = 1/2 < 1$, and the series $\sum a_n$ converges by the root test.

But if we use the ratio test here, lim sup value is 2 and lim inf value is 1/8.[8] The ratio test does not tell us anything about the convergence. Also note that the series converges to 2.

Prop 2.4.1 (Rearrangement) $a_n \geq 0$.[9] Suppose a bijection $r : \mathbb{N} \to \mathbb{N}$ exists.

1. $\displaystyle\sum_{n=1}^{\infty} a_n = s \iff \sum_{n=1}^{\infty} a_{r(n)} = s$
2. $\displaystyle\sum_{n=1}^{\infty} = \infty \iff \sum_{n=1}^{\infty} a_{r(n)} = s$

Proof.

1. ($\implies$) Let $t_n = \sum_{k=1}^{n} a_{r(k)}$. Then t_n is increasing and bounded by s. Thus t_n converges by MCT, and $\lim t_n \leq s$.
   $s_n = \sum_{k=1}^{n} a_k \leq \sum_{n=1}^{\infty} a_{r(n)} = t = \lim t_n$. ($a_n \geq 0$ was used here.)
   ($\impliedby$) Use $r^{-1}(n)$.

[7] The limit for the ratio test is much easier to calculate than the root test. That's why we use the ratio test.
[8] The ratios are: 2, 1/8, 2, 1/8 ...
[9] This is the important condition.

2. Contraposition of (1).

Prop 2.4.2 (Alternating Series Test) For a given sequence x_n, suppose the following holds.

- x_n is decreasing.
- $\lim x_n = 0$.

Then the series $\sum_{k=1}^{\infty}(-1)^{n-1}x_n$ is convergent.
Proof. Let $s_n = \sum_{k=1}^{n}(-1)^{k-1}x_k$. For $m < n$,

$$|s_n - s_m| = \left|(-1)^m x_{m+1} + \cdots + (-1)^{n-1}x_n\right| = |x_{m+1} - x_{m+2} + \cdots \pm x_n| \overset{(*)}{\in} [0, x_{m+1}]$$

$$\begin{aligned}(*) : x_{m+1} - x_{m+2} + \cdots + x_n &= (x_{m+1} - x_{m+2}) + \cdots + (x_{n-2} - x_{n-1}) + x_n \geq 0 \\ &= x_{m+1} - (x_{m+2} - x_{m+3}) - \cdots - (x_{n-1} - x_n) \leq x_{m+1}\end{aligned}$$

Check for the case with last term $-$.
Now, $\forall \epsilon > 0$, find N such that $|x_n| < \epsilon$ for $n \geq N$. Then for $n > m \geq N$, $|s_n - s_m| \leq x_{m+1} < \epsilon$.
Thus $\langle s_n \rangle$ is a Cauchy sequence and the given series converges.

Example. $a_n = (-1)^{n-1}/n$. $\sum a_n$ converges by alternating series test and converges to $\log 2$.

Remark. The rearrangement of the above example may not converge, or converge to a different value than $\log 2$.

Exam: 1.1 - 2.6

After the midterms we will be covering functions and continuity.
Chapter 1 has been about $\mathbb{R}$, and in Chapter 2, we have talked about subsets of $\mathbb{R}^n$.
2.1: What is $\mathbb{R}^n$? Vector Space, IPS, Metric Space, Normed Space...
2.2: Open, closed sets
2.3: Bounded sets and Cauchy sequences
(2.4: Convergence Tests)
2.5: Compact Sets
2.6: Connect Sets

April 10th, 2019

2.5 Compact Set

Definition. $\{U_i : i \in I\}$ (I is the index set, $U_i \subset \mathbb{R}^d$) is called "family of sets".

1. $\{U_i : i \in I\}$ is a **cover** of $K \subset \mathbb{R}^d \iff K \subset \bigcup_{i\in I} U_i$.
2. $\{U_i : i \in I\}$ is a **open cover** $\iff$ U_i are open for $\forall i$.
3. $J \subset I$, $\{U_i : i \in J\}$ is called a **subcover** of $\{U_i : i \in I\} \iff K \subset \bigcup_{i\in J} U_i$.

Definition. $K \subset \mathbb{R}^d$ is **compact** $\iff$ Any open cover of K has finite subcover.

Example.

