Discrete Mathematics HW5

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Problem 1

Proof. First, note that the *Theorem*: [Composition of bijections is a bijection]. Let $N = \{1, 2, \cdots, n\}$. Since $f : N \to N$ is a bijection, $f^k : N \to N$ is also a bijection. Let $X = \{f \mid f : N \to N, f \text{ is a bijection}\}$. Then $f^k \in X$ for ${}^\forall k \in \mathbb{Z}^+$. Note that by the definition of X, the number of elements of X is the same as the number of bijections from N to N, so $|X| = n!$. Although $|X|$ is finite, there are countably many f^k for $k \in \mathbb{Z}^+$. Therefore, by the *Pigeonhole Principle*, ${}^\exists i, j \in \mathbb{Z}^+$ such that $f^i = f^j$ even if $i \neq j$. $\square$

Problem 2

Before the start, it is important that understanding *Ramsey numbers*. This is based on *Complete graph with n nodes*: K_n. The concept(definition) of K_n is very easy, so I recommend looking it up and understanding it on Wikipedia(K_n is used after mid-term).
$R(m, n) = r$ means:

> Initial setting: Edges of K_r has only 2 colors: red or blue.
> For this K_r, we can find a K_m subgraph connected by red edges or a K_n subgraph connected by blue edges(not need to be both) without any counter example. Also, K_r is the minimum size: K_{r-1} should have some counter example.

(a) *Proof.* By the above definition, we can change the colors of edges: red to blue, blue to red. Then it is $R(n, m)$. Since both cases have the same number r(because the given graph is K_r), $R(m, n) = r = R(n, m)$.

□

(b) *Proof.* Assume $R(2, n) < n$. Let $R(2, n) = n - 1$. Then by the above definition, we have K_n subgraph in K_{n-1} if all edges are blue. But it is impossible. Therefore, $R(2, n) \geq n$. Suppose $R(2, n) = n$. We have 2 cases: [K_n has no red edges] and [K_n has at least one red edge].

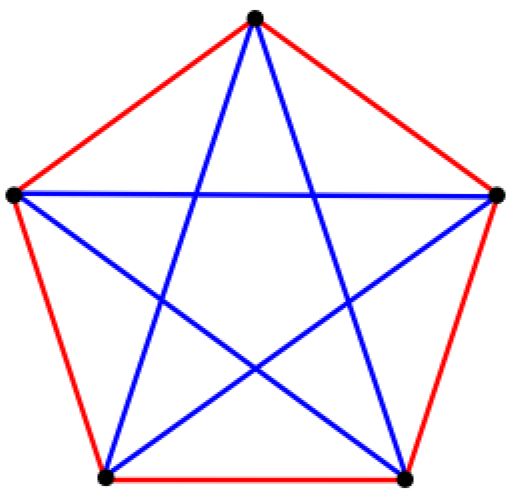
i) If all edges are blue, then it must have K_n subgraph: itself.

ii) If there is at least one red edge, then since K_2 needs only one red edge, it must have K_2 subgraph.

Therefore, $R(2, n) = n$ is true.

□

(c) *Proof.* Suppose $R(3, 3) = 5$. If we find a counter example, then done(because if $R(3, 3) = 5$ is false, then $R(3, 3) \leq 5$ is also false by the definition of subgraph). Here is the counter example. The below graph do not have a K_3 subgraph. Note that this is not the only one. Besides this, other counterexamples also exist.



□

Problem 3

Proof. For any two points $a_1 = (x_1, y_1)$ and $a_2 = (x_2, y_2)$, the midpoint $z = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$. If $\frac{x_1+x_2}{2}$ is an integer, then x_1 and x_2 are both even or odd. Similarly, y_1 and y_2 are both even or odd. All points with integer coordinates can be classified into four sets:

$$\begin{aligned}
A_1 &= \{(x, y) \mid x \text{ is even, } y \text{ is odd}\} \\
A_2 &= \{(x, y) \mid x \text{ is even, } y \text{ is even}\} \\
A_3 &= \{(x, y) \mid x \text{ is odd, } y \text{ is odd}\} \\
A_4 &= \{(x, y) \mid x \text{ is odd, } y \text{ is even}\}
\end{aligned}$$

If some A_i has at least two elements, then we can get midpoint with integer coordinates by picking two of them. Since five distinct points are given, by the *Pigeonhole Principle*, some A_i has at least two elements.
WARNING: When you classify them, using the word **group** is dangerous. Because there is a mathmatical structure called a group. Avoid using the word group unless it describes a real-life situation. We have a nice container called a set.

$\square$

Problem 4

For all problems, suppose that student choose all answers half-and-half.

(a) *Solution.* There is only one case: ${}_{10}\mathrm{C}_{10} = 1$(from 10 questions, select 10 correct answers). That case has $\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{0} = \left(\frac{1}{2}\right)^{10}$(10 corrects, 0 incorrects). Only about 0.098%.

$\square$

(b) *Solution.* Similarly, there is only one case: ${}_{10}\mathrm{C}_{0} = 1$(from 10 questions, select 0 correct answers). That case has $\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10}$(0 corrects, 10 incorrects). Only about 0.098%.

