해석개론 및 연습 1 과제 #4

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1. It is trivial that x is continuous on $\mathbb{R}$, thus finite dimensional polynomials are also continuous on $\mathbb{R}$, since polynomials are composed of sums and products of x.

(1) $(x+1)^3$ is continuous on $(-1,1)$, and never 0 in this interval. Thus $f(x) = 1/(x+1)^3$ is continuous on $(-1,1)$.
Consider $x_n = 1/n - 1$. We immediately observe that $-1 < 1/n - 1 \leq 0$, and $\lim_{n\to\infty} x_n = -1$. Thus $\langle x_n \rangle$ is a Cauchy sequence in $(-1,1)$. But $f(x_n) = n^3$, and the sequence diverges. Therefore $f(x)$ is not uniformly continuous on X.

(2) $x+3$ is continuous on $(0,\infty)$ and never 0 in this interval. Thus $f(x) = 1/(x+3)$ is continuous on $(0,\infty)$.
For given $\epsilon > 0$, Set $\delta = 9\epsilon$. For $x, y \in (0,\infty)$, we observe that

$$9 < xy + 3x + 3y + 9 = (x+3)(y+3)$$

then if $|x-y| < \delta$,

$$|f(x) - f(y)| = \left|\frac{1}{x+3} - \frac{1}{y+3}\right| = \frac{|x-y|}{(x+3)(y+3)} < \frac{9\epsilon}{9} = \epsilon$$

Thus $f(x)$ is uniformly continuous on X.

(3) x^2+1 is continuous on $\mathbb{R}$, and never 0. Thus $f(x) = 1/(x^2+1)$ is continuous on $\mathbb{R}$.
Given $\epsilon > 0$, set $\delta = \epsilon$. Since

$$x^2y^2 + x^2 + y^2 + 1 - x - y = x^2y^2 + \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \frac{1}{2} > 0$$

, the following directly follows.

$$\frac{x+y}{(x^2+1)(y^2+1)} < 1$$

Now, if $|x-y| < \delta = \epsilon$,

$$|f(x) - f(y)| = \left|\frac{1}{x^2+1} - \frac{1}{y^2+1}\right| = \frac{|x+y|\,|x-y|}{(x^2+1)(y^2+1)} < |x-y| < \epsilon$$

Therefore $f(x)$ is uniformly continuous on X.

(4) x^2+1 is continuous on $(0,\infty)$, and $\sqrt{x}$ is continuous on $(1,\infty)$. Thus their composition, $f(x) = \sqrt{x^2+1}$ is continuous on $(0,\infty)$.
It is trivial that

$$\sqrt{x^2+1} + \sqrt{y^2+1} - x - y > 0 \implies \frac{x+y}{\sqrt{x^2+1} + \sqrt{y^2+1}} < 1$$

so given $\forall\epsilon > 0$, if $|x-y| < \delta = \epsilon$, we have

$$|f(x) - f(y)| = \left|\sqrt{x^2+1} - \sqrt{y^2+1}\right| = \frac{(x+y)\,|x-y|}{\sqrt{x^2+1} + \sqrt{y^2+1}} < |x-y| < \epsilon$$

Therefore $f(x)$ is uniformly continuous on on X.

2. (1) ($\Longrightarrow$) Suppose $x_0 \in X' \backslash X$. Then there exists a sequence x_n in X that converges to x_0. Because f is uniformly continuous and x_n is a Cauchy sequence, $f(x_n)$ is also a Cauchy sequence. Thus $\lim_{n\to\infty} f(x_n) = \alpha \in \mathbb{R}$ exists. Define the continuous extension g of f by setting $g(x_0) = \alpha$.

Now we must check if $g(x_0)$ is well-defined. For any two sequence $\langle x_n \rangle, \langle y_n \rangle$ that converge to x_0, consider $\langle z_n \rangle = x_1, y_1, x_2, y_2, \ldots$. It is trivial that $z_n \to x_0$. Since $\langle z_n \rangle$ is a Cauchy sequence, $\langle f(z_n) \rangle$ is also a Cauchy sequence by uniform continuity of f. Let its limit be γ. Then $\langle f(x_n) \rangle, \langle f(y_n) \rangle$ is a subsequence of $\langle f(z_n) \rangle$, thus they both must converge to γ.

Now we must check if g is continuous on $\overline{X}$. For $x_0 \in \overline{X}$, there exists a sequence x_n in X that converges to x_0. Since $g(x_n) = f(x_n)$, $f(x_n)$ converges to $f(x_0)$ by continuity of f. Thus $g(x)$ is continuous extension of f to $\overline{X}$.

