

March 29th, 2019

Remark. \limsup is the limit of \sup . If \sup is easy to calculate, find \sup and take the limit.

Quiz 1 Solutions

#1. Given set A , $\text{int}(A)$, A' , determine whether the set is open or closed.

1. $A = \mathbb{N} \subset \mathbb{R}$. $\text{int}(A) = \emptyset$, $A' = \emptyset$, A is closed.
2. $\mathbb{Q} \subset \mathbb{R}$. $\text{int}(\mathbb{Q}) = \emptyset$, $\mathbb{Q}' = \mathbb{R}$, \mathbb{Q} is neither open nor closed.
3. $C = [0, 1] \cup (2, 3) \cap \{4\} \subset \mathbb{R}$. $\text{int}(C) = (0, 1) \cup (2, 3)$, $C' = [0, 1] \cup [2, 3]$, C is neither open nor closed.
4. $D = \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) : 0 \leq y \leq 1\} \subset \mathbb{R}^2$. $\text{int}(D) = \emptyset$, $D' = D \cup \{(0, y) : 0 \leq y \leq 1\}$, D is neither open nor closed. ($\because \text{int}D \neq D$, $\overline{D} \neq D$)

#2. Find a limit point of given set.

1. $A = \mathbb{Q} \subset \mathbb{R}$. 0 is a limit point. (Directly follows from Archimedes' principle)
2. $B = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. 0 is a limit point of B . (Also directly follows from Archimedes')
3. $C = \{2^{-n} + 3^{-m} : n, m \in \mathbb{N}\} \subset \mathbb{R}$. 0 is a limit point of C . Given $\epsilon > 0$, exists $N \in \mathbb{N}$ such that for $n, m \geq N$, $2^{-n} < \epsilon/2$, $3^{-m} < \epsilon/2$. Then $0 \neq 2^{-n} + 3^{-m} < \epsilon$.

#3. True or False? If false, find a counterexample.

1. $\overline{A \cup B} = \overline{A} \cup \overline{B}$ **True**
2. $\overline{A \cap B} = \overline{A} \cap \overline{B}$ **False**. Set $A = (0, 1)$, $B = (1, 2)$.
Correct Statement: $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$
3. $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$ **False**. Set $A = [0, 1]$, $B = [1, 2]$.
Correct Statement: $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$
4. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$ **True**

Thm. $A \subset B \implies \overline{A} \subset \overline{B}, \text{int}(A) \subset \text{int}(B).$

Proof.

- We need to show $A' \subset B'$. Let $x \in A'$.
 $\implies \forall \epsilon > 0, N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset.$
 $\implies \forall \epsilon > 0, N(x, \epsilon) \cap (B \setminus \{x\}) \neq \emptyset$
 $\implies x \in B'.$
- Let $x \in \text{int}(A)$
 $\implies \exists \epsilon > 0, N(x, \epsilon) \subset A \implies N(x, \epsilon) \subset B \implies x \in \text{int}(B).$

Proof of (c). $A, B \subset A \cup B$

$\implies \text{int}(A), \text{int}(B) \subset \text{int}(A \cup B).$ Thus $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$

Proof of (d). $A \cap B \subset A, B \implies \text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B).$ Thus $\text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B)$
Suppose $x \in \text{int}(A) \cap \text{int}(B).$ Then $\exists \epsilon_A, \epsilon_B > 0$ s.t. $N(x, \epsilon_A) \subset A, N(x, \epsilon_B) \subset B.$ Take $\epsilon = \min\{\epsilon_A, \epsilon_B\}/2.$ Then $N(x, \epsilon) \subset A, B.$ Therefore $N(x, \epsilon) \subset A \cap B, x \in \text{int}(A \cap B).$

Example. $A = \{(x, y) : x^2 + 2y^2 < 1\}.$ $\text{int}(A) = A, A' = \{(x, y) : x^2 + 2y^2 \leq 1\}.$

Suppose $(x_0, y_0) \in A.$ $x_0^2 + 2y_0^2 = 1 - \delta < 1$ for some $\delta > 0.$ By symmetry, let $x_0, y_0 > 0.$ From

$$(x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 = x_0^2 + 2y_0^2 + \epsilon(2x_0 + 4y_0 + 3\epsilon) < 1$$

, we want $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \delta.$ Set $\epsilon < 1/10.$ Then $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \epsilon(2x_0 + 4y_0 + 3) < \delta.$

Now set $\epsilon = \min \left\{ \frac{1}{2(2x_0 + 4y_0 + 3)}, \frac{1}{100} \right\} > 0.$

Then $|x - x_0| < \epsilon, |y - y_0| < \epsilon.$ $x_0^2 + 2y_0^2 < (x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 < 1.$ $N((x_0, y_0), \epsilon) \subset A.$

Interior points are limit points, and for the points (x_0, y_0) on the border, consider a sequence $(x_0 - 1/n, y_0 - 1/n).$ Then the elements are in A and they converge to $(x_0, y_0).$ Thus the border is also included in $A'.$

April 1st, 2019

$\text{int}A : x \in A \text{ s.t. } N(x, \epsilon) \subset A \text{ for some } \epsilon > 0.$

$A' : x \in \mathbb{R}^d \text{ s.t. } N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset \text{ for } \forall \epsilon > 0$

$\overline{A} : x \in \mathbb{R}^d \text{ s.t. } N(x, \epsilon) \cap A \neq \emptyset, \forall \epsilon > 0, \overline{A} = A \cup A'$

Example. $A = [0, 1) \cup \{2\}$. $1 \in A', 2 \notin A', 2 \in \overline{A}$

Prop 2.3.3 $x \in A' \implies N(x, \epsilon) \cap (A \setminus \{x\})$ 는 무한집합이다.

