

**March 29th, 2019**

**Remark.**  $\limsup$  is the limit of  $\sup$ . If  $\sup$  is easy to calculate, find  $\sup$  and take the limit.

### Quiz 1 Solutions

#1. Given set  $A$ ,  $\text{int}(A)$ ,  $A'$ , determine whether the set is open or closed.

- (1)  $A = \mathbb{N} \subset \mathbb{R}$ .  $\text{int}(A) = \emptyset$ ,  $A' = \emptyset$ ,  $A$  is closed.
- (2)  $\mathbb{Q} \subset \mathbb{R}$ .  $\text{int}(\mathbb{Q}) = \emptyset$ ,  $\mathbb{Q}' = \mathbb{R}$ ,  $\mathbb{Q}$  is neither open nor closed.
- (3)  $C = [0, 1] \cup (2, 3) \cap \{4\} \subset \mathbb{R}$ .  $\text{int}(C) = (0, 1) \cup (2, 3)$ ,  $C' = [0, 1] \cup [2, 3]$ ,  $C$  is neither open nor closed.
- (4)  $D = \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) : 0 \leq y \leq 1\} \subset \mathbb{R}^2$ .  $\text{int}(D) = \emptyset$ ,  $D' = D \cup \{(0, y) : 0 \leq y \leq 1\}$ ,  $D$  is neither open nor closed. ( $\because \text{int}D \neq D$ ,  $\overline{D} \neq D$ )

#2. Find a limit point of given set.

- (1)  $A = \mathbb{Q} \subset \mathbb{R}$ . 0 is a limit point. (Directly follows from Archimedes' principle)
- (2)  $B = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ . 0 is a limit point of  $B$ . (Also directly follows from Archimedes')
- (3)  $C = \{2^{-n} + 3^{-m} : n, m \in \mathbb{N}\} \subset \mathbb{R}$ . 0 is a limit point of  $C$ . Given  $\epsilon > 0$ , exists  $N \in \mathbb{N}$  such that for  $n, m \geq N$ ,  $2^{-n} < \epsilon/2$ ,  $3^{-m} < \epsilon/2$ . Then  $0 \neq 2^{-n} + 3^{-m} < \epsilon$ .

#3. True or False? If false, find a counterexample.

- (1)  $\overline{A \cup B} = \overline{A} \cup \overline{B}$  **True**
- (2)  $\overline{A \cap B} = \overline{A} \cap \overline{B}$  **False**. Set  $A = (0, 1)$ ,  $B = (1, 2)$ .  
**Correct Statement:**  $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$
- (3)  $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$  **False**. Set  $A = [0, 1]$ ,  $B = [1, 2]$ .  
**Correct Statement:**  $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$
- (4)  $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$  **True**

**Thm.**  $A \subset B \implies \overline{A} \subset \overline{B}, \text{int}(A) \subset \text{int}(B).$

**Proof.**

- We need to show  $A' \subset B'$ . Let  $x \in A'$ .  
 $\implies \forall \epsilon > 0, N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset.$   
 $\implies \forall \epsilon > 0, N(x, \epsilon) \cap (B \setminus \{x\}) \neq \emptyset$   
 $\implies x \in B'.$
- Let  $x \in \text{int}(A)$   
 $\implies \exists \epsilon > 0, N(x, \epsilon) \subset A \implies N(x, \epsilon) \subset B \implies x \in \text{int}(B).$

**Proof of (c).**  $A, B \subset A \cup B$

$\implies \text{int}(A), \text{int}(B) \subset \text{int}(A \cup B).$  Thus  $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$

**Proof of (d).**  $A \cap B \subset A, B \implies \text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B).$  Thus  $\text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B)$   
Suppose  $x \in \text{int}(A) \cap \text{int}(B).$  Then  $\exists \epsilon_A, \epsilon_B > 0$  s.t.  $N(x, \epsilon_A) \subset A, N(x, \epsilon_B) \subset B.$  Take  $\epsilon = \min\{\epsilon_A, \epsilon_B\}/2.$  Then  $N(x, \epsilon) \subset A, B.$  Therefore  $N(x, \epsilon) \subset A \cap B, x \in \text{int}(A \cap B).$

**Example.**  $A = \{(x, y) : x^2 + 2y^2 < 1\}.$   $\text{int}(A) = A, A' = \{(x, y) : x^2 + 2y^2 \leq 1\}.$

Suppose  $(x_0, y_0) \in A.$   $x_0^2 + 2y_0^2 = 1 - \delta < 1$  for some  $\delta > 0.$  By symmetry, let  $x_0, y_0 > 0.$  From

$$(x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 = x_0^2 + 2y_0^2 + \epsilon(2x_0 + 4y_0 + 3\epsilon) < 1$$

, we want  $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \delta.$  Set  $\epsilon < 1/10.$  Then  $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \epsilon(2x_0 + 4y_0 + 3) < \delta.$

Now set  $\epsilon = \min \left\{ \frac{1}{2(2x_0 + 4y_0 + 3)}, \frac{1}{100} \right\} > 0.$

Then  $|x - x_0| < \epsilon, |y - y_0| < \epsilon.$   $x_0^2 + 2y_0^2 < (x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 < 1.$   $N((x_0, y_0), \epsilon) \subset A.$

Interior points are limit points, and for the points  $(x_0, y_0)$  on the border, consider a sequence  $(x_0 - 1/n, y_0 - 1/n).$  Then the elements are in  $A$  and they converge to  $(x_0, y_0).$  Thus the border is also included in  $A'.$

April 1st, 2019

$\text{int}A : x \in A \text{ s.t. } N(x, \epsilon) \subset A \text{ for some } \epsilon > 0.$

$A' : x \in \mathbb{R}^d \text{ s.t. } N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset \text{ for } \forall \epsilon > 0$

$\overline{A} : x \in \mathbb{R}^d \text{ s.t. } N(x, \epsilon) \cap A \neq \emptyset, \forall \epsilon > 0, \overline{A} = A \cup A'$

**Example.**  $A = [0, 1) \cup \{2\}$ .  $1 \in A', 2 \notin A', 2 \in \overline{A}$

**Prop 2.3.3**  $x \in A' \implies N(x, \epsilon) \cap (A \setminus \{x\})$  는 무한집합이다.

**Proof.** 유한집합이라고 가정하자.  $N(x, \epsilon) \cap (A \setminus \{x\}) = \{x_1, \dots, x_n\}$  이라 할 수 있다. Set  $\delta = \min\{\|x - x_i\| : \forall i\}$ . Then  $N(x, \delta) \cap (A \setminus \{x\}) = \emptyset$ . 모순.

그래서 사실은 공집합이 아닌 것으로 정의했지만 **사실은** 무한집합이다.

**Remark.**  $A' \neq \emptyset \implies A$ 는 무한집합.

(대우)  $A$ 가 유한집합이면 극한점이 존재하지 않는다. (2.2 보기 4)

(역) 거짓.  $A = \{1, 2, \dots\}$  이면  $A' = \emptyset$ .

그러면 역이 언제 성립하나요? 다음 단원 내용!

**Definition.** Convergence in  $\mathbb{R}^d$

Let  $\langle x_n \rangle$  be a sequence in  $\mathbb{R}^d$ .

$$\lim_{n \rightarrow \infty} x_n = x \iff \forall \epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies \|x_n - x\| < \epsilon)$$

**Exercise.**  $x_n = (x_n^{(1)}, \dots), x = (x^{(1)}, \dots)$  일 때,  $x_n \rightarrow x \iff \forall i, x_n^{(i)} \rightarrow x^{(i)}$

**Notation.**  $A \subset \mathbb{R}^d; \langle x_n \rangle$  is a sequence in  $A \iff \forall n, x_n \in A$

**Theorem 2.2.2**

(1)  $x \in A' \iff \exists \langle x_n \rangle$  in  $A \setminus \{x\}$  such that  $x_n \rightarrow x$

(2)  $x \in \overline{A} \iff \exists \langle x_n \rangle$  in  $A$  such that  $x_n \rightarrow x$

**Proof.**

(1) ( $\implies$ )  $x_n \in N(x, \frac{1}{n}) \cap (A \setminus \{x\})$  이라 하자. (공집합이 아니므로 이러한 원소가 존재한다.)  
그러면  $\|x_n - x\| < 1/n$  이므로  $x_n$  은  $x$  로 수렴한다. 그리고  $x_n \in A \setminus \{x\}$  이므로 수열이  $A \setminus \{x\}$  에 있다.

(2) Left as exercise. Replace  $A \setminus \{x\}$  with  $A$ .

**Theorem 2.2.3.** The following are equivalent.

- (1)  $F$  is closed.
- (2)  $F' \subset F$ .
- (3)  $F = \overline{F}$
- (4) For a sequence  $\langle x_n \rangle$  in  $F$ ,  $\lim_{n \rightarrow \infty} x_n = x \implies x \in F$ .

**Proof.**

- (1)  $\iff$  (3) ( $\overline{F}$ : smallest closed set containing  $F$ .)
- (2)  $\iff$  (3) 은 자명.
- (1)  $\iff$  (4) by the above theorem. (Thm 2.2.2)

**Applications.**

- (1)  $A'$  is closed.

*Proof.* We want to show that  $(A')' \subset A'$ .

We want to show:  $x \in (A')' \implies x \in A'$ .

( $A'$  이 공집합이면 자명. 공집합이 아니라고 가정하고...)

Given  $\epsilon > 0$ ,  $N(x, \epsilon) \cap (A' \setminus \{x\}) \neq \emptyset$ . Take an element  $y \in A'$  from this set. Now set  $\delta = \min\{\|x - y\|, \epsilon - \|x - y\|\}$  then we have  $N(y, \delta) \cap (A \setminus \{y\}) \neq \emptyset$ . ( $\because y \in A'$ )  
 $z \in N(y, \delta) \cap (A \setminus \{y\})$  라 하자.

(a)  $z \in A \setminus \{y\} \subset A$ .

(b)  $\|x - z\| \leq \|x - y\| + \|y - z\| < \|x - y\| + \delta \leq \epsilon$  ( $z \in N(y, \delta)$ )

(c)  $\|x - z\| \geq \|x - y\| - \|y - z\| > \|x - y\| - \delta \geq 0$  (By the choice of  $\delta$ .) Thus  $x \neq z$ .

