

## 해석개론 및 연습 1 과제 #4

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1. It is trivial that  $x$  is continuous on  $\mathbb{R}$ , thus finite dimensional polynomials are also continuous on  $\mathbb{R}$ , since polynomials are composed of sums and products of  $x$ .

(1)  $(x+1)^3$  is continuous on  $(-1, 1)$ , and never 0 in this interval. Thus  $f(x) = 1/(x+1)^3$  is continuous on  $(-1, 1)$ .

Consider  $x_n = 1/n - 1$ . We immediately observe that  $-1 < 1/n - 1 \leq 0$ , and  $\lim_{n \rightarrow \infty} x_n = -1$ . Thus  $\langle x_n \rangle$  is a Cauchy sequence in  $(-1, 1)$ . But  $f(x_n) = n^3$ , and the sequence diverges. Therefore  $f(x)$  is not uniformly continuous on  $X$ .

(2)  $x+3$  is continuous on  $(0, \infty)$  and never 0 in this interval. Thus  $f(x) = 1/(x+3)$  is continuous on  $(0, \infty)$ .

For given  $\epsilon > 0$ , Set  $\delta = 9\epsilon$ . For  $x, y \in (0, \infty)$ , we observe that

$$9 < xy + 3x + 3y + 9 = (x+3)(y+3)$$

then if  $|x - y| < \delta$ ,

$$|f(x) - f(y)| = \left| \frac{1}{x+3} - \frac{1}{y+3} \right| = \frac{|x-y|}{(x+3)(y+3)} < \frac{9\epsilon}{9} = \epsilon$$

Thus  $f(x)$  is uniformly continuous on  $X$ .

(3)  $x^2 + 1$  is continuous on  $\mathbb{R}$ , and never 0. Thus  $f(x) = 1/(x^2 + 1)$  is continuous on  $\mathbb{R}$ .

Given  $\epsilon > 0$ , set  $\delta = \epsilon$ . Since

$$x^2y^2 + x^2 + y^2 + 1 - x - y = x^2y^2 + \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \frac{1}{2} > 0$$

, the following directly follows.

$$\frac{x+y}{(x^2+1)(y^2+1)} < 1$$

Now, if  $|x - y| < \delta = \epsilon$ ,

$$|f(x) - f(y)| = \left| \frac{1}{x^2+1} - \frac{1}{y^2+1} \right| = \frac{|x+y||x-y|}{(x^2+1)(y^2+1)} < |x-y| < \epsilon$$

Therefore  $f(x)$  is uniformly continuous on  $X$ .

(4)  $x^2 + 1$  is continuous on  $(0, \infty)$ , and  $\sqrt{x}$  is continuous on  $(1, \infty)$ . Thus their composition,  $f(x) = \sqrt{x^2 + 1}$  is continuous on  $(0, \infty)$ .

It is trivial that

$$\sqrt{x^2 + 1} + \sqrt{y^2 + 1} - x - y > 0 \implies \frac{x+y}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}} < 1$$

so given  $\forall \epsilon > 0$ , if  $|x - y| < \delta = \epsilon$ , we have

$$|f(x) - f(y)| = \left| \sqrt{x^2 + 1} - \sqrt{y^2 + 1} \right| = \frac{(x+y)|x-y|}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}} < |x-y| < \epsilon$$

Therefore  $f(x)$  is uniformly continuous on  $X$ .

2. (1) ( $\implies$ ) Suppose  $x_0 \in X' \setminus X$ . Then there exists a sequence  $x_n$  in  $X$  that converges to  $x_0$ . Because  $f$  is uniformly continuous and  $x_n$  is a Cauchy sequence,  $f(x_n)$  is also a Cauchy sequence. Thus  $\lim_{n \rightarrow \infty} f(x_n) = \alpha \in \mathbb{R}$  exists. Define the continuous extension  $g$  of  $f$  by setting  $g(x_0) = \alpha$ .

Now we must check if  $g(x_0)$  is well-defined. For any two sequence  $\langle x_n \rangle, \langle y_n \rangle$  that converge to  $x_0$ , consider  $\langle z_n \rangle = x_1, y_1, x_2, y_2, \dots$ . It is trivial that  $z_n \rightarrow x_0$ . Since  $\langle z_n \rangle$  is a Cauchy sequence,  $\langle f(z_n) \rangle$  is also a Cauchy sequence by uniform continuity of  $f$ . Let its limit be  $\gamma$ . Then  $\langle f(x_n) \rangle, \langle f(y_n) \rangle$  is a subsequence of  $\langle f(z_n) \rangle$ , thus they both must converge to  $\gamma$ .