Proof. 유한집합이라고 가정하자. $N(x, \epsilon) \cap (A \setminus \{x\}) = \{x_1, \dots, x_n\}$ 이라 할 수 있다. Set $\delta = \min\{\|x - x_i\| : \forall i\}$. Then $N(x, \delta) \cap (A \setminus \{x\}) = \emptyset$. 모순.

그래서 사실은 공집합이 아닌 것으로 정의했지만 **사실은** 무한집합이다.

Remark. $A' \neq \emptyset \implies A$ 는 무한집합.

(대우) A 가 유한집합이면 극한점이 존재하지 않는다. (2.2 보기 4)

(역) **거짓.** $A = \{1, 2, \dots\}$ 이면 $A' = \emptyset$.

그러면 역이 언제 성립하나요? 다음 단원 내용!

Definition. Convergence in \mathbb{R}^d

Let $\langle x_n \rangle$ be a sequence in \mathbb{R}^d .

$$\lim_{n \rightarrow \infty} x_n = x \iff \forall \epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies \|x_n - x\| < \epsilon)$$

Exercise. $x_n = (x_n^{(1)}, \dots), x = (x^{(1)}, \dots)$ 일 때, $x_n \rightarrow x \iff \forall i, x_n^{(i)} \rightarrow x^{(i)}$

Notation. $A \subset \mathbb{R}^d; \langle x_n \rangle$ is a sequence in $A \iff \forall n, x_n \in A$

Theorem 2.2.2

1. $x \in A' \iff \exists \langle x_n \rangle$ in $A \setminus \{x\}$ such that $x_n \rightarrow x$

2. $x \in \overline{A} \iff \exists \langle x_n \rangle$ in A such that $x_n \rightarrow x$

Proof.

1. (\implies) $x_n \in N(x, \frac{1}{n}) \cap (A \setminus \{x\})$ 이라 하자. (공집합이 아니므로 이러한 원소가 존재한다.)
그러면 $\|x_n - x\| < 1/n$ 이므로 x_n 은 x 로 수렴한다. 그리고 $x_n \in A \setminus \{x\}$ 이므로 수열이 $A \setminus \{x\}$ 에 있다.

2. Left as exercise. Replace $A \setminus \{x\}$ with A .

Theorem 2.2.3. The following are equivalent.

1. F is closed.
2. $F' \subset F$.
3. $F = \overline{F}$
4. For a sequence $\langle x_n \rangle$ in F , $\lim_{n \rightarrow \infty} x_n = x \implies x \in F$.

Proof.

- (1) \iff (3) (\overline{F} : smallest closed set containing F .)
 (2) \iff (3) 은 자명.
 (1) \iff (4) by the above theorem. (Thm 2.2.2)

Applications.

1. A' is closed.

Proof. We want to show that $(A')' \subset A'$.

We want to show: $x \in (A')' \implies x \in A'$.

(A' 이 공집합이면 자명. 공집합이 아니라고 가정하고...)

Given $\epsilon > 0$, $N(x, \epsilon) \cap (A' \setminus \{x\}) \neq \emptyset$. Take an element $y \in A'$ from this set. Now set $\delta = \min\{\|x - y\|, \epsilon - \|x - y\|\}$ then we have $N(y, \delta) \cap (A' \setminus \{y\}) \neq \emptyset$. ($\because y \in A'$)
 $z \in N(y, \delta) \cap (A' \setminus \{y\})$ 라 하자.

(a) $z \in A' \setminus \{y\} \subset A$.

(b) $\|x - z\| \leq \|x - y\| + \|y - z\| < \|x - y\| + \delta \leq \epsilon$ ($z \in N(y, \delta)$)

(c) $\|x - z\| \geq \|x - y\| - \|y - z\| > \|x - y\| - \delta \geq 0$ (By the choice of δ .) Thus $x \neq z$.

Therefore $z \in N(x, \epsilon)$ (by (b)), $z \in A' \setminus \{x\}$ (by (a), (c)).

$x \in A'$ since $N(x, \epsilon) \cap (A' \setminus \{x\})$ is not empty.

2. $A \subset \mathbb{R}$: closed and bounded $\implies \inf A = \min A$, $\sup A = \max A$. (Existence)

Proof. Let $\sup A = x \notin A$. ($\sup A \in A$ 이면 자명)

Claim. $x \in A'$.

Proof of Claim. $\forall \epsilon > 0$, $N(x, \epsilon) = (x - \epsilon, x + \epsilon)$

$x = \sup A$ 이므로 $x - \epsilon$ is not an upper bound.

$\exists y$ such that $y \in (x - \epsilon, x)$

$y \in N(x, \epsilon) \cap (A' \setminus \{x\}) \neq \emptyset$ 이므로 x 는 극한점.

따라서 $x \in A' \subset A$ (closed set 이므로 Thm 2.2.3 (2)) 모순.

$\sup A \in A$ 이므로 이 값이 최댓값이다.

2.3 유계집합과 코시수열

핵심: Thm 2.3.4, Thm 2.3.7

Definition. $\langle x_n \rangle$: 유계수열(bounded sequence) $\iff \exists M > 0$ s.t. $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Definition. $n_1 < n_2 < \dots$: sequence in \mathbb{N} 이라 하자. $\langle x_{n_k} \rangle_{k=1}^\infty = (x_{n_1}, x_{n_2}, \dots)$ 를 $\langle x_n \rangle$ 의 부분수열(subsequence)이라 한다.