($\Longleftarrow$) Since X is bounded, there exists a closed ball B such that $X \subset B$. Because $\overline{X}$ is the smallest closed set containing X, $\overline{X} \subset B$, and $\overline{X}$ is bounded. We know that $\overline{X}$ is closed, thus $\overline{X}$ is compact. By Heine's Theorem, the continuous extension g of f is uniformly continuous on $\overline{X}$. Now f is uniformly continuous since it is defined on a subset of the domain of g.

(2) Since $\overline{X} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 3\}$, define the continuous extension g of f by

$$g(x,y) = \begin{cases} f(x,y) & (x^2+y^2<3) \\ \sqrt{x^{2020}+y^{2020}+x^2+1} & (x^2+y^2=3, x=\sqrt{3}\cos\theta, y=\sqrt{3}\sin\theta) \end{cases}$$

(Such $\theta \in [0, 2\pi)$ exists) Now we show that $g(x, y)$ is continuous on $\overline{X}$. For $(x_0, y_0) \in X' \backslash X$, $x_0^2 + y_0^2 = 3$, set $x_0 = \sqrt{3}\cos\theta, y_0 = \sqrt{3}\sin\theta$. Define a sequence in X by

$$(x_n, y_n) = \left(\left(\sqrt{3} - \frac{1}{n}\right)\cos\theta, \left(\sqrt{3} - \frac{1}{n}\right)\sin\theta\right)$$

($x_n^2 + y_n^2 < 3$ can be easily checked), and it converges to (x_0, y_0). It can be easily seen that $g(x_n, y_n) \to g(x_0, y_0)$, because $1/n \to 0$. Thus $g(x)$ is a continuous extension of f to $\overline{X}$ and therefore f is uniformly continuous on X.

(3) ($\Longrightarrow$) Since f is uniformly continuous on (a, b), there exists a continuous extension g of f to $[a, b]$. Since $g(x)$ is continuous at $x = a$, $\forall\epsilon > 0$, $\exists\,\delta$ s.t. $x \in \overline{X}$, $|x - a| < \delta \implies |g(x) - g(a)| < \epsilon$. Observe that $x \in \overline{X}, |x - a| < \delta$ is equivalent to $x \in [a, a + \delta)$. Thus $(a, a + \delta) \subset [a, b]$, and we have

$$x \in (a, a+\delta) \implies x \in [a, a+\delta) \implies |g(x) - g(a)| = |f(x) - g(a)| < \epsilon$$

Now by definition, $\lim_{x\to a^+} f(x) = g(a)$.

Similarly, since $g(x)$ is continuous on $x = b$, $\forall\epsilon > 0$, $\exists\,\delta$ s.t. $x \in \overline{X}$, $|x - b| < \delta \implies |g(x) - g(b)| < \epsilon$. Observe that $x \in \overline{X}, |x - b| < \delta$ is equivalent to $x \in (b - \delta, b]$. Thus $(b - \delta, b) \subset [a, b]$, and we have

$$x \in (b-\delta, b) \implies x \in (b-\delta, b] \implies |g(x) - g(b)| = |f(x) - g(b)| < \epsilon$$

Now by definition, $\lim_{x\to b^-} f(x) = g(b)$.

3. Let the domain be $X = \mathbb{R}\backslash\{0\}$.

(1) $\lim\limits_{x\to 0-} \dfrac{\max\{x,0\}}{x} = 0.$
$\forall\epsilon > 0$, set $\delta = \epsilon$, $(-\delta, 0) \subset X$, and since $\max\{x,0\} = 0$ for all x in this interval, $\frac{\max\{x,0\}}{x} = 0 < \epsilon$.
$\lim\limits_{x\to 0+} \dfrac{\max\{x,0\}}{x} = 1.$
$\forall\epsilon > 0$, set $\delta = \epsilon$, $(0, \delta) \subset X$, and since $\max\{x,0\} = x$ for all x in this interval, $\left|\frac{\max\{x,0\}}{x} - 1\right| = 0 < \epsilon$.
Thus the wanted limit is -1.

(2) Given $\epsilon > 0$, set $\delta = \sqrt{\epsilon}$. $(0,\delta) \subset X$, and if $0 < x < \delta$, $0 < x^2 < \delta^2 = \epsilon$, then $0 < x^3/|x| < \epsilon$. Since $x > 0$, $\left|x^3/|x| - 0\right| < \epsilon$. $\therefore \lim\limits_{x\to 0+} \frac{x^3}{|x|} = 0$.