Therefore  $z \in N(x, \epsilon)$  (by (b)),  $z \in A \setminus \{x\}$  (by (a), (c)).

$x \in A'$  since  $N(x, \epsilon) \cap (A \setminus \{x\})$  is not empty.

- (2)  $A \subset \mathbb{R}$ : closed and bounded  $\implies \inf A = \min A$ ,  $\sup A = \max A$ . (Existence)

*Proof.* Let  $\sup A = x \notin A$ . ( $\sup A \in A$  이면 자명)

*Claim.*  $x \in A'$ .

*Proof of Claim.*  $\forall \epsilon > 0$ ,  $N(x, \epsilon) = (x - \epsilon, x + \epsilon)$

$x = \sup A$  이므로  $x - \epsilon$  is not an upper bound.

$\exists y$  such that  $y \in (x - \epsilon, x)$

$y \in N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset$  이므로  $x$  는 극한점.

따라서  $x \in A' \subset A$  (closed set 이므로 Thm 2.2.3 (2)) 모순.

$\sup A \in A$  이므로 이 값이 최댓값이다.

## 2.3 유계집합과 코시수열

핵심: Thm 2.3.4, Thm 2.3.7

**Definition.**  $\langle x_n \rangle$ : 유계수열(bounded sequence)  $\iff \exists M > 0$  s.t.  $\|x_n\| \leq M$  for all  $n \in \mathbb{N}$ .

**Definition.**  $n_1 < n_2 < \dots$  : sequence in  $\mathbb{N}$  이라 하자.  $\langle x_{n_k} \rangle_{k=1}^\infty = (x_{n_1}, x_{n_2}, \dots)$  를  $\langle x_n \rangle$ 의 부분수열(subsequence)이라 한다.

**Theorem 2.3.4** (Bolzano-Weierstrass Theorem)

If  $\langle x_n \rangle$  is bounded, there exists a convergent subsequence of  $\langle x_n \rangle$ .

**Idea of Proof.** Equivalent formulation for sets.

**Definition.** Set  $A$  is bounded  $\iff \exists M > 0$  such that  $\|x\| < M$  for all  $x \in A$ .

**Theorem 2.3.2** (Equivalent of 2.3.4)  $A$ 가 유계이고 무한집합이면,  $A' \neq \emptyset$ .

**Remark.**  $A' \neq \emptyset \implies A$ : 무한집합.

역이 성립하기 위해서는  $A$ 가 유계라는 조건이 필요하다.

극한점이 중요한 이유는 계속 수열과 관련이 있기 때문이다.

**Example.**  $A = \{1/n : n \in \mathbb{N}\}$  을 고려하는 것은 수열  $x_n = 1/n$  을 고려하는 것이나 마찬가지이다. 이 수열  $x_n$  이  $x$  로 수렴하는 것은  $A' = \{x\}$  와 동치이다. (Hence the name “limit point”) 이로부터  $x \in A' \iff$  Exists a subsequence of  $\langle x_n \rangle$  in  $A \setminus \{x\}$  converging to  $x$ .

### Proof of 2.3.2

(1) **Lemma 2.3.1** 축소구간정리 in  $\mathbb{R}^d$ .

$B$  is a closed box in  $\mathbb{R}^d \iff B = I_1 \times I_2 \times \dots \times I_d$ , where  $I_i = [a_i, b_i]$  for  $i = 1, \dots, d$ . ( $I_i$  is a closed and bounded interval.)

$$B_1 \supset B_2 \supset \dots \implies \bigcap_{n=1}^{\infty} B_n \neq \emptyset$$

**Proof.** 각 ‘좌표’  $I_i$  별로 1차원 축소구간정리를 적용하면 된다.

(2) **Divide and Conquer Strategy**

$B$ : Box 일 때,  $\text{diam}(B) = \sup\{\|x - y\| : x, y \in B\} = \sqrt{(a_1 - b_1)^2 + \dots + (a_d - b_d)^2}$

**Claim.** There exists closed boxes  $B_1, B_2, \dots$  s.t.

(a)  $B_1 \supset B_2 \supset \dots$

(b)  $\text{diam} B_n = \frac{1}{2^{n-1}} \text{diam} B_1$

(c)  $B_n \cap A$ : 무한집합

**Proof.** (Induction)  $n = 1$ ;  $B_1$ : 충분히 커서  $A \subset B_1$  인 box 를 잡으면 된다.

Suppose we have  $B_1, \dots, B_n$ ;  $B_n$ 을  $2^d$  등분하면 적어도 하나는  $A$ 의 원소를 무한개 포함하고 있다. 그 집합을  $B_{n+1}$  으로 잡는다. (비둘기집의 원리)

이제  $x \in \bigcap_{n=1}^{\infty} B_n$  으로 잡으면 (축소구간정리에 의해 잡을 수 있다)  $x \in A'$ . ( $A' \neq \emptyset$ )

$\because \forall \epsilon > 0$ ,  $\text{diam} B_n < \epsilon$  인  $N \in \mathbb{N}$  을 찾아  $n \geq N$  일 때 부등식이 성립하도록 할 수 있다.

이러한  $n$  들에 대하여  $B_n \subset N(x, \epsilon)$ . 그러면  $N(x, \epsilon) \cap (A \setminus \{x\}) \supset B_n \cap (A \setminus \{x\})$ .

April 3rd, 2019

우리가 지금 2.3 을 하고 있는데, 2 가지 중요한 결과가 있어요.

**Theorem 2.3.4**  $\langle x_n \rangle$  이 bounded 이면 수렴하는 부분수열을 갖는다.<sup>1</sup>

**Theorem 2.3.2**  $A$ 가 유계인 집합이고 무한집합이면 극한점을 가진다.  $A' \neq \emptyset$   
증명은 축소구간정리를 박스로 확장해가지고 분할 정복하면 된다.

**Recall 2.3.3**  $x \in A' \implies N(x, \epsilon) \cap (A \setminus \{x\})$  는 무한집합이다.

**Proof of 2.3.4.**  $A = \{x_1, x_2, \dots, x_n\}$  라고 하면 이 집합은 유계이다. (수열이 유계이므로)

(1)  $A$ 가 유한집합: 자명.

$\exists x$  such that  $x$  appears infinitely many times in  $\langle x_n \rangle$ . (PHP) 이 경우에는 부분수열을  $x, x, \dots$  로 잡으면 된다. 이는 수렴하는 부분수열이다.

(2)  $A$ 가 무한집합<sup>2</sup>

$A' \neq \emptyset$  이므로  $\alpha \in A'$  이라 하자.

**Claim.**  $\exists n_1 < n_2 < \dots$  such that  $\|x_{n_k} - \alpha\| < 1/k$ .

**Proof.** (첨자들이 증가하면서 가까워져야 한다는 것이 유일하게 tricky 한 부분이다. 귀납법을 사용하자.)  $k = 1$ :  $x_{n_1} \in N(\alpha, 1) \cap (A \setminus \{\alpha\})$  로 잡으면 된다.

$x_{n_1}, \dots, x_{n_k}$  를 잡았다고 가정:  $N(\alpha, \frac{1}{k+1}) \cap (A \setminus \{\alpha\})$  에서  $x_{n_{k+1}}$  를 잡아야 하는데 이 집합은 무한집합이다. (Recall 2.3.3) 이 집합에서 첨자가  $n_k$ 보다 큰 항이 반드시 존재하므로 그 중 하나를  $x_{n_{k+1}}$  이라 잡으면 된다.

따라서  $\lim_{k \rightarrow \infty} x_{n_k} = \alpha$  (Check as exercise)

**Application.** (Characterization of  $\limsup$  and  $\liminf$ )

$x_n$  이 bounded 이면,  $A = \{x : \exists \text{ subsequence of } x_n \text{ converging to } x\}$ . 이 때 Theorem 2.3.4에 의해  $A \neq \emptyset$  임을 증명하였다.

(1)  $A$ : closed and bounded  $\implies \max(A), \min(A)$  가 존재한다.

**Proof.**  $B = \{x_1, x_2, \dots\}$ ,  $C = \{\langle x_n \rangle \text{ 에 무한 번 나타나는 수} \}$  로 잡자.  $A = B' \cup C$ ,  $C \subset B$ ,  $C' \subset B'$  임을 확인해보라! 이를 이용하면  $B' \cup C = (B' \cup C') \cup C = B' \cup (C' \cup C) = B' \cup \overline{C}$  가 되어 닫힌집합의 합집합은 닫힌 집합이다.  $A$ 는 closed and bounded 이다.

(2)  $\limsup x_n = \max(A)$ ,  $\liminf x_n = \min(A)$

(부분수열이 가질 수 있는 극한값들 중 가장 큰 값이  $\limsup$ , 가장 작은 값이  $\liminf$ )

---

<sup>1</sup>증명이 가장 테크니컬 해요!

<sup>2</sup>이제 Thm 2.3.2 를 사용할 수 있다. 사실 경우를 나눈 것은 예외적인 case 를 처리하기 위한 것이었다.

**Proof.** Recall

$$\limsup x_n = \alpha \iff \begin{cases} \text{(i) } \forall \epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies x_n < \alpha + \epsilon) \\ \text{(ii) } \forall \epsilon > 0, x_n > \alpha - \epsilon \text{ for infinitely many } n \end{cases}$$

(a) 부분수열  $\langle x_{n_k} \rangle \rightarrow \beta$  이면 (i)에 의해  $k \geq N \implies x_{n_k} < \alpha + \epsilon$  이 되어  $\beta \leq \alpha + \epsilon$ .  $\beta \leq \alpha$ . 그러므로  $\max(A) \leq \alpha$  이다.

(b)  $\forall \epsilon > 0$ , (i), (ii)에 의해  $x_n \in (\alpha - \epsilon, \alpha + \epsilon)$  인  $n$  이 무한히 많다. 이 유계인 구간에 속하는 수열의 항들에 대해 부분수열을 잡아 (further subsequence)  $\gamma$  로 수렴하도록 할 수 있다. (Theorem 2.3.4) 그러면  $\langle x_{m_k} \rangle \rightarrow \gamma \in [\alpha - \epsilon, \alpha + \epsilon]$ . 따라서  $\alpha - \epsilon \leq \gamma \leq \max(A)$  가 되어  $\alpha \leq \max(A)$ .

따라서  $\max(A) = \alpha$ .

