

**March 29th, 2019**

**Remark.**  $\limsup$  is the limit of  $\sup$ . If  $\sup$  is easy to calculate, find  $\sup$  and take the limit.

### Quiz 1 Solutions

#1. Given set  $A$ ,  $\text{int}(A)$ ,  $A'$ , determine whether the set is open or closed.

- (1)  $A = \mathbb{N} \subset \mathbb{R}$ .  $\text{int}(A) = \emptyset$ ,  $A' = \emptyset$ ,  $A$  is closed.
- (2)  $\mathbb{Q} \subset \mathbb{R}$ .  $\text{int}(\mathbb{Q}) = \emptyset$ ,  $\mathbb{Q}' = \mathbb{R}$ ,  $\mathbb{Q}$  is neither open nor closed.
- (3)  $C = [0, 1] \cup (2, 3) \cap \{4\} \subset \mathbb{R}$ .  $\text{int}(C) = (0, 1) \cup (2, 3)$ ,  $C' = [0, 1] \cup [2, 3]$ ,  $C$  is neither open nor closed.
- (4)  $D = \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) : 0 \leq y \leq 1\} \subset \mathbb{R}^2$ .  $\text{int}(D) = \emptyset$ ,  $D' = D \cup \{(0, y) : 0 \leq y \leq 1\}$ ,  $D$  is neither open nor closed. ( $\because \text{int}D \neq D$ ,  $\overline{D} \neq D$ )

#2. Find a limit point of given set.

- (1)  $A = \mathbb{Q} \subset \mathbb{R}$ . 0 is a limit point. (Directly follows from Archimedes' principle)
- (2)  $B = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ . 0 is a limit point of  $B$ . (Also directly follows from Archimedes')
- (3)  $C = \{2^{-n} + 3^{-m} : n, m \in \mathbb{N}\} \subset \mathbb{R}$ . 0 is a limit point of  $C$ . Given  $\epsilon > 0$ , exists  $N \in \mathbb{N}$  such that for  $n, m \geq N$ ,  $2^{-n} < \epsilon/2$ ,  $3^{-m} < \epsilon/2$ . Then  $0 \neq 2^{-n} + 3^{-m} < \epsilon$ .

#3. True or False? If false, find a counterexample.

- (1)  $\overline{A \cup B} = \overline{A} \cup \overline{B}$  **True**
- (2)  $\overline{A \cap B} = \overline{A} \cap \overline{B}$  **False**. Set  $A = (0, 1)$ ,  $B = (1, 2)$ .  
**Correct Statement:**  $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$
- (3)  $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$  **False**. Set  $A = [0, 1]$ ,  $B = [1, 2]$ .  
**Correct Statement:**  $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$
- (4)  $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$  **True**

**Thm.**  $A \subset B \implies \overline{A} \subset \overline{B}, \text{int}(A) \subset \text{int}(B).$

**Proof.**

- We need to show  $A' \subset B'$ . Let  $x \in A'$ .  
 $\implies \forall \epsilon > 0, N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset.$   
 $\implies \forall \epsilon > 0, N(x, \epsilon) \cap (B \setminus \{x\}) \neq \emptyset$   
 $\implies x \in B'.$
- Let  $x \in \text{int}(A)$   
 $\implies \exists \epsilon > 0, N(x, \epsilon) \subset A \implies N(x, \epsilon) \subset B \implies x \in \text{int}(B).$

**Proof of (c).**  $A, B \subset A \cup B$

$\implies \text{int}(A), \text{int}(B) \subset \text{int}(A \cup B).$  Thus  $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$

**Proof of (d).**  $A \cap B \subset A, B \implies \text{int}(A \cap B) \subset \text{int}(A), \text{int}(B).$  Thus  $\text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B)$   
Suppose  $x \in \text{int}(A) \cap \text{int}(B).$  Then  $\exists \epsilon_A, \epsilon_B > 0$  s.t.  $N(x, \epsilon_A) \subset A, N(x, \epsilon_B) \subset B.$  Take  $\epsilon = \min\{\epsilon_A, \epsilon_B\}/2.$  Then  $N(x, \epsilon) \subset A, B.$  Therefore  $N(x, \epsilon) \subset A \cap B, x \in \text{int}(A \cap B).$

**Example.**  $A = \{(x, y) : x^2 + 2y^2 < 1\}.$   $\text{int}(A) = A, A' = \{(x, y) : x^2 + 2y^2 \leq 1\}.$

Suppose  $(x_0, y_0) \in A.$   $x_0^2 + 2y_0^2 = 1 - \delta < 1$  for some  $\delta > 0.$  By symmetry, let  $x_0, y_0 > 0.$  From

$$(x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 = x_0^2 + 2y_0^2 + \epsilon(2x_0 + 4y_0 + 3\epsilon) < 1$$

, we want  $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \delta.$  Set  $\epsilon < 1/10.$  Then  $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \epsilon(2x_0 + 4y_0 + 3) < \delta.$

Now set  $\epsilon = \min \left\{ \frac{1}{2(2x_0 + 4y_0 + 3)}, \frac{1}{100} \right\} > 0.$

Then  $|x - x_0| < \epsilon, |y - y_0| < \epsilon.$   $x_0^2 + 2y_0^2 < (x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 < 1.$   $N((x_0, y_0), \epsilon) \subset A.$

Interior points are limit points, and for the points  $(x_0, y_0)$  on the border, consider a sequence  $(x_0 - 1/n, y_0 - 1/n).$  Then the elements are in  $A$  and they converge to  $(x_0, y_0).$  Thus the border is also included in  $A'.$

April 1st, 2019

$\text{int}A : x \in A \text{ s.t. } N(x, \epsilon) \subset A \text{ for some } \epsilon > 0.$

$A' : x \in \mathbb{R}^d \text{ s.t. } N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset \text{ for } \forall \epsilon > 0$

$\overline{A} : x \in \mathbb{R}^d \text{ s.t. } N(x, \epsilon) \cap A \neq \emptyset, \forall \epsilon > 0, \overline{A} = A \cup A'$

**Example.**  $A = [0, 1) \cup \{2\}$ .  $1 \in A', 2 \notin A', 2 \in \overline{A}$

**Prop 2.3.3**  $x \in A' \implies N(x, \epsilon) \cap (A \setminus \{x\})$  는 무한집합이다.

**Proof.** 유한집합이라고 가정하자.  $N(x, \epsilon) \cap (A \setminus \{x\}) = \{x_1, \dots, x_n\}$  이라 할 수 있다. Set  $\delta = \min\{\|x - x_i\| : \forall i\}$ . Then  $N(x, \delta) \cap (A \setminus \{x\}) = \emptyset$ . 모순.

그래서 사실은 공집합이 아닌 것으로 정의했지만 **사실은** 무한집합이다.

**Remark.**  $A' \neq \emptyset \implies A$ 는 무한집합.

(대우)  $A$ 가 유한집합이면 극한점이 존재하지 않는다. (2.2 보기 4)

(역) 거짓.  $A = \{1, 2, \dots\}$  이면  $A' = \emptyset$ .

그러면 역이 언제 성립하나요? 다음 단원 내용!

**Definition.** Convergence in  $\mathbb{R}^d$

Let  $\langle x_n \rangle$  be a sequence in  $\mathbb{R}^d$ .

$$\lim_{n \rightarrow \infty} x_n = x \iff \forall \epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies \|x_n - x\| < \epsilon)$$

**Exercise.**  $x_n = (x_n^{(1)}, \dots), x = (x^{(1)}, \dots)$  일 때,  $x_n \rightarrow x \iff \forall i, x_n^{(i)} \rightarrow x^{(i)}$

**Notation.**  $A \subset \mathbb{R}^d; \langle x_n \rangle$  is a sequence in  $A \iff \forall n, x_n \in A$

**Theorem 2.2.2**

(1)  $x \in A' \iff \exists \langle x_n \rangle$  in  $A \setminus \{x\}$  such that  $x_n \rightarrow x$

(2)  $x \in \overline{A} \iff \exists \langle x_n \rangle$  in  $A$  such that  $x_n \rightarrow x$

**Proof.**

(1) ( $\implies$ )  $x_n \in N(x, \frac{1}{n}) \cap (A \setminus \{x\})$  이라 하자. (공집합이 아니므로 이러한 원소가 존재한다.)  
그러면  $\|x_n - x\| < 1/n$  이므로  $x_n$  은  $x$  로 수렴한다. 그리고  $x_n \in A \setminus \{x\}$  이므로 수열이  $A \setminus \{x\}$  에 있다.

(2) Left as exercise. Replace  $A \setminus \{x\}$  with  $A$ .

**Theorem 2.2.3.** The following are equivalent.

- (1)  $F$  is closed.
- (2)  $F' \subset F$ .
- (3)  $F = \overline{F}$
- (4) For a sequence  $\langle x_n \rangle$  in  $F$ ,  $\lim_{n \rightarrow \infty} x_n = x \implies x \in F$ .

**Proof.**

- (1)  $\iff$  (3) ( $\overline{F}$ : smallest closed set containing  $F$ .)
- (2)  $\iff$  (3) 은 자명.
- (1)  $\iff$  (4) by the above theorem. (Thm 2.2.2)

**Applications.**

- (1)  $A'$  is closed.

*Proof.* We want to show that  $(A')' \subset A'$ .

We want to show:  $x \in (A')' \implies x \in A'$ .

( $A'$  이 공집합이면 자명. 공집합이 아니라고 가정하고...)

Given  $\epsilon > 0$ ,  $N(x, \epsilon) \cap (A' \setminus \{x\}) \neq \emptyset$ . Take an element  $y \in A'$  from this set. Now set

$\delta = \min\{\|x - y\|, \epsilon - \|x - y\|\}$  then we have  $N(y, \delta) \cap (A \setminus \{y\}) \neq \emptyset$ . ( $\because y \in A'$ )

$z \in N(y, \delta) \cap (A \setminus \{y\})$  라 하자.

(a)  $z \in A \setminus \{y\} \subset A$ .

(b)  $\|x - z\| \leq \|x - y\| + \|y - z\| < \|x - y\| + \delta \leq \epsilon$  ( $z \in N(y, \delta)$ )

(c)  $\|x - z\| \geq \|x - y\| - \|y - z\| > \|x - y\| - \delta \geq 0$  (By the choice of  $\delta$ .) Thus  $x \neq z$ .

Therefore  $z \in N(x, \epsilon)$  (by (b)),  $z \in A \setminus \{x\}$  (by (a), (c)).

$x \in A'$  since  $N(x, \epsilon) \cap (A \setminus \{x\})$  is not empty.

- (2)  $A \subset \mathbb{R}$ : closed and bounded  $\implies \inf A = \min A$ ,  $\sup A = \max A$ . (Existence)

*Proof.* Let  $\sup A = x \notin A$ . ( $\sup A \in A$  이면 자명)

*Claim.*  $x \in A'$ .

*Proof of Claim.*  $\forall \epsilon > 0$ ,  $N(x, \epsilon) = (x - \epsilon, x + \epsilon)$

$x = \sup A$  이므로  $x - \epsilon$  is not an upper bound.

$\exists y$  such that  $y \in (x - \epsilon, x)$

$y \in N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset$  이므로  $x$  는 극한점.

따라서  $x \in A' \subset A$  (closed set 이므로 Thm 2.2.3 (2)) 모순.

$\sup A \in A$  이므로 이 값이 최댓값이다.

## 2.3 유계집합과 코시수열

핵심: Thm 2.3.4, Thm 2.3.7

**Definition.**  $\langle x_n \rangle$ : 유계수열(bounded sequence)  $\iff \exists M > 0$  s.t.  $\|x_n\| \leq M$  for all  $n \in \mathbb{N}$ .

**Definition.**  $n_1 < n_2 < \dots$  : sequence in  $\mathbb{N}$  이라 하자.  $\langle x_{n_k} \rangle_{k=1}^\infty = (x_{n_1}, x_{n_2}, \dots)$  를  $\langle x_n \rangle$ 의 부분수열(subsequence)이라 한다.

**Theorem 2.3.4** (Bolzano-Weierstrass Theorem)

If  $\langle x_n \rangle$  is bounded, there exists a convergent subsequence of  $\langle x_n \rangle$ .

**Idea of Proof.** Equivalent formulation for sets.

**Definition.** Set  $A$  is bounded  $\iff \exists M > 0$  such that  $\|x\| < M$  for all  $x \in A$ .

**Theorem 2.3.2** (Equivalent of 2.3.4)  $A$ 가 유계이고 무한집합이면,  $A' \neq \emptyset$ .

**Remark.**  $A' \neq \emptyset \implies A$ : 무한집합.

역이 성립하기 위해서는  $A$ 가 유계라는 조건이 필요하다.

극한점이 중요한 이유는 계속 수열과 관련이 있기 때문이다.

**Example.**  $A = \{1/n : n \in \mathbb{N}\}$  을 고려하는 것은 수열  $x_n = 1/n$  을 고려하는 것이나 마찬가지이다. 이 수열  $x_n$  이  $x$  로 수렴하는 것은  $A' = \{x\}$  와 동치이다. (Hence the name “limit point”) 이로부터  $x \in A' \iff$  Exists a subsequence of  $\langle x_n \rangle$  in  $A \setminus \{x\}$  converging to  $x$ .

### Proof of 2.3.2

(1) **Lemma 2.3.1** 축소구간정리 in  $\mathbb{R}^d$ .

$B$  is a closed box in  $\mathbb{R}^d \iff B = I_1 \times I_2 \times \dots \times I_d$ , where  $I_i = [a_i, b_i]$  for  $i = 1, \dots, d$ . ( $I_i$  is a closed and bounded interval.)

$$B_1 \supset B_2 \supset \dots \implies \bigcap_{n=1}^{\infty} B_n \neq \emptyset$$

**Proof.** 각 ‘좌표’  $I_i$  별로 1차원 축소구간정리를 적용하면 된다.

(2) **Divide and Conquer Strategy**

$B$ : Box 일 때,  $\text{diam}(B) = \sup\{\|x - y\| : x, y \in B\} = \sqrt{(a_1 - b_1)^2 + \dots + (a_d - b_d)^2}$

**Claim.** There exists closed boxes  $B_1, B_2, \dots$  s.t.

(a)  $B_1 \supset B_2 \supset \dots$

(b)  $\text{diam} B_n = \frac{1}{2^{n-1}} \text{diam} B_1$

(c)  $B_n \cap A$ : 무한집합

**Proof.** (Induction)  $n = 1$ ;  $B_1$ : 충분히 커서  $A \subset B_1$  인 box 를 잡으면 된다.

Suppose we have  $B_1, \dots, B_n$ ;  $B_n$ 을  $2^d$  등분하면 적어도 하나는  $A$ 의 원소를 무한개 포함하고 있다. 그 집합을  $B_{n+1}$  으로 잡는다. (비둘기집의 원리)

이제  $x \in \bigcap_{n=1}^{\infty} B_n$  으로 잡으면 (축소구간정리에 의해 잡을 수 있다)  $x \in A'$ . ( $A' \neq \emptyset$ )

$\because \forall \epsilon > 0$ ,  $\text{diam} B_n < \epsilon$  인  $N \in \mathbb{N}$  을 찾아  $n \geq N$  일 때 부등식이 성립하도록 할 수 있다.

이러한  $n$  들에 대하여  $B_n \subset N(x, \epsilon)$ . 그러면  $N(x, \epsilon) \cap (A \setminus \{x\}) \supset B_n \cap (A \setminus \{x\})$ .

April 3rd, 2019

우리가 지금 2.3 을 하고 있는데, 2 가지 중요한 결과가 있어요.

**Theorem 2.3.4**  $\langle x_n \rangle$  이 bounded 이면 수렴하는 부분수열을 갖는다.<sup>1</sup>

**Theorem 2.3.2**  $A$ 가 유계인 집합이고 무한집합이면 극한점을 가진다.  $A' \neq \emptyset$   
증명은 축소구간정리를 박스로 확장해가지고 분할 정복하면 된다.

**Recall 2.3.3**  $x \in A' \implies N(x, \epsilon) \cap (A \setminus \{x\})$  는 무한집합이다.

**Proof of 2.3.4.**  $A = \{x_1, x_2, \dots, x_n\}$  라고 하면 이 집합은 유계이다. (수열이 유계이므로)

(1)  $A$ 가 유한집합: 자명.

$\exists x$  such that  $x$  appears infinitely many times in  $\langle x_n \rangle$ . (PHP) 이 경우에는 부분수열을  $x, x, \dots$  로 잡으면 된다. 이는 수렴하는 부분수열이다.

(2)  $A$ 가 무한집합<sup>2</sup>

$A' \neq \emptyset$  이므로  $\alpha \in A'$  이라 하자.

**Claim.**  $\exists n_1 < n_2 < \dots$  such that  $\|x_{n_k} - \alpha\| < 1/k$ .

**Proof.** (첨자들이 증가하면서 가까워져야 한다는 것이 유일하게 tricky 한 부분이다. 귀납법을 사용하자.)  $k = 1$ :  $x_{n_1} \in N(\alpha, 1) \cap (A \setminus \{\alpha\})$  로 잡으면 된다.

$x_{n_1}, \dots, x_{n_k}$  를 잡았다고 가정:  $N(\alpha, \frac{1}{k+1}) \cap (A \setminus \{\alpha\})$  에서  $x_{n_{k+1}}$  를 잡아야 하는데 이 집합은 무한집합이다. (Recall 2.3.3) 이 집합에서 첨자가  $n_k$ 보다 큰 항이 반드시 존재하므로 그 중 하나를  $x_{n_{k+1}}$  이라 잡으면 된다.

따라서  $\lim_{k \rightarrow \infty} x_{n_k} = \alpha$  (Check as exercise)

**Application.** (Characterization of  $\limsup$  and  $\liminf$ )

$x_n$  이 bounded 이면,  $A = \{x : \exists \text{ subsequence of } x_n \text{ converging to } x\}$ . 이 때 Theorem 2.3.4에 의해  $A \neq \emptyset$  임을 증명하였다.

(1)  $A$ : closed and bounded  $\implies \max(A), \min(A)$  가 존재한다.

**Proof.**  $B = \{x_1, x_2, \dots\}$ ,  $C = \{\langle x_n \rangle \text{ 에 무한 번 나타나는 수} \}$  로 잡자.  $A = B' \cup C$ ,  $C \subset B$ ,  $C' \subset B'$  임을 확인해보라! 이를 이용하면  $B' \cup C = (B' \cup C') \cup C = B' \cup (C' \cup C) = B' \cup \overline{C}$  가 되어 닫힌집합의 합집합은 닫힌 집합이다.  $A$ 는 closed and bounded 이다.

(2)  $\limsup x_n = \max(A)$ ,  $\liminf x_n = \min(A)$

(부분수열이 가질 수 있는 극한값들 중 가장 큰 값이  $\limsup$ , 가장 작은 값이  $\liminf$ )

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<sup>1</sup>증명이 가장 테크니컬 해요!

<sup>2</sup>이제 Thm 2.3.2 를 사용할 수 있다. 사실 경우를 나눈 것은 예외적인 case 를 처리하기 위한 것이었다.

**Proof.** Recall

$$\limsup x_n = \alpha \iff \begin{cases} \text{(i) } \forall \epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies x_n < \alpha + \epsilon) \\ \text{(ii) } \forall \epsilon > 0, x_n > \alpha - \epsilon \text{ for infinitely many } n \end{cases}$$

- (a) 부분수열  $\langle x_{n_k} \rangle \rightarrow \beta$  이면 (i)에 의해  $k \geq N \implies x_{n_k} < \alpha + \epsilon$  이 되어  $\beta \leq \alpha + \epsilon$ .  $\beta \leq \alpha$ . 그러므로  $\max(A) \leq \alpha$  이다.
- (b)  $\forall \epsilon > 0$ , (i), (ii)에 의해  $x_n \in (\alpha - \epsilon, \alpha + \epsilon)$  인  $n$  이 무한히 많다. 이 유계인 구간에 속하는 수열의 항들에 대해 부분수열을 잡아 (further subsequence)  $\gamma$  로 수렴하도록 할 수 있다. (Theorem 2.3.4) 그러면  $\langle x_{m_k} \rangle \rightarrow \gamma \in [\alpha - \epsilon, \alpha + \epsilon]$ . 따라서  $\alpha - \epsilon \leq \gamma \leq \max(A)$  가 되어  $\alpha \leq \max(A)$ .

따라서  $\max(A) = \alpha$ .

**Definition.**  $\langle x_n \rangle$ : Cauchy Sequence  $\iff \forall \epsilon > 0, \exists N \text{ s.t. } [m, n \geq N \implies \|x_m - x_n\| < \epsilon]$

**Prop 2.3.6, Thm 2.3.8**  $\langle x_n \rangle$ : convergent  $\iff \langle x_n \rangle$ : Cauchy sequence<sup>3</sup>

**Proof.** ( $\implies$ ) 자명.  $\|x_m - x_n\| \leq \|x_m - \alpha\| + \|x_n - \alpha\| < \epsilon/2 + \epsilon/2 = \epsilon$  인  $m, n \geq N$  존재. ( $\impliedby$ ) 수렴 값이 없는 상태에서 증명해야 한다. 먼저 수렴 값을 찾아보자.

(1)  $\langle x_n \rangle$  is bounded.

**Proof.**  $\exists N$  s.t.  $\|x_m - x_n\| < 1$  for all  $m, n \geq N$ .

Set  $M = \max\{\|x_1\|, \dots, \|x_{N-1}\|, \|x_N\| + 1\}$ . ( $\|x_m\| < \|x_N\| + 1$ )

따라서  $\|x_n\| \leq M$  for all  $n \in \mathbb{N}$ .

(2) There exists a subsequence  $\langle x_{n_k} \rangle$  converging to some  $\alpha$ . (Thm 2.3.4)

(3)  $\langle x_n \rangle$  converges to  $\alpha$ .

**Proof.**  $\epsilon > 0$  에 대해,

(a) 코시 수열의 성질에 의해  $\exists N_1$  s.t.  $\|x_m - x_n\| < \epsilon/2$  for all  $m, n \geq N_1$ .

(b) 부분수열이  $\alpha$ 로 수렴하므로  $\exists N_2$  s.t.  $\|x_{n_k} - \alpha\| < \epsilon/2$  for all  $k \geq N_2$ .

Let  $N = \max\{N_1, N_2\}$ .  $n \geq N, n_N \geq n_{N_1} \geq N_1$  이므로,

$$n > N \implies \|x_n - \alpha\| \leq \|x_n - x_{n_N}\| + \|x_{n_N} - \alpha\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

---

<sup>3</sup>중간고사 전 까지 가장 중요한 정리.



**Remark.** 우리의 여정을 돌아보자.

(1) Archimedes' Principle 을 가정하면

Completeness Axiom  $\implies$  Monotone Convergence Theorem  $\implies$  축소구간정리  $\implies$   
Bolzano-Weierstrass Theorem  $\implies$  **Cauchy Convergent Theorem**<sup>4</sup>

(Exercise)  $\implies$  Completeness Axiom

(2) **Example.**  $X = C([0, 1])$ . (Set of functions that are continuous in  $[0, 1]$ ) How would we define  $\|f - g\|$ ?  $\int_0^1 |f(x) - g(x)| dx$  ?  $\max\{|f(x) - g(x)| : x \in [0, 1]\}$  ? Only the second choice gives completeness for  $X$ .