Now we must check if  $g$  is continuous on  $\overline{X}$ . For  $x_0 \in \overline{X}$ , there exists a sequence  $x_n$  in  $X$  that converges to  $x_0$ . Since  $g(x_n) = f(x_n)$ ,  $f(x_n)$  converges to  $f(x_0)$  by continuity of  $f$ . Thus  $g(x)$  is continuous extension of  $f$  to  $\overline{X}$ .

( $\impliedby$ ) Since  $X$  is bounded, there exists a closed ball  $B$  such that  $X \subset B$ . Because  $\overline{X}$  is the smallest closed set containing  $X$ ,  $\overline{X} \subset B$ , and  $\overline{X}$  is bounded. We know that  $\overline{X}$  is closed, thus  $\overline{X}$  is compact. By Heine's Theorem, the continuous extension  $g$  of  $f$  is uniformly continuous on  $\overline{X}$ . Now  $f$  is uniformly continuous since it is defined on a subset of the domain of  $g$ .

- (2) Since  $\overline{X} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 3\}$ , define the continuous extension  $g$  of  $f$  by

$$g(x, y) = \begin{cases} f(x, y) & (x^2 + y^2 < 3) \\ \sqrt{x^{2020} + y^{2020} + x^2 + 1} & (x^2 + y^2 = 3, x = \sqrt{3} \cos \theta, y = \sqrt{3} \sin \theta) \end{cases}$$

(Such  $\theta \in [0, 2\pi)$  exists) Now we show that  $g(x, y)$  is continuous on  $\overline{X}$ . For  $(x_0, y_0) \in X' \setminus X$ ,  $x_0^2 + y_0^2 = 3$ , set  $x_0 = \sqrt{3} \cos \theta, y_0 = \sqrt{3} \sin \theta$ . Define a sequence in  $X$  by

$$(x_n, y_n) = \left( \left( \sqrt{3} - \frac{1}{n} \right) \cos \theta, \left( \sqrt{3} - \frac{1}{n} \right) \sin \theta \right)$$

( $x_n^2 + y_n^2 < 3$  can be easily checked), and it converges to  $(x_0, y_0)$ . It can be easily seen that  $g(x_n, y_n) \rightarrow g(x_0, y_0)$ , because  $1/n \rightarrow 0$ . Thus  $g(x)$  is a continuous extension of  $f$  to  $\overline{X}$  and therefore  $f$  is uniformly continuous on  $X$ .

- (3) ( $\implies$ ) Since  $f$  is uniformly continuous on  $(a, b)$ , there exists a continuous extension  $g$  of  $f$  to  $[a, b]$ . Since  $g(x)$  is continuous at  $x = a$ ,  $\forall \epsilon > 0, \exists \delta$  s.t.  $x \in \overline{X}, |x - a| < \delta \implies |g(x) - g(a)| < \epsilon$ . Observe that  $x \in \overline{X}, |x - a| < \delta$  is equivalent to  $x \in [a, a + \delta)$ . Thus  $(a, a + \delta) \subset [a, b]$ , and we have

$$x \in (a, a + \delta) \implies x \in [a, a + \delta) \implies |g(x) - g(a)| = |f(x) - g(a)| < \epsilon$$

Now by definition,  $\lim_{x \rightarrow a^+} f(x) = g(a)$ .

Similarly, since  $g(x)$  is continuous on  $x = b$ ,  $\forall \epsilon > 0, \exists \delta$  s.t.  $x \in \overline{X}, |x - b| < \delta \implies |g(x) - g(b)| < \epsilon$ . Observe that  $x \in \overline{X}, |x - b| < \delta$  is equivalent to  $x \in (b - \delta, b]$ . Thus  $(b - \delta, b) \subset [a, b]$ , and we have

$$x \in (b - \delta, b) \implies x \in (b - \delta, b] \implies |g(x) - g(b)| = |f(x) - g(b)| < \epsilon$$

Now by definition,  $\lim_{x \rightarrow b^-} f(x) = g(b)$ .