Theorem 2.3.4 (Bolzano-Weierstrass Theorem)

If $\langle x_n \rangle$ is bounded, there exists a convergent subsequence of $\langle x_n \rangle$.

Idea of Proof. Equivalent formulation for sets.

Definition. Set A is bounded $\iff \exists M > 0$ such that $\|x\| < M$ for all $x \in A$.

Theorem 2.3.2 (Equivalent of 2.3.4) A 가 유계이고 무한집합이면, $A' \neq \emptyset$.

Remark. $A' \neq \emptyset \implies A$: 무한집합.

역이 성립하기 위해서는 A 가 유계라는 조건이 필요하다.

극한점이 중요한 이유는 계속 수열과 관련이 있기 때문이다.

Example. $A = \{1/n : n \in \mathbb{N}\}$ 을 고려하는 것은 수열 $x_n = 1/n$ 을 고려하는 것이나 마찬가지이다. 이 수열 x_n 이 x 로 수렴하는 것은 $A' = \{x\}$ 와 동치이다. (Hence the name “limit point”) 이로부터 $x \in A' \iff$ Exists a subsequence of $\langle x_n \rangle$ in $A \setminus \{x\}$ converging to x .

Proof of 2.3.2

1. Lemma 2.3.1 축소구간정리 in \mathbb{R}^d .

B is a closed box in $\mathbb{R}^d \iff B = I_1 \times I_2 \times \dots \times I_d$, where $I_i = [a_i, b_i]$ for $i = 1, \dots, d$. (I_i is a closed and bounded interval.)

$$B_1 \supset B_2 \supset \dots \implies \bigcap_{n=1}^\infty B_n \neq \emptyset$$

Proof. 각 ‘좌표’ I_i 별로 1차원 축소구간정리를 적용하면 된다.

2. Divide and Conquer Strategy

B : Box 일 때, $\text{diam}(B) = \sup\{\|x - y\| : x, y \in B\} = \sqrt{(a_1 - b_1)^2 + \dots + (a_d - b_d)^2}$

Claim. There exists closed boxes B_1, B_2, \dots s.t.

(a) $B_1 \supset B_2 \supset \dots$

(b) $\text{diam} B_n = \frac{1}{2^{n-1}} \text{diam} B_1$

(c) $B_n \cap A$: 무한집합

Proof. (Induction) $n = 1$; B_1 : 충분히 커서 $A \subset B_1$ 인 box 를 잡으면 된다.

Suppose we have B_1, \dots, B_n ; B_n 을 2^d 등분하면 적어도 하나는 A 의 원소를 무한개 포함하고 있다. 그 집합을 B_{n+1} 으로 잡는다. (비둘기집의 원리)

이제 $x \in \bigcap_{n=1}^{\infty} B_n$ 으로 잡으면 (축소구간정리에 의해 잡을 수 있다) $x \in A'$. ($A' \neq \emptyset$)

$\because \forall \epsilon > 0$, $\text{diam} B_n < \epsilon$ 인 $N \in \mathbb{N}$ 을 찾아 $n \geq N$ 일 때 부등식이 성립하도록 할 수 있다.

이러한 n 들에 대하여 $B_n \subset N(x, \epsilon)$. 그러면 $N(x, \epsilon) \cap (A \setminus \{x\}) \supset B_n \cap (A \setminus \{x\})$.

April 3rd, 2019

우리가 지금 2.3 을 하고 있는데, 2 가지 중요한 결과가 있어요.

Theorem 2.3.4 $\langle x_n \rangle$ 이 bounded 이면 수렴하는 부분수열을 갖는다.¹

Theorem 2.3.2 A 가 유계인 집합이고 무한집합이면 극한점을 가진다. $A' \neq \emptyset$
증명은 축소구간정리를 박스로 확장해가지고 분할 정복하면 된다.

Recall 2.3.3 $x \in A' \implies N(x, \epsilon) \cap (A \setminus \{x\})$ 는 무한집합이다.

Proof of 2.3.4. $A = \{x_1, x_2, \dots, x_n\}$ 라고 하면 이 집합은 유계이다. (수열이 유계이므로)

1. A 가 유한집합: 자명.

$\exists x$ such that x appears infinitely many times in $\langle x_n \rangle$. (PHP) 이 경우에는 부분수열을 x, x, \dots 로 잡으면 된다. 이는 수렴하는 부분수열이다.

2. A 가 무한집합²

$A' \neq \emptyset$ 이므로 $\alpha \in A'$ 이라 하자.

Claim. $\exists n_1 < n_2 < \dots$ such that $\|x_{n_k} - \alpha\| < 1/k$.

Proof. (첨자들이 증가하면서 가까워져야 한다는 것이 유일하게 tricky 한 부분이다. 귀납법을 사용하자.) $k = 1$: $x_{n_1} \in N(\alpha, 1) \cap (A \setminus \{\alpha\})$ 로 잡으면 된다.

x_{n_1}, \dots, x_{n_k} 를 잡았다고 가정: $N(\alpha, \frac{1}{k+1}) \cap (A \setminus \{\alpha\})$ 에서 $x_{n_{k+1}}$ 를 잡아야 하는데 이 집합은 무한집합이다. (Recall 2.3.3) 이 집합에서 첨자가 n_k 보다 큰 항이 반드시 존재하므로 그 중 하나를 $x_{n_{k+1}}$ 이라 잡으면 된다.