**Definition.**  $\langle x_n \rangle$ : Cauchy Sequence  $\iff \forall \epsilon > 0, \exists N \text{ s.t. } [m, n \geq N \implies \|x_m - x_n\| < \epsilon]$

**Prop 2.3.6, Thm 2.3.8**  $\langle x_n \rangle$ : convergent  $\iff \langle x_n \rangle$ : Cauchy sequence<sup>3</sup>

**Proof.** ( $\implies$ ) 자명.  $\|x_m - x_n\| \leq \|x_m - \alpha\| + \|x_n - \alpha\| < \epsilon/2 + \epsilon/2 = \epsilon$  인  $m, n \geq N$  존재.

( $\impliedby$ ) 수렴 값이 없는 상태에서 증명해야 한다. 먼저 수렴 값을 찾아보자.

(1)  $\langle x_n \rangle$  is bounded.

**Proof.**  $\exists N$  s.t.  $\|x_m - x_n\| < 1$  for all  $m, n \geq N$ .

Set  $M = \max\{\|x_1\|, \dots, \|x_{N-1}\|, \|x_N\| + 1\}$ . ( $\|x_m\| < \|x_N\| + 1$ )

따라서  $\|x_n\| \leq M$  for all  $n \in \mathbb{N}$ .

(2) There exists a subsequence  $\langle x_{n_k} \rangle$  converging to some  $\alpha$ . (Thm 2.3.4)

(3)  $\langle x_n \rangle$  converges to  $\alpha$ .

**Proof.**  $\epsilon > 0$  에 대해,

(a) 코시 수열의 성질에 의해  $\exists N_1$  s.t.  $\|x_m - x_n\| < \epsilon/2$  for all  $m, n \geq N_1$ .

(b) 부분수열이  $\alpha$ 로 수렴하므로  $\exists N_2$  s.t.  $\|x_{n_k} - \alpha\| < \epsilon/2$  for all  $k \geq N_2$ .

Let  $N = \max\{N_1, N_2\}$ .  $n \geq N, n_N \geq n_{N_1} \geq N_1$  이므로,

$$n > N \implies \|x_n - \alpha\| \leq \|x_n - x_{n_N}\| + \|x_{n_N} - \alpha\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

---

<sup>3</sup>중간고사 전 까지 가장 중요한 정리.



**Remark.** 우리의 여정을 돌아보자.

(1) Archimedes' Principle 을 가정하면

Completeness Axiom  $\implies$  Monotone Convergence Theorem  $\implies$  축소구간정리  $\implies$   
Bolzano-Weierstrass Theorem  $\implies$  **Cauchy Convergent Theorem**<sup>4</sup>

(Exercise)  $\implies$  Completeness Axiom

(2) **Example.**  $X = C([0, 1])$ . (Set of functions that are continuous in  $[0, 1]$ ) How would we define  $\|f - g\|$ ?  $\int_0^1 |f(x) - g(x)| dx$  ?  $\max\{|f(x) - g(x)| : x \in [0, 1]\}$  ? Only the second choice gives completeness for  $X$ .

(3) **Convergence Test** without limit value. (**Theorem 2.3.9**)

$\sum_{n=1}^{\infty} a_n$  is convergent  $\iff \forall \epsilon > 0, \exists N$  s.t.  $(n > m \geq N \implies |a_{m+1} + \cdots + a_n| < \epsilon)$

**Proof.** Trivial.

**Definition.**  $\sum a_n$  is **absolutely convergent**  $\iff \sum |a_n|$  is convergent

**Theorem.** An absolutely convergent series converges.

**Proof.** Suppose  $\sum |a_n|$  converges. For  $\forall \epsilon > 0$ , there exists  $N$  such that  $||a_{m+1}| + \cdots + |a_n|| < \epsilon$  for all  $m, n \geq N$ . Therefore, for  $m, n \geq N$ ,

$$|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \epsilon$$

and  $\sum a_n$  converges.

---

<sup>4</sup>In any metric spaces, this is the condition for completeness.

**April 5th, 2019**

**Theorem.**  $\overline{A \cup B} = \overline{A} \cup \overline{B}$

**Proof.** ( $\subset$ ) Trivial.

( $\supset$ )  $A \subset \overline{A}, B \subset \overline{B} \implies A \cup B \subset \overline{A} \cup \overline{B} \implies \overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$ . The closure of a closed set is itself.

**6. (2)**  $a_n = \cos \sqrt{2019 + n^2 \pi^2}$

Consider  $\delta > 0$ , such that

$$(n\pi - \delta)^2 < 2019 + n^2 \pi^2 < (n\pi + \delta)^2$$

$$-2n\pi < \frac{2019}{\delta} \pm \delta < 2n\pi$$

We can find large enough  $N$  such that the above inequality holds for  $n \geq N$ .

Now we want  $b_n = \sqrt{2019 + n^2 \pi^2}$  bounded by  $n\pi \pm \delta$ .

$n \geq N, n \text{ even} \implies n\pi - \delta < b_n < n\pi + \delta$

$\implies 1 \geq a_n > 1 - \epsilon$

$n \geq N, n \text{ odd} \implies -1 \leq a_n < -1 + \epsilon$

**Problem 2.3.5**

$$(1) \ x_{n+2} = \frac{x_n + x_{n+1}}{2}$$

$$(2) \ x_{n+1} = x_n + x(-1)^n \frac{1}{3n+1}$$

**Solution.**

(1) Write  $x_{n+2} - x_{n+1} = a(x_{n+1} - x_n)$  and observe that  $a = -1/2$ . Write as

$$x_n = x_{n-1} + \left(-\frac{1}{2}\right)^{n-2} (x_2 - x_1)$$

Then we have

$$x_n = x_2 + \sum_{k=1}^{n-2} \left(-\frac{1}{2}\right)^k (x_2 - x_1)$$

This series converges to  $\frac{2x_2 + x_1}{3}$ .

(2) This is an alternating series. Write as

$$x_n = x_1 + \sum_{k=1}^{n-1} (-1)^k \frac{x}{3n+1}$$

By alternating series test, the second summation term converges, and the series converges to  $x_1$ .

Since a converging sequence is a Cauchy sequence,  $x_1, x_2$  can be any real number.

April 8th, 2019

Section 2.3: Bolzano-Weierstrass Theorem, Cauchy Convergent Theorem

In section 2.4, we will be studying about Convergence Tests.

정

## 2.4 급수의 수렴판정

**Cor 2.3.9.**  $\sum_{n=1}^{\infty} a_n$  is convergent  $\iff s_n = \sum_{k=1}^n a_k, \langle s_n \rangle$  is convergent  $\iff \langle s_n \rangle$  is Cauchy.

(1)  $\sum_{n=1}^{\infty} a_n$  is convergent  $\implies \lim_{n \rightarrow \infty} a_n = 0$ .

(2)  $\sum_{n=1}^{\infty} |a_n|$  is convergent  $\implies \sum_{n=1}^{\infty} a_n$  convergent.

**Theorem 2.4.3** (Comparison Test) Suppose  $\sum b_n$  converges. If  $0 \leq a_n \leq b_n$  for all  $n \in \mathbb{N}$ ,  $\sum a_n$  converges.

**Proof** Let  $M = \sum b_n$ ,  $s_n = \sum_{k=1}^n a_k$ .  $s_n$  is increasing and  $s_n$  is bounded by  $M$ .  $s_n$  is convergent by Monotone Convergence Theorem.

**Theorem.** Suppose sequences  $a_n, b_n$  satisfy  $0 \leq |a_n| \leq b_n$ <sup>5</sup> and  $\sum b_n$  converges. Then  $\sum a_n$  is convergent.<sup>6</sup>

**Proof.** By comparison test and absolute convergence.

**Prop 2.4.4** (Root Test) Suppose  $\alpha = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ .

If  $\alpha < 1$ ,  $\sum a_n$  converges. If  $\alpha > 1$ ,  $\sum a_n$  diverges.

**Proof.**

(1)  $\alpha < 1$ . Take  $\epsilon > 0$  such that  $\alpha < \alpha + \epsilon < 1$ . Then there exists  $N$  such that  $|a_n|^{1/n} < \alpha + \epsilon$  for all  $n \geq N$ . Therefore  $|a_n| < (\alpha + \epsilon)^n$ . Since  $\alpha + \epsilon < 1$ ,  $\sum (\alpha + \epsilon)^n$  converges. Apply the comparison test to see that  $\sum a_n < \infty$ .

(2)  $\alpha > 1$ . Take  $\epsilon > 0$  such that  $\alpha > \alpha - \epsilon > 1$ . Then  $|a_n|^{1/n} > \alpha - \epsilon$  for infinitely many  $n$ . Then  $|a_n| > (\alpha - \epsilon)^n > 1$ . Therefore  $\lim a_n \neq 0$ .  $\sum a_n$  diverges.

**Prop 2.4.5** (Ratio Test) Suppose  $a_n \neq 0$ . Let  $\beta = \limsup |a_{n+1}/a_n|$ ,  $\gamma = \liminf |a_{n+1}/a_n|$ .

If  $\beta < 1$ ,  $\sum a_n$  converges. If  $\gamma > 1$ ,  $\sum a_n$  diverges.

**Proof.**

(1)  $\beta < 1$ . Take  $\epsilon > 0$  such that  $\beta < \beta + \epsilon < 1$ . Then  $\exists N$  s.t.  $|a_{n+1}/a_n| < \beta + \epsilon$  for  $n \geq N$ .  
 $\implies |a_n| = |a_N| |a_{N+1}/a_N| \cdots |a_n/a_{n-1}| < |a_N| (\beta + \epsilon)^{n-N}$ .  
Set  $b_n = |a_N| (\beta + \epsilon)^{n-N}$  and apply comparison test to see that  $\sum a_n < \infty$ .

---

<sup>5</sup>Note that this condition can fail for finitely many  $n$ .

<sup>6</sup> $a_n$  may be a very complex expression, but we want  $b_n$  to be simple, an expression we know that it is convergent.

- (2)  $\gamma > 1$ . Take  $\epsilon > 0$  such that  $\gamma > \gamma - \epsilon > 1$ . Then  $\exists N$  s.t.  $|a_{n+1}/a_n| > \gamma - \epsilon$  for  $n \geq N$ . Then we see that  $|a_n|$  is increasing for  $n \geq N$ . Thus  $a_n$  cannot converge to 0.  $\sum a_n$  is divergent.

