Discrete Mathematics HW12

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Problem 1

WARNING: You should only use the basic axioms of *Boolean Algebra*. You cannot use any other law without proof!

$$\text{(Associative law) } (x+y)+z = x+(y+z) \text{ and } x\cdot(y\cdot z) = (x\cdot y)\cdot z \tag{1}$$
$$\text{(Commutative law) } x+y = y+x \text{ and } x\cdot y = y\cdot x \tag{2}$$
$$\text{(Distributive law) } x\cdot(y+z) = (x\cdot y)+(x\cdot z) \text{ and } x+(y\cdot z) = (x+y)\cdot(x+z) \tag{3}$$
$$\text{(Identity element) } x+0 = x \text{ and } x\cdot 1 = x \tag{4}$$
$$\text{(Inverse) } x+x' = 1 \text{ and } x\cdot x' = 0 \tag{5}$$

(a) *Proof.*

$$\begin{aligned} x+x &= 1\cdot(x+x) && \text{by (3)} \\ &= (x+x')\cdot(x+x) && \text{by (5)} \\ &= x+x\cdot x' && \text{by (3)} \\ &= x+0 && \text{by (5)} \\ &= x && \text{by (4)} \end{aligned}$$

$$\begin{aligned} x\cdot x &= x\cdot x+0 && \text{by (4)} \\ &= x\cdot x+x\cdot x' && \text{by (5)} \\ &= x\cdot(x+x') && \text{by (3)} \\ &= x\cdot 1 && \text{by (5)} \\ &= x && \text{by (4)} \end{aligned}$$

□

(b) *Proof.*

$$\begin{aligned} x+1 &= x+(x+x') && \text{by (5)} \\ &= (x+x)+x' && \text{by (1)} \\ &= x+x' && \text{by (a)} \\ &= 1 && \text{by (5)} \end{aligned}$$

$$\begin{aligned} x\cdot 0 &= x\cdot(x\cdot x') && \text{by (5)} \\ &= (x\cdot x)\cdot x' && \text{by (1)} \\ &= x\cdot x' && \text{by (a)} \\ &= 0 && \text{by (5)} \end{aligned}$$

□

(c) *Proof.*

$$\begin{aligned} x + x \cdot y &= x \cdot 1 + x \cdot y && \text{by (4)} \\ &= x \cdot (1 + y) && \text{by (3)} \\ &= x \cdot 1 && \text{by (b)} \\ &= x && \text{by (4)} \end{aligned}$$

$$\begin{aligned} x \cdot (x + y) &= (x + 0) \cdot (x + y) && \text{by (5)} \\ &= x + 0 \cdot y && \text{by (3)} \\ &= x + 0 && \text{by (b)} \\ &= x && \text{by (4)} \end{aligned}$$

□

(d) *Proof.*

$$\begin{aligned} (x')' &= (x')' + 0 && \text{by (4)} \\ &= (x')' + x \cdot x' && \text{by (5)} \\ &= ((x')' + x) \cdot ((x')' + x') && \text{by (3)} \\ &= ((x')' + x) \cdot 1 && \text{by (5)} \\ &= ((x')' + x) \cdot (x + x') && \text{by (5)} \\ &= (x + (x')') \cdot (x + x') && \text{by (2)} \\ &= x + (x')' \cdot x' && \text{by (3)} \\ &= x + 0 && \text{by (5)} \\ &= x && \text{by (4)} \end{aligned}$$

□

(e) *Proof.*

$$\begin{aligned} 1' &= 1' \cdot 1 && \text{by (4)} \\ &= 0 && \text{by (5)} \end{aligned}$$

$$\begin{aligned} 0' &= (1')' && \text{by (e)} \\ &= 1 && \text{by (d)} \end{aligned}$$

□

(f) *Proof.* Note that the Inverse law(5) implies

If $x + y = 1$ and $x \cdot y = 0$, then $x' = y$

We use this.