따라서 $\lim_{k \rightarrow \infty} x_{n_k} = \alpha$ (Check as exercise)

Application. (Characterization of \limsup and \liminf)

x_n 이 bounded 이면, $A = \{x : \exists \text{ subsequence of } x_n \text{ converging to } x\}$. 이 때 Theorem 2.3.4에 의해 $A \neq \emptyset$ 임을 증명하였다.

1. A : closed and bounded $\implies \max(A), \min(A)$ 가 존재한다.

Proof. $B = \{x_1, x_2, \dots\}$, $C = \{\langle x_n \rangle \text{ 에 무한 번 나타나는 수} \}$ 로 잡자. $A = B' \cup C$, $C \subset B$, $C' \subset B'$ 임을 확인해보라! 이를 이용하면 $B' \cup C = (B' \cup C') \cup C = B' \cup (C' \cup C) = B' \cup \overline{C}$ 가 되어 닫힌집합의 합집합은 닫힌 집합이다. A 는 closed and bounded 이다.

2. $\limsup x_n = \max(A)$, $\liminf x_n = \min(A)$

(부분수열이 가질 수 있는 극한값들 중 가장 큰 값이 \limsup , 가장 작은 값이 \liminf)

¹증명이 가장 테크니컬 해요!

²이제 Thm 2.3.2 를 사용할 수 있다. 사실 경우를 나눈 것은 예외적인 case 를 처리하기 위한 것이었다.

Proof. Recall

$$\limsup x_n = \alpha \iff \begin{cases} \text{(i) } \forall \epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies x_n < \alpha + \epsilon) \\ \text{(ii) } \forall \epsilon > 0, x_n > \alpha - \epsilon \text{ for infinitely many } n \end{cases}$$

- (a) 부분수열 $\langle x_{n_k} \rangle \rightarrow \beta$ 이면 (i)에 의해 $k \geq N \implies x_{n_k} < \alpha + \epsilon$ 이 되어 $\beta \leq \alpha + \epsilon$. $\beta \leq \alpha$. 그러므로 $\max(A) \leq \alpha$ 이다.
- (b) $\forall \epsilon > 0$, (i), (ii)에 의해 $x_n \in (\alpha - \epsilon, \alpha + \epsilon)$ 인 n 이 무한히 많다. 이 유계인 구간에 속하는 수열의 항들에 대해 부분수열을 잡아 (further subsequence) γ 로 수렴하도록 할 수 있다. (Theorem 2.3.4) 그러면 $\langle x_{m_k} \rangle \rightarrow \gamma \in [\alpha - \epsilon, \alpha + \epsilon]$. 따라서 $\alpha - \epsilon \leq \gamma \leq \max(A)$ 가 되어 $\alpha \leq \max(A)$.

따라서 $\max(A) = \alpha$.

Definition. $\langle x_n \rangle$: Cauchy Sequence $\iff \forall \epsilon > 0, \exists N \text{ s.t. } [m, n \geq N \implies \|x_m - x_n\| < \epsilon]$

Prop 2.3.6, Thm 2.3.8 $\langle x_n \rangle$: convergent $\iff \langle x_n \rangle$: Cauchy sequence³

Proof. (\implies) 자명. $\|x_m - x_n\| \leq \|x_m - \alpha\| + \|x_n - \alpha\| < \epsilon/2 + \epsilon/2 = \epsilon$ 인 $m, n \geq N$ 존재. (\impliedby) 수렴 값이 없는 상태에서 증명해야 한다. 먼저 수렴 값을 찾아보자.

1. $\langle x_n \rangle$ is bounded.

Proof. $\exists N$ s.t. $\|x_m - x_n\| < 1$ for all $m, n \geq N$.

Set $M = \max\{\|x_1\|, \dots, \|x_{N-1}\|, \|x_N\| + 1\}$. ($\|x_m\| < \|x_N\| + 1$)

따라서 $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

2. There exists a subsequence $\langle x_{n_k} \rangle$ converging to some α . (Thm 2.3.4)
3. $\langle x_n \rangle$ converges to α .

Proof. $\epsilon > 0$ 에 대해,

- (a) 코시 수열의 성질에 의해 $\exists N_1$ s.t. $\|x_m - x_n\| < \epsilon/2$ for all $m, n \geq N_1$.
- (b) 부분수열이 α 로 수렴하므로 $\exists N_2$ s.t. $\|x_{n_k} - \alpha\| < \epsilon/2$ for all $k \geq N_2$.

Let $N = \max\{N_1, N_2\}$. $n \geq N, n_N \geq n_{N_1} \geq N_1$ 이므로,

$$n > N \implies \|x_n - \alpha\| \leq \|x_n - x_{n_N}\| + \|x_{n_N} - \alpha\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

³중간고사 전 까지 가장 중요한 정리.

Remark. 우리의 여정을 돌아보자.

1. Archimedes' Principle 을 가정하면

Completeness Axiom \implies Monotone Convergence Theorem \implies 축소구간정리 \implies
Bolzano-Weierstrass Theorem \implies **Cauchy Convergent Theorem**⁴
(Exercise) \implies Completeness Axiom

2. **Example.** $X = C([0, 1])$. (Set of functions that are continuous in $[0, 1]$) How would we define $\|f - g\|$? $\int_0^1 |f(x) - g(x)| dx$? $\max\{|f(x) - g(x)| : x \in [0, 1]\}$? Only the second choice gives completeness for X .

3. **Convergence Test** without limit value. (**Theorem 2.3.9**)

$\sum_{n=1}^{\infty} a_n$ is convergent $\iff \forall \epsilon > 0, \exists N$ s.t. $(n > m \geq N \implies |a_{m+1} + \cdots + a_n| < \epsilon)$

Proof. Trivial.