4. **(1)** **True**. Since f, g are uniformly continuous on X, $\forall\epsilon > 0$, $\exists\,\delta$ s.t. if $\|x - y\| < \delta$ for all $x, y \in X \implies \|f(x) - f(y)\| < \epsilon/2$ and $\|g(x) - g(y)\| < \epsilon/2$. We immediately have

$$\|f(x) + g(x) - f(y) - g(y)\| \leq \|f(x) - f(y)\| + \|g(x) - g(y)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

Thus $f + g$ is uniformly continuous on X.

(2) **False**. (Counterexample) $f(x) = g(x) = x$ defined on $\mathbb{R}$. f, g are uniformly continuous, but x^2 is not uniformly continuous (proof in textbook).

(3) **True**. We use the fact $\max\{f,g\} = \frac{f+g}{2} + \frac{|f-g|}{2}$. For a constant c, since cf is uniformly continuous if f is uniformly continuous, it is sufficient to show that $|f - g|$ is uniformly continuous, then the result directly follows by (1). Furthermore, because $f - g$ and $\|x\|$ are uniformly continuous, ($\forall\epsilon > 0$, set $\delta = \epsilon$. Then $\|x - y\| < \delta \implies \|\|x\| - \|y\|\| \leq \|x - y\| < \epsilon$) we show that their composition is uniformly continuous.

Claim. Suppose $f : X \to Y$, $g : Y \to Z$ are uniformly continuous. Then $g \circ f$ is uniformly continuous.
Proof. For given $\epsilon > 0$, $\exists\,\delta_1$ s.t. $\|x' - y'\| < \delta_1 \implies \|g(x') - g(y')\| < \epsilon$, for all $x', y' \in Y$. For this δ_1, $\exists\,\delta$ s.t. $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \delta_1$, for all $x, y \in X$. Then if $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \delta_1 \implies \|g(f(x)) - g(f(y))\| < \epsilon$. Thus $g \circ f$ is uniformly continuous.

5. Define $g(x) = x^{2016} - f(x)$ on $[0,1]$. Then since $x^{2016}, f(x)$ are continuous, its difference $g(x)$ is also continuous. Thus we have

$$g(0) = 0 - f(0) \leq 0 \leq 1 - f(1) = g(1)$$

and by IVT, there exists $x_0 \in [0,1]$ s.t. $g(x_0) = 0$. For this x_0, $f(x_0) = x_0^{2016}$.

6. **(1)** Suppose f is Hölder continuous. Given $\forall\epsilon > 0$, set $\delta = \left(\frac{\epsilon}{M}\right)^{1/\alpha}$. Since $y = x^\alpha$ $(\alpha > 0)$ is increasing, for all $x, y \in X$, if $\|x - y\| < \delta$, $M\|x - y\|^\alpha < \epsilon$. By the Hölder continuity condition, $\|f(x) - f(y)\| < M\|x - y\|^\alpha < \epsilon$, and f is uniformly continuous on X.

(2) For fixed $x, y \in X$, suppose $x < y$. Given $N \in \mathbb{N}$, define $x_0, \ldots, x_N$ as follows.

$$x_0 = x,\ x_1 = x_0 + 1 \cdot \frac{y-x}{N},\ \ldots,\ x_i = x_0 + i \cdot \frac{y-x}{N},\ \ldots,\ x_N = y$$

Then we have

$$\|f(x) - f(y)\| \le \sum_{i=0}^{N-1} \|f(x_i) - f(x_{i+1})\| \le \sum_{i=0}^{N-1} M \|x_i - x_{i+1}\|^\alpha = M \frac{\|x-y\|^\alpha}{N^{\alpha-1}}$$

As $N \to \infty$, $0 \le \|f(x) - f(y)\| \le \lim_{N\to\infty} M \frac{\|x-y\|^\alpha}{N^{\alpha-1}} = 0$. Thus $\|f(x) - f(y)\| = 0$ for all $x, y \in X$. Thus $f(x) = f(0)$, merely a constant function.