1. $\mathbb{N}$ is not compact. Set $U_k = (k - 1/2, k + 1/2)$, then $\{U_k : k \in \mathbb{N}\}$ is a (open) cover of $\mathbb{N}$. But there are no finite subcover.
2. $A = (0, 1)$ is not compact. Set $U_k = (1/k, 1)$, then because $\bigcup_{k=1}^{\infty} U_k = (0, 1)$, $\{U_k : k \in \mathbb{N}\}$ is a (open) cover of A. But there are no finite subcover. $\bigcup_{i=1}^{m} U_{k_i} = U_{k_m} = (1/k_m, 1)$, which cannot contain $(0, 1)$.
3. $A = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{R}^d$ is compact. $\{U_i : i \in I\}$ be a cover of A. There exists $i_1, \ldots, i_m \in I$ such that $a_k \in U_{i_k}$ for $k = 1, \ldots, m$. Then $\{U_{i_1}, U_{i_2}, \ldots, U_{i_m}\}$ is a finite subcover of A.

Main Theorem: **Heine-Borel Theorem**

$$K \text{ is compact} \iff K \text{ is bounded and closed.}$$

Remark.

1. This is a part of Thm 2.5.4
2. Proof: Prop 2.5.1, Thm 2.5.2, Prop 2.5.3
3. **Characterization of compact sets in** $\mathbb{R}^d$.[10]

[10] Compact Set 을 이 단순한 공간 안에서는 characterize 할 수 있다!

Proof.

$(\implies)$ (Prop 2.5.1)

1. *Is K bounded?*
   Set $U_k = N(0,k)$. Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d$. Thus $\{U_k : k \in \mathbb{N}\}$ is an open cover of K. There exists a finite subcover $U_{k_1}, \ldots, U_{k_m}$ $(k_1 < \cdots < k_m)$ of K. Then we have $K \subset \bigcup_{i=1}^{m} U_{k_i} = U_{k_m} = N(0, k_m)$. Therefore K is bounded.

2. *Is K closed?*
   Suppose $x \in K^C$. Set $U_k = \{y : \|y - x\| > 1/k\}$. Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d \backslash \{x\} \supset K$. (Open cover) There exists a finite subcover $U_{k_1}, \ldots, U_{k_m}$ of K. $K \subset \bigcup_{i=1}^{m} U_{k_i} = U_{k_m}$. Therefore $K^C \supset U_{k_m}^C = \{y : \|y - x\| \le 1/k_m\} \supset N(x, 1/k_m)$. Thus K^C is open, K is closed.

$(\impliedby)$

1. (Theorem 2.5.2) *Closed box is compact.*
   $B = I_1 \times \cdots \times I_d$, $I_i = [a_i, b_i]$. Let $\{U_i : i \in I\}$ is an open cover of B.
   (Contradiction) Suppose there is no finite subcover of B.
   Claim. There exists $B = B_1 \supset B_2 \supset \cdots$ (closed boxes) such that

   - $\mathrm{diam}(B_n) = \frac{1}{2^n}\mathrm{diam}(B_1)$
   - There is no finite subcover of $\{U_i : i \in I\}$ covering B_n.

   By Lemma 2.3.1, there exists $x \in \bigcap_{n=1}^{\infty} B_n$. Since $x \in B$, $\exists U_i$ such that $x \in U_i$. Then $\exists \epsilon > 0$ such that $N(x, \epsilon) \subset U_i$.[11] Set $\frac{1}{2^{n-1}}\mathrm{diam}(B_1) < \epsilon$.
   If $y \in B_n \implies \|x - y\| \le \mathrm{diam}(B_n) < \epsilon \implies y \in N(x, \epsilon)$. Then $B_n \subset N(x, \epsilon) \subset U_i$, contradiction.

2. *K: compact, $F \subset K$, F is closed $\implies$ F: compact.*
   Let $\{U_i : i \in I\}$ be an open cover of F. Then $\{U_i : i \in I\} \cup \{F^C\}$ is an open cover of K. Because K is compact, there exists a finite subcover of K. There are two cases.

   (a) $\{U_{i_1}, U_{i_2}, \ldots, U_{i_m}\}$: This is already a finite subcover of F.

   (b) $\{U_{i_1}, U_{i_2}, \ldots, U_{i_m}, F^C\}$: Since F^C does not cover F, U_{i_k} must cover F.

3. *Closed and bounded set is compact.*
   Suppose K is bounded and closed. There exists a closed box B that contains K. Thus B is compact by (1), K is a closed subset of B. Then by (2), K is compact.

Theorem 2.5.2 가 가장 non-trivial 한 부분이다.