Definition. $\sum a_n$ is **absolutely convergent** $\iff \sum |a_n|$ is convergent

Theorem. An absolutely convergent series converges.

Proof. Suppose $\sum |a_n|$ converges. For $\forall \epsilon > 0$, there exists N such that $||a_{m+1}| + \cdots + |a_n|| < \epsilon$ for all $m, n \geq N$. Therefore, for $m, n \geq N$,

$$|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \epsilon$$

and $\sum a_n$ converges.

⁴In any metric spaces, this is the condition for completeness.

April 5th, 2019

Theorem. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof. (\subset) Trivial.

(\supset) $A \subset \overline{A}, B \subset \overline{B} \implies A \cup B \subset \overline{A} \cup \overline{B} \implies \overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. The closure of a closed set is itself.

6. (2) $a_n = \cos \sqrt{2019 + n^2 \pi^2}$

Consider $\delta > 0$, such that

$$(n\pi - \delta)^2 < 2019 + n^2 \pi^2 < (n\pi + \delta)^2$$

$$-2n\pi < \frac{2019}{\delta} \pm \delta < 2n\pi$$

We can find large enough N such that the above inequality holds for $n \geq N$.

Now we want $b_n = \sqrt{2019 + n^2 \pi^2}$ bounded by $n\pi \pm \delta$.

$n \geq N, n \text{ even} \implies n\pi - \delta < b_n < n\pi + \delta$

$\implies 1 \geq a_n > 1 - \epsilon$

$n \geq N, n \text{ odd} \implies -1 \leq a_n < -1 + \epsilon$

Problem 2.3.5

$$(1) \ x_{n+2} = \frac{x_n + x_{n+1}}{2}$$

$$(2) \ x_{n+1} = x_n + x(-1)^n \frac{1}{3n+1}$$

Solution.

(1) Write $x_{n+2} - x_{n+1} = a(x_{n+1} - x_n)$ and observe that $a = -1/2$. Write as

$$x_n = x_{n-1} + \left(-\frac{1}{2}\right)^{n-2} (x_2 - x_1)$$

Then we have

$$x_n = x_2 + \sum_{k=1}^{n-2} \left(-\frac{1}{2}\right)^k (x_2 - x_1)$$

This series converges to $\frac{2x_2 + x_1}{3}$.

(2) This is an alternating series. Write as

$$x_n = x_1 + \sum_{k=1}^{n-1} (-1)^k \frac{x}{3n+1}$$

By alternating series test, the second summation term converges, and the series converges to x_1 .

Since a converging sequence is a Cauchy sequence, x_1, x_2 can be any real number.

April 8th, 2019

Section 2.3: Bolzano-Weierstrass Theorem, Cauchy Convergent Theorem

In section 2.4, we will be studying about Convergence Tests.

정

2.4 급수의 수렴판정

Cor 2.3.9. $\sum_{n=1}^{\infty} a_n$ is convergent $\iff s_n = \sum_{k=1}^n a_k, \langle s_n \rangle$ is convergent $\iff \langle s_n \rangle$ is Cauchy.

1. $\sum_{n=1}^{\infty} a_n$ is convergent $\implies \lim_{n \rightarrow \infty} a_n = 0$.
2. $\sum_{n=1}^{\infty} |a_n|$ is convergent $\implies \sum_{n=1}^{\infty} a_n$ convergent.

Theorem 2.4.3 (Comparison Test) Suppose $\sum b_n$ converges. If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$, $\sum a_n$ converges.

Proof Let $M = \sum b_n$, $s_n = \sum_{k=1}^n a_k$. s_n is increasing and s_n is bounded by M . s_n is convergent by Monotone Convergence Theorem.

Theorem. Suppose sequences a_n, b_n satisfy $0 \leq |a_n| \leq b_n$ ⁵ and $\sum b_n$ converges. Then $\sum a_n$ is convergent.⁶

Proof. By comparison test and absolute convergence.

Prop 2.4.4 (Root Test) Suppose $\alpha = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$.

If $\alpha < 1$, $\sum a_n$ converges. If $\alpha > 1$, $\sum a_n$ diverges.

Proof.

1. $\alpha < 1$. Take $\epsilon > 0$ such that $\alpha < \alpha + \epsilon < 1$. Then there exists N such that $|a_n|^{1/n} < \alpha + \epsilon$ for all $n \geq N$. Therefore $|a_n| < (\alpha + \epsilon)^n$. Since $\alpha + \epsilon < 1$, $\sum (\alpha + \epsilon)^n$ converges. Apply the comparison test to see that $\sum a_n < \infty$.
2. $\alpha > 1$. Take $\epsilon > 0$ such that $\alpha > \alpha - \epsilon > 1$. Then $|a_n|^{1/n} > \alpha - \epsilon$ for infinitely many n . Then $|a_n| > (\alpha - \epsilon)^n > 1$. Therefore $\lim a_n \neq 0$. $\sum a_n$ diverges.

Prop 2.4.5 (Ratio Test) Suppose $a_n \neq 0$. Let $\beta = \limsup |a_{n+1}/a_n|$, $\gamma = \liminf |a_{n+1}/a_n|$.

If $\beta < 1$, $\sum a_n$ converges. If $\beta > 1$, $\sum a_n$ diverges.

Proof.