(3) **Convergence Test** without limit value. (**Theorem 2.3.9**)

$\sum_{n=1}^{\infty} a_n$  is convergent  $\iff \forall \epsilon > 0, \exists N$  s.t.  $(n > m \geq N \implies |a_{m+1} + \cdots + a_n| < \epsilon)$

**Proof.** Trivial.

**Definition.**  $\sum a_n$  is **absolutely convergent**  $\iff \sum |a_n|$  is convergent

**Theorem.** An absolutely convergent series converges.

**Proof.** Suppose  $\sum |a_n|$  converges. For  $\forall \epsilon > 0$ , there exists  $N$  such that  $||a_{m+1}| + \cdots + |a_n|| < \epsilon$  for all  $m, n \geq N$ . Therefore, for  $m, n \geq N$ ,

$$|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \epsilon$$

and  $\sum a_n$  converges.

---

<sup>4</sup>In any metric spaces, this is the condition for completeness.

**April 5th, 2019**

**Theorem.**  $\overline{A \cup B} = \overline{A} \cup \overline{B}$

**Proof.** ( $\subset$ ) Trivial.

( $\supset$ )  $A \subset \overline{A}, B \subset \overline{B} \implies A \cup B \subset \overline{A} \cup \overline{B} \implies \overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$ . The closure of a closed set is itself.

**6. (2)**  $a_n = \cos \sqrt{2019 + n^2 \pi^2}$

Consider  $\delta > 0$ , such that

$$(n\pi - \delta)^2 < 2019 + n^2 \pi^2 < (n\pi + \delta)^2$$

$$-2n\pi < \frac{2019}{\delta} \pm \delta < 2n\pi$$

We can find large enough  $N$  such that the above inequality holds for  $n \geq N$ .

Now we want  $b_n = \sqrt{2019 + n^2 \pi^2}$  bounded by  $n\pi \pm \delta$ .

$n \geq N, n \text{ even} \implies n\pi - \delta < b_n < n\pi + \delta$

$\implies 1 \geq a_n > 1 - \epsilon$

$n \geq N, n \text{ odd} \implies -1 \leq a_n < -1 + \epsilon$

**Problem 2.3.5**

$$(1) \ x_{n+2} = \frac{x_n + x_{n+1}}{2}$$

$$(2) \ x_{n+1} = x_n + x(-1)^n \frac{1}{3n+1}$$

**Solution.**

(1) Write  $x_{n+2} - x_{n+1} = a(x_{n+1} - x_n)$  and observe that  $a = -1/2$ . Write as

$$x_n = x_{n-1} + \left(-\frac{1}{2}\right)^{n-2} (x_2 - x_1)$$

Then we have

$$x_n = x_2 + \sum_{k=1}^{n-2} \left(-\frac{1}{2}\right)^k (x_2 - x_1)$$

This series converges to  $\frac{2x_2 + x_1}{3}$ .

(2) This is an alternating series. Write as

$$x_n = x_1 + \sum_{k=1}^{n-1} (-1)^k \frac{x}{3n+1}$$

By alternating series test, the second summation term converges, and the series converges to  $x_1$ .

Since a converging sequence is a Cauchy sequence,  $x_1, x_2$  can be any real number.

April 8th, 2019

Section 2.3: Bolzano-Weierstrass Theorem, Cauchy Convergent Theorem

In section 2.4, we will be studying about Convergence Tests.

정

## 2.4 급수의 수렴판정

**Cor 2.3.9.**  $\sum_{n=1}^{\infty} a_n$  is convergent  $\iff s_n = \sum_{k=1}^n a_k, \langle s_n \rangle$  is convergent  $\iff \langle s_n \rangle$  is Cauchy.

(1)  $\sum_{n=1}^{\infty} a_n$  is convergent  $\implies \lim_{n \rightarrow \infty} a_n = 0$ .

(2)  $\sum_{n=1}^{\infty} |a_n|$  is convergent  $\implies \sum_{n=1}^{\infty} a_n$  convergent.

**Theorem 2.4.3** (Comparison Test) Suppose  $\sum b_n$  converges. If  $0 \leq a_n \leq b_n$  for all  $n \in \mathbb{N}$ ,  $\sum a_n$  converges.

**Proof** Let  $M = \sum b_n$ ,  $s_n = \sum_{k=1}^n a_k$ .  $s_n$  is increasing and  $s_n$  is bounded by  $M$ .  $s_n$  is convergent by Monotone Convergence Theorem.

**Theorem.** Suppose sequences  $a_n, b_n$  satisfy  $0 \leq |a_n| \leq b_n$ <sup>5</sup> and  $\sum b_n$  converges. Then  $\sum a_n$  is convergent.<sup>6</sup>

**Proof.** By comparison test and absolute convergence.

**Prop 2.4.4** (Root Test) Suppose  $\alpha = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ .

If  $\alpha < 1$ ,  $\sum a_n$  converges. If  $\alpha > 1$ ,  $\sum a_n$  diverges.

**Proof.**

(1)  $\alpha < 1$ . Take  $\epsilon > 0$  such that  $\alpha < \alpha + \epsilon < 1$ . Then there exists  $N$  such that  $|a_n|^{1/n} < \alpha + \epsilon$  for all  $n \geq N$ . Therefore  $|a_n| < (\alpha + \epsilon)^n$ . Since  $\alpha + \epsilon < 1$ ,  $\sum (\alpha + \epsilon)^n$  converges. Apply the comparison test to see that  $\sum a_n < \infty$ .

(2)  $\alpha > 1$ . Take  $\epsilon > 0$  such that  $\alpha > \alpha - \epsilon > 1$ . Then  $|a_n|^{1/n} > \alpha - \epsilon$  for infinitely many  $n$ . Then  $|a_n| > (\alpha - \epsilon)^n > 1$ . Therefore  $\lim a_n \neq 0$ .  $\sum a_n$  diverges.

**Prop 2.4.5** (Ratio Test) Suppose  $a_n \neq 0$ . Let  $\beta = \limsup |a_{n+1}/a_n|$ ,  $\gamma = \liminf |a_{n+1}/a_n|$ .

If  $\beta < 1$ ,  $\sum a_n$  converges. If  $\gamma > 1$ ,  $\sum a_n$  diverges.

**Proof.**

(1)  $\beta < 1$ . Take  $\epsilon > 0$  such that  $\beta < \beta + \epsilon < 1$ . Then  $\exists N$  s.t.  $|a_{n+1}/a_n| < \beta + \epsilon$  for  $n \geq N$ .  
 $\implies |a_n| = |a_N| |a_{N+1}/a_N| \cdots |a_n/a_{n-1}| < |a_N| (\beta + \epsilon)^{n-N}$ .  
Set  $b_n = |a_N| (\beta + \epsilon)^{n-N}$  and apply comparison test to see that  $\sum a_n < \infty$ .

---

<sup>5</sup>Note that this condition can fail for finitely many  $n$ .

<sup>6</sup> $a_n$  may be a very complex expression, but we want  $b_n$  to be simple, an expression we know that it is convergent.

- (2)  $\gamma > 1$ . Take  $\epsilon > 0$  such that  $\gamma > \gamma - \epsilon > 1$ . Then  $\exists N$  s.t.  $|a_{n+1}/a_n| > \gamma - \epsilon$  for  $n \geq N$ . Then we see that  $|a_n|$  is increasing for  $n \geq N$ . Thus  $a_n$  cannot converge to 0.  $\sum a_n$  is divergent.

**Remark.** If the above limits (ratio, root) exist, elementary tests can be applied. But if the limits turn out to be 1, the test fails. (ND: Non-Deterministic) Check it for  $\sum 1/n, \sum 1/n^2$ . Also, these are *weak tests*. For most of the series, the limit is 1. Moreover...

**Theorem 2.4.6** Suppose  $a_n \neq 0$ .

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|$$

Thus if the root test works, ratio test also works.<sup>7</sup>

**Proof.** We only need to prove the last inequality.

Let  $\beta = \limsup |a_{n+1}/a_n|$ ,  $\forall \epsilon > 0$ .  $\implies \exists N$  s.t.  $|a_{n+1}/a_n| \leq \beta + \epsilon$  for  $n \geq N$ . Then if  $n \geq N$ ,  $|a_n| \leq |a_N|(\beta + \epsilon)^{n-N}$ . (Similar to proof of 2.4.5) Then

$$|a_n|^{1/n} \leq (\beta + \epsilon) \left( \frac{|a_n|}{(\beta + \epsilon)^N} \right)^{1/n}$$

and take  $\limsup$  on both sides, then  $\limsup |a_n|^{1/n} \leq \beta + \epsilon$ .

**Example.**  $\langle a_n \rangle = \begin{cases} 1/2^n & n \text{ odd} \\ 1/2^{n-2} & n \text{ even} \end{cases}$

Check that  $\limsup |a_n|^{1/n} = 1/2 < 1$ , and the series  $\sum a_n$  converges by the root test.

But if we use the ratio test here,  $\limsup$  value is 2 and  $\liminf$  value is  $1/8$ .<sup>8</sup> The ratio test does not tell us anything about the convergence. Also note that the series converges to 2.

**Prop 2.4.1** (Rearrangement)  $a_n \geq 0$ .<sup>9</sup> Suppose a bijection  $r : \mathbb{N} \rightarrow \mathbb{N}$  exists.

$$(1) \sum_{n=1}^{\infty} a_n = s \iff \sum_{n=1}^{\infty} a_{r(n)} = s$$

$$(2) \sum_{n=1}^{\infty} a_n = \infty \iff \sum_{n=1}^{\infty} a_{r(n)} = \infty$$

**Proof.**

- (1) ( $\implies$ ) Let  $t_n = \sum_{k=1}^n a_{r(k)}$ . Then  $t_n$  is increasing and bounded by  $s$ . Thus  $t_n$  converges by MCT, and  $\lim t_n \leq s$ .

$$s = \sum_{k=1}^{\infty} a_k \leq \sum_{n=1}^{\infty} a_{r(n)} = t = \lim t_n. \quad (a_n \geq 0 \text{ was used here.})$$

$$(\iff) \text{ Use } r^{-1}(n).$$

<sup>7</sup>The limit for the ratio test is much easier to calculate than the root test. That's why we use the ratio test.

<sup>8</sup>The ratios are: 2,  $1/8$ , 2,  $1/8$  ...

<sup>9</sup>This is the important condition.

(2) Contraposition of (1).

**Prop 2.4.2** (Alternating Series Test) For a given sequence  $x_n$ , suppose the following holds.

- $x_n$  is decreasing.
- $\lim x_n = 0$ .

Then the series  $\sum_{k=1}^{\infty} (-1)^{k-1} x_k$  is convergent.

**Proof.** Let  $s_n = \sum_{k=1}^n (-1)^{k-1} x_k$ . For  $m < n$ ,

$$|s_n - s_m| = |(-1)^m x_{m+1} + \cdots + (-1)^{n-1} x_n| = |x_{m+1} - x_{m+2} + \cdots \pm x_n| \stackrel{(*)}{\in} [0, x_{m+1}]$$

$$(*) : x_{m+1} - x_{m+2} + \cdots + x_n = (x_{m+1} - x_{m+2}) + \cdots + (x_{n-2} - x_{n-1}) + x_n \geq 0$$

$$= x_{m+1} - (x_{m+2} - x_{m+3}) - \cdots - (x_{n-1} - x_n) \leq x_{m+1}$$

Check for the case with last term  $-$ .

Now,  $\forall \epsilon > 0$ , find  $N$  such that  $|x_n| < \epsilon$  for  $n \geq N$ . Then for  $n > m \geq N$ ,  $|s_n - s_m| \leq x_{m+1} < \epsilon$ .

Thus  $\langle s_n \rangle$  is a Cauchy sequence and the given series converges.

**Example.**  $a_n = (-1)^{n-1}/n$ .  $\sum a_n$  converges by alternating series test and converges to  $\log 2$ .

**Remark.** The rearrangement of the above example may not converge, or converge to a different value than  $\log 2$ .

Exam: 1.1 - 2.6

After the midterms we will be covering functions and continuity.

Chapter 1 has been about  $\mathbb{R}$ , and in Chapter 2, we have talked about subsets of  $\mathbb{R}^n$ .

2.1: What is  $\mathbb{R}^n$  ? Vector Space, IPS, Metric Space, Normed Space...

2.2: Open, closed sets

2.3: Bounded sets and Cauchy sequences

(2.4: Convergence Tests)

2.5: Compact Sets

2.6: Connected Sets

April 10th, 2019

## 2.5 Compact Set

**Definition.**  $\{U_i : i \in I\}$  ( $I$  is the index set,  $U_i \subset \mathbb{R}^d$ ) is called “family of sets”.

- (1)  $\{U_i : i \in I\}$  is a **cover** of  $K \subset \mathbb{R}^d \iff K \subset \bigcup_{i \in I} U_i$ .
- (2)  $\{U_i : i \in I\}$  is a **open cover**  $\iff U_i$  are open for  $\forall i$ .
- (3)  $J \subset I$ ,  $\{U_i : i \in J\}$  is called a **subcover** of  $\{U_i : i \in I\} \iff K \subset \bigcup_{i \in J} U_i$ .

**Definition.**  $K \subset \mathbb{R}^d$  is **compact**  $\iff$  Any open cover of  $K$  has finite subcover.

**Example.**

- (1)  $\mathbb{N}$  is not compact. Set  $U_k = (k - 1/2, k + 1/2)$ , then  $\{U_k : k \in \mathbb{N}\}$  is a (open) cover of  $\mathbb{N}$ . But there are no finite subcover.
- (2)  $A = (0, 1)$  is not compact. Set  $U_k = (1/k, 1)$ , then because  $\bigcup_{k=1}^{\infty} U_k = (0, 1)$ ,  $\{U_k : k \in \mathbb{N}\}$  is a (open) cover of  $A$ . But there are no finite subcover.  $\bigcup_{i=1}^m U_{k_i} = U_{k_m} = (1/k_m, 1)$ , which cannot contain  $(0, 1)$ .
- (3)  $A = \{a_1, a_2, \dots, a_m\} \subset \mathbb{R}^d$  is compact.  $\{U_i : i \in I\}$  be a cover of  $A$ . There exists  $i_1, \dots, i_m \in I$  such that  $a_k \in U_{i_k}$  for  $k = 1, \dots, m$ . Then  $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$  is a finite subcover of  $A$ .

Main Theorem: **Heine-Borel Theorem**

$$K \text{ is compact} \iff K \text{ is bounded and closed.}$$

**Remark.**

- (1) This is a part of Thm 2.5.4
- (2) Proof: Prop 2.5.1, Thm 2.5.2, Prop 2.5.3
- (3) **Characterization of compact sets in  $\mathbb{R}^d$ .**<sup>10</sup>

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<sup>10</sup>Compact Set 을 이 단순한 공간 안에서는 characterize 할 수 있다!

**Proof.**

( $\implies$ ) (Prop 2.5.1)

(1) *Is  $K$  bounded?*

Set  $U_k = N(0, k)$ . Then  $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d$ . Thus  $\{U_k : k \in \mathbb{N}\}$  is an open cover of  $K$ . There exists a finite subcover  $U_{k_1}, \dots, U_{k_m}$  ( $k_1 < \dots < k_m$ ) of  $K$ . Then we have  $K \subset \bigcup_{i=1}^m U_{k_i} = U_{k_m} = N(0, k_m)$ . Therefore  $K$  is bounded.

(2) *Is  $K$  closed?*

Suppose  $x \in K^C$ . Set  $U_k = \{y : \|y - x\| > 1/k\}$ . Then  $\bigcup_{k=1}^{\infty} U_k = \mathbb{R}^d \setminus \{x\} \supset K$ . (Open cover) There exists a finite subcover  $U_{k_1}, \dots, U_{k_m}$  of  $K$ .  $K \subset \bigcup_{i=1}^m U_{k_i} = U_{k_m}$ . Therefore  $K^C \supset U_{k_m}^C = \{y : \|y - x\| \leq 1/k_m\} \supset N(x, 1/k_m)$ . Thus  $K^C$  is open,  $K$  is closed.

( $\impliedby$ )

(1) (Theorem 2.5.2) *Closed box is compact.*

$B = I_1 \times \dots \times I_d$ ,  $I_i = [a_i, b_i]$ . Let  $\{U_i : i \in I\}$  is an open cover of  $B$ .

(Contradiction) Suppose there is no finite subcover of  $B$ .

**Claim.** There exists  $B = B_1 \supset B_2 \supset \dots$  (closed boxes) such that

- $\text{diam}(B_n) = \frac{1}{2^{n-1}} \text{diam}(B_1)$
- There is no finite subcover of  $\{U_i : i \in I\}$  covering  $B_n$ .

By Lemma 2.3.1, there exists  $x \in \bigcap_{n=1}^{\infty} B_n$ . Since  $x \in B$ ,  $\exists U_i$  such that  $x \in U_i$ . Then  $\exists \epsilon > 0$  such that  $N(x, \epsilon) \subset U_i$ .<sup>11</sup> Set  $\frac{1}{2^{n-1}} \text{diam}(B_1) < \epsilon$ .

If  $y \in B_n \implies \|x - y\| \leq \text{diam}(B_n) < \epsilon \implies y \in N(x, \epsilon)$ . Then  $B_n \subset N(x, \epsilon) \subset U_i$ , contradiction.

(2)  *$K$ : compact,  $F \subset K$ ,  $F$  is closed  $\implies F$ : compact.*

Let  $\{U_i : i \in I\}$  be an open cover of  $F$ . Then  $\{U_i : i \in I\} \cup \{F^C\}$  is an open cover of  $K$ . Because  $K$  is compact, there exists a finite subcover of  $K$ . There are two cases.

- (a)  $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$ : This is already a finite subcover of  $F$ .
- (b)  $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}, F^C\}$ : Since  $F^C$  does not cover  $F$ ,  $U_{i_k}$  must cover  $F$ .

(3) *Closed and bounded set is compact.*

Suppose  $K$  is bounded and closed. There exists a closed box  $B$  that contains  $K$ . Thus  $B$  is compact by (1),  $K$  is a closed subset of  $B$ . Then by (2),  $K$  is compact.

Theorem 2.5.2 가 가장 non-trivial 한 부분이다.

---

<sup>11</sup> $n$ 이 충분히 크면 ball 안에 box 가 들어가고 box 는  $U_i$  안에 있다? Claim 의 2번째에 모순.

**Theorem 2.5.4** The following are equivalent.

- (1)  $K$  is compact.
- (2)  $K$  is bounded and closed.
- (3) If  $A$  is an infinite subset of  $K$ ,  $\emptyset \neq A' \subset K$ .
- (4) For a sequence  $\langle x_n \rangle$  in  $K$ , there exists a convergent subsequence whose limit is in  $K$ .

**Proof.**

- (1)  $\iff$  (2) by Heine-Borel Theorem.
- (2)  $\implies$  (3) Suppose  $A$  is infinite and bounded. ( $A \subset K$ ) By Bolzano-Weierstrass,  $A' \neq \emptyset$ .  
 $A' \subset A' \cup A = \overline{A} \subset K$ . ( $\overline{A}$  is the smallest closed set containing  $A$ ,  $A \subset K$ .)
- (3)  $\implies$  (4) Let  $A = \{x_1, x_2, \dots\}$

(1) If  $A$  is finite, trivial. (Take a constant subsequence, which constant  $\in K$ .)

(2) If  $A$  is infinite,  $x \in A' \subset K$  by (3). ( $x \in A'$  by Thm 2.3.4)

(4)  $\implies$  (2)

(1)  $K$  is bounded.

(Contradiction) Suppose  $K$  is not bounded. Then  $\forall n \in \mathbb{N}$ , there exists  $x_n \in K$ ,  $\|x_n\| \geq n$ .  
There are no convergent subsequences, contradiction.

(2)  $K$  is closed.

(Contradiction) Suppose  $K$  is not closed.

(a)  $K$ : finite  $\rightarrow K$ : closed  $\rightarrow$  Contradiction.

(b)  $K$ : infinite  $\rightarrow K$ : infinite and bounded  $\xrightarrow{\text{B-W}} K' \neq \emptyset$

*Note.*  $K' \subset K \iff K$ : closed.

Then if  $K'$  is not a subset of  $K$ <sup>12</sup>, there exists  $x \in K' \setminus K$ . Since  $x \in K'$ , there exists a sequence  $\langle x_n \rangle$  in  $K \setminus \{x\}$  ( $= K$ )<sup>13</sup> converging to  $x$ . Thus for a subsequence of  $\langle x_n \rangle$ , its limit must be in  $K$ . But  $x$  is the only possible limit value.  $x \in K$ . Contradiction.

---

<sup>12</sup>Contraposition

<sup>13</sup> $x \notin K$



April 12th, 2019

**Problem 2.4.7** (바)  $\sum \frac{1}{n^p - n^q}$  ( $0 < q < p$ )

$0 < n^p - n^q \leq n^p$  이므로  $1/n^p \leq 1/(n^p - n^q)$  가 되어  $p \leq 1$  이면 발산한다.

충분히 큰  $N$ 에 대하여  $n \geq N$  일 때마다  $n^p - n^q \geq n^p/2$  가 되게 할수 있다. (이 때  $n^p/2 \geq n^q$  이므로  $n^{p-q} \geq 2$  가 되어  $N$  을 잡을 수 있다) 비교판정법에 의해 수렴한다.

**Problem 2.7.12** Given  $\langle a_n \rangle$  such that  $\lim a_n = a$ , show that  $\sigma_n = \frac{a_1 + \cdots + a_n}{n}$  also converges to  $a$ .

**Problem 2.7.13**  $r < 1$ ,  $\|x_{n+2} - x_{n+1}\| \leq r \|x_{n+1} - x_n\|$ . Show that  $\langle x_n \rangle$  is a Cauchy sequence.

**Proof.**  $\|x_{n+1} - x_n\| \leq r^{n-1} \|x_2 - x_1\| = r^{n-1} A$ , for  $A \in \mathbb{R}$ . Given  $\epsilon > 0$ , exists  $N$  such that for all  $n \geq N$ ,  $\|x_{n+1} - x_n\| < A r^{n-1} < \epsilon$ . Then we have

$$\begin{aligned} m > n \geq N \Rightarrow \|x_n - x_m\| &\leq \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq \|x_{n+1} - x_n\| (1 + r + r^2 + \cdots) < \frac{\epsilon}{1 - r} \end{aligned}$$

**Remark.** Counterexample for  $\|x_{n+2} - x_{n+1}\| < \|x_{n+1} - x_n\|$ .  $x_n = \sum_{k=1}^n \frac{1}{k}$

**Problem 2.7.14**  $x_n \rightarrow x$ ,  $A_k = \{x_i : i \geq k\}$ . Show that  $\bigcap_{k=1}^{\infty} \overline{A_k} = \{x\}$ .

**Proof.** Given  $\epsilon > 0$ , there exists  $N$  such that  $n \geq N \Rightarrow x_n \in (x - \epsilon, x + \epsilon)$ . Either  $x_n = x$ , or  $x_n \in (x - \epsilon, x + \epsilon) \setminus \{x\}$ . Thus  $x \in \overline{A_k}$  for all  $k$ .  $\{x\} \subset \bigcap_{k=1}^{\infty} \overline{A_k}$ .

For  $y \in \mathbb{R} \setminus \{x\}$ , we want to show that  $y \notin \bigcap_{k=1}^{\infty} \overline{A_k}$ . Then we want to find  $N$  such that  $y \notin \overline{A_N}$ . Since  $\|x - y\| > 0$ , set  $\epsilon = \frac{1}{3} \|x - y\|$ . There exists  $N$  such that  $\|x_n - x\| < \epsilon$ . Then  $\forall x_n \notin N(y, \epsilon)$ .  $\overline{A_N} = \{x_N, x_{N+1}, \dots\}$ , and  $y$  cannot be in  $\overline{A_N}$ .  $\{x\}^C \subset (\bigcap_{k=1}^{\infty} \overline{A_k})^C \Rightarrow \bigcap_{k=1}^{\infty} \overline{A_k} \subset \{x\}$ .