$$
\begin{aligned}
(x+y)+x'\cdot y' &= (x+y)\cdot 1 + x'\cdot y' && \text{by (4)}\\
&= (x+y)\cdot(x+x') + x'\cdot y' && \text{by (5)}\\
&= (x+y\cdot x') + x'\cdot y' && \text{by (3)}\\
&= x + (y\cdot x' + x'\cdot y') && \text{by (1)}\\
&= x + (x'\cdot y + x'\cdot y') && \text{by (2)}\\
&= x + x'\cdot(y+y') && \text{by (3)}\\
&= x + x'\cdot 1 && \text{by (5)}\\
&= x + x' && \text{by (4)}\\
&= 1 && \text{by (5)}
\end{aligned}
$$

$$
\begin{aligned}
(x+y)\cdot(x'\cdot y') &= x\cdot(x'\cdot y') + y\cdot(x'\cdot y') && \text{by (3)}\\
&= x\cdot(x'\cdot y') + y\cdot(y'\cdot x') && \text{by (2)}\\
&= (x\cdot x')\cdot y' + (y\cdot y')\cdot x' && \text{by (1)}\\
&= 0\cdot y' + 0\cdot x' && \text{by (5)}\\
&= 0+0 && \text{by (b)}\\
&= 0 && \text{by (a)}
\end{aligned}
$$

Therefore, $(x+y)' = x'\cdot y'$. Similarly,

$$
\begin{aligned}
x\cdot y + (x'+y') &= x\cdot y + (x'+y')\cdot 1 && \text{by (4)}\\
&= x\cdot y + (x'+y')\cdot(x'+x) && \text{by (5)}\\
&= x\cdot y + (x' + x\cdot y') && \text{by (3)}\\
&= (x\cdot y + x') + x\cdot y' && \text{by (1)}\\
&= (x' + x\cdot y) + x\cdot y' && \text{by (2)}\\
&= x' + (x\cdot y + x\cdot y') && \text{by (1)}\\
&= x' + x\cdot(y+y') && \text{by (3)}\\
&= x' + x\cdot 1 && \text{by (5)}\\
&= x' + x && \text{by (4)}\\
&= 1 && \text{by (5)}
\end{aligned}
$$

$$
\begin{aligned}
(x\cdot y)\cdot(x'+y') &= (x\cdot y)\cdot x' + (x\cdot y)\cdot y' && \text{by (3)}\\
&= (y\cdot x)\cdot x' + (x\cdot y)\cdot y' && \text{by (2)}\\
&= y\cdot(x\cdot x') + x\cdot(y\cdot y') && \text{by (1)}\\
&= y\cdot 0 + x\cdot 0 && \text{by (5)}\\
&= 0+0 && \text{by (b)}\\
&= 0 && \text{by (a)}
\end{aligned}
$$

Therefore, $(x\cdot y)' = x'+y'$.

$\square$

Problem 2

Solution.

$$\begin{aligned} f(w,x,y,z) = {} & (w \wedge x \wedge y \wedge z) \vee (w \wedge x \wedge \neg y \wedge z) \vee (w \wedge \neg x \wedge \neg y \wedge \neg z) \\ & \vee (\neg w \wedge x \wedge y \wedge z) \vee (\neg w \wedge \neg x \wedge y \wedge z) \end{aligned}$$

□

Problem 3

Proof. Let F be DNF of f. Then $F = m_1 \vee m_2 \vee \cdots \vee m_k$, where m_k is conjunctive form. Since $\neg m_k$ is disjunctive form and $\neg\vee = \wedge$, $\neg F$ is CNF.
The CNF form of the above problem is

$$\begin{aligned} f(w,x,y,z) = {} & (\neg w \vee \neg x \vee \neg y \vee z) \wedge (\neg w \vee \neg x \vee y \vee z) \wedge (\neg w \vee x \vee \neg y \vee \neg z) \\ & \wedge (\neg w \vee x \vee \neg y \vee z) \wedge (\neg w \vee x \vee y \vee z) \wedge (w \vee \neg x \vee \neg y \vee z) \wedge (w \vee \neg x \vee y \vee \neg z) \\ & \wedge (w \vee \neg x \vee y \vee z) \wedge (w \vee x \vee \neg y \vee z) \wedge (w \vee x \vee y \vee \neg z) \wedge (w \vee x \vee y \vee z) \end{aligned}$$

□

Problem 4

Proof. Note that the definition of NOR is 'not OR', i.e.

$$x \downarrow y = \neg(x \vee y)$$

First, claim that NOT is expressed by only NOR.

$$\neg p = \begin{cases} 0 & \text{if } p = 1 \\ 1 & \text{if } p = 0 \end{cases} = \begin{cases} p \downarrow p & \text{if } p = 1 \\ p \downarrow p & \text{if } p = 0 \end{cases} = p \downarrow p$$