1. $\beta < 1$. Take $\epsilon > 0$ such that $\beta < \beta + \epsilon < 1$. Then $\exists N$ s.t. $|a_{n+1}/a_n| < \beta + \epsilon$ for $n \geq N$.
 $\implies |a_n| = |a_N| |a_{N+1}/a_N| \cdots |a_n/a_{n-1}| < |a_N| (\beta + \epsilon)^{n-N}$.
Set $b_n = |a_N| (\beta + \epsilon)^{n-N}$ and apply comparison test to see that $\sum a_n < \infty$.

⁵Note that this condition can fail for finitely many n .

⁶ a_n may be a very complex expression, but we want b_n to be simple, an expression we know that it is convergent.

2. $\gamma > 1$. Take $\epsilon > 0$ such that $\gamma > \gamma - \epsilon > 1$. Then $\exists N$ s.t. $|a_{n+1}/a_n| > \gamma - \epsilon$ for $n \geq N$.

Then we see that $|a_n|$ is increasing for $n \geq N$. Thus a_n cannot converge to 0. $\sum a_n$ is divergent.

Remark. If the above limits (ratio, root) exist, elementary tests can be applied. But if the limits turn out to be 1, the test fails. (ND: Non-Deterministic) Check it for $\sum 1/n, \sum 1/n^2$.

Also, these are *weak tests*. For most of the series, the limit is 1. Moreover...

Theorem 2.4.6 Suppose $a_n \neq 0$.

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|$$

Thus if the root test works, ratio test also works.⁷

Proof. We only need to prove the last inequality.

Let $\beta = \limsup |a_{n+1}/a_n|$, $\forall \epsilon > 0$. $\implies \exists N$ s.t. $|a_{n+1}/a_n| \leq \beta + \epsilon$ for $n \geq N$. Then if $n \geq N$, $|a_n| \leq |a_N|(\beta + \epsilon)^{n-N}$. (Similar to proof of 2.4.5) Then

$$|a_n|^{1/n} \leq (\beta + \epsilon) \left(\frac{|a_n|}{(\beta + \epsilon)^N} \right)^{1/n}$$

and take \limsup on both sides, then $\limsup |a_n|^{1/n} \leq \beta + \epsilon$.

Example. $\langle a_n \rangle = \begin{cases} 1/2^n & n \text{ odd} \\ 1/2^{n-2} & n \text{ even} \end{cases}$

Check that $\limsup |a_n|^{1/n} = 1/2 < 1$, and the series $\sum a_n$ converges by the root test.

But if we use the ratio test here, \limsup value is 2 and \liminf value is $1/8$.⁸ The ratio test does not tell us anything about the convergence. Also note that the series converges to 2.

Prop 2.4.1 (Rearrangement) $a_n \geq 0$.⁹ Suppose a bijection $r : \mathbb{N} \rightarrow \mathbb{N}$ exists.

$$1. \sum_{n=1}^{\infty} a_n = s \iff \sum_{n=1}^{\infty} a_{r(n)} = s$$

$$2. \sum_{n=1}^{\infty} a_n = \infty \iff \sum_{n=1}^{\infty} a_{r(n)} = \infty$$

Proof.

1. (\implies) Let $t_n = \sum_{k=1}^n a_{r(k)}$. Then t_n is increasing and bounded by s . Thus t_n converges by MCT, and $\lim t_n \leq s$.

$$s = \sum_{k=1}^{\infty} a_k \leq \sum_{n=1}^{\infty} a_{r(n)} = t = \lim t_n. \quad (a_n \geq 0 \text{ was used here.})$$

(\impliedby) Use $r^{-1}(n)$.

⁷The limit for the ratio test is much easier to calculate than the root test. That's why we use the ratio test.

⁸The ratios are: 2, $1/8$, 2, $1/8$...

⁹This is the important condition.

2. Contraposition of (1).

Prop 2.4.2 (Alternating Series Test) For a given sequence x_n , suppose the following holds.

- x_n is decreasing.
- $\lim x_n = 0$.

Then the series $\sum_{k=1}^{\infty} (-1)^{k-1} x_k$ is convergent.

Proof. Let $s_n = \sum_{k=1}^n (-1)^{k-1} x_k$. For $m < n$,

$$|s_n - s_m| = |(-1)^m x_{m+1} + \cdots + (-1)^{n-1} x_n| = |x_{m+1} - x_{m+2} + \cdots \pm x_n| \stackrel{(*)}{\in} [0, x_{m+1}]$$

$$(*) : x_{m+1} - x_{m+2} + \cdots + x_n = (x_{m+1} - x_{m+2}) + \cdots + (x_{n-2} - x_{n-1}) + x_n \geq 0$$

$$= x_{m+1} - (x_{m+2} - x_{m+3}) - \cdots - (x_{n-1} - x_n) \leq x_{m+1}$$

Check for the case with last term $-$.

Now, $\forall \epsilon > 0$, find N such that $|x_n| < \epsilon$ for $n \geq N$. Then for $n > m \geq N$, $|s_n - s_m| \leq x_{m+1} < \epsilon$.

Thus $\langle s_n \rangle$ is a Cauchy sequence and the given series converges.

Example. $a_n = (-1)^{n-1}/n$. $\sum a_n$ converges by alternating series test and converges to $\log 2$.

Remark. The rearrangement of the above example may not converge, or converge to a different value than $\log 2$.

Exam: 1.1 - 2.6

After the midterms we will be covering functions and continuity.

Chapter 1 has been about \mathbb{R} , and in Chapter 2, we have talked about subsets of \mathbb{R}^n .

2.1: What is \mathbb{R}^n ? Vector Space, IPS, Metric Space, Normed Space...