$\square$

(c) *Solution.* There are ${}_{10}\mathrm{C}_{1} = 10$ cases(from 10 questions, select 1 correct answer). Each case has the probability $\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{9} = \left(\frac{1}{2}\right)^{10}$(1 correct, 9 incorrects). Therefore, the answer is $10\left(\frac{1}{2}\right)^{10} \simeq 0.98\%$.

$\square$

(d) *Solution.* There are ${}_{10}\mathrm{C}_{5}$ cases(from 10 questions, select 5 correct answers). Each case has the probability $\left(\frac{1}{2}\right)^{5}\left(\frac{1}{2}\right)^{5} = \left(\frac{1}{2}\right)^{10}$(5 corrects, 5 incorrects). Therefore, the answer is ${}_{10}\mathrm{C}_{5}\left(\frac{1}{2}\right)^{10} \simeq 24.61\%$.

$\square$

Problem 5

(a) *Solution.* There are 7 days per week. For each person, there is a $\frac{1}{7}$ chance of being born on any specific day. Also, we have ${}_7\mathrm{C}_1 = 7$ cases to choose that specific day. Therefore, $7 \times \frac{1}{7} \times \frac{1}{7} = \frac{1}{7}$

$\square$

(b) *Solution.* First, calculate the probability that all n people were born on different days of the week. The first person can be born on any of the 7 days of the week, so the probability is $\frac{7}{7}$. rom the second person onwards, they must be born on a different day than the previous person, which gives a probability of $\frac{6}{7}$. Continue this, then

$$\frac{7}{7} \times \frac{6}{7} \times \cdots \times \frac{7-(n-1)}{7}$$

Note that this is valid for $n \leq 7$. If $n \geq 8$, then that value is 0(If you apply the *Pigeonhole principle*, then the probability is 1 if $n \geq 8$). The answer is subtract it from 1.

$\square$

(c) *Solution.* From (b), just put $n = 1$ to 7. Here are the results.

- $n = 1 \Rightarrow P = 0$ (since there's only one person, the probability of two people being born on the same day is impossible, 0)
- $n = 2 \Rightarrow P = \frac{1}{7}$(note that this is the same as in (a))
- $n = 3 \Rightarrow P \simeq 0.388$
- $n = 4 \Rightarrow P \simeq 0.650$
- $n = 5 \Rightarrow P \simeq 0.850$
- $n = 6 \Rightarrow P \simeq 0.957$
- $n = 7 \Rightarrow P \simeq 0.994$
- $n \geq 8 \Rightarrow P = 1$

Therefore, we need at least 4 people.

$\square$

Problem 6

Solution. First, summarize the given information.

- From company, 60% success and 40% failure. $\Rightarrow P(\text{success}) = 0.6, P(\text{failure}) = 0.4$
- From predict, 70% success if success. $\Rightarrow P(\text{predict success} \mid \text{success}) = 0.7$
- From predict, 40% success if failure. $\Rightarrow P(\text{predict success} \mid \text{failure}) = 0.4$

We want to find success if predict success. i.e. $P(\text{success} \mid \text{predict success})$. Use *Bayes'theorem.*

$$\begin{aligned} P(\text{success} \mid \text{predict success}) &= \frac{P(\text{success} \cap \text{predict success})}{P(\text{predict success})} \\ &= \frac{P(\text{predict success} \mid \text{success}) \times P(\text{success})}{P(\text{predict success})} \end{aligned}$$

Note that $P(\text{predict success}) = P(\text{predict success} \mid \text{success}) \times P(\text{success}) + P(\text{predict success} \mid \text{failure}) \times P(\text{failure}) = 0.7 \times 0.6 + 0.4 \times 0.4 = 0.58$. Therefore, the answer is $\frac{0.7 \times 0.6}{0.58} \simeq 72.41\%$.
APPENDIX: You can use a table and explain brief. But **be careful**: table is just for supporting purposes only, not a logical explanation. Following is how to draw the table.

C is a company, R is a predict, S is a success, and F is a failure.

C

		S	F	total
P	S	?	?	?
	F	?	?	?
	total	?	?	1

From the given information, we get the following result:

C

		S	F	total
P	S	0.6×0.7	0.4×0.4	?
	F	?	?	?
	total	0.6	0.4	1

We have equalities for row-sum and column-sum. Therefore, we can get

		C: S	C: F	total
P	S	0.6×0.7	0.4×0.4	$0.6 \times 0.7 + 0.4 \times 0.4 = 0.58$
	F	$0.6 - 0.6 \times 0.7 = 0.6 \times 0.3$	$0.4 - 0.4 \times 0.4 = 0.4 \times 0.6$	?
	total	0.6	0.4	1

By the same way, we can fill remained cells.

		C: S	C: F	total
P	S	0.6×0.7	0.4×0.4	0.58
	F	0.6×0.3	0.4×0.6	$0.6 \times 0.3 + 0.4 \times 0.6 = 0.42 = 1 - 0.58$
	total	0.6	0.4	1

Here is brief explain.