**Remark.** If the above limits (ratio, root) exist, elementary tests can be applied. But if the limits turn out to be 1, the test fails. (ND: Non-Deterministic) Check it for  $\sum 1/n, \sum 1/n^2$ . Also, these are *weak tests*. For most of the series, the limit is 1. Moreover...

**Theorem 2.4.6** Suppose  $a_n \neq 0$ .

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|$$

Thus if the root test works, ratio test also works.<sup>7</sup>

**Proof.** We only need to prove the last inequality.

Let  $\beta = \limsup |a_{n+1}/a_n|$ ,  $\forall \epsilon > 0$ .  $\implies \exists N$  s.t.  $|a_{n+1}/a_n| \leq \beta + \epsilon$  for  $n \geq N$ . Then if  $n \geq N$ ,  $|a_n| \leq |a_N| (\beta + \epsilon)^{n-N}$ . (Similar to proof of 2.4.5) Then

$$|a_n|^{1/n} \leq (\beta + \epsilon) \left( \frac{|a_n|}{(\beta + \epsilon)^N} \right)^{1/n}$$

and take  $\limsup$  on both sides, then  $\limsup |a_n|^{1/n} \leq \beta + \epsilon$ .

**Example.**  $\langle a_n \rangle = \begin{cases} 1/2^n & n \text{ odd} \\ 1/2^{n-2} & n \text{ even} \end{cases}$

Check that  $\limsup |a_n|^{1/n} = 1/2 < 1$ , and the series  $\sum a_n$  converges by the root test.

But if we use the ratio test here,  $\limsup$  value is 2 and  $\liminf$  value is  $1/8$ .<sup>8</sup> The ratio test does not tell us anything about the convergence. Also note that the series converges to 2.

**Prop 2.4.1** (Rearrangement)  $a_n \geq 0$ .<sup>9</sup> Suppose a bijection  $r : \mathbb{N} \rightarrow \mathbb{N}$  exists.

$$(1) \sum_{n=1}^{\infty} a_n = s \iff \sum_{n=1}^{\infty} a_{r(n)} = s$$

$$(2) \sum_{n=1}^{\infty} a_n = \infty \iff \sum_{n=1}^{\infty} a_{r(n)} = \infty$$

**Proof.**

- (1) ( $\implies$ ) Let  $t_n = \sum_{k=1}^n a_{r(k)}$ . Then  $t_n$  is increasing and bounded by  $s$ . Thus  $t_n$  converges by MCT, and  $\lim t_n \leq s$ .

$$s_n = \sum_{k=1}^n a_k \leq \sum_{n=1}^{\infty} a_{r(n)} = t = \lim t_n. \quad (a_n \geq 0 \text{ was used here.})$$

$$(\iff) \text{ Use } r^{-1}(n).$$

<sup>7</sup>The limit for the ratio test is much easier to calculate than the root test. That's why we use the ratio test.

<sup>8</sup>The ratios are: 2,  $1/8$ , 2,  $1/8$  ...

<sup>9</sup>This is the important condition.

(2) Contraposition of (1).

**Prop 2.4.2** (Alternating Series Test) For a given sequence  $x_n$ , suppose the following holds.

- $x_n$  is decreasing.
- $\lim x_n = 0$ .

Then the series  $\sum_{k=1}^{\infty} (-1)^{k-1} x_k$  is convergent.

**Proof.** Let  $s_n = \sum_{k=1}^n (-1)^{k-1} x_k$ . For  $m < n$ ,

$$|s_n - s_m| = |(-1)^m x_{m+1} + \cdots + (-1)^{n-1} x_n| = |x_{m+1} - x_{m+2} + \cdots \pm x_n| \stackrel{(*)}{\in} [0, x_{m+1}]$$

$$\begin{aligned} (*) : x_{m+1} - x_{m+2} + \cdots + x_n &= (x_{m+1} - x_{m+2}) + \cdots + (x_{n-2} - x_{n-1}) + x_n \geq 0 \\ &= x_{m+1} - (x_{m+2} - x_{m+3}) - \cdots - (x_{n-1} - x_n) \leq x_{m+1} \end{aligned}$$

Check for the case with last term  $-$ .

Now,  $\forall \epsilon > 0$ , find  $N$  such that  $|x_n| < \epsilon$  for  $n \geq N$ . Then for  $n > m \geq N$ ,  $|s_n - s_m| \leq x_{m+1} < \epsilon$ .

Thus  $\langle s_n \rangle$  is a Cauchy sequence and the given series converges.

**Example.**  $a_n = (-1)^{n-1}/n$ .  $\sum a_n$  converges by alternating series test and converges to  $\log 2$ .

**Remark.** The rearrangement of the above example may not converge, or converge to a different value than  $\log 2$ .

Exam: 1.1 - 2.6

After the midterms we will be covering functions and continuity.

Chapter 1 has been about  $\mathbb{R}$ , and in Chapter 2, we have talked about subsets of  $\mathbb{R}^n$ .

2.1: What is  $\mathbb{R}^n$  ? Vector Space, IPS, Metric Space, Normed Space...

2.2: Open, closed sets

2.3: Bounded sets and Cauchy sequences

(2.4: Convergence Tests)

2.5: Compact Sets

2.6: Connected Sets

April 10th, 2019

## 2.5 Compact Set

**Definition.**  $\{U_i : i \in I\}$  ( $I$  is the index set,  $U_i \subset \mathbb{R}^d$ ) is called “family of sets”.

- (1)  $\{U_i : i \in I\}$  is a **cover** of  $K \subset \mathbb{R}^d \iff K \subset \bigcup_{i \in I} U_i$ .
- (2)  $\{U_i : i \in I\}$  is a **open cover**  $\iff U_i$  are open for  $\forall i$ .
- (3)  $J \subset I$ ,  $\{U_i : i \in J\}$  is called a **subcover** of  $\{U_i : i \in I\} \iff K \subset \bigcup_{i \in J} U_i$ .

**Definition.**  $K \subset \mathbb{R}^d$  is **compact**  $\iff$  Any open cover of  $K$  has finite subcover.

**Example.**

- (1)  $\mathbb{N}$  is not compact. Set  $U_k = (k - 1/2, k + 1/2)$ , then  $\{U_k : k \in \mathbb{N}\}$  is a (open) cover of  $\mathbb{N}$ . But there are no finite subcover.
- (2)  $A = (0, 1)$  is not compact. Set  $U_k = (1/k, 1)$ , then because  $\bigcup_{k=1}^{\infty} U_k = (0, 1)$ ,  $\{U_k : k \in \mathbb{N}\}$  is a (open) cover of  $A$ . But there are no finite subcover.  $\bigcup_{i=1}^m U_{k_i} = U_{k_m} = (1/k_m, 1)$ , which cannot contain  $(0, 1)$ .
- (3)  $A = \{a_1, a_2, \dots, a_m\} \subset \mathbb{R}^d$  is compact.  $\{U_i : i \in I\}$  be a cover of  $A$ . There exists  $i_1, \dots, i_m \in I$  such that  $a_k \in U_{i_k}$  for  $k = 1, \dots, m$ . Then  $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$  is a finite subcover of  $A$ .

Main Theorem: **Heine-Borel Theorem**

$$K \text{ is compact} \iff K \text{ is bounded and closed.}$$

**Remark.**

- (1) This is a part of Thm 2.5.4
- (2) Proof: Prop 2.5.1, Thm 2.5.2, Prop 2.5.3
- (3) **Characterization of compact sets in  $\mathbb{R}^d$ .**<sup>10</sup>

---

<sup>10</sup>Compact Set 을 이 단순한 공간 안에서는 characterize 할 수 있다!

**Proof.**

( $\implies$ ) (Prop 2.5.1)

(1) *Is  $K$  bounded?*

Set  $U_k = N(0, k)$ . Then  $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d$ . Thus  $\{U_k : k \in \mathbb{N}\}$  is an open cover of  $K$ . There exists a finite subcover  $U_{k_1}, \dots, U_{k_m}$  ( $k_1 < \dots < k_m$ ) of  $K$ . Then we have  $K \subset \bigcup_{i=1}^m U_{k_i} = U_{k_m} = N(0, k_m)$ . Therefore  $K$  is bounded.

(2) *Is  $K$  closed?*

Suppose  $x \in K^C$ . Set  $U_k = \{y : \|y - x\| > 1/k\}$ . Then  $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d \setminus \{x\} \supset K$ . (Open cover) There exists a finite subcover  $U_{k_1}, \dots, U_{k_m}$  of  $K$ .  $K \subset \bigcup_{i=1}^m U_{k_i} = U_{k_m}$ . Therefore  $K^C \supset U_{k_m}^C = \{y : \|y - x\| \leq 1/k_m\} \supset N(x, 1/k_m)$ . Thus  $K^C$  is open,  $K$  is closed.

( $\impliedby$ )

(1) (Theorem 2.5.2) *Closed box is compact.*

$B = I_1 \times \dots \times I_d$ ,  $I_i = [a_i, b_i]$ . Let  $\{U_i : i \in I\}$  is an open cover of  $B$ .

(Contradiction) Suppose there is no finite subcover of  $B$ .

**Claim.** There exists  $B = B_1 \supset B_2 \supset \dots$  (closed boxes) such that

- $\text{diam}(B_n) = \frac{1}{2^{n-1}} \text{diam}(B_1)$
- There is no finite subcover of  $\{U_i : i \in I\}$  covering  $B_n$ .

By Lemma 2.3.1, there exists  $x \in \bigcap_{n=1}^{\infty} B_n$ . Since  $x \in B$ ,  $\exists U_i$  such that  $x \in U_i$ . Then  $\exists \epsilon > 0$  such that  $N(x, \epsilon) \subset U_i$ .<sup>11</sup> Set  $\frac{1}{2^{n-1}} \text{diam}(B_1) < \epsilon$ .

If  $y \in B_n \implies \|x - y\| \leq \text{diam}(B_n) < \epsilon \implies y \in N(x, \epsilon)$ . Then  $B_n \subset N(x, \epsilon) \subset U_i$ , contradiction.

(2)  *$K$ : compact,  $F \subset K$ ,  $F$  is closed  $\implies F$ : compact.*

Let  $\{U_i : i \in I\}$  be an open cover of  $F$ . Then  $\{U_i : i \in I\} \cup \{F^C\}$  is an open cover of  $K$ . Because  $K$  is compact, there exists a finite subcover of  $K$ . There are two cases.

