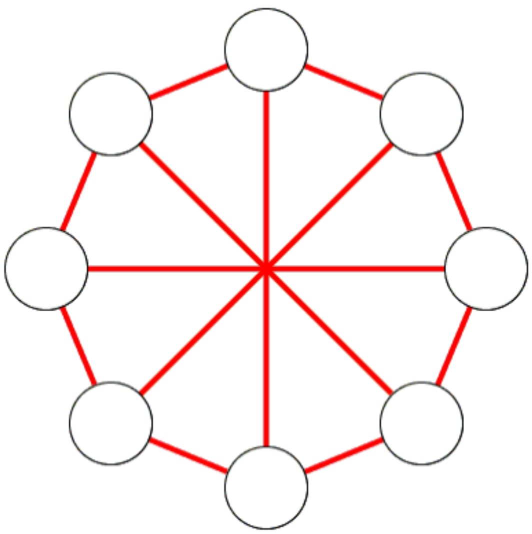
Discrete Mathematics HW8

20180617 You SeungWoo

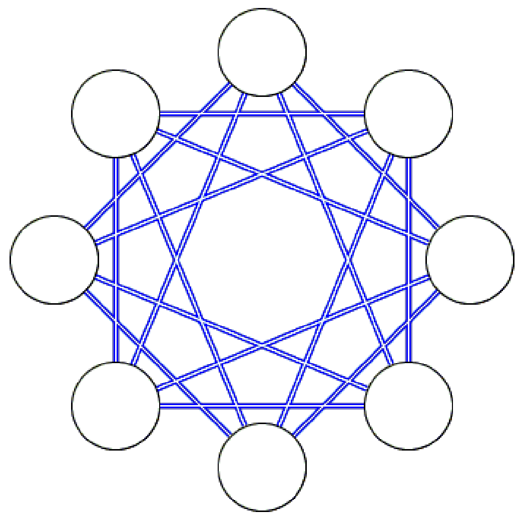
November 13, 2023

Problem 1

Solution. They are G and $\overline{G}$.



(a) G



(b) $\overline{G}$

Since G does not contain K_3 and $\overline{G}$ does not contain K_4, by the definition of *Ramsey number*, it is a counter example for $R(3,4) = 8$. Therefore, $R(3,4) > 8$.

$\square$

Problem 2

Proof. Skipped. Just try it as following:

- Divide inner, outer, and bridge edges.
- Show that inner and outer have no cycle with length ≤ 4, and we cannot make such cycle even if using bridge.

$\square$

Problem 3

Proof. Skipped. Check the additional file 1.1.30.

□

Problem 4

Solution. Skipped. Just check the followings if not isomorphic.

- Degree. All degrees of vertices should be conserved.
- Connection. All connection(weight of edges, degree of vertices, or else) are the same.
- Cycle. All cycles are preserved.

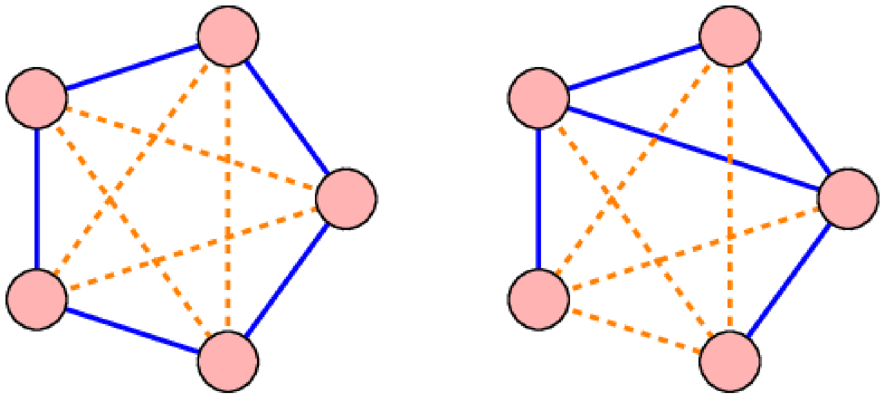
If you claim the isomorphism, then use the following form:

> Let $f : G \to H$ be a graph mapping with $f(a) = 1, f(b) = 2, \cdots$ (The alphabet is the node of G, and the number is the node of H. Lists ways to correspond nodes in G to nodes in H). Then $G \simeq H$ under f.

□

Problem 5

Solution.



□

Problem 6

Proof. First, note that the *Euler formula*: [For planer graph, $v-e+f=2$]. Suppose there is no cycle. Then $f=1$, so $v-e=1$.

$$\begin{aligned}
&4 \leq v \\
\Rightarrow\ &\frac{7}{2} \leq 4 \leq v \\
\Rightarrow\ &\frac{7}{3} \leq \frac{2}{3}v \\
\Rightarrow\ &v \leq \frac{2}{3}v - \frac{7}{3} + v \\
\Rightarrow\ &e = v - 1 \leq \frac{5}{3}v - \frac{10}{3}
\end{aligned}$$

If there are cycles, then since each $\deg f$ is at least 5(because the graph has no simple cycles of length ≤ 4), $2e = \sum \deg f \geq 5f$. By *Euler*, apply $f = 2 - v + e$.

$$\begin{aligned}
2e &\geq 5f = 5(2 - v + e) \\
-3e &\geq 10 - 5v \\
e &\leq \frac{5}{3}v - \frac{10}{3}
\end{aligned}$$