By the given information, we know that

- $P(\text{success}) = 0.6$
- $P(\text{failure}) = 0.4$
- $P(\text{success} \cap \text{predict success}) = 0.6 \times 0.7$
- $P(\text{failure} \cap \text{predict success}) = 0.4 \times 0.4$

We want to find $P(\text{success} \mid \text{predict success})$.

$$
\begin{aligned}
& P(\text{success} \mid \text{predict success}) \\
= & \frac{P(\text{success} \cap \text{predict success})}{P(\text{predict success})} \\
= & \frac{P(\text{success} \cap \text{predict success})}{P(\text{predict success} \cap \text{success}) + P(\text{predict success} \cap \text{failure})} \\
= & \frac{0.6 \times 0.7}{0.6 \times 0.7 + 0.4 \times 0.4} \\
= & \frac{42}{58}
\end{aligned}
$$

□

Problem 7

For this problem, I use the table method for explain. But you should explain like the APPENDIX in Problem 6(but do not need to expain how to make such table). The given statement makes the below table. Y is a bicycle, C is a car, B is a bus, E is an early, and L is a late.

	Y	C	B	total
E	?	?	?	?
L	$x_1 \times 0.05$	$x_2 \times 0.5$	$x_3 \times 0.2$	?
total	x_1	x_2	x_3	1

Also, we can fill other cells.

	Y	C	B	total
E	$x_1 \times 0.95$	$x_2 \times 0.5$	$x_3 \times 0.8$	$\sum$
L	$x_1 \times 0.05$	$x_2 \times 0.5$	$x_3 \times 0.2$	$\sum$
total	x_1	x_2	x_3	1

(a) *Solution.* The given information is $x_1 = x_2 = x_3 = \frac{1}{3}$.

	Y	C	B	total
E	$\frac{1}{3} \times 0.95$	$\frac{1}{3} \times 0.5$	$\frac{1}{3} \times 0.8$	$\frac{1}{3} \times 2.25$
L	$\frac{1}{3} \times 0.05$	$\frac{1}{3} \times 0.5$	$\frac{1}{3} \times 0.2$	$\frac{1}{3} \times 0.75$
total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

We want to find $P(\mathrm{C} \mid \mathrm{L})$.

$$\begin{aligned} & P(\mathrm{C} \mid \mathrm{L}) \\ = & \frac{P(\mathrm{L} \cap \mathrm{C})}{P(\mathrm{L})} \\ = & \frac{\frac{1}{3} \times 0.5}{\frac{1}{3} \times 0.75} = \frac{2}{3} \end{aligned}$$

$\square$

(b) *Solution.* The given information is $x_1 = 0.6, x_2 = 0.3, x_3 = 0.1$.

	Y	C	B	total
E	0.6×0.95	0.3×0.5	0.1×0.8	0.8
L	0.6×0.05	0.3×0.5	0.1×0.2	0.2
total	0.6	0.3	0.1	1

We want to find $P(\mathrm{C} \mid \mathrm{L})$.

$$\begin{aligned} & P(\mathrm{C} \mid \mathrm{L}) \\ = & \frac{P(\mathrm{L} \cap \mathrm{C})}{P(\mathrm{L})} \\ = & \frac{0.15}{0.2} = \frac{3}{4} \end{aligned}$$

□

Problem 8

Solution. First, summarize the given information. Let S be a spam, N be a not spam, E be a word 'enhancement', and H be a word 'herbal'.

- The number of S = 10,000
- The number of N = 5,000
- $P(\mathrm{E} \mid \mathrm{S}) = \frac{1{,}500}{10{,}000}$, $P(\mathrm{E} \mid \mathrm{N}) = \frac{20}{5{,}000}$
- $P(\mathrm{H} \mid \mathrm{S}) = \frac{800}{10{,}000}$, $P(\mathrm{H} \mid \mathrm{N}) = \frac{200}{5{,}000}$

We want to find $P(\mathrm{S} \mid \mathrm{E, H})$

$$\begin{aligned}
P(\mathrm{S} \mid \mathrm{E, H}) &= \frac{P(\mathrm{E, H} \mid \mathrm{S}) \times P(\mathrm{S})}{P(\mathrm{E, H})} \\
P(\mathrm{E, H} \mid \mathrm{S}) &= P(\mathrm{E} \mid \mathrm{S}) \times P(\mathrm{H} \mid \mathrm{S}) \\
P(\mathrm{E, H}) &= P(\mathrm{E, H} \mid \mathrm{S}) \times P(\mathrm{S}) + P(\mathrm{E, H} \mid \mathrm{N}) \times P(\mathrm{N}) \\
P(\mathrm{E, H} \mid \mathrm{N}) &= P(\mathrm{E} \mid \mathrm{N}) \times P(\mathrm{H} \mid \mathrm{N})
\end{aligned}$$

By the given information, we get

$$P(\mathrm{S} \mid \mathrm{E, H}) \simeq \frac{0.008}{0.0080533} \simeq 99.33\%$$

□