- (a)  $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$ : This is already a finite subcover of  $F$ .
- (b)  $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}, F^C\}$ : Since  $F^C$  does not cover  $F$ ,  $U_{i_k}$  must cover  $F$ .

(3) *Closed and bounded set is compact.*

Suppose  $K$  is bounded and closed. There exists a closed box  $B$  that contains  $K$ . Thus  $B$  is compact by (1),  $K$  is a closed subset of  $B$ . Then by (2),  $K$  is compact.

Theorem 2.5.2 가 가장 non-trivial 한 부분이다.

---

<sup>11</sup> $n$ 이 충분히 크면 ball 안에 box 가 들어가고 box 는  $U_i$  안에 있다? Claim 의 2번째에 모순.

**Theorem 2.5.4** The following are equivalent.

- (1)  $K$  is compact.
- (2)  $K$  is bounded and closed.
- (3) If  $A$  is an infinite subset of  $K$ ,  $\emptyset \neq A' \subset K$ .
- (4) For a sequence  $\langle x_n \rangle$  in  $K$ , there exists a convergent subsequence whose limit is in  $K$ .

**Proof.**

- (1)  $\iff$  (2) by Heine-Borel Theorem.
- (2)  $\implies$  (3) Suppose  $A$  is infinite and bounded. ( $A \subset K$ ) By Bolzano-Weierstrass,  $A' \neq \emptyset$ .  
 $A' \subset A' \cup A = \overline{A} \subset K$ . ( $\overline{A}$  is the smallest closed set containing  $A$ ,  $A \subset K$ .)
- (3)  $\implies$  (4) Let  $A = \{x_1, x_2, \dots\}$

(1) If  $A$  is finite, trivial. (Take a constant subsequence, which constant  $\in K$ .)

(2) If  $A$  is infinite,  $x \in A' \subset K$  by (3). ( $x \in A'$  by Thm 2.3.4)

(4)  $\implies$  (2)

(1)  $K$  is bounded.

(Contradiction) Suppose  $K$  is not bounded. Then  $\forall n \in \mathbb{N}$ , there exists  $x_n \in K$ ,  $\|x_n\| \geq n$ .  
There are no convergent subsequences, contradiction.

(2)  $K$  is closed.

(Contradiction) Suppose  $K$  is not closed.

(a)  $K$ : finite  $\rightarrow K$ : closed  $\rightarrow$  Contradiction.

(b)  $K$ : infinite  $\rightarrow K$ : infinite and bounded  $\xrightarrow{\text{B-W}} K' \neq \emptyset$

*Note.*  $K' \subset K \iff K$ : closed.

Then if  $K'$  is not a subset of  $K$ <sup>12</sup>, there exists  $x \in K' \setminus K$ . Since  $x \in K'$ , there exists a sequence  $\langle x_n \rangle$  in  $K \setminus \{x\}$  ( $= K$ )<sup>13</sup> converging to  $x$ . Thus for a subsequence of  $\langle x_n \rangle$ , its limit must be in  $K$ . But  $x$  is the only possible limit value.  $x \in K$ . Contradiction.

---

<sup>12</sup>Contraposition

<sup>13</sup> $x \notin K$



April 12th, 2019

**Problem 2.4.7** (바)  $\sum \frac{1}{n^p - n^q}$  ( $0 < q < p$ )

$0 < n^p - n^q \leq n^p$  이므로  $1/n^p \leq 1/(n^p - n^q)$  가 되어  $p \leq 1$  이면 발산한다.

충분히 큰  $N$ 에 대하여  $n \geq N$  일 때마다  $n^p - n^q \geq n^p/2$  가 되게 할수 있다. (이 때  $n^p/2 \geq n^q$  이므로  $n^{p-q} \geq 2$  가 되어  $N$  을 잡을 수 있다) 비교판정법에 의해 수렴한다.

**Problem 2.7.12** Given  $\langle a_n \rangle$  such that  $\lim a_n = a$ , show that  $\sigma_n = \frac{a_1 + \cdots + a_n}{n}$  also converges to  $a$ .

**Problem 2.7.13**  $r < 1$ ,  $\|x_{n+2} - x_{n+1}\| \leq r \|x_{n+1} - x_n\|$ . Show that  $\langle x_n \rangle$  is a Cauchy sequence.

**Proof.**  $\|x_{n+1} - x_n\| \leq r^{n-1} \|x_2 - x_1\| = r^{n-1} A$ , for  $A \in \mathbb{R}$ . Given  $\epsilon > 0$ , exists  $N$  such that for all  $n \geq N$ ,  $\|x_{n+1} - x_n\| < A r^{n-1} < \epsilon$ . Then we have

$$\begin{aligned} m > n \geq N \Rightarrow \|x_n - x_m\| &\leq \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq \|x_{n+1} - x_n\| (1 + r + r^2 + \cdots) < \frac{\epsilon}{1 - r} \end{aligned}$$

**Remark.** Counterexample for  $\|x_{n+2} - x_{n+1}\| < \|x_{n+1} - x_n\|$ .  $x_n = \sum_{k=1}^n \frac{1}{k}$

**Problem 2.7.14**  $x_n \rightarrow x$ ,  $A_k = \{x_i : i \geq k\}$ . Show that  $\bigcap_{k=1}^{\infty} \overline{A_k} = \{x\}$ .

**Proof.** Given  $\epsilon > 0$ , there exists  $N$  such that  $n \geq N \Rightarrow x_n \in (x - \epsilon, x + \epsilon)$ . Either  $x_n = x$ , or  $x_n \in (x - \epsilon, x + \epsilon) \setminus \{x\}$ . Thus  $x \in \overline{A_k}$  for all  $k$ .  $\{x\} \subset \bigcap_{k=1}^{\infty} \overline{A_k}$ .

For  $y \in \mathbb{R} \setminus \{x\}$ , we want to show that  $y \notin \bigcap_{k=1}^{\infty} \overline{A_k}$ . Then we want to find  $N$  such that  $y \notin \overline{A_N}$ . Since  $\|x - y\| > 0$ , set  $\epsilon = \frac{1}{3} \|x - y\|$ . There exists  $N$  such that  $\|x_n - x\| < \epsilon$ . Then  $\forall x_n \notin N(y, \epsilon)$ .  $\overline{A_N} = \{x_N, x_{N+1}, \dots\}$ , and  $y$  cannot be in  $\overline{A_N}$ .  $\{x\}^C \subset (\bigcap_{k=1}^{\infty} \overline{A_k})^C \Rightarrow \bigcap_{k=1}^{\infty} \overline{A_k} \subset \{x\}$ .

**Problem 2.7.15**  $\sum a_n$  converges absolutely.

(1)  $\sum a_n^2$

**Proof.**  $a_n^2 < |a_n|$  for large  $n$ . Converges by comparison test.

(2)  $\sum \frac{a_n}{1 + a_n}$

**Proof.** Since  $a_n \rightarrow 0$ , exists  $N$  such that  $n \geq N \Rightarrow |a_n| < 1/3$ . Then for  $n \geq N$ ,  $|1 + a_n| \geq 1 - |a_n| > 2/3 > 1/3$ ,  $1/|1 + a_n| < 3$ . We have  $\left| \frac{a_n}{1 + a_n} \right| < 3|a_n|$ . Converges by comparison test.

(3)  $\sum \frac{a_n^2}{1 + a_n^2}$

**Proof.** Trivial from 1, 2.

**April 15th, 2019**

$K$ : compact  $\iff$  Exists an open cover of  $K$  that has *finite* subcover.

**Theorem 2.5.4** (Heine-Borel) For  $\mathbb{R}^d$ ,  $K$ : compact  $\iff K$  is bounded and closed.

**Theorem 2.5.5** (Cantor's Intersection Theorem)<sup>14</sup>

Given family of **compact** sets  $\{K_i : i \in I\}$ , for all **finite**  $J \subset I$ ,  $\bigcap_{i \in J} K_i \neq \emptyset$ . Then

$$\bigcap_{i \in I} K_i \neq \emptyset$$

**Proof.** (Contradiction)  $\bigcap_{i \in I} K_i = \emptyset \implies \bigcup_{i \in I} K_i^C = \mathbb{R}^d$ . (Complement)

Take any  $K_a$  ( $a \in I$ ), then  $K_a \subset \bigcup_{i \in I} K_i^C (= \mathbb{R}^d) \implies \{K_i^C : i \in I\}$  is an open cover of  $K_a$ . Then there exists a finite subcover,  $\{K_i^C : i \in J\}$  ( $K_a$  is compact) Now we can write  $K_a \subset \bigcup_{i \in J} K_i^C$ . Take complement on both sides to get  $K_a^C \supset \bigcap_{i \in J} K_i$ . Then  $K_a \cap \bigcap_{i \in J} K_i = \emptyset$ , contradiction.

**Remark.** Let  $K_i = [a_i, b_i]$  (Compact in  $\mathbb{R}$ ) and set  $K_1 \supset K_2 \supset \dots$

$\implies$  For  $J = \{j_1, \dots, j_m\}$  ( $j_1 < \dots < j_m$ ),  $\bigcap_{i \in J} K_i = K_{j_m} \neq \emptyset$

$\implies \bigcap_{i=1}^{\infty} K_i \neq \emptyset$  (축소구간정리)

## 2.6 Connected Set

p46-p47 (Section 2.2)

**Definition.**  $X \subset \mathbb{R}^d$ ,  $x \in X$ . Define

$$N_X(x, r) = \{y \in X : \|y - x\| < r\} = N(x, \epsilon) \cap X$$

**Definition.**  $U \subset X$  is open in  $X \iff x \in U, \exists \epsilon > 0$  such that  $N_X(x, \epsilon) \subset U$ .

**Example.**

- $U = \{3\}$ .  $U$  is open in  $X = \mathbb{N}$ .  $N_{\mathbb{N}}(3, 1/10) = 3 \subset U$ . (But not open in  $\mathbb{R}$ )
- For  $X = [0, 10]$ ,  $U = [0, 1)$ .  $x \in U$ ,  $N(x, 1-x) = (2x-1, 1)$ , and this might not be subset of  $U$ . But

$$N_X(x, 1-x) = \begin{cases} (2x-1, 1) & (x > 1/2) \\ [0, 1) & (x \leq 1/2) \end{cases}$$

For both cases  $N_X(x, 1-x) \subset U$ .