**Problem 2.7.15**  $\sum a_n$  converges absolutely.

(1)  $\sum a_n^2$

**Proof.**  $a_n^2 < |a_n|$  for large  $n$ . Converges by comparison test.

(2)  $\sum \frac{a_n}{1 + a_n}$

**Proof.** Since  $a_n \rightarrow 0$ , exists  $N$  such that  $n \geq N \Rightarrow |a_n| < 1/3$ . Then for  $n \geq N$ ,  $|1 + a_n| \geq 1 - |a_n| > 2/3 > 1/3$ ,  $1/|1 + a_n| < 3$ . We have  $\left| \frac{a_n}{1 + a_n} \right| < 3|a_n|$ . Converges by comparison test.

(3)  $\sum \frac{a_n^2}{1 + a_n^2}$

**Proof.** Trivial from 1, 2.

**April 15th, 2019**

$K$ : compact  $\iff$  Exists an open cover of  $K$  that has *finite* subcover.

**Theorem 2.5.4** (Heine-Borel) For  $\mathbb{R}^d$ ,  $K$ : compact  $\iff K$  is bounded and closed.

**Theorem 2.5.5** (Cantor's Intersection Theorem)<sup>14</sup>

Given family of **compact** sets  $\{K_i : i \in I\}$ , for all **finite**  $J \subset I$ ,  $\bigcap_{i \in J} K_i \neq \emptyset$ . Then

$$\bigcap_{i \in I} K_i \neq \emptyset$$

**Proof.** (Contradiction)  $\bigcap_{i \in I} K_i = \emptyset \implies \bigcup_{i \in I} K_i^C = \mathbb{R}^d$ . (Complement)

Take any  $K_a$  ( $a \in I$ ), then  $K_a \subset \bigcup_{i \in I} K_i^C (= \mathbb{R}^d) \implies \{K_i^C : i \in I\}$  is an open cover of  $K_a$ . Then there exists a finite subcover,  $\{K_i^C : i \in J\}$  ( $K_a$  is compact) Now we can write  $K_a \subset \bigcup_{i \in J} K_i^C$ . Take complement on both sides to get  $K_a^C \supset \bigcap_{i \in J} K_i$ . Then  $K_a \cap \bigcap_{i \in J} K_i = \emptyset$ , contradiction.

**Remark.** Let  $K_i = [a_i, b_i]$  (Compact in  $\mathbb{R}$ ) and set  $K_1 \supset K_2 \supset \dots$

$\implies$  For  $J = \{j_1, \dots, j_m\}$  ( $j_1 < \dots < j_m$ ),  $\bigcap_{i \in J} K_i = K_{j_m} \neq \emptyset$

$\implies \bigcap_{i=1}^{\infty} K_i \neq \emptyset$  (축소구간정리)

## 2.6 Connected Set

p46-p47 (Section 2.2)

**Definition.**  $X \subset \mathbb{R}^d$ ,  $x \in X$ . Define

$$N_X(x, r) = \{y \in X : \|y - x\| < r\} = N(x, \epsilon) \cap X$$

**Definition.**  $U \subset X$  is open in  $X \iff x \in U, \exists \epsilon > 0$  such that  $N_X(x, \epsilon) \subset U$ .

**Example.**

- $U = \{3\}$ .  $U$  is open in  $X = \mathbb{N}$ .  $N_{\mathbb{N}}(3, 1/10) = 3 \subset U$ . (But not open in  $\mathbb{R}$ )
- For  $X = [0, 10]$ ,  $U = [0, 1)$ .  $x \in U$ ,  $N(x, 1-x) = (2x-1, 1)$ , and this might not be subset of  $U$ . But

$$N_X(x, 1-x) = \begin{cases} (2x-1, 1) & (x > 1/2) \\ [0, 1) & (x \leq 1/2) \end{cases}$$

For both cases  $N_X(x, 1-x) \subset U$ .

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<sup>14</sup>축소구간정리의 가장 일반적인 형태

**Prop 2.2.5**  $U$  is open in  $X \iff U = X \cap V$  for some open set  $V$  in  $\mathbb{R}^d$ .

**Remark.** First example:  $\{3\} = \mathbb{N} \cap (2.9, 3.1)$ , Second example:  $[0, 1] = [0, 10] \cap (-1, 1)$ .

Some references may write this definition as “*relatively*” open in  $X$ .

**Proof of 2.2.5**

( $\implies$ )  $x \in U$ ,  $\exists \epsilon_x > 0$  such that  $N_X(x, \epsilon_x) \subset U$ . Select  $V = \bigcup_{x \in U} N(x, \epsilon_x)$ , which is open.<sup>15</sup>

Then we have  $X \cap V = \bigcup_{x \in U} X \cap N(x, \epsilon_x) = \bigcup_{x \in U} N_X(x, \epsilon_x)$ , which is exactly equal to  $U$ .

( $\impliedby$ )  $x \in U = X \cap V \implies x \in V$ . Thus  $\exists \epsilon > 0$  such that  $N(x, \epsilon) \subset V$ . Then

$$N_X(x, \epsilon) = X \cap N(x, \epsilon) \subset X \cap V = U$$

Thus  $U$  is open in  $X$ .

**Cor.**  $U$ : open in  $X$ ,  $Y \subset X$ .  $\implies U \cap Y$ : open in  $Y$ .

**Proof.**  $U = X \cap V$  ( $V$ : open)  $\implies U \cap Y = X \cap V \cap Y = V \cap (X \cap Y) = V \cap Y$ .

**Definition.**  $S \subset \mathbb{R}^d$ : **disconnected**  $\iff$  There exists **non-empty** sets  $U, V$  such that

$$(1) \ U \cap V = \emptyset$$

$$(2) \ U \cup V = S$$

$$(3) \ U \text{ and } V \text{ are open in } S$$

$S \subset \mathbb{R}^d$ : **connected**  $\iff S$  is not disconnected.

**Question.** Find all  $A \subset \mathbb{R}^d$  such that  $A$  is open and closed.

**Proof.** The only possible sets are  $A = \emptyset, \mathbb{R}^d$ .

If  $A$  is open and closed  $\implies A$ : open,  $A^C$ : open. Then  $\mathbb{R}^d = A \cup A^C$ , and  $\mathbb{R}^d$  is disconnected.

But  $\mathbb{R}^d$  is connected. Contradiction if either  $A$  or  $A^C$  is empty.

**Theorem.** The following are equivalent for  $S \subset \mathbb{R}$ .

$$(1) \ S \text{ is connected.}$$

$$(2) \ \forall a, b \in S \text{ s.t. } a < b, \text{ and } c \in (a, b) \implies c \in S.$$

$$(3) \ S = [a, b] \text{ or } [a, b) \text{ or } (a, b] \text{ or } (a, b) \text{ (} a, b \text{ can be } \pm\infty)$$

---

<sup>15</sup> $N(x, \epsilon)$  is open and union of open sets are always open.

**Remark.** Prop 2.5.1 ( $1' \iff 2'$ ) + Discussion above ( $2 \iff 3$ )

**Proof.**

**(1  $\implies$  2)** (Contradiction) Assume  $a, b \in S, c \notin S$  for some  $a < c < b$ . Set  $U = (-\infty, c) \cap S$ ,  $V = (c, \infty) \cap S$ .  $U, V$  are non-empty.<sup>16</sup>  $U \cap V = \emptyset$  and  $U \cup V = S$ . (Note that  $c \notin S$ ) And  $U, V$  are open in  $S$ . (Prop 2.2.5) Then  $S$  is disconnected.

**(2  $\implies$  1)** (Contradiction) Assume  $S$  is disconnected. There exists  $U, V$  that satisfy the definition of disconnected set. For  $a \in U, b \in V$ , (WLOG  $a < b$ ). By (2),  $[a, b] \subset S$ .

Let  $c = \sup([a, b] \cap U)$ .

Case I)  $c \in U$ . Then  $c \neq b \implies c \in [a, b) = Y \implies c \in U \cap Y$ .

Since  $U$  is open in  $S$  and  $Y \subset S \implies U \cap Y$  is open in  $Y$ . (Cor of 2.2.5)

$\implies \exists \epsilon > 0$  such that  $N_Y(c, \epsilon) \subset U \cap Y \subset U \cap [a, b]$ .

$$Y \cap N(c, \epsilon) = [a, b) \cap (c - \epsilon, c + \epsilon) \supset [c, b) \cap [c, c + \epsilon) = [c, \min\{b, c + \epsilon\})$$

Therefore, we have

$$[c, \min\{b, c + \epsilon\}) \subset N_Y(c, \epsilon) \subset U \cap [a, b]$$

and since  $c$  was the supremum, contradiction.

Case II)  $c \in V$ . Similarly, contradiction.

**(2  $\implies$  3)**  $\inf S = u, \sup S = v$ . (If  $S$  is not bounded below,  $u = -\infty$ , if  $S$  is not bounded above,  $v = \infty$ ). Then if  $c \in (u, v) \implies c \in S$ . There exists  $a, b \in S$  such that  $u \leq a < c < b \leq v$ , meaning that  $S$  must be one of  $[u, v], [u, v), (u, v], (u, v)$ .

**(3  $\implies$  2)** Trivial.

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<sup>16</sup>Always check!  $a \in U, b \in V$ .

**April 17th, 2019**

**Definition.**  $S \subset \mathbb{R}^d$ : **disconnected**  $\iff$  There exists **non-empty** sets  $U, V$  such that

- (1)  $U \cap V = \emptyset$
- (2)  $U \cup V = S$
- (3)  $U$  and  $V$  are open in  $S$

Last time we characterized all connected sets of  $\mathbb{R}$ .

**Theorem 2.6.2** Suppose  $\{C_i : i \in I\}$  is a family of connected sets.<sup>17</sup>

$$\bigcap_{i \in I} C_i \neq \emptyset \implies \bigcup_{i \in I} C_i \text{ is connected}$$

**Proof.** (Routine) Assume  $C = \bigcup_{i \in I} C_i$  is disconnected.  $C$  can be decomposed into 2 sets  $U, V$  (that satisfy condition (1), (2), (3) from the definition). Let

$$U_i = C_i \cap U, \quad V_i = C_i \cap V \quad (\forall i)$$

then  $U_i, V_i$  are open in  $C_i$ .<sup>18</sup> Now  $U_i, V_i$  satisfy (2) and (3) for  $C_i$ . Since  $C_i$  is connected, (1) should not hold, in other words, either  $U_i$  or  $V_i$  must be  $\emptyset$ .

Define:  $I_1 = \{i \in I : U_i = \emptyset, V_i = C_i\}$ ,  $I_2 = \{i \in I : U_i = C_i, V_i = \emptyset\}$ . If  $I_1 = \emptyset \implies I_2 = I \implies V_i = \emptyset \ (\forall i) \implies V = \bigcup_{i \in I} V_i = \emptyset$ <sup>19</sup>, contradiction. Similarly if  $I_2 = \emptyset$ , contradiction.

Select  $i_1 \in I_1, i_2 \in I_2$ . Then  $C_{i_1} = V_{i_1} \subset V$ ,  $C_{i_2} = U_{i_2} \subset U$ . Therefore  $C_{i_1} \cap C_{i_2} = \emptyset$ . Contradiction.

**Example.**

(1)  $x, y \in \mathbb{R}^d$ ,  $[x, y] = \{tx + (1-t)y : t \in [0, 1]\}$  is connected. (Proof similar to Prop 2.6.1)

(2)  $N(x, r) = \bigcup_{y \in N(x, r)} [x, y]$  is connected by the theorem above. ( $\bigcap_{y \in N(x, r)} [x, y] = \{x\} \neq \emptyset$ )

(3)  $\mathbb{R}^d = \bigcup_{y \in \mathbb{R}^d} [0, y]$  is connected.

(4) Convex sets are connected.  $A = \bigcup_{y \in A} [x, y]$ .

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<sup>17</sup>활용 보다도 증명이 중요하니 꼭 기억해 두자.

<sup>18</sup> $U$ : open in  $X$  and  $Y \subset X \implies U \cap Y$ : open in  $Y$ .

<sup>19</sup>Check!

**Definition.** Set  $A$  is **convex**  $\iff x, y \in A \implies [x, y] \subset A$ .

**Comment.** Homework problem: Show that  $S = \{(x, y) : xy > 1\}$  is open.

**Proof.** 1. Show that  $N(z, \epsilon) \subset S$  for all  $z \in S$ .

2. Instead show that  $F = \{(x, y) : xy \leq 1\}$  is closed.

Use Thm 2.2.3 (4). Let  $(x_n, y_n)$  be a sequence in  $F$  that converges to  $(x, y)$ .

$$xy = \lim x_n \lim y_n = \lim x_n y_n \leq 1 \implies (x, y) \in F$$

**Example.**  $A \subset \mathbb{R}^n$ ,  $B \subset \mathbb{R}^m$ , define  $A \times B \subset \mathbb{R}^{n+m}$  as

$$A \times B = \{(x, y) : a \in A, b \in B\}$$

If  $m = n = 1$ ,  $A \times B$  is a rectangular box in  $\mathbb{R}^2$ .

If  $A, B$  is open/closed/compact/connected,  $A \times B$  is open/closed/compact/connected.

**Proof.**

- (1) (Open)  $(a, b) \in A \times B$ . There exists  $\epsilon_1, \epsilon_2 > 0$  such that  $N(a, \epsilon_1) \subset A$ ,  $N(b, \epsilon_2) \subset B$ . Choose  $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ . If  $(x, y) \in N((a, b), \epsilon) \subset \mathbb{R}^{n+m}$ ,<sup>20</sup> we have

$$\epsilon^2 > \|(x, y) - (a, b)\|^2 = \|x - a\|^2 + \|y - b\|^2$$

$$\|x - a\| < \epsilon < \epsilon_1 \text{ and } \|y - b\| < \epsilon < \epsilon_2. \quad x \in A, y \in B.$$

Therefore  $(x, y) \in A \times B$ , and  $N((a, b), \epsilon) \subset A \times B$ .

- (2) (Closed)  $(x_k, y_k)$ : sequence in  $A \times B$ . ( $x_k \in A, y_k \in B$ )  
Suppose  $(x_k, y_k) \rightarrow (x, y)$  ( $x_k \rightarrow x, y_k \rightarrow y$ ). Since  $A$  is closed and  $x_k$  is a sequence in  $A$ ,  $x \in A$ . Similarly,  $y \in B$ . Thus  $(x, y) \in A \times B$ , and  $A \times B$  is closed.

- (3) (Compact)  $A, B$  are closed and bounded. Closed is proven by (2).  
Since  $A, B$  are bounded,  $\exists M_1, M_2$  such that  $\|a\| \leq M_1$ ,  $\|b\| \leq M_2$  for all  $a \in A, b \in B$ .  
For all  $(a, b) \in A \times B$ ,

$$\|(a, b)\| = \sqrt{\|a\|^2 + \|b\|^2} \leq \sqrt{M_1^2 + M_2^2}$$

Therefore  $A \times B$  is bounded. Thus compact.

- (4) (Connected)  $a \in A \implies \{a\} \times B$  is connected.  $b \in B \implies A \times \{b\}$  is connected.

Proof. If the set is disconnected, exists  $\{a\} \times U, \{a\} \times V$  such that splits  $B$ .

Since  $(A \times \{b\}) \cap (\{a\} \times B) = \{(a, b)\} \neq \emptyset$ ,  $(A \times \{b\}) \cup (\{a\} \times B)$  is connected by Thm 2.6.2. Now fix  $a \in A$ , and define  $C_b = (A \times \{b\}) \cup (\{a\} \times B)$ .

Then  $\{C_b : b \in B\}$  is a family of connected sets, and  $\bigcap_{b \in B} C_b = \{a\} \times B \neq \emptyset$ .  $A \times B = \bigcup_{b \in B} C_b$  is connected by Thm 2.6.2.

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<sup>20</sup>Do not write as  $\mathbb{R}^{m+n}$ . First coordinate is  $n$ -dimension, second is  $m$ -dimension.

April 22nd, 2019

### 3. Continuous Functions

#### 3.1 Limit of a Function & Continuous Functions

특별한 언급이 없으면 다음과 같은 가정을 한다.<sup>21</sup>

$$f : X \rightarrow Y \quad (X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n)$$

**Definition.** For  $x_0 \in X'$ ,  $\lim_{x \rightarrow x_0} f(x) = y_0 \iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - y_0\| < \epsilon)$$

**Remark.** Why  $X'$ ?  $X = [0, 1] \cup \{2\}$ , consider  $f(x) = 2x$  on  $X$ .  $\lim_{x \rightarrow 2} f(x)$  is nonsense.

**Example.**

$$(1) f(x) = \begin{cases} x^2 & (x \neq 0) \\ 1 & (x = 0) \end{cases}, \lim_{x \rightarrow 0} f(x) = 0.^{22}$$

For  $\epsilon > 0$ , set  $\delta = \sqrt{\epsilon}$ . Then  $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| = |x^2| < \delta^2 = \epsilon$ .

$$(2) \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4. (X = \mathbb{R} \setminus \{2\}, Y = \mathbb{R}, 2 \in X')$$

For  $\epsilon > 0$ , set  $\delta = \epsilon$ . Then  $0 < |x - 2| < \delta \Rightarrow |f(x) - 4| = |x - 2| < \delta = \epsilon$ .

**Prop 3.1.1**  $f, g : X \rightarrow Y, x_0 \in X'^{23}$ . If  $\lim_{x \rightarrow x_0} f(x) = y_0, \lim_{x \rightarrow x_0} g(x) = z_0$ , then

$$(1) \lim_{x \rightarrow x_0} af(x) + bg(x) = ay_0 + bz_0$$

$$(2) \lim_{x \rightarrow x_0} f(x)g(x) = y_0z_0$$

$$(3) \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{y_0}{z_0} \quad (z_0 \neq 0)$$

연속을 3가지로 정의한다. 세 정의들이 서로 동치임을 이해하는 것이 중요하다.

**Definition.** Let  $f : X \rightarrow Y, x_0 \in X$ .  $f$  is **continuous** at  $x_0 \iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } (\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon)$$

**Remark.**  $\|x - x_0\| < \delta$  should be satisfied for  $x \in X$ . The  $0 <$  condition is omitted here since the inequality holds trivially for  $x_0$ .

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<sup>21</sup>지역이 중요하지 영역은 뭐...

<sup>22</sup>특별한 언급이 없으면  $X = f$  가 정의되는 곳,  $Y = \mathbb{R}^n$  으로 생각한다.

<sup>23</sup>책에  $X$ 로 되어있는데 이는 오타.

- (1)  $x_0 \in X'$ :  $f$  is continuous at  $x_0 \iff \lim_{x \rightarrow x_0} f(x) = f(x_0)$ .
- (2)  $x_0 \in X \setminus X'$  (isolated point):  $f$  is continuous at  $x_0$ .

**Definition.**

- (1)  $A \subset X, f : X \rightarrow Y$ . If  $f$  is continuous at  $x_0$  for all  $x \in A \implies f$  is continuous on  $A$ .
- (2) If  $f$  is continuous on  $X \implies f$  is continuous.

**Prop 3.1.3** The following are equivalent for  $f : X \rightarrow Y$ .

- (1)  $f$ : continuous at  $x_0 \in X$ .
- (2) If there exists a sequence  $\langle x_n \rangle$  in  $X$  converging to  $x_0 \implies \lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ .

**Proof.**

(1  $\implies$  2) Given  $\epsilon > 0$ ,

- (i)  $\exists \delta > 0$  s.t.  $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$
- (ii) Since  $x_n \rightarrow x_0, \exists N$  s.t. for  $n \geq N \implies \|x_n - x_0\| < \delta$ .

Therefore,  $n \geq N \implies \|x_n - x_0\| < \delta \implies \|f(x_n) - f(x_0)\| < \epsilon$ .

(2  $\implies$  1) (Contradiction) Suppose there exists  $\epsilon_0 > 0$  such that no  $\delta$  satisfies  $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon_0$ . (i.e. For all  $\delta > 0, \exists x \in X$  s.t.  $\|x - x_0\| < \delta$  and  $\|f(x) - f(x_0)\| \geq \epsilon_0$ )

Thus for all  $n \in \mathbb{N}$ , there exists  $x_n \in X$  s.t.  $\|x_n - x_0\| < 1/n$  and  $\|f(x_n) - f(x_0)\| \geq \epsilon_0$ . ( $\delta = 1/n$ ) Then we have  $\lim_{n \rightarrow \infty} x_n = x_0$ , but  $\lim_{n \rightarrow \infty} f(x_n) \neq f(x_0)$ . Contradiction.

**Definition.**  $f : X \rightarrow Y, A \subset X, B \subset Y$ . Define

$$f(A) = \{f(x) : x \in A\} \quad f^{-1}(B) = \{x \in X : f(x) \in B\}$$

**Remark.**

- (1)  $A \subseteq f^{-1}(f(A))$   
 $x \in A$  and let  $y = f(x)$ . Then  $y \in f(A)$ , thus  $x \in f^{-1}(f(A))$ .
- (2)  $f(f^{-1}(B)) \subseteq B$   
 $y \in f(f^{-1}(B))$  then  $y = f(x)$  for some  $x \in f^{-1}(B)$ . Thus we have  $x \in f^{-1}(B) \iff f(x) \in B. \therefore y = f(x) \in B$ .



Also remember the counterexamples where the equality does not hold. (1) doesn't hold if  $f$  is not injective, (2) doesn't hold if  $f$  is not surjective.

**Theorem 3.1.5** The following are equivalent for  $f : X \rightarrow Y$ .

- (1)  $f$  is continuous on  $X$ .
- (2)  $B$ : open set in  $Y \implies f^{-1}(B)$ : open in  $X$ .
- (3)  $B$ : closed in  $Y \implies f^{-1}(B)$ : closed in  $X$ .

**Proof.** (2  $\iff$  3) Trivial. Check  $f^{-1}(B^C)$ .

(1  $\implies$  2) Observation.  $f$  is continuous at  $x_0 \iff \forall \epsilon > 0, \delta > 0$  s.t.  $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$ . Re-write the last two inequality as  $x \in N_X(x, \delta)$  and  $f(x) \in N_Y(f(x_0), \epsilon)$ . Then continuity condition is equivalent to

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } f(N_X(x, \delta)) \subset N_Y(f(x_0), \epsilon)$$

Now suppose  $x_0 \in f^{-1}(B) \iff f(x_0) \in B$ . Since  $B$  is open, there exists  $\epsilon > 0$  s.t.  $N_Y(f(x_0), \epsilon) \subset B$ . Then there exists  $\delta > 0$  s.t.  $f(N_X(x_0, \delta)) \subset N_Y(f(x_0), \epsilon) \subset B$ . Take  $f^{-1}$  on both sides.  $N_X(x_0, \delta) \subset f^{-1}(f(N_X(x_0, \delta))) \subset f^{-1}(B)$ . Thus  $f^{-1}(B)$  is open in  $X$ .