[11] n이 충분히 크면 ball 안에 box 가 들어가고 box 는 U_i 안에 있다? Claim 의 2번째에 모순.

Theorem 2.5.4 The following are equivalent.

1. K is compact.

2. K is bounded and closed.

3. If A is an infinite subset of K, $\emptyset \neq A' \subset K$.

4. For a sequence $\langle x_n \rangle$ in K, there exists a convergent subsequence whose limit is in K.

Proof.
(1) $\iff$ (2) by Heine-Borel Theorem.
(2) $\implies$ (3) Suppose A is infinite and bounded. ($A \subset K$) By Bolzano-Weierstrass, $A' \neq \emptyset$. $A' \subset A' \cup A = \overline{A} \subset K$. ($\overline{A}$ is the smallest closed set containing A, $A \subset K$.)
(3) $\implies$ (4) Let $A = \{x_1, x_2, \dots\}$

1. If A is finite, trivial. (Take a constant subsequence, which constant $\in K$.)

2. If A is infinite, $x \in A' \subset K$ by (3). ($x \in A'$ by Thm 2.3.4)

(4) $\implies$ (2)

1. K is bounded.
   (Contradiction) Suppose K is not bounded. Then $\forall n \in \mathbb{N}$, there exists $x_n \in K$, $\|x_n\| \geq n$. There are no convergent subsequences, contradiction.

2. K is closed.
   (Contradiction) Suppose K is not closed.

   (a) K: finite $\to$ K: closed $\to$ Contradiction.
   (b) K: infinite $\to$ K: infinite and bounded $\overset{\text{B-W}}{\to}$ $K' \neq \emptyset$

   Note. $K' \subset K \iff K$: closed.
   Then if K' is not a subset of K[12], there exists $x \in K' \backslash K$. Since $x \in K'$, there exists a sequence $\langle x_n \rangle$ in $K \backslash \{x\}$ $(= K)$[13] converging to x. Thus for a subsequence of $\langle x_n \rangle$, its limit must be in K. But x is the only possible limit value. $x \in K$. Contradiction.

[12] Contraposition
[13] $x \notin K$

April 12th, 2019

Problem 2.4.7 (바) $\sum \frac{1}{n^p - n^q}$ $(0 < q < p)$
$0 < n^p - n^q \le n^p$ 이므로 $1/n^p \le 1/(n^p - n^q)$ 가 되어 $p \le 1$ 이면 발산한다.
충분히 큰 N에 대하여 $n \ge N$ 일 때마다 $n^p - n^q \ge n^p/2$ 가 되게 할수 있다. (이 때 $n^p/2 \ge n^q$ 이므로 $n^{p-q} \ge 2$ 가 되어 N 을 잡을 수 있다) 비교판정법에 의해 수렴한다.

Problem 2.7.12 Given $\langle a_n \rangle$ such that $\lim a_n = a$, show that $\sigma_n = \dfrac{a_1 + \cdots + a_n}{n}$ also converges to a.

Problem 2.7.13 $r < 1$, $\|x_{n+2} - x_{n+1}\| \le r \|x_{n+1} - x_n\|$. Show that $\langle x_n \rangle$ is a Cauchy sequence.
Proof. $\|x_{n+1} - x_n\| \le r^{n-1} \|x_2 - x_1\| = r^{n-1}A$, for $A \in \mathbb{R}$. Given $\epsilon > 0$, exists N such that for all $n \ge N$, $\|x_{n+1} - x_n\| < Ar^{n-1} < \epsilon$. Then we have

$$\begin{aligned} m > n \ge N \Rightarrow \|x_n - x_m\| &\le \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\| \\ &\le \|x_{n+1} - x_n\| (1 + r + r^2 + \cdots) < \frac{\epsilon}{1-r} \end{aligned}$$

Remark. Counterexample for $\|x_{n+2} - x_{n+1}\| < \|x_{n+1} - x_n\|$. $x_n = \sum_{k=1}^{n} \frac{1}{k}$