---

<sup>14</sup>축소구간정리의 가장 일반적인 형태

**Prop 2.2.5**  $U$  is open in  $X \iff U = X \cap V$  for some open set  $V$  in  $\mathbb{R}^d$ .

**Remark.** First example:  $\{3\} = \mathbb{N} \cap (2.9, 3.1)$ , Second example:  $[0, 1] = [0, 10] \cap (-1, 1)$ .

Some references may write this definition as “*relatively*” open in  $X$ .

**Proof of 2.2.5**

( $\implies$ )  $x \in U$ ,  $\exists \epsilon_x > 0$  such that  $N_X(x, \epsilon_x) \subset U$ . Select  $V = \bigcup_{x \in U} N(x, \epsilon_x)$ , which is open.<sup>15</sup>

Then we have  $X \cap V = \bigcup_{x \in U} X \cap N(x, \epsilon_x) = \bigcup_{x \in U} N_X(x, \epsilon_x)$ , which is exactly equal to  $U$ .

( $\impliedby$ )  $x \in U = X \cap V \implies x \in V$ . Thus  $\exists \epsilon > 0$  such that  $N(x, \epsilon) \subset V$ . Then

$$N_X(x, \epsilon) = X \cap N(x, \epsilon) \subset X \cap V = U$$

Thus  $U$  is open in  $X$ .

**Cor.**  $U$ : open in  $X$ ,  $Y \subset X$ .  $\implies U \cap Y$ : open in  $Y$ .

**Proof.**  $U = X \cap V$  ( $V$ : open)  $\implies U \cap Y = X \cap V \cap Y = V \cap (X \cap Y) = V \cap Y$ .

**Definition.**  $S \subset \mathbb{R}^d$ : **disconnected**  $\iff$  There exists **non-empty** sets  $U, V$  such that

$$(1) \ U \cap V = \emptyset$$

$$(2) \ U \cup V = S$$

$$(3) \ U \text{ and } V \text{ are open in } S$$

$S \subset \mathbb{R}^d$ : **connected**  $\iff S$  is not disconnected.

**Question.** Find all  $A \subset \mathbb{R}^d$  such that  $A$  is open and closed.

**Proof.** The only possible sets are  $A = \emptyset, \mathbb{R}^d$ .

If  $A$  is open and closed  $\implies A$ : open,  $A^C$ : open. Then  $\mathbb{R}^d = A \cup A^C$ , and  $\mathbb{R}^d$  is disconnected.

But  $\mathbb{R}^d$  is connected. Contradiction if either  $A$  or  $A^C$  is empty.

**Theorem.** The following are equivalent for  $S \subset \mathbb{R}$ .

$$(1) \ S \text{ is connected.}$$

$$(2) \ \forall a, b \in S \text{ s.t. } a < b, \text{ and } c \in (a, b) \implies c \in S.$$

$$(3) \ S = [a, b] \text{ or } [a, b) \text{ or } (a, b] \text{ or } (a, b) \text{ (} a, b \text{ can be } \pm\infty)$$

---

<sup>15</sup> $N(x, \epsilon)$  is open and union of open sets are always open.

**Remark.** Prop 2.5.1 ( $1' \iff 2'$ ) + Discussion above ( $2 \iff 3$ )

**Proof.**

**(1  $\implies$  2)** (Contradiction) Assume  $a, b \in S, c \notin S$  for some  $a < c < b$ . Set  $U = (-\infty, c) \cap S$ ,  $V = (c, \infty) \cap S$ .  $U, V$  are non-empty.<sup>16</sup>  $U \cap V = \emptyset$  and  $U \cup V = S$ . (Note that  $c \notin S$ ) And  $U, V$  are open in  $S$ . (Prop 2.2.5) Then  $S$  is disconnected.

**(2  $\implies$  1)** (Contradiction) Assume  $S$  is disconnected. There exists  $U, V$  that satisfy the definition of disconnected set. For  $a \in U, b \in V$ , (WLOG  $a < b$ ). By (2),  $[a, b] \subset S$ .

Let  $c = \sup([a, b] \cap U)$ .

Case I)  $c \in U$ . Then  $c \neq b \implies c \in [a, b) = Y \implies c \in U \cap Y$ .

Since  $U$  is open in  $S$  and  $Y \subset S \implies U \cap Y$  is open in  $Y$ . (Cor of 2.2.5)

$\implies \exists \epsilon > 0$  such that  $N_Y(c, \epsilon) \subset U \cap Y \subset U \cap [a, b]$ .

$$Y \cap N(c, \epsilon) = [a, b) \cap (c - \epsilon, c + \epsilon) \supset [c, b) \cap [c, c + \epsilon) = [c, \min\{b, c + \epsilon\})$$

Therefore, we have

$$[c, \min\{b, c + \epsilon\}) \subset N_Y(c, \epsilon) \subset U \cap [a, b]$$

and since  $c$  was the supremum, contradiction.

Case II)  $c \in V$ . Similarly, contradiction.

**(2  $\implies$  3)**  $\inf S = u, \sup S = v$ . (If  $S$  is not bounded below,  $u = -\infty$ , if  $S$  is not bounded above,  $v = \infty$ ). Then if  $c \in (u, v) \implies c \in S$ . There exists  $a, b \in S$  such that  $u \leq a < c < b \leq v$ , meaning that  $S$  must be one of  $[u, v], [u, v), (u, v], (u, v)$ .

**(3  $\implies$  2)** Trivial.

---

<sup>16</sup>Always check!  $a \in U, b \in V$ .

**April 17th, 2019**

**Definition.**  $S \subset \mathbb{R}^d$ : **disconnected**  $\iff$  There exists **non-empty** sets  $U, V$  such that

- (1)  $U \cap V = \emptyset$
- (2)  $U \cup V = S$
- (3)  $U$  and  $V$  are open in  $S$

Last time we characterized all connected sets of  $\mathbb{R}$ .

**Theorem 2.6.2** Suppose  $\{C_i : i \in I\}$  is a family of connected sets.<sup>17</sup>

$$\bigcap_{i \in I} C_i \neq \emptyset \implies \bigcup_{i \in I} C_i \text{ is connected}$$

**Proof.** (Routine) Assume  $C = \bigcup_{i \in I} C_i$  is disconnected.  $C$  can be decomposed into 2 sets  $U, V$  (that satisfy condition (1), (2), (3) from the definition). Let

$$U_i = C_i \cap U, \quad V_i = C_i \cap V \quad (\forall i)$$

then  $U_i, V_i$  are open in  $C_i$ .<sup>18</sup> Now  $U_i, V_i$  satisfy (2) and (3) for  $C_i$ . Since  $C_i$  is connected, (1) should not hold, in other words, either  $U_i$  or  $V_i$  must be  $\emptyset$ .

Define:  $I_1 = \{i \in I : U_i = \emptyset, V_i = C_i\}$ ,  $I_2 = \{i \in I : U_i = C_i, V_i = \emptyset\}$ . If  $I_1 = \emptyset \implies I_2 = I \implies V_i = \emptyset \ (\forall i) \implies V = \bigcup_{i \in I} V_i = \emptyset$ <sup>19</sup>, contradiction. Similarly if  $I_2 = \emptyset$ , contradiction.

Select  $i_1 \in I_1, i_2 \in I_2$ . Then  $C_{i_1} = V_{i_1} \subset V$ ,  $C_{i_2} = U_{i_2} \subset U$ . Therefore  $C_{i_1} \cap C_{i_2} = \emptyset$ . Contradiction.

**Example.**

(1)  $x, y \in \mathbb{R}^d$ ,  $[x, y] = \{tx + (1-t)y : t \in [0, 1]\}$  is connected. (Proof similar to Prop 2.6.1)

(2)  $N(x, r) = \bigcup_{y \in N(x, r)} [x, y]$  is connected by the theorem above. ( $\bigcap_{y \in N(x, r)} [x, y] = \{x\} \neq \emptyset$ )

(3)  $\mathbb{R}^d = \bigcup_{y \in \mathbb{R}^d} [0, y]$  is connected.

(4) Convex sets are connected.  $A = \bigcup_{y \in A} [x, y]$ .

---

<sup>17</sup>활용 보다도 증명이 중요하니 꼭 기억해 두자.

<sup>18</sup> $U$ : open in  $X$  and  $Y \subset X \implies U \cap Y$ : open in  $Y$ .

<sup>19</sup>Check!

**Definition.** Set  $A$  is **convex**  $\iff x, y \in A \implies [x, y] \subset A$ .

**Comment.** Homework problem: Show that  $S = \{(x, y) : xy > 1\}$  is open.

**Proof.** 1. Show that  $N(z, \epsilon) \subset S$  for all  $z \in S$ .

2. Instead show that  $F = \{(x, y) : xy \leq 1\}$  is closed.

Use Thm 2.2.3 (4). Let  $(x_n, y_n)$  be a sequence in  $F$  that converges to  $(x, y)$ .

$$xy = \lim x_n \lim y_n = \lim x_n y_n \leq 1 \implies (x, y) \in F$$

**Example.**  $A \subset \mathbb{R}^n$ ,  $B \subset \mathbb{R}^m$ , define  $A \times B \subset \mathbb{R}^{n+m}$  as

$$A \times B = \{(x, y) : a \in A, b \in B\}$$

If  $m = n = 1$ ,  $A \times B$  is a rectangular box in  $\mathbb{R}^2$ .

If  $A, B$  is open/closed/compact/connected,  $A \times B$  is open/closed/compact/connected.

**Proof.**

(1) (Open)  $(a, b) \in A \times B$ . There exists  $\epsilon_1, \epsilon_2 > 0$  such that  $N(a, \epsilon_1) \subset A$ ,  $N(b, \epsilon_2) \subset B$ .

Choose  $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ . If  $(x, y) \in N((a, b), \epsilon) \subset \mathbb{R}^{n+m}$ ,<sup>20</sup> we have

$$\epsilon^2 > \|(x, y) - (a, b)\|^2 = \|x - a\|^2 + \|y - b\|^2$$

$$\|x - a\| < \epsilon < \epsilon_1 \text{ and } \|y - b\| < \epsilon < \epsilon_2. \quad x \in A, y \in B.$$

Therefore  $(x, y) \in A \times B$ , and  $N((a, b), \epsilon) \subset A \times B$ .