3. Let the domain be  $X = \mathbb{R} \setminus \{0\}$ .

- (1)  $\lim_{x \rightarrow 0^-} \frac{\max\{x, 0\}}{x} = 0$ .  
 $\forall \epsilon > 0$ , set  $\delta = \epsilon$ ,  $(-\delta, 0) \subset X$ , and since  $\max\{x, 0\} = 0$  for all  $x$  in this interval,  
 $\frac{\max\{x, 0\}}{x} = 0 < \epsilon$ .  
 $\lim_{x \rightarrow 0^+} \frac{\max\{x, 0\}}{x} = 1$ .  
 $\forall \epsilon > 0$ , set  $\delta = \epsilon$ ,  $(0, \delta) \subset X$ , and since  $\max\{x, 0\} = x$  for all  $x$  in this interval,  
 $\left| \frac{\max\{x, 0\}}{x} - 1 \right| = 0 < \epsilon$ .

Thus the wanted limit is  $-1$ .

- (2) Given  $\epsilon > 0$ , set  $\delta = \sqrt{\epsilon}$ .  $(0, \delta) \subset X$ , and if  $0 < x < \delta$ ,  $0 < x^2 < \delta^2 = \epsilon$ , then  $0 < x^3/|x| < \epsilon$ .  
Since  $x > 0$ ,  $|x^3/|x| - 0| < \epsilon$ .  $\therefore \lim_{x \rightarrow 0^+} \frac{x^3}{|x|} = 0$ .

4. (1) **True.** Since  $f, g$  are uniformly continuous on  $X$ ,  $\forall \epsilon > 0$ ,  $\exists \delta$  s.t. if  $\|x - y\| < \delta$  for all  $x, y \in X \implies \|f(x) - f(y)\| < \epsilon/2$  and  $\|g(x) - g(y)\| < \epsilon/2$ . We immediately have

$$\|f(x) + g(x) - f(y) - g(y)\| \leq \|f(x) - f(y)\| + \|g(x) - g(y)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

Thus  $f + g$  is uniformly continuous on  $X$ .

- (2) **False.** (Counterexample)  $f(x) = g(x) = x$  defined on  $\mathbb{R}$ .  $f, g$  are uniformly continuous, but  $x^2$  is not uniformly continuous (proof in textbook).

- (3) **True.** We use the fact  $\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}$ . For a constant  $c$ , since  $cf$  is uniformly continuous if  $f$  is uniformly continuous, it is sufficient to show that  $|f - g|$  is uniformly continuous, then the result directly follows by (1). Furthermore, because  $f - g$  and  $\|x\|$  are uniformly continuous,  $(\forall \epsilon > 0$ , set  $\delta = \epsilon$ . Then  $\|x - y\| < \delta \implies \| \|x\| - \|y\| \| \leq \|x - y\| < \epsilon$ ) we show that their composition is uniformly continuous.

**Claim.** Suppose  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  are uniformly continuous. Then  $g \circ f$  is uniformly continuous.

**Proof.** For given  $\epsilon > 0$ ,  $\exists \delta_1$  s.t.  $\|x' - y'\| < \delta_1 \implies \|g(x') - g(y')\| < \epsilon$ , for all  $x', y' \in Y$ .  
For this  $\delta_1$ ,  $\exists \delta$  s.t.  $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \delta_1$ , for all  $x, y \in X$ . Then if  $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \delta_1 \implies \|g(f(x)) - g(f(y))\| < \epsilon$ . Thus  $g \circ f$  is uniformly continuous.

5. Define  $g(x) = x^{2016} - f(x)$  on  $[0, 1]$ . Then since  $x^{2016}, f(x)$  are continuous, its difference  $g(x)$  is also continuous. Thus we have

$$g(0) = 0 - f(0) \leq 0 \leq 1 - f(1) = g(1)$$

and by IVT, there exists  $x_0 \in [0, 1]$  s.t.  $g(x_0) = 0$ . For this  $x_0$ ,  $f(x_0) = x_0^{2016}$ .

6. (1) Suppose  $f$  is Hölder continuous. Given  $\forall \epsilon > 0$ , set  $\delta = \left(\frac{\epsilon}{M}\right)^{1/\alpha}$ . Since  $y = x^\alpha$  ( $\alpha > 0$ ) is increasing, for all  $x, y \in X$ , if  $\|x - y\| < \delta$ ,  $M \|x - y\|^\alpha < \epsilon$ . By the Hölder continuity condition,  $\|f(x) - f(y)\| < M \|x - y\|^\alpha < \epsilon$ , and  $f$  is uniformly continuous on  $X$ .