7. **(1)** Let $y \in f(\overline{A})$. Then there exists $x_0 \in \overline{A}$ s.t. $f(x_0) = y$. Given $\epsilon > 0$, since $x_0 \in \overline{A}$, there exists $\delta > 0$ s.t. $N_{\mathbb{R}^m}(x_0, \delta) \cap A \neq \emptyset$. Take an element x from $N_{\mathbb{R}^m}(x_0, \delta) \cap A$. Since $x \in A$, $f(x) \in f(A)$, and since $x \in N_{\mathbb{R}^m}(x_0, \delta)$, $f(x) \in N_{\mathbb{R}^n}(f(x_0), \epsilon)$ by continuity of f. Therefore we have $f(x) \in N_{\mathbb{R}^n}(f(x_0), \epsilon) \cap f(A) \neq \emptyset$, and $f(x_0) = y \in \overline{f(A)}$. $f(\overline{A}) \subset \overline{f(A)}$.

(2) False. Consider $f(x) = \frac{1}{x^2+1}$. f is continuous. Set $A = (0, \infty)$. Then $\overline{A} = [0, \infty)$, $f(\overline{A}) = (0, 1)$, while $\overline{f(A)} = [0, 1]$.

8. Define $f_A(x) = \mathrm{dist}(\{x\}, A)$, $f_B(x) = \mathrm{dist}(\{x\}, B)$. Then

$$f(x) = \frac{f_B(x)}{f_A(x) + f_B(x)}$$

is a function that satisfies the requirements. The following should be checked.

(i) Is $f : \mathbb{R}^d \to [0, 1]$?
The value of the dist function is always greater than equal to 0, and $f(x) \le 1$ is trivial. Also note that the denominator is never 0 since A, B are disjoint.

(ii) $x \in A \implies f(x) = 1$?
If $x \in A$, $f_A(x) = 0$, $f_B(x) > 0$. ($A \cap B = \emptyset$) Thus $f(x) = \frac{f_B(x)}{0 + f_B(x)} = 1$.

(iii) $x \in B \implies f(x) = 0$?
If $x \in B$, $f_B(x) = 0$, $f_A(x) > 0$. ($A \cap B = \emptyset$) Thus $f(x) = \frac{0}{f_A(x) + 0} = 0$.

(iv) Are f_A, f_B continuous ?
We will show that f_A is uniformly continuous. For $x, y \in \mathbb{R}^d$, and any $a \in A$, the following holds by triangle inequality.

$$\mathrm{dist}(\{x\}, \{a\}) \le \mathrm{dist}(\{x\}, \{y\}) + \mathrm{dist}(\{y\}, \{a\})$$

Taking infimum over all $a \in A$ gives

$$\mathrm{dist}(\{x\}, A) \le \mathrm{dist}(\{x\}, \{y\}) + \mathrm{dist}(\{y\}, A)$$

Switching roles for x, y will give us another inequality, and combining it with the above inequality will give us

$$|f_A(x) - f_A(y)| = |\mathrm{dist}(\{x\}, A) - \mathrm{dist}(\{x\}, A)| \le \mathrm{dist}(\{x\}, \{y\}) = \|x - y\|$$

Therefore, for all $\epsilon > 0$, setting $\delta = \epsilon$ will make f_A satisfy the definition of uniform continuity. Thus f_A is (uniformly) continuous. The proof is symmetric for f_B.

(v) Is f continuous ?

Since the denominator is never 0 and f_A, f_B are continuous, f is continuous.

9. ($\Longrightarrow$) Define a function $F : X \to X \times \mathbb{R}$ as $F(x) = (x, f(x))$. Then $F(X) = E$. To show compactness of E, we will show that F is continuous.
For $x \in X$, consider a sequence $\langle x_n \rangle$ in X converging to x. (X is closed) By the continuity of f, $\langle f(x_n) \rangle$ converges to $f(x)$. Therefore $\langle F(x_n) \rangle$ converges to $(x, f(x)) = F(x)$. Thus F is continuous, and because X is compact, its image $E = F(X)$ is also compact.
($\Longleftarrow$) Since E is compact, it is closed, and consider a sequence $(x_n, f(x_n))$ in E that converges to $(x, f(x))$. Then x_n is a sequence in X, and if $x_n \to x$, $x \in X$. (X is closed) Then we know that $f(x_n)$ must converge to $f(x)$ $(*)$, which implies continuity of f on X.
$(*)$ If $(x_n, f(x_n)) \to (x, f(x))$ and $x_n \to x$, then $f(x_n) \to f(x)$.