(2) (Closed)  $(x_k, y_k)$ : sequence in  $A \times B$ . ( $x_k \in A, y_k \in B$ )

Suppose  $(x_k, y_k) \rightarrow (x, y)$  ( $x_k \rightarrow x, y_k \rightarrow y$ ). Since  $A$  is closed and  $x_k$  is a sequence in  $A$ ,  $x \in A$ . Similarly,  $y \in B$ . Thus  $(x, y) \in A \times B$ , and  $A \times B$  is closed.

(3) (Compact)  $A, B$  are closed and bounded. Closed is proven by (2).

Since  $A, B$  are bounded,  $\exists M_1, M_2$  such that  $\|a\| \leq M_1$ ,  $\|b\| \leq M_2$  for all  $a \in A, b \in B$ .

For all  $(a, b) \in A \times B$ ,

$$\|(a, b)\| = \sqrt{\|a\|^2 + \|b\|^2} \leq \sqrt{M_1^2 + M_2^2}$$

Therefore  $A \times B$  is bounded. Thus compact.

(4) (Connected)  $a \in A \implies \{a\} \times B$  is connected.  $b \in B \implies A \times \{b\}$  is connected.

Proof. If the set is disconnected, exists  $\{a\} \times U, \{a\} \times V$  such that splits  $B$ .

Since  $(A \times \{b\}) \cap (\{a\} \times B) = \{(a, b)\} \neq \emptyset$ ,  $(A \times \{b\}) \cup (\{a\} \times B)$  is connected by Thm 2.6.2. Now fix  $a \in A$ , and define  $C_b = (A \times \{b\}) \cup (\{a\} \times B)$ .

Then  $\{C_b : b \in B\}$  is a family of connected sets, and  $\bigcap_{b \in B} C_b = \{a\} \times B \neq \emptyset$ .  $A \times B = \bigcup_{b \in B} C_b$  is connected by Thm 2.6.2.

---

<sup>20</sup>Do not write as  $\mathbb{R}^{m+n}$ . First coordinate is  $n$ -dimension, second is  $m$ -dimension.

April 22nd, 2019

### 3. Continuous Functions

#### 3.1 함수의 극한과 연속함수의 정의

특별한 언급이 없으면 다음과 같은 가정을 한다.<sup>21</sup>

$$f : X \rightarrow Y \quad (X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n)$$

**Definition.** For  $x_0 \in X'$ ,  $\lim_{x \rightarrow x_0} f(x) = y_0 \iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - y_0\| < \epsilon)$$

**Remark.** Why  $X'$ ?  $X = [0, 1] \cup \{2\}$ , consider  $f(x) = 2x$  on  $X$ .  $\lim_{x \rightarrow 2} f(x)$  is nonsense.

**Example.**

$$(1) f(x) = \begin{cases} x^2 & (x \neq 0) \\ 1 & (x = 0) \end{cases}, \lim_{x \rightarrow 0} f(x) = 0.^{22}$$

For  $\epsilon > 0$ , set  $\delta = \sqrt{\epsilon}$ . Then  $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| = |x^2| < \delta^2 = \epsilon$ .

$$(2) \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4. (X = \mathbb{R} \setminus \{2\}, Y = \mathbb{R}, 2 \in X')$$

For  $\epsilon > 0$ , set  $\delta = \epsilon$ . Then  $0 < |x - 2| < \delta \Rightarrow |f(x) - 4| = |x - 2| < \delta = \epsilon$ .

**Prop 3.1.1**  $f, g : X \rightarrow Y, x_0 \in X'^{23}$ . If  $\lim_{x \rightarrow x_0} f(x) = y_0, \lim_{x \rightarrow x_0} g(x) = z_0$ , then

$$(1) \lim_{x \rightarrow x_0} af(x) + bg(x) = ay_0 + bz_0$$

$$(2) \lim_{x \rightarrow x_0} f(x)g(x) = y_0z_0$$

$$(3) \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{y_0}{z_0} \quad (z_0 \neq 0)$$

연속을 3가지로 정의한다. 세 정의들이 서로 동치임을 이해하는 것이 중요하다.

**Definition.** Let  $f : X \rightarrow Y, x_0 \in X$ .  $f$  is **continuous** at  $x_0 \iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } (\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon)$$

**Remark.**  $\|x - x_0\| < \delta$  should be satisfied for  $x \in X$ . The  $0 <$  condition is omitted here since the inequality holds trivially for  $x_0$ .

---

<sup>21</sup>지역이 중요하지 영역은 뭐...

<sup>22</sup>특별한 언급이 없으면  $X = f$  가 정의되는 곳,  $Y = \mathbb{R}^n$  으로 생각한다.

<sup>23</sup>책에  $X$ 로 되어있는데 이는 오타.

- (1)  $x_0 \in X'$ :  $f$  is continuous at  $x_0 \iff \lim_{x \rightarrow x_0} f(x) = f(x_0)$ .
- (2)  $x_0 \in X \setminus X'$  (isolated point):  $f$  is continuous at  $x_0$ .

**Definition.**

- (1)  $A \subset X, f : X \rightarrow Y$ . If  $f$  is continuous at  $x_0$  for all  $x \in A \implies f$  is continuous on  $A$ .
- (2) If  $f$  is continuous on  $X \implies f$  is continuous.

**Prop 3.1.3** The following are equivalent for  $f : X \rightarrow Y$ .

- (1)  $f$ : continuous at  $x_0 \in X$ .
- (2) If there exists a sequence  $\langle x_n \rangle$  in  $X$  converging to  $x_0 \implies \lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ .

**Proof.**

(1  $\implies$  2) Given  $\epsilon > 0$ ,

- (i)  $\exists \delta > 0$  s.t.  $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$
- (ii) Since  $x_n \rightarrow x_0, \exists N$  s.t. for  $n \geq N \implies \|x_n - x_0\| < \delta$ .

Therefore,  $n \geq N \implies \|x_n - x_0\| < \delta \implies \|f(x_n) - f(x_0)\| < \epsilon$ .

(2  $\implies$  1) (Contradiction) Suppose there exists  $\epsilon_0 > 0$  such that no  $\delta$  satisfies  $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon_0$ . (i.e. For all  $\delta > 0, \exists x \in X$  s.t.  $\|x - x_0\| < \delta$  and  $\|f(x) - f(x_0)\| \geq \epsilon_0$ )

Thus for all  $n \in \mathbb{N}$ , there exists  $x_n \in X$  s.t.  $\|x_n - x_0\| < 1/n$  and  $\|f(x_n) - f(x_0)\| \geq \epsilon_0$ . ( $\delta = 1/n$ ) Then we have  $\lim_{n \rightarrow \infty} x_n = x_0$ , but  $\lim_{n \rightarrow \infty} f(x_n) \neq f(x_0)$ . Contradiction.

**Definition.**  $f : X \rightarrow Y, A \subset X, B \subset Y$ . Define

$$f(A) = \{f(x) : x \in A\} \quad f^{-1}(B) = \{x \in X : f(x) \in B\}$$

**Remark.**

- (1)  $f(f^{-1}(B)) \subseteq B$   
 $y \in f(f^{-1}(B))$  then  $y = f(x)$  for some  $x \in f^{-1}(B)$ . Thus we have  $x \in f^{-1}(B) \iff f(x) \in B. \therefore y = f(x) \in B$ .
- (2)  $f^{-1}(f(A)) \supseteq A$   
 Exercise.



Also remember the counterexamples where the equality does not hold. (1) doesn't hold if  $f$  is not surjective, (2) doesn't hold if  $f$  is not injective.

**Theorem 3.1.5** The following are equivalent for  $f : X \rightarrow Y$ .

- (1)  $f$  is continuous on  $X$ .
- (2)  $B$ : open set in  $Y \implies f^{-1}(B)$ : open in  $X$ .
- (3)  $B$ : closed in  $Y \implies f^{-1}(B)$ : closed in  $X$ .

**Proof.** (2  $\iff$  3) Trivial. Check  $f^{-1}(B^C)$ .

(1  $\implies$  2) Observation.  $f$  is continuous at  $x_0 \iff \forall \epsilon > 0, \delta > 0$  s.t.  $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$ . Re-write the last two inequality as  $x \in N_X(x, \delta)$  and  $f(x) \in N_Y(f(x_0), \epsilon)$ . Then continuity condition is equivalent to

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } f(N_X(x, \delta)) \subset N_Y(f(x_0), \epsilon)$$

Now suppose  $x_0 \in f^{-1}(B) \iff f(x_0) \in B$ . Since  $B$  is open, there exists  $\epsilon > 0$  s.t.  $N_Y(f(x_0), \epsilon) \subset B$ . Then there exists  $\delta > 0$  s.t.  $f(N_X(x_0, \delta)) \subset N_Y(f(x_0), \epsilon) \subset B$ . Take  $f^{-1}$  on both sides.  $N_X(x_0, \delta) \subset f^{-1}(f(N_X(x_0, \delta))) \subset f^{-1}(B)$ . Thus  $f^{-1}(B)$  is open in  $X$ .

(2  $\implies$  1)  $x_0 \in X, f(x_0) \in Y$ . Given  $\epsilon > 0$ ,  $N_Y(f(x_0), \epsilon)$  is open in  $Y$ . By (2),  $f^{-1}(N_Y(f(x_0), \epsilon))$  is open in  $X$ . Observe that this set always contains  $x_0$ . Then  $\exists \delta$  s.t.  $N_X(x_0, \delta) \subset f^{-1}(N_Y(f(x_0), \epsilon))$ . Now take  $f$  on both sides.  $f(N_X(x_0, \delta)) \subset f(f^{-1}(N_Y(f(x_0), \epsilon))) \subset N_Y(f(x_0), \epsilon)$ . Thus  $f$  is continuous at  $x_0$ .

April 24th, 2019

연속함수의 기본적 성질

**Prop 3.1.2** Suppose  $f, g : X \rightarrow \mathbb{R}^n$  are continuous on  $X$ .

- (1)  $af + bg$ : continuous
- (2)  $(n = 1) fg$ : continuous
- (3)  $\frac{f}{g}$ : continuous ( $g \neq 0$  on  $X$ )

**Proof.** (2) Given  $\epsilon > 0$ ,  $\exists \delta_1$  s.t.  $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\epsilon}{2|g(x_0)|+1}$ ,  $\exists \delta_2$  s.t.  $|x - x_0| < \delta_2 \implies |g(x) - g(x_0)| < \frac{\epsilon}{2(|f(x_0)| + \frac{\epsilon}{2|g(x_0)|+1})}$ . Then we have

$$\begin{aligned} |f(x)g(x) - f(x_0)g(x_0)| &= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\ &\leq |f(x)| |g(x) - g(x_0)| + |g(x_0)| |f(x) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Thus we have continuity.

