Discrete Mathematics HW2

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Problem 1

Proof. Proof by strong induction. For $P(j)$, consider $j = 1$. Then since $\frac{1}{b} < 1 \leq \sqrt{2}$ for $^\forall b \in \mathbb{Z}^+$, $\sqrt{2} \neq \frac{1}{b}$ for $^\forall b \in \mathbb{Z}^+$. So $j = 1$ is true.
Suppose $j = 1, 2, \cdots, n$ is true. Consider $j = n + 1$. We use contradiction. i.e. assume that:

$$P(j) \text{ is true for } j = 1, 2, \cdots, n \Rightarrow\ P(n+1) \text{ is false.} \tag{1}$$

It follows

$$\begin{aligned}
&P(n+1) \text{ is false.}\\
&\Rightarrow \sqrt{2} = \frac{n+1}{b} \text{ for some } b \in \mathbb{Z}^+\\
&\Rightarrow 2b^2 = (n+1)^2 \text{ for some } b \in \mathbb{Z}^+\\
&\Rightarrow (n+1)^2 \text{ is divisible by 2}\\
&\Rightarrow n+1 \text{ is divisible by 2}\\
&\Rightarrow n+1 = 2t \text{ for some } t \in \mathbb{Z}^+\\
&\Rightarrow 2b^2 = 4t^2 \text{ for some } b, t \in \mathbb{Z}^+\\
&\Rightarrow b^2 = 2t^2 \text{ for some } b, t \in \mathbb{Z}^+\\
&\Rightarrow b^2 \text{ is divisible by 2}\\
&\Rightarrow b \text{ is divisible by 2}\\
&\Rightarrow b = 2s \text{ for some } s \in \mathbb{Z}^+\\
&\Rightarrow \sqrt{2} = \frac{n+1}{b} = \frac{2t}{2s} = \frac{t}{s} \text{ for some } b, t, s \in \mathbb{Z}^+.
\end{aligned}$$

Since $n + 1 = 2t$, $t = \frac{n+1}{2} \leq n$ is true for $^\forall n \in \mathbb{Z}^+$. But it contradicts to the assumption (1). This implies the following:

$$P(j) \text{ is true for } j = 1, 2, \cdots, n \Rightarrow\ P(n+1) \text{ is true.}$$

$\square$

Problem 2 to 7

solution. They are very easy or useless problem, just for insight or challenge. Therefore, I skip them as I said in class.

□

Problem 8

(a) *solution.* Let $P(n) = 2\lg n + 4n + 3n\lg n$. Since $0 + 0 + 3n\lg n \leq P(n)$ for $n \geq 1$, $P(n) = \Omega(n\lg n)$. Since $P(n) \leq 2n\lg n + 4n\lg n + 3n\lg n = 9n\lg n$ for $n \geq 2$, $P(n) = O(n\lg n)$. Therefore, $P(n) = \Theta(n\lg n)$.

$\square$

(b) *solution.* Let $P(n) = \frac{(n^2+\lg n)(n+1)}{n+n^2}$ Note that $P(n) = \frac{n^2+\lg n}{n} = n + \frac{\lg n}{n}$ for $n \geq 1$. Since $n + 0 \leq P(n)$ for $n \geq 1$, $P(n) = \Omega(n)$. Since $\lg n \leq n^2$ for $n \geq 1$ (you don't have to prove this, but you can do by using induction! Try it.), $P(n) \leq n + \frac{n^2}{n} = n + n = 2n$ for $n \geq 1$, so $P(n) = O(n)$. Therefore, $P(n) = \Theta(n)$.

$\square$

(c) *solution.* Let $P(n) = 2 + 4 + 8 + 16 + \cdots + 2^n$. Note that $P(n) = \sum_{i=1}^{n} 2^i = \frac{2(2^n-1)}{2-1} = 2^{n+1} - 2$. Since $2 \leq 2^n$ for $n \geq 1$, $P(n) \geq 2 \cdot 2^n - 2^n = 2^n$ for $n \geq 1$, so $P(n) = \Omega(2^n)$. Since $P(n) \geq 2 \cdot 2^n + 0$, $P(n) = O(2^n)$. Therefore, $P(n) = \Theta(2^n)$.

$\square$