10. We first prove the following inequality.
Claim. For convex function $f : (a, b) \to \mathbb{R}$ and $a < x < y < z$,

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(z) - f(y)}{z - y}$$

Proof. Since f is convex, for $t \in (0, 1)$,

$$f(tx + (1-t)z) \le tf(x) + (1-t)f(z)$$

, and set $y = tx + (1-t)z$. Multiply $z - x$ on both sides, then

$$(z-x)f(y) \le t(z-x)f(x) + (1-t)(z-x)f(z) = (z-y)f(x) + (y-x)f(z)$$

Rearranging the terms gives

$$(z-y)(f(y) - f(x)) \le (y-x)(f(z) - f(y))$$

, which directly gives the inequality.

Suppose $x \in (a, b)$. Since (a, b) is open, select real numbers s.t. $x_0 < x_1 < x, y < x_2 < x_3$, and define $C = \max\{\frac{|f(x_1)-f(x_0)|}{x_1-x_0}, \frac{|f(x_3)-f(x_2)|}{x_3-x_2}\}$. Given $\epsilon > 0$, choose $\delta = \min\{\epsilon/C, x_2 - x_1\}$. If $x > y$, by **Claim**,

$$\frac{f(x) - f(y)}{x - y} \le \frac{f(x_2) - f(x)}{x_2 - x} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

and

$$\frac{f(x) - f(y)}{x - y} \ge \frac{f(y) - f(x_1)}{y - x_1} \ge \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

, therefore

$$\left| \frac{f(x) - f(y)}{x - y} \right| \le C = \max\left\{ \frac{|f(x_1) - f(x_0)|}{x_1 - x_0}, \frac{|f(x_3) - f(x_2)|}{x_3 - x_2} \right\}$$

and the inequality above can be shown similarly for $x < y$.
Therefore if $|x - y| < \delta$, $|f(x) - f(y)| \le C\,|x - y| \le C\dfrac{\epsilon}{C} = \epsilon$. $f(x)$ is continuous.

11. Since f is continuous, f has a maximum at $\alpha_x \in [a, x]$, by EVT. Then $f^*(x) = f(\alpha_x)$.
If $x_1 < x_2$, $f^*(x_2) = \sup\{f(y) : y \in [a, x_2]\} = \max\{f(a_{x_1}), \sup\{f(y) : y \in [x_1, x_2]\}\} \ge f(\alpha_{x_1}) = f^*(x_1)$. Therefore f^* is increasing.
For continuity, let $x_0 \in X = [a, b]$, and let $f^*(x_0) = M (= \sup\{f(y) : y \in [a, x_0]\})$.

Case 1. $f(x_0) < M$

Since f is continuous, for $\epsilon = M - f(x_0)$, there exists $\delta_1 > 0$ s.t. $|x - x_0| < \delta_1, x \in X \implies |f(x) - f(x_0)| < \epsilon$. Then $f(x) - f(x_0) < \epsilon \implies f(x) < M$, in this interval. Thus $|x - x_0| < \delta_1, x \in X \implies f^*(x) = M$.

Case 2. $f(x_0) = M$

For all $\epsilon > 0$, $\exists\, \delta_2 > 0$ s.t. $|x - x_0| < \delta_2, x \in X \implies |f(x) - f(x_0)| < \epsilon \implies M - \epsilon < f(x) < M + \epsilon$. Therefore we have $M - \epsilon < f^*(x) < M + \epsilon$, by the continuity of f. Now we have $|f^*(x) - f^*(x_0)| < \epsilon$.

For both cases, setting $\delta = \min\{\delta_1, \delta_2\}$ will give us $|f^*(x) - f^*(x_0)| < \epsilon$. Thus f^* is continuous.

12. Consider

$$f(x) = \frac{1}{x} \sin \frac{1}{x} \quad x \in (0, 1]$$

Since $1/x$ and $\sin x$ are continuous on $(0, 1]$, $\mathbb{R}$ respectively, f is also continuous on $(0, 1]$. Suppose f attains maximum value M at x_0. It is trivial that $M > 0$, because $M \geq f(2/\pi) = \pi/2 > 0$. From this result we also know that $\sin \frac{1}{x_0} > 0$.

Then for $x' = (1/x_0 + 2\pi)^{-1} (< 1)$, $f(x') = \left(\frac{1}{x_0} + 2\pi\right) \sin\left(\frac{1}{x_0} + 2\pi\right) = M + 2\pi \sin \frac{1}{x_0} > M$, contradicting the choice of M. Therefore f has no maximum. Similarly, if f attains minimum value m at x_1, we know that $\sin \frac{1}{x_1} < 0$, and setting $x' = (1/x_1 + 2\pi)^{-1}$ will let us arrive at a contradiction, which contradicts the choice of m. Thus f is continuous on $(0, 1]$ but has no minimum or maximum.