2.2: Open, closed sets

2.3: Bounded sets and Cauchy sequences

(2.4: Convergence Tests)

2.5: Compact Sets

2.6: Connected Sets

April 10th, 2019

2.5 Compact Set

Definition. $\{U_i : i \in I\}$ (I is the index set, $U_i \subset \mathbb{R}^d$) is called “family of sets”.

1. $\{U_i : i \in I\}$ is a **cover** of $K \subset \mathbb{R}^d \iff K \subset \bigcup_{i \in I} U_i$.
2. $\{U_i : i \in I\}$ is a **open cover** $\iff U_i$ are open for $\forall i$.
3. $J \subset I$, $\{U_i : i \in J\}$ is called a **subcover** of $\{U_i : i \in I\} \iff K \subset \bigcup_{i \in J} U_i$.

Definition. $K \subset \mathbb{R}^d$ is **compact** \iff Any open cover of K has finite subcover.

Example.

1. \mathbb{N} is not compact. Set $U_k = (k - 1/2, k + 1/2)$, then $\{U_k : k \in \mathbb{N}\}$ is a (open) cover of \mathbb{N} . But there are no finite subcover.
2. $A = (0, 1)$ is not compact. Set $U_k = (1/k, 1)$, then because $\bigcup_{k=1}^{\infty} U_k = (0, 1)$, $\{U_k : k \in \mathbb{N}\}$ is a (open) cover of A . But there are no finite subcover. $\bigcup_{i=1}^m U_{k_i} = U_{k_m} = (1/k_m, 1)$, which cannot contain $(0, 1)$.
3. $A = \{a_1, a_2, \dots, a_m\} \subset \mathbb{R}^d$ is compact. $\{U_i : i \in I\}$ be a cover of A . There exists $i_1, \dots, i_m \in I$ such that $a_k \in U_{i_k}$ for $k = 1, \dots, m$. Then $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$ is a finite subcover of A .

Main Theorem: **Heine-Borel Theorem**

$$K \text{ is compact} \iff K \text{ is bounded and closed.}$$

Remark.

1. This is a part of Thm 2.5.4
2. Proof: Prop 2.5.1, Thm 2.5.2, Prop 2.5.3
3. **Characterization of compact sets in \mathbb{R}^d .**¹⁰

¹⁰Compact Set 을 이 단순한 공간 안에서는 characterize 할 수 있다!

Proof.

(\implies) (Prop 2.5.1)

1. *Is K bounded?*

Set $U_k = N(0, k)$. Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d$. Thus $\{U_k : k \in \mathbb{N}\}$ is an open cover of K . There exists a finite subcover U_{k_1}, \dots, U_{k_m} ($k_1 < \dots < k_m$) of K . Then we have $K \subset \bigcup_{i=1}^m U_{k_i} = U_{k_m} = N(0, k_m)$. Therefore K is bounded.

2. *Is K closed?*

Suppose $x \in K^C$. Set $U_k = \{y : \|y - x\| > 1/k\}$. Then $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d \setminus \{x\} \supset K$. (Open cover) There exists a finite subcover U_{k_1}, \dots, U_{k_m} of K . $K \subset \bigcup_{i=1}^m U_{k_i} = U_{k_m}$. Therefore $K^C \supset U_{k_m}^C = \{y : \|y - x\| \leq 1/k_m\} \supset N(x, 1/k_m)$. Thus K^C is open, K is closed.

(\impliedby)

1. (Theorem 2.5.2) *Closed box is compact.*

$B = I_1 \times \dots \times I_d$, $I_i = [a_i, b_i]$. Let $\{U_i : i \in I\}$ is an open cover of B .

(Contradiction) Suppose there is no finite subcover of B .

Claim. There exists $B = B_1 \supset B_2 \supset \dots$ (closed boxes) such that

- $\text{diam}(B_n) = \frac{1}{2^n} \text{diam}(B_1)$
- There is no finite subcover of $\{U_i : i \in I\}$ covering B_n .

By Lemma 2.3.1, there exists $x \in \bigcap_{n=1}^{\infty} B_n$. Since $x \in B$, $\exists U_i$ such that $x \in U_i$. Then $\exists \epsilon > 0$ such that $N(x, \epsilon) \subset U_i$.¹¹ Set $\frac{1}{2^{n-1}} \text{diam}(B_1) < \epsilon$.

If $y \in B_n \implies \|x - y\| \leq \text{diam}(B_n) < \epsilon \implies y \in N(x, \epsilon)$. Then $B_n \subset N(x, \epsilon) \subset U_i$, contradiction.

2. *K : compact, $F \subset K$, F is closed $\implies F$: compact.*

Let $\{U_i : i \in I\}$ be an open cover of F . Then $\{U_i : i \in I\} \cup \{F^C\}$ is an open cover of K .

Because K is compact, there exists a finite subcover of K . There are two cases.

- (a) $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$: This is already a finite subcover of F .
- (b) $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}, F^C\}$: Since F^C does not cover F , U_{i_k} must cover F .

3. *Closed and bounded set is compact.*

Suppose K is bounded and closed. There exists a closed box B that contains K . Thus B is compact by (1), K is a closed subset of B . Then by (2), K is compact.

Theorem 2.5.2 가 가장 non-trivial 한 부분이다.

¹¹ n 이 충분히 크면 ball 안에 box 가 들어가고 box 는 U_i 안에 있다? Claim 의 2번째에 모순.

Theorem 2.5.4 The following are equivalent.

1. K is compact.
2. K is bounded and closed.
3. If A is an infinite subset of K , $\emptyset \neq A' \subset K$.
4. For a sequence $\langle x_n \rangle$ in K , there exists a convergent subsequence whose limit is in K .

Proof.