(2  $\implies$  1)  $x_0 \in X, f(x_0) \in Y$ . Given  $\epsilon > 0$ ,  $N_Y(f(x_0), \epsilon)$  is open in  $Y$ . By (2),  $f^{-1}(N_Y(f(x_0), \epsilon))$  is open in  $X$ . Observe that this set always contains  $x_0$ . Then  $\exists \delta$  s.t.  $N_X(x_0, \delta) \subset f^{-1}(N_Y(f(x_0), \epsilon))$ . Now take  $f$  on both sides.  $f(N_X(x_0, \delta)) \subset f(f^{-1}(N_Y(f(x_0), \epsilon))) \subset N_Y(f(x_0), \epsilon)$ . Thus  $f$  is continuous at  $x_0$ .

**April 24th, 2019**

연속함수의 기본적인 성질

**Prop 3.1.2** Suppose  $f, g : X \rightarrow \mathbb{R}^n$  are continuous on  $X$ .

- (1)  $af + bg$ : continuous
- (2)  $(n = 1)$   $fg$ : continuous
- (3)  $\frac{f}{g}$ : continuous ( $g \neq 0$  on  $X$ )

**Proof.** (2) Given  $\epsilon > 0$ ,  $\exists \delta_1$  s.t.  $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\epsilon}{2|g(x_0)|+1}$ ,  $\exists \delta_2$  s.t.  $|x - x_0| < \delta_2 \implies |g(x) - g(x_0)| < \frac{\epsilon}{2(|f(x_0)| + \frac{\epsilon}{2|g(x_0)|+1})}$ . Then we have

$$\begin{aligned} |f(x)g(x) - f(x_0)g(x_0)| &= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\ &\leq |f(x)| |g(x) - g(x_0)| + |g(x_0)| |f(x) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Thus we have continuity.

**Proof 2.** By sequential definition, exists  $\langle x_n \rangle \rightarrow x_0$  in  $X$  such that  $f(x_n) \rightarrow f(x_0), g(x_n) \rightarrow g(x_0)$ . Then we have  $f(x_n)g(x_n) \rightarrow f(x_0)g(x_0)$ .

**Prop 3.1.4** Suppose we have two continuous functions  $f : X \rightarrow Y, g : Y \rightarrow Z$ . If  $f$  is continuous at  $x_0 \in X$ , and if  $g$  is continuous at  $f(x_0)$ , then  $g \circ f$  is continuous at  $x_0$ .

**Proof.** Given  $\epsilon > 0$ ,  $\exists \delta_1 > 0$  s.t.  $\|y - f(x_0)\| < \delta_1 \implies \|g(y) - g(f(x_0))\| < \epsilon$ . Also,  $\exists \delta_2 > 0$  s.t.  $\|x - x_0\| < \delta_2 \implies \|f(x) - f(x_0)\| < \delta_1$ . Now we automatically have  $\|g(f(x)) - g(f(x_0))\| = \|(g \circ f)(x) - (g \circ f)(x_0)\| < \epsilon$ .

**Remark.** Suppose  $f$ : continuous  $X$ ,  $g$ : continuous on  $Y$  (or on  $f(X)$ ). Then  $g \circ f$  is continuous on  $X$ .

**Example.**

- (1) Polynomials are continuous. Use continuity of  $f(x) = x$ .
- (2)  $f(x) = \sqrt{x}$ .
- (3)  $f(x) = \sqrt{x^4 + 1}$  is continuous.

$$(4) f(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases} \text{ is not continuous.}$$

**Proof.**  $x_0 \in \mathbb{R}$ . Suppose there exists a sequence  $\langle x_n \rangle$  in  $\mathbb{Q}$  converging to  $x_0$ . Then  $\langle f(x_n) \rangle \rightarrow 1$ . ( $x_n = \lfloor nx_0 \rfloor / n$ ) But there also exists a sequence  $\langle x_n \rangle$  in  $\mathbb{R} \setminus \mathbb{Q}$  converging to  $x_0$ . Then  $\langle f(x_n) \rangle \rightarrow 0$ . ( $x_n = \lfloor \sqrt{2}nx_0 \rfloor / \sqrt{2}n$ )  $f(x)$  cannot be continuous anywhere.

### 3.2 Extreme Value Theorem & Intermediate Value Theorem

**Theorem 3.2.1** If  $f : X \rightarrow Y$  is continuous, surjective and  $X$ : compact, then  $Y$ : compact.

**Proof.** Suppose  $\{U_i : i \in I\}$  is an open cover of  $Y$ .  $V_i = U_i \cap Y$  is an open set in  $Y$ , and  $\{V_i : i \in I\}$  is also an open cover of  $Y$ . Consider  $\{f^{-1}(V_i) : i \in I\}$ , which is an open cover of  $X$ . Since  $X$  is compact, there exists a finite subcover  $\{f^{-1}(V_i) : i \in J\}$  ( $J \subset I$ ) of  $X$ . Then  $\{V_i : i \in J\}$  is a finite subcover of  $Y$ .

$$Y = f(X) = f\left(\bigcup_{i \in J} f^{-1}(V_i)\right) = \bigcup_{i \in J} f(f^{-1}(V_i)) \subset \bigcup_{i \in J} V_i$$

We have a finite subcover of  $Y$ . Thus  $Y$  is compact.

**Check.**  $\forall A \subset X$ .  $f$ : surjective  $\implies f(f^{-1}(A)) = A$ .  $f$ : injective  $\implies f^{-1}(f(A)) = A$ .

**Remark.**

(1)  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $f$ : continuous. If  $K \subset \mathbb{R}^m$  is compact,  $f(K)$  is compact.  
Set  $f : K \rightarrow f(K)$ .

(2) Image of compact set is compact.

**Cor 3.2.2** Suppose  $X$  is compact.  $f : X \rightarrow \mathbb{R} \implies f$  has maximum and minimum.

**Proof.** Set  $f : X \rightarrow f(X)$ , then  $f$  is surjective and  $f(X)$  is compact. Check that if  $K \subset \mathbb{R}$ ,  $K$ : compact, then  $\inf K, \sup K \in K$  and  $\inf K = \min K$ ,  $\sup K = \max K$ .

**Cor 3.2.4 (Extreme Value Theorem)** If  $f$  is a continuous function defined on  $[a, b]$ ,  $f$  has a maximum and minimum.

**Proof.**  $[a, b]$  is compact.

**Cor 3.2.3** Suppose  $X$  is compact and  $f : X \rightarrow \mathbb{R}$  is continuous. If  $f(x) > 0$  for all  $x \in X$ , then  $\exists \delta > 0$  s.t.  $f(x) \geq \delta > 0$  for all  $x \in X$ .

**Proof.** Let  $\delta = \min f(X) = f(u) > 0$  for some  $u$ .

**Remark.**  $X = [1, \infty)$ ,  $f(x) = 1/x$ . ( $X$  is not compact.)

**Cor 3.2.5** Suppose  $X$  is compact and  $f : X \rightarrow Y$  is bijective and continuous. Then  $f^{-1}$  is continuous.

**Check.**  $f : X \rightarrow Y$ .  $A \subset X, B \subset Y$ . Image:  $f(A)$ , pre-image:  $f^{-1}(B)$ . We must check if image of  $B$  on  $f^{-1}$  is equal to the pre-image of  $B$ . (Well-definedness!)

**April 26th, 2019**

Assignment 3.5 #3: Check and remember.

$$(2) \quad f\left(\bigcap_{i \in \mathcal{I}} A_i\right) \subset \bigcap_{i \in \mathcal{I}} f(A_i)$$

**Problem 3.1.2**  $f : X \rightarrow \mathbb{R}^n$ ,  $f(x) = (f_1(x), \dots, f_n(x))$  ( $x \in X$ ). The following are equivalent.

(1)  $f$  is continuous at  $x$ .

(2) For all  $i$ ,  $f_i : X \rightarrow \mathbb{R}$  is continuous at  $x$ .

**Proof.** (1  $\implies$  2)  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $\|y - x\| < \delta \implies \|f(y) - f(x)\| < \epsilon$ . Then we have  $\|f_i(y) - f_i(x)\| \leq \|f(y) - f(x)\| < \epsilon$ , for any  $i$ .

(2  $\implies$  1)  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $\|x - y\| < \delta \implies \|f_i(x) - f_i(y)\| < \epsilon/\sqrt{n}$ . Then

$$\|x - y\| < \delta \implies \|f(x) - f(y)\| = \sqrt{\sum_{i=1}^n \|f_i(x) - f_i(y)\|^2} < \sqrt{n \cdot \frac{\epsilon^2}{n}} = \epsilon$$

**Prop 3.1.2** (3)  $f, g$ : continuous  $\implies f/g$ : continuous ( $g \neq 0$  on  $X$ )

**Proof.**  $\forall \epsilon > 0, \exists \delta > 0$  s.t. for all  $x_0 \in X$ ,

$$|x - x_0| < \delta \implies |g(x) - g(x_0)| < \min\left\{\frac{1}{2}|g(x_0)|, \frac{1}{4}\frac{|g(x_0)|^2\epsilon}{|f(x_0)|+1}\right\}, |f(x) - f(x_0)| < \frac{1}{4}|g(x_0)|\epsilon.$$

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}\right| &\leq \frac{|g(x_0)| |f(x) - f(x_0)| + |f(x_0)| |g(x) - g(x_0)|}{|g(x)| |g(x_0)|} \\ &\leq \frac{|g(x_0)| \frac{1}{4}|g(x_0)|\epsilon + |f(x_0)| \frac{1}{4}\frac{|g(x_0)|^2\epsilon}{|f(x_0)|+1}}{\frac{1}{2}|g(x_0)|^2} < \frac{\frac{1}{4}|g(x_0)|^2\epsilon + \frac{1}{4}|g(x_0)|^2\epsilon}{\frac{1}{2}|g(x_0)|^2} = \epsilon \end{aligned}$$

**Example.**  $g(x) = \begin{cases} 0 & (x = 0, 1 \text{ or } x \in \mathbb{R} \setminus \mathbb{Q}) \\ 1/q & (x = p/q, \text{irreducible fraction}) \end{cases}$

(i)  $x_0 \in \mathbb{Q} \cap (0, 1)$  then  $g(x_0) > 0$ . Set  $\epsilon = \frac{1}{2}g(x_0) > 0$ . For all  $\delta > 0, \exists y \in \mathbb{Q}^C \cap [0, 1]$  s.t.  $|y - x_0| < \delta$ , but  $|g(y) - g(x_0)| = g(x_0) \geq \epsilon$ . Thus  $f$  is not continuous at  $x_0$ .

(ii)  $x_0 \in \mathbb{Q}^C \cup \{0, 1\}$ .  $g(x_0) = 0$ .  $\forall \epsilon > 0, \exists N \geq 1$  s.t.  $1/N < \epsilon$ . Then there are finitely many  $y$  such that  $g(y) \geq 1/N$ . ( $\frac{1}{N}, \frac{1}{N-1}, \dots, \frac{1}{2}$  is finite) Let them be  $y_1, \dots, y_k$  and set  $\delta = \min_{1 \leq i \leq k} |y_i - x_0| > 0$ . If  $\|y - x_0\| < \delta$ , then  $0 \leq g(y) < 1/N < \epsilon$ .  $|g(y) - g(x_0)| = g(y) < \epsilon$ .

**Problem 3.5.1**

(1)  $f(x) = 0, f(\mathbb{R}) = \{0\}$  (closed)

(3)  $f(x) = e^x, f(\mathbb{R}) = (0, \infty)$  (open)

April 29th, 2019

### 3.2 EVT & IVT

**Theorem 3.2.1** Suppose  $f : X \rightarrow Y$  is continuous and surjective.<sup>24</sup> If  $X$  is compact,  $Y$  is also compact.

**Remark.**  $f : X \rightarrow Y$  continuous,  $K \subset X : \text{compact} \implies f(K) : \text{compact}$ . Inverse does not hold. Consider  $f(x) = \sin x$ . Image is  $[0, 1]$  (compact), but pre-image is  $\mathbb{R}$  (not bounded).

**Definition.** Function  $f : X \rightarrow \mathbb{R}$  has **maximum**  $M$  if there exists  $u \in X$  s.t.  $f(u) = M$ , and  $\forall x \in X, f(x) \leq M$ .

**Cor 3.2.5** Suppose  $f : X \rightarrow Y$  is continuous and bijective. If  $X$  is compact,  $f^{-1} : Y \rightarrow X$  is continuous.<sup>25</sup>

**Proof.** Let  $f^{-1} = g : Y \rightarrow X$ . For any open set  $U$  in  $X$ , it is enough to show that  $g^{-1}(U)$  is open in  $Y$ . But  $g^{-1}(U) = (f^{-1})^{-1}(U) = f(U)$ . Check that  $Y \setminus f(U) = f(X \setminus U)$ . Since a closed subset of a compact set is compact,  $Y \setminus f(U) = f(X \setminus U)$  is compact, and hence closed in  $\mathbb{R}^d$ . Then  $f(U) = (Y \setminus f(U))^c \cap Y$  is open in  $Y$ .

**Example.**  $f : X = \{0\} \cup (1, 2) \rightarrow Y = [0, 1)$ .  $f(0) = 0$ ,  $f(x) = x - 1$  on  $(1, 2)$ . By definition,  $f$  is continuous on  $X$ . Consider  $f^{-1}$ .  $f^{-1}(0) = 0$ ,  $f^{-1}(x) = x + 1$  on  $(0, 1)$ .  $f^{-1}$  is not continuous.<sup>26</sup>

**Application.** (Distance between sets) Define dist as follows.

$$A, B \subset \mathbb{R}^d, \quad \text{dist}(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}$$

**Example.**  $A = \{(x, y) : x \leq 0\}$ ,  $B = \{(x, y) : xy \geq 1, x, y > 0\}$ .  $\text{dist}(A, B) \leq \|(0, n) - (\frac{1}{n}, n)\| = 1/n$  for all  $n$ . Thus  $\text{dist}(A, B) = 0$ .

**Theorem.**  $A : \text{compact}, B : \text{closed}. A \cap B = \emptyset \implies \text{dist}(A, B) > 0$ .

**Proof.**  $f : A \rightarrow \mathbb{R}, f(x) = \text{dist}(\{x\}, B)$  ( $x \in A$ ).

(i)  $f(x) > 0$  for all  $x \in A$ .

$\because N(x, \epsilon) \subset B^C$  (open)  $\implies \text{dist}(\{x\}, B) \geq \epsilon > 0$ .

(ii)  $f$ : continuous,  $b \in B$ . For  $x, y \in A$ ,  $\|x - b\| \leq \|x - y\| + \|y - b\|$ . Take infimum over  $b \in B$ . Then we have  $f(x) \leq \|x - y\| + f(y)$ . Similarly we have  $f(y) \leq \|x - y\| + f(x)$ . Hence  $\|f(x) - f(y)\| \leq \|x - y\|$ . (Continuity follows easily by setting  $\delta = \epsilon$ )

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<sup>24</sup>Not necessarily. Adjust  $Y$  to be  $f(X)$ .

<sup>25</sup>Thm 3.1.5 was about the pre-image of an open set. In this corollary, we must show that the image of an open set is also open.

<sup>26</sup>수학적으로 장난질 치는게 아니라 본질적인 의미가 있는 예시입니다.

**Lipschitz Continuous:**  $\|f(x) - f(y)\| \leq k \|x - y\|$  for some  $k \geq 0$  (Set  $\delta = \epsilon/k$  to show continuity)

**Contraction:** Lipschitz continuous and  $k = 1$ .

By Cor 3.2.3,  $\exists \delta > 0$  s.t.  $f(x) \geq \delta > 0$  for all  $x \in A$ . Then  $\text{dist}(A, B) \geq \delta > 0$ .

**Theorem 3.2.8** Suppose  $f : X \rightarrow Y$  is continuous and surjective. If  $X$  is connected,  $Y$  is also connected.

**Proof.**<sup>27</sup> (Contradiction) Assume  $Y$  is disconnected. Then there exists non-empty sets  $U, V$  that are open in  $Y$ , and  $U \cap V = \emptyset$ ,  $U \cup V = Y$ . Consider  $f^{-1}(U), f^{-1}(V)$ . We will show that  $X$  is disconnected. Since  $f$  is surjective,  $f^{-1}(U), f^{-1}(V)$  are non-empty. Decomposition conditions can be checked easily, (use theorems from assignment) and openness holds by continuity.

**Remark.** Suppose  $f : X \rightarrow Y$  is continuous. If  $C \subset X$  is connected,  $f(C)$  is also connected.

**Cor 3.2.9** Suppose  $f : I \rightarrow \mathbb{R}$  is continuous where  $I$  is any interval of  $\mathbb{R}$ . Then  $f(I)$  is also an interval and hence connected.<sup>28</sup>

**Cor 3.2.10 (Intermediate Value Theorem)** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. If  $\alpha$  is in between  $f(a)$  and  $f(b)$ ,<sup>29</sup> then  $\exists c \in [a, b]$  s.t.  $f(c) = \alpha$ .<sup>30</sup>

**Proof.**  $f([a, b])$  is an **interval** (Cor 3.2.9) which includes  $f(a), f(b)$ . Then it must include  $\alpha$ .<sup>31</sup>

**Cor 3.2.11** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then  $f([a, b])$  is a closed interval.

**Proof.**  $f([a, b])$  is an interval (Cor 3.2.9) and compact (Thm 3.2.1).

**Cor 3.2.12** Suppose  $f : [a, b] \rightarrow [a, b]$  is continuous. Then  $\exists c \in [a, b]$  s.t.  $f(c) = c$ . We call such  $c$  a fixed point.

**Proof.** Apply IVT on  $g(x) = x - f(x)$ , set  $\alpha = 0$ . Then we have

$$g(a) = a - f(a) \leq 0 = \alpha = 0 \leq b - f(b) = g(b)$$

and the result follows directly.

**Application. (Path-Connected Set)**

**Remark.**  $x, y \in \mathbb{R}^d \implies [x, y] = \{tx + (1 - t)y : 0 \leq t \leq 1\}$  (convex combination)

<sup>27</sup>책과 약간 다릅니다. 책의 증명도 읽어보세요.

<sup>28</sup>이런 집합을 구간으로만 이해를 하면 우리가 아무것도 못 해요. 그런데 애를 연결집합으로 이해하면 뭔가 할 것들이 생기고 여기서 중간값 정리가 바로 나오죠.

<sup>29</sup> $(f(a) - \alpha)(f(b) - \alpha) < 0$

<sup>30</sup>이 정리를 위해 달려온 것...

<sup>31</sup>구간은 볼록집합임을 이용해도  $\alpha$ 를 포함함을 보일 수 있다.

Set  $f : [0, 1] \rightarrow [x, y]$  as  $f(t) = tx + (1 - t)y$ . Then  $f$  is continuous. (Lipschitz continuity can be easily checked and  $f$  is surjective)

**Definition.** Let  $a, b \in \mathbb{R}$ ,  $a < b$ . Suppose  $f : [a, b] \rightarrow \mathbb{R}^d$  is continuous. Then  $f([a, b])$  is called a **path**.

**Remark.** Define  $f : [a, b] \rightarrow \mathbb{R}^3$  as  $f(t) = (\sin t, \cos t, \frac{1}{1+t^2})$  (Parameterized curve)  
Also note that a path is compact and connected. ( $[a, b]$  is compact and connected)

**Definition.**  $C \subset \mathbb{R}^d$  is called **path-connected** if for any  $x, y \in C$ , there exists a path **in**  $C$  connecting  $x$  and  $y$ .

**Theorem.** Path-connected  $\implies$  Connected

**Proof.** (Contradiction) Assume  $X$  is path-connected but disconnected. Then there exists sets  $U, V$  such that satisfy disconnectedness for  $X$ . Let  $x \in U$ ,  $y \in V$ . From path-connected condition, there exists  $f : [a, b] \rightarrow X$  s.t.  $f$  is continuous,  $f(a) = x$ , and  $f(b) = y$ . Let  $Y = f([a, b]) \subset X$ . Then  $Y$  can be decomposed into  $Y \cap U$  and  $Y \cap V$ . These two sets satisfy the disconnectedness condition, (check) hence  $Y$  is disconnected. But since paths are always connected, contradiction.

**Remark.** The converse of the above theorem is **false**. Consider  $f(x) = \sin \frac{1}{x}$  ( $x > 0$ ). Set  $A = \{(x, \sin \frac{1}{x}) : x \in (0, 1)\} \subset \mathbb{R}^2$ .  $A$  is a path and therefore connected.

But the problem arises when we consider  $\overline{A}$ . We can easily check that the closure of a connected set is connected. We can also check that  $\overline{A} = A \cup \{(0, t) : t \in [-1, 1]\}$ , which is not path-connected.<sup>32</sup>

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<sup>32</sup>We need a jump from  $x = 0$  to  $x > 0$ ...

May 1st, 2019

### 3.3 Uniform Continuity

**Definition.**  $f : X \rightarrow Y$  is **uniformly continuous**  $\iff \forall \epsilon > 0, \exists \delta > 0$  s.t.  $x, y \in X$ ,  $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$ .

**Remark.** “ $f : X \rightarrow Y$  is continuous at  $x_0 \in X$ ” meant that  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$ . In this definition,  $\delta$  was a function of  $x_0$ . But in the definition of uniform continuity,  $\delta$  is only dependent of  $\epsilon$ .

**Example.**

(1)  $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$  (Not uniformly continuous)

For  $\epsilon = 1$ , suppose we have  $\delta > 0$ . Set  $x = 1/\delta + \delta/2, y = 1/\delta$ . Then  $|x - y| = \delta/2 < \delta$ , but  $|f(x) - f(y)| = |x^2 - y^2| = 1 + \delta^2/4 > \epsilon$ .

(2)  $f : [0, 1] \rightarrow \mathbb{R}, f(x) = x^2$  (Uniformly continuous & Lipschitz continuous)<sup>33</sup>

Given  $\epsilon > 0, \delta = \epsilon/2$ . If  $|x - y| < \delta$  then  $|f(x) - f(y)| = |x + y| |x - y| < 2\delta = \epsilon$ .

(3) Lipschitz Continuity  $\implies$  Uniform Continuity

Suppose  $\forall x, y \in X, \exists k > 0$  s.t.  $\|f(x) - f(y)\| \leq k \|x - y\|$ . Then set  $\delta = \epsilon/k$  to show uniform continuity.

(4) **Lipschitz  $\implies$  Uniform  $\implies$  Continuous**

$f : [0, \infty) \rightarrow \mathbb{R}, f(x) = \sqrt{x}$ .

(a) Not Lipschitz continuous.

$|f(x) - f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq k |x - y|$  for all  $x, y \in X$ ? Impossible.

(b) Uniform continuous.

Set  $\delta = \epsilon^2$ .  $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|} < \sqrt{\delta} = \epsilon$

**Theorem 3.3.1** (Heine's Theorem) Suppose  $f : X \rightarrow Y$  is continuous. If  $X$  is compact,  $f$  is uniformly continuous.

**Proof.** Given  $\epsilon > 0, x \in X, \exists \delta(x) > 0$  s.t.  $\|y - x\| < \delta(x) \implies \|f(y) - f(x)\| < \epsilon/2$ .

Define  $U_x = N(x, \delta(x)/2)$ . Then  $\{U_x : x \in X\}$  is a open cover of  $X$ . By compactness, there exists a finite subcover  $\{U_{x_i}\}_{i=1}^n$ . Set  $\delta = \frac{1}{2} \min\{\delta(x_1), \dots, \delta(x_n)\}$ .

Suppose  $\|x - y\| < \delta$ . For some  $k, x \in U_{x_k}$ , and then  $y \in N(x_k, \delta(x_k))$ . This is because

$$\|x - x_k\| < \delta(x_k)/2, \quad \|y - x_k\| \leq \|y - x\| + \|x - x_k\| < \delta + \delta(x_k)/2 < \delta(x_k)$$

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<sup>33</sup>함수의 성질일 뿐만 아니라 domain 의 성질이기도 하다? Domain 도 중요한 역할을 한다.