Problem 2.7.14 $x_n \to x$, $A_k = \{x_i : i \ge k\}$. Show that $\bigcap_{k=1}^{\infty} \overline{A_k} = \{x\}$.
Proof. Given $\epsilon > 0$, there exists N such that $n \ge N \Rightarrow x_n \in (x - \epsilon, x + \epsilon)$. Either $x_n = x$, or $x_n \in (x - \epsilon, x + \epsilon) \backslash \{x\}$. Thus $x \in \overline{A_k}$ for all k. $\{x\} \subset \bigcap_{k=1}^{\infty} \overline{A_k}$.
For $y \in \mathbb{R} \backslash \{x\}$, we want to show that $y \notin \bigcap_{k=1}^{\infty} \overline{A_k}$. Then we want to find N such that $y \notin \overline{A_N}$. Since $\|x - y\| > 0$, set $\epsilon = \frac{1}{3}\|x - y\|$. There exists N such that $\|x_n - x\| < \epsilon$. Then $\forall x_n \notin N(y, \epsilon)$. $\overline{A_N} = \{x_N, x_{N+1}, \dots\}$, and y cannot be in $\overline{A_N}$. $\{x\}^C \subset \left(\bigcap_{k=1}^{\infty} \overline{A_k}\right)^C \Rightarrow \bigcap_{k=1}^{\infty} \overline{A_k} \subset \{x\}$.

Problem 2.7.15 $\sum a_n$ converges absolutely.

1. $\sum a_n^2$
   Proof. $a_n^2 < |a_n|$ for large n. Converges by comparison test.
2. $\sum \dfrac{a_n}{1 + a_n}$
   Proof. Since $a_n \to 0$, exists N such that $n \ge N \Rightarrow |a_n| < 1/3$. Then for $n \ge N$, $|1 + a_n| \ge 1 - |a_n| > 2/3 > 1/3$, $1/|1 + a_n| < 3$. We have $\left|\frac{a_n}{1+a_n}\right| < 3|a_n|$. Converges by comparison test.
3. $\sum \dfrac{a_n^2}{1 + a_n^2}$
   Proof. Trivial from 1, 2.

April 15th, 2019

K: compact $\iff$ Exists an open cover of K that has *finite* subcover.

Theorem 2.5.4 (Heine-Borel) For $\mathbb{R}^d$, K: compact $\iff$ K is bounded and closed.

Theorem 2.5.5 (Cantor's Intersection Theorem)[14]
Given family of **compact** sets $\{K_i : i \in I\}$, for all **finite** $J \subset I$, $\bigcap_{i\in J} K_i \neq \emptyset$. Then

$$\bigcap_{i\in I} K_i \neq \emptyset$$

Proof. (Contradiction) $\bigcap_{i\in I} K_i = \emptyset \implies \bigcup_{i\in I} K^C = \mathbb{R}^d$. (Complement)
Take any K_a ($a \in I$), then $K_a \subset \bigcup_{i\in I} K_i^C (= \mathbb{R}^d) \implies \{K_i^C : i \in I\}$ is an open cover of K_a. Then there exists a finite subcover, $\{K_i^C : i \in J\}$ (K_a is compact) Now we can write $K_a \subset \bigcup_{i\in J} K_i^C$. Take complement on both sides to get $K_a^C \supset \bigcap_{i\in J} K_i$. Then $K_a \cap \bigcap_{i\in J} K_i = \emptyset$, contradiction.

Remark. Let $K_i = [a_i, b_i]$ (Compact in $\mathbb{R}$) and set $K_1 \supset K_2 \subset \cdots$
$\implies$ For $J = \{j_1, \ldots, j_m\}$ ($j_1 < \cdots < j_m$), $\bigcap_{i\in J} K_i = K_{j_m} \neq \emptyset$
$\implies \bigcap_{i=1}^{\infty} K_i \neq \emptyset$ (축소구간정리)

2.6 Connected Set

p46-p47 (Section 2.2)

Definition. $X \subset \mathbb{R}^d$, $x \in X$. Define

$$N_X(x, r) = \{y \in X : \|y - x\| < r\} = N(x, \epsilon) \cap X$$

Definition. $U \subset X$ is open in X $\iff$ $x \in U, \exists\, \epsilon > 0$ such that $N_X(x, \epsilon) \subset U$.

Example.

- $U = \{3\}$. U is open in $X = \mathbb{N}$. $N_{\mathbb{N}}(3, 1/10) = 3 \subset U$. (But not open in $\mathbb{R}$)
- For $X = [0, 10]$, $U = [0, 1)$. $x \in U$, $N(x, 1-x) = (2x-1, 1)$, and this might not be subset of U. But

$$N_X(x, 1-x) = \begin{cases} (2x-1, 1) & (x > 1/2) \\ [0, 1) & (x \leq 1/2) \end{cases}$$

  For both cases $N_X(x, 1-x) \subset U$.

[14] 축소구간정리의 가장 일반적인 형태

Prop 2.2.5 U is open in $X \iff U = X \cap V$ for some open set V in $\mathbb{R}^d$.