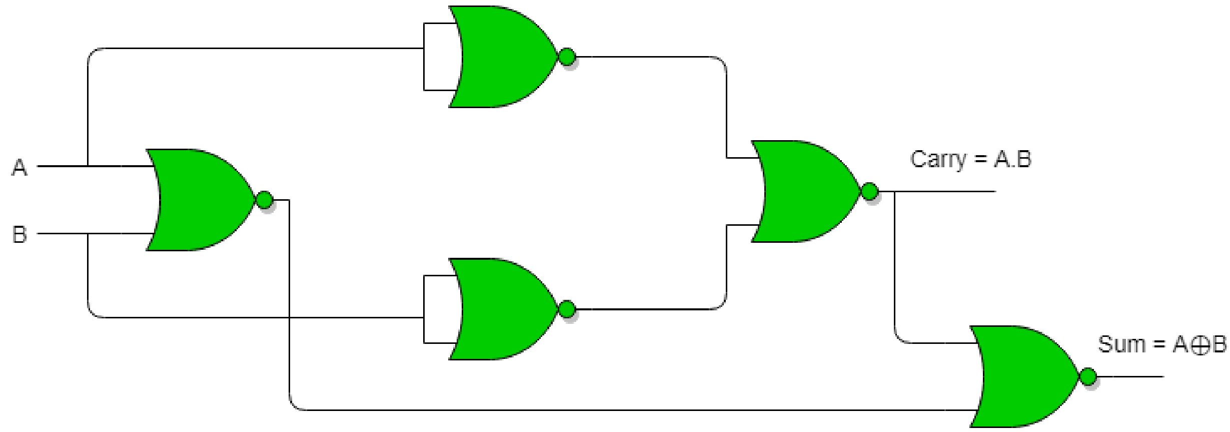
Next, claim that OR is expressed by only NOR.

$$\begin{aligned} p \vee q &= \neg\neg(p \vee q) = \neg(\neg(p \vee q)) \\ &= \neg(p \downarrow q) = (p \downarrow q) \downarrow (p \downarrow q) \end{aligned}$$

Therefore, since {OR, NOT} is functionally complete, {NOR} is functionally complete. $\square$

Problem 5

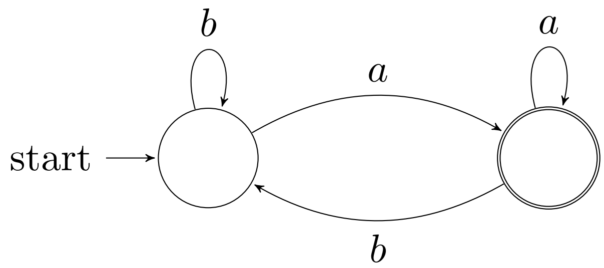
Solution. You can check the below circuit by writing all truth tables.