Then we have

$$\|f(x) - f(y)\| \leq \|f(x) - f(x_k)\| + \|f(x_k) - f(y)\| < \epsilon/2 + \epsilon/2 = \epsilon$$

by continuity of  $f$ . Thus  $f$  is uniformly continuous.

**Theorem 3.3.2** Suppose  $f : X \rightarrow Y$  is uniformly continuous. If  $\langle x_n \rangle$  is a Cauchy sequence in  $X$ ,  $\langle f(x_n) \rangle$  is also a Cauchy sequence.

**Proof.** Given  $\epsilon > 0$ ,  $\exists \delta > 0$  s.t.  $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$ . For this  $\delta$ ,  $\exists N$  s.t.  $m, n \geq N \implies \|x_m - x_n\| < \delta$ . Then we have

$$m, n \geq N \implies \|x_m - x_n\| < \delta \implies \|f(x_m) - f(x_n)\| < \epsilon$$

**Remark.** If  $f : X \rightarrow Y$  is continuous,  $\langle x_n \rangle \rightarrow x$  then  $\langle f(x_n) \rangle \rightarrow f(x)$ . In this case,  $\langle x_n \rangle, x$  must be in  $X$ ,  $\langle f(x_n) \rangle, f(x)$  must be in  $Y$ .

Consider  $f : (0, 1) \rightarrow \mathbb{R}$ ,  $f(x) = 1/x$ .  $x_n = 1/n$  converges, and is a Cauchy sequence. But  $f(x_n) = n$  is not Cauchy. The limit value of  $\langle x_n \rangle$  does not have to be in  $X$  for a uniform continuous function.

**Definition.** Suppose  $f : X \rightarrow Y$  is continuous,  $X \subset A, Y \subset B$ . If  $g : A \rightarrow B$  satisfies  $g(x) = f(x)$  for  $x \in X$ , and if  $g$  is continuous on  $A$ , we say that  $g$  is a **continuous extension** of  $f$  to  $A$ .

**Example.**

(1)  $f : (0, 1) \rightarrow \mathbb{R}$ ,  $f(x) = x$ .

Consider  $A = (0, 2)$ .  $g(x) = x$  on  $(0, 2)$  is a continuous extension,  $h(x) = x$  on  $(0, 1)$ ,  $h(x) = 1$  on  $[1, 2)$  is also a continuous extension.

Consider  $A = [0, 1]$ . Then  $g(0) = 0, g(1) = 1$ ,  $g(x) = x$  on  $(0, 1)$  is a unique continuous extension of  $f$ .

(2)  $f : (0, 1) \rightarrow \mathbb{R}$ ,  $f(x) = 1/x$ .

Consider  $A = [0, 1)$ . It is impossible to find a continuous extension.

**Cor 3.3.3** Suppose  $f : X \rightarrow Y$  is uniformly continuous. Then there exists a unique continuous extension of  $f$  to  $\overline{X}$ .<sup>34</sup>

**Proof.** Take  $x_0 \in \overline{X} \setminus X$ . Set  $g(x) = f(x)$  for  $x \in X$ . Now for  $g(x_0)$ , recall that  $x_0 \in \overline{X}$ , so there exists a sequence  $\langle x_n \rangle$  in  $X$  s.t.  $x_n \rightarrow x_0$ . Since  $\langle x_n \rangle$  is convergent,  $\langle x_n \rangle$  is Cauchy sequence and by Thm 3.3.2,  $\langle f(x_n) \rangle$  is also a Cauchy sequence. Thus  $\langle f(x_n) \rangle$  converges. Define  $g(x_0)$  as the limit of  $f(x_n)$ .

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<sup>34</sup> $Y$  is assumed to be extended to  $\mathbb{R}^d$ .

Now we must check if  $g(x_0)$  is well-defined. In other words: For any two sequence  $\langle x_n \rangle, \langle y_n \rangle$  that converge to  $x_0$ , does  $f(x_n), f(y_n)$  converge to the same value?

Consider  $\langle z_n \rangle = x_1, y_1, x_2, y_2, \dots$ . It is trivial that  $z_n \rightarrow x_0$ . Since  $\langle z_n \rangle$  is Cauchy,  $\langle f(z_n) \rangle$  is also Cauchy by uniform continuity. Let its limit be  $\gamma$ . Then  $\langle f(x_n) \rangle, \langle f(y_n) \rangle$  is a subsequence of  $\langle f(z_n) \rangle$ , thus they both must converge to  $\gamma$ . Uniqueness directly follows from this proof, and we can easily check that  $g$  is continuous.

May 8th, 2019

### 3.4 Monotone Function

For this section,  $f : X \rightarrow \mathbb{R}$ ,  $X \subset \mathbb{R}$ ,  $X$  is an interval.

**Definition.**  $f$  is **monotonically increasing** if  $x < y$  then  $f(x) \leq f(y)$ .<sup>35</sup>  $f$  is **monotonically decreasing** if  $x < y$  then  $f(x) \geq f(y)$ .

**Definition.**  $f$  is **increasing** if  $x < y$  then  $f(x) < f(y)$ , **decreasing** if  $x < y$  then  $f(x) > f(y)$ .

**Remark.** Monotonically increasing = Weakly increasing. Increasing = Strongly increasing.

**Example.**  $f(x) = \begin{cases} \sin \frac{1}{|x|} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$  has no left/right limits at  $x = 0$ .

**Definition.**  $f : X \rightarrow \mathbb{R}$ ,  $x_0 \in X$ ,  $\alpha \in \mathbb{R}$ .<sup>36</sup>

(1) (Right Limit)  $\lim_{x \rightarrow x_0+} f(x) = \alpha$ ,  $f(x_0+) = \alpha \iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } (x_0, x_0 + \delta) \subset X \text{ and } x \in (x_0, x_0 + \delta) \implies |f(x) - \alpha| < \epsilon$$

(2) (Left Limit)  $\lim_{x \rightarrow x_0-} f(x) = \alpha$ ,  $f(x_0-) = \alpha \iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } (x_0 - \delta, x_0) \subset X \text{ and } x \in (x_0 - \delta, x_0) \implies |f(x) - \alpha| < \epsilon$$

**Exercise.**  $\lim_{x \rightarrow x_0} f(x) = \alpha \iff f(x_0+) = f(x_0-) = \alpha$ .

**Definition.** (Infinite Limits)

(1)  $f(x_0+) = \infty \iff$

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } (x_0, x_0 + \delta) \subset X \text{ and } x \in (x_0, x_0 + \delta) \implies f(x) > M$$

(2)  $f(x_0+) = -\infty \iff$

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } (x_0, x_0 + \delta) \subset X \text{ and } x \in (x_0, x_0 + \delta) \implies f(x) < -M$$

**Remark.**  $x_0 \in \text{int}X$ , we define

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<sup>35</sup>Watch out for the " $\leq$ ".

<sup>36</sup> $(x_0, x_0 + \delta) \subset X$  condition is necessary. Consider  $X = [0, 1]$ , the right limit of  $x = 1$  can be any real number...

$$\lim_{x \rightarrow x_0} f(x) = \pm\infty \iff f(x_0+) = f(x_0-) = \pm\infty$$

**Theorem 3.4.1** Suppose  $f : X \rightarrow \mathbb{R}$  is monotone on  $X = (a, b)$ .

- (1)  $\forall x_0 \in (a, b) \implies$  Both  $f(x_0+), f(x_0-)$  exist.
- (2)  $f(a+), f(b-)$  exist.
- (3) For  $a < x < y < b$ , if  $f$  is monotonically increasing,

$$f(a+) \leq f(x-) \leq f(x) \leq f(x+) \leq f(y-) \leq f(y) \leq f(y+) \leq f(b-)$$

**Proof.** WLOG, suppose  $f$  is monotonically increasing.

- (1) Define  $\alpha = \inf\{f(t) : t \in (x_0, b)\}$ . (the set is bounded below by  $f(x_0)$ )

**Claim.**  $f(x_0+) = \alpha$ .

**Proof.**  $\forall \epsilon > 0, \exists x_1 \in (x_0, b)$  s.t.  $f(x_1) < \alpha + \epsilon$ . ( $\alpha$  is infimum) Now set  $\delta = x_1 - x_0$ . Then  $(x_0, x_0 + \delta) \subset X$ . For the second condition, if  $x \in (x_0, x_0 + \delta) = (x_0, x_1) \implies \alpha \leq f(x) \leq f(x_1) < \alpha + \epsilon$ . Thus  $|f(x) - \alpha| < \epsilon$ .

From the claim we have  $f(x_0+) = \inf\{f(t) : t \in (x_0, b)\}$ ,  $f(x_0-) = \sup\{f(t) : t \in (a, x_0)\}$

- (2) Define  $\alpha = \inf\{f(t) : t \in (a, b)\}$  if the set is bounded below,  $-\infty$  otherwise. Then we have  $f(a+) = \alpha$ . (Left as exercise)  
Also define  $\beta = \sup\{f(t) : t \in (a, b)\}$  if the set is bounded above,  $\infty$  otherwise. Then we have  $f(b-) = \beta$ .<sup>37</sup>

- (3) Trivial. Check  $f(x+) \leq f(y-)$ . ( $\frac{x+y}{2}$  is in both  $(x, b), (a, y)$ )

$$f(x+) = \inf\{f(t) : t \in (x, b)\} \leq f\left(\frac{x+y}{2}\right) \leq \sup\{f(t) : t \in (a, y)\} = f(y-)$$

**Cor 3.4.2** Suppose  $f : X \rightarrow \mathbb{R}$  is monotone and  $X$  is an interval. Define

$$D = \{x_0 \in X : f \text{ is discontinuous at } x_0\}$$

then  $D$  is finite or countable.

**Proof.** WLOG, suppose  $f$  is monotonically increasing.

Suppose  $x_0 \in D' = D \setminus \{\text{two endpoints of } X\}$ . By Thm 3.4.1, left, right limits at  $x_0$  exist, and  $f(x_0+) > f(x_0-)$ . (If equality holds,  $f$  is continuous at  $x_0$ )

Define  $g : D' \rightarrow \mathbb{Q}$  by  $g(x_0) = q_{x_0} \in (f(x_0-), f(x_0+))$  (any rational) Then  $g : D' \rightarrow g(D') \subset \mathbb{Q}$

---

<sup>37</sup>극한값이  $\infty$  인 경우도 존재한다고 표현하는가?

is bijective. Since  $g(D')$  is finite or countable (subset of  $\mathbb{Q}$ ),  $D'$  is also finite or countable.

**Theorem 3.4.3** Suppose  $f : X \rightarrow \mathbb{R}$  is continuous and  $X$  is an interval.<sup>38</sup> The following are equivalent.

- (1)  $f$  is injective.
- (2)  $f$  is strongly increasing or decreasing.

**Proof.** (책과 다름) (**2**  $\implies$  **1**) Trivial.

(**1**  $\implies$  **2**) Define  $D \subset \mathbb{R}^2$ ,  $D = \{(x, y) : x, y \in X, x < y\}$ .  $g : D \rightarrow \mathbb{R}$ ,  $g(x, y) = f(x) - f(y)$ .

- (1)  $D$  is connected. (Convex) (Check!)
- (2)  $g$  is continuous. (Trivial by sequence definition)

Thus  $g(D)$  is connected, and since it is a subset of  $\mathbb{R}$ ,  $g(D)$  is an interval. Also,  $0 \notin g(D)$  since  $x < y$  in the definition of  $D$  and  $f(x) - f(y)$  is never 0 by injectivity.

Hence  $g(D)$  is a subset of  $(0, \infty)$  or  $(-\infty, 0)$ . If  $g(D) \subset (0, \infty)$ ,  $f$  is decreasing.  $f$  is increasing for the second case.

**Remark.** Suppose  $f : X \rightarrow \mathbb{R}$  is continuous and  $X$  is an interval. If  $f$  is increasing (or decreasing),  $f : X \rightarrow f(X)$  is bijective, (injective by Thm 3.4.3) and  $f^{-1} : f(X) \rightarrow X$  is continuous.

**Proof.**  $\delta = \min\{f(x_0) - f(x_0 - \epsilon), f(x_0 + \epsilon) - f(x_0)\}$

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<sup>38</sup>Note that this is the first time supposing continuity.

May 13th, 2019

## 4. 미분가능함수의 성질

### 4.1 Differentiability

For this section, suppose  $f : I \rightarrow \mathbb{R}$ ,  $I = (a, b), (-\infty, b), (a, \infty), (-\infty, \infty)$ .

**Definition.**  $f$  is **differentiable** at  $x_0 \in I \iff$

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \alpha \in \mathbb{R}$$

**Remark.**

- (1) Denote  $\alpha = f'(x_0)$ . (**Derivative** of  $f$  at  $x_0$ )
- (2) Differentiability is defined point-wise.
- (3)  $f$  is differentiable on  $I \iff f$  is differentiable at all  $x_0 \in I$

**Prop 4.1.1** The following are equivalent for  $f : I \rightarrow \mathbb{R}$ ,  $x_0 \in I$ .

- (1)  $f$  is differentiable at  $x_0$ .
- (2)  $\exists \alpha \in \mathbb{R}, \exists \eta : N(0, \delta) \setminus \{0\} \rightarrow \mathbb{R}$  s.t.
  - (a)  $f(x_0 + h) - f(x_0) = \alpha h + |h| \cdot \eta(h)$ .<sup>39</sup>
  - (b)  $\lim_{h \rightarrow 0} \eta(h) = 0$ .

**Proof.** (1  $\implies$  2) Define

$$\eta(h) := \frac{f(x_0 + h) - f(x_0) - f'(x_0)h}{|h|} \quad (h \neq 0)$$

Now check if (b) is satisfied. Then

$$f(x_0 + h) - f(x_0) = f'(x_0)h + |h| \cdot \eta(h)$$

(2  $\implies$  1)

$$\frac{f(x_0 + h) - f(x_0)}{h} = \alpha + \frac{|h|}{h} \eta(h) \rightarrow \alpha = f'(x_0)$$

since  $||h| \eta(h)/h| \rightarrow 0$  as  $h \rightarrow 0$ .

**Example.** Define

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

---

<sup>39</sup> $|h|$  로 정의한 이유는 벡터 함수를 다루기 위함!

$f$  is differentiable at  $x = 0$ .<sup>40</sup>

**Proof.**  $f(h) - f(0) = h^2 \sin \frac{1}{h} - 0 = 0 \cdot h + |h| |h| \sin \frac{1}{h}$ , and set  $\eta(h) = |h| \sin \frac{1}{h}$ .

Note that

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

and  $f'$  is not continuous at 0.

**Definition.** Suppose  $n \in \mathbb{N}$ ,  $f : I \rightarrow \mathbb{R}$ .<sup>41</sup>

$$f \in C^n \iff f \text{ is differentiable } n \text{ times, } f^{(n)} \text{ is continuous on } I$$

**Remark.** Differentiable at  $x = x_0 \implies$  Continuous at  $x = x_0$ .

**Remark.**  $f$  is **nowhere differentiable** if  $f : I \rightarrow \mathbb{R}$  is continuous, and  $f$  is not differentiable at all  $x_0 \in I$ .  $f$  exists, and it describes Brownian motion.

**Prop 4.1.3** Suppose  $f, g : I \rightarrow \mathbb{R}$  are differentiable at  $x_0 \in I$ . Then  $f + g$ ,  $fg$ ,  $f/g$  are also differentiable at  $x_0$ , and

- (1)  $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
- (2)  $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
- (3)  $(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2} \quad (g(x_0) \neq 0)$

**Prop 4.1.4 (Chain Rule)** Suppose  $f : I \rightarrow J$ ,  $g : J \rightarrow \mathbb{R}$ ,  $x_0 \in I$ ,  $y_0 = f(x_0) \in J$ .

$f$  is differentiable at  $x_0$  and  $g$  is differentiable at  $y_0 \implies g \circ f$  is differentiable at  $x_0$ , and

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

**Proof.** By Prop 4.1.1, there exists  $\alpha(h), \beta(h)$  s.t.

$$g(y_0 + h) - g(y_0) = g'(y_0) \cdot h + |h| \alpha(h)$$

$$f(x_0 + h) - f(x_0) = f'(x_0) \cdot h + |h| \beta(h)$$

Then we have

$$\begin{aligned} g(f(x_0 + h)) - g(f(x_0)) &= g(y_0 + [f(x_0 + h) - f(x_0)]) - g(y_0) \\ &= g'(y_0)(f(x_0 + h) - f(x_0)) + |f(x_0 + h) - f(x_0)| \alpha(f(x_0 + h) - f(x_0)) \\ &= g'(f(x_0))(f'(x_0)h + |h| \beta(h)) \\ &\quad + |f(x_0 + h) - f(x_0)| \alpha(f(x_0 + h) - f(x_0)) \end{aligned}$$

<sup>40</sup>미분가능성의 장점을 거의 사용할 수 없는 (쓸데 없는) 함수...

<sup>41</sup> $f^{(n)}$ : 다들 아실테니까 정의 안하고 쓸게요!

Therefore we set

$$\eta(h) = \beta(h)g'(f(x_0)) + \left| \frac{f(x_0 + h) - f(x_0)}{h} \right| \alpha(f(x_0 + h) - f(x_0))$$

and check if  $\eta(h) \rightarrow 0$  as  $h \rightarrow 0$ . Use  $\lim_{h \rightarrow 0} \alpha(h) = \lim_{h \rightarrow 0} \beta(h) = 0$ .

**Remark.**

(1) In  $g(y_0 + h) - g(y_0) = g'(y_0) \cdot h + |h| \alpha(h)$ , 0 was not in the domain of  $\alpha$ . But defining  $\alpha(0) = 0$  will solve the problem.

(2) If  $f : [a, b] \rightarrow \mathbb{R}$  define right and left derivative at  $x = a, b$  as

$$f'(a) = \lim_{h \rightarrow 0^+} \frac{f(a + h) - f(a)}{h} \quad f'(b) = \lim_{h \rightarrow 0^-} \frac{f(b + h) - f(b)}{h}$$

if they exist.



May 15th, 2019

## 4.2 Mean Value Theorem

**Lemma 4.2.1 (Rolle's Theorem)** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f(a) = f(b)$ , there exists  $c \in (a, b)$  s.t.  $f'(c) = 0$ .

**Proof.**

(1) Maximum of  $f$  = Minimum of  $f = f(a) = f(b)$

$f$  is constant. Trivial.

(2) Maximum of  $f$  is not  $f(a), f(b)$

Suppose  $f$  attains maximum at  $x = c \in (a, b)$  Then  $f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$  must be 0. ( $\because f'_+(c) \leq 0$  and  $f'_-(c) \geq 0$ )

(3) Minimum of  $f$  is not  $f(a), f(b)$

(Proof is identical to that of (2))

**Theorem 4.2.2 (Cauchy's Mean Value Theorem)** Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists  $c \in (a, b)$  s.t.

$$(g(a) - g(b))f'(c) = (f(a) - f(b))g'(c)$$

**Proof.** Set  $h(x) = (g(a) - g(b))f(x) - (f(a) - f(b))g(x)$  and apply Rolle's Thm.

**Theorem 4.2.3 (Mean Value Theorem)** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists  $c \in (a, b)$  s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

**Proof.** Set  $g(x) = x$  in Cauchy's MVT.

**Theorem 4.2.5 (L'Hopital's Rule)** Suppose  $f, g : (a, b) \rightarrow \mathbb{R}$  are differentiable on  $(a, b)$ .

For  $x_0 \in (a, b)$ , if  $f(x_0) = g(x_0) = 0$  and  $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = \alpha$ , then  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \alpha$ .

**Proof.** Given  $\epsilon > 0$ , there exists  $\delta > 0$  s.t. if  $|x - x_0| < \delta$  then  $|f'(x)/g'(x) - \alpha| < \epsilon$ .

By Cauchy's MVT, there exists  $c_x$  in between  $x_0$  and  $x$  s.t.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c_x)}{g'(c_x)}$$

If  $|x - x_0| < \delta$ ,

$$\left| \frac{f(x)}{g(x)} - \alpha \right| = \left| \frac{f'(c_x)}{g'(c_x)} - \alpha \right| < \epsilon$$

since  $|c_x - x_0| < |x - x_0| < \delta$ .

### 4.3 Taylor Expansion

Suppose  $I$  is a closed interval, and  $a \in I$ .

**Theorem 4.3.1** Suppose  $f, g : I \rightarrow \mathbb{R} \in C^\infty(I)$ . If  $x \in \text{int}(I)$ , there exists  $c_x$  between  $a$  and  $x$  s.t.

$$\left( f(x) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k \right) g^{(n+1)}(c_x) = \left( g(x) - \sum_{k=0}^n \frac{g^{(k)}(a)}{k!} (x-a)^k \right) f^{(n+1)}(c_x)$$

**Proof.** Fix  $x$ . Define

$$F(t) = \sum_{k=0}^n \frac{f^{(k)}(t)}{k!} (x-t)^k$$

Then<sup>42</sup>

$$F'(t) = \sum_{k=0}^n \frac{f^{(k+1)}(t)}{k!} (x-t)^k + \sum_{k=1}^n \frac{f^{(k)}(t)}{k!} (-1)^k (x-t)^{k-1} = \frac{f^{(n+1)}(t)}{n!} (x-t)^n$$

Similarly define  $G(t)$  and calculate  $G'(t) = g^{(n+1)}(t)/n! \cdot (x-t)^n$ .

By Cauchy's MVT, there exists  $c_x$  between  $a$  and  $x$  s.t.

$$(F(x) - F(a))G'(c_x) = (G(x) - G(a))F'(c_x)$$

which simplifies to

$$(f(x) - F(a))g^{(n+1)}(c_x) \frac{(x-c_x)^n}{n!} = (g(x) - G(a))f^{(n+1)}(c_x) \frac{(x-c_x)^n}{n!}$$

and now the result directly follows.

**Remark.**

(1) Taylor Expansion (around  $a$ )

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

(2) (In the book)  $f, g \in C^n(I)$ , and  $f^{(n)}, g^{(n)}$  should be differentiable on  $\text{int}(I)$ .

(3) **(Taylor's Theorem)** Set  $g(x) = (x-a)^{n+1}$ .  $g^{(0)}(a) = \dots = g^{(n)}(a) = 0$ , but  $g^{(n+1)}(x) = (n+1)!$  (constant). Then we have

$$f(x) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k = f^{(n+1)}(c_x) \frac{(x-a)^{n+1}}{(n+1)!}$$

---

<sup>42</sup>Note the  $k=1$  in the second term.

**Prop 4.3.3** Suppose  $f : I \rightarrow \mathbb{R} \in C^\infty(I)$ .<sup>43</sup> For  $a, x \in I$ , define  $J$  as a interval with  $a, x$  as two endpoints. If there exists  $M > 0$  s.t.  $|f^{(n)}(y)| \leq M$  for  $\forall n \in \mathbb{N}, \forall y \in J$ ,<sup>44</sup> then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

**Proof.** Define

$$S_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

then we want to show that  $\lim_{n \rightarrow \infty} |S_n(x) - f(x)| = 0$ .

By Taylor's Theorem,  $\exists c_x \in J$  s.t.

$$|f(x) - S_n(x)| \leq |f^{(n+1)}(c_x)| \frac{|x-a|^{n+1}}{(n+1)!} \leq M \frac{|x-a|^{n+1}}{(n+1)!} \rightarrow 0$$

The last term converges to 0 since factorials increase faster than exponents.

