Discrete Mathematics HW1

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Problem 1

Solution. Just write down True-False table and check whether all result is true or not. Note that the following:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

For example, in (a): $((p \to q) \land \neg q) \to \neg p$,

p	q	$p \to q$	$\neg q$	$(p \to q) \land \neg q$	$\neg p$	$((p \to q) \land \neg q) \to \neg p$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Therefore, (a) is a tautology.

□

Problem 2

(a) *Proof.* Proof by induction. Consider $j = 1$. Then $H_2 = 1 + \frac{1}{2} \geq 1 + \frac{1}{2}$ is clearly true.
Suppose $j = n$ is true, consider $j = n + 1$. Then

$$\begin{aligned} H_{2^{n+1}} &= H_{2^n} + \sum_{i=1}^{2^n} \frac{1}{2^n + i} \\ &\geq 1 + \frac{n}{2} + \sum_{i=1}^{2^n} \frac{1}{2^n + i} \end{aligned}$$

If we show $\sum\limits_{i=1}^{2^n} \frac{1}{2^n+i} \geq \frac{1}{2}$, then the proof is done. Note that $\frac{1}{2^n+i} \geq \frac{1}{2^n \cdot 2}$ for $1 \leq i \leq 2^n$.

$$\begin{aligned} \sum_{i=1}^{2^n} \frac{1}{2^n + i} &\geq \sum_{i=1}^{2^n} \frac{1}{2^n \cdot 2} \\ &= 2^n \cdot \frac{1}{2^n \cdot 2} \\ &= \frac{1}{2} \end{aligned}$$

□

(b) *Proof.* Proof by induction. Let $P(j) = 7^{j+2} + 8^{2j+1}$. Consider $j = 1$. Then $7^{1+2} + 8^{2 \cdot 1+1} = 343 + 512 = 855 = 15 \cdot 57$ is clearly true.
Suppose $j = n$ is true, consider $j = n + 1$. Then

$$\begin{aligned} P(n+1) &= 7^{(n+1)+2} + 8^{2(n+1)+1} \\ &= 7 \cdot 7^{n+2} + 8^2 \cdot 8^{2n+1} \\ &= 7 \cdot 7^{n+2} + 8^2 \cdot 8^{2n+1} + 7 \cdot 8^{2n+1} - 7 \cdot 8^{2n+1} \\ &= 7 \cdot (7^{n+2} + 8^{2n+1}) + (8^2 - 7) \cdot 8^{2n+1} \\ &= 7 \cdot P(n) + 57 \cdot 8^{2n+1} \end{aligned}$$

Since $P(n)$ is divisible by 57, the proof is done.

□

Problem 3

Proof. Proof by induction. Let $P(k)$ = every $2^k \times 2^k$ checkerboard with one square removed can be tiled using right trominoes. Consider $k = 1$. Then it is easy to check that $P(1)$ is true(note that you need to write down the reason briefly).
Suppose $k = n$ is true, consider $k = n + 1$. Then $2^{n+1} \times 2^{n+1}$ checkerboard can be divided into 4 $2^n \times 2^n$ checkerboards. Since $P(n)$ is true, we can fill $2^n \times 2^n$ checkerboards with right tromino except only one tile per each. It means there exists 4 tiles in total.Then we can gather them in 2×2 place(For example, for each $2^n \times 2^n$ checkerboards, place remained tile in the corner so that it is in the center of the $2^{n+1} \times 2^{n+1}$ checkerboard). Then just put a right tromino in that place.

□

Problem 4

Proof. It is called *Helly's theorem.*

Proof by induction. We show that $\bigcap_{i=1}^{k} X_i \neq \emptyset$ for given condition. Consider $k = 4$. Define a_i be a common point which not include X_i. Then there exist two cases(This is called *Radon's theorem on Convex Set.* You can skip this if you explain the theorem well).

i) If some a_i make triangle, then WOLG, a_1 in $\triangle a_2a_3a_4$. Note that $a_2, a_3, a_4 \in X_1$ Since a_1 in $\triangle a_2a_3a_4$, a_1 should in X_1 by the convexity of X_1. Since $a_1 \in X_2, X_3, X_4$, a_1 is a common point of ${}^{\forall}X_i$.

ii) If they make a rectangle, then WOLG, let order as $\square a_1a_2a_3a_4$. Consider diagoals $\overline{a_1a_3}$ and $\overline{a_2a_4}$. They intersect in a point p. Since $p \in \overline{a_1a_3}$ and $p \in \overline{a_2a_4}$ and all X_i are convex, $p \in X_i$ for ${}^{\forall}i$.

Therefore, $k = 4$ is true.
Suppose $k = n$ is true, consider $k = n + 1$. Consider subset $Y_m = X_m \cap X_{n+1}, m = 1, 2, \cdots, n$. By the *intersection property*: intersection of convex sets is convex, Y_m is convex. Choose any Y_p, Y_q, Y_r. Then

$$Y_p \cap Y_q \cap Y_r = X_p \cap X_q \cap X_r \cap X_{n+1}$$

Since $k = 4$ is true, $Y_p \cap Y_q \cap Y_r \neq \emptyset$. This means that since $k = n$ is true for any convex sets $X_i, i = 1, 2, \cdots, n$, apply this for Y_m instead X_i, then we get $\bigcap_{m=1}^{n} Y_m \neq \emptyset$. Since $\bigcap_{m=1}^{n} Y_m = \bigcap_{i=1}^{n} X_i$, the proof is done.

$\square$

Problem 5

Proof. Proof by induction. Let k be a number of players. Consider $k = 1$. Then it is obvious, because that person is the only person, survivor.

Suppose it is true for $k = 2n - 1$ players. Consider $k = 2n + 1$ players. Choose two people who have the minimum distance. Then they are eliminated by each other. Since there remains $2n-1$ players, by induction, The argument is true.

$\square$

Problem 6

Proof. Proof by induction. Consider $s = 1$. i.e. $m = 1, n = 0$ or $m = 0, n = 1$. Then the knight can reach there in 3 moves(You need to show the reason briefly).
Suppose the knight can reach everywhere in finite sequene of moves when the size is s. Consider $s + 1$ size. It contains s-size board where knight can reach everywhere. So the knight can reach the expanded line in only one move at a distance of 2×1 or 1×2 from there.

□

Problem 7

Note that you must clearly explain the following:

- the strategy
- the remained shape at a specific turn
- the starter at that turn

(a) *Proof.* The strategy is select bottom-right $(n-1) \times (n-1)$ tiles at the first turn. Then only remain $(n-1) \times 1$ or $1 \times (n-1)$ tiles. If the opponent selects sub(or full) piece of one of them, then pick the same amount tiles from the other. Proof this by strong induction. Note that this strategy is clearly true if $n = 2$.
Consider $n = 2, 3, \cdots, N$ is true, consider $n = N + 1$. Since the remained shape after k turn(k is odd bigger than 1. i.e. start the opponent's turn) is the same as the after of the first turn(i.e. start the opponent's turn) of one of $i \times i$ board($2 \leq i \leq N$).

$\square$

(b) *Proof.* The strategy is select bottom-right 1×1 tile at the first turn. Then the opponent can only choose $1 \times k$ tiles($1 \leq k \leq n-1$) from upper or lower. If the opponent picks one, then just pick the same amount from the other. This is proved by strong induction similarly.

$\square$