Discrete Mathematics HW11

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Problem 1

Solution. Skipped. Very easy. Just following:

- For red box, write maximum value among its children.
- For yellow circle, write minimum value among its children.

□

Problem 2

Solution. Skipped. $\square$

Problem 3

(a) *Proof.* ($\Rightarrow$) Suppose the first player can always win Nim. There are two possible cases for his last turn:

- There is only one pile wtih m coins($m > 1$). Then he will take $m - 1$ coins to win at Nim.
- There are two piles: one coin and m coins($m > 1$). Then he will take m coins to win at Nim.

Note that other cases are impossible. Because if the number of piles ≥ 3, then the game needs at least 3-turn. This contradicts to the assumption: last turn. If there are two piles with n coins and m coins($n > 1$), then he will lose if the game end in 2-turn.
For the first case, he will take m coins instead $m - 1$ coins to win at Nim'. For the second case, also he will take $m - 1$ coins to win at Nim'.
($\Leftarrow$) Similar proof as above.

$\square$

(b) *Proof.* First, define the nim-sum operator $\oplus$

For each digit of binary numbers, apply XOR operation(modulo 2 operation)

For example, $1100 \oplus 1010 = 0110$.
For Nim, player A will win if he makes odd numbers of one-coin piles with the following rules:

Make nim-sum be 0 for all piles.

For example, let piles with 3, 4, and 5 coins be given. Then the nim-sum is $3 \oplus 4 \oplus 5 = 011_2 \oplus 100_2 \oplus 101_2 = 010_2 = 2$. To make nim-sum 0, remove 2 coins from 3-coin pile. Then $1 \oplus 4 \oplus 5 = 001_2 \oplus 100_2 \oplus 101_2 = 000_2 = 0$. This is always true because of the theorems:

- This is the winning strategy.
- We can always find the number of coins to make nim-sum 0.

The proof of these theorems is very difficult to understand, so just accept.
For Nim', just make nim-sum 0 at each turn. you don't have to consider making odd numbers of one-coin piles.

$\square$

Problem 4

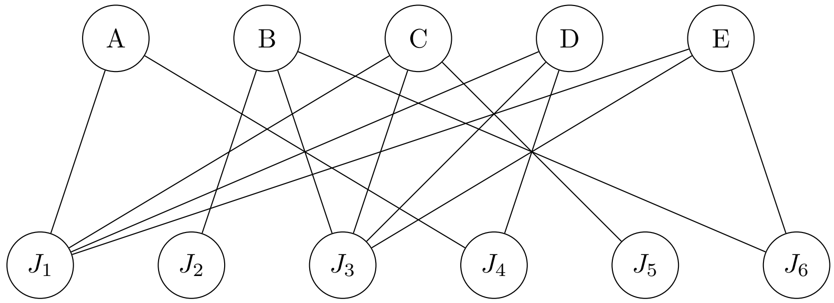
Solution. Skipped. Remember the definition of cut and flow. The answer is

- Maximum flow(=mincut) is 22
- This cut is (A, B) where $A = \{s, b, f, g, h, j, k, l, m\}$ and $B = \{c, d, e, i, n, t\}$

$\square$

Problem 5

(a) *Solution.*



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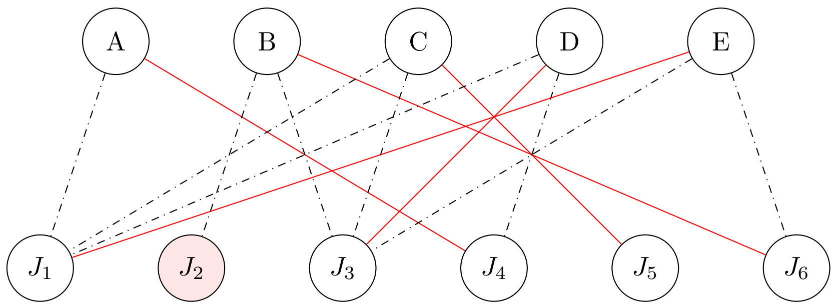
(b) *Solution.*



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(c) *Proof.* We cannot make complete(perfect) matching. There are two theorems.

- If perfect matching exists, then it has even nodes.
- (*Hall's theorem*) Let bipartite $G = (V_1 \cup V_2, E)$ be given. For $S \subseteq V_1$, define $N(S) = \{v \in V_2 \mid v \text{ is adjecent to } u \in S\}$. Then there is a perfect matching if and only if $|S| \leq |N(S)|$ for ${}^{\forall}S \subseteq V_1$.

The above graph has total 11 nodes. Also, let $S = V_1 = \{J_1, J_2, \cdots, J_6\}$. Then $N(S) = V_2 = \{A, B, C, D, E\}$, so $|S| > |N(S)|$.

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