**Example.**  $f(x) = \sin x$  satisfies the conditions of Prop 4.3.3, and calculating  $f^{(k)}(0)$  gives

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

**Example.**  $f(x) = e^x$ , at  $a = 0$ .  $x \in \mathbb{R}_{\geq 0}$ ,  $\{f^{(n)}(t) : t \in [0, x], n \in \mathbb{N}\}$  is bounded by  $e^x$ . Thus  $f(x) = \sum_{k=0}^{\infty} x^k/k!$  ( $x \geq 0$ )

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<sup>43</sup>Such functions are called **smooth**.

<sup>44</sup>이 조건은 매우 **과한** 조건이다.

May 20th, 2019

**Example.**  $f(x) = \log(1+x)$ ,  $I = [0, \infty) \xrightarrow{?} f(x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k}$

This cannot be done *yet*. (Chap 6)

**Definition.** Suppose  $f : X \rightarrow \mathbb{R}$  ( $X \subset \mathbb{R}^d$ ).

- (1)  $f$  has a **local maximum**  $f(x_0)$  at  $x_0$   
 $\iff$  Exists  $\delta > 0$  s.t.  $f(x_0) \geq f(x)$  for all  $x \in N(x_0, \delta) \cap X$
- (2)  $f$  has a **local minimum**  $f(x_0)$  at  $x_0$   
 $\iff$  Exists  $\delta > 0$  s.t.  $f(x_0) \leq f(x)$  for all  $x \in N(x_0, \delta) \cap X$

**Theorem.** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable and has local maximum (minimum) at  $c \in [a, b]$ .<sup>45</sup>

- (1) If  $c \in (a, b)$  then  $f'(c) = 0$ .
- (2) If  $c = a$ ,  $f'(a) \leq 0$  ( $\geq 0$ )
- (3) If  $c = b$ ,  $f'(b) \geq 0$  ( $\leq 0$ )

**Proof.** (1) : Compare left/right-hand limits. Since they must be the same,  $f'(c) = 0$ .

(2), (3) : Inspect right-hand and left-hand limits, respectively. Right-hand limit should be negative, left-hand limit should be positive.

**Remark.** Maximum (Minimum)  $\implies$  Local Maximum (Minimum)

**Recall.**

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

**Definition.** Suppose  $F : I \rightarrow \mathbb{R}$  is differentiable. If  $F' = f$ ,  $F$  is an **antiderivative** of  $f$ .

**Theorem 4.2.6 (Darboux's Theorem)** Suppose  $F : I \rightarrow \mathbb{R}$  is a differentiable function defined on a closed interval, and let  $F' = f$ . If  $a, b$  are points in  $I$  with  $a < b$  and  $f(a) < \alpha < f(b)$ , then there exists  $c \in (a, b)$  s.t.  $f(c) = \alpha$ .

**Proof.** Define  $G(x) = F(x) - \alpha x$ .  $G(x)$  is continuous and differentiable on  $I$  and has a minimum  $G(c)$ .  $G'(a) = F'(a) - \alpha = f(a) - \alpha < 0$ ,  $G'(b) = F'(b) - \alpha = f(b) - \alpha > 0$ . Since  $c$  is minimum, it must be a local minimum. If  $c = a$ ,  $G'(c) \geq 0$ , if  $c = b$ ,  $G'(c) \leq 0$ . Thus  $c \neq a, b$

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<sup>45</sup>Statements for local minimum in brackets.

and  $c \in (a, b)$ , therefore we have  $G'(c) = f(c) - \alpha = 0$ .

**Cor 4.2.7** Suppose  $F : I \rightarrow \mathbb{R}$  is a differentiable function and  $F' = f$ . For any interval  $J \subset I$ ,  $f(J)$  is also an interval.<sup>46</sup>

**Example.** Does  $f(x) = \begin{cases} x & (x < 0) \\ x + 1 & (x \geq 0) \end{cases}$  have an antiderivative ?

No.  $f([-1, 1]) = [-1, 0) \cup [1, 2]$ , which is not an interval.

---

<sup>46</sup>Intermediate value property 를 이용하여 구간의 상이 **연결집합**임을 보일 수 있었다!

$$\int_a^b f(x)dx$$

We learned about Riemann integrals, when  $f$  was continuous. There are two generalizations.

- Riemann-Stieltjes Integrals  $\int_a^b f(x)dg(x)$
- Lebesgue Integrals:  $\int_a^b f dm$  ( $m$ : measure) (Most general)

미분은 하면 할수록 함수가 안좋아져요, 그런데 적분은 하면 할수록 함수가 좋아져요!

## 5. 적분 가능 함수의 성질

### 5.1 Riemann Integrals <sup>47</sup>

**Definition.**

- (1)  $P$  is a **partition** of  $[a, b]$  if  $P \subset [a, b]$  is a finite subset and  $a, b \in P$ .
- (2)  $\mathcal{P}[a, b]$  is the **collection** of all partitions of  $[a, b]$ .

**Example.** Consider  $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ . Then we divided  $[a, b]$  into  $[x_0, x_1], \dots, [x_{n-1}, x_n]$ .

**Definition.** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is bounded.<sup>48</sup> Given  $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$ , define

$$m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\} \quad M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\}$$

then we define **lower/upper Riemann sums** as<sup>49</sup>

- (1) (Lower)  $L(f, P) = \sum_{i=1}^n (x_i - x_{i-1})m_i$
- (2) (Upper)  $U(f, P) = \sum_{i=1}^n (x_i - x_{i-1})M_i$

**Prop 5.1.1** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is bounded.

- (1)  $P, Q \in \mathcal{P}[a, b]$ , if  $P \subset Q$  ( $Q$  is a finer partition than  $P$ )

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

<sup>47</sup>If we define integration only with Riemann integrals, there aren't so many integrable functions.

<sup>48</sup> $\exists M \geq 0$  s.t.  $|f(x)| \leq M$  for all  $x \in [a, b]$ .

<sup>49</sup>We define it this way so that Riemann integrals can be defined also for non-continuous functions.

$$(2) \quad P, P' \in \mathcal{P}[a, b] \implies L(f, P) \leq U(f, P')$$

**Proof.** (1) : For partition  $P$ , consider an interval  $[x_i, x_{i+1}]$ . This interval adds  $M_{i+1}(x_{i+1} - x_i)$  to the upper sum  $U(f, P)$ . Meanwhile, in partition  $Q$ ,  $[x_i, x_{i+1}]$  can be considered as  $[y_a, y_b]$  for some  $a, b$  and this interval adds  $\sum_{j=a+1}^b M_j^Q(y_j - y_{j-1})$  to the upper sum  $U(f, Q)$ .

$$M_{i+1} = \sup\{f(t) : t \in [x_i, x_{i+1}]\} \quad M_j^Q = \sup\{f(t) : t \in [y_{j-1}, y_j]\}$$

If  $j = a + 1, \dots, b$ ,  $M_j^Q \leq M_{i+1}$ , and thus

$$\sum_{j=a+1}^b M_j^Q(y_j - y_{j-1}) \leq \sum_{j=a+1}^b M_{i+1}(y_j - y_{j-1}) = M_{i+1}(y_b - y_a) = M_{i+1}(x_{i+1} - x_i)$$

$$(2) : L(f, P) \leq L(f, P \cup P') \leq U(f, P \cup P') \leq U(f, P')$$

**Definition.** We define the following.

- Upper Integral  $\overline{\int_a^b} f = \inf\{U(f, P) : P \in \mathcal{P}[a, b]\}$
- Lower Integral  $\underline{\int_a^b} f = \sup\{L(f, P) : P \in \mathcal{P}[a, b]\}$

By Prop 5.1.1 (2),  $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ , and if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

we say that  $f$  is **Riemann integrable**.

May 22nd, 2019

## Review

$f : [a, b] \rightarrow \mathbb{R}$  is bounded.

$P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$

$$m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\} \quad M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\}$$

$$(1) \text{ (Lower) } L(f, P) = \sum_{i=1}^n (x_i - x_{i-1}) m_i$$

$$(2) \text{ (Upper) } U(f, P) = \sum_{i=1}^n (x_i - x_{i-1}) M_i$$

**Prop 5.1.1** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is bounded.

(1)  $P, Q \in \mathcal{P}[a, b]$ , if  $P \subset Q$  ( $Q$  is a finer partition than  $P$ )

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

(2)  $P, P' \in \mathcal{P}[a, b] \implies L(f, P) \leq U(f, P')$

Define

- Upper Integral  $\overline{\int_a^b} f = \inf\{U(f, P) : P \in \mathcal{P}[a, b]\}$
- Lower Integral  $\underline{\int_a^b} f = \sup\{L(f, P) : P \in \mathcal{P}[a, b]\}$

By Prop 5.1.1 (2),  $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ , and if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

we say that  $f$  is **Riemann integrable**.

**Example.**  $f : [0, 1] \rightarrow \mathbb{R}$ ,  $f(x) = \begin{cases} 2 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$

For any partition  $P$ ,  $M_i = 2$ ,  $m_i = 0$  for all  $i$ . Then  $U(f, P) = 2$ ,  $L(f, P) = 0$ , thus not Riemann Integrable.<sup>50</sup>

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<sup>50</sup>리만 적분의 약함을 보여주는 상징적인 예입니다.



**Remark.**  $\int_0^1 f(x)dx$  should be 0. Cardinality of  $\mathbb{R} \setminus \mathbb{Q}$  is larger than  $\mathbb{Q}$ .  $f$  is Lebesgue Integrable and the value is 0.

**Prop 5.1.2** The following are equivalent for bounded  $f : [a, b] \rightarrow \mathbb{R}$ .

- (1)  $f$  is Riemann Integrable.
- (2)  $\forall \epsilon > 0, \exists P \in \mathcal{P}[a, b]$  s.t.  $U(f, P) - L(f, P) < \epsilon$ .

**Proof.** (1  $\implies$  2) Suppose there exists partitions  $P_1, P_2 \in \mathcal{P}[a, b]$  s.t.

$$\overline{\int_a^b} f + \frac{\epsilon}{2} > U(f, P_1) \quad \underline{\int_a^b} f - \frac{\epsilon}{2} < L(f, P_2)$$

Since upper/lower integrals are equal, we have

$$L(f, P_2) \leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_1)$$

and then  $U(f, P_1 \cup P_2) - L(f, P_1 \cup P_2) < \epsilon$ .

(2  $\implies$  1) For all  $\epsilon > 0$ ,

$$\epsilon > U(f, P) - L(f, P) \geq \overline{\int_a^b} f - \underline{\int_a^b} f \geq 0$$

Thus upper/lower integrals must be same, and  $f$  is Riemann Integrable.

**Example.** Riemann Integrable Functions

- (1)  $f$ : Continuous
- (2)  $f$ : Monotone

$$(3) f(x) = \begin{cases} 0 & (0 \leq x < 1, 2 < x \leq 3) \\ 1 & (1 \leq x \leq 2) \end{cases}$$

Consider the partition

$$P = \left\{ 0, 1 - \frac{\epsilon}{5}, 1 + \frac{\epsilon}{5}, 2 - \frac{\epsilon}{5}, 2 + \frac{\epsilon}{5}, 3 \right\}$$

$$\text{Then } U(f, P) - L(f, P) = \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) = \frac{4}{5}\epsilon < \epsilon.$$

**Theorem 5.1.3** Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  is bounded and Riemann Integrable.

- (1)  $f + g$  is Riemann Integrable, and  $\int_a^b (f + g) = \int_a^b f + \int_a^b g$
- (2)  $\alpha \in \mathbb{R}$ ,  $\alpha f$  is Riemann Integrable, and  $\int_a^b \alpha f = \alpha \int_a^b f$

**Proof.**

(1) It is enough to show the following inequality.

$$\int_a^b f + \int_a^b g \leq \int_a^b (f + g) \leq \overline{\int_a^b (f + g)} \leq \overline{\int_a^b f} + \overline{\int_a^b g}$$

(a) For  $P = \{a = x_0 < \cdots < x_n = b\}$ , define the following

$$m_i^f = \inf\{f(t) : t \in [x_{i-1}, x_i]\}$$

$$m_i^g = \inf\{g(t) : t \in [x_{i-1}, x_i]\}$$

$$m_i^{f+g} = \inf\{(f+g)(t) : t \in [x_{i-1}, x_i]\}$$

Then we have<sup>51</sup>

$$m_i^{f+g} \geq m_i^f + m_i^g$$

(b) From the definition of lower Riemann sum, we have<sup>52</sup>

$$L(f+g, P) \geq L(f, P) + L(g, P)$$

(c)  $\forall \epsilon > 0$ , there exists  $P_1, P_2 \in \mathcal{P}[a, b]$  s.t.

$$L(f, P_1) > \int_a^b f - \frac{\epsilon}{2} \quad L(g, P_2) > \int_a^b g - \frac{\epsilon}{2}$$

(d)

$$\begin{aligned} \int_a^b (f+g) &\geq L(f+g, P_1 \cup P_2) \stackrel{(b)}{\geq} L(f, P_1 \cup P_2) + L(g, P_1 \cup P_2) \\ &\geq L(f, P_1) + L(g, P_2) \stackrel{(c)}{\geq} \int_a^b f + \int_a^b g - \epsilon \end{aligned}$$

Take  $\epsilon \rightarrow 0$  to prove the first inequality. (Last inequality can be proved similarly.)

(2) (a)  $\alpha > 0$ , then

$$U(\alpha f, P) = \alpha \cdot U(f, P) \quad L(\alpha f, P) = \alpha \cdot L(f, P)$$

thus

$$\overline{\int_a^b \alpha f} = \alpha \overline{\int_a^b f} \quad \underline{\int_a^b \alpha f} = \alpha \underline{\int_a^b f}$$

(b)  $\alpha < 0$ , then

$$U(\alpha f, P) = \alpha \cdot L(f, P) \quad L(\alpha f, P) = \alpha \cdot U(f, P)$$

thus

$$\overline{\int_a^b \alpha f} = \alpha \underline{\int_a^b f} \quad \underline{\int_a^b \alpha f} = \alpha \overline{\int_a^b f}$$

Thus Riemann Integrable in both cases.

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<sup>51</sup>각각을 최적화 한 것이 합쳐서 최적화 한 것보다 좋다.

<sup>52</sup>sup 을 양변에 취하는 시도는 실패한다.

**Theorem 5.1.4** Suppose  $f : [a, b] \rightarrow I$  is bounded and Riemann Integrable. Then for  $c \in (a, b)$

(1)  $f$  is Riemann Integrable on  $[a, c], [c, b]$ .

$$(2) \int_a^b f = \int_a^c f + \int_c^b f$$

**Proof.**

(1)  $\forall \epsilon > 0, \exists P \in \mathcal{P}[a, b]$  s.t.  $U(f, P) - L(f, P) < \epsilon$ . Suppose the partition is  $P = \{a = x_0 < x_1 < \dots < x_{l-1} \leq c \leq x_l < \dots < x_n = b\}$ . Define a partition  $Q = \{x_0 < x_1 < \dots < x_{l-1} \leq c\}$ . Then we have

$$U(f, Q) - L(f, Q) = \sum_{i=1}^{l-1} (M_i - m_i)(x_i - x_{i-1}) + (M'_l - m'_l)(c - x_{l-1})$$

$$U(f, P) - L(f, P) = \sum_{i=1}^{l-1} (M_i - m_i)(x_i - x_{i-1}) + (M_l - m_l)(x_l - x_{l-1}) + \sum_{i=l+1}^n (M_i - m_i)(x_i - x_{i-1})$$

Thus

$$U(f, Q) - L(f, Q) \leq U(f, P) - L(f, P) < \epsilon$$

and since  $Q \in \mathcal{P}[a, c]$ ,  $f$  is Riemann Integrable on  $[a, c]$  by Prop 5.1.2.

(2) It is enough to show that

$$\overline{\int_a^b f} = \overline{\int_a^c f} + \overline{\int_c^b f} \quad \underline{\int_a^b f} = \underline{\int_a^c f} + \underline{\int_c^b f}$$

We show the first equation.

( $\geq$ )  $\forall \epsilon > 0$ , exists  $Q \in \mathcal{P}[a, c]$ ,  $R \in \mathcal{P}[c, b]$  s.t.

$$\overline{\int_a^c f} + \frac{\epsilon}{2} > U(f, Q) \quad \overline{\int_c^b f} + \frac{\epsilon}{2} > U(f, R)$$

Then we have

$$\overline{\int_a^c f} + \overline{\int_c^b f} + \epsilon > U(f, Q) + U(f, R) = U(f, Q \cup R) \geq \overline{\int_a^b f}$$

( $\leq$ ) Define  $P = \{a = x_0 < x_1 < \dots < x_{l-1} \leq c \leq x_l < \dots < x_n = b\}$ . Define a partition  $Q = \{x_0 < x_1 < \dots < x_{l-1} \leq c\}$ ,  $R = \{c \leq x_l < \dots < x_n = b\}$ .

$\forall \epsilon > 0$ ,

$$\overline{\int_a^c f} + \overline{\int_c^b f} \leq U(f, Q) + U(f, R) = U(f, P \cup \{c\}) \leq U(f, P) \leq \overline{\int_a^b f} + \epsilon$$

(There exists  $P$  s.t. satisfy the last inequality)

**May 27th, 2019**

Currently: We are given bounded  $f : [a, b] \rightarrow \mathbb{R}$ . For  $P \in \mathcal{P}[a, b]$ , we defined  $U(f, P)$  and  $L(f, P)$ . Then we defined  $\overline{\int_a^b f}$  and  $\underline{\int_a^b f}$ , and  $f$  was Riemann Integrable when these two values were the same.

**Theorem 5.1.5** If  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann Integrable, then  $|f|$  is also Riemann Integrable. Also, the following holds.

$$\int_a^b |f| \leq \left| \int_a^b f \right|$$

**Proof.** From  $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$ , and for  $\epsilon > 0$ ,

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) < \epsilon$$

Thus  $|f|$  is integrable, and  $-|f| \leq f \leq |f|$  gives the inequality.

## 5.2 Riemann Integrable Functions

**Theorem 5.2.1** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then  $f$  is Riemann Integrable.

**Proof.** Given  $\epsilon > 0$ , our objective is finding a partition  $P$  s.t.  $U(f, P) - L(f, P) < \epsilon$ .

- (1) Our first observation is that  $f$  is uniformly continuous, since the domain is compact. Thus there exists  $\delta > 0$  s.t.

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}$$

- (2) Now we set a partition as  $P = \{a = x_0 < x_1 < \dots < x_n = b\}$  s.t.  $x_i - x_{i-1} < \delta$  for all  $i$ .
- (3) From EVT, for each closed interval  $[x_{i-1}, x_i]$ , there exists maximum and minimum  $f(u_i), f(v_i)$ . Thus  $M_i = f(u_i)$ ,  $m_i = f(v_i)$ .

- (4) Now we have

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^n (f(u_i) - f(v_i))(x_i - x_{i-1}) \\ &< \sum_{i=1}^n \frac{\epsilon}{b - a} (x_i - x_{i-1}) = \epsilon \end{aligned}$$

**Theorem 5.2.2** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is monotone. Then  $f$  is Riemann Integrable.

**Proof.** WLOG, suppose  $f$  is increasing.

Given  $\epsilon > 0$ , we want to find a partition  $P$ . Take  $n \in \mathbb{N}$  s.t.

$$n > \frac{(b - a)(f(b) - f(a))}{\epsilon}$$

Consider a partition as

$$x_i = a + \frac{b-a}{n}i \implies P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

Now

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \frac{b-a}{n} \\ &= \frac{b-a}{n} (f(x_n) - f(x_0)) = \frac{(b-a)(f(b) - f(a))}{n} < \epsilon \end{aligned}$$

**Definition.** For  $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$ , define the **norm** of  $P$  as<sup>53</sup>

$$\|P\| = \max_{1 \leq i \leq n} \{x_i - x_{i-1}\}$$

And we say that  $P$  is finer than  $Q$  if  $\|P\| \leq \|Q\|$ . Also, if  $P \subset Q$ ,  $\|Q\| \leq \|P\|$ .

**Definition. Riemann Sum**  $R(f, P)$  is defined as

$$R(f, P) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) \quad (t_i \in [x_{i-1}, x_i])$$

**Remark.**

$$(1) \quad R(f, P) = R(f, P, t_1, t_2, \dots, t_n)$$

$$(2)$$

$$U(f, P) = \sup_{t_1, \dots, t_n} R(f, P) \quad L(f, P) = \inf_{t_1, \dots, t_n} R(f, P)$$

$$(3)$$

$$L(f, P) \leq R(f, P) \leq U(f, P)$$

**Theorem 5.2.3** Characterization of Riemann Integral via Riemann sums.

The following are equivalent for bounded  $f : [a, b] \rightarrow \mathbb{R}$ .

$$(1) \quad f \text{ is Riemann Integrable and } \int_a^b f = A.$$

$$(2) \quad \forall \epsilon > 0, \exists \delta > 0 \text{ s.t.}$$

$$\|P\| < \delta \implies |R(f, P) - A| < \epsilon \quad (\forall t_1, \dots, t_n)$$

This is also written as  $\lim_{\|P\| \rightarrow 0} R(f, P) = A$ .

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<sup>53</sup>기준에 알고있던 norm 의 성질을 만족하지는 않는다. 좋은 이름은 아니다.

(3)  $\forall \epsilon > 0, \exists P_0 \in \mathcal{P}[a, b]$  s.t.

$$P \supset P_0 \implies |R(f, P) - A| < \epsilon$$

**Proof. (1  $\implies$  2)**

**Claim.**

$$(i) \quad \exists \delta_1 > 0 \text{ s.t. } \|P\| < \delta_1 \implies U(f, P) < A + \epsilon$$

$$(ii) \quad \exists \delta_2 > 0 \text{ s.t. } \|P\| < \delta_2 \implies L(f, P) > A - \epsilon$$

Setting  $\delta = \min\{\delta_1, \delta_2\}$  will prove (2) since

$$A - \epsilon < L(f, P) \leq R(f, P) \leq U(f, P) < A + \epsilon$$

**Proof of (i).** ((ii) is similar)

(1)  $f > 0$

$\exists P_0 \in \mathcal{P}[a, b]$  s.t.  $U(f, P_0) < A + \epsilon/2$  (By Riemann Integrability of  $f$ )

Set  $P_0 = \{a = x_0 < x_1 < \dots < x_n = b\}$ ,  $M$  as the upper bound of  $f$ . Now set

$$\delta_1 = \frac{\epsilon}{2Mn}$$

Now  $P = \{a = y_0 < y_1 < \dots < y_m = b\}$ , with  $\|P\| < \delta_1$ . Define

$$I = \{i : x_j \in (y_{i-1}, y_i) \text{ for some } j\} \quad J = \{i : [y_{i-1}, y_i] \subset [x_{j-1}, x_j] \text{ for some } j\}$$

Then

$$U(f, P) = \sum_{i \in I} \overbrace{M_i(y_i - y_{i-1})}^{\leq M \cdot \delta_1 \cdot n} + \sum_{i \in J} \overbrace{M_i(y_i - y_{i-1})}^{\leq U(f, P_0)} \leq U(f, P_0) + \delta_1 \cdot nM < A + \epsilon$$

(2) For general  $f$ : Set  $g = f + c$  where  $c$  is a positive constant large enough that  $g > 0$ .