**Proof 2.** By sequential definition, exists  $\langle x_n \rangle \rightarrow x_0$  in  $X$  such that  $f(x_n) \rightarrow f(x_0), g(x_n) \rightarrow g(x_0)$ . Then we have  $f(x_n)g(x_n) \rightarrow f(x_0)g(x_0)$ .

**Prop 3.1.4** Suppose we have two continuous functions  $f : X \rightarrow Y, g : Y \rightarrow Z$ . If  $f$  is continuous at  $x_0 \in X$ , and if  $g$  is continuous at  $f(x_0)$ , then  $g \circ f$  is continuous at  $x_0$ .

**Proof.** Given  $\epsilon > 0$ ,  $\exists \delta_1 > 0$  s.t.  $\|y - f(x_0)\| < \delta_1 \implies \|g(y) - g(f(x_0))\| < \epsilon$ . Also,  $\exists \delta_2 > 0$  s.t.  $\|x - x_0\| < \delta_2 \implies \|f(x) - f(x_0)\| < \delta_1$ . Now we automatically have  $\|g(f(x)) - g(f(x_0))\| = \|(g \circ f)(x) - (g \circ f)(x_0)\| < \epsilon$ .

**Remark.** Suppose  $f$ : continuous  $X$ ,  $g$ : continuous on  $Y$  (or on  $f(X)$ ). Then  $g \circ f$  is continuous on  $X$ .

**Example.**

- (1) Polynomials are continuous. Use continuity of  $f(x) = x$ .
- (2)  $f(x) = \sqrt{x}$ .<sup>24</sup>
- (3)  $f(x) = \sqrt{x^4 + 1}$  is continuous.

$$(4) f(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases} \text{ is not continuous.}$$

**Proof.**  $x_0 \in \mathbb{R}$ . Suppose there exists a sequence  $\langle x_n \rangle$  in  $\mathbb{Q}$  converging to  $x_0$ . Then  $\langle f(x_n) \rangle \rightarrow 1$ . ( $x_n = \lfloor nx_0 \rfloor / n$ ) But there also exists a sequence  $\langle x_n \rangle$  in  $\mathbb{R} \setminus \mathbb{Q}$  converging to  $x_0$ . Then  $\langle f(x_n) \rangle \rightarrow 0$ . ( $x_n = \lfloor \sqrt{2}nx_0 \rfloor / \sqrt{2}n$ )  $f(x)$  cannot be continuous anywhere.

---

<sup>24</sup>연속이지만 고른연속은 아닌 함수

### 3.2 최대최소정리와 중간값정리

**Theorem 3.2.1** Suppose  $f : X \rightarrow Y$  is surjective and  $X$  is compact. Then  $Y$  is compact.<sup>25</sup>

**Proof.** Suppose  $\{U_i : i \in I\}$  is an open cover of  $Y$ .  $V_i = U_i \cap Y$  is an open set in  $Y$ , and  $\{V_i : i \in I\}$  is also an open cover of  $Y$ . Consider  $\{f^{-1}(V_i) : i \in I\}$ , which is an open cover of  $X$ .<sup>26</sup> Since  $X$  is compact, there exists a finite subcover  $\{f^{-1}(V_i) : i \in J\}$  ( $J \subset I$ ) of  $X$ . Then  $\{V_i : i \in J\}$  is a finite subcover of  $Y$ .

$$Y = f(X) = f\left(\bigcup_{i \in J} f^{-1}(V_i)\right) = \bigcup_{i \in J} f(f^{-1}(V_i)) \subset \bigcup_{i \in J} V_i$$

We have a finite subcover of  $Y$ . Thus  $Y$  is compact.

**Check.**  $\forall A \subset X$ .  $f$ : surjective  $\implies f(f^{-1}(A)) = A$ .  $f$ : injective  $\implies f^{-1}(f(A)) = A$ .

**Remark.**

(1)  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $f$ : continuous. If  $K \subset \mathbb{R}^m$  is compact,  $f(K)$  is compact.  
Set  $f : K \rightarrow f(K)$ .

(2) Image of compact set is compact.

**Cor 3.2.2** Suppose  $X$  is compact.  $f : X \rightarrow \mathbb{R} \implies f$  has maximum and minimum.

**Proof.** Set  $f : X \rightarrow f(X)$ , then  $f$  is surjective and  $f(X)$  is compact. Check that if  $K \subset \mathbb{R}$ ,  $K$ : compact, then  $\inf K, \sup K \in K$  and  $\inf K = \min K, \sup K = \max K$ .

**Cor 3.2.4 (Extreme Value Theorem)** If  $f$  is a continuous function defined on  $[a, b]$ ,  $f$  has a maximum and minimum.

**Proof.**  $[a, b]$  is compact.

**Cor 3.2.3** Suppose  $X$  is compact and  $f : X \rightarrow \mathbb{R}$  is continuous. If  $f(x) > 0$  for all  $x \in X$ , then  $\exists \delta > 0$  s.t.  $f(x) \geq \delta > 0$  for all  $x \in X$ .

**Proof.** Let  $\delta = \min f(X) = f(u) > 0$  for some  $u$ .

**Remark.**  $X = [1, \infty)$ ,  $f(x) = 1/x$ . ( $X$  is not compact.)

**Cor 3.2.5** Suppose  $X$  is compact and  $f : X \rightarrow Y$  is bijective and continuous. Then  $f^{-1}$  is continuous.

**Check.**  $f : X \rightarrow Y$ .  $A \subset X, B \subset Y$ . Image:  $f(A)$ , pre-image:  $f^{-1}(B)$ . We must check if image of  $B$  on  $f^{-1}$  is equal to the pre-image of  $B$ . (Well-definedness!)

---

<sup>25</sup>연속성이 필요없나?

<sup>26</sup>Check at assignment 3.5.

**April 26th, 2019**

Assignment 3.5 #3: Check and remember.

$$(2) \quad f\left(\bigcap_{i \in \mathcal{I}} A_i\right) \subset \bigcap_{i \in \mathcal{I}} f(A_i)$$

**Problem 3.1.2**  $f : X \rightarrow \mathbb{R}^n$ ,  $f(x) = (f_1(x), \dots, f_n(x))$  ( $x \in X$ ). The following are equivalent.

(1)  $f$  is continuous at  $x$ .

(2) For all  $i$ ,  $f_i : X \rightarrow \mathbb{R}$  is continuous at  $x$ .

**Proof.** (1  $\implies$  2)  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $\|y - x\| < \delta \implies \|f(y) - f(x)\| < \epsilon$ . Then we have  $\|f_i(y) - f_i(x)\| \leq \|f(y) - f(x)\| < \epsilon$ , for any  $i$ .

(2  $\implies$  1)  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $\|x - y\| < \delta \implies \|f_i(x) - f_i(y)\| < \epsilon/\sqrt{n}$ . Then

$$\|x - y\| < \delta \implies \|f(x) - f(y)\| = \sqrt{\sum_{i=1}^n \|f_i(x) - f_i(y)\|^2} < \sqrt{n \cdot \frac{\epsilon^2}{n}} = \epsilon$$

**Prop 3.1.2** (3)  $f, g$ : continuous  $\implies f/g$ : continuous ( $g \neq 0$  on  $X$ )

**Proof.**  $\forall \epsilon > 0, \exists \delta > 0$  s.t. for all  $x_0 \in X$ ,

$$|x - x_0| < \delta \implies |g(x) - g(x_0)| < \min\left\{\frac{1}{2}|g(x_0)|, \frac{1}{4}\frac{|g(x_0)|^2\epsilon}{|f(x_0)|+1}\right\}, |f(x) - f(x_0)| < \frac{1}{4}|g(x_0)|\epsilon.$$

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}\right| &\leq \frac{|g(x_0)| |f(x) - f(x_0)| + |f(x_0)| |g(x) - g(x_0)|}{|g(x)| |g(x_0)|} \\ &\leq \frac{|g(x_0)| \frac{1}{4}|g(x_0)|\epsilon + |f(x_0)| \frac{1}{4}\frac{|g(x_0)|^2\epsilon}{|f(x_0)|+1}}{\frac{1}{2}|g(x_0)|^2} < \frac{\frac{1}{4}|g(x_0)|^2\epsilon + \frac{1}{4}|g(x_0)|^2\epsilon}{\frac{1}{2}|g(x_0)|^2} = \epsilon \end{aligned}$$

**Example.**  $g(x) = \begin{cases} 0 & (x = 0, 1 \text{ or } x \in \mathbb{R} \setminus \mathbb{Q}) \\ 1/q & (x = p/q, \text{irreducible fraction}) \end{cases}$

(i)  $x_0 \in \mathbb{Q} \cap (0, 1)$  then  $g(x_0) > 0$ . Set  $\epsilon = \frac{1}{2}g(x_0) > 0$ . For all  $\delta > 0, \exists y \in \mathbb{Q}^C \cap [0, 1]$  s.t.  $|y - x_0| < \delta$ , but  $|g(y) - g(x_0)| = g(x_0) \geq \epsilon$ . Thus  $f$  is not continuous at  $x_0$ .

(ii)  $x_0 \in \mathbb{Q}^C \cup \{0, 1\}$ .  $g(x_0) = 0$ .  $\forall \epsilon > 0, \exists N \geq 1$  s.t.  $1/N < \epsilon$ . Then there are finitely many  $y$  such that  $g(y) \geq 1/N$ . ( $\frac{1}{N}, \frac{1}{N-1}, \dots, \frac{1}{2}$  is finite) Let them be  $y_1, \dots, y_k$  and set  $\delta = \min_{1 \leq i \leq k} |y_i - x_0| > 0$ . If  $\|y - x_0\| < \delta$ , then  $0 \leq g(y) < 1/N < \epsilon$ .  $|g(y) - g(x_0)| = g(y) < \epsilon$ .

**Problem 3.5.1**

(1)  $f(x) = 0, f(\mathbb{R}) = \{0\}$  (closed)

(3)  $f(x) = e^x, f(\mathbb{R}) = (0, \infty)$  (open)