□

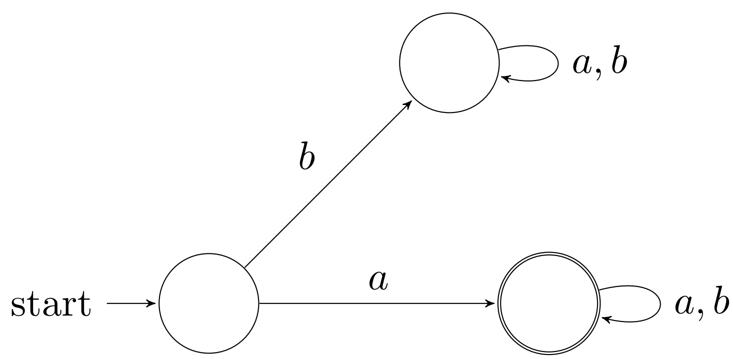
Problem 6

(a) *Solution.*



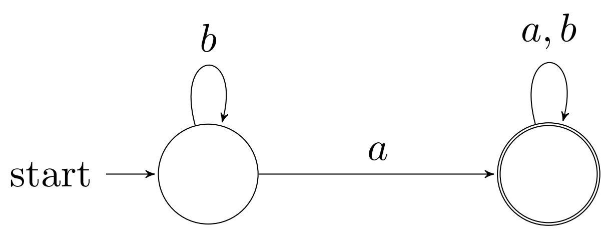
□

(b) *Solution.*



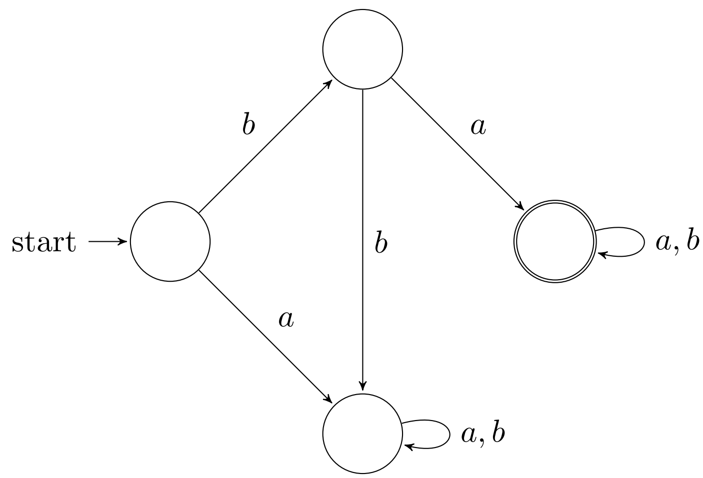
□

(c) *Solution.*



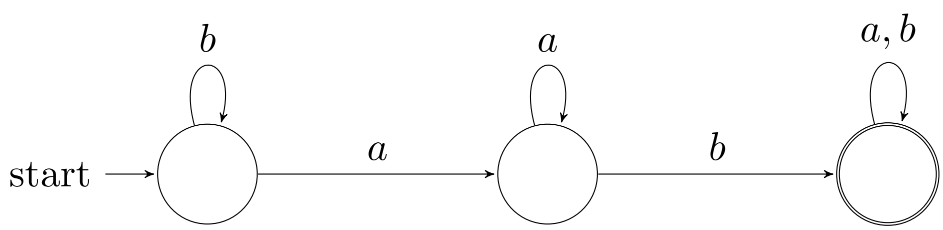
□

(d) *Solution.*



□

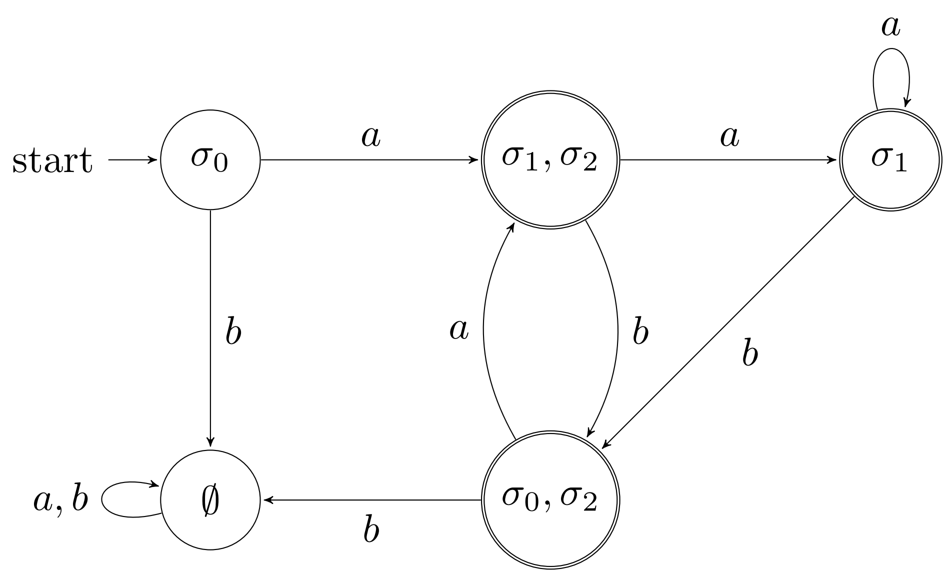
(e) *Solution.*



□

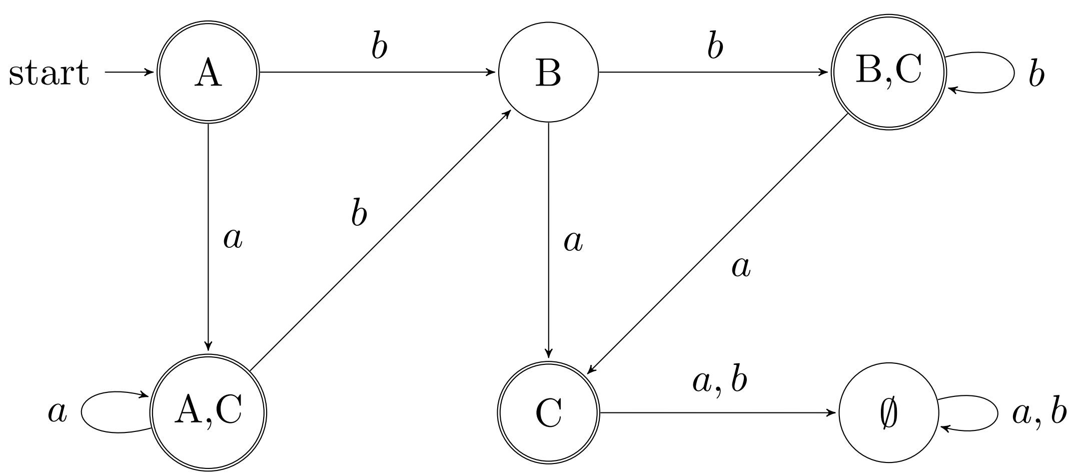
Problem 7

(a) *Solution.*



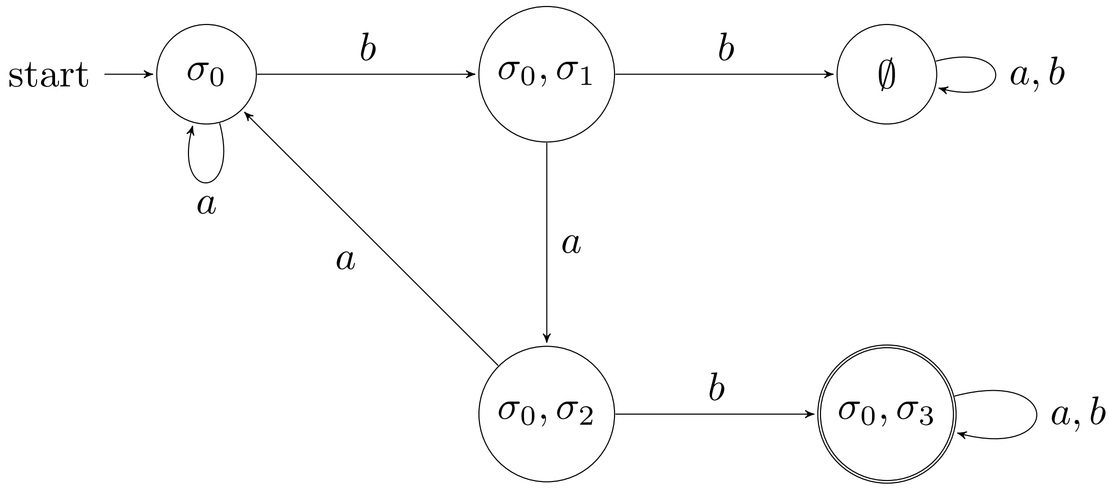
□

(b) *Solution.*



□

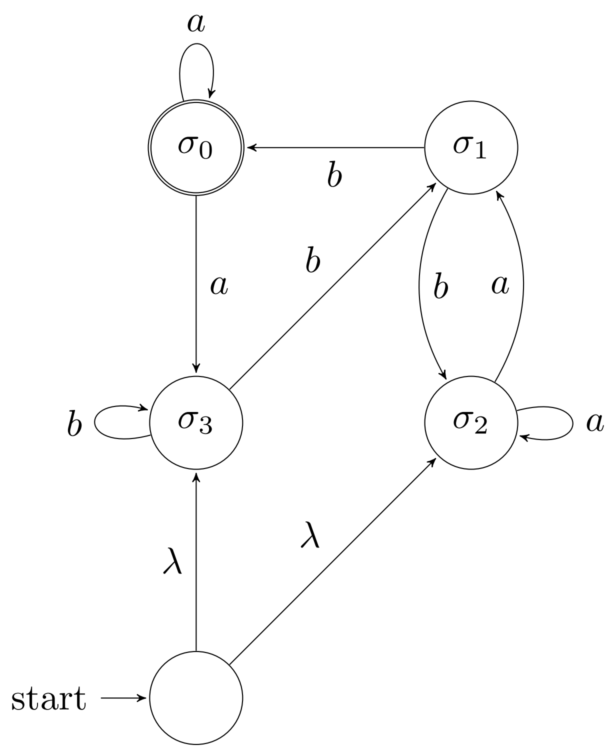
(c) *Solution.*



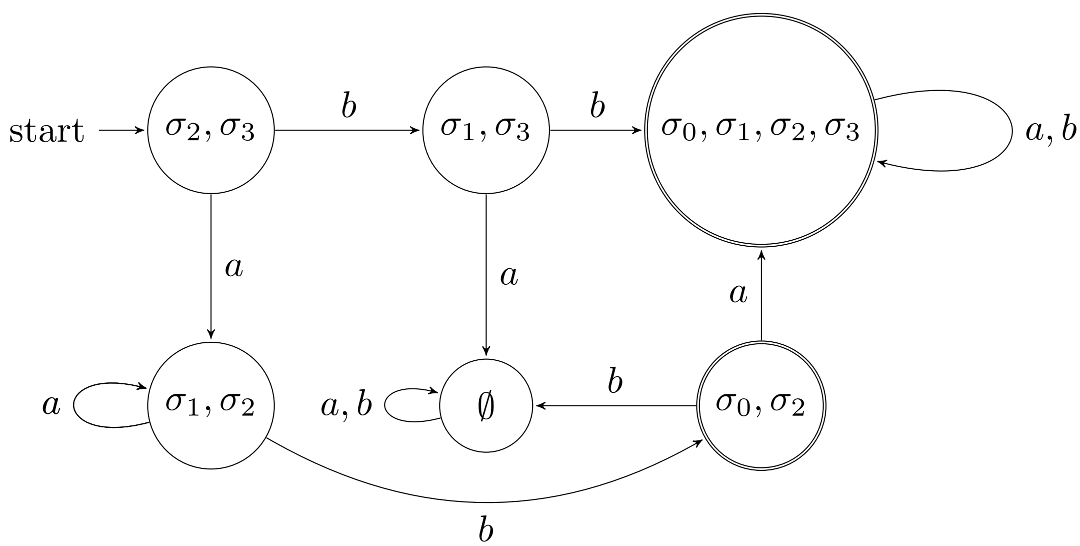
$\square$

Problem 8

Solution. This is non-deterministic finite-state automata(NFA).



Here, we use λ transition. This means transit the input state to connected states without changing input value. Thus, the deterministic finite-state automata(DFA) is



∎

Problem 9