Then  $\exists \delta_1$  s.t.

$$\|P\| < \delta_1 \implies U(g, P) < \int_a^b g + \epsilon \quad (*)$$

Note that

$$U(g, P) = \sum_{i=1}^n M_i^g(x_i - x_{i-1}) = \sum_{i=1}^n (M_i^f + c)(x_i - x_{i-1}) = U(f, P) + c(b - a)$$

Also

$$\int_a^b g = \int_a^b (f + c) = \int_a^b f + \int_a^b c = A + c(b - a)$$

Thus inequality (\*) is equivalent to

$$U(f, P) + c(b - a) < A + c(b - a) + \epsilon$$

and canceling  $c(b - a)$  gives the desired inequality.

**(2  $\implies$  3)** Let  $P_0$  be any partition s.t.  $\|P_0\| < \delta$ . If  $P_0 \subset P$ ,  $\|P\| \leq \|P_0\| < \delta$ . Therefore we have  $|R(f, P) - A| < \epsilon$ .

**(3  $\implies$  1)**  $\forall \epsilon > 0$ ,  $\exists P_0$  s.t.  $P_0 \subset P$  s.t.  $|R(f, P) - A| < \epsilon/3$ . Then

$$A - \frac{\epsilon}{3} < R(f, P) < A + \frac{\epsilon}{3}$$

Taking  $\inf_{t_1, \dots, t_n}$  and  $\sup_{t_1, \dots, t_n}$  on left/right inequalities respectively gives

$$U(f, P) \leq A + \frac{\epsilon}{3} \quad L(f, P) \geq A - \frac{\epsilon}{3}$$

Therefore

$$U(f, P) - L(f, P) \leq \frac{2\epsilon}{3} < \epsilon$$

and  $f$  is Riemann Integrable. Also,

$$A - \frac{\epsilon}{3} \leq L(f, P) \leq U(f, P) \leq A + \frac{\epsilon}{3}$$

We can infer that

$$A - \frac{\epsilon}{3} \leq \int_a^b f = \int_a^b f = \overline{\int_a^b f} \leq A + \frac{\epsilon}{3}$$

and taking  $\epsilon \rightarrow 0$  gives  $\int_a^b f = A$ .

**May 29th, 2019**

**Theorem 5.3.1 + 5.3.3 (Fundamental Theorem of Calculus)** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and Riemann Integrable.

- (1) Suppose  $F(x) = \int_a^x f(t)dt$ , and  $f$  is continuous at  $x_0$ . Then  $F$  is differentiable at  $x_0$  and  $F'(x_0) = f(x_0)$ .
- (2) If  $F' = f$  on  $[a, b]$ ,  $\int_a^b f(t)dt = F(b) - F(a)$ .

**Remark.**

- (1) (For 1) If  $f$  is continuous on  $[a, b]$ ,  $F' = f$  on  $[a, b]$ , and thus continuous functions have an antiderivative.
- (2) Consider  $f(x) = \begin{cases} 0 & (0 \leq x < 1) \\ 1 & (1 \leq x \leq 2) \end{cases}$  then  $F$  is not differentiable at  $x = 1$ .
- (3) (For 1)  $F$  is Lipschitz continuous.  
 $\because |f(x)| \leq M$ . For  $x > y$ ,

$$|F(x) - F(y)| = \left| \int_a^x f - \int_a^y f \right| = \left| \int_y^x f \right| \leq \int_y^x |f| \leq M(x - y)$$

**Proof.**

- (1)  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$ .

If  $x > x_0$ , we want to show that  $\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \rightarrow 0$ .

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &= \left| \frac{1}{x - x_0} \int_{x_0}^x f(t)dt - \frac{1}{x - x_0} \int_{x_0}^x f(x_0)dt \right| \\ &= \frac{1}{|x - x_0|} \left| \int_{x_0}^x (f(t) - f(x_0))dt \right| \\ &\leq \frac{1}{|x - x_0|} \int_{x_0}^x |f(t) - f(x_0)| dt \\ &< \frac{1}{|x - x_0|} \int_{x_0}^x \epsilon dt = \epsilon \quad (\because |t - x_0| < \delta \implies |f(t) - f(x_0)| < \epsilon) \end{aligned}$$

Therefore the right derivative of  $F$  at  $x_0$  is  $f(x_0)$ . The proof is similar for the left derivative.

- (2) Take any  $P = \{a = x_0 < x_1 < \dots < x_n = b\} \in \mathcal{P}[a, b]$ .

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) = \sum_{k=1}^n (F(x_k) - F(x_{k-1})) \\ &\stackrel{\text{MVT}}{=} \sum_{k=1}^n (x_k - x_{k-1}) f(t_k) \quad (\exists t_k \in (x_{k-1}, x_k)) \\ &= R(f, P) \end{aligned}$$

Now since  $f$  is Riemann Integrable,  $\int_a^b f(t)dt = F(b) - F(a)$



**Cor 5.3.2 (Mean Value Theorem for Integrals)** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then there exists  $c \in (a, b)$  such that

$$\frac{1}{b-a} \int_a^b f(t) dt = f(c)$$

**Proof.** Consider  $F(x) = \int_a^x f(t) dt$ .  $F$  is differentiable and apply MVT.

**Prop 5.3.4 (Substitution Rule)** Suppose  $g : [a, b] \rightarrow [c, d]$  is a  $C^1$ -function and  $f : [c, d] \rightarrow \mathbb{R}$  is continuous. Then

$$\int_{g(a)}^{g(b)} f(x) dx = \int_a^b f(g(t)) g'(t) dt$$

**Proof.**  $H(y) = \int_{g(a)}^y f(t) dt$ . Then  $H$  is differentiable and  $H' = f$ . Set

$$F_1(x) = \int_{g(a)}^{g(x)} f(t) dt = H(g(x)) \quad F_2(x) = \int_a^x f(g(t)) g'(t) dt$$

Then  $F_1'(x) = H'(g(x)) g'(x) = f(g(x)) g'(x) = F_2'(x)$ . Thus  $F_1(x) - F_2(x) = c$  (constant), and evaluating this at  $x = 0$  gives  $c = 0$ .

**Prop 5.3.5 (Integration by Parts)** Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  are  $C^1$ -functions. Then<sup>54</sup>

$$\int_a^b f(x) g'(x) dx = [f(x) g(x)]_a^b - \int_a^b f'(x) g(x) dx$$

**Proof.** Use  $(fg)' = fg' + f'g$ .

## 5.4 Function of Bounded Variation (BV function)

Given  $\alpha : [a, b] \rightarrow \mathbb{R}$ ,

$$\sum_{i=1}^n f(t_i) (\alpha(x_i) - \alpha(x_{i-1})) \xrightarrow{\|P\| \rightarrow 0} \int_a^b f d\alpha$$

If this limit exists,  $f$  is Stieltjes Integrable w.r.t  $\alpha$ . Here,  $\alpha$  must be at least of bounded variation.

**Definition.** For  $f : [a, b] \rightarrow \mathbb{R}$ ,  $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$ . Define

$$V(f, P) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$$

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<sup>54</sup> 모든 미분가능한 함수  $f$  에 대해 부분적분 식을 만족하면  $g'$  을  $g$  의 도함수로 정의하기도 한다. ‘미분 가능’의 범위를 넓히는 개념. 극한으로 정의하면 넓힐 방법이 없다...

and the **total variation** of  $f$  over  $[a, b]$  by

$$V_a^b(f) = \sup \{V(f, P) : P \in \mathcal{P}[a, b]\}$$

And  $f$  is said to be of **bounded variation** if the total variation is finite.  $V_a^b(f) < \infty$ .

**Example.**  $f(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$  is not BV. Consider

$$P_n = \left\{ 0 = x_0 < \frac{2}{(2n+1)\pi} < \frac{2}{(2n-1)\pi} < \cdots < \frac{2}{3\pi} < \frac{2}{\pi} < 1 \right\}$$

Then  $f\left(\frac{2}{(2k+1)\pi}\right) = \frac{2}{(2k+1)\pi}(-1)^k$  and

$$\left| f\left(\frac{2}{(2k+1)\pi}\right) - f\left(\frac{2}{(2k-1)\pi}\right) \right| = \frac{2}{(2k+1)\pi} + \frac{2}{(2k-1)\pi} > \frac{2}{(2k-1)\pi}$$

Then the total variation diverges.

$$V(f, P_n) > \frac{2}{(2n+1)\pi} + \frac{2}{(2n-1)\pi} + \cdots + \frac{2}{\pi} = \frac{2}{\pi} \left( 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} \right) \rightarrow \infty$$

.

**Example.**  $f : [a, b] \rightarrow \mathbb{R}$ .

(1)  $f$ : monotone  $\implies f$  is of bounded variation.

**Proof.** WLOG suppose  $f$  is increasing. Then  $V(f, P) = f(b) - f(a)$ .

(2)  $f$ : Lipschitz continuous  $\implies f$  is of bounded variation.

**Proof.**  $\exists M$  s.t.  $|f(x) - f(y)| \leq M|x - y|$ . Then  $V(f, P) \leq M(b - a)$ .

(3)  $f \in C^1$ ,  $f'$  is bounded  $\implies f$ : Lipschitz continuous  $\implies f$ : Bounded variation.

(4)  $f$ : continuous does not imply that  $f$  is of bounded variation. (Counterexample above)

**Lemma.** If  $f : [a, b] \rightarrow \mathbb{R}$  is of bounded variation,  $f$  is bounded.

**Proof.** Let  $x \in [a, b]$ .  $P = \{a, x, b\}$ .

$$|f(x)| \leq |f(a)| + |f(x) - f(a)| \leq |f(a)| + |f(x) - f(a)| + |f(b) - f(x)| = |f(a)| + V(f, P)$$

May 31st, 2019

**Theorem 5.3.1** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and Riemann integrable.

$$F(x) = \int_a^x f(t)dt \quad (a \leq x \leq b)$$

is uniformly continuous. If  $f$  is continuous then  $F$  is differentiable.

**Problem 5.3.1**  $F$ : differentiable does not imply that  $f$  is continuous.

**Problem 5.3.2**  $f(x) = \int_{x^2}^x \sqrt{1+t^2}dt \implies f'(x) = \sqrt{1+x^2} - 2x\sqrt{1+x^4}$

**Problem 5.6.2** If  $f, g$  are integrable,  $\max\{f, g\}, \min\{f, g\}$  are also integrable.

**Proof.** Use  $\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$ .

**Problem 5.6.3** If  $f, g$  are integrable,  $fg$  is integrable.

**Proof.**

- (1)  $0 \leq \sup\{|f(x)| : a \leq x \leq b\} = M < \infty$ . For given  $\epsilon > 0$ ,  $\exists P = \{a = x_0 < x_1 < \dots < x_n = b\}$  s.t.

$$\sum_{i=1}^n (x_i - x_{i-1})(M_i - m_i) < \frac{\epsilon}{2M+1}$$

Since

$$|f(x)^2 - f(y)^2| \leq |f(x) - f(y)| (|f(x)| + |f(y)|) \leq 2M |f(x) - f(y)|$$

, let  $\widetilde{M}_i, \widetilde{m}_i$  be supremum and infimum of  $f^2$  in  $[x_{i-1}, x_i]$ . Then

$$\widetilde{M}_i - \widetilde{m}_i \leq 2M(M_i - m_i)$$

Thus

$$\sum_{i=1}^n (x_i - x_{i-1})(\widetilde{M}_i - \widetilde{m}_i) \leq \sum_{i=1}^n (x_i - x_{i-1})2M(M_i - m_i) \leq 2M \cdot \frac{\epsilon}{2M+1} < \epsilon$$

and  $f^2$  is integrable.

- (2) Now write  $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$  to observe that  $fg$  is integrable.

**Problem 5.6.4**  $f : [a, b] \rightarrow \mathbb{R}$  is integrable. Prove that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{b-a}{n} f\left(a + \frac{b-a}{n}k\right) = \int_a^b f(x)dx$$

**Proof.**  $f$ : integrable.  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $\|P\| < \delta \implies \left| R(f, P) - \int_a^b f \right| < \epsilon$ .

Take  $N$  so that  $\frac{b-a}{N} < \delta$ . Then for  $n \geq N$ ,

$$\left| \sum_{k=1}^n \frac{b-a}{n} f\left(a + \frac{b-a}{n}k\right) - \int_a^b f \right| < \epsilon$$

Converse: False.  $f : [0, 1] \rightarrow \mathbb{R}$ .  $f(x) = 1$  if  $x \in \mathbb{Q}$ , 0 otherwise.  $f$  is not integrable, but the Riemann sum above equals 1.

### Problem

$$(1) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n}{n^2 + k^2} = \int_0^1 \frac{1}{1+t^2} dt = \frac{\pi}{4}$$

$$(2) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{k^2 + n^2}} = \int_0^1 \frac{1}{\sqrt{1+t^2}} dt = \sinh^{-1}(1)$$

**Problem 5.6.5**  $f : [0, 1] \rightarrow \mathbb{R}$ , continuous and  $f \geq 0$ . If  $\int_0^1 f(x) dx = 0$ , show that  $f \equiv 0$ .

**Proof.** (Contradiction) Suppose  $\exists a \in [0, 1]$  s.t.  $f(a) > 0$ .

For  $a \in (0, 1)$ ,  $\exists \delta > 0$  s.t.  $[a - \delta, a + \delta] \subset [0, 1]$  and  $|f(x) - f(a)| < \frac{f(a)}{2}$  if  $x \in [a - \delta, a + \delta]$ .

$$0 = \int_0^1 f = \int_0^{a-\delta} f + \int_{a-\delta}^{a+\delta} f + \int_{a+\delta}^1 f \geq 0 + \int_{a-\delta}^{a+\delta} \frac{f(a)}{2} + 0 = 2\delta \cdot \frac{f(a)}{2} > 0$$

**Problem 5.6.9**  $f : \mathbb{R} \rightarrow \mathbb{R}$ , continuous and bounded. If for all  $[a, b]$ ,  $\int_a^b f = 0$  then  $f \equiv 0$ .

**Proof.** Similar to 5.6.5. (Contradiction) WLOG  $f(a) > 0$  ...

**Problem 5.6.6**  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Show that

$$\lim_{n \rightarrow \infty} \left( \int_a^b |f(x)|^n dx \right)^{1/n} = \max\{|f(x)| : x \in [a, b]\}$$

**Proof.** WLOG  $f \geq 0$ . Let  $M = \max\{|f(x)| : x \in [a, b]\}$

$$(\leq) \text{ For all } n, \left( \int_a^b |f(x)|^n dx \right)^{1/n} \leq (M^n(b-a))^{1/n} = M(b-a)^{1/n}.$$

Take lim sup on both sides to get (LHS)  $\leq M$ .

$$(\geq) \exists c \in [a, b] \text{ s.t. } f(c) = M.$$

$\forall \epsilon > 0$ , we want to show that

$$\liminf_{n \rightarrow \infty} \left( \int_a^b |f(x)|^n dx \right)^{1/n} \geq M - \epsilon$$

$$\exists \delta > 0 \text{ s.t. } [c - \delta, c + \delta] \subset [a, b], \text{ and } x \in [c - \delta, c + \delta] \implies M - \epsilon \leq |f(x)| \leq M.$$

$$\left( \int_a^b |f(x)|^n dx \right)^{1/n} \geq \left( \int_{c-\delta}^{c+\delta} |f(x)|^n dx \right)^{1/n} \geq (2\delta)^{1/n} (M - \epsilon)$$

Take lim inf on both sides to show the desired inequality.

**June 3rd, 2019**

$f : [a, b] \rightarrow \mathbb{R}$ ,  $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \mathcal{P}[a, b]$

$$V(f, P) = V_a^b(f, P) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$$

If  $\{V(f, P) : P \in \mathcal{P}[a, b]\}$  is bounded above,  $f$  is a function of bounded variation. And we write the total variation of  $f$  over  $[a, b]$  as  $V(f) = V_a^b(f) = \sup\{V(f, P) : P \in \mathcal{P}[a, b]\}$

**Remark.**

(1) For two partitions  $P, Q$  s.t.  $P \subset Q$ , then  $V(f, P) \leq V(f, Q)$ .

$$(2) \quad f(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases} \text{ is not BV.}$$

(3)  $f \in C^1[a, b] \implies f$ : differentiable,  $f'$ : bounded  $\implies f$ : Lipschitz continuous  $\implies f$ : BV

(4)  $f$ : BV  $\implies f$ : bounded.

**Prop 5.4.1** Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  is BV. Also,  $\exists M_f, M_g$  s.t.  $|f| \leq M_f, |g| \leq M_g$ .<sup>55</sup>

(1)  $f + g$  is BV,  $V(f + g) \leq V(f) + V(g)$ .

(2)  $fg$  is BV,  $V(fg) \leq M_f \cdot V(g) + M_g \cdot V(f)$ .

(3)  $\alpha f$  is BV,  $V(\alpha f) = |\alpha| V(f)$ .

**Proof.**

(1) We know that  $f + g$  is BV by

$$\begin{aligned} V(f + g, P) &= \sum_{i=1}^n |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\ &\leq V(f, P) + V(g, P) \leq V(f) + V(g) \end{aligned}$$

and taking sup over all  $P \in \mathcal{P}[a, b]$  gives  $V(f + g) \leq V(f) + V(g)$ .

(2) Sum the following inequality from  $i = 1$  to  $n$ .

$$\begin{aligned} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| &= |f(x_i)(g(x_{i-1}) - g(x_{i-1})) + g(x_{i-1})(f(x_i) - f(x_{i-1}))| \\ &\leq M_f |g(x_i) - g(x_{i-1})| + M_g |f(x_i) - f(x_{i-1})| \end{aligned}$$

Thus  $V(fg, P) \leq M_f \cdot V(g, P) + M_g \cdot V(f, P) \leq M_f \cdot V(g) + M_g \cdot V(f)$  and  $fg$  is BV.

Taking sup over all  $P \in \mathcal{P}[a, b]$  gives  $V(fg) \leq M_f \cdot V(g) + M_g \cdot V(f)$ .

(3) Exercise.

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<sup>55</sup>Now we see that any linear combination of BV functions are BV.

**Prop 5.4.2** Suppose  $f : [a, b] \rightarrow \mathbb{R}$ ,  $c \in (a, b)$ . The following are equivalent.

- (1)  $f$  is of bounded variation on  $[a, b]$ .
- (2)  $f$  is of bounded variation on  $[a, c]$  and  $[c, b]$ .

Moreover, if (1), (2) both hold, then

$$V_a^b(f) = V_a^c(f) + V_c^b(f)$$

**Proof.**

- Show that (1)  $\implies$  [(2),  $V_a^c(f) + V_c^b(f) \leq V_a^b(f)$ ]

For  $Q \in \mathcal{P}[a, c]$ ,  $R \in \mathcal{P}[c, b]$  define  $P = Q \cup R \in \mathcal{P}[a, b]$ . By definition and (1),

$$V_a^c(f, Q) + V_c^b(f, R) = V_a^b(f, P) \leq V_a^b(f)$$

Since  $V(\cdot)$  is positive, (2) holds by

$$V_a^c(f, Q) \leq V_a^b(f) \quad V_c^b(f, R) \leq V_a^b(f)$$

and taking sup over partitions of  $[a, c]$ ,  $[c, b]$  will give the desired inequality.

- Show that (2)  $\implies$  [(1),  $V_a^c(f) + V_c^b(f) \geq V_a^b(f)$ ]

Let  $P = \{a = x_0 < x_1 < \dots < x_n = b\} \in \mathcal{P}[a, b]$ . set  $c \in [x_{l-1}, x_l]$ . Define

$$Q = \{a = x_0 < x_1 < \dots < x_{l-1} \leq c\} \in \mathcal{P}[a, c] \quad R = \{c \leq x_l < \dots < x_n = b\} \in \mathcal{P}[c, b]$$

Then

$$\begin{aligned} & V_a^c(f, Q) + V_c^b(f, R) \\ &= \sum_{i=1}^{l-1} |f(x_{i-1}) - f(x_i)| + |f(x_{l-1}) - f(c)| + |f(c) - f(x_l)| + \sum_{i=l+1}^n |f(x_{i-1}) - f(x_i)| \\ &\geq \sum_{1 \leq i \leq n, i \neq l} |f(x_{i-1}) - f(x_i)| + |f(x_{l-1}) - f(x_l)| \geq \sum_{i=1}^n |f(x_{i-1}) - f(x_i)| = V_a^b(f, P) \end{aligned}$$

$$V_a^b(f, P) \leq V_a^c(f, Q) + V_c^b(f, R) \leq V_a^c(f) + V_c^b(f)$$

Thus  $f$  is BV on  $[a, b]$  and  $V_a^b(f) \leq V_a^c(f) + V_c^b(f)$ .

**Theorem 5.4.2** The following are equivalent for  $f : [a, b] \rightarrow \mathbb{R}$ .

- (1)  $f$  is of bounded variation.
- (2) There exists monotonically increasing functions  $g, h : [a, b] \rightarrow \mathbb{R}$  s.t.  $f = g - h$ .

**Proof.** (2  $\implies$  1) Monotonic  $\implies$  BV. Thus  $g - f$  is BV.

(1  $\implies$  2) Consider  $g(x) = V_a^x(f)$  and  $h(x) = g(x) - f(x)$ . Then  $g$  is obviously monotonically increasing and  $f = g - h$ . Now we show that  $h$  is monotonically increasing.

$$\begin{aligned} h(y) - h(x) &= g(y) - g(x) - [f(y) - f(x)] = V_x^y(f) - [(f(y) - f(x))] \\ &\geq V_x^y(f, P) - [f(y) - f(x)] \geq |f(y) - f(x)| - [f(y) - f(x)] \geq 0 \end{aligned}$$

**Remark.**

- (1) In (2),  $g, h$  are not unique, and setting  $G(x) = g(x) + x$ ,  $H(x) = h(x) + x$  gives strictly increasing functions that satisfy  $f = G - H$ .
- (2) However,  $f = \widehat{g} - \widehat{h}$  and if  $\widehat{g}, \widehat{h}$  are monotonically increasing,  $\widehat{g}(a) = 0$ .  
Then  $\widehat{g}(x) \geq V_a^x(f)$  for all  $x \in [a, b]$ .

**Why is BV important?** 1. Length of Curve. 2. Stieltjes Integral.

**Definition. (Length of Curve)** For curve  $\alpha : [a, b] \rightarrow \mathbb{R}^m$ . For any partition  $P = \{a = x_0 < x_1 < \dots < x_n = b\} \in \mathcal{P}[a, b]$ , define

$$\Lambda(\alpha, P) = \sum_{i=1}^n \|\alpha(x_i) - \alpha(x_{i-1})\|$$

. If  $\{\Lambda(\alpha, P) : P \in \mathcal{P}[a, b]\}$  is bounded above, we define the supremum of this set as the **length of curve**  $\alpha$  and denote it as  $\Lambda(\alpha)$ .

**Theorem 5.4.4 + 5.4.5** Suppose  $\alpha : [a, b] \rightarrow \mathbb{R}^m$ ,  $\alpha(t) = (\alpha_1(t), \dots, \alpha_m(t))$ .

- (1)  $\Lambda(\alpha) < \infty \iff \alpha_i$  is BV for all  $i$ .
- (2) For all  $i$ , if  $\alpha_i \in C^1([a, b]) \implies \Lambda(\alpha) = \int_a^b \sqrt{\alpha_1'(t)^2 + \dots + \alpha_m'(t)^2} dt$

**Proof.**

- (1) We use that fact that

$$V(\alpha_i, P) \leq \Lambda(\alpha, P) = \sum_{i=1}^n \|\alpha(x_i) - \alpha(x_{i-1})\| \leq \sum_{j=1}^m \sum_{i=1}^n |\alpha_j(x_i) - \alpha_j(x_{i-1})| = \sum_{j=1}^m V(\alpha_j, P)$$

Thus if  $\Lambda(\alpha) < \infty$ ,  $V(\alpha_i, P) \leq \Lambda(\alpha)$  and  $\alpha_i$  is BV.