Remark. First example: $\{3\} = \mathbb{N} \cap (2.9, 3.1)$, Second example: $[0,1) = [0,10] \cap (-1,1)$.
Some references may write this definition as "*relatively*" open in X.

Proof of 2.2.5
$(\implies)$ $x \in U$, $\exists \epsilon_x > 0$ such that $N_X(x,\epsilon_x) \subset U$. Select $V = \bigcup_{x\in U} N(x,\epsilon_x)$, which is open.[15]
Then we have $X \cap V = \bigcup_{x\in U} X \cap N(x,\epsilon_x) = \bigcup_{x\in U} N_X(x,\epsilon_x)$, which is exactly equal to U.

$(\impliedby)$ $x \in U = X \cap V \implies x \in V$. Thus $\exists\, \epsilon > 0$ such that $N(x,\epsilon) \subset V$. Then

$$N_X(x,\epsilon) = X \cap N(x,\epsilon) \subset X \cap V = U$$

Thus U is open in X.

Cor. U: open in X, $Y \subset X$. $\implies U \cap Y$: open in Y.
Proof. $U = X \cap V$ (V: open) $\implies U \cap Y = X \cap V \cap Y = V \cap (X \cap Y) = V \cap Y$.

Definition. $S \subset \mathbb{R}^d$: **disconnected** $\iff$ Exists **non-empty** sets U, V such that U, V is a decomposition of S, and U, V are open in S.
$S \subset \mathbb{R}^d$: **connected** $\iff$ S is not disconnected.

Question. Find all $A \subset \mathbb{R}^d$ such that A is open and closed.
Proof. The only possible sets are $A = \emptyset, \mathbb{R}^d$.
If A is open and closed $\implies A$: open, A^C: open. Then $\mathbb{R}^d = A \cup A^C$, and $\mathbb{R}^d$ is disconnected. But $\mathbb{R}^d$ is connected. Contradiction if either A or A^C is empty.

Theorem. The following are equivalent for $S \subset \mathbb{R}$.

1. S is connected.
2. $\forall a, b \in S$ s.t. $a < b$, and $c \in (a,b) \implies c \in S$.
3. $S = [a,b]$ or $[a,b)$ or $(a,b]$ or (a,b) (a, b can be $\pm\infty$)

Remark. Prop 2.5.1 (1' $\iff$ 2') + Disscussion above ($2 \iff 3$)
Proof.
($1 \implies 2$) (Contradiction) Assume $a, b \in S, c \notin S$ for some $a < c < b$. Set $U = (-\infty, c) \cap S$, $V = (c, \infty) \cap S$. U, V are non-empty.[16] $U \cap V = \emptyset$ and $U \cup V = S$. (Note that $c \in S$) And U, V

[15] $N(x,\epsilon)$ is open and union of open sets are always open.
[16] Always check! $a \in U, b \in V$.

are open in S. (Prop 2.2.5) Then S is disconnected.

($2 \implies 1$) (Contradiction) Assume S is disconnected. There exists U, V that satisfy the definition of disconnected set. For $a \in U, b \in V$, (WLOG $a < b$). By (2), $[a, b] \subset S$.
Let $c = \sup([a, b] \cap U)$.
Case I) $c \in U$. Then $c \neq b \implies c \in [a, b) = Y \implies c \in U \cap Y$.
Since U is open in S and $Y \subset S \implies U \cap Y$ is open in Y. (Cor of 2.2.5)
$\implies \exists\, \epsilon > 0$ such that $N_Y(c, \epsilon) \subset U \cap Y \subset U \cap [a, b]$.

$$Y \cap N(c, \epsilon) = [a, b) \cap (c - \epsilon, c + \epsilon) \supset [c, b) \cap [c, c + \epsilon) = [c, \min\{b, c + \epsilon\})$$

Therefore, we have

$$[c, \min\{b, c + \epsilon\}) \subset N_Y(c, \epsilon) \subset U \cap [a, b]$$

and since c was the supremum, contradiction.
Case II) $c \in V$. Similarly, contradiction.

($2 \implies 3$) $\inf S = u$, $\sup S = v$. (If S is not bounded below, $u = -\infty$, if S is not bounded above, $v = \infty$). Then if $c \in (u, v) \implies c \in S$. There exists $a, b \in S$ such that $u \leq a < c < b \leq v$, meaning that S must be one of $[u, v], [u, v), (u, v], (u, v)$.

($3 \implies 2$) Trivial.