Proof. Assume L is a regular language. Fix $m \in \mathbb{Z}^+$ and let $w = a^m b a^m$. Then $w \in L$ with $|w| \geq m$. By *Pumping lemma*, $w = xyz$ with $|y| \geq 1$, $|xy| \leq m$, and $w_i = xy^i z \in L$ for $^\forall i \geq 0$.
Since $|xy| \leq m$ and w starts with a^m, y consists of only a. Let $|y| = k$ where $1 \leq k \leq m$. Then $w_0 = a^{m-k} b a^m$. But $w_0 \notin L$. This contradicts to the first assumption. Therefore, L is not a regular language.
WARNING: You use *Pumping lemma* only to prove L is not a regular language, NOT to prove L is a regular language. Also, the division form should not be fixed. This lemma does not tell the form. It means, if you claim that the statement is false because one of the xyz form does not satisfy *Pumping lemma*, then you will get 0 point. Therefore, if you want to use *Pumping lemma*, then use the following strategy:

- Choose any $w \in L$ with $|w| \geq m$. This choice does not deduct your points.
- Choose any i what you want or easy to prove. Its value does not have to be fixed to one value.
- Show that for any division form xyz, $w_i \notin L$ with your choice of w and i.

$\square$

Problem 10

Proof. Note that regular language is closed under concatenation: [if L_1 and L_2 are regular languages, then L_1L_2 is a regular language]. Thus, L^n is a regular language for $n \geq 1$. Also note that the kleene star notation is defined as $L^* = \{\epsilon\} \cup L \cup L^2 \cup \cdots$, where ϵ is empty expression. Since regular language is closed under union, L^* is a regular language(it is also true that regular language is closed under kleene star). The given language is the same as L^*, it is a regular.

APPENDIX: It is useful to prove L is a regular language. For regular language L_1 and L_2, followings are also regular(we say that closed under $\sim\sim$).

- $L_1 \cup L_2$(union)
- $L_1 \cap L_2$(intersection)
- L_1L_2(concatenation)
- $\overline{L_1}$(complementation)
- L_1^*(kleene star)
- L_1^R(reverse)
- L_1/L_2(right quotient)

□