(1) \iff (2) by Heine-Borel Theorem.

(2) \implies (3) Suppose A is infinite and bounded. ($A \subset K$) By Bolzano-Weierstrass, $A' \neq \emptyset$.
 $A' \subset A' \cup A = \overline{A} \subset K$. (\overline{A} is the smallest closed set containing A , $A \subset K$.)

(3) \implies (4) Let $A = \{x_1, x_2, \dots\}$

1. If A is finite, trivial. (Take a constant subsequence, which constant $\in K$.)

2. If A is infinite, $x \in A' \subset K$ by (3). ($x \in A'$ by Thm 2.3.4)

(4) \implies (2)

1. K is bounded.

(Contradiction) Suppose K is not bounded. Then $\forall n \in \mathbb{N}$, there exists $x_n \in K$, $\|x_n\| \geq n$.
There are no convergent subsequences, contradiction.

2. K is closed.

(Contradiction) Suppose K is not closed.

(a) K : finite $\rightarrow K$: closed \rightarrow Contradiction.

(b) K : infinite $\rightarrow K$: infinite and bounded $\xrightarrow{\text{B-W}} K' \neq \emptyset$

Note. $K' \subset K \iff K$: closed.

Then if K' is not a subset of K ¹², there exists $x \in K' \setminus K$. Since $x \in K'$, there exists a sequence $\langle x_n \rangle$ in $K \setminus \{x\}$ ($= K$)¹³ converging to x . Thus for a subsequence of $\langle x_n \rangle$, its limit must be in K . But x is the only possible limit value. $x \in K$. Contradiction.

¹²Contraposition

¹³ $x \notin K$

April 12th, 2019

Problem 2.4.7 (바) $\sum \frac{1}{n^p - n^q}$ ($0 < q < p$)

$0 < n^p - n^q \leq n^p$ 이므로 $1/n^p \leq 1/(n^p - n^q)$ 가 되어 $p \leq 1$ 이면 발산한다.

충분히 큰 N 에 대하여 $n \geq N$ 일 때마다 $n^p - n^q \geq n^p/2$ 가 되게 할수 있다. (이 때 $n^p/2 \geq n^q$ 이므로 $n^{p-q} \geq 2$ 가 되어 N 을 잡을 수 있다) 비교판정법에 의해 수렴한다.

Problem 2.7.12 Given $\langle a_n \rangle$ such that $\lim a_n = a$, show that $\sigma_n = \frac{a_1 + \cdots + a_n}{n}$ also converges to a .

Problem 2.7.13 $r < 1$, $\|x_{n+2} - x_{n+1}\| \leq r \|x_{n+1} - x_n\|$. Show that $\langle x_n \rangle$ is a Cauchy sequence.

Proof. $\|x_{n+1} - x_n\| \leq r^{n-1} \|x_2 - x_1\| = r^{n-1} A$, for $A \in \mathbb{R}$. Given $\epsilon > 0$, exists N such that for all $n \geq N$, $\|x_{n+1} - x_n\| < A r^{n-1} < \epsilon$. Then we have

$$\begin{aligned} m > n \geq N \Rightarrow \|x_n - x_m\| &\leq \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq \|x_{n+1} - x_n\| (1 + r + r^2 + \cdots) < \frac{\epsilon}{1 - r} \end{aligned}$$

Remark. Counterexample for $\|x_{n+2} - x_{n+1}\| < \|x_{n+1} - x_n\|$. $x_n = \sum_{k=1}^n \frac{1}{k}$

Problem 2.7.14 $x_n \rightarrow x$, $A_k = \{x_i : i \geq k\}$. Show that $\bigcap_{k=1}^{\infty} \overline{A_k} = \{x\}$.

Proof. Given $\epsilon > 0$, there exists N such that $n \geq N \Rightarrow x_n \in (x - \epsilon, x + \epsilon)$. Either $x_n = x$, or $x_n \in (x - \epsilon, x + \epsilon) \setminus \{x\}$. Thus $x \in \overline{A_k}$ for all k . $\{x\} \subset \bigcap_{k=1}^{\infty} \overline{A_k}$.

For $y \in \mathbb{R} \setminus \{x\}$, we want to show that $y \notin \bigcap_{k=1}^{\infty} \overline{A_k}$. Then we want to find N such that $y \notin \overline{A_N}$. Since $\|x - y\| > 0$, set $\epsilon = \frac{1}{3} \|x - y\|$. There exists N such that $\|x_n - x\| < \epsilon$. Then $\forall x_n \notin N(y, \epsilon)$. $\overline{A_N} = \{x_N, x_{N+1}, \dots\}$, and y cannot be in $\overline{A_N}$. $\{x\}^C \subset (\bigcap_{k=1}^{\infty} \overline{A_k})^C \Rightarrow \bigcap_{k=1}^{\infty} \overline{A_k} \subset \{x\}$.

Problem 2.7.15 $\sum a_n$ converges absolutely.

1. $\sum a_n^2$ converges.

Proof. $a_n^2 < |a_n|$ for large n . Converges by comparison test.

2. $\sum \frac{a_n}{1 + a_n}$

Proof. Since $a_n \rightarrow 0$, exists N such that $n \geq N \Rightarrow |a_n| < 1/3$. Then for $n \geq N$, $|1 + a_n| \geq 1 - |a_n| > 2/3 > 1/3$, $1/|1 + a_n| < 3$. We have $\left| \frac{a_n}{1 + a_n} \right| < 3|a_n|$. Converges by comparison test.

3. $\sum \frac{a_n^2}{1 + a_n^2}$

Proof. Trivial from 1, 2.