## Assignment 4

Due Date: 2019/05/16, 1:30 PM

**1.** Prove that the following functions  $f: X \to \mathbb{R}$  are continuous. Determine whether each function is uniformly continuous or not, and verify your answer.

(1) 
$$f(x) = \frac{1}{(x+1)^3}$$
 with  $X = (-1, 1)$ 

(2) 
$$f(x) = \frac{1}{x+3}$$
 with  $X = (0, \infty)$ 

(3) 
$$f(x) = \frac{1}{x^2+1}$$
 with  $X = \mathbb{R}$ .

(4) 
$$f(x) = \sqrt{x^2 + 1}$$
 with  $X = (0, \infty)$ .

- 2. Answer the following questions.
- (1) Suppose that  $f: X \to Y$  is a continuous function on X, and that the set X is bounded (in some Euclidean space). Then, prove that f is uniformly continuous on X if and only if f has a continuous extension to  $\overline{X}$ .
- (2) Use part (1), show that

$$f(x, y) = \sqrt{x^{2020} + y^{2020} + x^2 + 1}$$

is a uniformly continuous function on  $X = \{(x, y) : x^2 + y^2 < 3\}.$ 

(3) For a < b, suppose that  $f: (a, b) \to \mathbb{R}$  is a continuous function. Then, prove that f is uniformly continuous on (a, b) if and only if  $\lim_{x\to a^+} f(x)$  and  $\lim_{x\to b^-} f(x)$  exist.

1

3. Compute the following values and verify your answer rigorously.

(1) 
$$\lim_{x\to 0^-} \frac{\max\{x,0\}}{x} - \lim_{x\to 0^+} \frac{\max\{x,0\}}{x}$$

(2) 
$$\lim_{x\to 0^+} \frac{x^3}{|x|}$$

- **4.** For uniformly continuous functions  $f, g: X \to \mathbb{R}$ , prove or disprove the following statements:
- (1) f + g is uniformly continuous on X.
- (2) fg is uniformly continuous on X.
- (3)  $\max\{f, g\}$  is uniformly continuous on X.
- **5.** Let  $f:[0, 1] \to \mathbb{R}$  be a continuous function on [0, 1] and suppose that  $f(x) \in [0, 1]$  for all  $x \in [0, 1]$ . Prove that there exists  $x_0 \in [0, 1]$  such that  $f(x_0) = x_0^{2016}$ .
- **6.** A function  $f: X \to Y$  is called a Hölder continuous function on X if there exists  $\alpha > 0$  and M > 0 such that

$$||f(x) - f(y)|| \le M||x - y||^{\alpha}$$
 for all  $x, y \in X$ .

- (1) Prove that a Hölder continuous f on X is uniformly continuous (and hence continuous) on X.
- (2) Suppose that  $X = \mathbb{R}^d$  for some  $d \in \mathbb{N}$ , and that f is Hölder continuous on  $\mathbb{R}^d$  with some  $\alpha > 1$ . Prove that f is merely a constant function.
- 7. Suppose that  $f: \mathbb{R}^m \to \mathbb{R}^n$  is a continuous function.
- (1) Prove that  $f(\overline{A}) \subset \overline{f(A)}$  for all  $A \subset \mathbb{R}^m$ .
- (2) Prove or disprove that  $f(\overline{A}) = \overline{f(A)}$  for all  $A \subset \mathbb{R}^m$ .
- **8.** Suppose that A, B are disjoint closed subsets of  $\mathbb{R}^d$ , i.e.,  $A \cap B = \emptyset$ . Prove that there exists a continuous function  $f: \mathbb{R}^d \to [0, 1]$  such that f(x) = 1 for all  $x \in A$  and f(x) = 0 for all  $x \in B$ . (Hint. Use  $g(x) = \text{dist}(\{x\}, A)$ )

- **9.** Suppose that  $X \subset \mathbb{R}^m$  is a compact set and that  $f: X \to Y$  is a function for some  $Y \subset \mathbb{R}^n$ . Define a set  $E \subset \mathbb{R}^{m+n}$  by  $E = \{(x, f(x)) : x \in X\}$ . Prove that f is continuous on X if and only if E is a compact set in  $\mathbb{R}^{m+n}$ .
- **10.** A function  $f:(a, b) \to \mathbb{R}$  is called convex on (a, b) if and only if  $f(\lambda x + (1 \lambda)y) \le \lambda f(x) + (1 \lambda)f(y) \text{ for all } x, y \in (a, b) \text{ and } \lambda \in [0, 1].$

Prove that, all convex function f on (a, b) is continuous on (a, b).

**11.** Suppose that  $f:[a,b]\to\mathbb{R}$  is a continuous function on [a,b]. Prove that

$$f^*(x) = \sup\{f(y) : y \in [a, x]\}$$

is an increasing continuous function on [a, b].

**12.** Does there exists a continuous function on (0, 1], which does not have maximum and minimum? Prove your answer.