Assignment 3.5

Due Date: None

1. Let $X \subset \mathbb{R}^d$. Answer the following questions.

(1) For $x \in X$ and $r > 0$, verify that $N_X(x, r) = N(x, r) \cap X$.

(2) Prove that $\emptyset$ and X are both open and closed in X.

(3) Suppose that $\{U_i : i \in \mathcal{I}\}$ is a family of open sets in X. Prove that $\bigcup_{i\in\mathcal{I}} U_i$ is an open set in X.

(4) Suppose that $U_1, \ldots, U_n$ are open sets in X. Prove that $\bigcap_{i=1}^n U_i$ is an open set in X.

(5) Suppose that $\{F_i : i \in \mathcal{I}\}$ is a family of closed sets in X. Prove that $\bigcap_{i\in\mathcal{I}} F_i$ is a closed set in X.

(6) Suppose that $F_1, \ldots, F_n$ are closed sets in X. Prove that $\bigcup_{i=1}^n F_i$ is a closed set in X.

2. Let $X \subset \mathbb{R}^d$, and let $\{U_i : i \in \mathcal{I}\}$ be a family of open sets in X. Suppose that $\{U_i : i \in \mathcal{I}\}$ is a cover of X. If X is a compact set, prove that $\{U_i : i \in \mathcal{I}\}$ has a finite subcover of X. (In this sense, we can also say that $\{U_i : i \in \mathcal{I}\}$ is an *open cover of* X)

3. Let $f : X \to Y$ where X and Y are subsets of Euclidean spaces. Suppose that $\{A_i : i \in \mathcal{I}\}$ and $\{B_i : i \in \mathcal{I}\}$ are family of subsets of X and Y, respectively. Then, prove the following properties.

$$f(\cup_{i\in\mathcal{I}} A_i) = \bigcup_{i\in\mathcal{I}} f(A_i)$$

$$f(\cap_{i\in\mathcal{I}} A_i) \subset \bigcap_{i\in\mathcal{I}} f(A_i)$$

$$f^{-1}(\cup_{i\in\mathcal{I}} B_i) = \bigcup_{i\in\mathcal{I}} f^{-1}(B_i)$$

$$f^{-1}(\cap_{i\in\mathcal{I}} B_i) = \bigcap_{i\in\mathcal{I}} f^{-1}(B_i)$$