$\square$

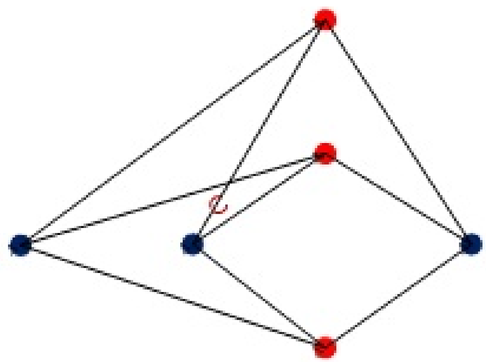
Problem 7

Proof. If Peterson Graph G is a planar, then it should satisfies the above problem because G has no cycle with length ≤ 4. G has $v = 10$ and $e = 15$. This gives $f = 2 - v + e = 7$ if G is a planar. However, from the above problem, we get $2e \geq 5f$, contradiction. Therefore, G is not planar.

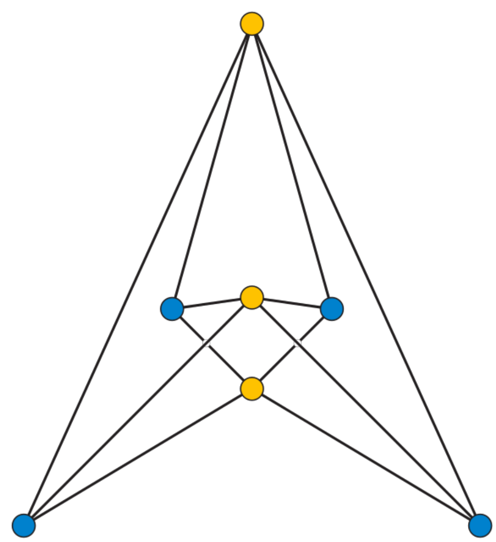
$\square$

Problem 8

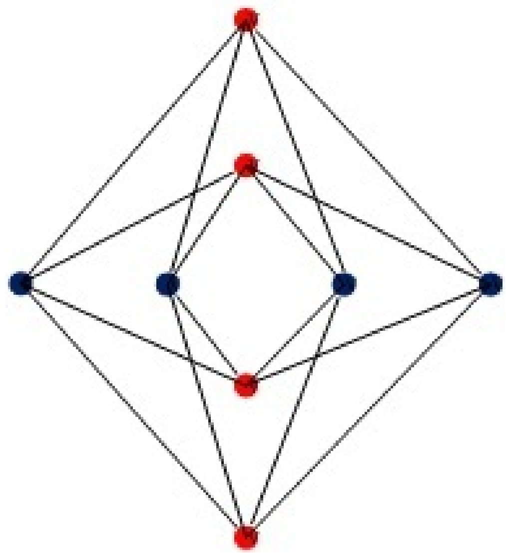
Proof. Denote that $X(G) = 1$ means the crossing number of G is 1.
We can find one example for $X(K_{3,3}) \leq 1$.



If we show that $X(K_{3,3}) \neq 0$ i.e. $K_{3,3}$ is not planar, then $X(K_{3,3}) = 1$. We can prove this by applying *Euler formula* as the Problem 6. Therefore, $X(K_{3,3}) = 1$.
Similarly, we can find $X(K_{3,4}) \leq 2$(note that $X(K_{4,3}) = X(K_{3,4})$).



Suppose $X(K_{3,4}) = 1$. Remove one blue vertex(in 4-node set) who occurs the crossing(note that crossing occurs between 4 vertices). Then there is no crossing, the graph forms $K_{3,3}$. i.e. $X(K_{3,3}) = 0$, contradiction. Therefore, $X(K_{3,4}) > 1$, so $X(K_{3,4}) = 2$.
Similarly, we can find $X(K_{4,4}) \leq 4$.



Suppose $X(K_{4,4}) = 3$. Note taht crossing needs at least 4 vertices. Since $K_{4,4}$ has 8 vertices, there is at least one vertex which occurs at least 2 crossing. It is impossible to create 3 crossings with only 8 nodes so that every node participates in the crossing only once. So choose the node who occurs 2 crossing and remove it. Then the graph is $K_{3,4}$ with 1 crossing, contradiction. Therefore, $X(K_{4,4}) > 3$, so $X(K_{4,4}) = 4$.

$\square$

Problem 9

Solution. Skipped. Just drawing and show that it is impossible with only 2 time zones(height of tree). The answer is it needs at least 3 time zones.

□

Appendix

For a graph G, the diameter of the graph G is the greatest distance between any pair of vertices in G. In other words, the diameter d is

$$d = \max_{(u,v) \in V \times V} d(u, v)$$

Show that the diameter of a graph is a graph invariant.

Proof. Suppose G has a diameter d with n nodes. Then there is a path from u to v which make d. Let $u = v_1$ and $v = v_k (1 \leq k \leq n)$. Note that this k is fixed. Consider another graph $H \simeq G$. Then there is $\{u_i\}_{ji=1}^{k} \subseteq H$ such that $u_i \mapsto v_i$(we can indexing in such way). Also, isomorphism preserves the edge-weight, this new path has the same cost. By the isomorphism, there is no higher path in H. Therefore, d is graph invariant.

$\square$