Also, if  $\alpha_i$  are BV,  $\Lambda(\alpha, P)$  is upper bounded by  $V(\alpha_j, P) \leq V(\alpha_j)$ . Thus  $\Lambda(\alpha)$  is finite.

- (2) Apply MVT for each component of  $\alpha(x_i) - \alpha(x_{i-1})$ .

$$\Lambda(\alpha, P) = \sum_{i=1}^n \|\alpha(x_i) - \alpha(x_{i-1})\| = \sum_{i=1}^n (x_i - x_{i-1}) \sqrt{\sum_{j=1}^m \alpha_j'(s_j)^2}$$

where  $s_j \in (x_{i-1}, x_i)$  for each  $j$ . Use uniform continuity to bound... (omitted here)

**June 5th, 2019**

### 5.5 Stieltjes Integral

$f, \alpha : [a, b] \rightarrow \mathbb{R}$ , we want to define  $\int f d\alpha$ . We define this for cases where  $\alpha$  is monotonically increasing, and of bounded variation.

$\alpha$ : Monotonically Increasing Case

**Definition.** Given bounded function  $f : [a, b] \rightarrow \mathbb{R}$ , monotonically increasing  $\alpha : [a, b] \rightarrow \mathbb{R}$ , and a partition  $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ , define

$$U(f, P, \alpha) = \sum_{i=1}^n M_i(\alpha(x_i) - \alpha(x_{i-1})) \quad L(f, P, \alpha) = \sum_{i=1}^n m_i(\alpha(x_i) - \alpha(x_{i-1}))$$

Also, Prop 5.1.1 holds.<sup>56</sup>

(1) For  $P, Q \in \mathcal{P}[a, b]$ , if  $P \subset Q$ ,

$$U(f, P, \alpha) \geq U(f, Q, \alpha) \geq L(f, Q, \alpha) \geq L(f, P, \alpha)$$

(2)  $P, Q \in \mathcal{P}[a, b] \implies U(f, P, \alpha) \geq L(f, Q, \alpha)$ .

**Proof.** (1): For  $t \in [x_{i-1}, x_i]$ , define  $X = [x_{i-1}, t]$  and  $Y = [t, x_i]$ . We only need to check

$$M_i(\alpha(x_i) - \alpha(x_{i-1})) \geq M_i^X(\alpha(t) - \alpha(x_{i-1})) + M_i^Y(\alpha(x_i) - \alpha(t))$$

This inequality holds because  $\alpha$  is monotonically increasing.

We can also define

$$\overline{\int_a^b} f d\alpha = \inf \{U(f, P, \alpha) : P \in \mathcal{P}[a, b]\} \quad \underline{\int_a^b} f d\alpha = \sup \{L(f, P, \alpha) : P \in \mathcal{P}[a, b]\}$$

and if these two values are the same,  $f$  is **Stieltjes Integrable** w.r.t.  $\alpha$ . We write  $f \in \mathcal{R}(\alpha)$ , and

$$\int_a^b f d\alpha = \int_a^b f(x) d\alpha(x) = \overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$$

**Theorem 5.5.1** Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  is bounded and given monotonically increasing  $\alpha, \beta : [a, b] \rightarrow \mathbb{R}$ ,  $c \in \mathbb{R}$ .

(1) If  $f, g \in \mathcal{R}(\alpha)$ ,  $f + g, cf \in \mathcal{R}(\alpha)$  and

$$\int_a^b (f + g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha \quad \int_a^b cf d\alpha = c \int_a^b f d\alpha$$

---

<sup>56</sup>Setting  $\alpha(x) = x$  will give the definition of Riemann Integrals.



(2)  $a < p < b$ . If  $f \in \mathcal{R}(\alpha)$  on  $[a, b] \iff f \in \mathcal{R}(\alpha)$  on  $[a, p]$  and  $[p, b]$ , and

$$\int_a^b f d\alpha = \int_a^p f d\alpha + \int_p^b f d\alpha$$

(3) If  $f \in \mathcal{R}(\alpha), \mathcal{R}(\beta)$ , then  $f \in \mathcal{R}(\alpha + \beta), \mathcal{R}(c\alpha)$  for  $c \geq 0$ . And

$$\int_a^b f d(\alpha + \beta) = \int_a^b f d\alpha + \int_a^b f d\beta \quad \int_a^b f d(c\alpha) = c \int_a^b f d\alpha$$

**Theorem 5.5.2 (1/2)** The following are equivalent for bounded  $f$  and monotonically increasing  $\alpha$ .

(1)  $f \in \mathcal{R}(\alpha)$

(2)  $\forall \epsilon > 0, \exists P \in \mathcal{P}[a, b]$  s.t.  $U(f, P, \alpha) - L(f, P, \alpha) < \epsilon$

Note that the above theorem could only be used for testing integrability. So to calculate the value of the integral, we define a Stieltjes Sum by

$$S(f, P, \alpha) = \sum_{i=1}^n f(t_i)(\alpha(x_i) - \alpha(x_{i-1})) \quad (t_i \in [x_{i-1}, x_i])$$

Also note that on the proof of 5.2.3 (2)  $\alpha(x) = x$  was heavily used.

**Theorem 5.5.2 (2/2)**

(1)  $\int_a^b f d\alpha = A$

(2)  $\forall \epsilon > 0, \exists P_0 \in \mathcal{P}[a, b]$  s.t.  $P_0 \subset P \implies |S(f, P, \alpha) - A| < \epsilon$

Why is Stieltjes integral important?

(1) Intermediate object between Riemann Integral and Lebesgue Integral.

(2) Ex.  $E(f(X)) = \int_a^b f dF$  ( $F$ : cumulative distribution function)

**Remark.** 5.5.2 (2/2)  $\implies \alpha : C^1$  then  $\int_a^b f d\alpha = \int_a^b f(x) \alpha'(x) dx$

**Example.**  $\alpha(x) = \begin{cases} 0 & (x \leq 0) \\ 1 & (x > 0) \end{cases}$ . Calculate  $\int_{-1}^1 f d\alpha$  for bounded  $f : [-1, 1] \rightarrow \mathbb{R}$ ,  $\lim_{x \rightarrow 0^+} f(x) = f(0)$ .

$\forall \epsilon > 0, \exists \delta > 0$  s.t.  $x \in [0, \delta] \implies |f(x) - f(0)| < \epsilon$  since  $\lim_{x \rightarrow 0^+} f(x) = f(0)$ .

Set  $P = \{-1, 0, \delta, 1\}$ . Check that

$$U(f, P, \alpha) = M_2 \quad L(f, P, \alpha) = m_2$$

and  $M_2 = \sup\{f(x) : x \in [0, \delta]\} < f(0) + \epsilon$ ,  $m_2 = \inf\{f(x) : x \in [0, \delta]\} > f(0) - \epsilon$ . Therefore  $U(f, P, \alpha) - L(f, P, \alpha) < 2\epsilon$  and  $f$  is Stieltjes Integrable. Take  $\epsilon \rightarrow 0$ . The answer is  $f(0)$ .

This is counter-intuitive... *Dirac delta function*...

$$\therefore \lim_{x \rightarrow 0^+} f(x) = f(0) \implies f \in \mathcal{R}(\alpha), \int_{-1}^1 f d\alpha = f(0)$$

**Remark.** Consider this as  $\int_{-1}^1 f(x)\alpha'(x)dx$ . For our  $\alpha$ ,  $\alpha'(x) = \infty$  at  $x = 0$ , 0 otherwise.

Also consider  $f(x) = 2$  for  $x \geq 0$ , 0 otherwise. Then setting  $x_{i-1} < 0 < x_i$ , with  $\|P\| < \delta$  will give  $S(f, P, \alpha) = f(t_i)$ , which might be either 0 or 2 depending on  $t_i$ 's sign. Thus we cannot say that  $\lim_{\|P\| \rightarrow 0} S(f, P, \alpha) = \int_a^b f d\alpha$ .

$\alpha$ : Bounded Variation Case

**Definition.** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and  $\alpha : [a, b] \rightarrow \mathbb{R}$  is of bounded variation. If  $\alpha = \alpha_1 - \alpha_2$  for some monotonically increasing functions  $\alpha_1, \alpha_2$ , and  $f \in \mathcal{R}(\alpha_1)$ ,  $f \in \mathcal{R}(\alpha_2)$

$$\implies \int_a^b f d\alpha = \int_a^b f d\alpha_1 - \int_a^b f d\alpha_2$$

Well-Definedness!

If  $\alpha = \alpha_1 - \alpha_2 = \beta_1 - \beta_2$ ,

$$\int_a^b f d\alpha_1 - \int_a^b f d\alpha_2 = \int_a^b f d\beta_1 - \int_a^b f d\beta_2$$

holds because of Thm 5.5.1 (1)

**Theorem 5.5.3** If  $f : [a, b] \rightarrow \mathbb{R}$  is **continuous** and  $\alpha : [a, b] \rightarrow \mathbb{R}$  is BV,  $f \in \mathcal{R}(\alpha)$ .

**Proof.** Enough to show for monotonically increasing  $\alpha$ .

For  $\epsilon > 0$ ,  $\exists \delta > 0$  s.t.  $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{M(\alpha(b) - \alpha(a))}$ , where  $M$  is an upper bound of  $|f|$ . For  $P \in \mathcal{P}[a, b]$  s.t.  $\|P\| < \delta$ ,

$$U(f, P, \alpha) - L(f, P, \alpha) = \sum_{i=1}^n (M_i - m_i)(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^n (f(u_i) - f(v_i))(\alpha(x_i) - \alpha(x_{i-1}))$$

for some  $u_i, v_i$ , since  $f$  is continuous. (EVT) But setting  $u_i, v_i \in [x_{i-1}, x_i] \implies |u_i - v_i| < \delta$ .

Thus

$$\leq \frac{\epsilon}{M(\alpha(b) - \alpha(a))} \sum_{i=1}^n (\alpha(x_i) - \alpha(x_{i-1})) < \frac{\epsilon}{M}$$

Now if  $\alpha$  is BV,  $\alpha = \alpha_1 - \alpha_2$  for monotonically increasing  $\alpha_1, \alpha_2$ . Then  $f \in \mathcal{R}(\alpha_1)$ ,  $f \in \mathcal{R}(\alpha_2)$ .

Then  $f \in \mathcal{R}(\alpha)$  and  $\int_a^b f d\alpha = \int_a^b f d\alpha_1 - \int_a^b f d\alpha_2$  by definition.

**June 7th, 2019**

**Remark.** If  $\alpha = \alpha_1 - \alpha_2$ , (BV)

$$S(f, P, \alpha) = \sum_{t_i \in [x_{i-1}, x_i]} f(t_i)(\alpha(x_i) - \alpha(x_{i-1})) = S(f, P, \alpha_1) - S(f, P, \alpha_2)$$

**Theorem 5.5.4** Thm 5.5.2 holds for BV function  $\alpha$ .<sup>57</sup>

**Proof.** Define  $V(x) = V_a^x(\alpha)$  as **variation of  $\alpha$  on  $[a, x]$** . Let  $\alpha = V - (V - \alpha)$  where  $V$  is monotonically increasing. Note that  $\alpha, V - \alpha$  are both monotonically increasing.

(1  $\implies$  2) Exists monotonically increasing  $\alpha_1, \alpha_2$  s.t.  $f \in \mathcal{R}(\alpha_1), f \in \mathcal{R}(\alpha_2)$ . Let  $\int_a^b f d\alpha_1 = A_1, \int_a^b f d\alpha_2 = A_2$ . Then  $A = A_1 - A_2$ . By Thm 5.5.2, there exists  $P_1$  s.t.  $|S(f, P, \alpha_1) - A_1| < \frac{\epsilon}{2}$  if  $P_1 \subset P$ . Also, there exists  $P_2$  s.t.  $|S(f, P, \alpha_2) - A_2| < \frac{\epsilon}{2}$  if  $P_2 \subset P$ . Now set  $P_0 = P_1 \cup P_2$ , and for  $P \supset P_0$ ,

$$|S(f, P, \alpha) - A| = |S(f, P, \alpha_1) - A_1 - (S(f, P, \alpha_2) - A_2)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(2  $\implies$  1) **Claim.** (2) implies

$$(i) \ f \in \mathcal{R}(V), \int_a^b f dV = A_1.$$

$$(ii) \ f \in \mathcal{R}(V - \alpha), \int_a^b f d(V - \alpha) = A_1 - A.$$

(i)  $\implies$  (ii):  $S(f, P, V - \alpha) = S(f, P, V) - S(f, P, \alpha)$ .  $\exists P_0$  s.t.  $|S(f, P, \alpha) - A| < \frac{\epsilon}{2}$  if  $P \supset P_0$  (By (2)). And  $\exists P_1$  s.t.  $|S(f, P, V) - A_1| < \frac{\epsilon}{2}$  if  $P \supset P_1$  (By (i), Thm 5.5.2). Set  $P_2 = P_0 \cup P_1$ , and for  $P \supset P_2$ ,

$$|S(f, P, V - \alpha) - (A_1 - A)| \leq |S(f, P, V) - A_1| + |S(f, P, \alpha) - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

Since  $V - \alpha$  is increasing, by Thm 5.5.2 (ii) holds.

(i), (ii)  $\implies$  (1):  $f \in \mathcal{R}(V), f \in \mathcal{R}(V - \alpha), V, V - \alpha$  are monotonically increasing. Thus  $f \in \mathcal{R}(V - (V - \alpha)) = \mathcal{R}(\alpha)$ , and

$$\int_a^b f d\alpha = \int_a^b f dV - \int_a^b f d(V - \alpha) = A_1 - (A_1 - A) = A$$

**Proof of (i).**  $\forall \epsilon > 0$ , we try to find a  $P$  s.t.  $U(f, P, V) - L(f, P, V) < \epsilon$ . Note that  $V(b) = V_a^b(\alpha)$ .

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<sup>57</sup>리만 적분에서는 norm 이 작기만 하면 되었는데, 스틸체스는 그렇지 않아요. 예를 들어  $\alpha$  가 불연속점을 가질 때 불연속을 포함하게 자르면 안 됐죠. 그니까 어떤 잘 썬는 partition  $P_0$  에 대해 그거 보다 저 잘 썰면 스틸체스합이 수렴한다는 뜻입니다.

There exists  $P_1$  s.t.  $P \supset P_1 \implies |V(\alpha, P) - V(\alpha)| < \epsilon'$ .<sup>58</sup>

By (2),  $\forall \epsilon > 0, \exists P_0$  s.t.  $|S(f, P, \alpha) - A| < \epsilon'$  if  $P \supset P_0$ . Set  $P = P_0 \cup P_1$ , then  $P \supset P_0$  and  $P \supset P_1$ .

$$\begin{aligned} U(f, P, V) - L(f, P, V) &= \sum (M_i - m_i)(V(x_i) - V(x_{i-1})) \\ &= \sum (M_i - m_i)(V(x_i) - V(x_{i-1}) - |\alpha(x_i) - \alpha(x_{i-1})|) \quad \cdots \quad c_1 \\ &\quad + \sum (M_i - m_i) |\alpha(x_i) - \alpha(x_{i-1})| \quad \cdots \quad c_2 \end{aligned}$$

Let  $M$  be an upper bound of  $|f|$  on  $[a, b]$ .

$$0 \leq c_1 \leq 2M \sum_{i=1}^n \{V(x_i) - V(x_{i-1}) - |\alpha(x_i) - \alpha(x_{i-1})|\} = 2M\{V_a^b(\alpha) - V(\alpha, P)\} < 2M\epsilon'$$

, since  $P \supset P_1$ . Now for  $c_2$ ,

$$M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\} \implies \exists u_i \text{ s.t. } f(u_i) > M_i - \epsilon'$$

$$m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\} \implies \exists v_i \text{ s.t. } f(v_i) < m_i + \epsilon'$$

therefore  $f(u_i) - f(v_i) > M_i - m_i - 2\epsilon'$  and  $M_i - m_i < f(u_i) - f(v_i) + 2\epsilon'$ . Define

$$(t_i, s_i) = \begin{cases} (u_i, v_i) & \text{if } \alpha(x_i) - \alpha(x_{i-1}) \geq 0 \\ (v_i, u_i) & \text{if } \alpha(x_i) - \alpha(x_{i-1}) < 0 \end{cases}$$

Then

$$\begin{aligned} c_2 &\leq \sum_{i=1}^n (f(u_i) - f(v_i) + 2\epsilon') |\alpha(x_i) - \alpha(x_{i-1})| \\ &= \sum_{i=1}^n (f(u_i) - f(v_i)) |\alpha(x_i) - \alpha(x_{i-1})| + 2\epsilon' V(\alpha, P) \\ &\leq \sum_{i=1}^n (f(t_i) - f(s_i)) |\alpha(x_i) - \alpha(x_{i-1})| + 2\epsilon' V(\alpha) \\ &= S_1(f, P, \alpha, t_1, \dots, t_n) - S_2(f, P, \alpha, s_1, \dots, s_n) + 2\epsilon' V(\alpha) \end{aligned}$$

Now we use (2), for  $P \supset P_0$ ,<sup>59</sup>

$$|S_1(f, P, \alpha) - S_2(f, P, \alpha)| \leq |S_1(f, P, \alpha) - A| + |S_2(f, P, \alpha) - A| < \epsilon' + \epsilon' = 2\epsilon'$$

Overall,

$$U(f, P, V) - L(f, P, V) = c_1 + c_2 \leq 2M\epsilon' + 2\epsilon' + 2\epsilon'V(\alpha) = \epsilon'(2M + 2 + 2V(\alpha))$$

$\epsilon = \epsilon'(2M + 2V(\alpha) + 2)$  will show what we wanted.

**Theorem 5.5.4** The following are equivalent for BV  $\alpha$ .

---

<sup>58</sup>Check as assignment.

<sup>59</sup>이  $P_0$  는 어디서 왔을까?

$$(1) f \in \mathcal{R}(\alpha), \int_a^b f d\alpha = A.$$

$$(2) \forall \epsilon > 0, \exists P_0 \in \mathcal{P}[a, b] \text{ s.t. } |S(f, P, \alpha) - A| < \epsilon \text{ for all } P \supset P_0.$$

**Remark.**  $f \in \mathcal{R}(\alpha) \implies f \in \mathcal{R}(V) !!$

**Theorem 5.5.5** Suppose  $\alpha \in C_1 : [a, b] \rightarrow \mathbb{R}$ . If  $f \in \mathcal{R}(\alpha)$ ,  $f\alpha'$  is Riemann Integrable, and

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx = \int_a^b f\alpha'$$

**Proof.**  $S(f, P, \alpha) = \sum_{i=1}^n f(t_i)(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^n f(t_i)\alpha'(s_i)(x_i - x_{i-1})$  for  $s_i \in (x_{i-1}, x_i)$  by MVT, and  $R(f\alpha', P) = \sum_{i=1}^n f(t_i)\alpha'(t_i)(x_i - x_{i-1})$

$$(1) \exists P_0 \text{ s.t. } P \supset P_0 \implies |S(f, P, \alpha) - A| < \frac{\epsilon}{2} \text{ (Thm 5.5.4)}$$

$$(2) \exists \delta > 0 \text{ s.t. } |x - y| < \delta \implies |\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{2M(b-a)} \text{ (Uniform continuity of } \alpha \text{ on } [a, b])$$

Set  $P_1$  as any superset of  $P_0$  s.t.  $\|P_1\| < \delta$ . If  $P = \{a = x_0 < x_1 < \dots < x_n = b\} \supset P_1$ ,

$$\begin{aligned} |R(f\alpha', P) - A| &\leq |R(f\alpha', P) - S(f, P, \alpha)| + |S(f, P, \alpha) - A| \\ &\leq \sum_{i=1}^n |f(t_i)| |\alpha'(s_i) - \alpha'(t_i)| (x_i - x_{i-1}) + \frac{\epsilon}{2} \\ &< M \frac{\epsilon}{2M(b-a)} \sum_{i=1}^n (x_i - x_{i-1}) + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

now  $f\alpha'$  is Riemann Integrable and the integral is equal to  $A$ .

**Example.**

$$(1) \int_{-1}^1 x^3 d(x^2) = \int_{-1}^1 x^3 \cdot 2x dx = \frac{4}{5}, \text{ since } x^2 \text{ is } C^1 \text{ (} \implies \text{Lipschitz} \implies \text{BV})$$

$$(2) \int_{-1}^1 x^3 d|x|.$$

$|x| = \alpha_1 - \alpha_2$  where

$$\alpha_1(x) = \begin{cases} 0 & (x < 0) \\ x & (x \geq 0) \end{cases} \quad \alpha_2(x) = \begin{cases} x & (x < 0) \\ 0 & (x \geq 0) \end{cases}$$

These are both increasing, then splitting the integral and a simple calculation yields  $\frac{1}{2}$ .

$$(3) \int_{-1}^1 x^3 d\alpha \text{ for } \alpha(x) = \begin{cases} -x & (-1 \leq x \leq 0) \\ x+1 & (0 < x \leq 1) \end{cases}.$$

Let  $\alpha = \beta_1 + \beta_2$  s.t.  $\beta_1(x) = |x|$  and  $\beta_2(x) = 1$  for  $x > 0$ , 0 otherwise.  $\beta_1, \beta_2$  are both BV,

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<sup>60</sup>You can show that  $|x|$  is Lipschitz continuous then it is BV.

and their sum  $\alpha$  is BV.

$$\int_{-1}^1 f d\alpha = \int_{-1}^1 f d\beta_1 + \int_{-1}^1 f d\beta_2 = \frac{1}{2} + f(0) = \frac{1}{2}$$

(Check the first equality for BV functions)

**Theorem 5.5.6** Suppose  $f, \alpha : [a, b] \rightarrow \mathbb{R}$  and  $f, \alpha$  is BV. If  $f \in \mathcal{R}(\alpha)$ , then  $\alpha \in \mathcal{R}(f)$  and

$$\int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b f d\alpha$$

**Proof.** Let  $\int_a^b f d\alpha = A$ . Then by Thm 5.5.4  $\forall \epsilon > 0, \exists P_0$  s.t.  $P \supset P_0 \implies |S(f, P, \alpha) - A| < \epsilon$ .

$$S(\alpha, P, f) = \sum_{i=1}^n \alpha(t_i)(f(x_i) - f(x_{i-1}))$$

Rewrite

$$f(b)\alpha(b) - f(a)\alpha(a) = \sum_{i=1}^n (\alpha(x_i)f(x_i) - \alpha(x_{i-1})f(x_{i-1}))$$

And for  $P \supset P_0$ ,

$$\begin{aligned} & |S(\alpha, P, f) - (f(b)\alpha(b) - f(a)\alpha(a) - A)| \\ &= \left| \sum_{i=1}^n f(x_i)[\alpha(x_i) - \alpha(t_i)] + \sum_{i=1}^n f(x_{i-1})[\alpha(t_i) - \alpha(x_{i-1})] - A \right| \\ &= |S(f, Q, \alpha) - A| < \epsilon \end{aligned}$$

where  $Q = \{a = x_0 \leq t_0 \leq x_1 \leq t_1 \leq x_2 \leq \dots \leq t_n \leq x_n = b\} = P \cup \{t_1, \dots, t_n\} \supset P \supset P_0$ .