**(2)** For fixed  $x, y \in X$ , suppose  $x < y$ . Given  $N \in \mathbb{N}$ , define  $x_0, \dots, x_N$  as follows.

$$x_0 = x, x_1 = x_0 + 1 \cdot \frac{y-x}{N}, \dots, x_i = x_0 + i \cdot \frac{y-x}{N}, \dots, x_N = y$$

Then we have

$$\|f(x) - f(y)\| \leq \sum_{i=0}^{N-1} \|f(x_i) - f(x_{i+1})\| \leq \sum_{i=0}^{N-1} M \|x_i - x_{i+1}\|^\alpha = M \frac{\|x - y\|^\alpha}{N^{\alpha-1}}$$

As  $N \rightarrow \infty$ ,  $0 \leq \|f(x) - f(y)\| \leq \lim_{N \rightarrow \infty} M \frac{\|x - y\|^\alpha}{N^{\alpha-1}} = 0$ . Thus  $\|f(x) - f(y)\| = 0$  for all  $x, y \in X$ . Thus  $f(x) = f(0)$ , merely a constant function.

**7. (1)** Let  $y \in f(\bar{A})$ . Then there exists  $x_0 \in \bar{A}$  s.t.  $f(x_0) = y$ . Given  $\epsilon > 0$ , since  $x_0 \in \bar{A}$ , there exists  $\delta > 0$  s.t.  $N_{\mathbb{R}^m}(x_0, \delta) \cap A \neq \emptyset$ . Take an element  $x$  from  $N_{\mathbb{R}^m}(x_0, \delta) \cap A$ . Since  $x \in A$ ,  $f(x) \in f(A)$ , and since  $x \in N_{\mathbb{R}^m}(x_0, \delta)$ ,  $f(x) \in N_{\mathbb{R}^n}(f(x_0), \epsilon)$  by continuity of  $f$ . Therefore we have  $f(x) \in N_{\mathbb{R}^n}(f(x_0), \epsilon) \cap f(A) \neq \emptyset$ , and  $f(x_0) = y \in \overline{f(A)}$ .  $f(\bar{A}) \subset \overline{f(A)}$ .

**(2)** False. Consider  $f(x) = \frac{1}{x^2+1}$ .  $f$  is continuous. Set  $A = (0, \infty)$ . Then  $\bar{A} = [0, \infty)$ ,  $f(\bar{A}) = (0, 1)$ , while  $\overline{f(A)} = [0, 1]$ .

**8.** (Urysohn's Lemma)

**9.**

**10.** We first prove the following inequality.

**Claim.** For convex function  $f : (a, b) \rightarrow \mathbb{R}$  and  $a < x < y < z$ ,

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y}$$

**Proof.** Since  $f$  is convex, for  $t \in (0, 1)$ ,

$$f(tx + (1-t)z) \leq tf(x) + (1-t)f(z)$$

, and set  $y = tx + (1-t)z$ . Multiply  $z - x$  on both sides, then

$$(z - x)f(y) \leq t(z - x)f(x) + (1-t)(z - x)f(z) = (z - y)f(x) + (y - x)f(z)$$

Rearranging the terms gives

$$(z - y)(f(y) - f(x)) \leq (y - x)(f(z) - f(y))$$

, which directly gives the inequality.

Suppose  $x \in (a, b)$ . Since  $(a, b)$  is open, select real numbers s.t.  $x_0 < x_1 < x, y < x_2 < x_3$ , and define  $C = \max\{\frac{|f(x_1) - f(x_0)|}{x_1 - x_0}, \frac{|f(x_3) - f(x_2)|}{x_3 - x_2}\}$ . Given  $\epsilon > 0$ , choose  $\delta = \min\{\epsilon/C, x_2 - x_1\}$ . If  $x > y$ , by **Claim**,

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x_2) - f(x)}{x_2 - x} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

and

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(y) - f(x_1)}{y - x_1} \geq \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

, therefore

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq C = \max \left\{ \frac{|f(x_1) - f(x_0)|}{x_1 - x_0}, \frac{|f(x_3) - f(x_2)|}{x_3 - x_2} \right\}$$

and the inequality above can be shown similarly for  $x < y$ .

Therefore if  $|x - y| < \delta$ ,  $|f(x) - f(y)| \leq C|x - y| \leq C\frac{\epsilon}{C} = \epsilon$ .  $f(x)$  is continuous.

**11.**

**12.**