Discrete Mathematics HW9

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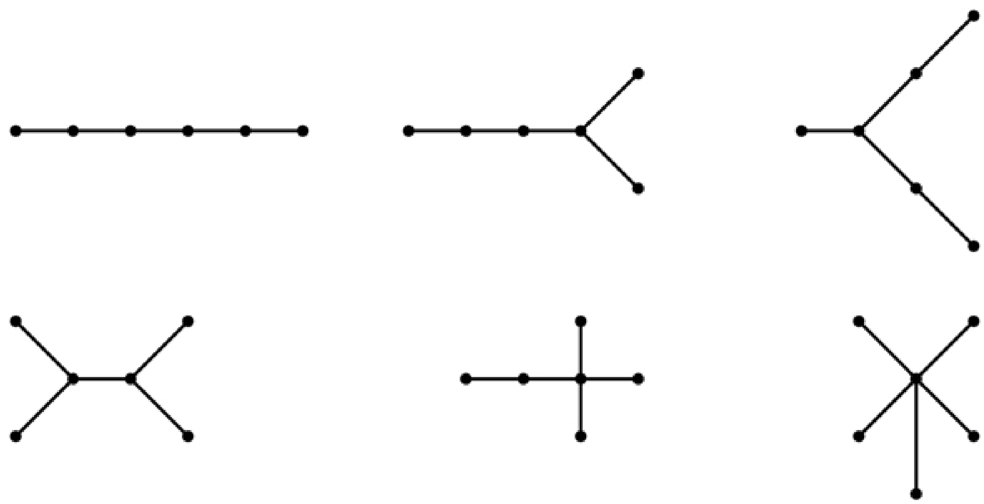
November 17, 2023

Problem 1

Solution.



□

Problem 2

Proof. By the definition of *full m-ary tree*, m should satisfy $m \mid (84 - 1) = m \mid 83$, because all nodes except the root node are grouped into chunks size of m. Since 83 is a prime, $m = 1$ or 83. If $m = 1$, then the height is 83. If $m = 83$, then the height is 1. Both cases are not suitable.

$\square$

Problem 3

(a) *Proof.* By the definition of *full m-ary tree*, m should satisfy $m \mid (81-1) = m \mid 80$.

$$m = 1, 2, 4, 5, 8, 10, 16, 20, 40, 80$$

Consider the number of possible chunks. Let r be the number of all chunks size of m. i.e. $r = \frac{N-1}{m}$, where N is the number of all nodes($N = 84$ in this problem). For a given height $h_i \geq 1$, the chunks in that height range from 1 to m^{h_i-1}(remember the definition of *full m-ary tree*). So for the given maximum height $h(= 4$ in this problem),

$$\begin{aligned}
&\sum_{k=1}^{h} 1 \leq r \leq \sum_{k=1}^{h} m^{k-1} \\
\Rightarrow\ & h \leq r = \frac{N-1}{m} \leq \sum_{k=1}^{h} m^{k-1} \\
\Rightarrow\ & mh \leq N-1 \leq \sum_{k=1}^{h} m^{k}
\end{aligned}$$

By the above inequality, $m \neq 1, 2, 40, 80$. Therefore,

$$m = 4, 5, 8, 10, 16, 20$$

□

(b) *Proof.* By the definition of *balanced full m-ary tree*, all chuncks in height 1 to $h-1$ should be filled and the last height can have the chuncks from 1 to m^{h-1}. So the range is changed into

$$\begin{aligned}
&\sum_{k=1}^{h-1} m^{k-1} + 1 \leq r \leq \sum_{k=1}^{h-1} m^{k-1} + m^{h-1} \\
\Rightarrow\ & \sum_{k=1}^{h-1} m^{k-1} + 1 \leq r = \frac{N-1}{m} \leq \sum_{k=1}^{h} m^{k-1} \\
\Rightarrow\ & \sum_{k=1}^{h-1} m^{k} + m \leq N-1 \leq \sum_{k=1}^{h} m^{k}
\end{aligned}$$

But all of m are impossible.

□

Problem 4

(a) *Solution.* Remove all leaf nodes until base-case. Then the remain is a center. Yellow tree has c, red tree has e as a center. But note that the number of center can be one or two.

$\square$

(b) *Proof.* Proof by (strong) induction. Let $T(k)$ be a tree with k nodes. Consider $T(k=1)$ and $T(k=2)$. Note that it has 2 base-case because of the problem statement. Both cases satisfy the statement clearly, they themselves are centers.
Suppose $k = 1, 2, \cdots, N$ is true. Consider $T(k = N+1)$. Note that $T(k \geq 3)$ has at least one internal node. Remove all leaf nodes in $T(N+1)$. Then it is also a tree with $1 \leq n \leq N$ nodes. By induction, $T(n)$ satisfies the statement. Add all leaf nodes to $T(n)$ while preserving the location, then it is $T(N+1)$ and has the same center as $T(n)$. Therefore, $T(N+1)$ also satisfies the statement.

$\square$

Problem 5

Solution. Skipped. Try it case-by-case. You only find one possible solution if exists.

$\square$

Problem 6

Proof. Let $G = (V, E)$ be the graph. Let $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ be two MSTs of G. If we show $T_1 = T_2$, then the proof is done. Note that consider V and E as sets.
Proof by contradiction. Assume $T_1 \neq T_2$. This means $E_1 \neq E_2$. Since $|E_1| = |V| - 1 = |E_2|$(by the definition of *tree*), $E_1 \neq E_2$ implies $E_1 - E_2 \neq \emptyset$ and $E_2 - E_1 \neq \emptyset$. WLOG, just consider the first-case: $E_1 - E_2 \neq \emptyset$. Let $e \in E_1 - E_2$. We can pick such edge e because of non-empty. Since $e \notin E_2$, the new subgraph $\Gamma_2 = (V, E_2 \cup \{e\})$ has a cycle C. Since all weights in G are distinct, there is a edge $e' \in C$ which has the largest weight. Claim that $e' \notin$ any MST.

> *Proof of the claim.* Proof by contradiction. If $e' \in T$ which is MST, then deleting e' will break T into two subtrees. Since e' is from the cycle C, we can pick another edge f which connects the two subetrees. By the definition of e', $e' > f$. This gives T is not MST because we can make a tree with smaller weight using f. Therefore, $e' \notin T$. □

Note that the above claim is called *cycle property.* Since $e' \in C$ and $C \subseteq E_2 \cup \{e\}$, $e' \in E_2 \cup \{e\}$. Here, we have two cases:

- If $e' = e$, then since $e \in E_1 - E_2 \subset E_1$, $e' \in E_1$. But $T_1 = (V, E_1)$ is a MST, contradicts to the claim.
- If $e' \in E_2$, but then $T_2 = (V, E_2)$ is a MST, contradicts to the claim.

Therefore, our assumption $T_1 \neq T_2$ is false.

□

Problem 7

Proof. Let T be a tree. Pick any $v \in T$ and paint it with a color c_1. Next, paint all the vertices adjecent to v with a color c_2. Again, paint all the vertices adjecent to them using c_1. Continue this until the end. This gives an optimal solution. Because, if not, then there is a connection between odd-step colorings(WOLG). Specifically, there are two cases.

- Connected between nodes in the same step. Let v_1 and v_2 are connected in the same step. Then there is a cycle $v_1 - \cdots - v - \cdots - v_2 - v_1$, contradiction.
- Connected between nodes in different (odd) steps. Let v_1 and v_2 are connected. Then there is a cycle $v_1 - \cdots - v - \cdots - v_2 - v_1$, contradiction.

APPENDIX: This problem shows [Trees are bipartite graphs].

□